WEAK FACTORIZATION SYSTEMS, MODEL CATEGORIES, AND THE HOMOTOPY THEOREM

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1. Examples of Weak Factorization Systems

One of the most important collection of tools and theories to have at hand in the study of categorical homotopy is that of weak factorization systems. Much of the underlying structure of a model category, and much of the way that homotopy works, lie in the fact that in model categories we have two distinct but entwined weak factorization systems at play that in a sense "localize" the weak equivalences in a model category (this is made precise by the Homotopy Theorem, which is presented in section 5). While we assume a basic familiarity with weak factorization systems (up to what is presented in [Rie08]), we will first discuss some examples of weak factorization systems, as well as present one particularly important class of weak factorization systems (the one presented in Propostion 1.7) that will be used in the proof of the projective model structure on $\mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$ (see section 3 for details).

Throughout this section, when given a category $\mathfrak{C}$ we define the classes $\mathcal{A} := \operatorname{Mor}\mathfrak{C}$, $\mathcal{I} = \operatorname{Iso}(\mathfrak{C})$, $\mathscr{M} := \{\mu \in \operatorname{Mor}\mathfrak{C} \mid \mu \text{ monic}\}$, $\mathscr{E} := \{\varepsilon \in \operatorname{Mor}\mathfrak{C} \mid \varepsilon \text{ regular epic}\}$. Furthermore, we define the class of morphisms $\mathcal{N}$ in $\mathbf{Set}$ as $\mathcal{N} := \{f \in \operatorname{Mor}(\mathbf{Set}) \mid f : \varnothing \to X, X \neq \varnothing\}$.

Example 1.1. Let $\mathfrak{C}$ be any category. Then $(\mathcal{A}, \mathcal{I})$ and $(\mathcal{I}, \mathcal{A})$ are both orthogonal factorization systems (and hence weak factorization systems) on $\mathfrak{C}$. We show only the $(\mathcal{A}, \mathcal{I})$ system; the other is dual. Note that any map is factorized by a map followed by an isomorphism, as the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f"'] & & Y \arrow[dl, equal] \\ & Y & \end{tikzcd}$$

shows. For the lifting property let

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] & Y \end{tikzcd}$$

be a commuting diagram with f a morphism and g and isomorphism. Then define the lifting $h : B \to X$ by[1] $h = vg^{-1}$. Then $fh = fvg^{-1} = ugg^{-1} = u$ and $hg = vg^{-1}g = v$, so f has the left lifting property with g. Note that the lift h is unique because g^{-1} is the unique inverse of g.
Finally for retracts note that the $\mathcal{A}$ case is trivial. The other case is given exactly by the isomorphism lemma (Lemma).

Example 1.2. The factorization system $(\mathscr{E}, \mathscr{M})$ is an orthogonal factorization system on any elementary topos $\mathcal{E}$.

Example 1.3. The system $(\mathscr{M}, \mathscr{E})$ is a weak factorization system in $\mathbf{Set}$ if we accept the Axiom of Choice. We see this by observing first that the prior example shows that epics and monics are both preserved under

Date: August 16, 2016 (Version 2).

[1]Of course we do the perverse thing throughout these notes and write our composition diagramatically, i.e., for morphisms $f : X \to Y$ and $g : Y \to Z$ the map $fg : X \to Z$ is the composite $X \xrightarrow{f} Y \xrightarrow{g} Z$.

retract. Now, since the injection $\iota_0 : A \to A \coprod B$ in **Set** is injective and hence monic we note that the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\iota_0"'] & & Y \\
& X \coprod Y \arrow[ur, "\langle f, \mathrm{id}_B \rangle"'] &
\end{tikzcd}$$

shows that given any morphism $f : X \to Y$ there is an $(\mathscr{M}, \mathscr{E})$ factorization of f. To see the lifting property consider the commuting diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "m"'] & X \arrow[d, "g"] \\
B \arrow[r, "v"'] & Y
\end{tikzcd}$$

with $m \in \mathscr{M}$ and $e \in \mathscr{E}$. If $A = \varnothing$ the statement is trivial, so assume otherwise. Now note that m monic implies that m is injective. With this in define the map $\theta : B \to X$ via

$$\theta(b) := \begin{cases} u(a) & \exists a \in A. f(a) = b \\ u(a') & \nexists a \in A. f(a) = b \end{cases}$$

for some fixed $a' \in A$ (note here is where the Axiom of Choice assumption comes in; we need to choose an element in A). Now by construction we have that $f\theta = u$ and $\theta g = ug = v$ and hence θ is a lift from f to g.

Example 1.4. The pairs $(\mathcal{A} \setminus \mathcal{N}, \mathcal{I} \cup \mathcal{N})$ and $(\mathscr{M} \setminus \mathcal{N}, \mathscr{E} \cup \mathcal{N})$ are weak factorizations in **Set**. The verifications of these facts are trivial and hence omitted.

We now show the most important example in this section. However, we will need two lemmas to do so: one follows from the general properties of Abelian and pre-Abelian categories (and hence is omitted), while the other is a property of retracts.

Lemma 1.5. *Let $\mathfrak{A}$ be an additive category. Then the coproduct injections $\iota : A \to A \oplus B$ are sections.*

Lemma 1.6. *Let $f : A \to B$ be a retract of the section $s : X \to Y$. Then f is a section.*

Proof. Begin by constructing the commutative diagram

$$\begin{tikzcd}
& X \arrow[rrr, shift left, "s"] \arrow[dr, "r'"] & & & Y \arrow[lll, shift left, "r"] \arrow[dr, "r''"] & \\
A \arrow[ur, "\ell'"] \arrow[rr, equal] & & A \arrow[r, "f"'] & B \arrow[ur, "\ell''"] \arrow[rr, equal] & & B
\end{tikzcd}$$

with r the right inverse of s given by f a retract of s. Define $g := \ell'' r r'$ and observe that

$$fg = f(\ell'' r r') = \mathrm{id}_A \, f \ell'' r r' = \ell' r' f \ell'' r r' = \ell' s r r' = \mathrm{id}_A \,.$$

Thus f is a section. ∎

Proposition 1.7. *Let $\mathfrak{A}$ be a pre-Abelian category and define $\mathcal{P} := \{P\}$ to be the family of objects in $\mathfrak{A}$ such that all the P have the property $\mathfrak{A}(P, -)$ is exact. Then if $\mathcal{CP} := \{\iota_0 : A \to A \oplus P \mid P \in \mathcal{P}\}$, the pair $(\mathcal{CP}, \mathscr{E})$ is a weak factorization system on $\mathfrak{A}$.*

Remark 1.8. Note that it is immediate from definition that $\mathfrak{A}(P, -)$ is an exact functor if and only if for all epimorphisms $\varepsilon : X \to Y$, whenever there is a map $g : P \to Y$ we can find a map $f : P \to X$ rendering the diagram

$$\begin{tikzcd}
P \arrow[d, dashed, "\exists f"'] \arrow[dr, "g"] & \\
X \arrow[r, two heads, "\varepsilon"'] & Y
\end{tikzcd}$$

commutative.

Proof. We first note that if $f : A \to B$ is any morphism in $\mathfrak{A}$ the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\iota_0"'] & & B \\ & A \oplus P \arrow[ur, "f\oplus 0"'] & \end{tikzcd}$$

provides a factorization of f.

We now show the lifting property of the weak factorization system. Let

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\iota_0"'] & X \arrow[d, two heads, "e"] \\ A \oplus P \arrow[r, "v"'] & Y \end{tikzcd}$$

be a commutative diagram with $P \in \mathcal{P}$ and $e \in \mathscr{E}$. Now form the commuting diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\iota_0"'] & X \arrow[d, two heads, "e"] \\ A \oplus P \arrow[r, "v"'] & Y \\ P \arrow[u, "\iota_1"] \arrow[ur, "\iota_1 v"'] & \end{tikzcd}$$

and use $\mathfrak{A}(P, -)$ exact to provide a lift $f : P \to X$ making $fe = \iota_1 v$. Set $h = u \oplus f$. Then

$$\iota_0 h = \iota_0(u \oplus f) = u$$

and

$$he = (u \oplus f)e = ue \oplus fe = \iota_0 v \oplus \iota_1 v = v.$$

Thus h provides the desired lift:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\iota_0"'] & X \arrow[d, two heads, "e"] \\ A \oplus P \arrow[r, "v"'] \arrow[ur, "h"] & Y \end{tikzcd}$$

We finally show the retract property. Consider the diagram retract diagram:

$$\begin{tikzcd} & X \arrow[rrr, "\iota_0"] \arrow[dr, "r"] & & & X \oplus P' \arrow[dr, "r'"] & \\ A \arrow[ur, "s"] \arrow[rr, equal] & & A \arrow[r, "f"'] & B \arrow[ur, "s'"] \arrow[rr, equal] & & B \end{tikzcd}$$

We will now show that f is a coproduct injection. By the prior lemmas we have that since ι_0 is a section, f is as well. Since $\iota_0 = rfs'$ and $rfs'\pi_1 = 0$ it follows that f is a coproduct injection.

Now consider the retract diagram

$$\begin{tikzcd} & A' \arrow[rrr, "\iota_0"] \arrow[dr, "r"] & & & A' \oplus P' \arrow[dr, "r'"] & \\ A \arrow[ur, "s"] \arrow[rr, equal] & & A \arrow[r, "\iota_0"'] & A \oplus B \arrow[ur, "s'"] \arrow[rr, equal] & & A \oplus B \end{tikzcd}$$

We must show that $B \in \mathcal{P}$. To do this let $\varepsilon : X \to Y$ be an epimorphism and let $g : B \to Y$ be a morphism. Use this to produce the induced diagram

$$\begin{tikzcd} & P' \arrow[dr, "r'_*"] \arrow[drr, "r'_* g"] & & X \arrow[d, two heads, "\varepsilon"] \\ B \arrow[ur, "s'_*"] \arrow[rr, equal] & & B \arrow[r, "g"'] & Y \end{tikzcd}$$

Since $P' \in \mathcal{P}$ there exists a map $h' : P \to X$ such that $h'\varepsilon = r'_* g$. Then $s'_* h' : B \to X$ is a morphism such that

$$s'_* h' \varepsilon = s'_* r'_* g = \mathrm{id}_B\, g = g,$$

providing a lift $h = s'_* h'$ making the diagram

$$\begin{tikzcd} P \arrow[d, dashed, "\exists h"'] \arrow[dr, "g"] & \\ X \arrow[r, two heads, "\varepsilon"'] & Y \end{tikzcd}$$

commute, giving $B \in \mathcal{P}$. This proves that $(\mathcal{CP}, \mathscr{E})$ is a weak factorization system on $\mathfrak{A}$. $\square$

2. Introducing Model Categories

Here we will introduce model categories. They will give us just enough structure to talk about homotopies on a category, and will provide many examples of the uses of weak factorization systems. Before we proceed, however, we shall take a convention used by Lurie to make our discussion (linguistically) simpler: we shall assume that we are working with some Grothendieck universe. Explicitly, we will fix some strongly inaccessable cardinal κ and say that anything of size $\lambda \leq \kappa$ is *small*, while everything of larger size is said to be *large*. This is a linguistic simplification, and the results we will prove will not depend in any fundamental way on the existance of the cardinal κ. Furthermore, many of the results we present in this section are facts about weak factorization systems in general; we provide them for the sake of completeness, but omit many of the proofs.

Definition 2.1. *Let $\mathfrak{C}$ be a category. We say that $\mathfrak{C}$ is a* model category *(or that $\mathfrak{C}$ has a* model structure *on it) if there are three classes of morphisms in $\mathfrak{C}$, called*

(1) *Cofibrations;*
(2) *Fibrations;*
(3) *Weak Equivalences*

such that the following axioms are satisfied:

(1) *$\mathfrak{C}$ is (small)-complete and (small)-cocomplete.*
(2) *If we have the diagram*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathfrak{C}$ such that any two of f, g, or fg are weak equivalences, then so is the third.

(3) *Let $f : X \to Y$ be a retraction of $g : X' \to Y'$, i.e., the diagram*

$$\begin{tikzcd} & X' \arrow[dr, "r"] \arrow[rrr, "g"] & & & Y' \arrow[dr, "r'"] & \\ X \arrow[ur, "\ell"] \arrow[rr, equal] & & X \arrow[r, "f"'] & Y \arrow[ur, "\ell'"] \arrow[rr, equal] & & Y \end{tikzcd}$$

commutes with $\ell r = \mathrm{id}_X$ and $\ell' r' = \mathrm{id}_Y$. Then if g is a (fibration, cofibration, weak equivalence), so is f.

(4) *Given a commuting square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

there is a cross-map $\theta : B \to X$ making the square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, "\theta"'] & Y \end{tikzcd}$$

commute if:

- *f is a cofibration and g is both a fibration and a weak equivalence;*
- *f is both a cofibration and a weak equivalence and g is a fibration.*

(5) *Any $\varphi \in \mathfrak{C}(X, Z)$ admits the factorizations*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

and

$$X \xrightarrow{f'} Y' \xrightarrow{g'} Z$$

such that f is a cofibration, g is a cofibartion and a weak equivalence, f' is a cofibration and a weak equivalence, and g' is a fibration; furthermore, the following square commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"'] \arrow[dr, "\varphi"] & Y \arrow[d, "g"] \\ Y' \arrow[r, "g'"'] & Z \end{tikzcd}$$

Remark 2.2. Note that an alternative, and equivalent, formulation of the definition above is that $\mathfrak{C}$ is (co)complete, $\mathscr{W}$ satisfies the 2-out-of-3 axiom, and that both $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$ and $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ are weak factorization systems.

Definition 2.3. *Let $\mathfrak{C}$ be a model category. We then say that a morphism $f \in \mathfrak{C}(X, Y)$ is a* trivial fibration *if it is a weak equivalence and a fibration; similarly f is a* trivial cofibration *if it is a weak equivalence and a cofibration.*

Definition 2.4. *Let $\mathfrak{C}$ be a model category and let $\perp$ be the initial object of $\mathfrak{C}$ with $\top$ the terminal object of $\mathfrak{C}$. We then say that an object X is* fibrant *if the unique map $\tau_X : X \to \top$ is a fibration and that X is* cofibrant *if the unique map $\iota_X : \perp \to X$ is a cofibration.*

Example 2.5. Let $\mathfrak{C}$ be any category. Then we can put a model structure, called the *trivial model*, on $\mathfrak{C}$ via the assignments:

- Cofibrations: All morhphisms $f \in \mathrm{Mor}(\mathfrak{C})$;
- Fibrations: All morphisms $g \in \mathrm{Mor}(\mathfrak{C})$;
- Weak Equivalences: All isomorphisms $h \in \mathrm{Mor}(\mathfrak{C})$.

Before we exhibit another example, it will be useful to have a structural result about these model structures. In particular, we need to know if we ever must give *all* the data of the cofibrations, fibrations, or the weak equivalences of a model structure in order to determine it, or if we can determine, say, the cofibrations in terms of the fibrations and weak equivalences (or the fibrations in terms of the cofibrations and weak equivalences). The short answer is yes, we can: it is sufficient to define any two of the three classes of maps in $\mathfrak{C}$ to give the third in a model structure. To prove this we need the following lemma, which allows us to characterize the fibrations and cofibrations of $\mathfrak{C}$ in terms of lifting properties by simply abusing the structure of the weak factorization systems underlying a model category. They are standard facts from the theory of weak factorization systems, but are provided here for the sake of completeness; as such we will not provided all of the proofs. We begin by recalling the definition of the left and right lifting properties before providing some basic properties of these categories.

Definition 2.6. *Let $\mathscr{C}$ and $\mathscr{D}$ be classes of morphisms in a category $\mathfrak{C}$. Then we say that $\mathscr{C}$ has the* left lifting property *with $\mathscr{D}$ if given any $f \in \mathfrak{C}(X, Y)$ with $f \in \mathscr{C}$ and any $g \in \mathfrak{C}(A, B)$ with $g \in \mathscr{D}$ such that the square*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \arrow[d, "g"] \\ Y \arrow[r] & B \end{tikzcd}$$

commutes, there is a map $\varphi : Y \to A$ making the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \arrow[d, "g"] \\ Y \arrow[r] \arrow[ur, "\varphi"] & B \end{tikzcd}$$

commute. Conversely, $\mathscr{D}$ is said to have the right lifting property *with $\mathscr{C}$.*

Lemma 2.7 (Retract Lemma)**.** *Let $\mathfrak{C}$ be any category and let $f : X \to Y$ be a retract of $g : X' \to Y'$. Then if g has the left lifting property with $h : A \to B$ (or the right lifting property) then f has the left lifting property (right lifting property) with h as well.*

Lemma 2.8 ([DS95], Lemma 2.7; The Isomorphism Lemma)**.** *Let $\mathfrak{C}$ be any category. Then if $f \in \mathfrak{C}(A, B)$ is a retract of an isomorphism $g \in \mathfrak{C}(X, Y)$, f is an isomorphisms.*

Proof. Let the retraction diagram

$$\begin{tikzcd} & X \arrow[rrr, shift left, "g"] \arrow[dr, "r"] & & & Y \arrow[lll, shift left, "g^{-1}"] \arrow[dr, "r'"] & \\ A \arrow[ur, "\ell"] \arrow[rr, equal] & & A \arrow[r, "f"'] & B \arrow[ur, "\ell'"] \arrow[rr, equal] & & A \end{tikzcd}$$

be given with g^{-1} the inverse of g. Set $h = \ell' g^{-1} r$. Then we calculate that

$$fh = f(\ell' g^{-1} r) = \ell(r f \ell') g^{-1} r = \ell g g^{-1} r = \mathrm{id}_A$$

and

$$hf = \ell' g^{-1} r f(\ell' r') = \ell' g^{-1}(r f \ell') r' = \ell' g^{-1} g r' = \mathrm{id}_B,$$

proving that f is indeed an isomorphism. □

Lemma 2.9. *Let $\mathfrak{C}$ be a model category and assume that $f \in \mathfrak{C}(X, Y)$ has the left lifting property with fibrations (trivial fibrations). Then f is a trivial cofibration (f is a cofibration).*

Corollary 2.10. *Assume that $\mathfrak{C}$ is a model category and $f \in \mathfrak{C}(X, Y)$ has the rihgt lifting property with cofibrations (trivial cofibrations). Then f is a trivial fibration (f is a fibration).*

Proof. Dualize the prior proposition. □

Proposition 2.11. *In a model category $\mathfrak{C}$ the cofibrations may be determined by the fibrations and weak equivalences or the fibrations may be given in terms of the cofibrations and weak equivalences.*

Remark 2.12. We do not prove this proposition because it

Proposition 2.13. *The class of weak equivalences in a model category may be given in terms of the fibrations and cofibrations.*

Proof. Begin by defining the trivial cofibrations to be the maps that have the left lifting property with fibrations and the trivial fibrations to be the maps that have the right lifting property with cofibrations. Then by the two-out-of three axiom and the factorization axiom we derive that the class of weak equivalences is the class of maps that admit a factorization $w = cf$, where c is a trivial cofibration and f is a trivial fibration. □

It will be frequently useful, in addition to being important for understanding the factorization structure on $\mathfrak{C}$, for us to determine which classes of maps in $\mathfrak{C}$ are preserved by pushouts and pullbacks. While this is a general property of weak factorization systmes, it is important to emphasize because we will use it frequently throughout the course of this article. We omit the proof (NOTE: OR WE CITE BEN'S NOTES), but the interested reader may find a proof of the more general fact in [Rie08] or of this fact in [DS95].

Lemma 2.14 ([DS95], Prop.3.14)**.** *Let $\mathfrak{C}$ be a model category. Then the following hold:*

(1) *Cofibrations and trivial cofibrations are preserved by pushouts;*
(2) *Fibrations and trivial fibrations are preserved by pullbacks.*

Theorem 2.15. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be model categories with an adjunction of functors*

$$(\eta, \varepsilon) : F \dashv G : \mathfrak{C} \to \mathfrak{D}\,.$$

Then TFAE:

(1) *The functor F takes cofibrations to cofibrations and trivial cofibrations to trivial cofibrations;*
(2) *The functor F takes cofibrations to cofibrations and the functor G takes fibrations to fibrations;*
(3) *The functor G takes fibrations to fibrations and trivial fibrations to trivial fibrations.*

Before we prove this theorem, I feel it is worthwhile to talk about our proof strategy. We will endeavor to show it by proving that the problem of determining when a square

$$\begin{tikzcd} FA \arrow[r] \arrow[d, "Ff"'] & X \arrow[d, "g"] \\ FB \arrow[r] & Y \end{tikzcd}$$

has a lift $\varphi : FB \to Y$ is equivalent to determining when a square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & GX \arrow[d, "Gg"] \\ B \arrow[r] & GY \end{tikzcd}$$

has a lift $\psi : B \to GX$. This phenomenon is covered in the following lemma, which essentially proves the theorem.

Lemma 2.16. *Let $(\eta, \varepsilon) : F \dashv G : \mathfrak{C} \to \mathfrak{D}$ be an adjunction on model categories $\mathfrak{C}$ and $\mathfrak{D}$. Furthermore, let $u \in \mathfrak{C}(A, B)$ and $v \in \mathfrak{D}(X, Y)$ be morphisms. Then Fu has the left lifting property with respect to v if and only if u has the left lifting property with Gv.*

Proof. We prove the $\implies$ direction only; the other follows from duality. Assume that any diagram of the form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"'] & GX \arrow[d, "Gv"] \\ B \arrow[r, "b"'] \arrow[ur, dashed, "\exists\varphi"] & GY \end{tikzcd}$$

has a lift $\varphi : B \to GX$ in $\mathfrak{C}$ and let the commuting diagram

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Fu"'] & X \arrow[d, "v"] \\ FB \arrow[r, "\beta"'] & Y \end{tikzcd}$$

be given in $\mathfrak{D}$. We will show that there is a lift $\psi : FB \to X$ making the diagram commute. To do this apply the functor G to the above diagram to produce the square

$$\begin{tikzcd} G(FA) \arrow[r, "G\alpha"] \arrow[d, "G(Fu)"'] & GX \arrow[d, "Gv"] \\ G(FB) \arrow[r, "G\beta"'] & GY \end{tikzcd}$$

and apply the universal property of adunjction to give the rectangle

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "u"'] & G(FA) \arrow[r, "G\alpha"] \arrow[d, "G(Fu)"'] & GX \arrow[d, "Gv"] \\ B \arrow[r, "\eta_B"'] & G(FB) \arrow[r, "G\beta"'] & GY \end{tikzcd}$$

Since this contracts to the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A G(\alpha)"] \arrow[d, "u"'] & GX \arrow[d, "Gv"] \\ B \arrow[r, "\eta_B G(\beta)"'] \arrow[ur, dashed, "\varphi"] & GY \end{tikzcd}$$

there is a lift $\varphi : B \to GX$ from u to Gv. Now apply the functor F to the square to produce the diagram

$$\begin{tikzcd} FA \arrow[r, "F(\eta_A G(\alpha))"] \arrow[d, "Fu"'] & F(GX) \arrow[r, "\varepsilon_X"] \arrow[d, "Gv"'] & X \arrow[d, "v"] \\ FB \arrow[r, "F(\eta_B G(\beta))"'] \arrow[ur, "F\varphi"] & F(GY) \arrow[r, "\varepsilon_Y"'] & Y \end{tikzcd}$$

where the maps ε_X and ε_Y are given by the couniversal property of adjunction. From the triangle identities of adjunction it follows that

$$F(\eta_A G(\alpha))\varepsilon_X = \alpha$$

and

$$F(\eta_B G(\beta))\varepsilon_Y = \beta$$

giving that the diagram

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Fu"'] & X \arrow[d, "v"] \\ FB \arrow[r, "\beta"'] \arrow[ur, "F(\varphi)\varepsilon_X"] & Y \end{tikzcd}$$

commutes. This shows that Fu has the left lifting property with v; dualizing appropriately completes the proof of the lemma. □

Proof of Thereom 2.15. Clear by repeated use of Lemma 2.16. □

Example 2.17 ([DS95])**.** Let $\mathfrak{C}$ be a model category and let $A \in \mathrm{Ob}(\mathfrak{C})$. We can then make the coslice category $A/\mathfrak{C}$ (i.e. the comma category $A \downarrow \mathfrak{C}$, in the notation of [ML98]) into a model category as follows:

Let $f : A \to X$ and $g : A \to Y$ be objects in $A/\mathfrak{C}$ and let $h \in A/\mathfrak{C}(X, Y)$. Then we define the model structure on :

(1) h is a *weak equivalence* in $A/\mathfrak{C}$ if $h \in \mathfrak{C}(X, Y)$ is a weak equivalence;
(2) h is a *cofibration* in $A/\mathfrak{C}$ if $h \in \mathfrak{C}(X, Y)$ is a cofibration.

Example 2.18. Let $\mathfrak{C}$ be a model category and let $A \in \mathrm{Ob}(\mathfrak{C})$. Then the slice category $\mathfrak{C}/A$ (i.e. $\mathfrak{C} \downarrow A$) may be made into a model category in the same way as $A/\mathfrak{C}$. That is, a map $h : (X \to A) \to (Y \to A)$ in the slice category is said to be:

(1) a *weak equivalence* if h is a weak equivalence in $\mathbb{C}$;
(2) a *cofibration* if h is a cofibration in $\mathfrak{C}$.

3. The Projective Model Structure on Chain Complexes of R-Modules

We wish to describe here the model structure on the category of (left) R modules, where R is a unital ring. In order to do this we will need some quick preliminaries and facts from homological algebra. While we will be using the conventions of [DS95], most of the algebraic facts we present can be found through an Abelian categorical approach in [HS97], or ring theoretically in [Lam99]. Throughout this section we fix a unital ring R and fix the category **R-Mod** of left R-modules. We use without proof that **R-Mod** is **Set**-(co)complete and that **R-Mod** is an Abelian category.

Definition 3.1. *Let $(A_n)_{n\geq 0}$ be a family of left R-modules. If there is a map $\partial_{n-1} \in \mathbf{R\text{-}Mod}(A_n, A_{n-1})$ for all natural numbers $n \geq 1$ such that $\partial_n\partial_{n-1} = 0$, then we say the pair $(A_n, \partial_n)_{n\geq 0}$ is a* nonnegatively graded sequence of left R-modules *and write $A_\bullet = (A_n, \partial_n)$. Furthermore, we say that a sequence $A_\bullet$ is* exact at A_n *for $n \in \mathbb{N}$ if $\ker \partial_{n-1} = \operatorname{im} \partial_n$.*

Definition 3.2. *Define the category $\mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$ as follows:*

(1) *Objects: nonnegatively graded sequences of left R-modules $A_\bullet$;*
(2) *Morphism: Given sequences $A_\bullet = (A_n, \partial_n)$ and $B_\bullet = (B_n, \delta_n)$, we say that a collection of maps $f_n \in \mathbf{R\text{-}Mod}(A_n, B_n)$, $n \geq 0$, is a morphism of sequences if for every $n \geq 0$ the diagram*

$$\begin{tikzcd} A_{n+1} \arrow[r, "\partial_n"] \arrow[d, "f_{n+1}"'] & A_n \arrow[d, "f_n"] \\ B_{n+1} \arrow[r, "\delta_n"'] & B_n \end{tikzcd}$$

of left R-modules commutes.

Identities are then pointwise identity maps and composition corresponds to stacking commuting squares. That $\mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$ is a category is clear. However, as a quick remark, we will adopt the convetion used in [HS97] in which we denote a map of chain complexes $f_\bullet : A_\bullet \to B_\bullet$ instead by a bolded font $\mathbf{f} : A_\bullet \to B_\bullet$. This is simply a notational convention that we use in order to simplify indeces in the general caclulation of limits and colimits (so that we do not have some notational nonsense of the form $f_{\bullet ij}$ for i, j members of some index set).

Lemma 3.3. *The category $\mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$ has all* **Set***-indexed limits and colimits.*

Proof. We prove only the existence of colimits; limits are shown dually. Recall that since **R-Mod** is an Abelian category, **R-Mod** has a zero object (which we will write as 0) and all **Set**-indexed limits and colimits. To see that $\mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$ has a zero object, let $0_\bullet$ be the (nonnegatively graded) sequence such that $0_n = 0$ for all $n \in \mathbb{N}$ and $\delta_n : 0_{n+1} \to 0_n$ is the identity map. This sequence is easily seen to be the zero object in $\mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$ and hence constitutes the empty product and coproduct in $\mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$.

We now show the existence of all coequalizers. Recall that in any category of (left, right, two-sided) R-modules, the coequalizer of two maps $f, g : A \to B$ is the difference cokernel of $f - g$, i.e., $\mathrm{Coeq}(f, g) = (\mathrm{Coker}(f - g), \mathrm{c}(f - g))$, where

$$\mathrm{Coker}(f - g) = \frac{B}{\operatorname{im}(f - g)}$$

and the map $\mathrm{c}(f - g)$ is the canonical projection

$$\mathrm{c}(f - g) = \pi_{\operatorname{im}(f-g)} : B \to \frac{B}{\operatorname{im}(f - g)}.$$

Now let $\mathbf{f}, \mathbf{g} \in \mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})(A_\bullet, B_\bullet)$ be two maps of chain compleces and let $\mathbf{h} \in \mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})(B_\bullet, C_\bullet)$ be given with $A_\bullet = (A_n, \partial_n), B_\bullet = (B_n, \delta_n)$, and $C_\bullet = (C_n, \mathrm{d}_n)$. Then for each $n \in \mathbb{N}$ we can produce the object and map pair $(\operatorname{Coker}(f_n - g_n), \mathrm{c}(f_n - g_n))$ and construct the commuting diagram

$$
\begin{tikzcd}
A_n \arrow[r, shift left, "f_n"] \arrow[r, shift right, "g_n"'] & B_n \arrow[r, "\mathrm{c}(f_n-g_n)"] \arrow[dr, "h_n"'] & \operatorname{Coker}(f_n - g_n) \arrow[d, dashed, "\exists! k_n"] \\
 & & C_n
\end{tikzcd}
$$

with $k_n : \operatorname{Coker}(f_n - g_n) \to C_n$ a unique map of left R-modules given by the universal property of the cokernel. Now, since $f_n\delta_{n-1} = \partial_{n-1}f_{n-1}$ and $g_n\delta_{n-1} = \partial_{n-1}g_{n-1}$ there is an induced map $\hat{\delta}_n$ on the factor modules $\operatorname{Coker}(f_n - g_n) \to \operatorname{Coker}(f_{n-1} - g_{n-1})$ given by applying δ_{n-1} to B_n and then taking the corresponding projection; that is, $\hat{\delta}_{n-1}$ is the map making the square

$$
\begin{tikzcd}
B_n \arrow[r, "\delta_{n-1}"] \arrow[d, "\mathrm{c}(f_n-g_n)"'] & B_{n-1} \arrow[d, "\mathrm{c}(f_{n-1}-g_{n-1})"] \\
B_n/\operatorname{im}(f_n - g_n) \arrow[r, "\hat{\delta}_{n-1}"'] & B_{n-1}/\operatorname{im}(f_{n-1} - g_{n-1})
\end{tikzcd}
$$

commute. The collection $\operatorname{Coker}(f - g)_\bullet = (\operatorname{Coker}(f_n - g_n), \hat{\delta}_n)$ makes a cokernel sequence with $\mathbf{k} \in \mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})(\operatorname{Coker}_\bullet, C_\bullet)$. Write $\operatorname{Coker}(f_n - g_n) =: \operatorname{Coker}(f - g)_n$ and $\mathrm{c}(f_n - g_n) =: \mathrm{c}(f - g)_n$. Then by construction the maps k_n satisfy the equalities $k_n\mathrm{d}_{n-1} = \hat{\delta}_{n-1}k_{n-1}$, making $\mathbf{k} = (k_n)_{n\geq 0}$ into a morphism of compleces $\mathbf{k} : \operatorname{Coker}(f - g)_\bullet \to C_\bullet$. Furthermore, for every $n \geq 1$ we have a commuting diagram of the form:

$$
\begin{tikzcd}
 & A_n \arrow[rr, shift left, "f_n"] \arrow[rr, shift right, "g_n"'] \arrow[dl, "\partial_{n-1}"'] & & B_n \arrow[rr, "\mathrm{c}(f-g)_n"] \arrow[dl, "\delta_{n-1}"'] \arrow[drr, "h_n"] & & \operatorname{Coker}(f-g)_n \arrow[d, dashed, "k_n"] \arrow[dl, "\hat{\delta}_{n-1}"'] \\
A_{n-1} \arrow[rr, shift left, "f_{n-1}"] \arrow[rr, shift right, "g_{n-1}"'] & & B_{n-1} \arrow[rr, "\mathrm{c}(f-g)_{n-1}"'] \arrow[drr, "h_{n-1}"'] & & \operatorname{Coker}(f-g)_{n-1} \arrow[d, dashed, "k_{n-1}"] & C_n \arrow[dl, "\mathrm{d}_{n-1}"] \\
 & & & & C_{n-1} &
\end{tikzcd}
$$

Thus $\operatorname{Coker}(f - g)_\bullet$ is the coequalizer of $\mathbf{f}$ and $\mathbf{g}$.

Finally we must show that for any nonempty set I, the coproduct

$$\bigoplus_{i\in I} A_{\bullet i}$$

exists when $A_{\bullet i}$ is an object in $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$ for every $i \in I$. Begin by observing that for every $n \in \mathbb{N}$ the left R-module

$$\bigoplus_{i\in I} A_{in}$$

exists with canonical injections, for every $j \in I$, $\iota_{jn} : A_{jn} \to \bigoplus A_{in}$. When $n \geq 1$, each A_{in} has a differential $\partial_{i(n-1)} : A_{in} \to A_{i(n-1)}$. Consequently, the comparison map $\langle \partial_{i(n-1)} \rangle_{i\in I}$ produces a map $\bigoplus A_{in} \to \bigoplus A_{i(n-1)}$. Furthermore, by the property $\partial_{in}\partial_{i(n-1)} = 0$, we calculate that

$$\langle \partial_{in} \rangle_{i\in I} \langle \partial_{i(n-1)} \rangle_{i\in I} = \langle 0 \rangle_{i\in I} = 0,$$

proving that

$$\bigoplus_{i\in I} A_{i\bullet} = \left(\bigoplus_{i\in I} A_{in}, \langle \partial_{in} \rangle_{i\in I} \right)$$

is a sequence in $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$. From here the universal property is immediate by taking pointwise unique maps. This shows that $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$ is $\mathbf{Set}$ cocomplete, which proves the lemma after taking the dual result. $\square$

Remark The proof above carries over verbatim to the category $\mathbf{Ch}(\mathbf{R}-\mathbf{Mod})$ of $\mathbb{Z}$-graded chain compleces (here our sequences are allowed to extend infinitely in both directions); there was nothing special about using nonnegatively graded sequences in the **Set**-indexed (co)completeness of $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$.

We now need to define the classes of morphisms that will constitute our cofibrations, fibrations, and weak equivalences. While we could appeal to (PROP/LEMMA; FIND THIS OUT BY BACKCHECKING REFERENCES) ¡NUBMER¿, I feel in this case it is useful to see a full definition (so that we can see that checking the full detail is frequently awful). This will also help highlight the fact that while cofibrations and fibrations may feel like some sort of dual concept, they need not be strict dual objects in their ambient categories. Before we do this, however, we recall the definition/theorem that describes projective modules.

Definition 3.4 ([Lam99], p.21-22)**.** *Let P be a left R-module. We then say that P is* projective *if P satisfies any of the following equivalent conditions:*

(1) *P is a direct summand of a free R-module;*
(2) *If $\varepsilon : A \to P$ is an epimorphism of R-modules, then ε has a section $s : P \to A$;*
(3) *The functor $\mathbf{R}\text{-}\mathbf{Mod}(P, -)$ is exact.*

Proposition 3.5 (An Equivalent Notion of Projective)**.** *Let P be a left R-module for R a ring (not necessarily with identity). Then the following hold:*

(1) *P is projective;*
(2) *For every epimorphism $\varepsilon : A \to B$ whenever there is a map $f : P \to B$ there exists a map $\overline{f} : P \to A$ making the diagram*

$$\begin{tikzcd} P \arrow[d, "\overline{f}"'] \arrow[dr, "f"] & \\ A \arrow[r, two heads, "\varepsilon"] & B \end{tikzcd}$$

commute.

Remark: This is the first real place where having the assumption that R is unital is useful. We want to invoke the theorem that states that an R-module F is free if and only if it is a direct sum of copies of R because this then gives us that projective modules are characterized by being direct sums of direct summands of R. However, if R is nonunital then free modules are poorly behaved. While the characterization given above remains true, we cannot simply say that all direct sums of direct summands of a (nonunital) ring R comprise all possible projective modules. Instead, we have to worry about the fact that free modules of a nonunital ring are free unital modules over the unitization of R, that is, free unital modules over the universal unital ring containing R. In this sense the projective modules actually arise as direct summands of the *unitization* of R, and not simply direct summands of R, as we may hope.

Before we continue to describe the model structure on $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$, we must speak of the homology associated to a chain complex, as it is necessary to describe the weak equivalences of $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$. Formally, for each natural number $n \in \mathbb{N}$ there is a homology functor $H_n : \mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{R}\text{-}\mathbf{Mod}$ that records an algebraic analogue of the structure of n-dimensional holes in the complex $A_\bullet$. That is, for each natural number $n \in \mathbb{N}$ define the *n-cycles of a complex $A_\bullet$* as

$$Z_n(A_\bullet) := \begin{cases} A_0 & \text{if n} = 0, \\ \ker(\partial_n) & \text{if } n \geq 1 \end{cases}$$

and the *n-boundaries of $A_\bullet$* via the assignment

$$B_n(A_\bullet) := \operatorname{im}(\delta_n)$$

for all $n \in \mathbb{N}$. Note that because $A_\bullet$ is a sequence and the fact that $\partial_{n+1}\partial_n = 0$ for all $n \in \mathbb{N}$, $\operatorname{im}\partial_n$ is a left R-submodule of $\ker \partial_{n-1}$ for all $n \geq 1$ (and trivially $\operatorname{im}\partial_0$ is a submodule of A_0). Thus for every $n \in \mathbb{N}$ the factor module Z_n/B_n is well-defined; it is in fact through this insight and the fact that this factor module behave that we arrive at the homology functor.

Definition 3.6. *Let $n \in \mathbb{N}$ be a natural number. Then define the functor*

$$H_n : \mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{R}\text{-}\mathbf{Mod}$$

via the assignment

$$H_n(A_\bullet) := \frac{Z_n(A_\bullet)}{B_n(A_\bullet)}$$

and adapting on maps appropriately. The resulting functor H_n is called the n-th homology functor*.*

It is now pertinent to speak about what is meant by "adapting maps appropriately." Since $\mathbf{f}$ is a morphism of chain complexes, $f_n : A_n \to C_n$ is a morphism of left R-modules. Restricting f_n to the submodule $\ker \partial_{n-1}$ of A_n (or to $\ker \partial_0$ of A_0 if $n = 0$) then produces a map $\mathrm{res}(f_n) \in \mathbf{R\text{-}Mod}(Z_n(A_\bullet), Z_n(C_\bullet))$. Because of the fact that $A_\bullet$ and $C_\bullet$ are sequences, it then follows that there is an induced map

$$f_* : H_n(A_\bullet) \to H_n(C_\bullet).$$

It is worth observing that $H_n(A_\bullet) = 0$ if and only if $A_\bullet$ is exact at A_n. Before moving on we provide an important property of homology with respect to exact sequences without proof; the interested reader can see either [HS97] or [ML98] for a proof.

Theorem 3.7. *Let R be a unital ring and let*

$$0 \longrightarrow A_\bullet \xrightarrow{\mathbf{f}} B \xrightarrow{\mathbf{g}} C \longrightarrow 0$$

be an exact sequence in $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$. Then there is a long exact sequence of the form

$$\cdots \xrightarrow{\delta_n} H_n(A_\bullet) \xrightarrow{H_n(\mathbf{f})} H_n(B_\bullet) \xrightarrow{H_n(\mathbf{g})} H_n(C_\bullet) \xrightarrow{\delta_{n-1}} \cdots \longrightarrow H_0(C_\bullet) \longrightarrow 0$$

where the connecting homomorphisms δ_n are given by the Snake Lemma (see Lemma A.8 for details).

The long exact homology sequence tells us that we can extend H_n into a functor

$$H_*(A_\bullet) : \mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod}) \to \mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$$

by defining the homology complex $H_*(A_\bullet)$ to have $H_n(A_\bullet)$ as its n-th term and the connecting homomorphisms $\delta_n : H_{n+1}(A_\bullet) \to H_n(A_\bullet)$ as the differentials. There is also an understanding here to be made as homology as a left derived functor of a certain right exact functor, but we will not go into this description; the interested reader may find this in [HS97]. Instead we move on to define the cofibrations, fibrations, and weak equivalences in order to introduce and study the model structure on $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$.

Definition 3.8. *We say that a morphism $\mathbf{f} \in \mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})(A_\bullet, B_\bullet)$ is:*

(1) *a* weak equivalence *if $H_k(\mathbf{f}) : H_k(A_\bullet) \to H_k(B_\bullet)$ is an isomorphism of left R-modules for all $n \geq 0$;*
(2) *a* cofibration *if for each $n \geq 0$ the map $f_n : A_n \to B_n$ is monic and $\mathrm{c}(f_n)$ is a projective R-module;*
(3) *a* fibration *if $f_n : A_n \to B_n$ is epic for every $n \geq 1$.*

Proposition 3.9. *Let $\mathbf{g} : C_\bullet \to D_\bullet$ be a morphism in $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$ and let $\mathbf{f} : A_\bullet \to B_\bullet$ be a retract of $\mathbf{g}$. Then if $\mathbf{g}$ is a cofibration (weak equivalence, fibration), so is $\mathbf{f}$.*

Proof. We begin with a general observation about epics and monics in the category **R-Mod**: Since **R-Mod** is an Abelian category, the system $(\mathscr{E}, \mathscr{M})$ of epics and monics are an Orthogonal Factorization System on **R-Mod**; consequently $(\mathscr{E}, \mathscr{M})$ is a Weak Factorization System as well. This yields the fact that $\mathscr{E}$ and $\mathscr{M}$ are both preserved by retracts in **R-Mod**.

Assume that $\mathbf{g}$ is a cofibration, i.e., that $g_n : C_n \to D_n$ is monic for every $n \in \mathbb{N}$. Then since f_n is a retract of g_n for all $n \in \mathbb{N}$, f_n is monic because monics are preserved by retracts. Thus f_n is monic for every $n \in \mathbb{N}$ and $\mathbf{f}$ is a cofibration.

Assume that $\mathbf{g}$ is a weak equivalence. Then the induced map $(g_n)_* : H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism for all $n \in \mathbb{N}$. Produce the retract diagram

$$\begin{tikzcd}
 & H_nC_\bullet \arrow[rrrr, "(g_n)_*"] \arrow[dr] & & & & H_nD_\bullet \arrow[dr] & \\
H_nA_\bullet \arrow[ur] \arrow[rr, equal] & & H_nA_\bullet \arrow[r, "(f_n)_*"'] & H_nB_\bullet \arrow[urr] \arrow[rrr, equal] & & & H_nB_\bullet
\end{tikzcd}$$

and recall by Proposition 2.8 that isomorphisms are preserved by retract. Thus $(f_n)_*$ is an isomorphism and $\mathbf{f}$ is a weak equivalence.

Finally assume that $\mathbf{g}$ is a fibration. Then when $n = 0$ there is nothing to verify (as the retract of a morphism in **R-Mod** is again a morphism), so let $n \geq 1$. Then g_n is epic, and so f_n is preserved by retraction. Thus f_n is epic for all $n \geq 1$ and so is a fibration. $\square$

We will first prove that the cofibrations $\mathscr{C}$ have the left lifting property with trivial fibrations and that any morphism of chain complexes may be factored into a cofibration followed by a trivial fibration. We will primarily follow [DS95] here, but we will fill some rather annoying blanks left in the proofs.

Theorem 3.10. *Let $\mathbf{f} : A_\bullet \to B_\bullet$ be a cofibration and $\mathbf{g} : X_\bullet \to Y_\bullet$ be a trivial fibration. Then $\mathbf{f}$ has the Left Lifting Property with $\mathbf{g}$.*

As a notational remark we will abuse notation throughout this proof by writing all differentials associated to a sequence by the notation ∂ subject tot he convention $\partial_n : A_{n+1} \to A_n$ for all $n \in \mathbb{N}$. This will be to simplify the equations we write and hopefully will not result in any confusion.

Proof. We first make an observation about g_0 that will be important. Since $H_0X_\bullet = X_0/\operatorname{im}\partial_0$ and $H_0Y_\bullet = Y_0/\operatorname{im}\partial_0$ the diagram

$$\begin{tikzcd}
X_1 \arrow[r, "\partial_0"] \arrow[d, "g_1"'] & X_0 \arrow[r] \arrow[d, "g_0"'] & H_0X_\bullet \arrow[r] \arrow[d, "(g_0)_*"', "\cong"] & 0 \arrow[d, equal] \\
Y_1 \arrow[r, "\partial_0"'] & Y_0 \arrow[r] & H_0Y_\bullet \arrow[r] & 0
\end{tikzcd}$$

commutes with top and bottom rows exact and g_1 epic by assumption. Since $(g_0)_*$ is an isomorphism it is both epic and monic; similarly with $\mathrm{id}_0 : 0 \to 0$. Thus the diagram satisfies the hypotheses of the Epic Four Lemma (Lemma A.7) and so g_0 is epic as well. Consequently $\mathbf{g}$ is epic and so the sequence

$$0 \longrightarrow \operatorname{Ker}(\mathbf{g})_\bullet \xrightarrow{\mathbf{i}} X_\bullet \xrightarrow{\mathbf{g}} Y_\bullet \longrightarrow 0$$

is exact. Write $K_\bullet = \operatorname{Ker}(\mathbf{g})_\bullet$ from here on. Then appealing to the long exact homology sequence to produce the exact sequence

$$\cdots \longrightarrow H_nK_\bullet \longrightarrow H_nX_\bullet \xrightarrow{H_n(\mathbf{g})} H_nY_\bullet \longrightarrow H_{n-1}K_\bullet \longrightarrow \cdots \longrightarrow H_0Y_\bullet$$

and using that $H_n(\mathbf{g})$ is an isomorphism for all $n \in \mathbb{N}$ shows that $H_nK_\bullet = 0$ for all $n \in \mathbb{N}$.

Assume we have a commutative diagram of chain complexes:

$$\begin{tikzcd}
A_\bullet \arrow[r, "\mathbf{u}"] \arrow[d, "\mathbf{f}"'] & X_\bullet \arrow[d, "\mathbf{g}"] \\
B_\bullet \arrow[r, "\mathbf{v}"'] & Y_\bullet
\end{tikzcd}$$

We now prove the lifts $h_k : B_k \to X_k$ for all $k \geq 0$ by induction. Set $k = 0$ and note that since $\mathbf{f}$ is a cofibration each map $f_k : A_k \to B_k$ is monic and $\operatorname{Coker} f_k$ is projective. Consequently the sequence

$$0 \longrightarrow A_0 \xrightarrow{f_0} B_0 \xrightarrow{\mathrm{c}(f_0)} \operatorname{Coker} f_0 \longrightarrow 0$$

is short exact, with $c(f_0)$ denoting the canonical cokernel map onto $\operatorname{Coker} f_0$. Because $\operatorname{Coker} f_0$ is projective, the sequence splits and $B_0 \cong A_0 \oplus \operatorname{Coker} f_0$. Since $(\mathcal{CP}, \mathscr{E})$ is a weak factorization system on $\mathbf{Ch}_{\mathrm{n}\geq 0}(\textbf{R-Mod})$ there is a lift $h_0 : A_0 \oplus \operatorname{Coker} f_0$ making the diagram

$$\begin{tikzcd}
A_0 \arrow[r, "u_0"] \arrow[d, "f_0"'] & X_0 \arrow[d, "g_0"] \\
A_0 \oplus \operatorname{Coker} f_0 \arrow[r, "v_0"'] \arrow[ur, "h_0"] & Y_0
\end{tikzcd}$$

commute, providing us with a map $B_0 \to X_0$ after precomposing h_0 with the isomorphism $\theta : B_0 \to A_0 \oplus \operatorname{Coker} f_0$.

We now proceed with the induction. Assume that there exists a $k \geq 1$ in $\mathbb{N}$ such that for all $0 \leq \ell < k$ we have maps $h_\ell : B_\ell \to X_\ell$ satisfying the system of equations

$$\begin{cases} h_{\ell+1}\partial_\ell = \partial_\ell h_\ell & 0 \leq \ell < k-1; \\ h_\ell g_\ell = v_\ell & 0 \leq \ell < k; \\ f_\ell h_\ell = u_\ell & 0 \leq \ell < k. \end{cases}$$

Once again use that f_k is monic and $\operatorname{Coker} f_k$ is projective to provide an isomorphism $B_k \cong A_k \oplus \operatorname{Coker} f_k$ derived from the short exact sequence

$$0 \longrightarrow A_k \xrightarrow{f_k} B_k \longrightarrow \operatorname{Coker} f_k \longrightarrow 0$$

Since we have the commuting square

$$\begin{tikzcd} A_k \arrow[r, "u_k"] \arrow[d, "f_k"'] & X_k \arrow[d, two heads, "g_k"] \\ B_k \arrow[r, "v_k"'] & Y_k \\ A_k \oplus \operatorname{Coker} f_k \arrow[u, "\theta_k", "\cong"'] \arrow[ur, "\theta_k v_k"'] & \end{tikzcd}$$

and g_k is epic, there is a map $a_k : \operatorname{Coker} f_k \to X_k$ making the diagram

$$\begin{tikzcd} \operatorname{Coker} f_k \arrow[d, "a_k"'] \arrow[dr, "\theta_k v_k"] & \\ X_k \arrow[r, two heads, "g_k"'] & Y_k \end{tikzcd}$$

commute. Define $t_k : A_k \oplus \operatorname{Coker} f_k \to X_k$ via the comparison map $t_k = f_k \oplus a_k$. By construction it then follows that

$$t_k g_k = v_k$$

and

$$f_k g_k = u_k.$$

From here we will construct a map on $A_k \oplus \operatorname{Coker} f_k$ satisfying all three properties above; precomposing with the isomorphism θ_k^{-1} will provide the desired map on B_k. So define the difference map $D_k : A_k \oplus \operatorname{Coker} f_k$ via the assignment

$$D_k := t_k\partial_{k-1} - \partial_{k-1}h_{k-1}$$

so that D_k measures the degree that t_k fails to satisfy the first equation. We then calculate that

$$D_k\partial_{k-2} = (t_k\partial_{k-1} - \partial_{k-1}h_{k-1})\partial_{k-2} = t_k\partial_{k-1}\partial_{k-2} - \partial_{k-1}h_{k-1}\partial_{k-2} = 0$$

because h commutes with the differential operators. Furthermore, since $t_k g_k = v_k$ commute with the differentials we derive that

$$D_k g_{k-1} = t_k\partial_{k-1}g_{k-1} - \partial_{k-1}h_{k-1}g_{k-1} = t_k g_k \partial_{k-1} - \partial_{k-1}h_{k-1}g_{k-1} = v_k\partial_{k-1} - \partial_{k-1}v_{k-1} = 0$$

and similarly that $f_k D_k = 0$. Consequently D_k induces a map

$$D_k' : \operatorname{Coker} f_k \to Z_{k-1}K_\bullet.$$

Because $K_\bullet$ is exact, the map $\partial_{k-1} : K_k \to Z_{k-1}K_\bullet$ is epic. Consequently there the diagram

$$\begin{tikzcd} \operatorname{Coker} f_k \arrow[d, "D_k''"'] \arrow[dr, "D_k'"] & \\ K_k \arrow[r, two heads, "\partial_{k-1}"'] & Z_{k-1}K_\bullet \end{tikzcd}$$

commutes with D''_k existing by $\operatorname{Coker} f_k$ projective. Let $\pi_{\operatorname{im} f_k}$ be the canonical projection $B_k \to \operatorname{Coker} f_k = B_k / \operatorname{im} f_k$ and define D'''_k to be the composite map

$$D'''_k = \pi_{\operatorname{im} f_k} D''_k i_k$$

so that the diagram

$$\begin{tikzcd} B_k \arrow[r, "\pi_{\operatorname{im} f_k}"] \arrow[rrd] & \operatorname{Coker} f_k \arrow[r, "D''_k"] & K_k \arrow[d, "i_k"] \\ & & X_k \end{tikzcd}$$

commutes. Set $h_k := t_k - D'''_k$. We then verify that

$$h_k \partial_{k-1} = (t_k - D'''_k)\partial_{k-1} = \partial_{k-1} h_{k-1},$$

while

$$f_k h_k = f_k(t_k - D'''k) = f_k t_k - 0 = u_k,$$

and

$$h_k g_k = (t_k - D'''_k) g_k = t_k g_k - D'{\prime\prime}_k g_k = v_k - 0 = v_k.$$

Thus h_k makes the cube

$$\begin{tikzcd} A_k \arrow[rr, "u_k"] \arrow[dr, "\partial_{k-1}"] \arrow[dd, "f_k"'] & & X_k \arrow[dr, "\partial_k"] \arrow[dd, "g_k"] & \\ & A_{k-1} \arrow[rr, "u_{k-1}"] \arrow[dd, "f_{k-1}"] & & X_{k-1} \arrow[dd, "g_{k-1}"] \\ B_k \arrow[rr, "v_k"] \arrow[dr, "\partial_{k-1}"'] \arrow[uurr, "h_k"] & & Y_k \arrow[dr, "\partial_{k-1}"] & \\ & B_{k-1} \arrow[rr, "v_{k-1}"'] \arrow[uurr, "h_{k-1}"] & & Y_{k-1} \end{tikzcd}$$

commute. By the principle of mathematical induction there then exists a map h_n for every $n \in \mathbb{N}$ such that taking $\mathbf{h} = (h_n)_{n \in \mathbb{N}}$ provides a morphism of chain complexes. Consequently $\mathbf{h}$ makes the diagram

$$\begin{tikzcd} A_\bullet \arrow[r, "\mathbf{u}"] \arrow[d, "\mathbf{f}"'] & X_\bullet \arrow[d, "\mathbf{g}"] \\ B_\bullet \arrow[r, "\mathbf{v}"'] \arrow[ur, "\mathbf{h}"] & Y_\bullet \end{tikzcd}$$

commute in $\mathbf{Ch}_{n \geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$, as was to be shown. This proves the theorem. □

Before we prove the second lifting condition we need some lemmas and definitions. We will introduce the module theoretic notion of an n-disk for $n \geq 1$ and provide some important results on "projective disks" in $\mathbf{Ch}_{n \geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$. The structural importance of these disks" will be characterized in Proposition 3.12 and used extensively in Lemma

Definition 3.11. [DS95] *Let A be any left R-module and let $n \geq 1$. Then define the n-disk on A as the complex*

$$D_n A_k := \begin{cases} A & k = n, n-1; \\ 0 & k \neq n, n-1. \end{cases}$$

with differential given by

$$\partial_k := \begin{cases} \mathrm{id}_A & k = n-1; \\ 0 & k \neq n-1. \end{cases}$$

Proposition 3.12 ([DS95]). *Let $n \geq 1$ in $\mathbb{N}$ and let A be any left R-module. Define the functor*

$$D_n : \mathbf{R\text{-}Mod} \to \mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$$

by sending every left R-module A to the complex D_nA and sending the morphism $f \in \mathbf{R\text{-}Mod}(A, B)$ to the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & B \arrow[r, equal] & B \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Furthermore, define the functor

$$C_n : \mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod}) \to \mathbf{R\text{-}Mod}$$

by sending a chain complex $A_\bullet = (A_n, \partial_n)$ to A_n and sending a map $\mathbf{f} = (f_n) \in \mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})(A_\bullet, B_\bullet)$ to $f_n : A_n \to B_n$. Then there is an adjunction

$$D_n \dashv C_n : \mathbf{R\text{-}Mod} \to \mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$$

with unit $\eta = \mathrm{id}$.

Proof. Since $C_n(D_n(A)) = A$ we see quickly that $\eta_A = \mathrm{id}_A$, as was claimed. Now let M_n be the n-th term in the chain complex $M_\bullet = (M_n, \partial_n) \in \mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$ and assume that we have a commuting diagram of left R-modules:

$$\begin{tikzcd}
A \arrow[r, equal, "\eta_A"] \arrow[dr, "g"'] & A \arrow[d, dashed] \\
& M_n
\end{tikzcd}$$

We must find a unique map of chain complexes $\mathbf{f} : D_nA \to M_\bullet$ for which $C_n(\mathbf{f})$ fills in the downwards arrow. Begin by defining the map $\mathbf{f} : D_nA \to M_\bullet$ via:

$$f_k := \begin{cases} g & k = n \\ \partial_{n-1} g & k = n-1 \\ 0 & k \neq n, n-1 \end{cases}$$

A routine verification shows that $C_n(\mathbf{f}) = g$ and that $\mathbf{f}$ is the unique such map of chain complexes that provides the dotted arrow

$$\begin{tikzcd}
A \arrow[r, equal, "\eta_A"] \arrow[dr, "g"'] & A \arrow[d, dashed, "\exists! C_n(\mathbf{f})"] \\
& M_n
\end{tikzcd}$$

This proves the proposition. $\square$

Lemma 3.13. *Let P be a projective left R-module and let $\varepsilon : A_\bullet \to B_\bullet$ be a morphism of chain complexes epic in degrees $n \geq 1$. Then any morphism $\mathbf{g} : D_nP \to B_\bullet$ has a lift $\mathbf{f} : D_nP \to A_\bullet$.*

Proof. Consider the cheese-wedge” diagram:

$$\begin{tikzcd}
& \cdots \arrow[r] & 0 \arrow[r] & P \arrow[rr, equal] \arrow[d, "g_n"] \arrow[ddl] & & P \arrow[d, "g_n\partial_{n-1}"] \arrow[ddl] \\
& \cdots \arrow[r] & B_{n+1} \arrow[r] & B_n \arrow[rr, "\partial_{n-1}"'] & & B_{n-1} \\
\cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[ur, "\varepsilon_{n+1}"] & A_n \arrow[rr, "\partial_{n-1}"'] \arrow[ur, "\varepsilon_n"'] & & A_{n-1} \arrow[ur, "\varepsilon_{n-1}"'] &
\end{tikzcd}$$

Since ε_n is epic, the map f_n exists by the fact that P is projective. Post composing f_n with ∂_{n-1} provides the desired f_{n-1} and gives the nontrivial maps appearing in $\mathbf{f}$. Furthermore it is trivial to see that $\mathbf{f}\varepsilon = \mathbf{g}$, which completes the proof. $\square$

Corollary 3.14. *Let $\{P_i \mid i \in I\}$ be a family of projective left R-modules and let $\{n_i \mid n_i \geq 1, n_i \in \mathbb{N}, i \in I\}$ be a family of natural numbers. Then if $\varepsilon : A_\bullet \to B_\bullet$ is epic in positive degrees and there is a morphism*

$$\mathbf{g} : \bigoplus_{i \in I} D_{n_i} P_i \to B_\bullet,$$

there is a lift

$$\mathbf{f} : \bigoplus_{i \in I} D_{n_i} P_i \to A_\bullet$$

making $\mathbf{f}\varepsilon = \mathbf{g}$.

Proof. Use Lemma 3.13 to produce the individual lifting maps $\mathbf{f}_i : D_{n_i} P_i \to A_\bullet$. By Proposition A.9 the pointwise sum of any of the objects of $\bigoplus D_{n_i} P_i$ is a projective module with pointwise lifts the comparison maps $\langle f_{ki} \rangle_{i \in I}$. Taking $\mathbf{f} = (\langle \mathbf{f}_{ki} \rangle_{i \in I})$ then provides the desired lift. □

Lemma 3.15 ([DS95])**.** *Let $P_\bullet \in \mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$ be an exact chain complex such that for all $n \in \mathbb{N}$ the module P_n is projective. Then $Z_n P_\bullet$ is projective for all $n \in \mathbb{N}$ and we have the isomorphism of chain complexes*

$$P_\bullet \cong \bigoplus_{n \geq 1} D_n(Z_{n-1} P_\bullet).$$

Proof. Begin by observing that $P_\bullet$ exact implies immediately that $H_n P_\bullet = 0$; consequently $B_n P_\bullet = \operatorname{im} \partial_n = Z_n P_\bullet$ for all $n \in \mathbb{N}$. In particular, for $n = 0$ we have the equality

$$\operatorname{im} \partial_0 = B_0 P_\bullet = Z_0 P_\bullet = P_0$$

and so ∂_0 is epic. Take the exact sequence:

$$\cdots \qquad P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_0} P_0$$

By P_0 projective we derive that

$$P_1 \cong \operatorname{im} \partial_1 \oplus \operatorname{im} \partial_0 \cong B_1 P_\bullet \oplus B_0 P_\bullet \cong Z_1 P_\bullet \oplus P_0.$$

Since P_1 is projective and direct summands of projective modules are projective, $B_1 P_\bullet = Z_1 P_\bullet$ is projective as well. Proceeding as above we use the projectivity of $Z_1 P_\bullet$ to derive that

$$P_2 \cong \operatorname{im} \partial_2 \oplus \operatorname{im} \partial_1 \cong B_2 P_\bullet \oplus B_1 P_\bullet \cong Z_2 P_\bullet \oplus Z_1 P_\bullet.$$

Once again the fact that P_2 is projective and that direct summands of projective modules are projective gives that $Z_2 P_\bullet$ is projective. Proceeding in this way for all $n \geq 1$ gives that all $Z_{n-1} P_\bullet$ are projective and provides the desired isomorphism

$$P_\bullet \cong \bigoplus_{n \geq 1} D_n(Z_{n-1} P_\bullet).$$

This completes the proof of the lemma. □

These lemmas give us the tools necessary to show that the trivial cofibrations in $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$ have the left lifting property with the fibrations in $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$.

Theorem 3.16. *The trivial cofibrations have the left lifting property with the fibrations in $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$.*

Proof. Begin by assuming that we have the commuting diagram of chain complexes

$$\begin{tikzcd} A_\bullet \arrow[r, "\mathbf{u}"] \arrow[d, "\mathbf{f}"'] & X_\bullet \arrow[d, "\mathbf{g}"] \\ B_\bullet \arrow[r, "\mathbf{v}"'] & Y_\bullet \end{tikzcd}$$

with $\mathbf{f}$ a trivial cofibration and $\mathbf{g}$ a fibration. Then $\mathbf{f}$ is monic with $\operatorname{Coker} f_n$ a projective module for every $n \in \mathbb{N}$. Since the sequence

$$0 \longrightarrow A_\bullet \xrightarrow{\mathbf{f}} B_\bullet \longrightarrow \operatorname{Coker} \mathbf{f} \longrightarrow 0$$

is exact, produce the long exact homology sequence

$$\cdots \longrightarrow H_n A_\bullet \xrightarrow[\cong]{(f_n)_*} H_n B_\bullet \longrightarrow H_n \operatorname{Coker} \mathbf{f} \xrightarrow{\delta_{n-1}} H_{n-1} A_\bullet \xrightarrow[\cong]{(f_{n-1})_*} \cdots$$

and use it to derive that $H_n \operatorname{Coker} \mathbf{f} = 0$ for all $n \in \mathbb{N}$. This gives us that $\operatorname{Coker} \mathbf{f}$ is exact and consequently satisfies the hypotheses of Lemma 3.15, and consequently is isomorphic to a direct sum of n-disks. However, it then satisfies the hypotheses of Lemma 3.14; the lifting property allows us to provide a splitting of the sequence

$$0 \longrightarrow A_\bullet \xrightarrow{\mathbf{f}} B_\bullet \longrightarrow \operatorname{Coker} \mathbf{f} \longrightarrow 0$$

giving an isomorphism $B_\bullet \cong A_\bullet \oplus \operatorname{Coker} \mathbf{f}$. Use this splitting to find a map $\nu : \operatorname{Coker} \mathbf{f} \to Y_n$ such that $\nu = \mathbf{v}\big|_{\operatorname{Coker} \mathbf{f}}$. Since each $\operatorname{Coker} f_n$ is projective and g_n is epic in positive degrees, we invoke Corollary 3.14 to provide a lift in every degree $k_n : \operatorname{Coker} f_n \to X_n$ for which $k_n g_n = \nu_n$. Thus define the map

$$\mathbf{h} : A \oplus \operatorname{Coker} \mathbf{f}$$

by taking

$$\mathbf{h} := \mathbf{u} \oplus \mathbf{k}.$$

Then a routine verification shows that this provides the desired lift and hence proves that the diagram

$$\begin{tikzcd} A_\bullet \arrow[r, "\mathbf{u}"] \arrow[d, "\mathbf{f}"'] & X_\bullet \arrow[d, "\mathbf{g}"] \\ B_\bullet \arrow[r, "\mathbf{v}"'] \arrow[ur, "\mathbf{h}"] & Y_\bullet \end{tikzcd}$$

commutes in $\mathbf{Ch}_{n\geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$ (abusing notation slightly where $\mathbf{h} : B_\bullet \to X_\bullet$ actually means precomposing the $\mathbf{h}$ we defined with the isomorphism $B_\bullet \to A_\bullet \oplus \operatorname{Coker} \mathbf{f}$). This proves the theorem. □

We move on here to discuss the actual factorization of maps in $\mathbf{Ch}_{n\geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$. We will use the Gluing Construction provided by the small object argument (with our regular cardinal as $\kappa = \aleph_0$ in this case), which will allow us to factor the maps based solely on a lifting property. We will recall many of these constructions for the case $\kappa = \aleph_0$ in Appendix

Definition 3.17. *Define the n-sphere associated to a left R-module A to be the complex determined by*

$$(S^n A)_k := \begin{cases} 0 & k \neq n; \\ A & k = n. \end{cases}$$

with the obvious differentials.

Lemma 3.18. *Let $\mathbf{g} : X_\bullet \to Y_\bullet$ be a morphism in $\mathbf{Ch}_{n\geq 0}(\mathbf{R}\text{-}\mathbf{Mod})$. Then $\mathbf{g}$ is a fibration iff $\mathbf{g}$ has the right lifting property with respect to the maps $0 \to D_n R$ for $n \geq 1$.*

Proof. Begin by observing from the adjunction on D_n we have

$$\mathbf{Ch}_{n\geq 0}(\mathbf{R}\text{-}\mathbf{Mod})(D_n R, Y_\bullet) \cong \mathbf{R}\text{-}\mathbf{Mod}(R, Y_n) \cong Y_n.$$

From here it follows that if $\mathbf{g}$ is a fibration then we can find an appropriate lift immediately, while if $\mathbf{g}$ has the right lifting property with $0 \to D_n R$ for all $n \geq 1$, then it must be epic in all positive degrees.

□

Lemma 3.19. *Let $\mathbf{g} : X_\bullet \to Y_\bullet$ be a morphism of chain complexes. Then $\mathbf{g}$ is a trivial fibration if and only if the induced map $\theta_n = \langle g_n, \partial_{n-1}\rangle$ from the pullback*

$$\begin{tikzcd}
A_n \arrow[dr, dashed, "\theta_n"] \arrow[drr, bend left, "\partial_{n-1}"] \arrow[ddr, bend right, "g_n"'] & & \\
& B_n \times_{Z_{n-1}B_\bullet} Z_{n-1}A_\bullet \arrow[r, "p_{n0}"] \arrow[d, "p_{n1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z_{n-1}A_\bullet \arrow[d, "f_{n-1}"] \\
& B_n \arrow[r, "\partial_{n-1}"'] & Z_{n-1}B_\bullet
\end{tikzcd}$$

is an epimorphism of left R-modules.

Proof. The proof of the lemma is a routine diagram chase and so is omitted. □

Lemma 3.20. *Let $\mathbf{g} : X_\bullet \to Y_\bullet$ be a morphism of chain complexes. Then $\mathbf{g}$ is a trivial fibration if and only if $\mathbf{g}$ has the right lifting property with respect to all the maps $\{S^{n-1}R \xrightarrow{\iota_n} D_nR\}_{n\geq 0}$, where $S^{-1}R = 0 = D_0R$.*

Proof. The proof is constructive. Begin by observing that a morphism $\mathbf{u} : S^nR \to X_\bullet$ is a selection of a submodule of $Z_nX_\bullet$, which is easily seen from the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "\partial_n"] & X_n \arrow[r, "\partial_{n-1}"] & X_{n-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & R \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Consequently there is a natural isomorphism

$$\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})(S^nR, X_\bullet) \cong Z_nX_\bullet$$

for every $n \in \mathbb{N}$. Now let $S(\iota_n, \mathbf{g})$ be the left R-module of all pairs of maps $(\mathbf{u}, \mathbf{v})$ that make the digram

$$\begin{tikzcd}
S^{n-1}R \arrow[r, "\mathbf{u}"] \arrow[d, "\iota_n"'] & X_\bullet \arrow[d, "\mathbf{g}"] \\
D_nR \arrow[r, "\mathbf{v}"'] & Y_\bullet
\end{tikzcd}$$

commute. Then $S(\iota_n, \mathbf{g})$ is a submodule of $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})(S^{n-1}R, X_\bullet) \times \mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})(D_nR, Y_\bullet)$. Thus there is a natural homomorphism

$$\varepsilon_n : \mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})(D_nR, X_\bullet) \to S(\iota_n, \mathbf{g})$$

which is isomorphic through Proposition 3.12 to a map

$$\langle g_n, \partial_{n-1}\rangle = \theta_n : X_n \to Y_n \times_{Z_{n-1}Y_\bullet} Z_{n-1}X_\bullet.$$

It is then easy to see through Lemma 3.19 that $\mathbf{g}$ is a trivial fibration if and only if there is a lift $\ell_n : D_nR \to X_\bullet$ for all $n \in \mathbb{N}$. □

Proposition 3.21. *Let $\mathbf{f} : X_\bullet \to Y_\bullet$ be a morphism in $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$. Then $\mathbf{f}$ can be factorized by a cofibration followed by a trivial fibration.*

Proof. Let $\mathscr{A}$ be the set of maps

$$\mathscr{A} := \{\iota_n : S^{n-1}R \to D_nR \mid n \geq 0\}$$

and produce the factorization of chain complexes

$$\begin{tikzcd} X_\bullet \arrow[r, "\mathbf{i}_\infty"] \arrow[dr, "\mathbf{f}"'] & G^\infty(\mathscr{A}, \mathbf{f}) \arrow[d, "\mathbf{f}_\infty"] \\ & Y_\bullet \end{tikzcd}$$

given by the Gluing Construction of Definition B.4. From Example B.3, Lemma 3.20, and Proposition B.5 it follows that $\mathbf{f}_\infty$ is a trivial fibration. We must now show that $\mathbf{i}_\infty$ is a cofibration. Begin by observing that at each degree $n \in \mathbb{N}$ for every $k \in \mathbb{N}$, $G^{k+1}(\mathscr{A}, \mathbf{f})$ is the direct sum of $G^k(\mathscr{A}, \mathbf{f})$ with some copies of R (which is a free R-module and hence projective as well as the cokernel of the injection $\mathbf{i}_{k+1}$). Taking the colimit gives $G^\infty(\mathscr{A}, \mathbf{f})_n$ as the direct sum of X_n with some copies of R, which are free (and hence projective). It is then trivial to see that at each such n the cokernel of $\mathbf{i}_\infty$ is the direct sum of the copies of R and hence is projective. This shows that $\mathbf{i}_\infty$ is a cofibration and completes the proof. $\square$

Proposition 3.22. [DS95] *Let $\mathbf{f} : X_\bullet \to Y_\bullet$ be a morphism in $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$. Then $\mathbf{f}$ can be factorized by a trivial cofibration followed by fibration.*

Proof. We begin by letting $\mathscr{D}$ be the set of maps

$$\mathscr{D} := \{0 \to D_n R \mid n \geq 1\}.$$

From the Gluing Constrution of Definition B.4 produce the factorization of chain complexes

$$\begin{tikzcd} X_\bullet \arrow[r, "\mathbf{i}_\infty"] \arrow[dr, "\mathbf{f}"'] & G^\infty(\mathscr{D}, \mathbf{f}) \arrow[d, "\mathbf{f}_\infty"] \\ & Y_\bullet \end{tikzcd}$$

By Example B.3, Proposition B.5, and Lemma 3.18 it follows that $\mathbf{f}_\infty$ is a fibration. Now, from the construction of $G^k(\mathscr{D}, \mathbf{f})$ and the fact that we are lifting against $\mathscr{D}$ we have that at each degree $n \in \mathbb{N}$

$$G^\infty(\mathscr{D}, \mathbf{f})_n = X_n \oplus \left(\bigoplus_{m \geq 1} \bigoplus_{i \in X_n} D_{m_i} R \right).$$

Note that this gives $\mathbf{i}_\infty$ is the coproduct injection $X_\bullet \to G^\infty(\mathscr{D}, \mathbf{f})$. Certainly $\operatorname{Coker} \mathbf{i}_\infty$ is projective and $\mathbf{i}_\infty$ is monic, so we need only check that $H_n(\mathbf{i}_\infty)$ is an isomorphism for every $n \in \mathbb{N}$. To do this observe that for every $n \in \mathbb{N}$ we have the isomorphisms

$$\begin{aligned} H_n\left(G^\infty(\mathscr{D}, \mathbf{f})\right) &\cong H_n \left(X_\bullet \oplus \bigoplus_{m \geq 1} \bigoplus_{i \in X_m} D_{m_i} R \right) \cong H_n X_\bullet \oplus H_n \left(\bigoplus_{m \geq 1} \bigoplus_{i \in X_m} D_{m_i} R \right) \\ &\cong H_n X_\bullet \oplus \bigoplus_{m \geq 1} \bigoplus_{i \in X_m} H_n(D_{m_i} R) \cong H_n X_\bullet \end{aligned}$$

implying that $(\mathbf{i}_\infty)_n$ is an isomorphism for every $n \in \mathbb{N}$. This shows that $\mathbf{i}_\infty$ is a trivial cofibration and proves the proposition. $\square$

Remark 3.23. We can adapt this proof in the case of a general Abelian category with enough projectives by taking some projective resolution of each X_n in terms of a projective generator P and then proceeding in a similar fashion (namely by lifting against injections $0 \to D_n P$).

Theorem 3.24. *There is a model structure on $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$ given by the following classes of maps:*

1. *Weak Equivalences: Degreewise homology isomorphisms;*
2. *Cofibrations: Monomorphisms in every degree with projective cokernel;*
3. *Fibrations: Maps epic in every positive degree.*

Proof. By Lemma 3.3 we have that $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$ is (co)complete. Proposition 3.9 shows that each class of maps is preserved by retracts. The two-out-of-three axiom on weak equivalences is clear by composition of isomorphisms. Theorem 3.10 shows that cofibrations have the left lifting property with trivial fibrations, while Theorem 3.16 shows that trivial cofibrations have the left lifting property with respect to fibrations. Proposition 3.21 shows the cofibration/trivial fibration factorization while Proposition 3.22 shows the trivial cofibration/fibration factorization. This verifies the axioms of a model category and hence proves the theorem. $\square$

Remark 3.25. The model structure constructed in the prior theorem is called the *Projective Model Structure* on chain complexes.

Remark 3.26. We should observe that in the course of proving this model structure we never once used any special properties of the category of R-modules, as opposed to any other Abelian category. In fact, by generalizing appropriately and with a certain amount of care we can show that there is a model structure on the category of nonnegatively graded chain complexes of objects in *any* Abelian category $\mathfrak{A}$ that has enough projectives.

Remark 3.27. We can dualize the model structure provided by showing that whenever R is a ring of unity there is a model structure on the category of nonnegatively graded cochain complexes given by the classes of maps:

(1) Weak Equivalences are maps $\mathbf{f} : A^\bullet \to B^\bullet$ that induce degree-wise isomorphisms $H^n(\mathbf{f}) : H^n A^\bullet \to H^n B^\bullet$;
(2) Cofibrations are monomorphisms in degrees $n > 1$;
(3) Fibrations are epimiorphisms in every degree that have injective kernel.

This model structure is called the *Injective Model Structure* on cochain complexes. As in the projective module, if $\mathfrak{A}$ is an Abelian category that has enough injectives then there is an injective model structure on $\mathfrak{A}$.

4. Homotopy and Model Categories

In this section we will show that model categories provide us with the correct categorical notion of homotopy, capturing essence of homotopy on **Top** and on $\mathbf{Ch}_{\mathrm{n}\geq 0}(\mathbf{R\text{-}Mod})$ and allowing us to adapt this essence to other, perhaps more exotic or less traditional, situations. We will begin by providing some of the basic constructions and lemmas before moving forward to provide an introduction to categorical homotopy. Our first step will be to introduce cylinder and path objects before providing some of their basic properties.

Definition 4.1. *Let $\mathfrak{C}$ be a model category and let $X \in \mathrm{Ob}(\mathfrak{C})$. We then say that an object C of $\mathfrak{C}$ is a* cylinder object of X *if there is a map*

$$f : X \coprod X \to C$$

and a map

$$g : C \to X$$

such that the composition $fg : X \coprod X \to X$ is the codiagonal ∇, i.e., the diagram

$$\begin{tikzcd} X \coprod X \arrow[rr, "\nabla"] \arrow[dr, "f"'] & & X \\ & C \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. Furthermore, we say that a cylinder object C is:

(1) *a* good cylinder object *if the map $f : X \coprod X \to C$ is a cofibration;*
(2) *a* very good cylinder object *if C is a good cylinder object and if the map $g : C \to X$ is a trivial fibration.*

Definition 4.2 (Dual of Definition 4.1)**.** *Let $\mathfrak{C}$ be a model category and $Y \in \mathrm{Ob}(\mathfrak{C})$ an object. We then say that an object P of $\mathfrak{C}$ is a* path object of Y *if there is a map $p : P \to Y \times Y$ and $q : Y \to P$ with q a weak equivalence such that the following diagram commutes:*

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[dr, "\Delta"'] & P \arrow[d, "p"] \\ & Y \times Y \end{tikzcd}$$

Moreover, we say that a path object P is:

(1) *a* good path object *if the map p is a fibration;*
(2) *a* very good path object *if P is a good path object and the map q is a trivial cofibration.*

Proposition 4.3. *If $\mathfrak{C}$ is a model category, then every object $X \in \mathrm{Ob}(\mathfrak{C})$ admits at least one very good cylinder object and at least one very good path object.*

Proof. We prove only the existence of cylinder objects; the existence of path objects is given by duality. Consider the map $\nabla : X \coprod X \to X$. By axiom (5) of Definition 2.1 we have that there is some object $C \in \mathrm{Ob}(\mathfrak{C})$ with a cofibration $c : X \coprod X \to C$ and a trivial fibration $f : C \to X$ making the triangle

$$\begin{tikzcd} X \coprod X \arrow[r, "c"] \arrow[dr, "\nabla"'] & C \arrow[d, "f"] \\ & X \end{tikzcd}$$

commute. But then C is a cylinder object of X and we are done. □

Observe that for any object $A \in \mathrm{Ob}\,\mathfrak{C}$ that there are maps $i_0, i_1 : A \to C_A$ for any cylinder object $c : A \coprod A \to C_A$ of A defined via the compositions

$$\begin{tikzcd} A \arrow[r, shift left, "\iota_0"] \arrow[r, shift right, "\iota_1"'] & A \coprod A \arrow[r, "c"] & C_A \end{tikzcd}$$

Dually, we have morphisms $p_0, p_1 : P_X \to X$, for any object X and path object $p : P_X \to X \times X$ given by the compositions

$$\begin{tikzcd} P_X \arrow[r, "p"] & X \times X \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] & X \end{tikzcd}$$

It will be of frequent technical use to know about the relationship between these maps and the objects A and C_A (dually between P_X and X). A best case scenario exists when A is cofibrant, which tells us that A and C_A are weakly equivalent. This is captured in the following lemma, which we then dualize to describe when P_X is weakly equivalent to X.

Lemma 4.4 (Lemma 4.4 of [DS95])**.** *Let $\mathfrak{C}$ be a model category and let A be cofibrant. Then if $A \coprod A \xrightarrow{c_A} C_A$ is a good cylinder object for A, the maps $i_0, i_1 : A \to C_A$ are trivial cofibrations in $\mathfrak{C}$.*

Proof. Since A is cofibrant the unique map $\iota_A : \bot \to A$ is a cofibration. Now write the coproduct $A \coprod A$ as the pushout:

$$\begin{tikzcd} \bot \arrow[r, dashed, "\exists! \iota_A"] \arrow[d, dashed, "\exists! \iota_A"'] & A \arrow[d, "\iota_0"] \\ A \arrow[r, "\iota_1"'] & A \coprod A \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

Invoking Lemma 2.14 gives that both ι_0 and ι_1 are cofibrations. Recall that since $\mathscr{C}$ is a saturated class of morphisms, it is composition closed. Consequently, since $c_A : A \coprod A \to C_A$ is a cofibration we have that the compositions $i_0 = \iota_0 c_A$ and $i_1 = \iota_1 c_A$ are as well.

To see that i_0 and i_1 are weak equivalences recall that $\iota_0 \nabla = \mathrm{id}_A = \iota_1 \nabla$ and that the diagram

$$\begin{tikzcd} A \coprod A \arrow[r, "c_A"] \arrow[dr, "\nabla"'] & C_A \arrow[d, "c'_A"] \\ & A \end{tikzcd}$$

commutes with c'_A a weak equivalence. Thus we have that

$$i_0 c'_A = (\iota_0 c_A)c'_A = \iota_0(\nabla) = \mathrm{id}_A = \iota_1(\nabla) = (\iota_1 c_A)c'_A = i_1 c'_A$$

and so by the two-out-of-three axiom i_0 and i_1 are weak equivalences. This completes the proof that i_0 and i_1 are trivial cofibrations. □

Corollary 4.5. *Let $\mathfrak{C}$ be a model category and let X be a fibrant object. Then if P_X is a good path object of X, the maps $p_0, p_1 : P_X \to X$ defined by the compositions*

$$\begin{tikzcd} P_X \arrow[r, "p_X"] & X \times X \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] & X \end{tikzcd}$$

are weak equivalences.

Proof. Dualize the proof of Lemma 4.4 □

We now define the notion of a left homotopy relation in $\mathfrak{C}$. While we would wish to have a naïve two-sided homotopy from the beginning, it is unfortunately not quite possible if we want homotopy to be an equivalence relation on the hom-set $\mathfrak{C}(A, X)$. Consequently we need to keep in mind the notion of a left deformation and a right deformation before we can describe when these deformations describe the same action.

Definition 4.6. *Let $\mathfrak{C}$ be a model category. We then say that two parallel morphisms $f, g : X \to Y$ have a* left homotopy from f to g *whenever there is a cylinder object C_X of X such that there exists a map $h : C_X \to Y$ making the diagram*

$$\begin{tikzcd} X \coprod X \arrow[rr, "\langle f,g \rangle"] \arrow[dr, "c"'] & & Y \\ & C_X \arrow[ur, "h"'] & \end{tikzcd}$$

commute. We call h a left homotopy from f to g *and write $f \,{}_\ell\!\simeq g$ in this case. Furthermore, we say that h is a* good left homotopy *if C_X is a good cylinder object and that h is a* very good left homotopy *if C_X is a very good cylinder object.*

At this point it seems like the notion of homotopy depends heavily on the choice of cylinder object, and that it may be poorly behaved (in the sense that it may not be a very good or even good homotopy). However, we need not worry; every left homotopy $f \,{}_\ell\!\simeq g$ has a *good* left homotopy $f \,{}_\ell\!\simeq g$, and a very good homotopy from f to g if the codomain of f and g is fibrant.

Proposition 4.7. *Let $\mathfrak{C}$ be a model category, A and X objects of $\mathfrak{C}$, and $f \,{}_\ell\!\simeq g : A \to X$. Then there exists a good left homotopy $f \,{}_\ell\!\simeq g$, and if X is fibrant then this homotopy may be taken to be very good.*

Proof. Throughout this proof we let $d : A \coprod A \to D_A$ denote the cylinder object of the left homotopy $f \,{}_\ell\!\simeq g$ with homotopy $h : D_A \to X$ and let $d' : D_A \to A$ denote the given map such that $dd' = \nabla$. We then immediately derive that the good homotopy is given by giving the map $d : A \coprod A \to D_A$ a $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$ factorization

$$\begin{tikzcd} A \coprod A \arrow[rr, "d"] \arrow[dr, "c'"'] & & D_A \\ & C_D \arrow[ur, "w'"'] & \end{tikzcd}$$

We then easily check that C_D is a good cylinder object for A and that $w'h$ is a left homotopy from f to g.

Assume now that X is fibrant; we shall proceed to show that there is a very good left homotopy from f to g. Begin by factorizing d' as

$$\begin{tikzcd} C_A \arrow[r, "\alpha"] \arrow[dr, "d'"'] & Z \arrow[d, "\beta"] \\ & A \end{tikzcd}$$

for some object Z in $\mathfrak{C}$ for which $\alpha \in \mathscr{C}$ and $\beta \in \mathscr{W} \cap \mathscr{F}$. This then allows us to form the commuting square

$$\begin{tikzcd} A \coprod A \arrow[r, "d\alpha"] \arrow[d, "c"'] & Z \arrow[d, "\beta"] \\ C_A \arrow[r, "c'"'] & A \end{tikzcd}$$

in $\mathfrak{C}$, where C_A is a very good cylinder object of A given by providing ∇ with a $(\mathscr{C}, \mathscr{W} \cap \mathscr{F})$ factorization. Since $c \in \mathscr{C}$ and $\beta \in \mathscr{W} \cap \mathscr{F}$, there is a lift $\varphi : C_A \to Z$ which makes the following diagram commute:

$$\begin{tikzcd} A \coprod A \arrow[r, "d\alpha"] \arrow[d, "c"'] & Z \arrow[d, "\beta"] \\ C_A \arrow[r, "c'"'] \arrow[ur, "\varphi"] & A \end{tikzcd}$$

Since $d' = \alpha\beta$ is a weak equivalence and β is a weak equivalence by construction, the two-out-of-three axiom implies that α is a weak equivalence as well. This allows us to conclude that α is a trivial cofibration.

Recall that since X is fibrant the unique map $\tau_X : X \to \top$ in $\mathfrak{C}$ is a fibration and consider the commuting square:

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "\alpha"'] & X \arrow[d, dashed, "\exists! \tau_X"] \\ Z \arrow[r, dashed, "\exists! \tau_Z"'] & \top \end{tikzcd}$$

Since α is a trivial cofibration and τ_X is a fibration there is a lift $\psi : Z \to X$ making the square

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "\alpha"'] & X \arrow[d, dashed, "\exists! \tau_X"] \\ Z \arrow[r, dashed, "\exists! \tau_Z"'] \arrow[ur, "\psi"] & \top \end{tikzcd}$$

commute. Then $\varphi\psi$ is a morphism with $\varphi\psi \in \mathfrak{C}(C_A, X)$ making the diagram

$$\begin{tikzcd} A \coprod A \arrow[r, "d"] \arrow[d, "c"'] & C \arrow[r, "\alpha"'] \arrow[rr, bend left, "h"] & Z \arrow[r, "k"'] \arrow[d, "\beta"'] & X \arrow[d, dashed, "\exists! \tau_X"] \\ C_0 \arrow[rr, "c'"'] \arrow[urr, "\varphi"'] & & A \arrow[r, dashed, "\exists! \tau_A"'] & \top \end{tikzcd}$$

commute; in particular we have that

$$(c\varphi)\psi = (d\alpha)\psi = d(\alpha\psi) = dh = \langle f, g \rangle.$$

Therefore the diagram

$$\begin{tikzcd} A \coprod A \arrow[rr, "c"] \arrow[dr, "\langle f,g \rangle"'] & & C_0 \arrow[dl, "\varphi k"] \\ & X & \end{tikzcd}$$

commutes in $\mathfrak{C}$, proving that $\varphi\psi$ is a very good left homotopy $f\,{}_{\ell}{\simeq}\,g$ defined over C_A. $\square$

This proposition will be of frequent use as we proceed, as it will allow us to either replace a homotopy with a good homotopy or a very good homotopy. We now proceed to discover that whenever A is a cofibrant object in $\mathfrak{C}$, then left homotopy is an equivalence relation on $\mathfrak{C}(A, X)$. This allows us to intuit that in a model category having a cofibrant object gives us just enough structure to have left deformations of maps as an equivalence relation on hom-sets with A as the domain.

Lemma 4.8. *Let A be a cofibrant object in $\mathfrak{C}$. Then the relation ${}_{\ell}{\simeq}$ is an equivalence relation on $\mathfrak{C}(A, X)$.*

Proof. We first prove the reflexivity of ${}_{\ell}{\simeq}$. Note that since A is a cylinder object over itself via the factorization

$$\begin{tikzcd} A\coprod A \arrow[rr, "\nabla"] \arrow[dr, "\nabla"'] & & A \\ & A \arrow[ur, equal] & \end{tikzcd}$$

it follows that the diagram

$$\begin{tikzcd} A\coprod A \arrow[rr, "{\langle f,f\rangle}"] \arrow[dr, "\nabla"'] & & X \\ & A \arrow[ur, "f"'] & \end{tikzcd}$$

commutes, showing that $f\,{}_{\ell}{\simeq}\,f$ for all $f \in \mathfrak{C}(A, X)$.

We now show that ${}_{\ell}{\simeq}$ is symmetric. Define the morphism $s : A\coprod A \to A\coprod A$ via the assignment $s = \langle \iota_1, \iota_0\rangle$. We then derive the identity $\langle g, f\rangle = \langle f, g\rangle s$ and observe that from here it is immediate that whenever $f\,{}_{\ell}{\simeq}\,g$ we have $g\,{}_{\ell}{\simeq}\,f$.

We finally show that ${}_{\ell}{\simeq}$ is transitive. Assume that $f\,{}_{\ell}{\simeq}\,g$ and $g\,{}_{\ell}{\simeq}\,k$ with good left homotopies (the existence of which justified by Proposition 4.7)

$$\begin{tikzcd} A\coprod A \arrow[rr, "{\langle f,g\rangle}"] \arrow[dr, "{c=\langle i_0,i_1\rangle}"'] & & X \\ & C_A \arrow[ur, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} A\coprod A \arrow[rr, "{\langle g,k\rangle}"] \arrow[dr, "{d=\langle i_0',i_1'\rangle}"'] & & X \\ & D_A \arrow[ur, "h'"'] & \end{tikzcd}$$

Now consider the pushout:

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_0'"'] & C_A \arrow[d, "p_1"] \\ D_A \arrow[r, "p_0"'] & E_A \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The diagram

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_0'"'] & C_A \arrow[d, "p_1"] \arrow[ddr, bend left, "c'"] & \\ D_A \arrow[r, "p_0"'] \arrow[drr, bend right, "d'"'] & E_A \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists! e'"] & \\ & & A \end{tikzcd}$$

shows that e' is a weak equivalence (by using Lemma 4.4 to give i_0' and i_1 as trivial cofibrations, using Lemma 2.14 to give that p_0 and p_1 are trivial cofibrations, and finally invoking the two-out-of-three axiom) with $\langle i_0' p_0, i_1 p_1 \rangle e' = \nabla$, proving that E_A is a cylinder object of A. Producing the diagram

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[d, "i_0'"'] & C_A \arrow[d, "p_1"] \arrow[ddr, bend left, "h"] & \\
D_A \arrow[r, "p_0"'] \arrow[drr, bend right, "h'"'] & E_A \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists ! h''"] & \\
 & & X
\end{tikzcd}$$

allows us to check that $\langle i_0' p_0, i_1 p_1 \rangle h'' = \langle f, k \rangle$, showing that $f_\ell \simeq k$ and hence proving the lemma. $\square$

Definition 4.9. *Let $A, B \in \operatorname{Ob} \mathfrak{C}$. Then define the set*

$$_\ell\pi(A, B)$$

to be the set of equivalence classes of the equivalence relation on $\mathfrak{C}(A, B)$ generated by the relation $_\ell \simeq$.

Remark 4.10. Note that we use the terminology "generated by" here because if A is not cofibrant then $_\ell \simeq$ need not be an equivalence relation; we used fundamentally in the proof that the injections $i_n : A \to C_A$ are trivial cofibrations, which happens in general only when A is cofibrant. It is also worth observing that when A is cofibrant that $_\ell\pi(A, B) := \mathfrak{C}(A, B)/_\ell \simeq$.

Lemma 4.11. *Let A be cofibrant and let $p : X \to Y$ be a trivial fibration. Then the map*

$$p_* : {_\ell\pi}(A, X) \to {_\ell\pi}(A, Y)$$

given by $[f] \mapsto [fp]$ is an isomorphism in **Set** *(or in whatever enrichment of* **Set** *we care to work).*

Proof. We first show that p_* is well-defined. Assume that $f_\ell \simeq g$ through the cylinder object C_A and the map h. Then the commuting diagram

$$\begin{tikzcd}
A \coprod A \arrow[rr, "\langle f, g \rangle"] \arrow[dr, "c"'] & & X \arrow[r, "p"] & Y \\
 & C_A \arrow[ur, "h"] \arrow[urr, "hp"'] & &
\end{tikzcd}$$

implies that

$$\begin{tikzcd}
A \coprod A \arrow[rr, "\langle fp, gp \rangle"] \arrow[dr, "c"'] & & Y \\
 & C_A \arrow[ur, "hp"'] &
\end{tikzcd}$$

commutes, giving $[fp] = [gp]$.

We now show that p_* is injective. Let $f, g \in \mathfrak{C}(A, X)$, assume that $fp_\ell \simeq gp$, and choose a good left homotopy $h : fp \to gp$. Then let $\varphi : C_A \to X$ be a lift in the diagram

$$\begin{tikzcd}
A \coprod A \arrow[r, "\langle f, g \rangle"] \arrow[d, "c"'] & X \arrow[d, "p"] \\
C_A \arrow[r, "h"'] \arrow[ur, dashed, "\exists \varphi"] & Y
\end{tikzcd}$$

provided by the fact that $c \in \mathscr{C}$ and $p \in \mathscr{W} \cap \mathscr{F}$. Then $c\varphi = \langle f, g \rangle$, so $f_\ell \simeq g$ via the good left homotopy φ.

Finally we show that p_* is surjective. Let $[g] \in {_\ell\pi}(A, Y)$ and consider the lift $\varphi : A \to X$ appearing in the diagram:

$$\begin{tikzcd}
\perp \arrow[r, dashed, "\exists ! \iota_X"] \arrow[d, dashed, "\exists ! \iota_A"'] & X \arrow[d, "p"] \\
A \arrow[r, "g"'] \arrow[ur, dashed, "\exists \varphi"] & Y
\end{tikzcd}$$

Note that φ exists because $\iota_A \in \mathscr{C}$ and $p \in \mathscr{W} \cap \mathscr{F}$. Then we note

$$p_*[\varphi] = [\varphi p] = [g]$$

which completes the proof. □

Corollary 4.12. *If A, X, Y are objects in $\mathfrak{C}$ with $p : X \to Y$, then $p_* : {}_\ell\pi(A, X) \to {}_\ell\pi(A, Y)$ is a morphism of sets. If p is a trivial fibration then p_* is monic and if p is a trivial fibration while A is cofibrant then p_* is epic.*

Lemma 4.13. *Let X be fibrant in $\mathfrak{C}$ and let $f {}_\ell\simeq g : A \to X$. Then if $k : B \to A$ is any morphism $kf {}_\ell\simeq kg$.*

Proof. By Proposition 4.7 let $A \coprod A \xrightarrow{c} C_A \xrightarrow{w} A$ be a very good cylinder object for a very good left homotopy $h : f \to g$. Moreover, let $B \coprod B \xrightarrow{b} C_B \xrightarrow{w'} B$ be a very good cylinder object for B given by taking a $(\mathscr{C}, \mathscr{F} \cap \mathscr{W})$ factorization of ∇. Then there is a map $\varphi : C_B \to C_A$ making the diagram

$$\begin{tikzcd}
B \coprod B \arrow[r, "\langle k,k \rangle"] \arrow[d, "b"'] & A \coprod A \arrow[r, "c"] & C_A \arrow[d, "w"] \\
C_B \arrow[r, "w'"'] \arrow[urr, dashed, "\exists \varphi"] & B \arrow[r, "k"'] & A
\end{tikzcd}$$

commute with the existence of φ given by $b \in \mathscr{C}$ and $w \in \mathscr{W} \cap \mathscr{F}$. Now we calculate that

$$b\varphi h = \langle k, k \rangle ch = \langle k, k \rangle \langle f, g \rangle = \langle f, g \rangle,$$

implying that the diagram

$$\begin{tikzcd}
B \coprod B \arrow[rr, "\langle kf,kg \rangle"] \arrow[dr, "b"'] & & X \\
& C_B \arrow[ur, "\varphi h"'] &
\end{tikzcd}$$

commutes. Thus φh is a left homotopy $kf {}_\ell\simeq kg$ and we are done □

Lemma 4.14. *Let X be a fibrant object and let $A, B \in \operatorname{Ob} \mathfrak{C}$. Then composition $A \to B \to X$ in $\mathfrak{C}$ induces a map*

$${}_\ell\pi(A, B) \times {}_\ell\pi(B, X) \to {}_\ell\pi(A, X)$$

given by

$$([f], [g]) \mapsto [fg].$$

Remark 4.15. It is worth observing there that we are *not* assuming that A and B are cofibrant. Consequently even when $[f] = [f']$ in ${}_\ell\pi(A, B)$ or ${}_\ell\pi(B, X)$, it need not be true that there is a left homotopy $f \to f'$. However, since the equivalence relation on ${}_\ell\pi(A, B)$ is *induced* by ${}_\ell\simeq$, it is eunough to prove the lemma when there is an actual homotopy $f {}_\ell\simeq f'$.

Proof. Let $f {}_\ell\simeq g \in {}_\ell\pi(A, B)$ and let $a {}_\ell\simeq b \in {}_\ell\pi(B, X)$. We will then show that composition induces a morphism by showing that $fa {}_\ell\simeq gb$; to show this it is sufficient to show that $fa {}_\ell\simeq fb$ and that $fa {}_\ell\simeq ga$. The first case is covered by Lemma 4.13, so we need only show the second case. To see this consider that the proof of Lemma 4.8 shows that post composition is a morphism in **Set** and hence that $fa {}_\ell\simeq ga$ whenever $f {}_\ell\simeq g$. This proves the Lemma. □

We now dualize essentially the entire section thus far to give a description of right homotopy. While conceptually these results do not necessarily need explicity mention, it will aid in our organization of these ideas to write them down and have them at hand for whenever (if ever) we need to use them (especially since the choice of working with either a path or a cylinder object is frequently a matter of taste in the homotopy category). We will need both of these concepts to define and work with homotopy in general, so let us proceed.

Definition 4.16. *Let $f, g : A \to X$ in $\mathfrak{C}$. Then we say that f* is right homotopic to g, *written $f \simeq_r g$, if there exits a path object $X \times X \xrightarrow{q} P_X \xrightarrow{p} X$ for X such that there is a map $h : X \to P_X$ (the right homotopy $h : f \to g$) such that the following diagram commutes:*

$$\begin{tikzcd} A \arrow[rr, "\langle f,g \rangle"] \arrow[dr, "h"'] & & X \times X \\ & P_X \arrow[ur, "p"'] & \end{tikzcd}$$

Proposition 4.17. *If $f \simeq_r g : A \to X$, then there exists a good right homotopy $h' : f \to g$. Furthermore, if A is cofibrant then there exists a very good right homotopy $h'' : f \to g$.*

Proof. Dualize the proof of Proposition 4.7. □

Lemma 4.18. *If X is fibrant then the relation $\simeq_r$ on $\mathfrak{C}(A, X)$ is an equivalence relation.*

Proof. Dualize the proof of Lemma 4.8. □

Definition 4.19. *Let $A, X \in \operatorname{Ob}\mathfrak{C}$. The define the set $\pi_r(A, X)$ to be the set of equivalence classes on $\mathfrak{C}(A, X)$ induced by the relation $\simeq_r$.*

Lemma 4.20. *Let X be fibrant and let $c : A \to B$ be a trivial cofibration. Then the map*

$$c^* : \pi_r(B, X) \to \pi_r(A, X)$$

given by

$$[f] \mapsto [cf]$$

is an isomorphism in **Set**.

Proof. Dualize the proof of Lemma 4.11. □

Corollary 4.21. *If $a \in \mathfrak{C}(A, B)$ then $a^* : \pi_r(B, X) \to \pi_r(A, X)$ is a morphism of sets.*

Lemma 4.22. *If $f \simeq_r g : A \to X$ for a cofibrant object A and $k \in \mathfrak{C}(X, Y)$, then $fk \simeq_r gk$.*

Proof. Dualize the proof of 4.13. □

Lemma 4.23. *Let A be a cofibrant object in $\mathfrak{C}$. Then composition $A \to X \to Y$ in $\mathfrak{C}$ induces a map*

$$\pi_r(A, X) \times \pi_r(X, Y) \to \pi_r(A, Y)$$

given by

$$([f], [g]) \mapsto [fg].$$

Proof. Dualize the proof of 4.14. □

Proposition 4.24. *Let $f, g : A \to X$. Then:*

(1) *if A is cofibrant and $f_\ell \simeq g$, $f \simeq_r g$;*
(2) *if X is fibrant and $f \simeq_r g$, $f_\ell \simeq g$.*

Proof. We prove only condition 1; the second follows from duality. Assume that A is cofibrant $f_\ell \simeq g$, and let $A \coprod A \xrightarrow{c} C_A \xrightarrow{w} A$ be a good cylinder object for A with good left homotopy $h : f \to g$ (i.e., $h : C_A \to X$).

Since A is cofibrant, Lemma 4.4 gives us that both maps $i_0, i_1 : A \to C_A$ are trivial cofibrations. Now let $X \xrightarrow{q} P_X \xrightarrow{p} X \times X$ be a good path object of X. Then observe that the diagram

$$\begin{tikzcd}
A \arrow[r, "fq"] \arrow[d, "i_0"'] & P_X \arrow[d, "p"] \\
C_A \arrow[r, "\langle wf, h\rangle"'] \arrow[ur, dashed, "\exists k"] & X \times X
\end{tikzcd}$$

commutes by construction with the existence of the lift $k : C_A \to P_X$ given by $i_0 \in \mathscr{C} \cap \mathscr{W}$ and $p \in \mathscr{F}$. Define $h' := i_1 k : A \to P_X$. Then we calculate that

$$i_1 kp = i_1 \langle wf, h\rangle = \langle i_1 wf, i_1 h\rangle = \langle f, g\rangle$$

giving that h' is a right homotopy from f to g. $\square$

We note from the above proposition that if A is cofibrant and X is fibrant, any maps $f, g : A \to X$ have the property that $f_\ell \simeq g$ if and only if $f \simeq_r g$, which says that in some way the homotopy between the two objects does not depend in any way on the sidedness of any particular homotpy. Furthermore, since A is cofibrant and X is fibrant we can always find a particular very good cylinder object to define our left homtopies over or, dually, a very good path object to define our right homotopies over.

Definition 4.25. *Let A be cofibrant in $\mathfrak{C}$ and X fibrant in $\mathfrak{C}$. Then we say that if there is either $f \simeq_r g : A \to X$ or if $f_\ell \simeq g : A \to X$ then f and g are said to be* homotopic *and we write $f \simeq g$. Moreover, we write $\pi(A, X)$ as the set of equivalence classes of $\mathfrak{C}(A, X)$ under the relation $\simeq$.*

Proposition 4.26. *Let A and X be objects in $\mathfrak{C}$ that are both fibrant and cofibrant. Then $f : A \to X$ is a weak equivalence if and only if there exists a morphism $g : X \to A$ such that $fg \simeq \mathrm{id}_A$ and $gf \simeq \mathrm{id}_X$.*

Definition 4.27. *Any two morphisms $f : A \to X$ and $g : X \to A$ such that $fg \simeq \mathrm{id}_A$ and $gf \simeq \mathrm{id}_X$ are called* homotopy inverses.

Proof of Proposition 4.26. $\implies$: Assume that f is a weak equivalence and give f a $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ factorization

$$A \xrightarrow{c} Z \xrightarrow{p} X.$$

The two-out-of-three axiom implies immediately that p is a trivial fibration. Now form the two (dual) diagrams

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "c"'] & A \arrow[d, dashed, "\exists ! \tau_A"] \\
Z \arrow[r, dashed, "\exists ! \tau_Z"'] & \top
\end{tikzcd}
\qquad
\begin{tikzcd}
\bot \arrow[r, dashed, "\exists ! \iota_Z"] \arrow[d, dashed, "\exists ! \iota_X"'] & Z \arrow[d, "p"] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

Since A is fibrant the map $\tau_A \in \mathscr{F}$ and there is a retract $r : Z \to A$ with left inverse c; similarly X cofbirant implies that $\iota_X \in \mathscr{C}$ and hence there is a section $s : X \to Z$ with right inverse p. Using that A is cofibrant and invoking Lemma 4.20 gives that $c^* : \pi_r(Z, Z) \to \pi_r(A, C)$ is an isomorphism. Furthermore,

$$c^*[rc] = [crc] = [c]$$

giving $rc \simeq_r \mathrm{id}_Z$. Thus $cr_\ell \simeq \mathrm{id}_A$ and $rc \simeq_r \mathrm{id}_Z$. Dually we derive that $sp \simeq_r \mathrm{id}_X$ and $ps_\ell \simeq \mathrm{id}_Z$. Consequently taking $g = sr$ gives a morphism for which $fg \simeq \mathrm{id}_A$ and $gf \simeq \mathrm{id}_X$.
$\impliedby$: Assume that f has a homotopy inverse g. Then give f a $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ factorization $A \xrightarrow{c} Z \xrightarrow{p} X$. Since $f = cp$ and c is a trivial cofibration, it is sufficient to prove that p is a trivial fibration in order to show that f is a weak equivalence. So observe that from the diagrams

$$\begin{tikzcd}
\bot \arrow[r, dashed, "\exists ! \iota_A"] \arrow[dr, dashed, "\exists ! \iota_Z"'] & A \arrow[d, "c"] \\
 & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "p"] \arrow[dr, dashed, "\exists ! \tau_Z"'] & X \arrow[d, dashed, "\exists ! \tau_X"] \\
 & \top
\end{tikzcd}$$

and the fact that $\mathscr{C}$ an $\mathscr{F}$ are saturated it follows that Z is both fibrant and cofibrant. Now let $h : C_X \to X$ be a homotopy $h : gf \to \mathrm{id}_X$ and consider the commuting diagram

$$\begin{tikzcd} X \arrow[r, "gc"] \arrow[d, "i_0"'] & Z \arrow[d, "p"] \\ C_X \arrow[r, "h"'] \arrow[ur, dashed, "\exists k"] & X \end{tikzcd}$$

with the existence of k given by i_0 a trivial cofibration (by Lemma 4.4) and p a fibration. Consider the map $s = i_1 k : X \to Z$. Then we have immediately that

$$sp = i_1 kp = i_1 h = \mathrm{id}_X \,.$$

Now note that since $c : A \to Z$ is a weak equivalence, by $\implies$ proof we know that c has a homotopy inverse $r : Z \to Z$. Furthermore, since $f = cp$, left multiplication by r gives the homotopies

$$rf \simeq rcp \simeq \mathrm{id}_A\, p = p$$

while the homotopy identity $i_1 \simeq i_0$ implies that

$$s = i_1 k \simeq i_0 k = gc.$$

Combining these gives the identities

$$ps \simeq rfs \simeq rfgc \simeq r\,\mathrm{id}_A\, c \simeq rc \simeq \mathrm{id}_Z,$$

implying that $ps \in \mathscr{W}$. Now the commutativity of the diagram

$$\begin{tikzcd} & Z \arrow[rrr, "ps"] \arrow[dl, equal] \arrow[dr, equal] & & & Z \arrow[dr, "p"] & \\ Z \arrow[rr, equal] & & Z \arrow[r, "p"'] & X \arrow[ur, "s"] \arrow[rr, equal] & & X \end{tikzcd}$$

implies that p is a retract of ps. Using that $\mathscr{W}$ is closed to retracts shows that p is a weak equivalence; invoking the two-out-of-three axiom on $f = cp$ shows that f is a weak equivalence and proves the Proposition. ☐

These numerous lemmas give us the technical tools in order to describe, construct, and prove what the homotopy category associated to a model category is. Intuitively we would now want to see for each object X some sort of "natural" way of finding a fibrant/cofibrant object associated to X that has hom-set $\pi(X, Y)$. It turns out that we can do exactly this by finding what are called (co)fibrant replacements and then showing that these replacements depend only up to left (or right) homotopy on maps. In the category of R-modules these cofibrant replacements may be thought of as finding projective resolutions, for example. To take on the task of constructing the homotopy category we need to consider six particular categories associated to a model category:

Definition 4.28 ([DS95])**.** *Let $\mathfrak{C}$ be a model category. Then define:*

(1) *$\mathfrak{C}_C$ to be the full subcategory of $\mathfrak{C}$ generated by the cofibrant objects of $\mathfrak{C}$, i.e., the objects of $\mathfrak{C}_C$ are the cofibrant objects of $\mathfrak{C}$ and $\mathfrak{C}_C(X, Y) = \mathfrak{C}(X, Y)$ whenever X and Y are cofibrant;*
(2) *$\mathfrak{C}_F$ to be the full subcategory of $\mathfrak{C}$ generated by the fibrant objects of $\mathfrak{C}$;*
(3) *$\mathfrak{C}_{CF}$ to be the full subcategory of $\mathfrak{C}$ generated by the objects both fibrant and cofibrant;*
(4) *$\pi\,\mathfrak{C}_C$: the category whose objects are cofibrant objects of $\mathfrak{C}$ and $\pi\,\mathfrak{C}_C(X, Y) := \pi_r(X, Y)$;*
(5) *$\pi\,\mathfrak{C}_F$: the category whose objects are fibrant objects of $\mathfrak{C}$ and $\pi\,\mathfrak{C}_F(X, Y) := {}_\ell\pi(X, Y)$;*
(6) *$\pi\,\mathfrak{C}_{CF}$: the category whose objects are both fibrant and cofibrant in $\mathfrak{C}$ and $\pi\,\mathfrak{C}_{CF}(X, Y) := \pi(X, Y)$.*

Remark 4.29. Note that the fact that the categories described in $(4) - (6)$ are proved to be categories by the various lemmas we have shown prior to this. We do note explicitly prove any of those facts, but identities will be the classes of maps $[\mathrm{id}_X]$ while composition will be given by $[f][g] \mapsto [fg]$ whenever f and g are composable in $\mathfrak{C}$. The interested reader can simply pour over the various lemmas and use them as necessary.

We now go about the task of finding (co)fibrant objects that will be used to replace objects on the homotopy level. We will ask that these (co)fibrant objects be weakly equivalent to the original objects we start with so that they are isomorphic on the homotopy level, i.e., so that there is a homotopy inverse between them. The way we do this is as follows:

Definition 4.30 ([DS95])**.** *Let $\mathfrak{C}$ be a model category and let $X \in \operatorname{Ob}\mathfrak{C}$ be arbitrary. Then give the unique map $\perp \to X$ a $(\mathscr{C}, \mathscr{F} \cap \mathscr{W})$ factorization*

$$\begin{tikzcd} \perp \arrow[r, dashed] \arrow[dr, dashed] & QX \arrow[d, "p_X"] \\ & X \end{tikzcd}$$

and the unique map $X \to \top$ a $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ factorization

$$\begin{tikzcd} \top & RX \arrow[l, dashed] \\ & X \arrow[u, "i_X"'] \arrow[ul, dashed] \end{tikzcd}$$

If X is cofibrant we take $QX := X$ and if X is fibrant we take $RX := X$. We say that QX is the cofibrant replacement *of X and that RX is the* fibrant replacement *of X.*

While this definition makes sense and seems to be a useful way of replacing X by a cofibrant or fibrant object weakly equivalent to X, we should ask how canonical these replacements are. Does their existence depend deeply on the particular factorization we chose, or is it "good enough" in the sense that it depends only up to homotopy?

Lemma 4.31 ([DS95])**.** *Let $f : X \to Y$ be a morphism in $\mathfrak{C}$. Then there exists a morphism $\widetilde{f} : QX \to QY$ such that the diagram*

$$\begin{tikzcd} QX \arrow[r, "\widetilde{f}"] \arrow[d, "p_X"'] & QY \arrow[d, "p_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes in $\mathfrak{C}$. The morphism $\widetilde{f}$ depends only up to left or right homotopy on f and $\widetilde{f} \in \mathscr{W}$ if and only if $f \in \mathscr{W}$. Moreover, if Y is fibrant then $\widetilde{f}$ depends up to left or right homotopy on $[f] \in \pi_r(X, Y)$.

Proof. Begin by considering the diagram

$$\begin{tikzcd} \perp \arrow[r, dashed] \arrow[d, dashed] & QY \arrow[d, "p_y"] \\ QX \arrow[ur, dashed, "\exists \widetilde{f}"] \arrow[r, "p_X f"'] \arrow[d, "p_X"'] & Y \\ X \arrow[ur, "f"'] & \end{tikzcd}$$

with the existence of $\widetilde{f}$ given by QX cofibrant and p_y a trivial fibration. Since post composition of maps induces a bijection on left homotopy hom-sets with cofibrant domain we see right away that $\widetilde{f}$ depends on f only up to left homotopy. By Proposition 4.24 and the fact that QX is cofibrant we have that any left homotopy is a right homotopy as well. If Y is fibrant then we apply Lemma 4.14 to derive that $\widetilde{f}$ depends only up to left or right homotopy on $[f]$. Finally for the weak equivalence statement, since

$$\widetilde{f} p_Y = p_X f$$

and both p_X and p_Y are trivial fibrations we see that by the two-out-of-three axiom that $\widetilde{f} \in \mathscr{W}$ if and only if $f \in \mathscr{W}$. $\square$

By duality we obtain the following result:

Lemma 4.32 ([DS95]). *If $f : X \to Y$ is a morphism in $\mathfrak{C}$ there exists a morphism $\widehat{f} : RX \to RY$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] & Y \arrow[d, "i_Y"] \\ RX \arrow[r, "\widehat{f}"'] & RY \end{tikzcd}$$

commutes in $\mathfrak{C}$. The map $\widehat{f}$ depends only up to left or right homotopy on f and $f \in \mathscr{W}$ if and only if $\widehat{f} \in \mathscr{W}$. Furthermore, if X is cofibrant then $\widehat{f}$ depends only up to left or right homotopy on $[f] \in {}_{\ell}\pi(X, Y)$.

Remark 4.33. While these squares make it *almost* look like the collection $i = \{i_X \mid X \in \operatorname{Ob}\mathfrak{C}\}$ and $p = \{p_X \mid X \in \operatorname{Ob}\mathfrak{C}\}$ are natural transformations $p : Q \to \mathrm{id}_{\mathfrak{C}}$ and $i : \mathrm{id}_{\mathfrak{C}} \to R$, there are problems when we try to regard R and Q as functors: we must *choose* a lift $\widetilde{f}$ (or $\widehat{f}$) for every morphism $f : X \to Y$, and the nature of this choice leads to issues with the well-definition of Q and R. It is for that reason that we instead interpret Q and R as functors into their respective $\pi\,\mathfrak{C}$ categories, as in this case we need not worry about a choice of lift; we instead map to the homotopy class of the lift.

Definition 4.34. *Let $\mathfrak{C}$ be a model category. Then define the functor $Q : \mathfrak{C} \to \pi\,\mathfrak{C}_C$ via the assignments*

$$X \mapsto QX$$

and

$$f \mapsto \left[\widetilde{f}\right] \in \pi_r(QX, QY).$$

Dually, define the functor $R : \mathfrak{C} \to \pi\,\mathfrak{C}_F$ via the assignments

$$X \mapsto RX$$

and

$$f \mapsto \left[\widehat{f}\right] \in {}_{\ell}\pi(RX, RY).$$

Proposition 4.35. *The mappings Q and R are functors.*

Proof. We prove only the claim about Q; the claim about R will follow analogously. Begin by observing that if we produce the lift $\widetilde{\mathrm{id}_X}$ from the diagram

$$\begin{tikzcd} QX \arrow[r, "\widetilde{\mathrm{id}_X}"] \arrow[d, "p_X"'] & QX \arrow[d, "p_X"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

then by Lemma 4.31 we have that $\widetilde{\mathrm{id}_X} \simeq_r \mathrm{id}_X$, which shows $Q(\mathrm{id}_X) = [\mathrm{id}_X] = \mathrm{id}_{QX}$. Furthermore, Lemma 4.23 states that composition in $\mathfrak{C}$ induces a map on right homotopy groups. Thus $Q(f)Q(g) = [f][g] = [fg] = Q(fg)$ and so Q is a functor. Dualizing these facts by using the appropriate lemmas gives that R is a functor and proves the lemma. □

Remark 4.36. Note that if we restrict the functor Q to $\mathfrak{C}_F$ we obtain a functor $Q\big|_F : \mathfrak{C}_F \to \pi\,\mathfrak{C}_{CF}$; consequently $Q\big|_F$ induces a functor $Q' : \pi\,\mathfrak{C}_F \to \pi\,\mathfrak{C}_{CF}$ by first taking the induced map $\mathfrak{C}_F \to \pi\,\mathfrak{C}_F$ by sending $X \mapsto X$ for every fibrant object and $f : X \to Y$ maps to $[f] : X \to Y \in {}_{\ell}\pi(X, Y)$ and then sending $X \to QX$ and adapting maps into their appropriate two-sided homotopy classes (i.e., $[f] \mapsto [f] \in \pi(QX, QY)$). Dualizing this process shows us that if we restrict R to $\mathfrak{C}_C$ then we obtain a functor $R\big|_C : \mathfrak{C}_C \to \pi\,\mathfrak{C}_{CF}$ and induce a functor $R' : \pi\,\mathfrak{C}_C \to \pi\,\mathfrak{C}_{CF}$ by sending $X \mapsto RX$ and $[f] \mapsto [f] \in \pi(RX, RY)$.

This brings us to define the homotopy category of a model category. This category will have as objects the objects of $\mathfrak{C}$, but will capture appropriately the *two-sided* homotopies of maps between objects in a canonical way.

Definition 4.37. *Let $\mathfrak{C}$ be a model category. We then define the* homotopy category of $\mathfrak{C}$ *to be the category* $\mathrm{h}\mathfrak{C}$ *defined by:*

- $\mathrm{Ob}\,\mathrm{h}\mathfrak{C} := \mathrm{Ob}\,\mathfrak{C}$;
- $\mathrm{h}\mathfrak{C}(X,Y) := \pi\,\mathfrak{C}_{CF}(R'(QX), R'(QY)) = \pi(R(QX), R(QY))$.

Remark 4.38. In this definition we did something unnatural: we *chose* to take the homotopy category $\mathrm{h}\mathfrak{C}$ as having maps $\pi(R(QX), R(QY))$; what if we instead chose to take the morphisms to be $\pi(Q(RX), Q(RY))$? If we define the category $\mathrm{h}'\,\mathfrak{C}$ to be defined as above save that $\mathrm{h}'\,\mathfrak{C}(X,Y) = \pi(Q(RX), Q(RY))$, we can show that $\mathrm{h}\mathfrak{C} \cong \mathrm{h}'\,\mathfrak{C}$ uniquely through the Homotopy Theorem (this is in fact the content of Corollary 5.5). We choose to use the definition of $\mathrm{h}\mathfrak{C}$ given here so that we may closely follow [DS95], but it is worthwhile to observe that if we chose to use $\mathrm{h}'\,\mathfrak{C}$ then all results would carry over through a dual process.

Proposition 4.39 ([DS95])**.** *If $\mathfrak{C}$ is a model category then there is a canonical functor $\gamma : \mathfrak{C} \to \mathrm{h}\mathfrak{C}$ given by the mapping $X \mapsto X$ for all objects $X \in \mathrm{Ob}\,\mathfrak{C}$ and $f \mapsto R'(Q(f))$ for all $f \in \mathrm{Mor}\,\mathfrak{C}$.*

We now seek to understand the structure of the maps in $\mathrm{h}\mathfrak{C}$. Since the objects in $\mathrm{h}\mathfrak{C}$ are the objects in $\mathfrak{C}$ and the maps are homotopy classes $\pi(R(QX), R(QY))$, we would like to know how we can write an arbitrary morphism in $\mathrm{h}\mathfrak{C}$ and exactly what relationship $f \in \mathrm{h}\mathfrak{C}(X,Y)$ has with some $f \in \mathfrak{C}(X,Y)$.

Proposition 4.40 ([DS95])**.** *Let $\mathfrak{C}$ be a model category. Then a morphism $f \in \mathscr{W}$ if and only if $f \in \mathrm{Iso}(\mathrm{h}\mathfrak{C})$. Furthermore, the morphisms in $\mathrm{h}\mathfrak{C}$ are generated under composition by the images of morphisms in $\mathfrak{C}$ under γ and inverses of maps in $\mathscr{W}$ under γ.*

Proof. We begin by proving the first claim. If $f \in \mathscr{W}$ then $R'(Q(f))$ is represented by a map $\widehat{f} : R(QX) \to R(QY) \in \mathscr{W}$. By Proposition 4.26 $\widehat{f}$ has a homotopy inverse $g : R'(QY) \to R'(QX)$, showing that $[f]$ is invertible in $\mathrm{h}\mathfrak{C}$ with $[g] = \gamma(f)^{-1}$. On the other hand, if $\gamma(f)$ is invertible then then the representative map $\widehat{f} : R(QX) \to R(QY)$ has a homotopy inverse; again invoking Proposition 4.26 shows that $\widehat{f} \in \mathscr{W}$. Now consider the diagram:

$$\begin{tikzcd}
R(QX) \arrow[r, "\widehat{f}"] & R(QY) \\
QX \arrow[u, "i_{QX}"] \arrow[r, "\widetilde{f}"] \arrow[d, "p_X"'] & QY \arrow[u, "i_{QY}"'] \arrow[d, "p_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The two-out-of-three axiom gives that since i_{QX}, i_{QY}, and $\widehat{f}$ are all weak equivalences, so is $\widetilde{f}$; using this same derivation for p_X, p_Y, and $\widetilde{f}$ gives that $f \in \mathscr{W}$.

Observe that in $\mathrm{h}\mathfrak{C}$ we have that for all objects $X \in \mathrm{Ob}\,\mathfrak{C}$ that $X \cong R(QX)$ through the map $\gamma(p_X)^{-1}\gamma(i_{QX})$. Recall that since $\mathrm{h}\mathfrak{C}(X,Y) = \pi(R(QX), R(QY))$ and $\pi(R(QX), R(QY))$ is a quotient set of $\mathfrak{C}(R(QX), R(QY))$, the functor γ induces an epimorphism

$$\gamma_{XY} : \mathfrak{C}(R(QX), R(QY)) \to \mathrm{h}\mathfrak{C}(R(QX), R(QY)).$$

Let $f \in \mathrm{h}\mathfrak{C}(X,Y)$. It then follows from the above decomposition and epimorphism that we can give the map f the representation

$$\gamma(p_X)^{-1}\gamma(i_{QX})\gamma\left(\widehat{f}\right)\gamma(i_{QY})^{-1}\gamma(p_Y),$$

where $\widehat{f}$ is the representative map $R(QX) \to R(QY)$ of f. This proves the proposition. $\square$

Lemma 4.41. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories such that $\mathrm{Ob}\,\mathfrak{C} = \mathrm{Ob}\,\mathfrak{D}$, there is a functor $\gamma : \mathfrak{C} \to \mathfrak{D}$ that is the identity on objects, and every morphism g in $\mathfrak{D}$ may be written as a composition*

$$g = \prod_{i=1}^{n} \gamma(f_i)^{e_i}$$

where $e_i \in \{-1, 1\}$, e_i is allowed to take the value -1 only if γf_i is invertible, $1 \leq n \in \mathbb{N}$, and $f_i \in \operatorname{Mor} \mathfrak{C}$ for all i. Then if there are functors $F, G : \mathfrak{C} \to \mathfrak{D}$ such that there is a natural transformation

$$\begin{tikzcd} \mathfrak{D} \arrow[r, bend left, "\gamma F"] \arrow[r, bend right, "\gamma G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathfrak{A} \end{tikzcd}$$

then α may also be regarded as a natural transformation $\alpha : F \to G$.

Proof. Begin by observing that the fact $\alpha : \gamma F \to \gamma G$ is a natural transformation implies that when $h = \gamma(f) \in \mathfrak{D}(X, Y)$ (or when $h = \gamma(f)^{-1}$) for some $f \in \operatorname{Mor} \mathfrak{C}$, we have that the diagram

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Fh"'] & GX \arrow[d, "Gh"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

commutes. Now write

$$h = \prod_{i=1}^{n} \gamma(f_i)^{e_i}$$

so that

$$F(h) = F\left(\prod_{i=1}^{n} \gamma(f_i)^{e_i}\right) = \prod_{i=1}^{n} F\left(\gamma(f_i)^{e_i}\right)$$

and

$$G(h) = G\left(\prod_{i=1}^{n} \gamma(f_i)^{e_i}\right) = \prod_{i=1}^{n} G\left(\gamma(f_i)^{e_i}\right).$$

Then we can "stack diagrams" using this equivalence and square-wise commutativity to give that the diagram

$$\begin{tikzcd} FX \arrow[r, "F(\gamma(f_1)^{e_1})"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "="] & FX_1 \arrow[r, "F(\gamma(f_2)^{e_2})"] \arrow[d, "\alpha_{X_1}"] \arrow[dr, phantom, "="] & \cdots \arrow[r, "F(\gamma(f_{n-1})^{e_{n-1}})"] \arrow[d, phantom, "\vdots"] \arrow[dr, phantom, "="] & FX_{n-1} \arrow[r, "F(\gamma(f_n)^{e_n})"] \arrow[d, "\alpha_{X_{n-1}}"'] \arrow[dr, phantom, "="] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "G(\gamma(f_1)^{e_1})"'] & GX_1 \arrow[r, "F(\gamma(f_2)^{e_2})"'] & \cdots \arrow[r, "G(\gamma(f_{n-2})^{e_{n-1}})"'] & GX_{n-1} \arrow[r, "G(\gamma(f_n)^{e_n})"'] & GY \end{tikzcd}$$

gives that the diagram

$$\begin{tikzcd} FX \arrow[r, "Fh"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "="] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gh"'] & GY \end{tikzcd}$$

commutes. Thus $\alpha : F \to G$ is a natural transformation. $\square$

Corollary 4.42. *If $\mathfrak{C}$ is a model category and $F, G : \mathrm{h}\mathfrak{C} \to \mathfrak{D}$ with a natural transformation*

$$\begin{tikzcd} \mathfrak{C} \arrow[r, bend left, "\gamma F"] \arrow[r, bend right, "\gamma G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathfrak{D} \end{tikzcd}$$

then α may be regarded as a natural transformation $\alpha : F \to G$.

5. Localization of Categories and the Homotopy Theorem

One of the important intuitions about the homotopy of a category is that the homotopy category inverts the weak equivalences of a model category $\mathfrak{C}$. This section seeks to describe the perhaps surprising result that the model structure provides exactly enough structure to produce a homotopy category solely from localization. The other reason is to provide some algebro-geometric intuition as to what homotopy is doing; as localization works for rings and modules, the universal property extends to categories and provides a useful formal analogy. The primary resource for localizations is [Hir09]. However, we do not need the full power of the localization of categories he develops, so we parallel the exposition in [DS95] instead; to go into full detail about localization would make this already long article a good deal longer.

Definition 5.1. *Let $\mathfrak{C}$ be any category and let $S \subseteq \operatorname{Mor}\mathfrak{C}$ be a subclass of morphisms in $\mathfrak{C}$. Then the* localization of $\mathfrak{C}$ at S *is the category, denoted $S^{-1}\mathfrak{C}$, equipped with a natural functor $\lambda_S : \mathfrak{C} \to S^{-1}\mathfrak{C}$ such that if $G : \mathfrak{C} \to \mathfrak{D}$ is any functor satisfying the condition $f \in S$ implies that $G(f)$ is an isomorphism in $\mathfrak{D}$, then there exists a unique functor $\overline{G} : \mathfrak{C} \to \mathfrak{D}$ such that the triangle*

$$\begin{tikzcd} \mathfrak{C} \arrow[rr, "G"] \arrow[dr, "\lambda_S"'] & & \mathfrak{D} \\ & S^{-1}\mathfrak{C} \arrow[ur, "\overline{G}"'] & \end{tikzcd}$$

commutes.

This lemma is useful for studying functors between model categories that preserve certain subclasses of the class $\mathcal{W}$ of weak equivalences in $\mathfrak{C}$. We will see that this lemma greatly simplifies

Lemma 5.2 (Ken Brown's Lemma; [?Hovey]). *Let $\mathfrak{C}$ be a model category and $\mathfrak{D}$ be a category with a class of weak equivalences $\mathcal{W} \subseteq \operatorname{Mor}\mathfrak{D}$ satisfying the two-out-of-three axiom. Then:*

1. *If F takes trivial cofibrations between cofibrant objects to weak equivaelnces, then it maps all weak equivalences between cofibrant objects to weak equivalences in $\mathfrak{D}$.*
2. *If F takes trivial fibrations between fibrant objects to weak equivalences, then it maps all weak equivalences between fibrant objects to weak equivalences in $\mathfrak{D}$.*

Proof. We prove only statement 1; the other follows from duality. Begin by assuming that $A, B \in \mathfrak{C}$ are cofibrant and that $w \in \mathfrak{C}(A, B)$ has $w \in \mathscr{W}$. Then give the map $\langle w, \mathrm{id}_B\rangle : A \coprod B \to B$ a $(\mathscr{C}, \mathscr{F} \cap \mathscr{W})$ factorization

$$\begin{tikzcd} A\coprod B \arrow[r, "c"] \arrow[dr, "\langle w,\mathrm{id}_B\rangle"'] & C \arrow[d, "p"] \\ & B \end{tikzcd}$$

It follows from Lemma 2.14 that since A and B are cofibrant their injections $\iota_0 : A \to A\coprod B$ and $\iota_1 : B \to A\coprod B$ are cofibrations. Since $c : A\coprod B \to C$ is a cofibration and $\mathscr{C}$ is saturated, $\iota_0 c$ and $\iota_1 c$ are both members of $\mathscr{C}$. Since $\mathrm{id}_B, w \in \mathscr{W}$ (note that $\mathrm{id}_B \in \mathscr{W}$ by $\mathscr{W}$ saturated), and we have the identities

$$w = \iota_0\langle w, \mathrm{id}_B\rangle = \iota_0 cp = (\iota_0 c)p$$

and

$$\mathrm{id}_B = \iota_1\langle w, \mathrm{id}_B\rangle = \iota_1 cp = (\iota_1 c)p$$

we have from the two-out-of-three axiom that $\iota_0 c, \iota_1 c \in \mathscr{W}$. Thus $\iota_0 c, \iota_1 c \in \mathscr{W} \cap \mathscr{C}$. By assumption $F(\iota_0 c), F(\iota_1 c) \in \mathcal{W}$. Now

$$\mathrm{id}_{FB} = F(\mathrm{id}_B) = F(\iota_1 cp) = F(\iota_1 c)F(p)$$

so by the two-out-of-three axiom $F(p) \in \mathcal{W}$. Thus the composite $F(\iota_0 c)F(p) = F(\iota_0 cp) = F(w) \in \mathcal{W}$ and the lemma is proved. $\square$

We will move on from here to prove the Homotopy Theorem, which will show that $\mathrm{h}\mathfrak{C}$ is isomorphic to the localization of $\mathfrak{C}$ at $\mathscr{W}$ (with localization γ), which will be the culmination of this article and provide an extremely useful and important tool for understanding $(\infty, 1)$-categories. However, before we can do this we need a quick lemma that will be used in the proof of the Homotopy Theorem.

Lemma 5.3 ([DS95])**.** *Let $\mathfrak{C}$ be a model category and let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor such that $F(\mathscr{W}) \subseteq Iso(\mathfrak{D})$. If $f\,_\ell\!\simeq g : A \to X$ or if $f \simeq_r g : A \to X$ then $F(f) = F(g)$.*

Proof. We prove the case in which $f\,_\ell\!\simeq g$; the other will follow from duality. Begin by choosing a good left homotopy $h : C_A \to X$ from f to g over the good cylinder object $A \coprod A \xrightarrow{c} C_A \xrightarrow{w} A$; note $c \in \mathscr{C}$. Because $i_0 w = \iota_0 c w = \mathrm{id}_A = \iota_1 c w = i_1 w$ it follows that

$$F(i_0)F(w) = F(i_0 w) = F(i_1 w) = F(i_1)F(w).$$

Since $w \in \mathscr{W}$, $F(w)$ is an isomorphism in $\mathfrak{D}$, and the equality $F(i_1)F(w) = F(i_0)F(w)$ implies that $F(i_0) = F(i_1)$. Then we calculate that

$$F(f) = F(i_0 h) = F(i_0)F(h) = F(i_1)F(h) = F(i_1 h) = F(g)$$

and the lemma is proved. □

Theorem 5.4 (Homotopy Theorem; [DS95])**.** *Let $\mathfrak{C}$ be a model category and let $\mathscr{W}$ be the class of weak equivalences in $\mathscr{W}$. Then there is a unique isomorphism*

$$\mathscr{W}^{-1}\,\mathfrak{C} \cong \mathrm{h}\mathfrak{C}.$$

Proof. We show the theorem by first proving that the functor $\gamma : \mathfrak{C} \to \mathrm{h}\mathfrak{C}$ is a localization of $\mathfrak{C}$ at $\mathscr{W}$; from there the conclusion is trivial by considering the diagrams:

$$\begin{tikzcd} \mathfrak{C} \arrow[rr, "\gamma"] \arrow[dr, "\lambda_{\mathscr{W}}"'] & & \mathrm{h}\mathfrak{C} \\ & \mathscr{W}^{-1}\,\mathfrak{C} \arrow[ur, dashed, "\exists! F"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C} \arrow[rr, "\lambda_{\mathscr{W}}"] \arrow[dr, "\gamma"'] & & \mathscr{W}^{-1}\,\mathfrak{C} \\ & \mathrm{h}\mathfrak{C} \arrow[ur, dashed, "\exists! G"'] & \end{tikzcd}$$

In order to show that $\mathrm{h}\mathfrak{C}$ is the localization of $\mathfrak{C}$ at $\mathscr{W}$ we need to first show that γ sends weak equivalences to isomorphisms; however, this is taken care of by Proposition 4.40, so we need to instead show that there if there is a functor $F : \mathfrak{C} \to \mathfrak{D}$ sending weak equivalences to isomorphisms then there is unique functor $\overline{F} : \mathrm{h}\mathfrak{C} \to \mathfrak{D}$ making $\gamma\overline{F} = F$.

We begin this task by first constructing $\overline{F}$. Since $\mathrm{Ob}\,\mathfrak{C} = \mathrm{Ob}\,\mathrm{h}\mathfrak{C}$, we define $\overline{F}X := FX$ for all objects $X \in \mathrm{Ob}\,\mathrm{h}\mathfrak{C}$. To see how to define morphisms we let $f \in \mathrm{h}\mathfrak{C}(X,Y)$ and choose a representative morphism $\widehat{f} : R(QX) \to R(QY)$. By Lemma 5.3 the image $F\left(\widehat{f}\right)$ depends only on the homotopy class $\left[\widehat{f}\right] \in \pi(R(QX), R(QY))$. Since $\widehat{f}$ is well-defined up to homotopy with f, it then follows that $F\left(\widehat{f}\right)$ depends only on f. This allows us to define

$$\overline{F}(f) := F(p_X)^{-1}F(i_{QX})F\left(\widehat{f}\right)F(i_{QY})^{-1}F(p_Y),$$

which is consequently well-defined. We now show that $\overline{F}$ is a functor. It follows from the fact that F sends weak equivalences to isomorphisms that

$$\overline{F}(\mathrm{id}_X) = F(p_X)^{-1}F(i_{QX})F\left(\widehat{\mathrm{id}_X}\right)F(i_{QY})^{-1}F(p_Y) = F(\mathrm{id}_X) = \mathrm{id}_{FX} = \mathrm{id}_{\overline{F}X}.$$

Furthermore, if $f \in \mathrm{h}\mathfrak{C}(X,Y)$ and $g \in \mathrm{h}\mathfrak{C}(Y,Z)$ then

$$\begin{aligned} \overline{F}(f)\overline{F}(g) &= F(p_X)^{-1}F(i_{QX})F\left(\widehat{f}\right)F(i_{QY})^{-1}F(p_Y)F(p_Y)^{-1}F(i_{QY})F\left(\widehat{g}\right)F(i_{QZ})^{-1}F(p_Z) \\ &= F(p_X)^{-1}F(i_{QX})F\left(\widehat{f}\widehat{g}\right)F(i_{QZ})^{-1}F(p_Z) = F(p_X)^{-1}F(i_{QX})F\left(\widehat{fg}\right)F(i_{QZ})^{-1}F(p_Z) \\ &= \overline{F}(fg). \end{aligned}$$

Thus $\overline{F}$ is a functor. To see that $\gamma\overline{F} = F$ let $f = \gamma(h)$ for some $h : X \to Y$ and let $\widehat{f} : R(QX) \to R(QY)$ be a representative map of f. From Lemmas 4.31 and 4.32 we see that after potentially altering $\widehat{f}$ by a right homotopy we can produce a commuting diagram

$$\begin{tikzcd}
X \arrow[r, "h"] & Y \\
QX \arrow[u, "p_X"] \arrow[r, "\widetilde{h}"] \arrow[d, "i_{QX}"'] & QY \arrow[u, "p_Y"'] \arrow[d, "i_{QY}"] \\
R(QX) \arrow[r, "\widehat{f}"'] & R(QY)
\end{tikzcd}$$

Applying the functor F gives

$$F(h) = F(p_X)^{-1}F(i_{QX})F\left(\widehat{f}\right)F(i_{QY})^{-1}F(p_Y) = \overline{F}(f) = \overline{F}(\gamma(h))$$

and proves that $\gamma\overline{F} = F$.

We finally prove that $\overline{F}$ is the unique functor for which $\gamma\overline{F} = F$. To do this let $G : \mathrm{h}\mathfrak{C} \to \mathfrak{D}$ be a functor making the diagram

$$\begin{tikzcd}
\mathfrak{C} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & & \mathfrak{D} \\
& \mathrm{h}\mathfrak{C} \arrow[ur, shift left, "G"] \arrow[ur, shift right, "\overline{F}"'] &
\end{tikzcd}$$

commute. Note that $GX = FX = \overline{F}X$ for all objects $X \in \mathrm{h}\mathfrak{C}$. If $f \in \mathrm{h}\mathfrak{C}(X,Y)$ with representative $f : R(QX) \to R(QY)$ we note that

$$G(f) = G\left(\gamma(p_X)^{-1}\gamma(i_{QX})\gamma\left(\widehat{f}\right)\gamma(i_{QY})^{-1}\gamma(p_y)\right) = F(p_X)^{-1}F(i_{QX})F\left(\widehat{f}\right)F(i_{QY})^{-1}F(p_Y) = \overline{F}(f),$$

proving that $\overline{F} = G$. □

Corollary 5.5. *The categories* $\mathrm{h}\mathfrak{C}$ *and* $\mathrm{h}'\,\mathfrak{C}$ *are uniquely isomorphic.*

Proof. It follows from the Homotopy Theorem that $\mathrm{h}\mathfrak{C} \cong \mathscr{W}^{-1}\,\mathfrak{C}$ uniquely; dualizing gives a unique isomorphism $\mathrm{h}'\,\mathfrak{C} \cong \mathscr{W}^{-1}\,\mathfrak{C}$. Composing these isomorphisms gives a unique isomorphism $\mathrm{h}\mathfrak{C} \cong \mathrm{h}'\,\mathfrak{C}$ and hence proves the corollary. □

Appendix A. Homological Algebraic Results

Two of the most important lemmas in homological algebra are the five lemmas. We present them here for the purpose of completeness, but without proof. When we speak of an Abelian category here we mean an Abelian category in the sense of Freyd. Explicitly we mean the following:

Definition A.1 ([HS97])**.** *A category $\mathfrak{A}$ is said to be* Abelian *if the following conditions hold:*

1. *$\mathfrak{A}$ has a zero object;*
2. *$\mathfrak{A}$ has all finite biproducts;*
3. *Every morphism f in $\mathfrak{A}$ has a kernel and cokernel;*
4. *Every monic in $\mathfrak{A}$ is the kernel of its cokernel;*
5. *Every epic in $\mathfrak{A}$ is the cokernel of its kernel;*
6. *$\mathfrak{A}$ has a factorization system $(\mathscr{E}, \mathscr{M})$ where $\mathscr{E}$ is the class of all epics and $\mathscr{M}$ is the class of all monics, subject to the condition that the natural map* $\operatorname{coker}(\ker f) \to \ker(\operatorname{coker} f)$ *is an isomorphism.*

These categories have brief expositions in both [HS97] and [ML98], but are treated fully in [Fre64]. Abelian categories are the categories where classical homological algebra occurs, and consequently we provide the diagram lemmas below in their full generality. The reason these lemmas hold for arbitrary Abelian categories is because of the Mitchell-Freyd Embedding Theorem, which says that any small Abelian category may be fully, faithfully, and exactly embedded into a module category for some unital ring R. Effectively it means that any lemma involving only a *finite* diagram may be proved module-theoretically.

Definition A.2. *A category $\mathfrak{A}$ is said to be* pre-Abelian *if and only if $\mathfrak{A}$ is an Abelian category, but the natural maps* $\operatorname{Ker}\operatorname{Coker} f \to \operatorname{Coker}\operatorname{Ker} f$ *need not be an isomorphism.*

Example A.3. The category **Ban** of Banach spaces over the field $\mathbb{C}$ is a pre-Abelian category that is not Abelian. In general the image of the natural map $\operatorname{Ker}\operatorname{Coker} f \to \operatorname{Coker}\operatorname{Ker} f$ is only a dense subspace. Another example is seen in the category **TopAb** of Abelian Topological Groups; this category is pre-Abelian but not Abelian.

Lemma A.4 (Short Five Lemma; [ML98], p.202)**.** *Let $\mathfrak{A}$ be an Abelian category and let*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

be a commuting diagram with top and bottom rows exact. Then if f and h are isomorphisms (monics, epics), so is g.

Lemma A.5 (Five Lemma; [HS97])**.** *Let $\mathfrak{A}$ be an Abelian category and let*

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d, "f_1"'] & A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_3"'] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[d, "f_4"] \\
B_0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4
\end{tikzcd}$$

be a commuting diagram with top and bottom rows exact. If f_0 is epic, f_4 is monic, and both of f_1 and f_3 are isomorphisms then f_2 is an isomorphism as well.

Of frequent use are the two four lemmas. Combining them actually proves the Five Lemma in full generality.

Lemma A.6 (Monic Four Lemma)**.** *Let $\mathfrak{A}$ be an Abelian category and let*

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d, "f_0"'] & A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[d, "f_3"] \\
B_0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3
\end{tikzcd}$$

be a commuting diagram with exact rows. If f_0 is epic and both f_1 and f_3 are monic then f_2 is monic as well.

Lemma A.7 (Epic Four Lemma). *Let $\mathfrak{A}$ be an Abelian category and let*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[d, "f_4"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \end{tikzcd}$$

be a commuting diagram with exact rows. If f_4 is monic and both f_1 and f_3 are epic then f_2 is epic as well.

This leads us to the so-called Snake Lemma. The Snake Lemma allows us to, given some exact sequences, relate and connect the resulting kernel and cokernel sequences in an exact way. It can even be used in proving the long exact homology sequence by providing the connecting homomorphisms.

Lemma A.8 (Snake Lemma; [HS97]). *Let $\mathfrak{A}$ be an Abelian category and assume that the diagram*

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z & \end{tikzcd}$$

commutes with both rows exact. Then there is a morphism $\delta \in \mathfrak{A}(\mathrm{Ker}\, h, \mathrm{Coker}\, f)$ making the resulting sequence

$$\mathrm{Ker}\, f \longrightarrow \mathrm{Ker}\, g \longrightarrow \mathrm{Ker}\, h \xrightarrow{\delta} \mathrm{Coker}\, f \longrightarrow \mathrm{Coker}\, g \longrightarrow \mathrm{Coker}\, h$$

exact.

Proposition A.9. *Let $\{P_i \mid i \in I\}$ be a family of projective left R-modules for a ring (not necessarily unital). Then the module*

$$\bigoplus_{i \in I} P_i$$

is projective.

It is worth remarking that this says any sum (coproduct) of projective modules is again projective. Consequently we can characterize projective modules over any unital ring R as *direct sums of direct summands of R.*

Appendix B. The Small Object Argument for $\mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$

In this appendix we recall the gluing construction (without proof) and provide two important results. Information on these constructions may be found in general in [Rie08] and (briefly and without a focus on weak factorization systems) in [DS95].

Definition B.1 ([Rie08]). *Let κ be a regular cardinal. We then say that an object A of a category $\mathfrak{C}$ is* κ-small *if for every ordinal $\lambda \leq \kappa$ and functor $F : \lambda \to \mathfrak{C}$ the natural map*

$$\varinjlim_{\alpha < \lambda} \mathfrak{C}(A, F\alpha) \to \mathfrak{C}\left(A, \varinjlim_{\alpha < \lambda} F\alpha\right)$$

is an isomporphism.

Definition B.2 ([DS95], p.33). *An object A of a category $\mathfrak{C}$ is said to be* sequentially small *if for every functor $F : \mathbb{N} \to \mathfrak{C}$ the natural map*

$$\varinjlim_{n \in \mathbb{N}} \mathfrak{C}(A, Fn) \to \mathfrak{C}\left(A, \varinjlim_{n \in \mathbb{N}} Fn\right)$$

is an isomorphism.

Example B.3. A set S is sequentially small if and only if it is finite.
An R-module A is sequentially small if and only if it has finite presentation.
A nonnegatively graded chain complex $A_\bullet$ in $\mathbf{Ch}_{n\geq 0}(\mathbf{R\text{-}Mod})$ is sequentially small if and only if a finite number of terms A_n are nonzero and all A_n have finite presentation.

The next construction is more to present and introduce notation than to introduce new theory. I have provided it for the sake of completeness and to reduce reference chasing.

Definition B.4 (The Gluing Construction)**.** *Let* $\mathscr{A} := \{f_i : A_i \to B_i \mid i \in I\}$ *be a set of maps in a cocomplete category* $\mathfrak{C}$ *and let* $g \in \mathfrak{C}(X, Y)$*. Let* S_i *be the set of all maps* (h, k) *that make the diagram*

$$\begin{tikzcd} A_i \arrow[r, "h"] \arrow[d, "f_i"'] & X \arrow[d, "g"] \\ B_i \arrow[r, "k"'] & Y \end{tikzcd}$$

commute. Then the first gluing construction $G^1(\mathscr{A}, g)$ *is defined to be the object of the pushout diagram*

$$\begin{tikzcd} \coprod_{i\in I} \coprod_{(h,k)\in S_i} A_i \arrow[r, "\langle h\rangle_{i\in I,(h,k)\in S_i}"] \arrow[d, "\langle f_i\rangle_{i\in I}"'] & X \arrow[d, "i_1"] \\ \coprod_{i\in I} \coprod_{(h,k)\in S_i} B_i \arrow[r, "\langle k\rangle_{i\in I,(h,k)\in S_i}"'] & G^1(\mathscr{A}, g) \end{tikzcd}$$

Note that there is a unique map induced by the pushout $g_1 : G^1(\mathscr{A}, g) \to Y$ *for which* $\iota_1 g_1 = g$*. Proceed in this manner inductively to produce objects, for all* $k > 1$, $G^k(\mathscr{A}, g) := G^1(\mathscr{A}, g_{k-1})$ *where* $g_k := (g_{k-1})_1$*. Then produce a diagram of the form*

$$\begin{tikzcd} G^0(\mathscr{A}, g) := X \arrow[r, "i_1"] \arrow[d, "g"'] & G^1(\mathscr{A}, g) \arrow[r, "i_2"] \arrow[d, "g_1"'] & G^2(\mathscr{A}, g) \arrow[r, "i_3"] \arrow[d, "g_2"'] & \cdots \arrow[r] & G^k(\mathscr{A}, g) \arrow[r, "i_{k+1}"] \arrow[d, "g_k"'] & \cdots \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[r, equal] & \cdots \arrow[r, equal] & Y \arrow[r, equal] & \cdots \end{tikzcd}$$

and define the Infinite Gluing Construction *to be the colimit*

$$G^\infty(\mathscr{A}, g) := \varinjlim_{n\in\mathbb{N}} G^n(\mathscr{A}, g).$$

Note that the infinite gluing construction $G^\infty(\mathscr{A}, g)$ comes equipped with maps $i_\infty : X \to G^\infty(\mathscr{A}, g)$ and $g_\infty : G^\infty(\mathscr{A}, g) \to Y$ for which $g = i_\infty g_\infty$.

Proposition B.5 ([Rie08],p.xy)**.** *Let* $\mathscr{A}$ *be a set of maps*

$$\mathscr{A} := \{f_i \in \mathfrak{C}(A_i, B_i) \mid i \in I\}$$

for objects A_i, B_i *in a cocomplete category* $\mathfrak{C}$ *with a map* $g : X \to Y$*. Then if* A_i *is sequentially small in* $\mathfrak{C}$ *for every* $i \in I$ *the map* $g_\infty : G^\infty(\mathscr{A}, g) \to Y$ *has the right lifting property with every map* $f_i \in \mathscr{A}$*.*

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