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Tempered Distributions and the Fourier Transform of a Distribution

Geoff Vooys

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Definitions and Whatnot

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Definition

Let $\mathcal{C}_c^\infty(\mathbb{R})$ denote the set of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ on compact support.

Definition

A distribution on $\mathbb{R}$ is a linear map

$$T : \mathcal{C}_c^\infty(\mathbb{R}) \to \mathbb{C}$$

such that whenever the sequence (g_n) converges to $g \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\lim_{n\to\infty} T(g_n) = T(g).$$

Definitions and Whatnot

Definition

A function $f : \mathbb{R} \to \mathbb{C}$ is said to be Schwartz if and only if f is inifinitely differentiable and if for all $m, n \in \mathbb{N}$ the number

$$\sigma_{m,n}(f) := \sup_{x \in \mathbb{R}} \{|x^m f^{(n)}(x)|\}$$

exists and is finite, i.e., $\sigma_{m,n}(f) \in \mathbb{R}$ for all $m, n \in \mathbb{N}$.

Definition

The set

$$\mathcal{S}(\mathbb{R}) := \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid \sigma_{m,n}(f) \in \mathbb{R}; m, n \in \mathbb{N}\}$$

is the space of Schwartz functions on $\mathbb{R}$.

May the Schwartz be With You

Theorem

$\mathcal{S}(\mathbb{R})$ is a commutative $\mathbb{C}$-algebra with pointwise multiplication.

Proof

Define the $\mathbb{C}$ action $\circ$ on $\mathcal{S}(\mathbb{R})$ by, for $a \in \mathbb{C}$ and $f \in \mathcal{S}(\mathbb{R})$

$$a \circ f(x) = af(x).$$

The fact that af is again Schwartz and that addition in $\mathcal{S}(\mathbb{R})$ respects the sup condition is not difficult to see; as such we need only worry that multiplication works out. Let $f, g \in \mathcal{S}(\mathbb{R})$. Then a routine induction gives for all $n \in \mathbb{N}^{\times}$ the formula

$$D^n(fg(x)) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k}(f) D^k(g)$$

where D denotes the differential operator. Because both f and g are infinitely differentiable, it follows that their product is by the formula above.

May the Schwartz Continue to be With You

Proof, cont.

Now let $m \in \mathbb{N}$ be arbitrary and consider the following:

$$
\begin{aligned}
|x^m D^n(fg)| &\leq |x^{2m} D^n(fg)| = \left| x^{2m} \sum_{k=0}^{n} \binom{n}{k} D^{n-k} f(x) D^k g(x) \right| \\
&\leq \sum_{k=0}^{n} \binom{n}{k} \left| x^{2m} D^{n-k} f(x) D^k g(x) \right| = \sum_{k=0}^{n} \binom{n}{k} \left| x^m D^{n-k} f(x) x^m D^k g(x) \right| \\
&\leq \sum_{k=0}^{n} \binom{n}{k} \sup_{x \in \mathbb{R}} |x^m D^{n-k} f(x)| \sup_{y \in \mathbb{R}} |y^m D^k g(y)| \\
&= \sum_{k=0}^{n} \binom{n}{k} \sigma_{m,n-k}(f) \sigma_{m,k}(g).
\end{aligned}
$$

This shows that the product of two Schwartz functions is again Schwartz. Commutativity is immediate and we are done. $\square$

Theorem

$\mathcal{C}_c^\infty(\mathbb{R})$ is a proper subspace of $\mathcal{S}(\mathbb{R})$.

Proof

Let $f \in \mathcal{C}_c^\infty(\mathbb{R})$. Then f and all its derivatives are bounded over $\mathbb{R}$. Let $A \subseteq \mathbb{R}$ be a compact interval with $\operatorname{supp}(f) \subseteq A$. Then for any $m, n \in \mathbb{N}$ we find that

$$\sigma_{m,n}(f) = \sup_{x\in\mathbb{R}}\{|x^m D^n f(x)|\} = \sup_{x\in A}\{|x^m D^n f(x)|\} \in \mathbb{R}.$$

Thusly we see that $\mathcal{C}_c^\infty \subseteq \mathcal{S}(\mathbb{R})$. To see that the containment is proper, consider the Gaussian kernel

$$f(x) := e^{-x^2}.$$

Then f is Schwartz over $\mathbb{R}$. However, e^{-x^2} is positive everywhere in $\mathbb{R}$, and so is not in $\mathcal{C}_c^\infty(\mathbb{R})$. This completes the proof. $\square$

Definition

We say that a sequence of functions $(f_k) \subseteq \mathcal{S}(\mathbb{R})$ converges to some $f \in \mathcal{S}(\mathbb{R})$ if, for all $m, n \in \mathbb{N}$,

$$\lim_{k\to\infty} \sigma_{m,n}(f - f_k) = 0.$$

Lemma

$\mathcal{S}(\mathbb{R})$ is a subspace of $L^2_{bc}(\mathbb{R})$. Furthermore, if (f_n) is a sequence of functions $f_n \in \mathcal{S}(\mathbb{R})$ such that $(f_n) \to f$ for some $f \in \mathcal{S}(\mathbb{R})$, then

$$\lim_{n\to\infty} \|f - f_n\|_2 = 0.$$

Tempered Distributions

Definition

Let $T : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ be a linear functional such that whenever (f_n) is a sequence in $\mathcal{S}(\mathbb{R})$ with limit f,

$$\lim_{n\to\infty} T(f_n) = T(f).$$

Such a functional T is called a Tempered Distribution and we denote the space of all tempered distributions by $\mathcal{S}(\mathbb{R})'$.

Remark

We note now that because $\mathcal{C}_c^\infty(\mathbb{R}) \subsetneq \mathcal{S}(\mathbb{R})$, whenever (g_k) is a sequence of functions in $\mathcal{C}_c^\infty(\mathbb{R})$ with limit $g \in \mathcal{C}_c^\infty(\mathbb{R})$, then any tempered distribution T satisfies $T(g_n) \to T(g)$. As such, after a restriction to $\mathcal{C}_c^\infty(\mathbb{R})$, the functional T is a distribution in the normal sense.

Rhostriction Maps and Injectivity

Proposition

The restriction map

$$\rho : \mathcal{S}(\mathbb{R})' \to \mathcal{C}_c^\infty(\mathbb{R})'$$

given by

$$T \mapsto T\big|_{\mathcal{C}_c^\infty(\mathbb{R})}$$

is injective.

Proof

Because both $\mathcal{C}_c^\infty(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ are $\mathbb{C}$-vector spaces, their corresponding dual spaces, $\mathcal{C}_c^\infty(\mathbb{R})'$ and $\mathcal{S}(\mathbb{R})'$ are as well; as such, in order to prove injectivity, we need to show that the kernel of ρ is trivial.
Aside: Personally I would like to do this by using the contravariant functor $\mathrm{Hom}_{\mathfrak{C}}(\cdot, \mathbb{C})$ and the homological characterisation of the dual space: $\mathcal{S}(\mathbb{R})' = \mathrm{Hom}_{\mathfrak{C}}(\mathcal{S}(\mathbb{R}), \mathbb{C})$ and $\mathcal{C}_c^\infty(\mathbb{R})' = \mathrm{Hom}_{\mathfrak{C}}(\mathcal{C}_c^\infty(\mathbb{R}), \mathbb{C})$. Sadly, I could not pin down the exact category that $\mathfrak{C}$ would need to be, and so we instead complete the proof in a more conventional way.

Proof, cont.

Since we are showing $\ker\rho$ is trivial, let $T \in \ker\rho$ and assume that $\rho(T) = 0 = T\big|_{\mathcal{C}_c^\infty(\mathbb{R})}$. Let $f \in \mathcal{S}(\mathbb{R})$. We will now show that $T(f) = 0$ so that T is the zero functional. To do this let $\varphi : \mathbb{R} \to [0,1]$ be a smooth function so that

$$\varphi(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 1 \end{cases}.$$

Define the functions χ_n for all positive $n \in \mathbb{N}^\times$ by

$$\chi_n(x) := \varphi(x+n)\varphi(n-x).$$

It then follows, $\chi_n \in \mathcal{C}_\mathrm{c}^\infty(\mathbb{R})$ and satisfies $\chi_n(x) = 1$ whenever $|x| \leq n-1$. We now construct a sequence that converges to f and will hence prove the result. Let $n \in \mathbb{N}^\times$ and define the sequence of functions (f_n) by

$$f_n(x) := \chi_n(x) f(x).$$

Then each $f_n \in \mathcal{C}_\mathrm{c}^\infty(\mathbb{R})$ and the sequence converges to $f \in \mathcal{S}(\mathbb{R})$ (!). Taking limits then completes the proof. $\square$.

Filling in a Detail

Question

Why is it that the sequence (f_n) converges to $f \in \mathcal{S}(\mathbb{R})$?

Answer

Unfortunately the analysis here is messy. Let us consider the difference $f - f_k$ for some arbitrary $k \in \mathbb{N}^{\times}$. In order for $(f_k) \to f$, we need that each number

$$\sigma_{m,n}(f - f_k) \to 0$$

as $k \to \infty$ for any $m, n \in \mathbb{N}$. To see this let $\varphi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R})$ as in the previous proof and let $f_k = \chi_k f$. Then for any $m, n \in \mathbb{N}$ we have

$$\begin{aligned}\sigma_{m,n}(f - f_k) &= \sup_{x\in\mathbb{R}} \{|x^m D^n (f - f_k)|\} = \sup_{x\in\mathbb{R}} \{|x^m D^n f - x^m D^n f_k|\} \\ &= \sup_{x\in\mathbb{R}} \{|x^m D^n f - x^m D^n \chi_k f|\} = \sup_{x\in\mathbb{R}} \{|x^m D^n f - x^m D^n (\varphi(x+k)\varphi(k-x)) f|\} \\ &= \sup_{x\in\mathbb{R}} \left\{ \left| x^m D^n f - x^m \sum_{i=0}^{n} \binom{n}{i} D^{n-i}(\varphi(x+k)\varphi(k-x)) D^i f \right| \right\}.\end{aligned}$$

Answer, cont

Continuing, we can show that

$$\sigma_{m,n}(f) = \sup_{x\in\mathbb{R}} \left\{ \left| x^m D^n f - x^m \sum_{i=0}^{n} \binom{n}{i} D^{n-i}(\varphi(x+k)\varphi(k-x)) D^i f \right| \right\}$$

$$= \sup_{x\in\mathbb{R}} \left\{ \left| x^m D^n f - x^m \sum_{i=0}^{n} \binom{n}{i} \left(\sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} D^{n-i-j}\varphi D^j \varphi \right) D^i f \right| \right\}$$

$$= \sup_{x\in\mathbb{R}} \left\{ \left| x^m D^n f - \sum_{i=0}^{n} \binom{n}{i} x^m D^i f \left(\sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} D^{n-i-j}\varphi D^j \varphi \right) \right| \right\}.$$

Each term $x^m D^i f$ exists and is finite because $f \in \mathcal{S}(\mathbb{R})$. All we need worry about is whether or not the monster term explodes or not. However, $\varphi : \mathbb{R} \to [0,1]$, so the derivatives of φ that appear have small maxima; furthermore, because of the alternating sign in the power of -1, the sum may be made sufficiently small by giving a bound depending on k, and we may claim that $\sigma_{m,n}(f - f_k) \to 0$ as $k \to \infty$. This justifies (somewhat) the hole left in the proof of the restriction being injective.

Induced Tempered Distributions

Proposition

Let φ be a function locally integrable on $\mathbb{R}$ such that there exists some $n \in \mathbb{N}$ such that the integral

$$\int_{-\infty}^{\infty} \frac{|\varphi(x)|}{1+x^{2n}} \mathrm{d}x < \infty.$$

Then for every $f \in \mathcal{S}(\mathbb{R})$ the integral

$$I_\varphi(f) := \int_{-\infty}^{\infty} \varphi(x) f(x) \mathrm{d}x$$

converges and induces a tempered distribution given by $f \mapsto I_\varphi(f)$.

Proof

We claim that the convergence of the integral $I_\varphi(f)$ is ``clear." Let $C = \int_{-\infty}^{\infty} |\varphi(x)|/(1+x^{2n})\mathrm{d}x$ and WOLOG take $C > 0$. Then let (f_m) be a sequence of functions in $\mathcal{S}(\mathbb{R})$ that converges to $f \in \mathcal{S}(\mathbb{R})$. For any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that whenever $m \geq N$ we have

$$|f - f_m| < \frac{\varepsilon}{C(1+x^{2n})}.$$

For all $m \geq N$ we then derive that (using the triangle inequality for integrals)

$$\begin{aligned} |I_\varphi(f) - I_\varphi(f_m)| &\leq \int_{-\infty}^{\infty} |\varphi||f - f_m(x)|\mathrm{d}x \\ &< \frac{\varepsilon}{C}\int_{-\infty}^{\infty} \frac{|\varphi(x)|}{1+x^{2n}}\mathrm{d}x = \varepsilon. \end{aligned}$$

This completes the proof. $\square$

Embedding L^2 into $\mathcal{S}(\mathbb{R})'$

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Theorem

The map $\iota : L^2_{bc}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})'$ is injective and extends to a natural embedding $L^2 \to \mathcal{S}(\mathbb{R})'$.

Proof

Let $\varphi \in L^2_{bc}(\mathbb{R})$ such that $I_\varphi = 0$. Then for any $f \in \mathcal{S}(\mathbb{R})$ we have

$$\langle \varphi, \overline{f} \rangle = I_\varphi(\overline{f}) = 0$$

and so by the density of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$, it follows that $\varphi = 0$ and so $L^2_{bc}(\mathbb{R})$ embeds into $\mathcal{S}(\mathbb{R})$. To see that ι may be extended, let (φ_n) be a Cauchy sequence in $L^2_{bc}(\mathbb{R})$ with limit $\varphi \in L^2(\mathbb{R})$ and let $f \in \mathcal{S}(\mathbb{R})$. Then since $\mathcal{S}(\mathbb{R})$ is a subspace of $L^2_{bc}(\mathbb{R})$ we may take the L^2 inner product of φ_n with the conjugate of f to derive that

$$\langle \varphi_n, \overline{f} \rangle = \int_{-\infty}^{\infty} \varphi_n(x) f(x) \mathrm{d}x = I_{\varphi_n}(f).$$

Extending the Embedding Proof

Proof, cont

As such we find that for any $m, n \in \mathbb{N}$

$$|I_{\varphi_m}(f) - I_{\varphi_n}(f)| = \left|\langle \varphi_m - \varphi_n, \overline{f}\rangle\right| \leq \|\varphi_m - \varphi_n\|_2 \|f\|_2$$

by the Cauchy-Schwarz inequality. Because the sequence (φ_n) is Cauchy, the above estimate implies that sequence (I_{φ_n}) is also Cauchy. The fact that the convergence of $(I_{\varphi_n}) \to I_\varphi$ does not depend on any particular choice of a sequence $(\varphi_n) \to \varphi$ is clear.
We now verify that I_φ is indeed a distribution. To see this let $(f_k) \to f$ for functions $f_k, f \in \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned} |I_\varphi(f) - I_\varphi(f_k)| &= |I_\varphi(f - f_k) = \lim_{n\to\infty} |I_{\varphi_n}(f - f_k)| \\ &= \lim_{n\to\infty} |\langle \varphi_n, \overline{f - f_k}\rangle| \leq \lim_{n\to\infty} \|\varphi_n\|_2 \|f - f_k\|_2 = \|\varphi\|_2 \|f - f_k\|_2. \end{aligned}$$

By a lemma previously discussed, the final term converges to 0 as $k \to \infty$, showing that I_φ is a tempered distribution. This completes the proof and shows that $L^2(\mathbb{R})$ may be naturally embedded in $\mathcal{S}(\mathbb{R})'$. $\square$

Remark

Note that the proof we just completed shows that there exists a commutative diagram of the form

$$\begin{tikzcd} L^2(\mathbb{R}) \arrow[r, "\hat{\iota}"] & \mathcal{S}(\mathbb{R})' \\ \mathcal{S}(\mathbb{R}) \arrow[u] \arrow[r] & L^2_{bc}(\mathbb{R}) \arrow[ul] \arrow[u, "\iota"'] \end{tikzcd}$$

Things should generalize by playing the same game that we just did over any complete field K of characteristic zero, provided we alter the codomain of our Schwartz, L^2, and L^2_{bc} functions to be the algebraic and topological completion of K and not just ``plain old'' $\mathbb{C}$. However, I would not really know how to go about showing this, so this is idle speculation at this point.

Tools for Defining the Fourier Transform of a Distribution

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Consider the Following

Let $f \in \mathcal{S}(\mathbb{R})$ and recall that the Fourier inversion formula tells us that

$$\mathscr{F}^2(f)(x) = f(-x).$$

We now ask ourselves if we may interchange the order taking complex conjugation and the Fourier transform. While the two functions do not commute, we can get fairly close to commutativity; in particular, we derive that

$$\mathscr{F}(\overline{f})(x) = \int_{-\infty}^{\infty} \overline{f(y)} e^{-2\pi i x y} \mathrm{d}y = \overline{\int_{-\infty}^{\infty} f(y) e^{2\pi i x y} \mathrm{d}y} = \overline{\mathscr{F}(f)(-x)}$$

and so

$$\mathscr{F}^3(\overline{f}) = \overline{\mathscr{F}(f)}.$$

This leads us to consider that for any $\varphi, f \in \mathcal{S}(\mathbb{R})$,

$$I_{\mathscr{F}(\varphi)}(f) = \langle \mathscr{F}(\varphi), \overline{f} \rangle = \langle \mathscr{F}^2(\varphi), \mathscr{F}(\overline{f}) \rangle = \langle \varphi, \mathscr{F}^3(\overline{f}) \rangle = \langle \varphi, \overline{\mathscr{F}(f)} \rangle = I_\varphi(\mathscr{F}(f)).$$

The Fourier Transform of a Distribution

Definition

Let $T \in \mathcal{S}(\mathbb{R})'$. Then we define the Fourier Transform of T to be

$$\mathscr{F}(T) := T(\mathscr{F}(f)).$$

Proposition

Let $T \in \mathcal{S}(\mathbb{R})'$ and let $f \in \mathcal{S}(\mathbb{R})$. Then

$$\mathscr{F}^2(T(f)) = T(\mathscr{F}^2(f)).$$

Proof (it's kinda cute)

Let everything be given as above. Then

$$\mathscr{F}^2(T(f)) = \mathscr{F}(T(\mathscr{F}^3(f)) = T(\mathscr{F}^6(f)) = T(\mathscr{F}^2(f)).$$

This completes the proof. □

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Definition

We say that a function $f \in \mathcal{C}^\infty(\mathbb{R})$ has moderate growth if for all $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}} \left\{ \frac{|D^n f(x)|}{1+x^{2m}} \right\}.$$

Proposition

Let f be a function of moderate growth and let $g \in \mathcal{S}(\mathbb{R})$. Then $fg(x) := f(x)g(x) \in \mathcal{S}(\mathbb{R})$ and if $(g_k) \to g \in \mathcal{S}(\mathbb{R})$, then the sequence $(fg_k) = (f(x)g_k(x)) \to fg$.

Proof of the Proposition

Proof

Let f, g, and (g_k) be given as above. We omit the proof of the fact that $fg \in \mathcal{S}(\mathbb{R})$ because it is analogous to showing that $\mathcal{S}(\mathbb{R})$ is a $\mathbb{C}$-algebra (the multiplication component, in particular). We now show the convergence of the sequence. Let $m, n \in \mathbb{N}$ and let $\varepsilon > 0$ be given. Observe that for m and n we have

$$\begin{aligned} |x^m D^n(fg)| &= \left| x^m \sum_{i=0}^{n} \binom{n}{i} D^{n-i} f D^i g \right| \\ &\leq |x^m| \sum_{i=0}^{n} \binom{n}{i} |D^{n-i} f(x)||D^i g(x)| \\ &\leq C|x|^m \left(1 + x^{2M}\right) \sum_{i=0}^{n} \binom{n}{i} |D^i g(x)|, \end{aligned}$$

for some constant $C \in \mathbb{R}_+$ and $M \in \mathbb{N}$ depending only on f and n.

The Proof of the Proposition, Continued

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Proof, cont.

Through the convergence of (g_k), we can find an $N \in \mathbb{N}$ such that whenever $k \geq N$ we have

$$C \sum_{i=0}^{n} \binom{n}{i} (\sigma_{m,n-i}(g - g_k) - \sigma_{m+2M,n-i}(g - g_k)) < \varepsilon$$

and so, by the convergence of $(g_k) \to g$, it follows that we may pick our N so that

$$\sigma_{m,n}(fg - fg_k) < \varepsilon,$$

completing the proof. $\square$

Definition

Let f be a function of moderate growth and $T \in \mathcal{S}(\mathbb{R})'$. Then we define the product fT as

$$fT(g) := T(fg).$$

Properties of the Distribution Transform

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Theorem

Let $S, T \in \mathcal{S}(\mathbb{R})'$. Then we have the following:

1. If $S(x) = -2\pi i x T(x)$, then $D(\mathscr{F}(T)) = \mathscr{F}(S)$, where D denotes the derivative, i.e., it is the differential operator.
2. $\mathscr{F}(DT)(y) = 2\pi i y \, \mathscr{F}(T)(y)$.

Remark

Recall that S and T are not, techinically speaking, functions of a real variable x. However, the notation is convenient, and so we shall just deal with the abuse as necessary. We must also observe that we have not defined differentiation for tempered distributions! However, the same game that we played with normal distributions may be carried out for tempered distributions and so we take the derivative to be the same thing; i.e., we define the derivative DT of a tempered distribution as

$$DT(f) := -T(Df).$$

Proof of the Theorem

Proof

Let S and T be given as in the statement of the theorem and let $f \in \mathcal{S}(\mathbb{R})$. We pretend like orders are natural and (naturally) prove condition (1) first.
Let $S(x) = -2\pi i x T(x)$. Then we find that

$$\begin{aligned} D(\mathscr{F}(T))(f) &= -\mathscr{F}(T)(Df) = -T(\mathscr{F}(Df)) = -T(2\pi i y \mathscr{F}(f)) \\ &= S(\mathscr{F}(f)) = \mathscr{F}(S)(f). \end{aligned}$$

For part (2) we note that

$$\begin{aligned} \mathscr{F}(DT)(f) &= DT(\mathscr{F}(f)) = -T(D(\mathscr{F}(f))) = T(\mathscr{F}(2\pi i x f(x))) \\ &= \mathscr{F}(T)(2\pi i x f(x)) = 2\pi i y \mathscr{F}(T)(f). \end{aligned}$$

This completes the proof. $\square$