The Mitchell-Freyd Embedding Theorem

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1 Introduction

One of the most fruitful areas of study in mathematics lies in the study of module theory. The study of such objects has been extremely fruitful in the study of ring theory, cohomology (and consequently homology) theories, algebraic geometry, general homological algebra, and even in analysis through the study of D-modules and Lie algebra cohomology. The far reaches and usefulness of the study of modules suggests that there is some categorical structure underlying the theory of modules that may explain their widespread utility. What is readily seen to be important about modules is that if we fix a ring R, then the collection of all left (and dually all right) R modules forms a category when we take the morphisms to be R-linear transformations. A further study of these categories, as done in [Freyd], shows that all such categories satisfy some properties that make them behave "categorically analogously" to the category of Abelian groups, which we will write as **Ab**. Taking these properties and generalizing them appropriately leads to the study of Abelian categories, which has been extremely useful, among other places, in the study of algebraic geometry and the study of sheaves of $\mathcal{O}_X$ modules, where $X = (X, \mathcal{O}_X)$ is a scheme and $\mathcal{O}_X$ is the structure sheaf of X. However, upon generalizing to these categories, we must worry about whether or not we have generalized so far that the methods and "flavor" of studying modules may be preserved in studying Abelian categories.

The whole purpose of this paper is to provide the perhaps surprising result that tells us that we have not generalized "too far" at all! We will be presenting as a culmination of our efforts the Mitchell-Freyd Embedding Theorem, which states simply that there is an exact full and faithful functor from any small Abelian category into the category **R-Mod** for some unital but not necessarily commutative ring. This result is important, for two reasons: (1) if one is content to work with only small Abelian categories, they may think of them as "module subcategories," and (2) that if one is only interested in a finite diagram, they can embed it into module categories and chase elements around the diagram to their heart's content. With this in mind let us proceed to explore the wonder of the Embedding Theorem by first understanding where Abelian categories come from.

2 Preadditive Categories and Additive Categories: Leading up to Abelian Categories

We begin our exploration of the Embedding Theorem by first looking at the categories that, while slightly more general than the Abelian categories we will study, are important for founding Abelian categories. The first of these are what are called **Ab**-categories, or if we follow [MacLane$_2$], are called preadditive categories. They will play an analogous role to the theory of Abelian categories as what a nonunital rig would play to the theory of a unital ring.

Definition 2.1. *Let $\mathfrak{C}$ be a category. Then if for every pair of objects $A, B \in \mathrm{Ob}(\mathfrak{C})$ the hom-set $\mathfrak{C}(A, B)$ may be given the structure of an Abelian group with bilinear composition (i.e. whenever we have the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

then we have $f(g+g')h = fgh + fg'h$) we say that $\mathfrak{C}$ is a preadditive category*.*

It is worth remarking that the above definition may be simply restated as *preadditive categories are* **Ab***-enriched categories*. While that terminology is more concise and arguably more elegant, we have chosen to follow the terminology used in [MacLane$_2$] because it more clearly indicates the relation of **Ab**-categories with additive categories and Abelian categories.

Example 2.1. Let A be an Abelian group and let $\mathfrak{A}$ be a category with one object in which the hom-set $\mathfrak{A}(*,*)$ may be made into the Abelian group A. Then $\mathfrak{A}$ is a preadditive category. Note that if A is a group with cardinality more than 1 that $\mathfrak{A}$ does not have an initial or terminal object.

Example 2.2. In the same way we can thing of a group G as a category with one object $\mathbb{G}$, we can think of a unital ring as a preadditive category $\mathfrak{R}$ as follows. Let $\mathfrak{R}$ have a single object $*$ and hom-set $\mathfrak{R}(*,*) = R$. Then $\mathfrak{R}(*,*)$ is a monoid with respect to composition, and so we interpret the composition of maps as multiplication in R. Then the addition in R makes $\mathfrak{R}(*,*)$ into an Abelian group and hence $\mathfrak{R}$ is a preadditive category. Note that $\mathfrak{R}$ has no zero object if $|R| \geq 2$.

While these categories appear in nature, they do not have all the structure that we insist upon. In particular, these categories do not necessarily have zero objects; for instance, writing a ring R as a category $\mathfrak{R}$ is an example of these when the cardinality of R is greater than one because there are nonunique maps $r : * \to *$. Because modules have "the" zero module, we require a little more structure than simply allowing our categories to be **Ab**-enriched.

Definition 2.2. *A* zero object *in a category $\mathfrak{C}$ is an object Z such that Z is both initial and terminal.*

Definition 2.3. *A category $\mathfrak{A}$ is said to be* additive *if $\mathfrak{A}$ is preadditive, $\mathfrak{A}$ has a zero object (which we will denote as 0 or Z depending on context), and $\mathfrak{A}$ has finite products.*

It is immediate from the definitions given above that the zero map

$$\begin{tikzcd} A \arrow[rr, "0"] \arrow[dr, "\exists! 0_A"'] & & B \\ & 0 \arrow[ur, "\exists! 0_B"'] & \end{tikzcd}$$

is the additive identity in any of the hom-sets $\mathfrak{A}(A,B)$ for any objects $A, B \in \mathrm{Ob}(\mathfrak{A})$. One worrying aspect of the definition we have given, especially to those familiar with module categories, is that we have not insisted upon any biproduct structure whatsoever. Worry not, however, as Abelian categories do have coproducts; in fact, they are "direct sums" in the sense that they are finite biproducts in any additive category $\mathfrak{A}$. We will even construct the (finite) coproducts in $\mathfrak{A}$ by building them first as a finite biproduct (and by following the proofs outlined in [Hilton]).

Proposition 2.1 (Lemma II.9.2 of [Hilton])**.** *Let $A, B \in \mathrm{Ob}(\mathfrak{A})$ and let $\iota_0 : A \to A \times B$ and $\iota_1 : B \to A \times B$ be maps such that the diagrams*

$$\begin{tikzcd} A \arrow[r, "\iota_0"] \arrow[dr, equal] & A \times B \arrow[d, "\pi_0"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\iota_1"] \arrow[dr, equal] & A \times B \arrow[d, "\pi_B"] \\ & B \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, "\iota_0"] \arrow[dr, "0"'] & A \times B \arrow[d, "\pi_1"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\iota_1"] \arrow[dr, "0"'] & A \times B \arrow[d, "\pi_0"] \\ & A \end{tikzcd}$$

commute in $\mathfrak{A}$. Then $\pi_0\iota_0 + \pi_1\iota_1 = \mathrm{id}_{A\times B}$.

Proof. We first calculate that by the bilinearity of composition in $\mathfrak{A}$ we have

$$(\pi_0\iota_0 + \pi_1\iota_1)\pi_0 = \pi_0\iota_0\pi_0 + \pi_1\iota_1\pi_0 = \pi_0 + 0 = \pi_0$$

because $\iota_0\pi_0 = \mathrm{id}_A$ and $\iota_1\pi_0 = 0$. A similar calculation shows that $(\pi_0\iota_0+\pi_1\iota_1)\pi_1 = \pi_1$. Write $\rho = \pi_0\iota_0+\pi_1\iota_1$ for the purpose of readability and note that the calculations we have done then produce the commuting diagram:

$$\begin{tikzcd}
 & A\times B \arrow[dl, bend right, "(\pi_0\iota_0+\pi_1\iota_1)\pi_0"'] \arrow[d, shift right, "\mathrm{id}_{A\times B}"'] \arrow[d, shift left, "\rho"] \arrow[dr, bend left, "(\pi_0\iota_0+\pi_1\iota_1)\pi_1"] & \\
A & A\times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B
\end{tikzcd}$$

Hence by the universal property of the product, $\pi_0\iota_0 + \pi_1\iota_1 = \mathrm{id}_{A\times B}$. ☐

Proposition 2.2 (Proposition II.9.1 of [Hilton])**.** *Let $\iota_0 : A \to A\times B$ and $\iota_1 : B \to A\times B$ be given as above. Then $(A\times B, \iota_0, \iota_1)$ is the coproduct in $\mathfrak{A}$.*

Proof. Begin by letting $\varphi : A \to C$ and $\psi : B \to C$ be morphisms in $\mathfrak{A}$. Now defin

$$(\varphi, \psi) : A \times B \to C$$

by

$$(\varphi, \psi) = \pi_0\varphi + \pi_1\psi.$$

Then we calculate that

$$\iota_0(\varphi, \psi) = \iota_0\pi_0\varphi + \iota_0\pi_1\psi = \varphi + 0 = \varphi$$

and similarly $\iota_1(\varphi, \psi) = \psi$. Thus the diagram

$$\begin{tikzcd}
 & C & \\
A \arrow[ur, bend left, "\varphi"] \arrow[r, "\iota_0"'] & A\times B \arrow[u, "(\varphi,\psi)"'] & B \arrow[ul, bend right, "\psi"'] \arrow[l, "\iota_1"]
\end{tikzcd}$$

commutes. To see the uniquness of (φ, ψ) let $\theta : A\times B \to C$ be given with $\iota_0\theta = \varphi$ and $\iota_1\theta = \psi$. Then by Proposition 2.1 we have

$$\theta = (\mathrm{id}_{A\times B})\theta = (\pi_0\iota_0 + \pi_1\iota_1)\theta = \pi_0\iota_0\theta + \pi_1\iota_1\theta = \pi_0\varphi + \pi_1\psi = (\varphi, \psi).$$

This shows that $(A\times B, \iota_0, \iota_1)$ is the coproduct of A and B in $\mathfrak{A}$ and completes the proof of the proposition. ☐

From here on we will write $A\oplus B$ in place of either $A\times B$ or $A \coprod B$ both to keep in the notational tradition of [Hilton], [MacLane$_1$], and [MacLane$_2$], as well as to emphasize that these objects are both products *and* sums in our categories $\mathfrak{A}$. In general infinite products differ from coproducts, however, so there we will write the product notation as appropriate, but we will stick with the direct sum notation to denote infinite coproducts.

It is natural at this point to ask if there are functors between additive categories that preserve additive structure. That is, we ask if there are functors $F : \mathfrak{A} \to \mathfrak{B}$ such that F takes zero objects to zero objects, F takes biproducts to biproducts, and F induces a homomorphism of Abelian groups on homsets. It turns out that in order to determine this problem we need only to answer whether F preserves the product structure, whether it preserves the coproduct (sum) structure, or whether the induced map $\mathfrak{A}(A, B) \to \mathfrak{B}(FA, FB)$ is a homomorphism. This is captured in the following proposition, whose proof, although instructive in how to use the properties of additive categories, is omitted due to spacial restrictions.

Proposition 2.3 (Proposition II.9.5 of [Hilton])**.** *Let $F : \mathfrak{A} \to \mathfrak{B}$ be a functor of additive categories $\mathfrak{A}$ and $\mathfrak{B}$. Then the following are equivalent:*

1. *F preserves binary sums;*

2. *F preserves binary products;*

3. *For any objects $A, B \in \mathrm{Ob}(\mathfrak{A})$, the induced map $F : \mathfrak{A}(A, B) \to \mathfrak{B}(FA, FB)$ is a homomorphism of Abelian groups.*

Proof. See pages 77-78 of [Hilton] for details. ☐

Definition 2.4. *Any functor $F : \mathfrak{A} \to \mathfrak{B}$ of additive categories $\mathfrak{A}$ and $\mathfrak{B}$ that satisfies the conditions of Proposition 2.3 is said to be an* additive functor. *The set (class) of all additive functors from $\mathfrak{A}$ to $\mathfrak{B}$ will be denoted by* $\mathrm{Add}(\mathfrak{A}, \mathfrak{B})$.

We now move forward to discuss and construct equalizers and coequalizers in additive categories. In order to study limits and colimits within additive categories, it is useful to know what these structures are. Our strategy shall be to construct a simply described class of equalizers (the kernels) and from there build all equalizers; dualizing this process will build coequalizers from cokernels.

Beginning in (hopefully) familiar territory, we wish to guide our intuition for what kernels should be by considering them in module categories. Within this theory, kernels are defined as the set of all elements in a module A that map to 0 under a homomorphism $f : A \to B$. That simply means that the kernel and its injection are an equalizer of the zero map 0 and the morphism f! Generalizing this phenomenon tells us exactly the role we need for kernels to play in an arbitrary additive category, should they exist.

Definition 2.5. *Let $\mathfrak{A}$ be an additive category. Then a* kernel *in $\mathfrak{A}$ of the map $f : A \to B$ is the equalizer of f and $0 : A \to B$. That is, if $g : C \to A$ is any map with $gf = g0 = 0$ then there is a unique map $\theta : C \to K$ making the diagram*

$$\begin{tikzcd} K \arrow[r, "k"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \\ C \arrow[u, dashed, "\exists!\theta"] \arrow[ur, "g"'] & & \end{tikzcd}$$

commute in $\mathfrak{A}$. We will write the kernel of f as $K = \mathrm{Ker}\, f$ and the map $k = \ker f$. Dually the cokernel of f is the coequalizer of $(f, 0)$ and is written as $\mathrm{Coker}\, f$ *with natural map* $c = \mathrm{coker}\, f$.

Proposition 2.4. *Let (E, e) be the equalizer of $f, g : A \to B$ in an additive category $\mathfrak{A}$. Then (E, e) is the kernel of $f - g$.*

Proof. Assume that we have the equalizer (E, e) and let K be an object in $\mathfrak{A}$ with a morphism $k : K \to A$ making the diagram

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \\ K \arrow[u, dashed, "\exists!\theta"] \arrow[ur, "k"'] & & \end{tikzcd}$$

commute. Now observe that

$$ef = eg \iff ef - eg = 0 \iff e(f - g) = 0$$

and similarly

$$kf = kg \iff kf - kg = 0 \iff k(f - g) = 0.$$

Note that since θ is the unique map $\theta : K \to E$ making $\theta e = k$ we have that the diagram

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f-g"] \arrow[r, shift right, "0"'] & B \\ K \arrow[u, dashed, "\exists!\theta"] \arrow[ur, "k"'] & & \end{tikzcd}$$

is an equalizer in $\mathfrak{A}$. Thus (E, e) is the kernel of $f - g$ and we are done. $\square$

The above proposition shows us that in additive categories the equalizers are the kernels, and the coequalizers are the cokernels. Because additive categories all have finite biproducts (with us interpreting the zero object as the nullary biproduct), in order to see if additive categories are finite (co)complete, we need to see if they admit (co)kernels. Unfortunately this question is answered in the negative, as the example below will show. This forces us to work with additive categories that admit all kernels and cokernels.

Example 2.3. Let R be a PID (principal ideal domain) that is not a field and consider the category $\mathbf{FreeR} - \mathbf{Mod}$ of free R modules. Then it is easily checked that $\mathbf{FreeR} - \mathbf{Mod}$ has a zero object (the free module generated by the empty set), is **Ab**-enriched, has bilinear composition of maps, and admits products. Thus $\mathbf{FreeR} - \mathbf{Mod}$ is additive. Because all submodules of a free module over a PID are free (see Theorem IV.6.1 of [Hungerford] for details fo the proof) we see readily that $\mathbf{FreeR} - \mathbf{Mod}$ admits all kernels, as kernels of modules are submodules. Consequently we need only check if cokernels exists. To see that they do not let $\mathfrak{p} = (\pi)$ be a nonzero prime ideal of R. Then the map

$$f : R \to \mathfrak{p}R$$

given by

$$a \mapsto \pi a$$

is a homomorphism of free R-modules. Note that $\mathfrak{p}R \cong \mathfrak{p}$ is free because $\mathfrak{p} \trianglelefteq R$ is a submodule of R, and hence a subobject of R in $\mathbf{FreeR} - \mathbf{Mod}$. Observe now that

$$\operatorname{Coker} f = \frac{R}{f(R)} = \frac{R}{\mathfrak{p}}$$

Because $R/\mathfrak{p}$ is a residue integral domain of R, we have that $R/\mathfrak{p}$ is not a free R-module. Thus $\mathbf{FreeR} - \mathbf{Mod}$ does not admit all cokernels.

This example shows us that we require more structure on our additive categories to ensure that we have all kernels and cokernels. The culmination of this structure with the addendum that all monics be kernels, all epics be cokernels, and an $(\mathscr{E}, \mathscr{M})$ factorization, will be an Abelian category, which will have all the tools that describe, categorically at least, **R-Mod** for any ring R. We will proceed from here to define these structures and study some of their basic properties, which will then lead us to the Embedding Theorem.

3 Abelian Categories and the Structure Thereof

At this point we need to be formally introduced to Abelian categories. While we have already met them in spirit (as they are all additive categories), we need to meet them for who they are, not for the party to which they belong. Essentially the Abelian categories comprise the most structurally rich of the additive categories; they have all equalizers, coequalizers, the monics are exactly the kernels, the epics are exactly the cokernels, and every map has a cokernel-kernel factorization. It turns out that this description completely sets them aside from their nonAbelian additive brethren, and gives us the categorical power we need to get our hands on the "correct" generalization of module categories. Without further ado, here are Abelian categories.

Definition 3.1 ([Hilton], p.78)**.** *Let $\mathfrak{A}$ be an additive category. Then $\mathfrak{A}$ is an* Abelian category *if and only if:*

1. *Every map in $\mathfrak{A}$ has a kernel and a cokernel;*
2. *Every monic in $\mathfrak{A}$ is the kernel of its cokernel;*
3. *Every epic in $\mathfrak{A}$ is the cokernel of its kernel;*

4. $\mathfrak{A}$ *has an* $(\mathscr{E}, \mathscr{M})$ *factorization system, where* $\mathscr{E}$ *is the class of all epics in* $\mathfrak{A}$ *and* $\mathscr{M}$ *is the class of all monics in* $\mathfrak{A}$.

Note that because every epic in an Abelian category is a cokernel and every monic is a kernel, and moreover that maps may be factorized by a cokernel-kernel pair, we can develop a concrete way of determining what the "image" of a map $f : A \to B$ is that works with the image of a morphism of R-modules $g : M \to N$. In particular, we observe that in classical module theory we have the cokernel of a map $g : M \to N$ is given by

$$\operatorname{Coker} f = \frac{N}{g(M)}.$$

Consequently we can identify the image of g with

$$\operatorname{im} g = g(M) \cong \operatorname{Ker}(\operatorname{Coker} g).$$

Because all Abelian categories have all kernels and cokernels, we can give the definitions of image in exactly this way! While the definition we give makes sense in any additive category that has all kernels and cokernels, we shall provide a proposition afterwards that shows why we have waited until now to define the image and coimage of a map.

Definition 3.2. *Let* $f : A \to B$ *be a morphism in an Abelian category* $\mathfrak{A}$. *Then the* image *of* f*, denoted* $\operatorname{im}(f)$ *or* $\operatorname{im} f$*, is defined by*

$$\operatorname{im} f := \operatorname{Ker}(\operatorname{Coker} f)$$

Similarly, the coimage *of* f *is defined as*

$$\operatorname{coim} f := \operatorname{Coker}(\operatorname{Ker} f)$$

Proposition 3.1. *In an Abelian category* $\mathfrak{A}$*, given any map* $f : A \to B$ *we have an isomorphism*

$$\operatorname{im} f = \operatorname{Ker}(\operatorname{Coker} f) \cong \operatorname{Coker}(\operatorname{Ker} f) = \operatorname{coim} f.$$

Before we proceed with some examples of Abelian categories and start getting our hands on the work leading up to the Embedding Theorem, I think it is pertinent to consider some of the important structural aspects of Abelian categories, like exact sequences and exact functors. I feel introducing these first is important for two reasons: (1) the properties we are introducing are properties of *Abelian* categories that rely heavily on the definitions we have given above and consequences thereof, so it is useful to see them to distinguish Abelian categories from additive categories; and (2) I feel like what we are about to introduce is so important to the nature of Abelian categories that to see examples without thinking of exactness is to somehow "miss the point" of Abelian categories.

Definition 3.3. *Let* $\mathfrak{A}$ *be an Abelian category. We then say that a* sequence *in* $\mathfrak{A}$ *is a diagram, either infinite of the form*

$$\cdots \xrightarrow{\partial_{n-2}} A_{n-1} \xrightarrow{\partial_{n-1}} A_n \xrightarrow{\partial_n} A_{n+1} \xrightarrow{\partial_{n+1}} \cdots$$

or finite of the form

$$A_0 \xrightarrow{\partial_0} \cdots \xrightarrow{\partial_{n-1}} A_n$$

such that $\partial_n \partial_{n+1} = 0$*, whenever such maps exist. We will frequently write* $A_\bullet = (A_\bullet, \partial_\bullet)$ *to denote a family of objects*

$$A_\bullet := \{A_n \mid n \in \mathbb{Z}, A_n \in \operatorname{Ob}(\mathfrak{A})\}$$

and a family of maps

$$\partial_\bullet := \{\partial_n \in \mathfrak{A}(A_n, A_{n+1}) \mid n \in \mathbb{Z}, \partial_n \partial_{n+1} = 0\}$$

to denote a sequence whose objects are the A_n *and maps the* ∂_n.

Definition 3.4. *Let $A_\bullet = (A_\bullet, \partial_\bullet)$ be a sequence in the Abelian category $\mathfrak{A}$. We then say that $A_\bullet$ is exact at the object A_n if and only if, given the diagram*

$$\cdots \xrightarrow{\partial_{n-2}} A_{n-1} \xrightarrow{\partial_{n-1}} A_n \xrightarrow{\partial_n} A_{n+1} \xrightarrow{\partial_{n+1}} \cdots$$

we have that $\operatorname{Ker} \partial_n = \operatorname{im}(\partial_{n-1})$. *We say that the sequence is exact if and only if it is exact at every object A_n.*

Definition 3.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Abelian categories. We then say that a functor $F : \mathfrak{A} \to \mathfrak{B}$ is left (right) exact if given an exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

$\mathfrak{A}$, the resulting sequence

$$0 \longrightarrow FA \xrightarrow{F(f)} FB \xrightarrow{F(g)} FC \longrightarrow 0$$

is exact at FA and FB (at FB and FC) in $\mathfrak{B}$. A functor that is both left and right exact is simply said to be exact.

The notions of exactness and (left/right) exact functors are extremely important to Abelian categories, as they illustrate a certain amount of "ideal" internal structure within the category. Whether or not a functor F is (left/right) exact will tell us how well this structure is preserved under the mapping F. In fact the whole notion of (co)homology comes down to making a (left) right exact functor exact by patching in an infinite chain of what are called (right) left derived functors on the inexact side of the sequence in such a way that the new infinite sequence is exact at every object; more details on derived functors may be found in Chapter IV of [Hilton] or Chapter XII of [MacLane$_2$].

Example 3.1. Let $\mathfrak{C}$ be a small category and let $\mathfrak{A}$ be an Abelian category. Then the category $\mathfrak{A}^{\mathfrak{C}}$ is Abelian (the proof of this is not hard and is analogous to showing that $\mathbf{Set}^{\mathfrak{C}^{op}}$ is (co)complete for a small category; simply build up the objects in $\mathfrak{A}^{\mathfrak{C}}$ locally from the objects $F(U)$, for $U \in \mathrm{Ob}(\mathfrak{C})$, and then abuse the fact that $\mathfrak{A}$ is an Abelian category).

Example 3.2. This example is extremely important to us in the proof of the Embedding Theorem. Let $\mathfrak{A}$ be an Abelian category and define a functor $\rho : \mathfrak{A}^{op} \to \mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ given by, for all objects $A \in \mathrm{Ob}(\mathfrak{A})$ and maps $f : A \to B$ in $\mathfrak{A}$,

$$A \mapsto \mathfrak{A}(A, -)$$

and

$$f \mapsto \mathfrak{A}(f, -) := f^*.$$

Definition 3.6. *The functor ρ defined above is called the* representation functor*.*

We now wish to talk about some of the structure of Abelian categories that will be important in the course of the proof of the Embedding Theorem. While we do not wish to give an exhaustive account of every theorem involving a pullback or pushout in an Abelian category, we do wish to present some of the structure of pushouts and pullbacks. In particular, the theorems and propositions we present do not only showcase the behavior of Abelian categories, they also provide us with some important tools useful for proving the Embedding Theorem.

Proposition 3.2 ([Freyd], Theorem 2.52)**.** *Let $\mathfrak{A}$ be an Abelian category and consider the pullback diagram*

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & C \end{tikzcd}$$

Then if $(\operatorname{Ker} p_1, \ker p_1)$ *is a kernel of* $p_1 : P \to B$ *we have that* $(\operatorname{Ker} p_1, (\ker p_1); p_0)$ *is a kernel of* $\alpha : A \to C$. *In particular α is monic if and only if p_0 is monic.*

Proof. Let $X \in \mathrm{Ob}(\mathfrak{A})$ be an object with a map $f : X \to A$ such that $f\alpha = 0$. Then the diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "f"'] & & \\
& P \arrow[r, "p_1"'] \arrow[d, "p_0"] & B \arrow[d, "\beta"] \\
& A \arrow[r, "\alpha"'] & C
\end{tikzcd}$$

commutes in $\mathfrak{A}$, giving a unique map $\theta : X \to P$ making $\theta p_0 = f$ and $\theta p_1 = 0$. But then θ equalizes 0 and p_0; consequently there is a unique map $\eta : X \to \mathrm{Ker}\, p_1$ making the diagram

$$\begin{tikzcd}
\mathrm{Ker}\, p_1 \arrow[r, "\ker p_1"] & P \arrow[r, shift left, "p_1"] \arrow[r, shift right, "0"'] & B \\
X \arrow[u, dashed, "\exists!\eta"] \arrow[ur, "\theta"'] & &
\end{tikzcd}$$

commute. Thus $\eta \ker p_1 = \theta$ and hence

$$f = \theta p_0 = \eta; (\ker p_1); p_0$$

with the composition

$$0 = f\alpha = (\eta; (\ker p_1); p_0)\alpha.$$

As such the diagram

$$\begin{tikzcd}
\mathrm{Ker}\, p_1 \arrow[r, "(\ker p_1);p_0"] & A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "0"'] & C \\
X \arrow[u, dashed, "\exists!\eta"] \arrow[ur, "f"'] & &
\end{tikzcd}$$

commutes and is an equalizer diagram, proving that $(\mathrm{Ker}\, p_1, (\ker p_1); p_0)$ is a kernel to α. For the second claim note that in any category if α is monic so is p_1. If p_1 is monic then $\mathrm{Ker}\, p_1 = 0$. But then since $\mathrm{Ker}\, p_1 = 0$ is the zero object in $\mathfrak{A}$ we have $(\ker p_1); p_0 = 0$ and so α is monic as well. $\square$

We now provide statement of two propositions appearing in [Freyd]. They are important results that we will use in the course of our proof of the Embedding Theorem, but their proofs are omitted due to spacial constraints. The interested reader may find their proofs on pages 52 and 53 of [Freyd].

Proposition 3.3 (Proposition 2.53 of [Freyd]). *Let $\mathfrak{A}$ be an Abelian category and consider the square*

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "\alpha"] \\
B \arrow[r, "\beta"'] & P
\end{tikzcd}$$

Write $\langle f, g\rangle : C \to A \oplus B$ to be the canonical comparison map and $\langle \alpha, \beta\rangle : A \oplus B \to P$ to be the canonical pairing map. Then:

1. *$\langle f, g\rangle\langle \alpha, \beta\rangle = 0$ if and only if the square commutes;*
2. *The sequence*

$$0 \longrightarrow C \xrightarrow{\langle f,g\rangle} A \oplus B \xrightarrow{\langle \alpha,\beta\rangle} P$$

   is exact if and only if the square is a pullback square;

3. The sequence

$$C \xrightarrow{\langle f,g\rangle} A \oplus B \xrightarrow{\langle \alpha,\beta\rangle} 0$$

is exact if and only if the square is a pushout square;

4. The sequence

$$0 \longrightarrow C \xrightarrow{\langle f,g\rangle} A \oplus B \xrightarrow{\langle \alpha,\beta\rangle} P \longrightarrow 0$$

is exact if and only if (C, f, g) is a pullback and (P, α, β) is a pushout.

Proposition 3.4 (Proposition 2.54∗ of [Freyd])**.** *Let $\mathfrak{A}$ be an Abelian category and let*

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

be a pushout square. Then if f is monic so is β.

In order understand Abelian categories further we now need to consider what it means for these categories to have objects that are generators (and dually cogenerators). Intuitively, we would like it to means that the covariant (contravariant) hom-functors beginning at the object must be embeddings, i.e., faithful. Having these objects at hand is important in many of the statements we will consider, so a introduction at this point is only wise.

Definition 3.7. *Let $\mathfrak{A}$ be an Abelian category. We then say that an object G is a* generator *of $\mathfrak{A}$ if and only if the functor $\mathfrak{A}(G, -) : \mathfrak{A} \to \mathbf{Ab}$ is faithful. Dually, we say that an object C is a* cogenerator *of $\mathfrak{A}$ if and only if the functor $\mathfrak{A}(-, C) : \mathfrak{A} \to \mathbf{Ab}$ is faithful.*

Definition 3.8. *Let $\mathfrak{A}$ be an Abelian category. We then say that an object P is* projective *if and only if the functor $\mathfrak{A}(P, -)$ is exact. Dually, an object I of $\mathfrak{A}$ is injective if and only if $\mathfrak{A}(-, I)$ is exact.*

Definition 3.9. *An object $P \in \mathrm{Ob}(\mathfrak{A})$, for Abelian category $\mathfrak{A}$, is said to be a* projective generator *if $\mathfrak{A}(P, -) : \mathfrak{A} \to \mathbf{Ab}$ is exact and faithful. Dually, an object $I \in \mathrm{Ob}(\mathfrak{A})$ is said to be an* injective cogenerator *if the functor $\mathfrak{A}(-, I) : \mathfrak{A}^{op} \to \mathbf{Ab}$ is exact and faithful.*

These definitions encode the information of and generalize the notions of projective and injective modules and (co)generating objects. In fact, it is the content of Mitchell's Theorem (Theorem 4.44 of [Freyd]) that tells us that in a **Set**-complete and **Set**-cocomplete Abelian category (an Abelian category with **Set**-indexed products and coproducts) with a projective generator, we can essentially treat projective generators as elements and complete any theorems in these categories that rely only on finite diagram chases by selecting appropriate morphisms out of the projective generator and adapting them appropriately. This phenomenon is perhaps justified by the following three propositions.

Proposition 3.5 (Proposition 3.35 of [Freyd])**.** *If $\mathfrak{A}$ is an Abelian category with a generator G, then the family of subobjects of any object A (the family of monic morhpsims $\{\mu_i : A_i \to A\}$ in $\mathfrak{A}$) is a set.*

Proof. Since G is a generator we have that $\mathfrak{A}(G, A_i)$ and $\mathfrak{A}(G, A)$ are all sets. Since μ_i is monic, the induced map $(\mu_i)_* = \mathfrak{A}(G, \mu_i)$ is an injective morphism of Abelian groups. Thus for all such A_i we have

$$\mathfrak{A}(G, A_i) \subseteq \mathfrak{A}(G, A).$$

Hence the family $\{\mu_i : A_i \to A\}$ is a set. □

Proposition 3.6 (Proposition 3.36 of [Freyd])**.** *Let $\mathfrak{A}$ be a* **Set***-cocomplete Abelian category. Then G is a generator in $\mathfrak{A}$ if and only if for every $A \in \mathfrak{A}$ the map*

$$\pi : \bigoplus_{\mathfrak{A}(G,A)} G \to A$$

given by, for all $f \in \mathfrak{A}(G, A)$, the assignment making the diagram

$$\begin{tikzcd} G \arrow[r, "\iota_f"] \arrow[dr, "f"'] & \bigoplus_{\mathfrak{A}(G,A)} G \arrow[d, "\pi"] \\ & A \end{tikzcd}$$

commute in $\mathfrak{A}$.

Proof. Omitted. ☐

Proposition 3.7 (Proposition 3.37 of [Freyd])**.** *Let $\mathfrak{A}$ be a* **Set***-complete and* **Set***-cocomplete Abelian category with a generator. Then every object in $\mathfrak{A}$ bmay be embedded in an injective object if and only if $\mathfrak{A}$ has an injective cogenerator.*

Proof. $\Longrightarrow$: Let C be an injective cogenerator for $\mathfrak{A}$ and let $A \in \mathrm{Ob}(\mathfrak{A})$ be arbitrary. Then by the dual of Proposition 3.6 we have that the map ι making the diagram, for all $g \in \mathfrak{A}(A, C)$,

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "g"'] & \prod_{\mathfrak{A}(A,C)} C \arrow[d, "\pi_g"] \\ & C \end{tikzcd}$$

commute is dual to the π described above. Thus $\iota : A \to \prod_{\mathfrak{A}(A,C)} C$ is monic and so $\prod_{\mathfrak{A}(A,C)} C$ is injective.
$\Longleftarrow$: Assume that G is a generator for $\mathfrak{A}$ and let $\{P_i \mid P_i = \operatorname{Coker} \mu_i, \mu_i : A_i \to G \text{ monic}\}$. By Proposition 3.5, the collection of all the P_i is a set. Now define

$$P := \prod P_i$$

and let $\mu : P \to I$ be a monic with I injective. To see that I is actualy a cogenerator let $f : A \to B$ be a nonzero map in $\mathfrak{A}$. Since G is a generator, we can find a $g : G \to A$ such that $gf : G \to A$ is nonzero. Fix one such g. Now write $C = \operatorname{im} gf$ and let $m : \operatorname{im} gf \to B$ be the natural monic. Let ν be a monic factoring as

$$\begin{tikzcd} \operatorname{im} gf \arrow[rr, "\nu"] \arrow[dr, "p"'] & & I \\ & P \arrow[ur, "\mu"'] & \end{tikzcd}$$

where p is some map induced by gf. Then by the injectivity of I we can find a map $\varphi : B \to I$ making

$$m\varphi = p\mu.$$

Now since

$$gf\varphi = fgm\varphi \neq 0$$

we conclude that $f\varphi \neq 0$ and so I is a cogenerator of $\mathfrak{A}$. Since I is injective we are done. ☐

From here we will move from studying Abelian categories and their structure, as we could fill an entire book just on the such topics, into doing the work leading up to the Embedding Theorem. While we will primarily follow the order of discourse as presented in [Aly], the proof of the Embedding Theorem and almost all of our relevant lemmas and propositions are found within [Freyd]; our use of the reference [Aly] is simply because the language is a little more modern, despite the fact that [Aly] follows the route to the Embedding Theorem paved in [Freyd]. Our change in tone will simply represent a more modern perspective and attempt to place things in more current language.

4 The Mitchell-Freyd Embedding Theorem

Perhaps the most important tool moving forward is the Yoneda Embedding. Because we are already talking about representability (especially with the representation functor ρ), it is be beyond pertinent to have about special representable functors, especially in the context of Abelian categories. We begin with the Yoneda Lemma (although the proof is omitted; it may be found on page 61 of [MacLane$_1$]) and proceed from there to determine when functors are exact in $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$.

Lemma 4.1 (The Yoneda Lemma; cf. [MacLane$_1$], p.61)**.** *Let $\mathfrak{C}$ be a category with small hom-sets and let $F : \mathfrak{C} \to \mathbf{Set}$ be a functor with $A \in \mathrm{Ob}(\mathfrak{C})$ given and fixed. Then there is a bijection*

$$y : \mathbf{Nat}(\mathfrak{C}(A, -), F) \cong F(A)$$

which sends each natrual transformation

$$\begin{tikzcd} \mathfrak{C} \arrow[r, bend left, "\mathfrak{C}(A,-)"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Set} \end{tikzcd}$$

to the element $\alpha_A(\mathrm{id}_A)$.

Proof. See page 61 of [MacLane$_1$] for details. □

An important corollary of the Yoneda Lemma is the following: it lets us completely determine the natural transformations between the functors $\mathfrak{C}(A, -) \to \mathfrak{C}(B, -)$ by maps $f : A \to B$. This is very important, as it allows us to conclude that the functor ρ is an embedding!

Corollary 4.1 ([MacLane$_1$], p.61)**.** *Let $\mathfrak{C}$ be a category that satisfies the statement of the Yoneda Lemma and let $A, B \in \mathrm{Ob}(\mathfrak{C})$. Then if α is a natural transformation*

$$\begin{tikzcd} \mathfrak{C} \arrow[r, bend left, "\mathfrak{C}(A,-)"] \arrow[r, bend right, "\mathfrak{C}(B,-)"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Set} \end{tikzcd}$$

then $\alpha = \mathfrak{C}(f, -) = f^*$ for some unique $f : B \to A$ in $\mathfrak{C}$.

Corollary 4.2 (Theorem 5.36 of [Freyd])**.** *The functor $\rho : \mathfrak{A} \to \mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ is full and faithful.*

Proof. By construction of ρ and from the Yoneda Lemma it follows that for any $A, B \in \mathrm{Ob}(\mathfrak{A})$ we have the natural equivalence

$$\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})(\rho(A), \rho(B)) \cong \mathfrak{A}^{op}(A, B) \cong \mathfrak{A}(B, A).$$

□

It is worth noting that this corollary shows that the functors $\rho(A) = \mathfrak{A}(-, A)$ are full and faithful. Thus ρ describes a contravariant embedding of $\mathfrak{A}$ into $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$.

Proposition 4.1 ([Aly], Theorem 4)**.** *Let $A, B,$ and C be objects in an Abelian category $\mathfrak{A}$ with small homsets. Then if we have maps $f : A \to B$ and $g : B \to C$ and if for every $D \in \mathrm{Ob}(\mathfrak{A})$ we have that the diagram*

$$\mathfrak{A}(D,A) \xrightarrow{f_*} \mathfrak{A}(D,B) \xrightarrow{g_*} \mathfrak{A}(D,C)$$

is exact, then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact.

Proof. First set $D = A$ and note that by the Yoneda Lemma

$$fg = 0 = g_*(f_*(\mathrm{id}_A)),$$

hence giving $\mathrm{im}\, f \subseteq \mathrm{Ker}\, g$. Taking now $D = \mathrm{Ker}\, g$ we have an inclusion $\iota : \mathrm{Ker}\, g \to B$ such that $\mathfrak{A}(D,g)(\iota) - g \circ \iota = 0$. Thus we can find an $\alpha \in \mathfrak{A}(\ker g, A)$ for which $\iota = \mathfrak{A}(D,f)(\alpha) = f \circ \alpha$. Thus $\mathrm{Ker}\, g = \mathrm{im}\, \iota \subseteq \mathrm{im}\, f$ and so $\mathrm{Ker}\, g = \mathrm{im}\, f$. Consequently the sequence is exact and we are done. $\square$

Proposition 4.2. *Let $A, B,$ and C be objects in an Abelian category $\mathfrak{A}$. Then if we have maps $f : A \to B$ and $g : B \to C$ and if for every $D \in \mathrm{Ob}(\mathfrak{A})$ we have that the diagram*

$$0 \longrightarrow \mathfrak{A}(C,D) \xrightarrow{g^*} \mathfrak{A}(B,D) \xrightarrow{f^*} \mathfrak{A}(A,D)$$

is exact, then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact in $\mathfrak{A}$.

Proof. Dualize the proof of Proposition 4.1. $\square$

These propositions we have just presented (as well as the celebrated Yoneda Lemma) provide us with the tools to present an important theorem. In particular, we can state that if a functor $F \in \mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ is faithful and preserves monics, then it is exact! This allows us to construct exact functors by providing us with useful criteria: a functor simply must preserve monics and be faithful.

Theorem 4.1 (Theorem 7.11 of [Freyd])**.** *Let $F \in \mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ for some Abelian category $\mathfrak{A}$. Then if F preserves monics and if F is faithful it is exact.*

Proof. Begin by letting

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence in $\mathfrak{A}$. Then since ρ is contravariant we have the reflected sequence

$$0 \longrightarrow \mathfrak{A}(C,-) \xrightarrow{g^*} \mathfrak{A}(B,-) \xrightarrow{f^*} \mathfrak{A}(A,-)$$

in $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$. Since the sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact we have that the sequence

$$0 \longrightarrow \mathfrak{A}(C,D) \xrightarrow{g^*} \mathfrak{A}(B,D) \xrightarrow{f^*} \mathfrak{A}(A,D)$$

is exact in $\mathbf{Ab}$ for every $D \in \mathrm{Ob}(\mathfrak{A})$; invoking Proposition 4.2 allows us to conclude the sequence $0 \to \rho(C) \xrightarrow{g^*} \rho(B) \xrightarrow{f^*} \rho(A)$ is exact in $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$. Because a sequence of functors

$$G \longrightarrow G' \longrightarrow G''$$

in $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ is exact if and only if the sequence

$$G(U) \longrightarrow G'(U) \longrightarrow G''(U)$$

is exact for every $U \in \mathrm{Ob}(\mathfrak{A})$, we conclude that the functor $\mathbf{Nat}(-, F) : \mathrm{Add}(\mathfrak{A}, \mathbf{Ab}) \to \mathbf{Ab}$ is exact. So apply the (contravariant) functor $\mathbf{Nat}(-, F)$ and produce the exact sequence

$$\mathbf{Nat}(\rho(A), F) \xrightarrow{f^{**}} \mathbf{Nat}(\rho(B), F) \xrightarrow{g^{**}} \mathbf{Nat}(\rho(C), F) \longrightarrow 0$$

in $\mathbf{Ab}$. By the Yoneda Lemma we have a natural isomorphism $\mathbf{Nat}(\rho(X), F) \cong F(X)$ for all $X \in \mathrm{Ob}(\mathfrak{A})$. Thus we produce the naturally isomorphic sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

in $\mathbf{Ab}$. Because isomorphisms preserve exactness, this shows that F is right exact.

We now observe that a right exact functor is exact if and only if it preserves all monics, i.e., it sends kernels to kernels. So, since F is faithful and it preserves monics we must have that it sends left exact sequences to left exact sequences. Thus F is exact. □

We now will show that the category of all left-exact functors of $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ is Abelian and that it admits injective cogenerators. This will then allow us to embed every small Ablian category fully and faithfully (through the representation functor ρ) into a complete Abelian category $\mathfrak{A}$ with an injective cogenerator, which is the first big step in our proof of the Embedding Theorem. To show that the subcategory of all left exact functors in $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ has injective cogenerators we need to consider two types of special monic morphisms.

Definition 4.1. *Let $\mathfrak{C}$ be a category and let $\{\mu_i : A_i \to A \mid i \in I\}$ be a family of monics indexed by I in $\mathfrak{C}$. Then we say that a monic $\mu : A' \to A$ is the* intersection *of the μ_i if and only if for every $i \in I$ there exists a unique $\nu_i : A' \to A_i$ making the diagram*

$$\begin{tikzcd} A' \arrow[rr, "\mu"] \arrow[dr, dashed, "\exists! \nu_i"'] & & A \\ & A_i \arrow[ur, "\mu_i"'] & \end{tikzcd}$$

commute in $\mathfrak{C}$. Furthermore, if $f : A \to B$ is any morphism factoring through each μ_i, then it must factor uniquely through μ.

Definition 4.2. *Let $\mathfrak{C}$ be a category and $\mu : A \to B$ be a monic. We then say that μ is an essential extension of A if and only if for every nonzero monic $\mu' : A' \to B$ the intersection of μ and μ' is nonzero.*

Proposition 4.3 ("Essential Lemma" 7.12 of [Freyd])**.** *Let $\mu : F \to G$ be an essential extension in* $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$. *Then if F is monic so is G.*

Proof. Assume for the purpose of deriving a contradiction that G is not monic. Then there exists a monic $f : A' \to A$ in $\mathfrak{A}$ such that $G(f) : G(A') \to G(A)$ is not monic in $\mathbf{Ab}$. Find an $x \in G(A')$ for which $x \neq 0$ and and $G(f)(x) = 0$. We will now construct a subfunctor E of G "generated" by x. Define, for every $B \in \mathrm{Ob}(\mathfrak{A})$,

$$E(B) := \{y \in G(B) \mid \exists\, g \in \mathfrak{A}(A', B) \text{ such that } G(g)(x) = y\}.$$

Then for any $h : B' \to B$ we have

$$G(h)\,(E(B')) \subseteq E(B)$$

and hence we may define $E(h)$ by an appropriate restriction. Now observe that by construction the functor E is the image of the natural transformation

$$\eta : \rho(A) \to G$$

defined by $\eta(\mathrm{id}_A) = x$; consequently each object $E(B)$ is a subgroup of $G(B)$ and hence $E : \mathfrak{A} \to \mathbf{Ab}$ is a functor.

By the construction of E and from the fact that $x \neq 0$ we know that E is not the zero functor. Furthermore, because $\mu : F \to G$ is essential and since there is a natural monic $\iota : E \to G$ given by injection of subgroups on each object of $\mathfrak{A}$, we have that the intersection of F and E is nonzero. Since the intersection is nontrivial, find a $C \in \mathrm{Ob}(\mathfrak{A})$ for which $E(C) \cap F(C) \neq 0$. Since $E(C) \cap F(C) \neq 0$, there is a $y \in E(C) \cap F(C)$ with $y \neq 0$; furthermore, from the construction of E we can find a map $g : A' \to C$ such that $y = G(k)(x)$. Now let (P, p_0, p_1) be the pushout of $A \xleftarrow{f} A' \xrightarrow{g} C$ and consider the diagram

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "p_0"] \\ C \arrow[r, "p_1"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By Proposition 3.4, since f is monic the map p_1 is monic as well. Since F is a monic functor we have

$$F(p_1)(y) \neq 0$$

and hence

$$0 \neq G(p_1)(y) = G(p_1)(G(g)(x)) = G(p_0 \circ f) = G(p_0)(G(f)(x)) = 0.$$

This is an obvious contradiction, leading us to the conclusion that G must be monic. $\square$

This "Essential Lemma," as Freyd calls it (and I enjoy the pun, so I will call it this as well) completes the essential work needed to embed an Abelian category fully, faithfully, and exactly into its additive functor category. The method in which we do this will invoke the subcategory of all left-exact functors from $\mathfrak{A}$ to $\mathbf{Ab}$ and show that the representation functor ρ is a full and faithful functor from $\mathfrak{A}$ to the subcategory of all left exact functors. The work that we are about to do is simply to show exactly this; it is a large task, but it is necessary.

Definition 4.3. *Let $\mathfrak{A}$ be an Abelian category and let $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$ be the full subcategory of* $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ *comprised of all left exact functors and natural transformations between them.*

Lemma 4.2. *The category* $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ *is Abelian and both* **Set***-cocomplete and* **Set***-complete. Furthermore, direct sums and products in* $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ *are exact.*

Sketch. We only sketch this proof. The fact that $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ is Abelian is clear; as is standard, we simply build our desired properties locally via the presheaf assignment

$$A \mapsto F(A)$$

and then using that all the properties we desire exist at each object in $\mathbf{Ab}$ in a functorial way. The group law on $\mathbf{Nat}(F, G)$ for any $F, G \in \mathrm{Ob}(\mathrm{Add}(\mathfrak{A}, \mathbf{Ab}))$ is given by the local assignment

$$\alpha + \beta := \{\alpha_A + \beta_A \mid A \in \mathrm{Ob}(\mathfrak{A})\}.$$

The proof that $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ is both **Set**-complete and **Set**-cocomplete follows from the fact that $\mathbf{Ab}$ is both **Set**-complete and **Set**-cocomplete (the arbitrary product in $\mathbf{Ab}$ is the arbitrary group theoretic product and the arbitrary coproduct is the the infinite direct sum of Abelian groups); in particular, we build the sum and product of a family of functors $\{F_i \mid i \in I\}$ via the presheaf

$$A \mapsto \prod_{i \in I} F_i(A)$$

or

$$A \mapsto \bigoplus_{i \in I} F_i(A).$$

The fact that products and sums in $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ are exact follows similarly and is reasonably clear. $\square$

Theorem 4.2 (Theorem 7.32 of [Freyd])**.** *The category $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$ is $\mathbf{Set}$-complete and $\mathbf{Set}$-cocomplete and has an injective cogenerator.*

Proof. The products and sums in $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$ are constructed in a straightforward manner and are the sums and products that already appear in $\mathrm{Add}(\mathfrak{A}, \mathbf{Ab})$ (as exact functors are in particular left exact).

Consider the product

$$\prod_{A \in \mathrm{Ob}(\mathfrak{A})} \rho(A) = \prod_{A \in \mathrm{Ob}(\mathfrak{A})} \mathfrak{A}(A, -).$$

Because all the functors $\mathfrak{A}(A, -)$ have already been seen to be left exact by Lemma 4.2 we see that their product

$$P = \prod_{A \in \mathrm{Ob}(\mathfrak{A})} \rho(A) = \prod_{A \in \mathrm{Ob}(\mathfrak{A})} \mathfrak{A}(A, -)$$

is as well. Furthermore, P is also a generator for $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$ by the Yoneda Lemma and Corollary 4.2. Thus invoking Proposition 3.7 gives us that $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$ has an injective cogenerator. $\square$

Theorem 4.3 (Theorem 7.33 of [Freyd])**.** *The representation functor $\rho : \mathfrak{A} \to \mathcal{L}(\mathfrak{A}, \mathbf{Ab})$ is an exact and full embedding.*

Proof. By Corollary 4.2 we already know that ρ is full and faithful. To see that it is exact let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence in $\mathfrak{A}$. We need to show that the sequence

$$0 \longrightarrow \rho(C) \longrightarrow \rho(B) \longrightarrow \rho(A) \longrightarrow 0$$

is exact in $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$. To see this not ethat such a sequnece is exact if and only if the sequence

$$0 \longrightarrow \mathbf{Nat}(\rho(C), I) \longrightarrow \mathbf{Nat}(\rho(B), I) \longrightarrow \mathbf{Nat}(\rho(A), I) \longrightarrow 0$$

is exact for some injective cogenerator I in $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$. However, invoking the Yoneda Lemma shows that for some generator G the above sequence is naturally isomorphic to the sequence

$$0 \longrightarrow I(C) \longrightarrow I(B) \longrightarrow I(A) \longrightarrow 0$$

which is exact if and only if I is exact. However, since $\rho \to I$ is an essential extension by the Essential Lemma we know that I is exact. Thus we conclude that ρ is an exact full and faithful embedding of $\mathfrak{A}$ into $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$. $\square$

This completes the first big step in the proof of the embedding theorem: we have embedded an Abelian category fully and faithfully into its category of left exact additive functors! All we need to do now is show that this embedding extends from a small Abelian category to some module category. This is done through Mitchell's Theorem, which tells us that for every $\mathbf{Set}$-complete and $\mathbf{Set}$-cocomplete Abelian category $\mathfrak{A}$ with a projective generator, each small Abelian subcategory $\mathfrak{B} \hookrightarrow \mathfrak{A}$ may be embedded into some module category. This comprises our second big step in proving the Embedding Theorem. The proof we provide is essentially the proof as given in [Aly] and [Freyd].

Definition 4.4. *Let $\mathfrak{A}$ be an Abelian category and let $\mathfrak{B}$ be a subcategory of $\mathfrak{A}$. We then say that $\mathfrak{B}$ is* exact *if the inclusion functor $\iota : \mathfrak{B} \to \mathfrak{A}$ is exact.*

Theorem 4.4 (Mitchell; Theorem 4.44 of [Freyd])**.** *Let $\mathfrak{A}$ be a complete Abelian category with a projective generator P. Then for every small Abelian category $\mathfrak{B}$ of $\mathfrak{A}$ ther eis a ring of unity R and an exact fully faithful functor $F : \mathfrak{B} \to$ **R-Mod**.*

Proof. Let $\mathfrak{B}$ be a small, exact, and full subcategory of $\mathfrak{A}$ and let P be a projective generator for $\mathfrak{A}$. Then for each object $B \in \mathrm{Ob}(\mathfrak{B})$ consider the natural morphism

$$\varepsilon : \bigoplus_{\mathfrak{A}(P,B)} P \to B.$$

Defining ε as the morphism π in Proposition 3.6 gives that ε the map defineed by the commutative diagram, for every $f \in \mathfrak{A}(P, B)$:

$$\begin{tikzcd} P \arrow[r, "\iota_f"] \arrow[dr, "f"'] & \bigoplus\limits_{\mathfrak{A}(P,B)} P \arrow[d, "\varepsilon"] \\ & B \end{tikzcd}$$

Proposition 3.6 gives that that P being a generator makes the map well-defined; P being projective makes it epic. Now define

$$I := \bigcup_{B \in \mathrm{Ob}(\mathfrak{B})} \mathfrak{A}(P, B)$$

and set

$$G := \bigoplus_{i \in I} P.$$

Then G is a generator with an epic morphism $\varepsilon_B : G \to B$ for every $B \in \mathrm{Ob}(\mathfrak{B})$. To see that G is projective note first that each direct summand of G is projective. So let $f : A \to A'$ be an epic in $\mathfrak{A}$ and note that by the projectivity of each direct summand, call it G_i, there are maps $g_i' : G_i \to A'$ and a map $g_i : G_i \to A$ with $g_i' = g_i f$. Taking the comparison maps $\langle g_i' \rangle_{i \in I}$ and $\langle g_i \rangle_{i \in I}$ produce maps

$$\langle g_i \rangle_{i \in I} : G = \bigoplus_{i \in I} G_i \to A$$

and

$$\langle g_i' \rangle_{i \in I} : G \to A'$$

making the diagram

$$\begin{tikzcd} G \arrow[d, "\langle g_i \rangle_{i \in I}"'] \arrow[dr, "\langle g_i' \rangle_{i \in I}"] & \\ A \arrow[r, "f"'] & A' \end{tikzcd}$$

commute, proving that G is projective.

Now let $R := \mathfrak{A}(G, G)$ and note that R is a unital ring. Then for every $A \in \mathrm{Ob}(\mathfrak{A})$ the hom-set $\mathfrak{A}(P, A)$ has a canonical left R-module structure given by left-composing morphisms. Note that this forms the commuting diagram of maps, for every $f \in \mathfrak{A}(P, A)$ and $\varphi \in R$:

$$G \xrightarrow{\varphi} G \xrightarrow{f} A$$

Now let $f : A \to B$ be a morphism in $\mathfrak{A}$. Then the induced map $f_* : \mathfrak{A}(G, A) \to \mathfrak{A}(G, B)$ is R-linear, as any map $\varphi \in R$ produces the commuting diagram

$$G \xrightarrow{\varphi} G \longrightarrow A \xrightarrow{f} B$$

This shows that there is a canoncial map $F : \mathfrak{A} \to \textbf{R-Mod}$ given via the assignments

$$A \mapsto \mathfrak{A}(G, A)$$

on objects and

$$f \mapsto f_* = \mathfrak{A}(G, f)$$

for maps $f \in \mathfrak{A}(A, B)$. Because G is a projective generator this functor F is a both exact and fully faithful on $\mathfrak{A}$. Furthermore, because $\mathfrak{B}$ is exact we need only to show that the restriction of F to $\mathfrak{B}$ is full in order to show that $F\big|_{\mathfrak{B}}$ is exact and fully faithful.

To show that $F\big|_{\mathfrak{B}}$ is full we must prove that for any map $f \in \textbf{R-Mod}(FA, FB)$ there is a $g \in \mathfrak{B}(A, B)$ for which $F(g) = g_* = f$. To do this let

$$0 \longrightarrow K \xrightarrow{k} G \xrightarrow{h} A \longrightarrow 0$$

and

$$G \xrightarrow{r} B \longrightarrow 0$$

be exact sequences in $\mathfrak{A}$. By construction $F(G) = R$. Applying the functor F then produces the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & FK \arrow[r, "F(k)"] & R \arrow[r, "F(h)"] \arrow[d, "v"'] & FA \arrow[r] \arrow[d, "f"] & 0 \\
 & & R \arrow[r, "F(r)"'] & FB \arrow[r] & 0
\end{tikzcd}$$

where the existence of the map v is given by the fact that R is projective in **R-Mod**. Furthermore, in order to make this diagram commute h must be an automorphism. Because ring automorphisms correspond to left multiplication by a unit $a \in R$, consequently we have that $v(s) = as$ for some fixed $a \in R$. Since $a \in R = \mathfrak{A}(G, G)$ we have the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] & G \arrow[r, "h"] \arrow[d, "a"'] & A \arrow[r] & 0 \\
 & & G \arrow[r, "r"'] & B \arrow[r] & 0
\end{tikzcd}$$

in $\mathfrak{A}$. Since $F(k)vF(r) = 0$ and because F is faithful we find that $kar = 0$. Thus there is a map $g \in \mathfrak{A}(A, B) = \mathfrak{B}(A, B)$ making the diagram

$$\begin{tikzcd}
G \arrow[r, "h"] \arrow[d, "a"'] & A \arrow[d, "g"] \\
G \arrow[r, "r"'] & B
\end{tikzcd}$$

commute; consequently the diagram

$$\begin{tikzcd}
R \arrow[r, "F(h)"] \arrow[d, "v"'] & FA \arrow[d, "f"] \\
R \arrow[r, "F(r)"'] & FB
\end{tikzcd}$$

commutes in **R-Mod** with $g_* = f$. This proves that $F\big|_{\mathfrak{B}}$ is full and hence that there is a full and faithful exact embedding of $\mathfrak{B}$ into a module category. $\square$

This provides us with the last ingredient for proving the Embedding Theorem. The final task is to combine our results in such a way as to show that a small Abelian category has an embedding into a module category. Luckily this is not a Herculean task anymore! We have shown from Theorem 4.3 that any Abelian category $\mathfrak{A}$ (and small Abelian categories in particular) have contravariant full, exact, and faithful embeddings into the **Set**-complete and **Set**-cocomplete Abelian category $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$ *and* that $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$ has an injective cogenerator. The strategy now is to take the dual category and note that $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})^{op}$ is **Set**-complete and **Set**-cocomplete with a projective generator; thus Mitchell's Theorem applies! By proceeding in this way we prove the Embedding Theorem.

Theorem 4.5 (Mitchell-Freyd Embedding Theorem; Theorem 7.34 of [Freyd])**.** *Let $\mathfrak{A}$ be a small Abelian category. Then there exists a unital ring such that there is a full and faithful functor $F : \mathfrak{A} \to$ **R-Mod** such that F preserves exact sequences.*

Proof. Let $\mathfrak{A}$ be a small Abelian category and note that by Theorems 4.2 and 4.3 the category $\mathcal{L}(\mathfrak{A}, \mathbf{Ab})$ is **Set**-complete, **Set**-cocomplete, has an injective cogenerator, and the map $\rho : \mathfrak{A} \to \mathcal{L}(\mathfrak{A}, \mathbf{Ab})$ is an exact full and faithful contravariant embedding. Now let $o : \mathcal{L}(\mathfrak{A}, \mathbf{Ab}) \to \mathcal{L}(\mathfrak{A}, \mathbf{Ab})^{op}$ be the canonical opposite functor and post-compose ρ with o. Then ρo is an exact full and faithful covariant embedding into a **Set**-complete and **Set**-cocomplete Abelian category with a projective generator. Consequently Mitchell's Theorem applies; invoking Theorem 4.4 gives that we can embed $o(\rho(\mathfrak{A}))$ exactly, fully, and faithfully into a category **R-Mod** of R-modules for some unitary ring R; call this embedding F. Since the composition of embeddings is an embedding, the composition of full functors is full, and the composition of exact functors is exact, we have that

$$\rho o F : \mathfrak{A} \to \mathbf{R}\text{-}\mathbf{Mod}$$

is an exact full and faithful embedding. This completes the proof of the theorem. $\square$

5 Conclusion

We have no proved, with a great deal of effort, the Mitchell-Freyd Embedding Theorem! The beauty of this Theorem lies in the fact that it allows us to prove theorems involving only finite diagrams in Abelian categories by simply picking elements and then doing a diagram chase. Two such celebrated and famous results (as famous as something from homological algebra can be, anyway) are the Five Lemma and the Snake Lemma. We will state them, but not prove them, because it feels like a sin to introduce Abelian categories without at some point introducing these two lemmas.

Lemma 5.1 (Five Lemma)**.** *Let $\mathfrak{A}$ be an Abelian category and consider the commuting diagram*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "\varphi_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "\varphi_2"'] & A_3 \arrow[r, "f_3"] \arrow[d, "\varphi_3"'] & A_4 \arrow[r, "f_4"] \arrow[d, "\varphi_4"] & A_5 \arrow[d, "\varphi_5"] \\
B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5
\end{tikzcd}$$

with top and bottom rows exact. If φ_1 is epic, if φ_5 is monic, and if both φ_2 and φ_4 are isomorphisms, then φ_3 is an isomorphism as well.

Lemma 5.2 (Snake Lemma)**.** *Let $\mathfrak{A}$ be an Abelian category and consider the commuting diagram*

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' &
\end{tikzcd}$$

with top and bottom row exact. Then there is a morphism $\delta : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$ *making the sequence*

$$\operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}\beta \longrightarrow \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \longrightarrow \operatorname{Coker}\beta \longrightarrow \operatorname{Coker}\gamma$$

exact in $\mathfrak{A}$*.*

The usefulness of these two Lemmas, especially in proving each other and in the study of the various cohomology theories, is beyond measure. The fact that we can greatly simplify the proofs and use of these Lemmas simply because they rely only upon finite diagrams is amazing! Moreover, being able to justify the study of module theory as the study of small Abelian categories is fantastic; it tells us that homological algebra and its theory of derived functors and module theory are really just category theory. This provides us with good heuristics as to why the study of modules and "generalized modules" are so useful and important in many areas of mathematics: it's because they are particularly well-behaved categories! It also gives us justification in saying that in order to study modules, one may as well just study Abelian categories or a special subclass of Abelain categories.

Thank you, dear reader, for taking part in this journey of discovery of Abelian categories and the Mitchell-Freyd Embedding Theorem. This perhaps not small introduction, I hope, was at least a faithful and exact introduction to the subject of Abelian categories and to the proof of the celebrated Embedding Theorem. While this was an introduction to but a small module of the theory of Abelian categories, but we can use it to embed a certain amount of understanding into the general case.

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