TMA: LaTeX101-sample

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Update for 2024/11/08 v1.12

This document is provided as an example of how to use the **tma** package. With the 8th November 2024 version of the package there has been a rewrite of the majority of the package along with the renaming of a couple of the commands. `\C`, representing the set of complex numbers, has been renamed to `\Complex`, and `\vec{}`, for setting traditional vectors, has been renamed to `\vect{}`. In both instances, the renaming has been done to avoid naming clashes from other packages.
To use the options start your document with:

```
\documentclass[a4paper,12pt]{article}
\usepackage[OPTION]{tma}

\myname{...
```

Where **OPTION** is one of the following

[**roman**] Questions numbered as 1, 1(i), 1(i)(a)...

[**alph**] *Default* Questions numbered as 1, 1(a), 1(a)(i)...

The 26th October 2016 version of the package added two further options which allow extended referencing and the adding of bookmarks for questions into pdf output.

[**cleveref**] Prepares the questions, qparts and subqparts to be referenced using the `cleveref` package.

[**pdfbookmark**] Automatically adds pdf bookmarks for each question, qpart and subqpart. This uses the `hyperref` package.

The 8th November 2024 version added one further option.

[**legacy**] reactivate the old conflicting commands of `\C` and `\vec`. This is intended for use **only** when using the current package with an old document.

See the `notes.pdf` file for further details of available options.

Q 1

(i) We have $1 = 10^0$ and $1+2+3+4 = 10^1$. Prove that there are no other powers of ten which are the sum of the first n integers.

We have:

$$\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$$

Let

$$\frac{(n)(n+1)}{2} = 10^x$$
$$\Rightarrow (n)(n+1) = 2^{x+1}5^x$$

Now, either n is odd, or $n+1$ is odd.

Consider the case where n is odd:
By the Fundamental Theorem of Arithmetic, n can only have the prime factors 2 or 5. Since it is odd, it can only be a perfect power of 5. Now, $n+1$ also can only have the prime factors of 2 or 5. If n is divisible by 5, then $n+1$ is not divisible by 5. Therefore $n+1$ is a perfect power of 2. Therefore:

$$n = 5^x \qquad \text{and} \qquad n+1 = 2^{x+1}$$
$$\Rightarrow x = 0$$

(for any higher x, $5^x \gg 2^{x+1}$)

$$\Rightarrow n = 1$$

Now consider the case where $n+1$ is odd:
By similar arguments to above, $n+1$ must be a perfect power of 5 and n must be a perfect power of 2.

$$n = 2^{x+1} \qquad \text{and} \qquad n+1 = 5^x$$
$$\Rightarrow x = 1$$

(for any higher x, $5^x \gg 2^{x+1}$)

$$\Rightarrow n = 4$$

Therefore $n = 1$ and $n = 4$ are the only solutions to the original problem. ☻

(iii) (a) Show that:

$$\sum_{x=1}^{n} x(x+1) = \frac{n(n+1)(n+2)}{3}$$

Let

$$f(n) = \frac{n(n+1)(n+2)}{3}$$

Now, adding the n+1 term to the above

$$\begin{aligned} f(n) + (n+1)(n+2) &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \\ &= \frac{(n^3 + 3n^2 + 2n+)}{3} + n^2 + 3n + 2 \\ &= \frac{(n^3 + 6n^2 + 11n + 6)}{3} \\ &= \frac{((n+1)(n+2)(n+3))}{3} \\ &= f(n+1) \end{aligned}$$

Therefore, if $f(n)$ is valid, then so is $f(n+1)$.

Since $1 \times 2 = 2 = \frac{1\times2\times3}{3} = f(1)$, then $f(n)$ is valid for all $n \geq 1$. ☻

(b) Show that:

$$\sum_{x=1}^{n} x^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Let

$$f(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Now, adding the n+1 term to the above

$$\begin{aligned}
f(n)+(n+1)^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}+(n+1)^4 \\
&= \frac{1}{30}(6n^5+15n^4+10n^3-n)+(n^4+4n^3+6n^2+4n+1) \\
&= \frac{1}{30}(6n^5+15n^4+10n^3-n+30n^4+120n^3+180n^2+120n+30) \\
&= \frac{1}{30}(6n^5+45n^4+130n^3+180n^2+119n+30) \qquad (1.1)
\end{aligned}$$

Now,

$$\begin{aligned}
f(n+1) &= \frac{(n+1)\big((n+1)+1\big)\big(2(n+1)+1\big)\big(3(n+1)^2+3(n+1)-1\big)}{30} \\
&= \frac{1}{30}(n+1)(n+2)(2n+3)(3n^2+9n+5) \\
&= \frac{1}{30}(6n^5+45n^4+130n^3+180n^2+119n+30) \qquad (1.2)
\end{aligned}$$

Comparing equation (1.1) with equation (1.2) we see that

$$f(n)+(n+1)^4=f(n+1)$$

Therefore, if $f(n)$ is valid, then so is $f(n+1)$.

Since $1^4=1=\frac{1\times2\times3\times5}{30}=f(1)$, then $f(n)$ is valid for all $n\geq 1$. ☻

Q 3

Find the general solution of the equation

$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+y=x^2 \qquad (3.1)$$

The auxillary equation is

$$3\lambda^2+4\lambda+1=0 \qquad (3.2)$$

which factorises to

$$(\lambda+1)(3\lambda+1)=0 \qquad (3.3)$$

and so has solutions

$$\lambda=-1 \text{ and } \lambda=-\frac{1}{3} \qquad (3.4)$$

As both roots are real and distinct, the complementary function is

$$y_c=C\mathrm{e}^{-x}+D\mathrm{e}^{-\frac{1}{3}x} \qquad (3.5)$$

Now, let us find the particular integral. As the right hand side of equation 3.1 is x^2, our trial solution is the polynomial

$$y_p = px^2 + qx + r \tag{3.6}$$

$$\Rightarrow \frac{\mathrm{d}y_p}{\mathrm{d}x} = 2px + q \tag{3.7}$$

$$\Rightarrow \frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} = 2p \tag{3.8}$$

Substituting the trial particular integral into equation 3.1

$$6p + 8px + 4q + px^2 + qx + r = x^2 \tag{3.9}$$

$$\Rightarrow px^2 + (8p + q)x + (6p + 4q + r) = x^2 \tag{3.10}$$

$$\Rightarrow p = 1, \quad q = -8, \quad r = 26 \tag{3.11}$$

Thus the particular integral is

$$y_p = x^2 - 8x + 26 \tag{3.12}$$

and combining equation 3.5 with equation 3.12, by the rule of superposition, we get the general solution of equation 3.1 to be

$$y = C\mathrm{e}^{-x} + D\mathrm{e}^{-\frac{1}{3}x} + x^2 - 8x + 26 \tag{3.13}$$