INFINITY CATEGORIES WITH A SIMPLICIAL FLAVOR

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1. The Category $\mathbf{\Delta}$

While the purpose of these notes is very much to give an introduction as to what is happening in $(\infty, 1)$-categories (and to a certain degree in $(\infty, 0)$-categories), the approach to higher category that we take will be that of what are deemed *quasi-categories*. Because of this we will need to familiarize ourselves with simplicial sets, and consequently with the category $\mathbf{\Delta}_0$ of finite ordinals (which contains a "universal" monoid). Because this category is important to the theory of strict monoidal categories and the theory of monads, we begin by reviewing some of the relevant definitions (as well as giving a connection of the theory of simplicial sets with the theory of monads).

Definition 1.1 ([ML98])**.** *A category $\mathfrak{C}$ is said to be a* strict monoidal categoyr *if there is a bifunctor $\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ and an object E of $\mathfrak{C}$ (which we will call the unit of $\mathfrak{C}$) such that the diagrams*

$$\begin{tikzcd}
\mathfrak{C} \otimes \mathfrak{C} \otimes \mathfrak{C} \arrow[r, "\langle \otimes, \mathrm{id}_{\mathfrak{C}} \rangle"] \arrow[d, "\langle \mathrm{id}_{\mathfrak{C}}, \otimes \rangle"'] & \mathfrak{C} \times \mathfrak{C} \arrow[d, "\otimes"] \\
\mathfrak{C} \times \mathfrak{C} \arrow[r, "\otimes"'] & \mathfrak{C}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathfrak{C} \arrow[r, "(-) \times E"] \arrow[d, "E \times (-)"'] \arrow[dr, equal] & \mathfrak{C} \otimes \mathfrak{C} \arrow[d, "\otimes"] \\
\mathfrak{C} \otimes \mathfrak{C} \arrow[r, "\otimes"'] & \mathfrak{C}
\end{tikzcd}$$

both commute.

We will write $(\mathfrak{C}, \otimes, E)$ to emphasize the data required of a strict monoidal category. Note that the second commuative diagram implies that

$$E \otimes A = A = A \otimes E$$

for every $A \in \operatorname{Ob} \mathfrak{C}$.

Definition 1.2 ([ML98])**.** *Let $(\mathfrak{C}, \otimes, E)$ and $(\mathfrak{D}, \oplus, I)$ be strict monoidal categories. We then say that $F : \mathfrak{C} \to \mathfrak{D}$ is a* morphism of strict monoidal categories *if F is a functor such that for all objects $A, B \in \operatorname{Ob} \mathfrak{C}$ and all morphisms $f, g \in \operatorname{Mor} \mathfrak{C}$ we have:*

$$\begin{aligned} F(A \otimes B) &= FA \oplus FB \\ F(f \otimes g) &= F(f) \oplus F(g) \\ F(E) &= I \end{aligned}$$

Definition 1.3 ([ML98])**.** *Let $(\mathfrak{C}, \otimes, E)$ be a strict monoidal category. We then say that a* monoid in $\mathfrak{C}$ *is an object M of $\mathfrak{C}$ together with morphisms $\mu : M \otimes M \to M$ (the "multiplication" of the monoid) and a map*

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$\eta : E \to M$ such that the diagrams

$$\begin{tikzcd}
M \otimes M \otimes M \arrow[r, "\mathrm{id}_M \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{id}_M"'] & M \otimes M \arrow[d, "\mu"] \\
M \otimes M \arrow[r, "\otimes"'] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
E \otimes M \arrow[r, "\eta \otimes \mathrm{id}_M"] \arrow[dr, equal] & M \otimes M \arrow[d, "\mu"'] & M \otimes E \arrow[l, "\mathrm{id}_M \otimes \eta"'] \arrow[dl, equal] \\
& M &
\end{tikzcd}$$

both commute.

Definition 1.4 ([ML98])**.** *Let $\mathbf{\Delta}_0$ be the category defined as follows:*

- *Objects: Linearly ordered sets $[n] = \{0, \cdots, n-1\}$ for all $n \in \mathbb{N}$ where $[0] := \varnothing$;*
- *Morphisms: Weakly order preserving morphisms, i.e., maps $f : [m] \to [n]$ where $k \leq \ell$ implies $f(k) \leq f(\ell)$.*

Composition and identity maps are defined in the obvious way. Furthermore, since $\mathbf{\Delta}_0$ is a subcategory of **Set** there is a $(\mathscr{E}, \mathscr{M})$ (epic, monic) orthogonal factorization system on $\mathbf{\Delta}_0$.

Proposition 1.1. *The category $\mathbf{\Delta}_0$ is a strict monoidal category.*

Proof. In order to prove this we must find a bifunctor and a unit. Define the functor $\oplus : \mathbf{\Delta}_0 \times \mathbf{\Delta}_0 \to \mathbf{\Delta}_0$ by

$$[m] \oplus [n] := [m+n]$$

and, for all maps $f : [m] \to [k]$ and $g : [n] \to [\ell]$,

$$(f \oplus g)(i) := \begin{cases} f(i) & 0 \leq i \leq m-1 \\ k + f(i-m) & m \leq i \leq m+n-1 \end{cases}$$

Then it is trivial to verify that $\oplus$ is a bifunctor. To verify the first commuting diagram note that

$$[n] \oplus [m] \oplus [\ell] = [n+m] \oplus [\ell] = [n+m+\ell] = [n] \oplus [m+\ell] = [n] \oplus [m] \oplus [\ell].$$

For the morphisms note that it follows similarly that $(f \oplus g) \oplus h = f \oplus (g \oplus h)$.
Now consider that

$$[0] \oplus [n] = [0+n] = [n] = [n+0] = [n] \oplus [0]$$

showing that $[0]$ is a unit to $(\mathbf{\Delta}_0, \oplus)$. This proves that with $E = [0]$ the second diagram commutes, proving that $(\mathbf{\Delta}_0, \oplus, [0])$ is a strict monoidal category. $\square$

Observe that since $[1] = \{0\}$, $[1]$ is terminal in $\mathbf{\Delta}_0$. Consequently there are unique maps $\eta : [0] \to [1]$ and $\mu : [2] = [1] \oplus [1] \to [1]$ that satisfy the monoid axioms, making $([1], \mu, \eta)$ into a monoid in $\mathbf{\Delta}_0$. From here on we write the unique map $\tau_{[n]} : [n] \to [1]$ as $\mu^{(n)} : [n] \to [1]$. Then $\mu^{(0)} = \eta$, $\mu^{(1)} = \mathrm{id}_{[1]}$, and $\mu^{(2)} = \mu$. From the fact that $[1]$ is terminal it follows that if $1 \leq i \leq n$ then

$$\left(\bigoplus_{i=0}^{n} \mu^{(k_i)} \right) \mu^{(n)} = \mu^{(k_1 + \cdots + k_n)};$$

that is, the diagram

$$\begin{tikzcd}
{[k_1] \oplus \cdots \oplus [k_n]} \arrow[r, equal] \arrow[d, "\mu^{(k_1)} \oplus \cdots \oplus \mu^{(k_n)}"'] & {[k_1 + \cdots + k_n]} \arrow[d, "\mu^{(k_1 + \cdots + k_n)}"] \\
{[n]} \arrow[r, "\mu^{(n)}"'] & {[1]}
\end{tikzcd}$$

commutes in $\mathbf{\Delta}_0$.

We now show that any morphism in $\mathbf{\Delta}_0$ is a sum of $\mu^{(k_i)}$ maps. To do this let $f : [m] \to [n]$ be any morphism in $\mathbf{\Delta}_0$ and for all $0 \le i \le n-1$ define

$$m_i := \left|f^{-1}(i)\right|$$

so that m_i is the size of the preimage of f above $i \in [n]$. Note that since the sum of the m_i's must cover $[m]$ we have

$$m = \sum_{i=0}^{n-1} m_i$$

and that it is possible that we could have $m_i = 0$ for some (but not all) $i \in [n]$. From these facts it follows that

$$f = \bigoplus_{i=0}^{n-1} \mu^{(m_i)}$$

and furthermore that this decomposition is unique. From this we get the following proposition immediately:

Proposition 1.2 ([ML98]). *Let $(\mathfrak{C}, \otimes, E)$ be a strict monoidal category and (M, μ', η') a monoid in $\mathfrak{C}$. Then there is a unique morphism of strict monoidal categories such that the diagram*

$$\begin{tikzcd}
{[0]} \arrow[r, "\eta"] \arrow[d, "F"'] & {[1]} \arrow[d, "F"'] & {[2] = [1] \oplus [1]} \arrow[l, "\mu"'] \arrow[d, "F"] \\
E \arrow[r, "\eta'"'] & M & M^{(2)} = M \otimes M \arrow[l, "\mu'"]
\end{tikzcd}$$

commutes, i.e., we have the equalities:

$$F(\eta) = \eta'$$
$$F(\mu) = \mu'$$
$$F([1]) = M$$

Sketch. We only sketch the proof. Since F is a morphism of strict monoidal functors we have immediately that $F([0]) = E$. From the fact that $F([1]) = M$ and that F preserves tensors (preserves bifunctors) we force the assignment, for all $n \ge 1$,

$$F([n]) = F\left(\bigoplus_{i=1}^{n}[1]\right) = \bigotimes_{i=1}^{n} F([1]) = \bigotimes_{i=1}^{n} M = M^{(n)}.$$

The assertions that $F(\eta) = \eta'$ and $F(\mu) = \mu'$ and the unique decomposition of maps

$$f = \bigoplus_{i=1}^{n} \mu^{(m_i)}$$

in $\mathbf{\Delta}_0$ force that

$$F(f) = F\left(\bigoplus_{i=1}^{n} \mu^{(m_i)}\right) = \bigotimes_{i=1}^{n} F(\mu^{(m_i)}) = \bigotimes_{i=1}^{n} (\mu')^{(m_i)}$$

uniquely determines the assignment of F on maps. $\square$

We now move forward to give an alternative description of the maps in $\mathbf{\Delta}_0$ that is more consistent with what is traditionally described in the literature (such as the descriptions in [GJ09], [GS07], [JT08], and [Lur09]) that are used to consrtruct much of the theory of simplicial sets and simplicial categories. We begin by defining two important classes of maps: the cofaces and codegeneracies.

Definition 1.5 ([Lur09]). *Let $n \in \mathbb{N}$ and $0 \le k \le n$. Then define the map $d^k : [n] \to [n+1]$ via the assignment*

$$d^k(\ell) := \begin{cases} \ell & \ell < k, \\ \ell + 1 & \ell \ge k. \end{cases}$$

and, for $n \geq 1$, $s^k : [n+1] \to [n]$ *via the assignment*

$$s^k(\ell) := \begin{cases} \ell & \ell \leq k, \\ \ell - 1 & \ell > k. \end{cases}$$

Lemma 1.1 ([JT08],[ML98])**.** *In* $\mathbf{\Delta}_0$ *any map* $f : [n] \to [m]$ *has a unique representation*

$$s^{j_1} s^{j_2} \cdots s^{j_k} d^{i_\ell} d^{i_{\ell-1}} \cdots d^{i_1}$$

where the indicies satisfy $n - k + \ell = m$ *and*

$$0 \leq j_k < j_{k-1} < \cdots < j_1 < n-1 \qquad 0 \leq i_\ell < i_{\ell-1} < \cdots < i_1 < m.$$

Proof. We first observe that any map f is determined by its image $S \subseteq [m]$ and hence by the $i \in [n]$ for which $i \notin \mathrm{im}(f)$. Order all such i by size so that $0 \leq i_\ell < i_{\ell-1} < \cdots < i_1 < m$. Now determine all $j \in [n]$ such that $f(j) = f(j+1)$ and order them by size so that $0 \leq j_k < j_{k-1} < \cdots < j_1 < m-1$. Then we find $n - k + \ell = m$ and a quick calculation verifies that

$$f = s^{j_1} s^{j_2} \cdots s^{j_k} d^{i_\ell} d^{i_{\ell-1}} \cdots d^{i_1}.$$

The uniqueness of this factorization follows from the fact that the composition of all s^j's is epic and the composition of all d^i's is monic; consequently the factorization above is the unique $(\mathscr{E}, \mathscr{M})$ factorization of f. $\square$

Another important property of the coface and codegeneracy maps is that they satisfy some important equational relations. These relations may be used to commute functions of the form $d^i s^j$ into the form $s^k d^\ell$ for some k and ℓ. The equations we present below are frequently called the *cosimplicial identities.*

Lemma 1.2 ([JT08])**.** *Let* $n \geq 1$ *and let* $[n] \in \mathbf{\Delta}_0$*. Then:*

$$\begin{cases} d^i d^j = d^{j-1} d^i & i < j \\ s^i s^j = s^{j+1} s^i & i \leq j \\ d^i s^j = s^{j-1} d^i & i < j \\ d^i s^j = \mathrm{id} & i = j, j+1 \\ d^i s^j = s^j d^{i-1} & i > j+1 \end{cases}$$

The proof of the lemma is a routine calculation and so is omitted. However, from this lemma and the lemma prior we immediately derive the following proposition:

Proposition 1.3 ([ML98])**.** *The category* $\mathbf{\Delta}_0$ *is generated by the* d^i *and* s^j *maps subject to the relations in Lemma 1.2.*

We now move to work with the full subcategory $\mathbf{\Delta}$ of nonempty linearly ordered sets in $\mathbf{\Delta}_0$. We build the theory of simplicial categories from $\mathbf{\Delta}$, as this is the subcategory of $\mathbf{\Delta}$ in which there are coface and codegeneracy maps for all $n \in \mathbb{N}$, as opposed to situation in which there are no codegeneracy maps $s^i : [1] \to [0]$ because the empty set recieves no maps with nonempty domain.

Definition 1.6 (Lurie, p.821)**.** *Let* $\mathbf{\Delta}$ *be the category defined as:*

(1) *Objects: Linearly ordered sets* $[n] = \{0, \cdots, n\}$ *(one for each* $n \in \mathbb{N}$*).*
(2) *Morphisms: Order-preserving functions of sets (non-strict)* $[m] \to [n]$*, i.e., if* $k \leq \ell$ *for* $k, \ell \in [n]$*, then* $f : [n] \to [m]$ *if* $f(k) \leq f(\ell)$ *in* $[m]$*.*

We then say that $\mathbf{\Delta}$ *is called the category of* combinatorial simpleces*.*

Lemma 1.3 ([JT08])**.** *Let* $\mathfrak{C}$ *be a category and let*

$$\begin{tikzcd} A \arrow[r, shift left, "p_4"] \arrow[d, shift right, "p_3"'] & B \arrow[l, shift left, "s_4"] \arrow[d, shift left, "p_1"] \\ C \arrow[u, shift right, "s_3"'] \arrow[r, shift right, "p_2"'] & D \arrow[l, shift right, "s_2"'] \arrow[u, shift left, "s_1"] \end{tikzcd}$$

be a commutative diagram. Then if the identites

$$s_1 s_4 = s_2 s_3$$
$$p_4 p_1 = p_3 p_2$$
$$s_3 p_4 = p_2 s_1$$
$$s_i p_i = \mathrm{id}$$

and either

$$s_4 p_3 = p_1 s_2$$

or $p_1 = p_2$, $s_1 = s_2$, *and* $s_4 p_3 = \mathrm{id}$. *Then the square of* s *maps is an absolute pullback and the square of* p *maps is an absolute pushout.*

It is worth remarking that in [JT08] the square above is described as an *equational* absolute pullback or pushout. All that is meant in that case is that the pushout/pullback is determined by the equational relations on the maps, just like how certain analytic functions in complex analysis can be determined by functional equations.

Proof. We prove only the pushout; the other case is handled by duality. Begin by letting $g_1 : B \to X$ and $g_2 : C \to X$ be morphisms such that the diagram

$$\begin{tikzcd} & & X \\ B \arrow[r, "p_1"] \arrow[urr, bend left, "g_1"] & D & \\ A \arrow[u, "p_4"] \arrow[r, "p_3"'] & C \arrow[u, "p_2"'] \arrow[uur, bend right, "g_2"'] & \end{tikzcd}$$

commutes. Assume that $s_4 p_3 = p_1 s_2$ and observe that

$$g_1 = s_4 p_4 = s_4 p_3 g_2$$

so that

$$s_1 g_1 = s_1 s_4 p_3 g_2 = s_2 s_3 p_3 g_2 = s_2 g_2$$

as morphisms $D \to X$. This is unique, as if $f : D \to X$ was a morphism for which $p_1 f = g_1$ and $p_2 f = g_2$ then we would have

$$s_1 g_1 = s_1 p_1 f = f$$

and

$$s_2 g_2 = s_2 p_2 f = f,$$

establishing the uniqueness of $s_1 g_1$. On the other hand, if $p_1 = p_2$, $s_1 = s_2$, and $s_4 p_3 = \mathrm{id}$ then

$$g_1 = s_4 p_3 g_2 = g_2$$

giving that $s_1 g_1 = s_2 g_2$. This is again unique by the same reasoning, which proves the lemma. $\square$

From this it follows that

$$\begin{tikzcd} {[n+1]} \arrow[r, "s^i"] \arrow[d, "s^j"'] & {[n]} \arrow[d, equal] \\ {[n]} \arrow[r, equal] \arrow[ur, phantom, "\ulcorner"] & {[n]} \end{tikzcd}$$

is a pushout in $\mathbf{\Delta}$ because s^i is surjective and equational because s^i has a section. While absolute pushouts always exist, it is not necessarily true that the absolute pullback of injections exist in $\mathbf{\Delta}$. For instance, the pullback of

$$\begin{tikzcd} & {[0]} \arrow[d, "d^0"] \\ {[0]} \arrow[r, "d^1"'] & {[1]} \end{tikzcd}$$

does not exist in $\mathbf{\Delta}$ because there is no empty object in $\mathbf{\Delta}$. These observations lead us to the following: proposition:

Theorem 1.1 ([JT08]). *$\mathbf{\Delta}$ has absolute pushouts of surjections and absolute pullbacks of injections with nonempty intersection.*

2. Simplicial Sets, Geometric Realization, and the Quillen Model Structure on $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$

We now move from introducing the theory of the category $\mathbf{\Delta}$ to the theory of simplicial sets and simplicial categories. We need this at hand to discuss the model structure of simplicial sets so that we may understand the homotopy theory of $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ (and of **Top** through an adjunction) given by the (Quillen) model structure on the category of simplicial sets. This adjunction and intuition is crucial to the formalism underlying ∞-categories, so we need at hand a good understanding of simplicial sets and simplicial categories.

Definition 2.1 (Lurie, p.821). *Let $\mathfrak{C}$ be any category. We then say that a* simplicial object of $\mathfrak{C}$ *is a presheaf $F \in [\mathbf{\Delta}^{op}, \mathfrak{C}]$.*

While this definition is quite general, the most important simplicial category we will use is the category of simplicial sets, $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$. We will now give some notational conventions and formalism of simplical categories that will make our discussion much less cumbersome.

Definition 2.2. *Let $X \in [\mathbf{\Delta}^{op}, \mathfrak{C}]$ for some category $\mathfrak{C}$. We will then write the simplicial object X as the following data:*

1. *An object $X_n := X([n])$ for every $n \in \mathbb{N}$;*
2. *The map $f^* = X(f)$ for every $f \in \mathbf{\Delta}$.*

Furthermore, for every $0 \leq k \leq n$, $n \in \mathbb{N}$, define the k-th face map *as $d_k := X(d^k) : X_n \to X_{n+1}$ and the* k-th degeneracy map *as $s_k := X(s^k) : X_{n+1} \to X_n$.*

Definition 2.3. *Let X be a simplicial object in a category $\mathfrak{C}$. We then say that an element $x \in X_n$ is an* n-simplex *in X.*

Definition 2.4. *We say that an n-simplex $x \in X_n$, for a simplicial set X, is* degenerate *if there is a natural number $m < n$, a surjection $\varepsilon : [n] \to [m]$, and an m-simplex $y \in X_m$ such that $\varepsilon^*(y) = x$.*

There are some various important identities that the face and degeneracy maps satisfy. These relations will be frequently of use in practice for calculating the nature of a map $f^* : X_n \to X_m$. Note that they all follow from dualizing the relations described in Lemma 1.2.

Lemma 2.1 ([GS07]). *Let X be a simplicial object in a category $\mathfrak{C}$. Then the following hold:*

$$\begin{aligned} d_k d_\ell &= d_\ell d_{k-1} && \ell < k \\ s_k d_\ell &= d_\ell s_{k-1} && \ell < k \\ s_k d_\ell &= \mathrm{id}_{X_n} && \ell = k, k+1 \\ s_k d_\ell &= d_{\ell-1} s_k && \ell > k+1 \\ s_k s_\ell &= s_\ell s_{k+1} && \ell \leq k \end{aligned}$$

From here we move to discuss the model structure on $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$, as this is what we will primarily be interested in. We begin by recalling some facts about presheaves and coends, and then to define a the geometric realization functor $|\cdot| : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Top}$.

Lemma 2.2. *Let $\mathfrak{C}$ be a category. Then $[\mathfrak{C}^{op}, \mathbf{Set}]$ is* **Set**-*(co)complete.*

Proof (sketch). The proof of this fact is largely an exercise. If we have a family $\{F_i \mid F_i \in [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}], i \in I\}$ of presheaves over $\mathfrak{C}$ with natural transformations $\{\alpha_{ij} : F_i \to F_j \mid i, j \in I\}$, then for every $U \in \mathfrak{C}$ the objects

$$\varprojlim F_i U$$

and

$$\varinjlim F_i U$$

both exist. A quick check shows that the presheaf defined by either

$$U \mapsto \varprojlim F_i U$$

or

$$U \mapsto \varinjlim F_i U$$

has the desired universal property of a limit or colimit, respectively. □

Corollary 2.1. *The category* $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ *is (co)complete.*

Definition 2.5. *A presheaf* $\rho \in [\mathfrak{C}^{op}, \mathbf{Set}]$ *is said to be* representable *if there is a natural isomorphism*

$$\rho \cong \mathfrak{C}(-, U)$$

for some object $U \in \operatorname{Ob} \mathfrak{C}$*.*

We now recall that if $\mathfrak{C}$ is a category with **Set**-indexed coproducts and if $X \in \operatorname{Ob} \mathbf{Set}$ and $A \in \operatorname{Ob} \mathfrak{C}$, we define the *copower* of X and A to be

$$X \odot A := \coprod_{x \in X} A.$$

Lemma 2.3 (The Density Lemma)**.** *Any presheaf* $F \in [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ *is the colimit of representable presheaves.*

We prove the above lemma by use of coend calculus. We take for granted that coends are colimits and the dinatural calculus involving the functors

$$F(-) := \int^{A \in \mathfrak{C}} FA \odot \mathfrak{C}(-, A).$$

Proof. Let F and G be presheaves in $[\mathfrak{C}^{\mathrm{op}}, \mathbf{Set}]$. We will show that there is a natural bijection

$$\mathbf{Nat}(F, G) \cong \mathbf{Nat}\left(\int^{X \in \mathfrak{C}} FX \odot \mathfrak{C}(-, X), G\right).$$

Begin by observing that any natural transformation

$$\alpha : \int^{X \in \mathfrak{C}} F(X) \odot \mathfrak{C}(-, X) \implies G$$

is a dinatural transformation. So pick such an α and consider that α is a family of transformations

$$\alpha_{A,B} : FA \odot \mathfrak{C}(B, A) \to GB$$

dinatural in A and natural in B. These transformations are in natural bijective correspondence with transformations

$$\beta_{A,B} : FA \to \mathbf{Set}(\mathfrak{C}(B, A), GA)$$

which are dinatural in B and natural in A. Moreover, there is a natural bijection between these maps with natural transformations

$$\gamma_A : FA \to \mathbf{Nat}(\mathfrak{C}(-, A), G)$$

which, after invoking the Yoneda Lemma to provide the natural isomorphism $G \cong \mathbf{Nat}(\mathfrak{C}(-, A), G)$ provides the desired natural bijection

$$\mathbf{Nat}\left(\int^{X \in \mathfrak{C}} FX \odot \mathfrak{C}(-, X), G\right) \cong \mathbf{Nat}(F, G).$$

Setting $G = F$ and again invoking the Yoneda Lemma provides the natural isomorphism

$$\int^{X\in\mathfrak{C}} FX \odot \mathfrak{C}(-,X) \cong F(-).$$

Because coends are colimits we have completed the proof of the lemma. $\square$

Definition 2.6. *Let $n \in \mathbb{N}$. Then define the representable presheaves in $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ as*

$$\Delta^n := \mathbf{\Delta}(-,[n]).$$

Corollary 2.2. *Let $X \in [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ be a presheaf. Then there is a natural isomorphism*

$$X \cong \int^{[n]\in\mathbf{\Delta}} X_n \odot \Delta^n.$$

Proof. By the Density Lemma we have a natural isomorphism

$$X \cong \int^{[n]\in\mathbf{\Delta}} X_n \times \mathbf{\Delta}(-,[n]).$$

Since $\mathbf{\Delta}(-,[n]) = \Delta^n$ we are done. $\square$

Corollary 2.3. *Let $\mathfrak{C}$ be any concrete category with all* **Set***-indexed coproducts. Then any presheaf $X \in [\mathbf{\Delta}^{op}, \mathfrak{C}]$ has a natural isomorphism of presheaves*

$$X \cong \int^{[n]\in\mathbf{\Delta}} X_n \odot \Delta^n.$$

Definition 2.7. *For all representable presheaves $\Delta^n \in [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ define the topological spaces $|\Delta^n|$ as*

$$|\Delta^n| := \left\{ \mathbf{v} \in [0,1]^{n+1} \subseteq \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} v_i = 1 \right\} = \mathrm{ConvHull}\{\mathbf{e}_i\}_{i=1}^{n+1}$$

where the set $E = \{\mathbf{e}_i \mid 1 \le i \le n+1\}$ denotes the standard basis on $\mathbb{R}^{n+1}$ and $\mathrm{ConvHull}\, S$ denotes the convex hull of the points in S. That is, $\mathrm{ConvHull}(S)$ contains all convex combinations of points in S. The sets $|\Delta^n|$ are called Topological n-Simplexes.

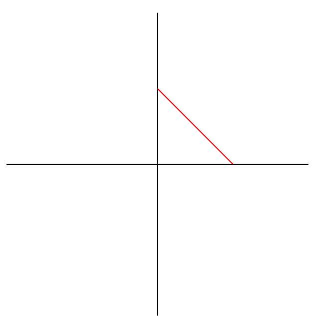
Example 2.1. We list and provide pictures of the first three topological n-simpleces.

$n = 0$: The 0-simplex $|\Delta^0|$ is the set

$$\mathrm{ConvHull}(\{\mathbf{e}_1\}) = \{1\} \subseteq \mathbb{R}\,.$$

$n = 1$: The 1-simplex $|\Delta^1|$ is the set

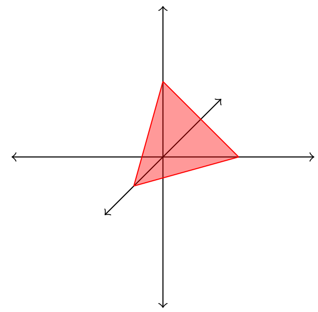
$$\mathrm{ConvHull}(\{\mathbf{e}_1, \mathbf{e}_2\}) = \left\{ t\begin{pmatrix}1\\0\end{pmatrix} + (1-t)\begin{pmatrix}0\\1\end{pmatrix} \;\middle|\; t \in [0,1] \right\}.$$



That is $|\Delta^1|$ is the line segment connecting $(1,0)$ to $(0,1)$.

$n = 2$: The 2-simplex is the set

$$\mathrm{ConvHull}\left(\{\mathbf{e}_i\}_{i=1}^3\right) = \left\{ \theta_1\begin{pmatrix}1\\0\\0\end{pmatrix} + \theta_2\begin{pmatrix}0\\1\\0\end{pmatrix} + \theta_3\begin{pmatrix}0\\0\\1\end{pmatrix} \;\middle|\; \theta_1,\theta_2,\theta_3 \in [0,1]; \sum_{i=1}^{3}\theta_i = 1 \right\}.$$



That is $|\Delta^2|$ is the triangle with vertices $\mathbf{e}_i$ for $1 \leq i \leq 3$.

It is easy to see that if $f_* : \Delta^n \to \Delta^m$ is any natural transformation of presheaves (induced by a map $f : [n] \to [m]$) that the map $|f_*| : |\Delta^n| \to |\Delta^m|$, induced by applying f to the vertices of $|\Delta^n|$ and then extending linearly through the rest of the simplex, is a continuous homomorphism of topological spaces. This makes $|\cdot|$ into a functor when restricted to representable presheaves. We will extend this result via the observation that **Top** is (co)complete and through Corollary 2.2.

Definition 2.8. *Define the* geometric realization *and* singular complex *functors* $|\cdot| : [\mathbf{\Delta}^{\text{op}}, \mathbf{Set}] \to \mathbf{Top}$ *and* $\operatorname{Sing} : \mathbf{Top} \to [\mathbf{\Delta}^{\text{op}}, \mathbf{Set}]$ *via the assignments, for* $K \in [\mathbf{\Delta}^{\text{op}}, \mathbf{Set}]$ *and* $X \in \mathbf{Top}$*,*

$$K \mapsto \int^{[n]\in\mathbf{\Delta}} X_n \odot |\Delta^n| \in \mathbf{Top}$$

and

$$\operatorname{Sing}(X)_n := \mathbf{Top}(|\Delta^n|, X)$$

for all $n \in \mathbb{N}$ *with apporpriate adaptation of maps.*

Theorem 2.1 ([GS07])**.** *There is an adjunction*

$$|\cdot| \dashv \operatorname{Sing} : [\mathbf{\Delta}^{\text{op}}, \mathbf{Set}] \to \mathbf{Top}\,.$$

Proof. The proof is by end and coend calculus. Observe that if K is a simplicial set and X is a topological space we have the chain of natural isomorphisms

$$[\mathbf{\Delta}^{\text{op}}, \mathbf{Set}](K, \operatorname{Sing} X) \cong \int_{[n]\in\mathbf{\Delta}} \mathbf{Set}(K_n, \operatorname{Sing}(X)_n) = \int_{[n]\in\mathbf{\Delta}} \mathbf{Set}(K_n, \mathbf{Top}(|\Delta^n|, X))$$

$$\cong \int_{[n]\in\mathbf{\Delta}} \mathbf{Top}(K_n \odot |\Delta^n|, X) \cong \mathbf{Top}\left(\int^{[n]\in\mathbf{\Delta}} K_n \odot |\Delta^n|, X\right) = \mathbf{Top}(|K|, X).$$

This proves the adjunction. $\square$

We now provide an important example of a simplicial set associated to any category $\mathfrak{C}$. It will be very important in the formalism of $(\infty, 1)$-categories, but we provide it as an example now because it provides us with useful and intuitive examples of how simplicial sets are related to categories.

Example 2.2. Let $\mathfrak{C}$ be any small category. Then we define the simplicial set $N\,\mathfrak{C}$, the *nerve of* $\mathfrak{C}$, defined as follows: for all $n \in \mathbb{N}$ we define the set

$$N(\mathfrak{C})_n := \{f_1 \cdots f_n \mid f_i \in \mathfrak{C}(X_{i-1}, X_i), 1 \leq i \leq n\}$$

of all composable chains of morphisms of length n in $\mathfrak{C}$. Then the face maps d_i, for $0 \leq i \leq n-1$, are given by sending a chain

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$$

to the chain

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} A_n.$$

The degeneracy maps s_i are given by mapping to the chain

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} A_i = A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} A_n$$

We now move to some of the more important simplicial sets that will be crucial in describing the Quillen model structure. We will need to know of the horns and boundaries associated to a complex in order to describe and discuss the fibrations, cofibrations, weak equivalences, and the nature of ∞-categories and Kan compleces. We begin by providing a fairly concrete definition of the boundary of a simplex, and then move forward to show how it may be realized as a coequalizer.

Definition 2.9. *Let $n \in \mathbb{N}$ and define the* boundary of Δ^n, *denoted $\partial\Delta^n$, to be given as follows: for all $m \in \mathbb{N}$ define*

$$(\partial\Delta^n)_m := \{f : [m] \to [n] \mid f \text{ not surjective}\}$$

and setting $(\partial\Delta^n)(f) = \Delta^n(f) = f^*$ for every morphism f in $\mathbf{\Delta}$.

Note that if $m \leq n$ then $(\partial\Delta^n)_m = (\Delta^n)_m$. An equivalent formulation of the boundary is as follows: if we let be the presheaf given by

$$\partial_i\Delta^n := \operatorname{im}((d^i)_*)$$

for every $0 \leq i \leq n$, then we can write the

$$\partial\Delta^n = \bigcup_{i=0}^{n} \partial_i\Delta^n.$$

Note that the above union states that the boundary of Δ^n is the simultaneous union of all $(n-1)$ subsimpleces in Δ^n, as we would expect.
We now proceed to show that the boundary of Δ^n is a coequalizer. However, in order to do this we will need to discuss an important lemma, the Eilenberg-Zilbern Lemma, and then develop the notion of what the n-dimensional skeleton of a simplical complex is.

Lemma 2.4 (Eilenberg-Zilber; [JT08])**.** *For all n-simpleces $x \in X_n$ there exists a unique surjection $\varepsilon : [m] \to [n]$ and a unique nondegenerate $y \in X_m$ such that $x = \varepsilon^*(y)$.*

Proof. In order to establish the existence of the pair (ε, y) we note the following: if $x \in X_n$ is nondegenerate then the surjection $\mathrm{id}_{[n]} : [n] \to [n]$ is a surjection for which $(\mathrm{id}_{[n]})^*(x) = \mathrm{id}_{X_n}(x) = x$. If x is degenerate, then find the number of edges in x that are trivial arrows; say there are $k > 0$ trivial sides to x. Then we can find a surjection $\varepsilon : [n] \to [n-k]$ such that there is a nondegenerate $(n-k)$ simplex y whose vertices are exactly the non-repeating vertices of x; then $\varepsilon^*(y) = x$.

We now show that if (ε, y) and (ε', y') are epimorphisms and nondegenerate simpleces such that $\varepsilon(y) = x\varepsilon'(y')$. Then since the pushout

$$\begin{tikzcd} {[n]} \arrow[r, "\varepsilon"] \arrow[d, "\varepsilon'"'] & {[m]} \arrow[d, "p_0"] \\ {[m']} \arrow[r, "p_1"'] & {[k]} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is absolute by Lemma 1.3, the diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "\varepsilon_*"] \arrow[d, "(\varepsilon')_*"'] & \Delta^m \arrow[d, "(p_0)_*"] \\ \Delta^{m'} \arrow[r, "(p_1)_*"'] & \Delta^k \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout in $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$. Now, since $y \in X([m]) = X_m$ and $y' \in X([m']) = X_{m'}$ the Yoneda Lemma gives, through a natural bijection $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Delta^m, X) \cong X([m])$, that there is a characterization

$$\widehat{y} : \Delta^m \to X$$

and

$$\widehat{y'} : \Delta^{m'} \to X$$

where $\widehat{y}$ is the Yoneda Lemma bijection applied to y and $\widehat{y}'$ is the Yoneda Lemma bijection applied to y'. Furthermore, these natural transformations make the diagram

$$\begin{tikzcd}
\Delta^n \arrow[r, "\varepsilon_*"] \arrow[d, "(\varepsilon')_*"'] & \Delta^m \arrow[d, "(p_0)_*"'] \arrow[ddr, bend left, "\widehat{y}"] & \\
\Delta^{m'} \arrow[r, "(p_1)_*"'] \arrow[drr, bend right, "\widehat{y}'"'] & \Delta^k \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & X
\end{tikzcd}$$

commute because $\varepsilon^*(y) = x = (\varepsilon')^*(y')$. Consequently there is a unique map $z : \Delta^k \to X$ making the diagram

$$\begin{tikzcd}
\Delta^n \arrow[r, "\varepsilon_*"] \arrow[d, "(\varepsilon')_*"'] & \Delta^m \arrow[d, "(p_0)_*"'] \arrow[ddr, bend left, "\widehat{y}"] & \\
\Delta^{m'} \arrow[r, "(p_1)_*"'] \arrow[drr, bend right, "\widehat{y}'"'] & \Delta^k \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "z"] & \\
& & X
\end{tikzcd}$$

commute, given by the universal property of the pushout. This then implies that $\widehat{y} = z^{op}(p_0)^*$ and $\widehat{y}' = z^{op}(p_1)^*$. Since y and y' are nondegenerate it then follows that

$$p_0 = \mathrm{id} = p_1, \quad y = y', \quad \varepsilon = \varepsilon'$$

and hence we must have that (ε, y) are both unique. This completes the proof of the lemma. $\square$

We proceed by defining, for all $n \in \mathbb{N}$, the category $\mathbf{\Delta}_n$ to be the full subcategory of $\mathbf{\Delta}$ where $[k] \in \mathbf{\Delta}_n$ if and only if $k \leq n$. The inclusion $\mathbf{\Delta}_n \to \mathbf{\Delta}$ induces a restriction functor

$$\mathrm{tr}_n : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{\Delta}_{\mathrm{n}}^{\mathrm{op}}, \mathbf{Set}]$$

which is given by truncating a simplicial set X at X_n. Because tr_n is the functor induced by an inclusion, it has a left adjoint $\mathrm{sk}_n : [\mathbf{\Delta}_{\mathrm{n}}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ given by

$$\mathrm{sk}_n X := \int^{[m] \in \mathbf{\Delta}_n} X_m \odot \Delta^m = \varinjlim_{\substack{m \leq n \\ \Delta^m \to X}} \Delta^m$$

The fact that these functors are an adjoint pair follows from ninja coend calculus and the fact that sk_n is the left Kan extension of tr_n. Furthermore, since $\mathbf{\Delta}_n \to \mathbf{\Delta}$ is full and faithful there is an isomorphism

$$X_m \cong (\mathrm{sk}_n X)_m$$

for every $0 \leq m \leq n$, proving that

$$\mathrm{tr}_n(\mathrm{sk}_n X) \cong X \in [\mathbf{\Delta}_{\mathrm{n}}^{\mathrm{op}}, \mathbf{Set}] .$$

Definition 2.10. *Let $n \in \mathbb{N}$. We then define the n-*skeleton of a simplicial set X*, $\mathrm{Sk}_n X$, to be the simplicial set*

$$\mathrm{Sk}_n X := \mathrm{sk}_n(\mathrm{tr}_n X).$$

Definition 2.11. *We say that a simplicial set X has dimension $\leq n$ if $X = \mathrm{Sk}_n X$.*

Note that it is immediate from the definitions that we have an ascending chain of inclusions

$$\mathrm{Sk}_0 X \subseteq \mathrm{Sk}_1 X \subseteq \cdots \subseteq \mathrm{Sk}_n X \subseteq \mathrm{Sk}_{n+1} X \subseteq \cdots$$

and that

$$X = \bigcup_{n \in \mathbb{N}} \mathrm{Sk}_n X.$$

Furthermore, it is clear from the coend definition that if $m \leq n$ then $(\mathrm{Sk}_n X)_m \cong X_m$ while if $m > n$ then $(\mathrm{Sk}_n X)_m$ contains only degenerate simpleces.

Proposition 2.1 ([JT08]). *The counit*

$$\varepsilon_X : \mathrm{Sk}_n X \to X$$

is monic for all simplicial sets X.

Proof. The proof we provide here is a categorical alternative to the combinatorial proof provided in [JT08]. Begin by noting that

$$\mathrm{Sk}_n X = \mathrm{sk}_n(\mathrm{tr}_n) = \int^{[m]\in\mathbf{\Delta}_n} X_m \odot \Delta^m \cong \operatorname{colim}_{\substack{m\leq n \\ \Delta^m\to X}} \Delta^m$$

so, for all $k \leq n$ we have the equality

$$(\mathrm{Sk}_n X)_k = X_k,$$

where the counit identifies this equality. Now, whenever $k > n$ consider from the Yondea Lemma that there is a natural isomorphism

$$(\mathrm{Sk}_n X)_k = (\mathrm{Sk}_n X)([k]) \cong \mathbf{Nat}(\Delta^k, X) = [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Delta^k, X).$$

However, from the coend we see that for any such morphism $f : \Delta^k \to \mathrm{Sk}_n X$ must have some $m \leq n$ such that there is a factorization

$$\begin{tikzcd} \Delta^k \arrow[rr, "f"] \arrow[dr] & & \mathrm{Sk}_n X \\ & \Delta^m \arrow[ur] & \end{tikzcd}$$

Consequently $(\mathrm{Sk}_n X)_k$ contains only the degenerate k-simpleces of X and the counit ε simply has $\varepsilon_k : (\mathrm{Sk}_n X)_k \to X_k$ identify the degenerate k-simpleces of X and embeds them. □

We now move to show how we may build the n-skeleton of a simplicial set X from the $(n-1)$-skeleton and attaching appropriately. Begin this by letting $e(X)_n$, for $n \geq 0$, be the set of all nondegenerate n-simpleces of X and noting that for all $n \in \mathbb{N}$, $\mathrm{Sk}_n X = e(X)_n \cup \mathrm{Sk}_{n-1} X$ where we define $\mathrm{Sk}_{-1} X := \varnothing$. For all $n \geq 0$ now consider that the diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \Delta^n \arrow[d] \\ \mathrm{Sk}_{n-1} X \arrow[r] & \mathrm{Sk}_n X \end{tikzcd}$$

commutes where both horizontal arrows are the standard embeddings and $\partial\Delta^0 := \varnothing$. This allows us to construct the commuting diagram

$$\begin{tikzcd} \coprod_{x\in e(X)_n} \partial\Delta^n \arrow[r] \arrow[d] & \coprod_{x\in e(X)_n} \Delta^n \arrow[d] \\ \mathrm{Sk}_{n-1} X \arrow[r] & \mathrm{Sk}_n X \end{tikzcd}$$

We claim that this diagram is a pushout that allows us to construct the n-skeleton out of the $(n-1)$-skeleton.

Proposition 2.2 ([JT08]). *Let X be a simplicial set. Then the diagram*

$$\begin{tikzcd} \coprod_{x\in e(X)_n} \partial\Delta^n \arrow[r] \arrow[d] & \coprod_{x\in e(X)_n} \Delta^n \arrow[d] \\ \mathrm{Sk}_{n-1} \arrow[r] & \mathrm{Sk}_n X \end{tikzcd}$$

is a pushout.

Proof. Note that since each of the simplicial sets $\partial\Delta^n$, Δ^n, $\mathrm{Sk}_{n-1} X$, and $\mathrm{Sk}_n X$ have dimension $\leq n$, we only need to verify that whenever $m \leq n$ the diagram

$$\begin{tikzcd}
\coprod\limits_{x \in e(X)_n} (\partial\Delta^n)_m \arrow[r] \arrow[d] & \coprod\limits_{x \in e(X)_n} (\Delta^n)_m \arrow[d] \\
(\mathrm{Sk}_{n-1} X)_m \arrow[r] & (\mathrm{Sk}_n X)_m
\end{tikzcd}$$

is a pushout in **Set**. If $m \leq n-1$ then $(\partial\Delta^n)_m \cong (\Delta^n)_m$ and $(\mathrm{Sk}_{n-1} X)_m \cong (\mathrm{Sk}_n X)_m$. In these cases the fact that the diagram is a pushout is clear. If $m = n$ then $(\Delta^n)_n \setminus (\partial\Delta^n)_n = \{\mathrm{id}_{[n]}$, which implies that

$$\left(\coprod_{x \in e(X)_n} (\Delta^n)_n\right) \setminus \left(\coprod_{x \in e(X)_n} (\partial\Delta^n)_n\right) \cong \coprod_{x \in e(X)_n} (\Delta^n)_n \setminus (\partial\Delta^n)_n = \coprod_{x \in e(X)_n} \{\mathrm{id}_{[n]}\} \cong e(X)_n.$$

However, since $(\mathrm{Sk}_n X)_n = (\mathrm{Sk}_{n-1} X)_n \cup e(X)_n$ the commuting diagram is a pushout and we are done. $\square$

Note that this proposition can be used to show that $|X|$ is a CW-complex for all simplicial sets X. While we do not prove this fact, it is done both in [JT08] and [GJ09]. We do instead cite it, as it shows that the adjunction $|\cdot| \dashv \mathrm{Sing} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Top}$ has essential image the category **CGHaus**. This is important to keep in mind, for while **Top** is not Cartesian Closed (and $|X \times Y|$ need not be homeomorphic to $|X| \times |Y|$ in **Top**), the category **CGHaus** *is* Cartesian Closed, and $|X \times Y| \cong |X| \times_{Ke} |Y|$, where the second product is the Kelley product of spaces, i.e., the product in **CGHaus**.

Theorem 2.2 ([GJ09])**.** *If* $X \in [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ *then* $|X| \in \mathbf{CGHaus}$.

Theorem 2.3 ([GJ09])**.** *The functors* $\mathrm{Sing} : \mathbf{CGHaus} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ *and* $|\cdot| : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{CGHaus}$ *commute with finite products.*

The basic idea of the proof is to embed all the representable functors Δ^n into the skeletons $\mathrm{Sk}_n X$ and then pass through $|\cdot|$ to show that $|\mathrm{Sk}_n X|$ is produced from $|\mathrm{Sk}_{n-1} X|$ by attaching a appropriate n-cells. We now move on to show that $\partial\Delta^n$ is a coequalizer. Begin by considering the diagram, for all $0 \leq i < j \leq n$

$$\begin{tikzcd}
\Delta^{n-2} \arrow[r, "(d^{j-1})_*"] \arrow[d, "\iota_{ij}"'] & \Delta^{n-1} \arrow[d, "\iota_i"] & & \\
\coprod\limits_{0 \leq i < j \leq n} \Delta^{n-2} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \coprod\limits_{i=0}^{n} \Delta^{n-1} \arrow[rr] \arrow[dr, "\theta"'] & & \mathrm{Coeq}(f,g) \arrow[dl, dashed] \\
\Delta^{n-2} \arrow[r, "(d^i)_*"'] \arrow[u, "\iota_{ij}"] & \Delta^{n-1} \arrow[u, "\iota_j"'] & \Delta^n &
\end{tikzcd}$$

where f, g, and θ are the comparison maps determined by the equations:

$$\iota_{ij} f = (d^{j-1})_* \iota_i$$
$$\iota_{ij} g = d^i \iota_j$$
$$\iota_i \theta = d^i$$

Note that they make the diagram commute. Now, since

$$\begin{tikzcd}
{[n-2]} \arrow[r, "d^{j-1}"] \arrow[d, "d^i"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[n-1]} \arrow[d, "d^i"] \\
{[n-1]} \arrow[r, "d^j"'] & {[n]}
\end{tikzcd}$$

is an absolute pullback whenever $0 \leq i < j \leq n$, the diagram

$$\begin{tikzcd}
\Delta^{n-2} \arrow[r, "(d^{j-1})_*"] \arrow[d, "(d^i)_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^{n-1} \arrow[d, "(d^i)_*"] \\
\Delta^{n-1} \arrow[r, "(d^j)_*"'] & \Delta^n
\end{tikzcd}$$

is a pullback in $[\mathbf{\Delta}^{\text{op}}, \mathbf{Set}]$. Then it follows from the fact that $\partial\Delta^n$ is the union of all $(n-1)$-simpleces of Δ^n that the coequalizer $\text{Coeq}(f, g)$ that appears in the diagram satisfies the equality

$$\text{Coeq}(f, g) = \partial\Delta^n.$$

Another important simplicial set is derived from a diagram that is analogous to the coequalizer appearing above. These are the *horns* of Δ^n, and

Definition 2.12. *For all $0 \leq k \leq n$ we define the k-*th horn *of Δ^n, denoted Λ^n_k, to be the coequalizer in the diagram:*

$$\begin{tikzcd}
\Delta^{n-2} \arrow[r, "(d^{j-1})_*"] \arrow[d, "\iota_{ij}"'] & \Delta^{n-1} \arrow[d, "\iota_i"] & & \\
\coprod\limits_{\substack{0\leq i<j\leq n \\ j\neq k}} \Delta^{n-2} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \coprod\limits_{i=0}^{n} \Delta^{n-1} \arrow[rr] \arrow[dr, "\theta"'] & & \text{Coeq}(f, g) \arrow[dl, dashed] \\
\Delta^{n-2} \arrow[r, "(d^i)_*"'] \arrow[u, "\iota_{ij}"] & \Delta^{n-1} \arrow[u, "\iota_j"'] & \Delta^n &
\end{tikzcd}$$

where f, g, and θ, are all defined analogously as in the coequalizer construction of $\partial\Delta^n$.

Note that, as in the case of the boundaries, we have can define the horns in terms of unions of $(n-1)$-simpleces. In fact, we can give an equality of simplicial sets.

$$\Lambda^n_k = \bigcup_{\substack{i=0 \\ i\neq k}}^{n} \partial_i \Delta^n$$

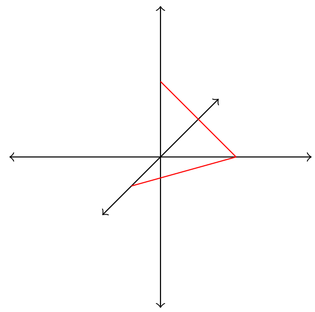
Recall that since $|\cdot|$ is a left adjoint it preserves colimits; consequently the above coequalizer construction shows that

$$|\partial\Delta^n| \cong \partial|\Delta^n|;$$

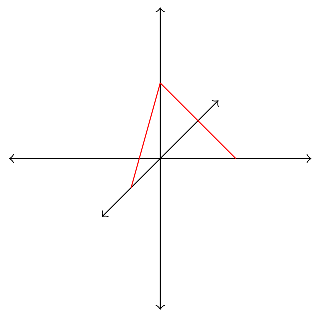
we cannot assert direct equality because the coequalizer in **Top** maps $|\partial\Delta^n|$ to the $(n-1)$-sphere bounding Δ^n, which after invoking the homeomorphism $\mathbb{S}^{n-1} \cong \partial|\Delta^n|$ carries the intuition over. This useful observation allows us to think of the boundary of a complex as the boundary of simplex.

If Λ^n_k is the k-th horn of Δ^n, then $|\Lambda^n_k|$ is formed by taking the simplex $|\Delta^n|$, deleting the interior, and then removing the face of $|\Delta^n|$ opposite the vertex k.

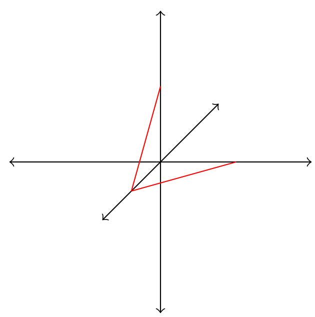
Example 2.3. Let $n = 2$. Then we sketch the realizations $|\Lambda^2_k|$ for $k = 0, 1, 2$ by considering the diagrams: The horn $|\Lambda^2_0|$ is given by



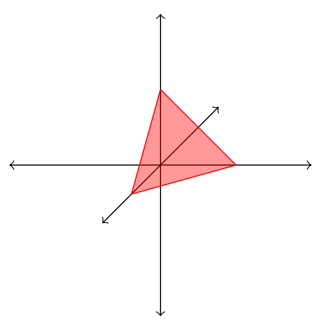
while $|\Lambda^2_1|$ is given by



and finally $|\Lambda^2_2|$ is given by



where we treat $\mathbf{e}_{i+1}$ as the i-th vertex. Note that all these horns naturally embed into the simplex $|\Delta^2|$:



We now proceed to describe the model structure on $[\mathbf{\Delta}^{\text{op}}, \mathbf{Set}]$. It will be of frequent use in the formalism of ∞-categories and will provide us with perhaps the most ubiquitous example of both a Quillen functor and a Quillen equivalence in the sense that through the adjunction $|\cdot| \dashv \mathrm{Sing}$ we will develop a model structure on **Top** based on the construction in $[\mathbf{\Delta}^{\text{op}}, \mathbf{Set}]$.

Definition 2.13. *Let $f : X \to Y$ be a morphism in $[\mathbf{\Delta}^{\text{op}}, \mathbf{Set}]$. We then say that f is:*

- *A* weak equivalence *if $|f| : |X| \to |Y|$ induces isomorphisms of sets $\pi_0(|X|, x_0) \to \pi_0(|Y|, \langle f\rangle(x_0))$ and an isomorphism of homotopy groups*

$$\pi_n\big(|X|, x_0\big) \to \pi_n\big(|Y|, |f|(x_0)\big)$$

for all $n \geq 1$ and all $x_0 \in |X|$;

- *A* cofibration *if $f_n : X_n \to Y_n$ is a monomorphism for all $n \in \mathbb{N}$;*
- *A* fibration *if and only if there is a lift $\varphi : \Delta^n \to X$ making the diagram*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, "\iota"'] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\varphi"] & Y \end{tikzcd}$$

for all k and all n.

Definition 2.14. *Let $p : X \to Y$ be a simplicial set. We say that p is a* Kan fibration *if p has the right lifting property with all injections $\{\Lambda^n_k \to \Delta^n\}$.*

Definition 2.15. *Let X be a fibrant simplicial set. Then X is a* Kan complex*.*

Definition 2.16. *Let $p : X \to Y$ be a morphism of spaces. We say that p is a* Serre Fibration *if and only if given a diagram of the form*

$$\begin{tikzcd} |\Lambda^n_k| \arrow[r] \arrow[d, "|\iota|"'] & X \arrow[d, "p"] \\ |\Delta^n| \arrow[r] \arrow[ur, dashed, "\varphi"] & Y \end{tikzcd}$$

there is a lift $\varphi : |\Delta^n| \to X$ rendering the diagram commutative.

A lemma in my model category theory notes states that if we have an adjunction $F \dashv G : \mathfrak{C} \to \mathfrak{D}$, then the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & GX \arrow[d, "Gv"] \\ B \arrow[r] \arrow[ur, dashed, "\varphi"] & GY \end{tikzcd}$$

has a lift φ if and only if the diagram

$$\begin{tikzcd} FA \arrow[r] \arrow[d, "Fu"'] & X \arrow[d, "v"] \\ FB \arrow[r] \arrow[ur, dashed, "\psi"] & Y \end{tikzcd}$$

has a lift ψ. It then follows immediately from that lemma and definition that $p : X \to Y$ is a Serre fibration if and only if $\mathrm{Sing}(p) : \mathrm{Sing}\, X \to \mathrm{Sing}\, Y$ is a Kan fibration. This is the first step in defining the Quillen model structure on **Top**, as it allows us to generate a model structure on **Top** based on the model structure in $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$. We proceed from here to show prove the that $(\mathscr{C}, \mathscr{F} \cap \mathscr{W})$ and $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ are weak factorization systems.

Definition 2.17. *Define the following three classes of morphisms in $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$:*

(1)
$$\mathcal{A}_1 := \{\Lambda^n_k \to \Delta^n \mid 0 \leq k \leq n, n \in \mathbb{N}\};$$

(2)
$$\mathcal{A}_2 := \left\{ (\Delta^n \times \{0\}) \coprod_{\partial\Delta^n \times \{0\}} (\partial\Delta^n) \to \Delta^n \times \Delta^1 \,\middle|\, n \in \mathbb{N} \right\};$$

(3)
$$\mathcal{A}_3 := \left\{ (B \times \{0\}) \coprod_{A \times \{0\}} (A \times \Delta^1) \,\middle|\, A \subseteq B; A, B \in [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \right\}.$$

Theorem 2.4 ([Lur09],[GJ09])**.** *Let $\mathscr{A}_1$ be the saturation of $\mathcal{A}_1$, $\mathscr{A}_2$ the saturation of $\mathcal{A}_2$, and $\mathscr{A}_3$ the saturation of $\mathcal{A}_3$. Then $\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}_3$ and $\mathscr{A}_1$ has the left lifting property with $\mathscr{F}$.*

The proof we provide comes in two parts: the first is adapted from [Lur09], while the final part is from [GJ09].

Proof. We begin with the observation that $\mathscr{A}_2 = \mathscr{A}_3$ because since every simplicial set B has the form

$$B \cong \int^{[n]\in\mathbf{\Delta}} B_n \odot \Delta^n$$

we have $\mathcal{A}_2 \subseteq \mathcal{A}_3$; the other containment of classes of morphisms follows from the fact that every map in $\mathcal{A}_3$ is an iterated pushout of maps in $\mathcal{A}_2$.

We now observe that the $(n+1)$-simpleces of $\Delta^n \times \Delta^1$ are indexed by order preserving maps $[n+1] \to [n] \times [1]$, where $[n] \times [1] = \{0, \cdots, n\} \times \{0, 1\}$ is regarded as an object in the category **Poset** of posets with the lexicographic order. In particular the nondegenerate $(n+1)$-simpleces of $\Delta^n \times \Delta^1$ are described by the pushforward of the poset maps

$$\sigma_k : [n+1] \to [n] \times [1]$$

given by:

$$\sigma_k(\ell) := \begin{cases} (\ell, 0) & \ell \leq k \\ (\ell - 1, 1) & \ell > k \end{cases}$$

In order to show that $\mathscr{A}_2 \subseteq \mathscr{A}_1$ we proceed by decending induction on k with respect to a collection $\{X_k \mid 0 \leq k \leq n, n \in \mathbb{N}\}$ of simplicial subsets of $\Delta^n \times \Delta^1$. Begin by defining

$$X_{n+1} := (\Delta^n \times \{0\}) \coprod_{\partial\Delta^n \times \{0\}} \left(\partial\Delta^n \times \Delta^1\right).$$

To proceed with thie induction assume let $0 \leq k \leq n$ and assume that X_{k+1} is defined. Let X_k be given as the pushout

$$X_k := X_{k+1} \coprod_{\Lambda^n_k} \Delta^{n+1}$$

so that X_k is regarded as a subset of $\Delta^n \times \Delta^1$ by unioning X_{k+1} with σ_k and the faces of σ_k. Note we have from this construction that

$$X_0 = \Delta^n \times \Delta^1.$$

Since each of the inclusions

$$X_{n+1} \to X_n \to \cdots \to X_1 \to X_0$$

arise as pushouts over maps in $\mathcal{A}_1$, each inclusion is in $\mathscr{A}_1$. Thus it follows that the inclusion

$$X_{n+1} \to X_0 \in \mathscr{A}_1.$$

We now show that $\mathscr{A}_1 \subseteq \mathscr{A}_3$ by proving that. Let $0 \leq k < n$ and define $j_* : \Delta^n \to \Delta^n \times \Delta^1$ to be the pushforward of the inclusion defined by

$$[n] \cong [n] \times \{1\} \to [n] \times [1]$$

and $(r_k)_*$ to be the pushforward of the map $r_k : [n] \times [1] \to [n]$ defined by

$$r_k(\ell, 0) := \begin{cases} \ell & m \neq k+1 \\ k & \ell = k+1 \end{cases}$$

and

$$r_k(\ell, 1) = \ell.$$

Then $r_k(j(\ell)) = r_k(\ell, 1) = \ell$ so that there is a retraction diagram

$$\begin{tikzcd} & \Delta^n \times \Delta^1 \arrow[dr, "(r_k)_*"] & \\ \Delta^n \arrow[ur, "j_*"] \arrow[rr, equal] & & \Delta^n \end{tikzcd}$$

Now consider the injection

$$i : \Lambda_k^n \cong \Lambda_k^n \times \{1\} \to (\Delta^n \times 0\}) \coprod_{\Lambda_k^n \times \{0\}} \left(\Lambda_k^n \times \Delta^1\right)$$

and the induced map

$$s_k : (\Delta^n \times \{0\}) \coprod_{\Lambda_k^n \times \{0\}} \to \Lambda_k^n$$

given by restriction of r_k. This then induces a retraction diagram

$$\begin{tikzcd}
 & (\Delta^n \times \{0\}) \coprod_{\Lambda_k^n \times \{0\}} \left(\Lambda_k^n \times \Delta^1\right) \arrow[dr, "s_k"] \arrow[rrr] & & & \Delta^n \times \Delta^1 \arrow[dr, "(r_k)_*"] & \\
\Lambda_k^n \arrow[ur, "i"] \arrow[rr, equal] & & \Lambda_k^n \arrow[r] & \Delta^n \arrow[ur, "j_*"] \arrow[rr, equal] & & \Delta^n
\end{tikzcd}$$

To complete the proof we need only to show that $\Lambda_n^n \to \Delta^n$ is a retract as in the prior cases. However, this is done in an analogous fashion as above by using the pushforward of the map $v_k : [n] \times [1] \to [n]$ defined by

$$v_k(\ell, 0) = n$$

and

$$v_k(\ell, 1) = \ell$$

to produce a retraction diagram

$$\begin{tikzcd}
 & (\Delta^n \times \{0\}) \coprod_{\Lambda_n^n \times \{0\}} \left(\Lambda_n^n \times \Delta^1\right) \arrow[dr, "s_k"] \arrow[rrr] & & & \Delta^n \times \Delta^1 \arrow[dr, "(r_k)_*"] & \\
\Lambda_n^n \arrow[ur, "i"] \arrow[rr, equal] & & \Lambda_n^n \arrow[r] & \Delta^n \arrow[ur, "j_*"] \arrow[rr, equal] & & \Delta^n
\end{tikzcd}$$

This shows that $\mathscr{A}_1 \subseteq \mathscr{A}_3$ and hence proves that $\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}_3$. For the final statement of the theorem it follows from general abstract nonsense of saturated classes of maps on a weak factorization system (see [Rie08], for instance, for details) that $\mathscr{A}_1$ has the left lifting property with $\mathscr{F}$. □

Definition 2.18. *The class of maps $\mathscr{A}_1$ is the class of* Anodyne Extensions*.*

Lemma 2.5. *A morphism of simplicial sets is a trivial cofibration if and only if it is an anodyne extension.*

Proof. To prove this it will be sufficient to show that the maps $\Lambda_k^n \to \Delta^n$ are trivial cofibrations. Since $\Lambda_k^n \to \Delta^n$ is clearly monic, we need only show that the induced map

$$|\iota| : |\Lambda_k^n| \to |\Delta^n|$$

is a homotopy equivalence. However, it follows from Example A.1 that $|\Lambda_k^n|$ is a strong deformation retract of $|\Delta^n|$. Thus we have that

$$|\iota|_* : \pi_n\left(|\Lambda_k^n|, x_0\right) \to \pi_n(|\Delta^n|, |\iota|(x_0))$$

is an isomorphism for every $n \in \mathbb{N}$ and for all $x_0 \in |\Lambda_k^n|$. □

Corollary 2.4. *The trivial cofibrations have the left lifting property with the fibrations in $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$.*

Proposition 2.3. *There is a $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ factorization of any morphism $f : X \to Y$ in $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$.*

Proof. Let $f : X \to Y$ be a morphism of simplicial sets. Then consider the factorization

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i_\infty"'] & & Y \\
 & G^\infty(\mathcal{A}_1, f) \arrow[ur, "p_\infty"'] &
\end{tikzcd}$$

provided by the small object argument. By Theorem 2.4 we have that p_∞ is a fibration. Since each of the objects $\Lambda_k^n \to \Delta^n$ is small in $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ it follows that i_∞ is a trivial cofibartion and we are done. □

Corollary 2.5. *The pair $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ constitutes a weak factorization system on $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$.*

Corollary 2.6 ([GJ09])**.** *Let $f : A \to B$ be an anodyne extension and $g : X \to Y$ an arbitrary inclusion. Then the induced map*

$$(A \times Y) \coprod_{A \times X} (B \times X) \to B \times Y$$

is an anodyne extension.

In order to finish constructing the model structure on $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ we need to prove that a morphism $p : X \to Y$ is a trivial fibration if and only if it has the right lifting property with the maps $\partial\Delta^n \to \Delta^n$, and then that we can give arbitrary morphisms a $(\mathscr{C}, \mathscr{F} \cap \mathscr{W})$ factorization (although this will be done with the small object arument so it will be easy). Note that this will tell us that the monomorphisms in $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ are generated by the injections $\partial\Delta^n \to \Delta^n$.

One method of proving that trivial fibrations are exactly the maps with the right lifting property against the saturation of $\{\partial\Delta^n \to \Delta^n\}$ requires us to show that $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ is Cartesian Closed. While we will not prove the lifting property in this way, we will provide the proof of this fact. This fact will provide us with good intuition as to why we should care about using simplicial data to describe the logical data of type theory and homotopy theory.

Proposition 2.4. *The category $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ is Cartesian Closed.*

This fact follows easily from the fact that $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ is a category of presheaves from a small category into **Set**, hence is a (Grothedieck) topos, and so gives (more or less) for free that $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ is Cartesian Closed. We provide a proof here that uses the Kan extension calculus we have used throughout this set of notes and to keep our perspective consistent.

Proof. Begin by letting $X \in \mathrm{Ob}\,[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ be fixed. Then there is a functor $F : \mathbf{\Delta} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ defined by

$$[n] \mapsto \Delta^n \times X$$

for all objects $[n] \in \mathrm{Ob}\,\mathbf{\Delta}$ and

$$f : [n] \to [m] \mapsto \langle f_*, \mathrm{id}_X \rangle : \Delta^n \times X \to \Delta^m \times X$$

for all morphisms $f \in \mathrm{Mor}\,\mathbf{\Delta}$. It follows from inspection that F is the composition of the Yoneda embedding $y : \mathbf{\Delta} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ and the functor $L := (-) \times X : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$; consequently the diagram

$$\begin{tikzcd} & {[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]} \arrow[dr, dashed, "L:=(-)\times X"] & \\ \mathbf{\Delta} \arrow[ur, "y"] \arrow[rr, "F"'] & & {[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]} \end{tikzcd}$$

commutes. From the theory of Kan extensions (see [Rie11]) it follows that the functor $[X, -] = R : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ given by defining, for all simplicial sets Y and all $n \in \mathbb{N}$,

$$R(Y)_n = [X, Y]_n := [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](X \times \Delta^n, Y).$$

The face and degeneracy maps then take the form of precomposition by the morphisms

$$\langle \mathrm{id}_X, d^i \rangle$$

and

$$\langle \mathrm{id}_X, s^i \rangle,$$

that is, $d_i := \big(\langle \mathrm{id}_X, d^i \rangle\big)^*$ and $s_i := \big(\langle \mathrm{id}_X, s^i \rangle\big)^*$. It then follows that since L is the left Kan extension of F along y, R is the right adjoint of L and we are done. □

We now must show the second half of the Quillen model structure on $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$. As stated earlier, we will now show that the monics are generated as a saturated class $\mathscr{C} = \mathscr{M}$ by the set $\{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$.

Lemma 2.6 ([JT08])**.** *The set $\{\partial\Delta^n \to \Delta^n\}$ generates $\mathscr{M} = \mathscr{C}$ as a saturated class.*

Proof. It is easy to see that $\mathcal{S}_{\{\partial\Delta^n \to \Delta^n\}}$, the saturation of $\{\partial\Delta^n \to \Delta^n\}$ is a subclass of the class of monomorphisms because $\partial\Delta^n \to \Delta^n$ is monic for all $n \in \mathbb{N}$. Consequently, to prove the lemma we will show that any monomorphism $\mu : A \to X$ may be derived from a series of pushouts and retractions of boundary maps. Begin by defining

$$e(X \setminus A)_n := \{x \in X_n, x \notin A_n \mid x \text{ nondegenerate}\}$$

and noting that Proposition 2.2 is easily adapted to give us that the diagram

$$\begin{tikzcd} \coprod\limits_{x\in e(X\setminus A_n)} \partial\Delta^n \arrow[r] \arrow[d] & \coprod\limits_{x\in e(X\setminus A)_n} \Delta^n \arrow[d] \\ \mathrm{Sk}_{n-1}(X)\cup A \arrow[r] & \mathrm{Sk}_n(X)\cup A \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout for $n \geq 0$. Since $\mu : A \to X$ is monic it follows that

$$X = \varinjlim_{n\geq -1} (\mathrm{Sk}_n(X) \cup A).$$

Furthermore, note that

$$\mathrm{Sk}_{-1}(X) \cup A = \varnothing \cup A = A$$

so that the inclusion map

$$\mu_{-1} : \mathrm{Sk}\,{-1}(X) \cup A \to \varinjlim_{n\geq -1} (\mathrm{Sk}_n(X) \cup A)$$

is actually the map $\mu : A \to X$. But then $\mu \in \mathcal{S}_{\{\partial\Delta^n \to \Delta^n\}}$ and we are done. $\square$

Lemma 2.7 ([GJ09])**.** *If $\{\partial\Delta^n \to \Delta^n\}$ has the left lifting property against a map $p : X \to Y$, then p is a trivial fibration.*

We only provide a sketch of the lifting property. The standard proof involves developing the theory of simplicial homotopy groups, showing an equality with topological homotopy groups, and then proving that p is a weak equivalence. We instead sketch the intuition behind the result.

Sketch. By Lemma 2.6 it follows that since $\{\partial\Delta^n \to \Delta^n\}$ has the left lifting property with p, so does $\mathscr{C}$; consequently the set $\{\Lambda^n_k \to \Delta^n\}$ has the left lifting property with p, giving that p is a fibration. To see that p is a homotopy equivalence recall that $|\partial\Delta^n| = \mathbb{S}^{n-1}$ and $|\Delta^n| \cong D_n$. Thus having a lift

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists\varphi"] & Y \end{tikzcd}$$

implies that the diagram

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r] \arrow[d] & {|X|} \arrow[d] \\ D_n \arrow[r] \arrow[ur, "\theta|\varphi|"] & {|Y|} \end{tikzcd}$$

has a lift in **Top**, where $\theta : D_n \to |\Delta^n|$ is a homoeomorphism. Recall that $\pi_n(X, x) = \mathbf{Top}_*(\mathbb{S}^n, X)/\simeq$. It then follows from the diagram

$$\begin{tikzcd} \mathbb{S}^{n} \arrow[r] \arrow[d] & {|X|} \arrow[d, "|p|"] \\ D_{n+1} \arrow[r] \arrow[ur] & {|Y|} \end{tikzcd}$$

that whenever we have an n-loop in $|X|$ starting and ending at x_0, then we may fill the interior about it and contract the loop. But then using that "filling" and then mapping through $|p|$ is with mapping of $D_{n+1} \to |Y|$ tells us that the homotopy type of $|p|(x)$ is the same as the homotopy type of x, proving $|p|_* : \pi_n(|X|, x) \to \pi_n(|Y|, |p|(x))$ is an isomorphism for all $n \in \mathbb{N}$. $\square$

Corollary 2.7. *The class of trivial fibrations has the right lifting property with cofibrations.*

Proposition 2.5. *There is a $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$ factorization of any map $f : X \to Y$ between preseaves.*

Proof. Consider the factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i_\infty"'] & & Y \\ & G^\infty(\{\partial\Delta^n \to \Delta^n\}, f) \arrow[ur, "p_\infty"'] & \end{tikzcd}$$

provided by the Small Object Argument. Since each of $\partial\Delta^n$ and Δ^n are sequentially small the map p_∞ has the right lifting property with $\{\partial^n \to \Delta^n\}$ and hence is a trivial fibration. The fact that i_∞ is monic follows from the fact that it arises as a series of successive pushouts along $\{\partial\Delta^n \to \Delta^n\}$ along its skeleta. □

This gives us all the tools to prove that the classes of maps we have defined indeed provide a model structure on $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$. At this point the model structure comes down to collecting the various faccts and theorems we have proved and developed regarding simplicial sets.

Theorem 2.5. *The maps defined in Definition 2.13 make* $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ *into a model category.*

Proof. The fact that $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ is (co)complete is Corollary 2.1. The two-out-of-three axiom comes from the fact that isomorphisms in any category satisfy the two-out-of-three axiom. Since $\mathscr{M} = \mathscr{C}$ is a saturated class by Lemma 2.6, it is retract closed. The fact that $\mathscr{W}$ is retract closed follows from the Isomorphism Lemma, while $\mathscr{F}$ is retract closed because if a map p is defined by a lifting property then any of its retracts also have that lifting property. The $(\mathscr{C}, \mathscr{F} \cap \mathscr{W})$ lifting property is Corollary 2.7 while the $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ lifting property is Lemma 2.5. The $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$-factorization is Proposition 2.5 while the $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ factorization is Proposition 2.3. This proves the Quillen model structure on **Top**. □

We end this section by defining a model structure on **Top** in such a way that the adjunction $|\cdot| \dashv \mathrm{Sing}$ is a Quillen equivalence. This is done as follows:

Definition 2.19. *Define the following classes of maps in* **Top***:*

- $\mathscr{C}$*: The saturated class of* $\{|\partial\Delta^n| \to |\Delta^n|\}$*;*
- $\mathscr{W}$*: The class of maps* $f : X \to Y$ *such that* f *the induced maps*

$$\pi_n(X, x) \to \pi_n(Y, f(x))$$

  are isomorphisms for all $n \geq 0$ *and all* $x \in X$*;*
- $\mathscr{F}$*: The class of all Serre Fibrations.*

This collection of maps forms a model structure on **Top**, which is proved by simply carrying out all our prior results through the adjunction. It then follows easily that $|\cdot| \dashv \mathrm{Sing}$ is a Quillen equivalence. This equivalence will important in our discussion of ∞-categories. In particular, we will need the following result, which we do not prove here:

Theorem 2.6 ([GJ09])**.** *The unit and counit of the adjunction* $|\cdot| \dashv \mathrm{Sing} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Top}$ *are weak equivalences of simplicial sets and topological spaces, respectively.*

An important corollary of the above theorem and the Quillen equivalence $|\cdot| \dashv \mathrm{Sing} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Top}$ is that the homotopy categories that arise from $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ and **Top** by inverting the weak equivalences, $\mathrm{h}[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ and $\mathrm{h}\mathbf{Top}$, are equivalent. This tells us that the homotopy theory of simplicial sets is equivalent to the homotopy theory of topological spaces, and consequently gives us a rich approach to homotopy theory.

3. A Gentle Introduction to Higher Category Theory

Example 3.1 ([Lur09])**.** We start with an example that will influence how we can motivate and think of ∞-categories. Begin by letting X be a topological space. Then for every $n \in \mathbb{N}$ we associate a category $\pi_{\leq n} X$ as follows: The objects of $\pi_{\leq n} X$ are the points of X and the 1-morphisms are paths $\gamma : [0, 1] \to X$ starting at x and ending at y, the 2-morphisms are homotopies, the 3-morphisms are homotopies between homotopies, and so on. If $n = 1$ we capture the usual fundamental groupoid $\pi_1 X$ of X; consequently we say that the category $\pi_{\leq n}X$ is the *n-groupoid of X* (as opposed simply to an n-category) because every k-morphism, $0 \leq k \leq n$, of $\pi_{\leq n}X$ has a homotopy inverse.

The above example is quite useful because it gives us some wonderful geometric intuition behind how n-morphisms should behave: they should act as if they are homotopies between $(n-1)$-morphisms when they are invertible and natural transformations between $(n-1)$ morphisms if they are not invertible. From this example we see that would want the category $\pi_{\leq\infty}X$ to have n-morphisms for every $n \in \mathbb{N}$.

In order to give a more concrete introduction of ∞-categories we will need to describe what n-categories are. We begin by defining a 0-category to be a category $\mathfrak{C}$ with only identity morphisms (and hence is a set or a class depending on whether or not $\mathfrak{C}$ is small or large). A 1-category is a category $\mathfrak{C}$ in the usual sense. A 2-category $\mathfrak{C}$ is a category with "morphisms between morphisms." That is, we take a 2-category to be a category $\mathfrak{C}$ *enriched over* **Cat**. This means that for every pair of objects A, B we define the 2-morphisms of $\mathfrak{C}$ to be the morphisms in $\mathfrak{C}(A,B)$. Furthermore, this enrichment means that for all objects A, B, C with $A \to B \to C$ there are *composition functors* $c_{A,B,C} : \mathfrak{C}(A,B) \times \mathfrak{C}(B,C) \to \mathfrak{C}(A,C)$ that send $(f,g) \mapsto fg$. However, this formalism is not flexible enough for homotopy theory, as we have asserted that since $\mathfrak{C}(A,B)$ is a category for all $A, B \in \operatorname{Ob}\mathfrak{C}$ the associativity of composition must be *strict*; that is, we assert that given any $A, B, C, D \in \operatorname{Ob}\mathfrak{C}$ we have that the diagram

$$\begin{tikzcd}
\mathfrak{C}(A,B) \times \mathfrak{C}(B,C) \times \mathfrak{C}(C,D) \arrow[r, "\langle c_{A,B,C}, \mathrm{id}_{\mathfrak{C}(C,D)} \rangle"] \arrow[d, "\langle \mathrm{id}_{\mathfrak{C}(A,B)}, c_{B,C,D} \rangle"'] & \mathfrak{C}(A,C) \times \mathfrak{C}(C,D) \arrow[d, "c_{A,C,D}"] \\
\mathfrak{C}(A,B) \times \mathfrak{C}(B,D) \arrow[r, "c_{A,B,D}"'] & \mathfrak{C}(A,D)
\end{tikzcd}$$

commutes on the nose, as it were. Unfortunately, many structures that arise naturally (such as homotopical structures) only commute *up to isomorphism*, natural in A, B, C, and D, instead of up to strict commutativity.

This process allows us to (perhaps) naïvely define the n-categories for $n \geq 1$. We can take an n-category to be a category $\mathfrak{C}$ enriched in $(n-1)$-categories; however, this formalism leads to many issues involving strict associativity versus weak associativity, and also in the complexity of the objects themselves. For instance, the explicit notion of a weak 3-category is almost too complicated to be useful in practice. Fortunately, we will manage to avoid many of these pains and grievences if we simply assert that most of the higher morphisms (for a given value of most") are invertible. This leads us to a definition:

Definition 3.1 ([Lur09])**.** *Let $\mathfrak{C}$ be an ∞-category. We then say that $\mathfrak{C}$ is an (∞, n)-category if, for all $k \geq n+1$, all k-morphisms are invertible.*

Definition 3.2. *Let $\mathfrak{C}$ be an $(\infty, 0)$-category. Then $\mathfrak{C}$ is an* ∞-groupoid*.*

Example 3.2 ([Lur09])**.** Let X be a topological space. We can then form an ∞-groupoid $\pi_{\leq\infty}X$ associated to X as we obtained the n-fundamental groupoids of X:

- Objects: points $x \in X$;
- Morphisms: $\gamma \in \pi_{\leq\infty}X(x,y)$ are (continuous) paths $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$;
- 2-morphisms: homotopies h between paths $\gamma, \gamma' \in \pi_{\leq\infty}X(x,y)$;
- n-morphisms: homotopies between $(n-1)$-morphisms

where we let $n \in \mathbb{N}$ be arbitrary.

Example 3.3 ([Uni13])**.** Let $\mathcal{U}$ be a universe of dependent type theory with identity types. We can then regard a type $A : \mathcal{U}$ as an ∞-groupoid as follows:

- Objects: terms $a : A$;
- Morphisms: paths (proofs) $p : \mathrm{Id}_A(x,y)$;
- 2-morphisms: paths $q : \mathrm{Id}_{\mathrm{Id}_A}(p,p')$;

and so on through all $n \in \mathbb{N}$; that is an n-morphism is a path in the identity type connecting two $(n-1)$-morphisms p and q.

Remark 3.1. Here we make an agreement for the rest of our discussion of higher category theory: We will refer to all $(\infty,1)$-categories $\mathfrak{C}$ as *∞-categories*.

The terminology ∞-groupoid comes from the fact that $(\infty, 0)$-categories are ∞-categories in which all one and higher morphisms are invertible. These categories are extensively studied: Example 3.2 shows that ∞-groupoids arise in the study of algebraic topology while Example 3.3 shows that ∞-groupoids arise in the study of (dependent) type theory. Thus the study of ∞-categories is important to the study and understanding of homotopy type theory, category theory, algebraic topology, and their interconnections.

In analogy with defining an n-category as an category enriched over the (large) category of $(n-1)$ categories, we have a definition suggested: we can define an ∞-category to be a category $\mathfrak{C}$ enriched over the category ∞-**Grpd** of ∞-groupoids. This means that we would want an ∞-category to be a category such that for every pair of objects A, B, the set (class) of maps $\mathfrak{C}(A, B)$ should record the *entire* homotopy type of maps between A and B. One way to do this is to define an ∞-category to be a category with an ∞-groupoid $\mathrm{Map}_{\mathfrak{C}}(A, B)$ that can be associated to a topological space. However, we are led to the following problem: do we assert that the associativity of maps is *strict* or do we only assert associativity up to coherent homotopy? Fortunately we need not worry, as it is shown in [Lur09] that any ∞-category with a weak associativity law is equivalent to an ∞-category with strict associativity. This leads us to define the following important class of categories:

Definition 3.3 ([Lur09]). *Let* **TopCat** *denote the category of all categories enriched over* **CGHaus***, the category of compactly generated, weakly Hausdorff spaces. Then we say that any object $\mathfrak{C} \in \mathrm{Ob}\,$***TopCat** *is a* Topological Category*.*

One can use the above definition as a foundation for the theory of ∞-categories, but while this definition gives clear geometric intuition of what we wish ∞-categories to capture, it is technically difficult to work with. For instance, many constructions in higher category theory (especially in algebraic topology) are only associative up to coherent homotopy. In order to remain in the world **TopCat** we must then "straighten" maps in order to produce a strictly associative composition law in all levels. While it is shown in [Lur09] that this may always be done, we will instead work with a theory of ∞-categories based on the theory of simplicial sets that more readily allows us to work with maps associative up to (coherent) homotopy. The theory we develop will be equivalent to that induced by topological categories by using the equivalence $\mathrm{h}\mathbf{Top} \cong \mathrm{h}\,[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ induced by the adjunction $|\cdot| \dashv \mathrm{Sing} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Top}$ described in the prior section. Note that the homotopy category $\mathrm{h}\mathfrak{C}$ is introduced and discussed towards the end of my notes on model categories, as well as in [DS95], [Hov99], and [Hir09].

Definition 3.4 ([Lur09]). *Let K be a simplicial set such that for all $n \in \mathbb{N}$ and all $0 < k < n$, there is a lift $\varphi : \Delta^n \to X$ making the diagram*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, dashed] \\ \Delta^n \arrow[r, dashed] \arrow[ur, dashed, "\exists\varphi"] & * \end{tikzcd}$$

commute. We then say that X is a weak Kan complex*.*

Definition 3.5 ([Lur09]). *An ∞-category is a weak Kan complex.*

Note that the above definition is different from K being fibrant, as we only have lifts guaranteed for the *inner* horns $\Lambda^n_k \to \Delta^n$. However, it will be important for the theory of ∞-categories to know which weak Kan compleces are isomorphic to the nerves of (small) categories (recall the nerve of a small category from Example 2.2). This will allow us to draw a better analogy between higher category theory and traditional category by seeing exactly when a simplicial set is capturing the data of a small category. The proof below is quite intuitive, as it is essentially done by showing how to construct a category from a certain simplicial set, as well as how to construct certain lifting properties from a nerve.

Proposition 3.1 ([Lur09]). *Let K be a simplicial set. Then the following are equivalent:*

(1) *There exists a small category $\mathfrak{C}$ such that $\mathrm{N}\,\mathfrak{C} \cong K$;*

(2) *For all $n \in \mathbb{N}$ and all $0 < k < n$ given a commuting square*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, dashed, "\exists!"] \\ \Delta^n \arrow[r, dashed, "\exists!"'] \arrow[ur, dashed, "\exists!\varphi"] & * \end{tikzcd}$$

there exists a unique *lift $\varphi : \Delta^n \to X$ rendering the entire diagram commutative.*

Remark 3.2. Before we begin with the proof, there is some notation that will be useful to codify beforehand. If $\{a_0, a_2, \cdots, a_m\}$ is a linearly ordered subset of $[n]$ for some $n \in \mathbb{N}$, then we use the notation

$$\Delta^{\{a_0, \cdots, a_m\}}$$

to denote the simplicial set $\Delta^{\{a_0,\cdots,a_m\}} \cong \Delta^m$ with vertices $a_0, \cdots, a_m$. For example, if $\{0, 2\} \subseteq [2]$ then we view $\Delta^{\{0,2\}} \cong \Delta^1$ as the representable functor given by considering all maps into the 1-simplex:

$$0 \longrightarrow 2$$

Proof. (1) $\implies$ (2): Let $K = \mathrm{N}\,\mathfrak{C}$ for some small category $\mathfrak{C}$ and assume that there is a morphism $f_0 : \Lambda^n_k \to K$. For every $0 \le i \le n$ let $X_i \in \mathrm{Ob}\,\mathfrak{C}$ be the image of the vertex $\{i\} \subseteq \Lambda^n_k \to K$. Then if $0 < i \le n$ define the map

$$g_i : X_{i-1} \to X_i$$

to be the morphism determined by the restriction of f_0 to the simplex $\Delta^{\{i-1,i\}} \cong \Delta^1$, i.e., to be the map induced by the morphism

$$f_0\big|_{\Delta^{\{i-1,i\}}} : \Delta^{\{i-1,i\}} \to X.$$

Then the composable chain

$$X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} X_n$$

determines a map $f : \Delta^n \to K$ and hence an n-simplex in $\mathrm{N}\,\mathfrak{C}$. Note that if there is any other map $f' : \Delta^n \to K$ making the diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f_0"] \arrow[d] & K \\ \Delta^n \arrow[ur, shift left, "f"] \arrow[ur, shift right, "f'"'] & \end{tikzcd}$$

commute, then f' must produce the same chain of maps in $\mathfrak{C}$, giving $f = f'$. Consequently if we can show that f is the desired lift we are done.

Now let $0 \le \ell \le n$. In order to show that f is a lift of f_0, we must show that whenever $\ell \ne k$ we have that

$$f\big|_{\Delta^{\{0,\cdots,\ell-1,\ell+1,\cdots,n\}}} = f_0\big|_{\Delta^{\{0,\cdots,\ell-1,\ell+1,\cdots,n\}}}.$$

However, this is equivalent to proving that whenever j, j' are adjacent elements in the linearly ordered set $\{0, \cdots, \ell-1, \ell+1, \cdots, n\} \subseteq [n]$, then $f|_{\Delta^{\{j,j'\}}} = f_0\big|_{\Delta^{\{j,j'\}}}$. However, if j, j' are adjacent in $[n]$ and $j, j' \ne \ell$ then by construction we are done (and in particular we are done for $j = 0$ and $j' = n$) so take $j = \ell - 1$ and $j' = \ell + 1$ for $0 < j < n$. If $n - 2$ then $j = 1 = k$ and we have a contradiction; assume $n \ge 3$ and either $j = \ell - 1 > 0$ or $j' = \ell + 1 < n$; WOLOG assume that $\ell - 1 > 0$. But then

$$\Delta^{\{\ell-1,\ell+1\}} \subseteq \Delta^{\{1,\cdots,n\}}$$

so from the condition $\ell = 0$ we are done.

(2) $\implies$ (1): Assume that for every map $\Lambda^n_k \to K$ there is a unique lift $f : \Delta^n \to K$ making the diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & K \\ \Delta^n \arrow[ur, dashed, "\exists! f"'] & \end{tikzcd}$$

commute. We now claim that we can construct a category $\mathfrak{C}$ from K via:

- Objects: Vertices (elements) of $K_0 \cong \mathbf{Nat}(\Delta^0, K)$;

- Morphisms: For $x, y \in \operatorname{Ob}\mathfrak{C}$, we define
$$\mathfrak{C}(x,y) := \{e \in \mathbf{Nat}(\Delta^1, K) \mid e|\{0\} = x, e|_{\{1\}} = y\};$$
- Identities: The identity map $x \to x$ is the edge given by the composition
$$\Delta^1 \longrightarrow \Delta^0 \xrightarrow{e} X$$
where $\Delta^0 \xrightarrow{e} X$ picks out the vertex x;
- Composition: If $f : x \to y$ and $g : y \to z$ are morphisms in $\mathfrak{C}$ then they produce a map
$$\sigma_0 : \Lambda^2_1 \to K;$$
extend this uniquely to a map $\sigma : \Delta^2 \to K$ with, writing $\iota : \Lambda^2_1 \to \Delta^2$, $\sigma_0 = \iota\sigma$ Then we regard $fg : x \to z$ as the edge given by the composition:
$$\Delta^1 \xrightarrow{\cong} \Delta^{\{0,2\}} \longrightarrow \Delta^2 \xrightarrow{\sigma} K$$

We must show that this does indeed construct a category. To do this note that if $f : x \to y$ and $g : z \to x$ are morphisms in $\mathfrak{C}$ then the 2-simplices σ and τ defined by

$$\begin{tikzcd} & x \arrow[dr, "f"] \arrow[dl, equal] & \\ x \arrow[rr, "f"'] & & y \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, equal] & \\ z \arrow[ur, "g"] \arrow[rr, "g"'] & & x \end{tikzcd}$$

have $s_0(\sigma) = f$ and $s_1(\tau) = g$. Thus σ and τ "witness" that $\mathrm{id}_x f = f$ and $g \,\mathrm{id}_x = g$, proving that the identity maps are indeed left and right units.

We now show that composition is associative. Begin by letting $f : x \to y, g : y \to z$, and $h : z \to w$ be given as morphisms in $\mathfrak{C}$. Now produce the 2-simplex σ_{012} defined by

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "fg"'] & & z \end{tikzcd}$$

and a 2-simplex σ_{123} given by:

$$\begin{tikzcd} & z \arrow[dr, "h"] & \\ y \arrow[ur, "g"] \arrow[rr, "gh"'] & & w \end{tikzcd}$$

Now find a 2-simplex σ_{023} corresponding to the diagram

$$\begin{tikzcd} & z \arrow[dr, "h"] & \\ x \arrow[ur, "fg"] \arrow[rr, "(fg)h"'] & & w \end{tikzcd}$$

so that gluing these three 2-simpleces provides a morphism $\varphi : \Lambda^3_2 \to K$. Since K admits unique lifts from morphisms $\varphi : \Lambda^3_2 \to K$, there is a lift $\psi : \Delta^3 \to K$ of φ. But then the composition
$$\Delta^2 \cong \Delta^{\{0,1,3\}} \hookrightarrow \Delta^3 \to K$$
fills in the face

$$\begin{tikzcd} & y \arrow[dr, "gh"] & \\ x \arrow[ur, "f"] \arrow[rr, "f(gh)"'] & & w \end{tikzcd}$$

and hence witnesses the associativity $(fg)h = f(gh)$.

From construction we now have that $\mathfrak{C}$ is a category and that there is a map $f : K \to \mathrm{N}\,\mathfrak{C}$. We must show that this is an isomorphism; to do this we proceed by induction . Begin by observing that our construction of $\mathfrak{C}$ provides isomorphisms

$$[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Delta^n, K) \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Delta^n, \mathrm{N}\,\mathfrak{C}).$$

for $n = 0, 1$. To complete the induction assume that $n \geq 2$ and let $0 < k < n$ be an integer. We can then produce the commuting square:

$$\begin{tikzcd}
{[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Delta^n, K)} \arrow[r] \arrow[d] & {[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Delta^n, \mathrm{N}\,\mathfrak{C})} \arrow[d] \\
{[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Lambda^n_k, K)} \arrow[r] & {[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Lambda^n_k, \mathrm{N}\,\mathfrak{C})}
\end{tikzcd}$$

Because both K and $\mathrm{N}\,\mathfrak{C}$ produce unique lifts against the inner horn injections, both vertical arrows in the commuting square are isomorphisms. It then follows from the inductive hypothesis that the bottom horizontal arrow is an isomorphism, giving by the two-out-of-three axiom that the top horizontal arrow is an isomorphism. This proves that $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Delta^n, K) \cong [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Delta^n, \mathrm{N}\,\mathfrak{C})$, and hence by the principle of mathematical induction we are done. $\square$

Corollary 3.1. *A simplicial set is a Kan complex if and only if it is isomorphic to the nerve* $\mathrm{N}\,G$ *for some groupoid* G*.*

Remark 3.3. I feel that it is important at this moment to defend our choice to use weak Kan compleces as ∞-categories. Note that while this proposition shows us that ∞-categories capture classical category theory (via the identification $\mathfrak{C} \sim \mathrm{N}\,\mathfrak{C}$), we can see immediately that it is unnatural to assert that the outer horns admit extensions to their respective simpleces; if they did then both $\Lambda^2_2 \to X$ and $\Lambda^2_0 \to X$ would admit an "filler" along their missing edges; in particular, if we have the simpleces

$$\begin{tikzcd}
 & B \arrow[dr, "f"] & \\
A \arrow[rr, equal] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B & \\
A \arrow[ur, "g"] \arrow[rr, equal] & & A
\end{tikzcd}$$

given by injecting the outer horns Λ^2_2 and Λ^2_0 into the nerve $\mathrm{N}\,\mathfrak{C}$ of a small category $\mathfrak{C}$, then we could extend (in this case uniquely) the simpleces to fill in the diagrams to the form

$$\begin{tikzcd}
 & B \arrow[dr, "f"] & \\
A \arrow[ur, dashed] \arrow[rr, equal] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[dr, dashed] & \\
A \arrow[ur, "g"] \arrow[rr, equal] & & A
\end{tikzcd}$$

and hence arrive at $\mathfrak{C}$ being a groupoid (and $\mathrm{N}\,\mathfrak{C}$ being an ∞-groupoid). Furthermore, since many of these structures arise "homotopically" it is also unnatural to assert that the lift $\Delta^n \to K$ be unique; in particular, if we are working with the fundamental groupoid of a space X, then composition is given by concatenation of paths. However, if we have paths $\gamma, \gamma' : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$, $\gamma'(0) = y$, and $\gamma'(1) = z$ for $x, y, z \in X$, then both the composite paths

$$p(t) := \begin{cases} \gamma(2t) & 0 \leq t \leq 1/2; \\ \gamma'(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

and

$$q(t) := \begin{cases} \gamma(3t) & 0 \leq t \leq 1/3 \\ \gamma'\left(\frac{3t-1}{2}\right) & 1/3 \leq t \leq 1 \end{cases}$$

provide maps $\gamma\gamma' : x \to z$. However, since $p \simeq q$, they produce the same map in πX, albeit only up to homotopy. Making the lifts $\Delta^n \to X$ unique would be the same as selecting a single path p in each case, once and for all, which is clearly unnatural and strange to impose.

We can see here that simplicial sets are indeed a good formalism for ∞-categories. They capture and describe ∞-groupoids (through Kan compleces), normal category theory (through the identification of a category with its nerve and Proposition 3.1). Furthermore, Proposition 3.1 shows that when we use weak Kan compleces as a formalism for ∞-categories, we are guaranteed that they behave as we would expect an ∞-category to behave (the extension condition tells us exactly that we can compose maps at all levels), while the simplicial set underlying the category captures the necessary combinatorial data to make sure that we have all n-morphisms and a flexible way of dealing with situations being either strictly associative or weakly associative. The only big problem with this notion of an ∞-category is that in order to capture the intuition of a type being an ∞-groupoid, we need to find a model for type theory in the category $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ and then check that it indeed interprets types as Kan compleces; one such way to do this is described by Streicher in [Str14], but we do not go into the details of the construction he gives here.

Example 3.4. We now give a way to produce the fundamental ∞-groupoid as alluded to in the discussion of $\pi_{\leq\infty}X$: Since the simplicial set $\operatorname{Sing} X$ of any topological space X is fibrant (and hence a Kan complex), it is an ∞-groupoid. Thus we define the fundamental ∞-groupoid of X $\pi_{\leq\infty}X$ as $\operatorname{Sing} X$.

We now move to define simplicial categories in order to provide a more concrete comparison between the geometry and topology of a simplicial set (and hence of an ∞-category) and the homotopy theory underlying it. Here we will use the theory of simplicial categories in order to provide a very concrete way of seeing that topological categories provide the same homotopy theory as that provided by simplicially, giving us that we can use either formalism safely as a foundation for higher category theory.

Definition 3.6. *Let $\mathfrak{C}$ be a small category. We then say that $\mathfrak{C}$ is a* simplicial category *if it is a simplicial object in* $\mathbf{Cat}$*, i.e., if it is a category enriched over* $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$.

Definition 3.7. *Let $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ denote the category of simplicial categories, i.e., of presheaves $F : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}$.*

From the adjunction $|\cdot| \dashv \operatorname{Sing} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Top}$ we can determine a geometric realization functor $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{TopCat}$ which will also be denoted as $|\cdot| : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{TopCat}$. It is defined as follows:

- $\operatorname{Ob}|\mathfrak{C}| = \operatorname{Ob}\mathfrak{C}$;
- If $A, B \in \operatorname{Ob}\mathfrak{C}$, then:
$$|\mathfrak{C}|(A, B) := |\mathfrak{C}(A, B)|;$$
- Composition of morphisms is given by applying the geometric realization functor to $\mathfrak{C}(A, B) \times \mathfrak{C}(B, C) \to \mathfrak{C}(A, C)$.

Note that because $|\cdot| : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{CGHaus}$ preserves finite products we have that in the above definition $|\mathfrak{C}(A, B) \times \mathfrak{C}(B, C)| \cong |\mathfrak{C}(A, B)| \times |\mathfrak{C}(B, C)|$ so the composition is well-defined and indeed produces a functor $|\cdot| : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{TopCat}$. Dually, we can produce a functor $\operatorname{Sing} : \mathbf{TopCat} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ by applying the singular complex functor to all mapping spaces $\mathfrak{C}(A, B)$. This will produce an adjunction $|\cdot| \dashv \operatorname{Sing} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{TopCat}$, which is a formal consequence of the adjunction $|\cdot| \dashv \operatorname{Sing} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Top}$. We will use this adjunction to show that the theory of topological categories actually does induce a theory of ∞-categories where $\mathfrak{C}(A, B)$ is an ∞-groupoid (and hence a category capturing the entire homotopy type between A and B) for all $A, B \in \operatorname{Ob}\mathfrak{C}$.

We begin examining this idea by first analyzing when simplicial categories may be seen to be "homotopically equivalent" to topological categories. Luckily this is not hard with the adjunction at hand: we simply need a notion of discussing when the (singular compleces of the) hom-spaces underlying a topological category are homotopically equivalent to (geometric realizations of the) simplicial hom-sets underlying a simplicial category. This is done first by defining what we mean by homotopy category, and then proceeding with the equivalence $\mathrm{h}\mathbf{Top} \cong \mathrm{h}\,[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ in mind:

Definition 3.8. *Let $\mathfrak{C}$ be a topological category. We then define the* homotopy category of $\mathfrak{C}$ *to be the category* $\mathrm{h}\mathfrak{C}$ *defined by:*

- *Objects: the objects of* $\mathrm{h}\mathfrak{C}$ *are the objects of* $\mathfrak{C}$*;*
- *Morphisms: for all* $X, Y \in \operatorname{Ob}\mathrm{h}\mathfrak{C}$ *define* $\mathrm{h}\mathfrak{C}(X, Y) := \pi(X, Y)$*;*
- *Composition: given by the assignment* $([f], [g]) \mapsto [fg]$.

Note that the homotopy category $\mathrm{h}\mathfrak{C}$ describes $\mathfrak{C}$ as a category enriched over the homotopy category $\mathrm{h}\mathbf{Top}$; this allows us to describe when two topological categories are weakly equivalent. In particular, we say that two topological categories are *weakly equivalent* if there is a functor $F : \mathfrak{C} \to \mathfrak{D}$ such that the induced map

$$F_* : \mathrm{h}\mathfrak{C} \to \mathrm{h}\mathfrak{D}$$

is an isomorphism of $\mathrm{h}\mathbf{Top}$-enriched categories. More explicitly, we require that:

(1) For all $X, Y \in \operatorname{Ob}\mathfrak{C}$ the induced map

$$\mathfrak{C}(X,Y) \to \mathfrak{D}(FX,FY)$$

is a homotopy equivalence of topological spaces;

(2) Every object $X \in \operatorname{Ob}\mathfrak{D}$ is isomorphic in $\mathrm{h}\mathfrak{D}$ to FA for some $A \in \operatorname{Ob}\mathfrak{C}$.

It follows immediately from the equivalence $\mathrm{h}\mathbf{Top} \cong \mathrm{h}\left[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}\right]$ described by the Quillen equivalence $|\cdot| \dashv \operatorname{Sing}$ that there is a notion of homotopy category $\mathrm{h}\mathfrak{C}$ of a simplicial category $\mathfrak{C}$ attained in the analogous way. This is codified in the following definition:

Definition 3.9. *Let $\mathfrak{C}$ be a simplicial category. Then the* homotopy category of $\mathfrak{C}$ *is the category $\mathrm{h}\mathfrak{C}$ whose objects are those of $\mathfrak{C}$ and morphism classes are those attained by inverting homotopies, i.e.,* $\mathrm{h}\mathfrak{C}(X,Y) := \pi(X,Y)$ *for all $X, Y \in \operatorname{Ob}\mathfrak{C}$.*

We observe that from the adjunction we not only have that there is an equivalence $\mathrm{h}\mathbf{Top} \cong \mathrm{h}\left[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}\right]$, but also canonical isomorphisms

$$\mathrm{h}\mathfrak{C} \cong \mathrm{h}|\mathfrak{C}|$$

and

$$\mathrm{h}\mathfrak{D} \cong \mathrm{h}\operatorname{Sing}\mathfrak{D}$$

for every $\mathfrak{C} \in \operatorname{Ob}\left[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}\right]$ and every $\mathfrak{D} \in \operatorname{Ob}\mathbf{TopCat}$.

From this natural isomorphism we now need a functorial manner in which to relate simplicial categories to simplicial sets (which we will then use to relate the theory of simplicial categories to topological categories). Observe that if we regard the linearly ordered set $[n]$ as a category then then we have an identification of the nerve of a small category via

$$[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]([n], \mathrm{N}\,\mathfrak{C}) = \mathbf{Cat}([n], \mathfrak{C}).$$

While this makes sense for simplicial categories $\mathfrak{C}$ as well as for general small categories, it makes no mention or use of the simplicial structure on $\mathfrak{C}$. In particular, the nerve $\mathrm{N}\,\mathfrak{C}$ records strictly associative data of the category $\mathfrak{C}$. In order to have a nerve record ∞-categorical data we will need to somehow "thicken" the structure of N, as well make this "thickening" functorial.

Definition 3.10 ([Lur09])**.** *Let $J \in \operatorname{Ob}\mathbf{FinLinOrd}$, the category of finite linearly ordered set. Then we may construct a simplicial category $\mathbb{C}\left[\Delta^J\right]$ via the assignment:*

- *The objects of $\mathbb{C}\left[\Delta^J\right]$ are elements $j \in J$;*
- *If $i, j \in J$ define the simplicial hom-set by:*

$$\mathbb{C}\left[\Delta^J\right](i,j) := \begin{cases} \varnothing & j > i \\ N(P_{i,j}) & i \leq j \end{cases}$$

where $N(P_{i,j})$ is the poset

$$P_{i,j} := \{I \subseteq J \mid \min(I) = i, \max(I) = j\}.$$

- *Composition*

$$\prod_{j=1}^{n} \mathbb{C}\left[\Delta^J\right](i_{j-1}, i_j) \to \mathbb{C}\left[\Delta^J\right](i_0, i_n)$$

is induced by the poset map

$$\prod_{j=1}^{n} P_{i_{j-1},i_j} \to P_{i_0,i_n}$$

given by

$$(I_1, \cdots, I_n) \mapsto \bigcup_{j=1}^{n} I_j.$$

Note that we call the simplicial set $\mathbb{C}\left[\Delta^n\right](i,j)$ a cube because the simplicial set $P_{i,j}$ has 2^{j-i-1} elements and hence may be identified with the subsets of $\{i+1, \cdots, j-1\}$ whenever $i \leq j$. Thus there is an isomorphism of $\mathbb{C}\left[\Delta^n\right](i,j)$ with

$$\mathbb{C}\left[\Delta^n\right](i,j) \cong \{f : \{i+1, \cdots, j-1\} \to \{0,1\} \mid \text{ if } k \leq \ell, f(k) \leq f(\ell)\} \cong \prod_{i=1}^{j-i-1} \Delta^1.$$

We call this set a simplicial $n-1$ cube because under realization we have

$$\left|\mathbb{C}\left[\Delta^n\right]\right| \cong \left|\prod_{i=1}^{n-1} \Delta^1\right| \cong \prod_{i=1}^{n-1} |\Delta^1| = [0,1]^{n-1}.$$

There is an important difference with the categories $[n]$ and $\mathbb{C}\left[\Delta^n\right]$ for all $n \in \mathbb{N}$. While the objects of $[n]$ and $\mathbb{C}\left[\Delta^n\right]$ are the same, the morphisms of $[n]$ are all unique. In particular, there is a unique map $f : i \to j$ for all $i \leq j$ in $[n]$. This uniqueness implies that $f_{ij} f_{jk} = f_{ik}$ for all $i \leq j \leq k$ in $[n]$. In the simplicial set $\mathbb{C}\left[\Delta^n\right](i,j)$ there is a vertex $p_{ij} \in \mathbb{C}\left[\Delta^n\right](i,j)$ corresponding to the set $\{i,j\} \in P_{i,j}$. Whenever $i = j = k$ we have $p_{ij} p_{jk} = p_{ik}$, but whenever $i \neq j$ or $j \neq k$ we have that

$$p_{ij} p_{jk} \neq p_{ik};$$

this is easily seen by

$$p_{ij} p_{jk} = \{i,j\} \cup \{j,k\} = \{i,j,k\}.$$

In general, if $i_0 < i_1 \cdots < i_n$ then the collection of all compositions

$$p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$$

constitute the distinct vertices of the cube $\left|\mathbb{C}\left[\Delta^n\right]\right|$. While these vertices are not at all equivalent, they are all canonically homotopic to each other. This allows us to think of $\mathbb{C}\left[\Delta^n\right]$ as a "thickening" of $[n]$ by dropping the strict associativity conditions on $[n]$.

Definition 3.11. *Let $f \in \mathbf{FinLinOrd}(J, J')$. Then define the morphism $\mathbb{C}[f] : \mathbb{C}\left[\Delta^J\right] \to \mathbb{C}\left[\Delta^{J'}\right]$ by:*

- *$\mathbb{C}[f](i) = f(i)$ for all $i \in J$;*
- *If $i \leq j$ in J, then the map $\mathbb{C}[f] : \mathbb{C}\left[\Delta^J\right](i,j) \to \mathbb{C}[f]\left[\Delta^{J'}\right](f(i), f(j))$ is taken to be the nerve of*

  $$P_{i,j} \to P_{f(i),f(j)}$$

  induced by the mapping $I \mapsto f(I)$.

This above definition makes $\mathbb{C}[-] : \mathbf{\Delta} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ into a functor via

$$[n] \mapsto \mathbb{C}[\Delta^n]$$

and $[f] \mapsto \mathbb{C}[f]$. However, from this it is easy to see that we can extend $\mathbb{C}[-]$ into a functor $\mathbb{C}[-] : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$. In particular, we make the following definition:

Definition 3.12. *Define the functor $\mathbb{C}[-] : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ to be the left Kan extension in the diagram*

$$\begin{tikzcd}
 & {[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]} \arrow[dr, dashed, "Lan_y := \mathbb{C}[-]"] & \\
\mathbf{\Delta} \arrow[ur, "y"] \arrow[rr, "\mathbb{C}[-]"'] & & {[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]}
\end{tikzcd}$$

where y is the Yoneda embedding $y : \mathbf{\Delta} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$.

Note that this gives $\mathbb{C}[-] : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ is defined on all simplicial sets K via the coend

$$\mathbb{C}[K] = \int^{[n]\in\mathbf{\Delta}} K_n \odot \mathbb{C}\left[\Delta^n\right].$$

As in the case of simplicial sets, $\mathbb{C}[-]$ has a right adjoint, which we define to be the simplicial nerve of a simplicial category $\mathfrak{C}$.

Definition 3.13 ([Lur09]). *Let $\mathfrak{C}$ be a simplicial category. Then the* simplicial nerve of $\mathfrak{C}$ *is the simplicial set whose n-simpleces are given by*

$$\mathrm{SN}(\mathfrak{C})_n := [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]\left(\mathbb{C}\left[\Delta^n\right], \mathfrak{C}\right).$$

Definition 3.14 ([Lur09]). *Let $\mathfrak{C}$ be a topological category. We then define the* topological nerve *of $\mathfrak{C}$ to be the simplicial set*

$$\mathfrak{N}\,\mathfrak{C} := \mathrm{SN}\left(\mathrm{Sing}\,\mathfrak{C}\right).$$

From this definition we have the immediate proposition below. It follows again from coend calculus and is completely analogous to the proof of Theorem 2.1, and hence is omitted.

Proposition 3.2. *There is an adjunction $\mathbb{C}\left[-\right] \dashv \mathrm{SN} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$.*

From this construction we can think of the category $\mathbb{C}[X]$ as the simplicial category generated freely by X and the simplicial set X. Furthermore, the functor $\mathbb{C}[\sigma] : \mathbb{C}\left[\Delta^n\right] \to \mathbb{C}[X]$ induced by a simplex $\sigma : \Delta^n \to X$ can be thought of as an n-homotopy coherent diagram in $\mathbb{C}[X]$. These constructions provide us with some key results for the theory of ∞-categories, as well as showing us that ∞-categories may be modeled as simplicial categories, at least up to homotopy equivalence.

Proposition 3.3 ([Lur09]). *Let $\mathfrak{C}$ be a simplicial category such that for all objects $A, B \in \mathrm{Ob}\,\mathfrak{C}$ the simplicial set $\mathfrak{C}(A, B)$ is a Kan complex. Then the simplicial nerve of $\mathfrak{C}$ is an ∞-category.*

Proof. We must show that for all $0 < k < n$, $n \in \mathbb{N}$, there is a lift $f : \Delta^n \to \mathrm{SN}\,\mathfrak{C}$ in the diagram:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathrm{SN}\,\mathfrak{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

from the adjunction $\mathbb{C}[-] \dashv \mathrm{SN}(-)$ we see that this is equivalent to showing a lift in every diagram of the form

$$\begin{tikzcd} \mathbb{C}\left[\Lambda^n_k\right] \arrow[r] \arrow[d] & \mathfrak{C} \\ \mathbb{C}\left[\Delta^n\right] \arrow[ur, dashed] & \end{tikzcd}$$

Observe now that $\mathrm{Ob}\,\mathbb{C}\left[\Lambda^n_k\right] = \{0, \cdots, n\} = \mathrm{Ob}\,\mathbb{C}\left[\Delta^n\right]$. Furthermore, whenever $0 \leq i \leq j \leq n$, since $k \neq 0, n$ whenever $j \neq 0$ and $k \neq n$ we have the equivalence of simplicial hom-sets

$$\mathbb{C}\left[\Delta^n\right](i, j) = \mathbb{C}\left[\Lambda^n_k\right](i, j).$$

Now consider that solving the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d] & \mathrm{SN}\,\mathfrak{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

reduces to solving the lifting problem

$$\begin{tikzcd} \mathbb{C}\left[\Lambda^n_k\right](0, n) \arrow[r, "\mathbb{C}[f]"] \arrow[d] & \mathfrak{C}(f(0), f(n)) \\ \mathbb{C}\left[\Delta^n\right](0, n) \arrow[ur, dashed] & \end{tikzcd}$$

Since $\mathrm{SN}\,\mathfrak{C}$ is a Kan complex, we need to show that $\mathbb{C}\left[\Lambda^n_k\right](0,n) \to \mathbb{C}\left[\Delta^n\right](0,n)$ is an anodyne extension. To do this recall that we have an identification of $\mathbb{C}\left[\Delta^n\right]$ as the cube

$$\mathbb{C}\left[\Delta^n\right](0,n) \cong \prod_{i=1}^{n-1} \Delta^1.$$

Under this identification we see that $\mathbb{C}\left[\Lambda^n_k\right]$ can be identified as a simplicial subset of $\prod_{i=1}^{n-1} \Delta^1$ given by deleting the interior of the cube and then removing one of its faces. This shows that the injection is anodyne and hence probes the proposition. $\square$

Corollary 3.2 ([Lur09]). *If $F : \mathfrak{C} \to \mathfrak{D}$ is a functor between simplical categories for which the induced morphisms of simplicial sets $\mathfrak{C}(A,B) \to \mathfrak{D}(FA,FB)$ are Kan fibrations, then the map $\mathrm{SN}\,\mathfrak{C} \to \mathrm{SN}\,\mathfrak{D}$ has the right lifting property against the inner horn injections $\{\Lambda^n_k \mid 0 < k < n, n \in \mathbb{N}\}$.*

Corollary 3.3 ([Lur09]). *If $\mathfrak{C}$ is a topological category then the topological nerve $\mathbb{C}\left[\mathrm{Sing}\,\mathfrak{C}\right]$ is an ∞-category.*

We now provide a theorem connecting the topological theory of ∞-categories without proof. It states that any topological category is weakly equivalent to the realization of a topological nerve of a topological category. The proof of the theorem is beyond the scope of this article, so we instead refer the interested reader §2.2.4 and §2.2.5 of [Lur09] for a more detailed discussion and proof of the theorem.

Theorem 3.1 ([Lur09]). *Let $\mathfrak{C}$ be a topological category and let $X, Y \in \mathrm{Ob}\,\mathfrak{C}$ be objects. Then the counit*

$$|\mathbb{C}\left[\mathfrak{N}\,\mathfrak{C}\right]| \to \mathfrak{C}(X,Y)$$

is a homotopy equivalence of spaces.

Definition 3.15 ([Lur09]). *Let X and Y be simplicial sets. We say that the* homotopy category of X *is the homotopy category*

$$\mathrm{h}\,X := \mathrm{h}\,\mathbb{C}\left[X\right].$$

Moreover, we say that $f : X \to Y$ is a categorical equivalence *if the map*

$$f_* : \mathrm{h}\,X \to \mathrm{h}\,Y$$

is a homotopy equivalence of simplicial categories.

From these definitions we see that any of the maps between topological categories do preserve the equality we desire on ∞-categories. Explicitly, from definition and the adjunctions at play we derive that $f : X \to Y$ is a categorical equivalence of simplicial sets if and only if $\mathbb{C}[f] : \mathbb{C}[X] \to \mathbb{C}[Y]$ is a homotopy equivalence if and only if $|\mathbb{C}[f]| : |\mathbb{C}[X]| \to |\mathbb{C}[Y]|$ is a homotopy equivalence. This implies that while not every simplicial set is an ∞-category, it is categorically equivalent to the nerve of some topological category, which *is* an ∞-category. In particular, we have the following diagram of adjunctions which each describe (homotopy) equivalent theories of ∞-categories:

$$\begin{tikzcd}
{[\boldsymbol{\Delta}^{\mathrm{op}}, \mathbf{Cat}]} \arrow[rr, shift left, "|\cdot|"] \arrow[dr, shift left, "\mathrm{SN}"] & & \mathbf{TopCat} \arrow[ll, shift left, "\mathrm{Sing}"] \arrow[dl, shift right, "\mathfrak{N}"'] \\
& {[\boldsymbol{\Delta}^{\mathrm{op}}, \mathbf{Set}]} \arrow[ul, shift left, "\mathbb{C}[\cdot]"] \arrow[ur, shift right, "|\mathbb{C}[\cdot]|"'] &
\end{tikzcd}$$

It is in this sense simplicial sets provide an important framework in which to found the theory of ∞-categories, as the provide the right setting to give the meaning behind "categories enriched over homotopy types." Furthermore, since type theory may be interpreted into simplicial sets, and since the hom-spaces of an ∞-category record important geometric data (when viewed as topological spaces), we see that a further use and study of these objects will aid us in connecting the logical and geometric connections between (dependent) type-theoretic logic and algebraic topology.

Appendix A. Topological Preliminaries and Facts

Because ∞-categories are intimately tied to the theory of topology, I feel it is important to have some of the more key topological definitions and notions readily at hand. While some elementary topology is assumed, most of what we provide here can be thought of as a look at some of the algebraic topology underlying the theory of simplicial sets. We begin by discussing a minimal amount of theory needed to describe CW compleces. Throughout this appendix we define

$$D_n := \{\mathbf{v} \in \mathbb{R}^n \ : \ \|\mathbf{v}\| \leq 1\}.$$

Definition A.1 ([Rot88])**.** *A* topological n-cell*, for natural numbers n, is defined to be a topological space c for which*

$$c \cong B_1(\mathbf{0}) = \{\mathbf{v} \in \mathbb{R}^n \ : \ \|\mathbf{v}\| < 1\} = D_n \setminus \mathbb{S}^{n-1}.$$

Definition A.2. *We define the* dimension of an n-cell e *to be*

$$\dim(e) := \min\{n \in \mathbb{N} \mid e \cong B_1(\mathbf{0}) \subseteq \mathbb{R}^n\}.$$

Definition A.3 ([Rot88])**.** *Let $E = \{e_i \mid i \in I\}$ be a family of cells such that*

$$X = \coprod_{i \in I} e_i.$$

Then for every $n \in \mathbb{N}$ we define the n-skeleton of X *to be the set*

$$X^{(n)} := \bigcup\{e \in E \mid \dim(e) \leq n\}.$$

It is immediate from the above definition that there is an increasing chain of subsets

$$X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \cdots$$

and that

$$X = \bigcup_{n \in \mathbb{N}} X^{(n)}.$$

Definition A.4 ([Rot88])**.** *Let X be a set covered by subsets A_i where $\mathscr{A} := \{A_i \mid i \in I\}$ denotes the covering family of subsets. Assume that:*

1. *Each set A_i is a topological space;*
2. *For each pair of indexes $i, j \in I$ the subspaces topology on $A_i \cap A_j$ is the same whether we regard $A_i \cap A_j$ as a subspace of A_i or if we regard $A_i \cap A_j$ as a subspace of A_j;*
3. *For each pair of indexes $i, j \in I$ the subset $A_i \cap A_j$ is closed in both A_i and A_j.*

Then we define the weak topology on X generated by the A_i *to be the topology whose closed sets are exactly the subsets $S \subseteq X$ for which $A_i \cap S$ is closed in A_i for every $i \in I$.*

Definition A.5. *A* CW complex *is an ordered triple $(X, E, \mathscr{F})$ where X is a Hausdorff space, $E = \{e_i \mid i \in I\}$ is a family of topological cells, and*

$$\mathscr{F} := \left\{f_e : D_{\dim(e)} \to e \cup X^{\dim(e)-1} \mid e \in E\right\}$$

is a family of continuous morphisms such that:

1. *X has*
$$X = \coprod_{e \in E} e;$$
2. *For all n-cells the map f_e has $f_e\big|_{D_n \setminus \mathbb{S}^{n-1}}$ is a homeomorphism onto e;*
3. *If $e \in E$ then*
$$\overline{e} \subseteq \bigcup_{i=1}^{m} e_i$$
for some $e_i \in E$;
4. *X has the weak topology generated generated by the family of sets $\{\overline{e} \mid e \in E\}$.*

Definition A.6. *A topological space X is* compactly generated *if X has the weak topology generated by its compact subsets.*

Proposition A.1. *Any CW complex is compactly generated.*

Proposition A.2. *The collection,* **CGHaus***, given by:*

1. *Objects: compactly generated Hausdorff spaces;*
2. *Morphisms: continuous functions*

is a subcategory of **Top**.

Proposition A.3. *The category* **CGHaus** *has finite products given by the Kelley product of spaces.*

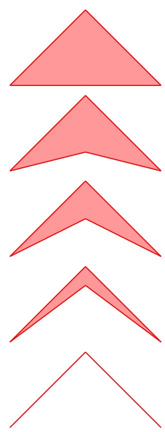
Proposition A.4. *The category* **CGHaus** *is Cartesian closed.*

Definition A.7 ([Rot88])**.** *Let X be a topological space and $A \subseteq X$ with $\iota : A \to X$ the standard injection. We then say that A* is a strong deformation retract of X *if there exists a continuous morphism* $;H : X \times [0,1] \to X$ *such that, for all $x \in X$, $a \in A$, and $t \in [0,1]$ the following hold:*

1. $h(x, 0) = x$*;*
2. $h(x, 1) \in A$*;*
3. $h(a, t) = a$*.*

If we instead insist on the assignment $h(a, 1) = a$ then we say that A is a deformation retract of X*.*

Example A.1. For all $0 \le k \le n$, the horn $|\Lambda_k^n|$ is a strong deformation retract of $|\Delta^n|$. This can be seen by the sequence of diagrams:



which describes the desired homotopy $h : |\Delta^n| \times [0,1] \to |\Delta^n|$.

Appendix B. Homotopy Groups in **Top**

Because it is important to the development of the model structure on $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$, for the sake of completeness we provide here the standard (Quillen) model structure on the category **Top** and the basic construction of the topological homotopy groups $\pi_n(X, x_0)$ for $n \ge 0$. We begin by constructing the n-homotopy groups (or, in the case $n = 0$, the path components of (X, x_0)) and from there moving on to describing the Quillen model structure on **Top**.

Definition B.1. *Define the* topological n-sphere*, for all $n \in \mathbb{N}$, to be the set*

$$\mathbb{S}^n := \{\mathbf{v} \in \mathbb{R}^{n+1} \;:\; \|\mathbf{v}\| = 1\},$$

given its subspace topology under the identification $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, where $\mathbb{R}^{n+1}$ has the standard Euclidean topology.

Definition B.2. *Let (X, x_0) be a pointed topological space, i.e., an object in the coslice category $\{*\}\backslash$* **Top** *and let $n \in \mathbb{N}$. Then define the set $\pi_n(X, x_0)$ to be the set*

$$\pi_n(X, x_0) := \{*\}\backslash\,\mathbf{Top}(\mathbb{S}^n, X)/\simeq$$

where $\simeq$ is the equivalence relation induced by defining $f \simeq g$ if there is a map $h : \mathbb{S}^n \to X$ making the diagram

$$\begin{tikzcd}
\mathbb{S}^n \coprod \mathbb{S}^n \arrow[rr, "\langle f,g \rangle"] \arrow[dr, "c"'] & & X \\
& \mathbb{S}^n \times [0,1] \arrow[ur, "h"'] &
\end{tikzcd}$$

where the map $c : \mathbb{S}^n \coprod \mathbb{S}^n$ identifies the disjoint union

$$\mathbb{S}^n \coprod \mathbb{S}^n := (\mathbb{S}^n \times \{0\}) \cup (\mathbb{S}^n \times \{1\})$$

as a subspace of $\mathbb{S}^n \times [0,1]$.

Note that the above definition captures the classical notion of an n-dimensional homotopy of maps $f \to g$. To see this we consider the case $n = 1$ and show that it is the same as the classical notion of homotopy as developed in, for example, [Rot88].

Example B.1. Let X be a topological space and let $f, g : \mathbb{S}^1 \to X$ be continuous maps such that the diagram

$$\begin{tikzcd}
\mathbb{S}^1 \coprod \mathbb{S}^1 \arrow[rr, "\langle f,g \rangle"] \arrow[dr, "c"'] & & X \\
& \mathbb{S}^1 \times [0,1] \arrow[ur, "h"'] &
\end{tikzcd}$$

commutes in **Top** for some morphism h of spaces. But then $\langle f, g\rangle = ch$ implies that

$$f = \iota_0 \langle f, g\rangle = \iota_0 ch$$

which gives that

$$h(c(\iota_0(z))) = h(c(z,0)) = h(z,0) = f(z);$$

dually we have

$$g = \iota_1 \langle f, g\rangle = \iota_1 ch$$

which implies

$$h(c(\iota_1(z))) = h(c(z,1)) = h(z,1) = g(z).$$

Thus h is a homotopy of spaces in the classical sense.

Conversely let $f, g : \mathbb{S}^1 \to X$ be continuous maps of spaces such that there exists a continuous map $h : \mathbb{S}^1 \times [0,1] \to X$ with the property that

$$h(z,0) = f(z)$$

and

$$h(z,1) = g(z).$$

This then implies that if $c : \mathbb{S}^1 \coprod \mathbb{S}^1 \to \mathbb{S}^1 \times [0,1]$ is the injection that identifies $\mathbb{S}^1 \coprod \mathbb{S}^1$ as a subspace of $\mathbb{S}^1 \times [0,1]$ and if $z \in \mathbb{S}^1$ we have

$$((\langle f, g\rangle \circ \iota_0)(z) = f(z) = h(z,0) = h(c(z,0))$$

while

$$((\langle f, g\rangle \circ \iota_1)(z) = g(z) = h(z,1) = h(c(z,1)).$$

Thus we have that the squares

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[rr, "f"] \arrow[d, "\iota_0"'] & & X \\
\mathbb{S}^1 \coprod \mathbb{S}^1 \arrow[r, "c"'] & \mathbb{S}^1 \times [0,1] \arrow[ur, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S}^1 \arrow[rr, "g"] \arrow[d, "\iota_1"'] & & X \\
\mathbb{S}^1 \coprod \mathbb{S}^1 \arrow[r, "c"'] & \mathbb{S}^1 \times [0,1] \arrow[ur, "h"'] &
\end{tikzcd}$$

both commute, implying that

$$\begin{tikzcd} \mathbb{S}^1 \coprod \mathbb{S}^1 \arrow[rr, "\langle f,g \rangle"] \arrow[dr, "c"'] & & X \\ & \mathbb{S}^1 \times [0,1] \arrow[ur, "h"'] & \end{tikzcd}$$

commutes in **Top**.

Theorem B.1. *If $n \geq 1$ then $\pi_n(X, x)$ is a group.*

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