CMPS 130 Homework 1

Textbook Exercises

0.1 a. The set of odd natural numbers
b. The set of even integers
c. The set of even natural numbers
d. The set of natural numbers divisible by 6
e. The set of palindromes over the alphabet $\{0,1\}$
f. The empty set

0.2 a. $\{1, 10, 100\}$
b. $\{n \in \mathbb{Z} \mid n > 5\}$ or $\{6, 7, 8, \ldots\}$
c. $\{n \in \mathbb{N} \mid n < 5\}$ or $\{1, 2, 3, 4\}$
d. $\{aba\}$
e. $\{\varepsilon\}$
f. $\{\}$ or $\emptyset$

0.3 a. No
b. Yes
c. $A \cup B = \{x, y, z\} = A$
d. $A \cap B = \{x, y\} = B$
e. $A \times B = \{(x, x), (x, y), (y, x), (y, y), (z, x), (z, y)\}$
f. $\{\{\}, \{x\}, \{y\}, \{x, y\}\}$

0.4. $|A \times B| = |A| \cdot |B| = ab$

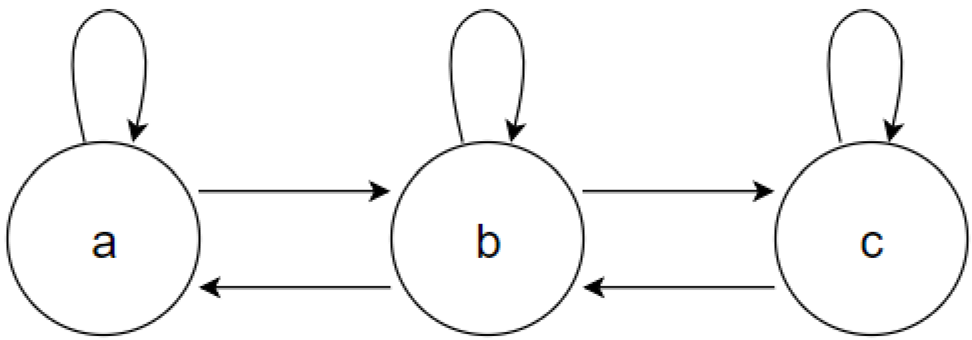
0.5. $|\mathcal{P}(C)| = 2^{|C|} = 2^{c}$

0.6 a. $f(2) = 7$
b. Domain $= X = \{1, 2, 3, 4, 5\}$
Range $= \{6, 7\}$
(Note: Codomain $= Y = \{6, 7, 8, 9, 10\}$
c. $g(2, 10) = 6$

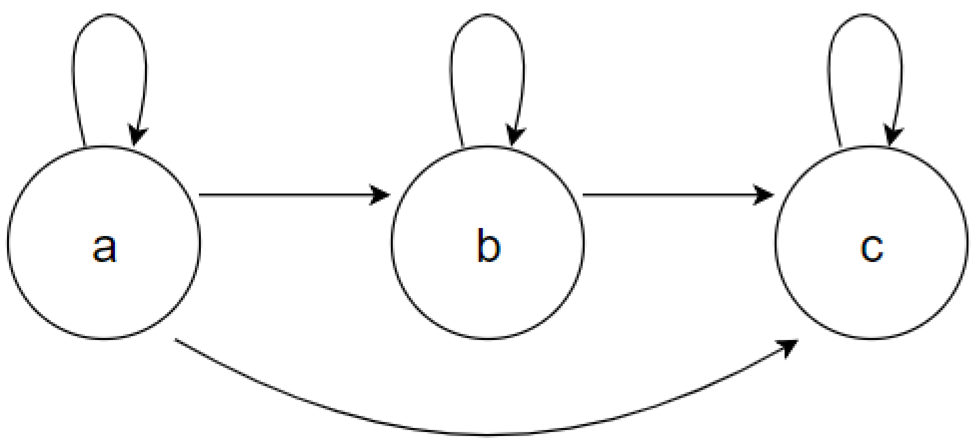
d. Domain $= X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$
Range $= Y = \{6, 7, 8, 9, 10\}$
(Note: Range = Codomain in this case}

e. $g(4, f(4)) = g(4, 7) = 8$

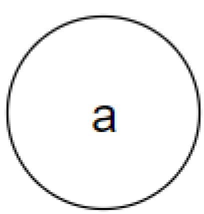
0.7 a.



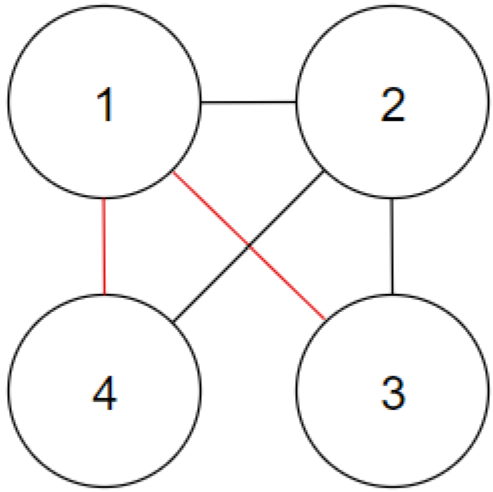
b.



c.



0.8.



n	deg(n)
1	3
2	3
3	2
4	2

One possible path from 3 to 4 is highlighted in red.

0.9. $G = (V, E)$
$V = \{1, 2, 3, 4, 5, 6\}$
$E = \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}\}$

0.10. The flaw in this proof occurs when both sides of the equation are divided by (a-b). You cannot do this when a=b, because then you would be dividing by 0.

0.12. The flaw in this proof occurs during the assertion that H_1 containing horses of the same color and H_2 containing horses of the same color implies that H contains horses of the same color. In the case where H has 2 horses, H_1 contains only one horse and H_2 contains the other horse. Subsets H_1 and H_2 clearly only have one color each, but do not necessarily have to be the same color.

0.13. Every (undirected) graph with two or more nodes contains two nodes that have equal degrees.
Proof:

- The maximum degree of a node is $|V| - 1$ (a node that is connected to every other node in the graph).
- The minimum degree of a node is 0 (a node that is not connected to any other node).
- It is impossible for a graph to have a node with degree $|V| - 1$ and a node with degree 0. A node that is connected to every other node in the graph implies that there is no node that is not connected to any other node and vice versa.
- The degrees of each node can be in the range $\{1, 2, 3, \ldots, |V| - 1\}$ or $\{0, 1, 2, \ldots, |V| - 2\}$. In either case, there are at most $|V| - 1$ possible degrees.
- Since there are $|V|$ nodes and at most $|V| - 1$ possible degrees, the pigeonhole principle tells us that at least two nodes in a graph have equal degrees.

Extra Problems

- $(\overline{A \cap B}) = (\overline{A} \cup \overline{B})$
  Proof:
  - $(\overline{A \cap B}) \subseteq (\overline{A} \cup \overline{B})$
    If x is an element of $(\overline{A \cap B})$, then x is not in $(A \cap B)$, by the definition of the complement of a set. If x is not in A and B (definition of the intersection of two sets), then x is either not in A, or not in B. In other words, x is an element of $(\overline{A} \cup \overline{B})$.
  - $(\overline{A} \cup \overline{B}) \subseteq (\overline{A \cap B})$
    If x is an element of $(\overline{A} \cup \overline{B})$, then x is an element of $\overline{A}$ or $\overline{B}$, by the definition of the union of two sets. If x is not in A or not in B (definition of the complement of a set), then x is not in A and B. In other words, x is an element of $(\overline{A \cap B})$.
  - Since $(\overline{A \cap B}) \subseteq (\overline{A} \cup \overline{B})$ and $(\overline{A} \cup \overline{B}) \subseteq (\overline{A \cap B})$, $(\overline{A \cap B}) = (\overline{A} \cup \overline{B})$.

- The set of odd natural numbers is countable

$n \in \mathbb{N}$	$m \in$ odd natural numbers
1	1
2	3
3	5
⋮	⋮

  $m = 2n - 1$
  (The above shows that the set of odd natural numbers has a one to one correspondence with the natural numbers, and is therefore countable)

- The set of odd integers is countable

$n \in \mathbb{N}$	$m \in$ odd integers
1	1
2	-1

3	3
4	-3
5	5
6	-5
⋮	⋮

$$m = \begin{cases} n, & \text{if n is odd} \\ 1 - n, & \text{if n is even} \end{cases}$$

(The above shows that the set of odd integers has a one to one correspondence with the natural numbers, and is therefore countable)

- For all $n \in \mathbb{N}$, $\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$
  Proof (by induction):
  - Base case ($n = 1$): $\overline{A_1} = \overline{A_1}$
  - Induction hypothesis: Assume for $k \geq 1$ that $\overline{\bigcap_{i=1}^{k} A_i} = \bigcup_{i=1}^{k} \overline{A_i}$
  - Inductive step: We now need to show that $\overline{\bigcap_{i=1}^{k+1} A_i} = \bigcup_{i=1}^{k+1} \overline{A_i}$
    - $\overline{\bigcap_{i=1}^{k+1} A_i} = \overline{S_k \cap A_{k+1}}$ where $S_k = \bigcap_{i=1}^{k} A_i$
    - $\overline{S_k \cap A_{k+1}} = \overline{S_k} \cup \overline{A_{k+1}}$ by De Morgan's law for two sets (proved earlier in this homework)
    - $\overline{S_k} \cup \overline{A_{k+1}} = \bigcup_{i=1}^{k} \overline{A_i} \cup \overline{A_{k+1}}$ by the induction hypothesis
    - $\bigcup_{i=1}^{k} \overline{A_i} \cup \overline{A_{k+1}} = \bigcup_{i=1}^{k+1} \overline{A_i}$

- Every non-empty finite set of positive integers contains a least element.
  Proof (by induction):
  - Base case: If a set has one element, that element is the least element of the set.
  - Induction hypothesis: Assume that any set with k integers has a least element.
  - Inductive step: We must now show that a set with k+1 integers (S_{k+1}) contains a least element. Take any subset of

size k (S_k) from this set. By the induction hypothesis, this subset has a least element (least(S_k)). Now compare this least element with the element not included in this subset (x). Because elements of a set are unique and unique integers are either less than or greater than each other, either least(S_k) or x are smaller than the other.

- Case 1: least(S_k) $<$ x
  In this case, least(S_k) is the least element of S_{k+1} because it is smaller than every other element in S_{k+1}.
- Case 2: x $<$ least(S_k)
  In this case, x is the least element of S_{k+1} because the transitivity of the $<$ relation implies that it is smaller than every other element in S_{k+1}.