Notes on Stacks and Stuff

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1 Sites and Sheaves on Sites

We begin by describing some basic definitions, notations, and properties involving the functor category

$$[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] := \mathbf{Cat}(\mathscr{C}, \mathbf{Set}),$$

as well as some notation used throughout this chapter.

Definition 1.1. A *Presheaf (of sets) on a category* $\mathscr{C}$ is a functor $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$, and the *presheaf category over* $\mathscr{C}$ is the category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.

Proposition 1.2. *If* $\mathscr{C}$ *is a category then* $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ *is a locally small topos. In particular,* $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ *is complete, cocomplete, and Cartesian closed.*

An important aspect of this fact is that limits and colimits in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ are calculated pointwise (locally). In particular, if $\{P_i \mid i \in I\}$ is a family of presheaves and $\{\alpha_{ij} : P_i \to P_j \mid i, j \in I\}$ is a family of morphisms (by definition, natural transformations) between presheaves, then the limit

$$\varprojlim_{i \in I} P_i$$

is calculated by defining, for all $U \in \operatorname{Ob} \mathscr{C}$,

$$\left(\varprojlim_{i \in I} P_i\right)(U) := \varprojlim_{i \in I}(P_i(U)).$$

Calculating colimits is, of course, done in the same manner, and the right adjoint $[P, -]$ of the product functor $(-) \times P$ on $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is induced by the equation, for all $U \in \operatorname{Ob} \mathscr{C}$,

$$[P, F](U) := [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]\big(\mathscr{C}(-, U) \times P, F\big).$$

Definition 1.3. If $\mathscr{C}$ is a category and $U \in \operatorname{Ob} \mathscr{C}$, define the set

$$\operatorname{Codom} U := \{\varphi \in \operatorname{Mor} \mathscr{C} \mid \operatorname{Codom} \varphi = U\} \subseteq \operatorname{Mor} \mathscr{C}\,.$$

Definition 1.4. The covariant power-set functor on $\mathbf{Set}$ will be denoted

$$\mathcal{P} : \mathbf{Set} \to \mathbf{Set}\,.$$

1.1 Sites, Sieves, and Sheaves

The axioms of a cover we will insist upon will come from three fairly reasonable geometric insights:

1. If $\varphi : V \to U$ is an isomorphism in $\mathscr{C}$, then it had better be a cover;

2. If $\{\varphi_i : U_i \to U \mid i \in I\}$ is a cover of U and, for each $i \in I$, $\{\psi_{ij} : V_{ij} \to U_i \mid j \in J_i\}$ is a cover of U_i, then the set of composite maps
$$\{\varphi_i \circ \psi_{ij} : V_{ij} \to U \mid i \in I, j \in J_i\}$$
had better be a cover of U;

3. If $\{\varphi_i : U_i \to U \mid i \in I\}$ is a cover of U and if $\rho : V \to U$ is any map, then when we consider all intersections
$$\begin{tikzcd} V \times_U U_i \arrow[r, "\pi_{2,i}"] \arrow[d, "\pi_{1,i}"'] & U_i \arrow[d, "\varphi_i"] \\ V \arrow[r, "\rho"'] & U \end{tikzcd}$$
simultaneously, we had better have that $\{\pi_{1,i} : V \times_U U_i \to V \mid i \in I\}$ is a cover of V.

Definition 1.5 ([3], [5])**.** A *Grothendieck pretopology τ (or a basis τ to a Grothendieck topology)* on a category $\mathscr{C}$ with fibre products is a collection of sets $\tau(U) \subseteq \mathcal{P}^2(\operatorname{Codom} U)$ (called basic covers of U) such that:

1. For all isomorphisms φ in $\mathscr{C}$ with $\operatorname{Codom}(\varphi) = U$, the set $\{\varphi : V \to U\} \in \tau(U)$;

2. If $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ and if $\{\psi_{ij} : V_{ij} \to U_i \mid j \in J_i\} \in \tau(U_i)$ for all $i \in I$, then
$$\{\varphi_i \circ \psi_{ij} : V_{ij} \to U \mid i \in I, j \in J_i\} \in \tau(U);$$

3. If $\rho \in \mathscr{C}(V, U)$ and if $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ then $\{\pi_{1,i} : V \times_U U_i \to V \mid i \in I\} \in \tau(V)$, where $\pi_{1,i}$ is the projection in the pullback:
$$\begin{tikzcd} V \times_U U_i \arrow[r, "\pi_{2,i}"] \arrow[d, "\pi_{1,i}"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_i \arrow[d, "\varphi_i"] \\ V \arrow[r, "\rho"'] & U \end{tikzcd}$$

Example 1.6. Let $\mathscr{C} := \mathbf{Open}(X)$ be the open lattice of a topological space X. Then define a pretopology τ on $\mathbf{Open}(X)$ by saying that $\{U_i \to U \mid i \in I\} \in \tau(U)$ if and only if
$$\bigcup_{i \in I} U_i = U.$$
We can easily check conditions (1) and (2): Since the only isomorphisms in a poset category are the identity, (1) follows from the fact that
$$U = \bigcup_{i \in \{i\}} U,$$
while if $\{U_i \to U \mid i \in I\} \in \tau(U)$ and for all $i \in I, \{U_{ij} \to U_i \mid j \in J_i\} \in \tau(U_i)$ then
$$\bigcup_{i \in I} \bigcup_{j \in J_i} U_{ij} = \bigcup_{i \in I} \left(\bigcup_{j \in J_i} U_{ij} \right) = \bigcup_{i \in I} U_i = U$$

shows that $\{U_{ij} \to U \mid i \in I, j \in J_i\} \in \tau(U)$. Finally, (3) holds from the following: If $\{U_i \to U \mid i \in I\} \in \tau(U)$ and if $V \to U$ is an arrow in $\mathbf{Open}(X)$, then $U_i \times_U V = U \cap V$. Moreover, the arrow $U_i \times_U V \to V$ is a subset inclusion of $U \cap V \to V$, so since the U_i cover U by De Morgan's Laws we have that

$$\bigcup_{i\in I}(U_i \times_U V) = \bigcup_{i \in I}(U_i \cap V) = \left(\bigcup_{i\in I} U_i\right) \cap V = U \cap V.$$

However, since $V \to U$ is an arrow in $\mathbf{Open}(X)$, $V \subseteq U$ and so $V \cap U = V$ and hence $\{U_i \times_U V \to V \mid i \in I\} \in \tau(V)$.

Example 1.7. Let $\mathscr{C} = \mathbf{AffSch}$ be the category of affine schemes and define a pretopology $f\acute{e}t$ on $\mathscr{C}$ by saying that a collection of morphisms of schemes $\{\varphi_i : U_i \to U \mid i \in I\} \in f\acute{e}t(U)$ if and only if the map

$$[\varphi_i]_{i\in I} : \coprod_{i \in I} U_i \to U$$

is surjective (so the φ_i are jointly surjective) and such that each φ_i is finite étale. This is the *Finite Étale* pretopology on **AffSch**. We can also define the Étale pretopology, *ét*, on **AffSch** by taking covers to be jointly surjective étale maps of schemes.

Example 1.8. Let $\mathscr{C} = \mathbf{Sch}$ and define the flat pretopology $fppf$ (for "fidèlement plat de présentation finie," meaning faithfully flat and of finite presentation) to be given by saying that if $U \in \operatorname{Ob}\mathbf{Sch}$ is affine, then $\{\varphi_i : U_i \to U \mid i \in I\} \in fppf(U)$ if and only if the φ_i are jointly surjective, each X_i is affine, and each φ_i is flat and finitely presented. For arbitrary schemes X, we say that $\{\varphi_i : X_i \to X \mid i \in I\} \in fppf(U)$ if and only if the cover happens to be an fppf cover after base changing to an open affine subscheme of X.

Now that we have described pretopologies, and even seen a few examples, we will show how to define sheaves on these pretopologies. We can do this largely in the same way that we define sheaves in the case of a topological space, but we will see that pretopologies have an unfortunate imprecision: Sometimes it is the case that distinct pretopologies give rise to the same categories of sheaves. This can lead to some frustrating consequences, but after we show some examples of distinct pretopologies giving rise to the same sheaves, we will show how to avoid this imprecision: Through Grothendieck topologies.

Definition 1.9. We say that a presheaf F on $\mathscr{C}$ is a *sheaf with respect to the pretopology* τ if given any covering family $\{U_i \xrightarrow{\varphi_i} U \mid i \in I\} \in \tau(U)$, the diagram

$$\begin{tikzcd} FU \arrow[r, "\langle F\varphi_i\rangle_{i\in I}"] & \prod\limits_{i\in I} FU_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{i,j\in I} F(U_i \times_U U_j) \end{tikzcd}$$

is an equalizer in **Set**.

Note that the morphisms p and q come from the following pairing maps: If i and j are any fixed indeces of I, then there are maps $(\pi_1)_{ij} : U_i \times_U U_j \to U_i$ and $(\pi_2)_{ij} : U_i \times_U U_j \to U_j$; iterating over all such pairs allows us to consider the diagrams

$$\begin{tikzcd} FU_i \arrow[r, "F(\pi_1)_{ij}"] & F(U_i \times_U U_j) \\ \prod\limits_{i\in I} FU_i \arrow[u, "\pi_i"] \arrow[r, "\langle F(\pi_1)_{ij}\rangle_{i\in I}"'] & \prod\limits_{i,j\in I} F(U_i \times_U U_j) \arrow[u, "\pi_{ij}"'] \end{tikzcd}$$

and

$$\begin{tikzcd}
FU_j \arrow[r, "F(\pi_2)_{ij}"] & F(U_i \times_U U_j) \\
\prod\limits_{i\in I} FU_i \arrow[u, "\pi_2"] \arrow[r, "\langle F(\pi_2)_{ij}\rangle_{j\in I}"'] & \prod\limits_{i,j\in I} F(U_I \times_U U_j) \arrow[u, "\pi_{ij}"']
\end{tikzcd}$$

The two of these together induce p and q as

$$p := \langle F(\pi_1)_{ij} \circ \pi_i\rangle_{i,j\in I}$$

and

$$q := \langle F(\pi_2)_{ij} \circ \pi_j\rangle_{i,j\in I},$$

respectively.

Definition 1.10. A morphism of sheaves F and G on a pretopology τ is simply a natural transformation $\varphi : F \to G$ in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$. The category full subcategory of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ of all τ-sheaves is denoted by $\mathbf{Shv}(\mathscr{C}, \tau)$.

Example 1.11. Let τ be a Grothendieck topology on $\mathscr{C}$ and define a new pretopology τ' by saying, for $U \in \operatorname{Ob}\mathscr{C}$, that $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau'(U)$ if and only if there exists an index set J_i for each $i \in I$ there is a set of morphisms $\{\psi_{ij} : U_{ij} \to U_i \mid j \in J_i\}$ such that the composites $\{\varphi_i \circ \psi_{ij} : U_{ij} \to U \mid i \in I, j \in J_i\} \in \tau(U)$. One can then show that a τ-sheaf is a τ'-sheaf, and vice-versa.

Example 1.12. If $\mathscr{C} = \mathbf{AffSch}$, consider the *fppf* pretopology given above, i.e., τ is defined by saying that $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ if and only if the φ_i are jointly surjective and each φ_i is flat and finitely presented. Then we define a pretopology τ' by saying that $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau'(U)$ if and only if $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ and each φ_i is quasi-finite. Then τ and τ' generate the same sheaf categories by Corollary 17.16.2 of [2].

As the above example shows, it can be quite annoying to try and work with sheaves of pretopologies because it can be the case that some very different looking covers (in the sense that quasi-finiteness, for example, is a reasonably strong assumption to ask) can generate the same sheaves. However, if one moves to work with Grothendieck topologies, one removes this frustration altogether! However, to get to Grothendieck topologies, we first need to learn about sieves.

Definition 1.13. A *Sieve S on an Object* $U \in \operatorname{Ob}\mathscr{C}$ is a subfunctor of $\mathbf{y}(U) := \mathscr{C}(-, U)$.

Remark 1.14. We will use some somewhat abusive notation and such when working with sieves; however, this abuse is common in the literature and does make the manipulation easier, and we will use it consistently throughout these notes. A sieve S on $U \in \operatorname{Ob}\mathscr{C}$ can be described as a set of morphisms, all with codomain U, such that if $f : V \to U \in S$ and $g : W \to V$, then $f \circ g \in S$, i.e., the following deduction holds:

$$\frac{f \in S \qquad g \in \operatorname{Mor}\mathscr{C} \,.\, \operatorname{Dom}(f) = \operatorname{Codom}(g)}{f \circ g \in S}$$

To see why, consider that since S is a subfunctor of $\mathscr{C}(-, U)$, for all $V \in \operatorname{Ob}\mathscr{C}$, $S(V) \subseteq \mathscr{C}(V, U)$ and for all $\varphi : W \to V$, there is a commuting diagram

$$\begin{tikzcd}
S(W) \arrow[r, "i_W"] & \mathscr{C}(W, U) \\
S(V) \arrow[u, "\varphi^*"] \arrow[r, "i_V"'] & \mathscr{C}(V, U) \arrow[u, "\varphi^*"']
\end{tikzcd}$$

where the natural transformation i is induced by the subset inclusion $S(X) \subseteq \mathscr{C}(X,U)$ for all $X \in \mathrm{Ob}\,\mathscr{C}$, and φ^* acts by pre-composition by φ. Defining, and perversely abusing notation in the process,

$$S := \bigcup_{V \in \mathrm{Ob}\,\mathscr{C}} S(V)$$

then allows us to see from the naturality square above that if $f \in S$, then for all $g \in \mathrm{Mor}\,\mathscr{C}$ with $\mathrm{Codom}(g) = \mathrm{Dom}(f)$, $f \circ g \in S$.

On the other hand, it is possible to build a subfunctor of $\mathscr{C}(-,U)$ from a pre-composition closed subset of $\mathrm{Codom}\,U$ by taking $S(V) := \{\varphi \in S \mid \mathrm{Dom}(\varphi) = V\}$, which shows that sieves and pre-composition closed subsets of $\mathrm{Codom}\,U$ are one and the same. Of course, in this way there is an equality

$$\frac{\mathscr{C}(-,U)}{\overline{\mathrm{Codom}\,U}}$$

but we will use the functor $\mathscr{C}(-,U)$ when we want to emphasize the functorial definition of a sieve and $\mathrm{Codom}\,U$ when we want to use the more set-theoretic definition of a sieve.

Definition 1.15. If S is a sieve on $U \in \mathrm{Ob}\,\mathscr{C}$ and if $\rho \in \mathscr{C}(V,U)$, then the *pullback sieve* $\rho^*(S)$ is the set

$$\rho^*(S) := \{\varphi \in \mathrm{Mor}\,\mathscr{C} \mid \mathrm{Codom}(\varphi) = V, \rho \circ \varphi \in S\}.$$

Lemma 1.16. *If S is a sieve on U and if $\rho \in \mathscr{C}(V,U)$ then $\rho^*(S)$ is a sieve on V.*

Proof. To see this note that if $\rho : V \to U$ is a morphism in $\mathscr{C}$, then there is an induced pullback diagram

$$\begin{tikzcd}
\rho^* S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d] \\
\mathscr{C}(-,V) \arrow[r, "\mathscr{C}(-,\rho)"'] & \mathscr{C}(-,U)
\end{tikzcd}$$

in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$. Because the morphism $S \to \mathscr{C}(-,U)$ is monic, so is the morphism $\rho^*(S) \to \mathscr{C}(-,V)$, which implies that $\rho^*(S)$ is a subfunctor of $\mathscr{C}(-,V)$ and hence a sieve on V. $\square$

Definition 1.17 ([3],[5])**.** A *Grothendieck Topology* J on a category $\mathscr{C}$ is a collection, for all $U \in \mathrm{Ob}\,\mathscr{C}$, of *covering sieves* on U called $J(U)$ such that:

1. The maximal sieve
$$\mathrm{Codom}\,U \in J(U);$$

2. If $S \in J(U)$ and if $\rho : V \to U$, the $\rho^*(S) \in J(V)$;

3. If $S \in J(U)$ and if R is any sieve on U such that for all $\varphi : \mathrm{Dom}\,\varphi \to U \in S$, $\varphi^*(R) \in J(\mathrm{Dom}\,\varphi)$, then $R \in J(U)$.

Remark 1.18. The definition of a Grothendieck and its covering sieves can be rephrased in terms of arrows. In this way we think of a sieve S on U covering a morphism $\rho : V \to U$ if $\rho \in S$. In this way we can rephrase a Grothendieck topology as a collection of covering sieves $S \in J(U)$, for all $U \in \mathrm{Ob}\,\mathscr{C}$, such that:

1. If S is a sieve on U and if $f : V \to U \in S$, then S covers f;

2. If S covers $f : V \to Y$, it covers any composite $f \circ g : W \to U$, where $g : W \to V$.

3. If S covers $f : V \to U$ and R is a sieve on U which covers every $\rho \in S$, then R covers f.

Remark 1.19. One useful aspect of a Grothendieck topology is that we no longer have to assume that the underlying category $\mathscr{C}$ has fibre products. This means that we can naïvely find more Grothendieck topologies than we can find pretopologies, although a different approach to fix the pretopologies-require-pullbacks problem is to embed $\mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ through the Yoneda Embedding $\mathbf{y}$ and then place a pretopology on $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ instead.

Definition 1.20. A *site* is a choice of category $\mathscr{C}$ and Grothendieck topology J on $\mathscr{C}$, and is denoted $(\mathscr{C}, J)$.

We say that if $(\mathscr{C}, J)$ is a site and if $R, S \in J(U)$ for some $U \in \operatorname{Ob}\mathscr{C}$ then a sieve refines R and S if $T \in J(U)$ and $T \subseteq R, T \subseteq S$. The next proposition we will describe shows that any two covering sieves on an object $U \in \operatorname{Ob}\mathscr{C}$ have a common refinement which is also a covering sieve of U.

Proposition 1.21. *If $(\mathscr{C}, J)$ is a site and $R, S \in J(U)$, then $R \cap S$ is a sieve and $R \cap S \in J(U)$.*

Proof. The fact that $R \cap S$ is a sieve is immediate, while the fact that $R \cap S \in J(U)$ may be deduced from the third axiom for Grothendieck topologies. □

Example 1.22. If $\mathscr{C}$ is a category, the trivial topology on $\mathscr{C}$ has the property that $S \in J(U)$ if and only if $S = \operatorname{Codom} U$. Evidently, J is the smallest topology on $\mathscr{C}$.

One unfortunate aspect of the axioms of a Grothendieck topology, as opposed to a Grothendieck pretopology, is that at a surface level they seem to have nothing, or at least very little, to do with each other. However, we will now show how to build a Grothendieck topology from a pretopology, as well as how to find a pretopology that generates a given topology (when the underlying category has fibre products, anyway).

Definition 1.23. Let $\mathscr{C}$ be a category with fibre products and let τ be a pretopology on $\mathscr{C}$. Now define a collection of covering sieves J on $\mathscr{C}$ by saying that, for some object $U \in \operatorname{Ob}\mathscr{C}$, a sieve $S \in J(U)$ if and only if there exists a cover $K \in \tau(U)$ such that $K \subseteq S$. The collection J is then said to be the *Grothendieck topology generated by τ*.

Theorem 1.24. *The collection J defined in Definition 1.23 is a Grothendieck topology on $\mathscr{C}$.*

Proof. (1): Begin by observing that since $\{\mathrm{id}_U : U \to U\}$ is an isomorphism, $\{\mathrm{id}_U\} \in \tau(U)$, so $\tau(U) \neq \varnothing$. Moreover, if $\operatorname{Codom} U$ is the maximal sieve on U, then $\{\mathrm{id}_U\} \subseteq \operatorname{Codom} U$, so $\operatorname{Codom} U \in J(A)$.

(2): Let $S \in J(U)$ and let $\rho \in \mathscr{C}(V, U)$. Find a cover $C := \{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ such that $K \subseteq S$. Then define C' to be the cover

$$C' := \{\pi_{1,i} : V \times_U U_i \to V \mid i \in I\},$$

where the pullback morphisms come from the diagram:

$$\begin{tikzcd} V \times_U U_i \arrow[r, "\pi_{2,i}"] \arrow[d, "\pi_{1,i}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & U_i \arrow[d, "\varphi_i"] \\ V \arrow[r, "\rho"'] & U \end{tikzcd}$$

Moreover, note that $C' \in \tau(V)$ and that from the commutativity of the pullback diagram we have that

$$\rho \circ \pi_{1,i} = \varphi_i \circ \pi_{2,i} \in S.$$

Thus each $\pi_{1,i} \in \rho^*(S)$ so $C' \subseteq \rho^*(S)$, proving that $\rho^*(S) \in J(V)$.

(3): Let $U \in \operatorname{Ob}\mathscr{C}$ and assume that R is a sieve on U such that there exists an $S \in J(U)$ with the property that for all $\rho \in S, \rho^*(R) \in J(\operatorname{Dom}\rho)$. So, since $S \in J(U)$, there exists a cover

$$K := \{\varphi_i : U_i \to U \mid i \in I\} \in \tau U$$

with $K \subseteq S$. Then, for all $i \in I$, we have that $\varphi_i^*(R) \in J(U_i)$; for each fixed $i \in I$, fix a cover

$$K_i := \{\varphi_{ij} : U_{ij} \to U_i \mid j \in J_i\} \in \tau U_i$$

such that $K_i \subseteq \varphi_i^*(R)$. Observe on one hand that it follows by definition that for each $\varphi_{ij} \in K_i, \varphi_i \circ \varphi_{ij} \in R$, while because each $K_i \in \tau U_i$ and because $K \in \tau U$, the cover

$$C := \{\varphi_i \circ \varphi_{ij} : U_{ij} \to U \mid i \in I, j \in J_i\} \in \tau U.$$

Thus, because each composite $\varphi_i \circ \varphi_{ij} \in R$, $C \subseteq R$ and so $R \in J(U)$. □

Now, in order to show how to give a maximal pretopology that generates a Grothendieck topology, assume that τ is a pretopology on a category $\mathscr{C}$ with fibre products, and let $C \in \tau(U)$. Now define

$$(C) := \{\varphi \circ \psi \mid \varphi \in C, \operatorname{Dom}\varphi = \operatorname{Codom}\psi\};$$

it is easy to show that (C) is a sieve, and that (C) is intimately related to C.

Definition 1.25. If τ is a pretopology on $\mathscr{C}$ and $C \in \tau(U)$ for some $U \in \operatorname{Ob}\mathscr{C}$, then we say that (C) is *the sieve on U generated by C.*

Proposition 1.26. *Let $\mathscr{C}$ be a category with fibre products and let $(\mathscr{C}, J)$ be a site. Then there exists a unique maximal pretopology τ which generates J.*

Sketch. Define the pretopology τ by saying that a cover $C := \{\varphi_i : U_i \to U \mid i \in I\}$ satisfies the rule

$$C \in \tau(U) \iff (C) \in J(U).$$

Showing that τ is a pretopology is straightforward using the three axioms of the topology J. Furthermore, the maximality of τ with respect to inclusion is also straightforward. □

Example 1.27. If $\mathscr{C} = \textbf{AffSch}$, then the finite étale, étale, and fppf topologies are the topologies on $\mathscr{C}$ generated by the corresponding pretopologies given in Examples 1.7 and 1.8.

We will now move on from discussing pretopologies and topologies to discuss the sheaves over top them. Just like how a sheaf on a space respects the fact that the inclusions $U \cap V \to U$ and $U \cap V \to V$ should be in a very real sense "the same," we would like it to be the case for sheaves on a topology to be the same on the "inclusions" $\pi_1 : U_i \times_U U_j \to U_i$ and $\pi_2 : U_i \times_U U_j \to U_j$ whenever the maps $\varphi_I : U_i \to U$ and $\varphi_j : U_j \to U$ are covered by a sieve $S \in J(U)$. However, this description unfortunately has the requirement that $\mathscr{C}$ admit fibre products, and we should instead be able to do this in any category. As such, we are going to work with covering sieves $S \in J(U)$ and work with gluing conditions that use the fact that $f \circ g \in S$ for all $f \in S$ and $g \in \operatorname{Mor}\mathscr{C}$ with $\operatorname{Dom} f = \operatorname{Codom} g$ to generalize how to deal with intersections in categories without fibre products.

In order to examine how this general gluing is best gone about, assume that $(\mathscr{C}, J)$ is a site and let $S \in J(U)$ be a covering sieve of U. Then, for any presheaf $P \in [\mathscr{C}^{\mathrm{op}}, \textbf{Set}]$ we can consider the object

$$\prod_{f \in S} P(\operatorname{Dom} f),$$

which should play the role of considering the product of a family PU_i for a cover $\{U_i \to U \mid i \in I\}$. We now, perhaps unfortunately, need to replace the fibre product induced intersections in order to proceed with defining general sheaves on J. However, this is not as daunting as one may expect; since S is a sieve, for all $g \in \operatorname{Mor}\mathscr{C}$, whenever $f \in S$ if $\operatorname{Dom} f = \operatorname{Codom} g$ then $f \circ g \in S$. Thus we should make sure that if a morphism $f \in S$ also is equal to a composite $f' \circ g$ for $f' \in S$ and some $g \in \operatorname{Mor}\mathscr{C}$, the action of P had better be the same on its "gluings." That is, we should consider the doubly indexed product

$$\prod_{\substack{f \in S, g \in \operatorname{Mor}\mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P(\operatorname{Dom} g)$$

in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, together with two maps from the product of the $P(\operatorname{Dom} f)$, for $f \in S$, that encapsulate the fact that P had better not destroy inclusions. To how to go about this we will construct two parallel morphisms

$$\prod_{f \in S} P(\operatorname{Dom} f) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{\substack{f \in S, g \in \operatorname{Mor} \mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P(\operatorname{Dom} g).$$

where p and q are induced in two different ways that had best be the same in order for P to respect gluings of covers.

In order to see how to define the maps p and q, assume that an element

$$(x_f)_{f \in S} \in \prod_{f \in S} P(\operatorname{Dom} f)$$

is given. On one hand, we can define a map simply based on consider the action on the $P(\operatorname{Dom} f)$ induced by the sieve axiom; that is, define p by the equation

$$p(x_f)_{f \in S} := (x_{f \circ g})_{f \in S, g \in \operatorname{Mor} \mathscr{C}, \operatorname{Dom} f = \operatorname{Codom} g}.$$

The map q, on the other hand, is defined by the action of $P(g)$ on each of the x_f; that is, whenever $g \in \operatorname{Mor} \mathscr{C}$ with $\operatorname{Dom} f = \operatorname{Codom} g$, there is a map $P(g) : P(\operatorname{Dom} f) \to P(\operatorname{Dom} g)$; taking the pairing map indexed over all such pairs then gives a morphism which is defined by the equation

$$q(x_f)_{f \in S} := \big(P(g)(x_f)\big)_{f \in S, g \in \operatorname{Mor} \mathscr{C}, \operatorname{Dom} f = \operatorname{Codom} g}.$$

In this way, if S is a covering sieve on U, we can produce a **likely noncommutative** diagram

$$PU \xrightarrow{e} \prod_{f \in S} P(\operatorname{Dom} f) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{\substack{f \in S, g \in \operatorname{Mor} \mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P \operatorname{Dom} g$$

where e is simply the pairing map of the presheaf morphism attached to each morphism in S, i.e.,

$$e = \langle P(f) \rangle_{f \in S}.$$

In order for P to be a sheaf, it had better be the case that the actions of p and q be the same on PU. However, this is the same thing as saying that the diagram above is an equalizer! This motivates our definition of a sheaf on a general Grothendieck topology, and shows how it is constructed.

Definition 1.28. Let $(\mathscr{C}, J)$ be a site. We then say that a presheaf $P \in [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a *sheaf on the J-topology, or a J-sheaf,* if for every $U \in \operatorname{Ob} \mathscr{C}$ and for every $S \in J(U)$, the induced diagram

$$PU \xrightarrow{e} \prod_{f \in S} P(\operatorname{Dom} f) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{\substack{f \in S, g \in \operatorname{Mor} \mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P \operatorname{Dom} g$$

is an equalizer in **Set**.

Remark 1.29. This remark serves to connect the definition of sheaf given classically (cf. [3] and [5]) to the equalizer definition given above, as well as so that we can use some of that classical theory to prove various things about sheaves in general. In the classical language, a sheaf was defined as a presheaf P such that whenever S was a sieve on $U \in \operatorname{Ob} \mathscr{C}$, given a natural transformation $\alpha : S \to P$, P was a sheaf if there was a unique natural transformation $\beta : \mathscr{C}(-, U) \to P$ making the diagram

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d] & P \\ \mathscr{C}(-,U) \arrow[ur, dashed, "\exists ! \beta"'] & \end{tikzcd}$$

commute.

The reasons that our two definitions of sheaf are equivalent are as follows: If S is a sieve on $U \in \operatorname{Ob}\mathscr{C}$ and P is a presheaf on $\mathscr{C}$, a natural transformation $\alpha : S \to P$ is a collection of morphisms (by the Yoneda Lemma) assigning, for each $f \in S$ an element $x_f \in P(\operatorname{Dom} f)$ such that if $g \in \mathscr{C}(W, \operatorname{Dom} f)$ then $P(g)(x_f) = x_{f \circ g}$. That is to say, a natural transformation α allows us to infer

$$\frac{V \xrightarrow{f} U \in S(V) \subseteq \mathscr{C}(V, U)}{x_f \in P(V), x_f = \alpha(f)}$$

such that:

$$\frac{W \xrightarrow{g} V \in \operatorname{Mor}\mathscr{C} \qquad V \xrightarrow{f} U \in S}{P(g)(x_f) = x_{f \circ g}}$$

In this way a unique extension of $\alpha : S \to P$ to a natural transformation $\mathscr{C}(-, U) \to P$ is the same as a unique element $x \in P(U)$ such that if $f \in S$ then

$$P(f)(x) = x_f.$$

Given such a condition, it then follows that if such an x is given, $e(x) = (x_f)_{f \in S}$ *and*

$$\begin{aligned}(p \circ e)(x) &= p(e(x)) = p(x_f)_{f \in S} = (x_{f \circ g})_{f \in S, g \in \operatorname{Mor}\mathscr{C}, \operatorname{Dom} f = \operatorname{Codom} g} \\ &= (P(g)(x_f))_{f \in S, g \in \operatorname{Mor}\mathscr{C}, \operatorname{Dom} f = \operatorname{Codom} g} = q(x_f)_{f \in S} \\ &= (q \circ e)(x),\end{aligned}$$

while the universal property of the equalizer follows from the uniqueness of the extension. Similarly, if we have sheaf we have defined, the unique (because equalizers in **Set** are monic, of course) lift of the $(x_f)_{f \in S} = e(x)$, for some $x \in P(U)$, defines the extension of the natural transformation $\alpha : S \to P$ induced by the sheaf axiom and the morphism p: $(x_f)_{f \in S} := (\alpha(f))_{f \in S}$.

It is unfortunate, but one difficulty that comes with working with the full weight of sheaves on a site $(\mathscr{C}, J)$ is that it can be quite awkward and very complicated to check that for all covering sieves S on $U \in \operatorname{Ob}\mathscr{C}$, the diagram

$$PU \xrightarrow{e} \prod_{f \in S} P(\operatorname{Dom} f) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{\substack{f \in S, g \in \operatorname{Mor}\mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P \operatorname{Dom} g$$

not only commutes, but is an equalizer. This is especially the case when you know very little about general covering sieves, and instead only know things about certain covers in a given pretopology τ. However, in the case where you do have a pretopology τ generating J, one's work is greatly simplified: You only need to check on the covers of $\tau(U)$ instead. This is the content of our next proposition, which has a simple and straightforward proof that only requires one to unravel the definitions.

Theorem 1.30. *Let $\mathscr{C}$ be a category with fibre products and let $(\mathscr{C}, J)$ be a site generated by the pretopology τ. Then F is a J-sheaf if and only if for all covers $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$, the diagram*

$$FU \xrightarrow{e} \prod_{i \in I} FU_i \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j \in I} F(U_i \times_U U_j)$$

is an equalizer.

Remark 1.31. In the proof of the theorem above, we will frequently be referring to the pullbacks $U_i \times_U U_j$ arising from morphisms $\varphi_i : U_i \to U$ and $\varphi_j : U_j \to U$ in a cover $\{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$. Thus, when we write the morphisms $\pi_{1,ij} : U_i \times_U U_j \to U_i$ and $\pi_{2,ij} : U_i \times_U U_j \to U_j$ we mean the first and second projections coming from the diagram

$$\begin{tikzcd}
U \times_U U_j \arrow[r, "\pi_{2,ij}"] \arrow[d, "\pi_{1,ij}"'] & U_j \arrow[d, "\varphi_j"] \\
U_i \arrow[r, "\varphi_i"'] & U
\end{tikzcd}$$

in $\mathscr{C}$.

Proof. $\Longrightarrow$: Assume that F is a J-sheaf and let $C := \{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ be a cover of U. Now find a collection of elements $\{x_i \in FU_i \mid i \in I\}$ such that for all $i, j \in I$ the equation

$$F(\pi_{1,ij})(x_i) = F(\pi_{2,ij})(x_j)$$

holds. Now consider the sieve

$$(C) = \{\varphi_i \circ \psi \mid i \in I, \psi \in \operatorname{Mor}\mathscr{C}\} = \{\rho : V \to U \mid \exists\, i \in I,\, \exists\, \psi \in \operatorname{Mor}\mathscr{C}\,.\, \rho = \varphi_i \circ \psi\},$$

which is a J-sieve because τ generates J. We will now show that the desired diagram is an equalizer via consideration of this sieve.

Define a morphism $\alpha : (C) \to F$ in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ by, for all $\rho \in (C)$,

$$\alpha(\rho) := F(\psi)(x_i)$$

whenever $\rho = \varphi_i \circ \rho$. To see that this is well-defined, assume that $\varphi_i \circ \psi = \rho = \varphi_j \circ \psi'$ for some $i, j \in I$ and some $\psi, \psi' \in \operatorname{Mor}\mathscr{C}$. Then, from the universal property of the pullback, there exists a unique morphism $\theta : \operatorname{Dom}\rho \to U_i \times_U U_j$ making the diagram

$$\begin{tikzcd}
\operatorname{Dom}\rho \arrow[dr, dashed, "\exists!\theta"] \arrow[drr, bend left, "\psi'"] \arrow[ddr, bend right, "\psi"'] & & \\
& U_i \times_U U_j \arrow[r, "\pi_{2,ij}"'] \arrow[d, "\pi_{1,ij}"] & U_j \arrow[d, "\varphi_j"] \\
& U_i \arrow[r, "\varphi_i"'] & U
\end{tikzcd}$$

commute in $\mathscr{C}$. However, it then follows that $\psi = \pi_{1,ij} \circ \theta$ and $\psi' = \pi_{2,ij} \circ \theta$. This then allows us to calculate that

$$F(\psi)(x_i) = F(\pi_{1,ij} \circ \theta)(x_i) = F(\theta)\big(F(\pi_{1,ij})(x_i)\big) = F(\theta)\big(F(\pi_{2,ij})(x_j)\big) = F(\pi_{2,ij} \circ \theta)(x_j) = F(\psi')(x_j),$$

from whence it follows that α is well-defined. Thus, since α is a morphism and F is a sheaf, because the diagram

$$\begin{tikzcd}
FU \arrow[r, "e"] & \prod\limits_{f \in (C)} F(\operatorname{Dom} f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{\substack{f \in (C), g \in \operatorname{Mor}\mathscr{C} \\ \operatorname{Codom} g = \operatorname{Dom} f}} F(\operatorname{Dom} g)
\end{tikzcd}$$

is an equalizer, there exists a unique $x \in FU$ such that

$$F(\rho)(x) = \alpha(\rho)$$

for all $\rho \in (C)$. Moreover, since $\varphi_i \in (C)$ for all $i \in I$,

$$F(\varphi_i)(x) = x_i.$$

To see this is the unique such element of FU, assume that $y \in FU$ such that $F(\varphi_i)(y) = x_i$ for all $i \in I$. Then for any $\rho \in (C)$ we have that $\rho = \varphi_i \circ \psi$ for some $i \in I$ and for some ψ, so it follows that

$$F(\rho)(y) = F(\varphi_i \circ \psi)(y) = F(\psi)(x_i) = \alpha(\rho) = F(\rho)(x).$$

Thus, using that e is monic (because it is an equalizer), we derive that $x = y$ and so the diagram

$$FU \xrightarrow{e} \prod_{i\in I} FU_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j\in I} F(U_i \times_U U_j)$$

is an equalizer.

$\Longleftarrow$: Fix an arbitrary $S \in J(U)$ and find a cover $C := \{\varphi_i : U_i \to U \mid i \in I\} \in \tau(U)$ such that $C \subseteq S$. Find a morphism $\alpha : S \to F$ and write $\alpha(\rho) = y_\rho$ for all $\rho \in S$. Then, by the naturality of α, we have that

$$F(\pi_{1,ij})(\alpha(\varphi_i)) = F(\pi_{1,ij})(y_{\varphi_i}) = F(\pi_{2,ij})(y_{\varphi_j}) = F(\pi_{2,ij})(\alpha(\varphi_j))$$

for all $i, j \in I$. Thus there exists a unique $x \in FU$ such that for all $i \in I$, the equation

$$F(\varphi_i)(x) = y_{\varphi_i}$$

holds.

We now need only to show that for all $\rho \in S$, $F(\rho)(x) = \alpha(\rho)$. So, fix a $\rho \in S$ and consider the pullbacks

$$\begin{tikzcd}
\operatorname{Dom}\rho \times_U U_i \arrow[r, "\pi_{2,i\rho}"] \arrow[d, "\pi_{1,i\rho}"'] & U_i \arrow[d, "\varphi_i"] \\
\operatorname{Dom}\rho \arrow[r, "\rho"'] & U
\end{tikzcd}$$

Because C is a cover, by the pullback axiom for pretopologies, the set

$$C' := \{\pi_{1,i\rho} : \operatorname{Dom}\rho \times_U U_i \to \operatorname{Dom}\rho \mid i \in I\} \in \tau(\operatorname{Dom}\rho).$$

Thus, for all $\varphi_i \in C$ we have that

$$\begin{aligned}
F(\rho \circ \pi_{1,i\rho})(x) &= F(\varphi_i \circ \pi_{2,i\rho})(x) = F(\pi_{2,i\rho})\big(F(\varphi_i)(x)\big) = F(\pi_{2,ij})(\alpha(\varphi_i)) \\
&= \alpha(\varphi_i \circ \pi_{2,ij}) = \alpha(\rho \circ \pi_{1,i\rho}) = F(\pi_{1,i\rho})(\alpha(\rho)).
\end{aligned}$$

Now fix $i, j \in I$ and consider the pullback P_{ij}

$$\begin{tikzcd}
P_{ij} \arrow[r, "\widetilde{\pi}_{2,ij}"] \arrow[d, "\widetilde{\pi}_{1,ij}"'] & \operatorname{Dom}\rho \times_U U_j \arrow[d, "\pi_{1,j\rho}"] \\
\operatorname{Dom}\rho \times_U U_i \arrow[r, "\pi_{1,i\rho}"'] & \operatorname{Dom}\rho
\end{tikzcd}$$

in $\mathscr{C}$. This diagram allows us to compute that

$$F(\widetilde{\pi}_{1,ij})\big(F(\rho \circ \pi_{1,i\rho})(y)\big) = F(\widetilde{\pi}_{2,ij})\big(F(\rho \circ \pi_{1,j\rho})(y)\big).$$

which in turn, because $C' \in \tau(\operatorname{Dom}\rho)$, allows us to conclude that the diagram

$$F(\operatorname{Dom}\rho) \xrightarrow{e_\rho} \prod_{i\in I} F(\operatorname{Dom}\rho \times_U U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j\in I} F(P_{ij})$$

is an equalizer. However, this implies that there is a unique factorization making the diagram

$$\begin{tikzcd} F(\operatorname{Dom}\rho) \arrow[r, "e_\rho"] & \prod\limits_{i\in I} F(\operatorname{Dom}\rho \times_U U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{i,j\in I} F(P_{ij}) \\ FU \arrow[u, dashed, "\exists !"] \arrow[ur] & & \end{tikzcd}$$

and so it must follow that $F(\rho)(y) = \alpha(\rho)$, which allows us to conclude that the diagram

$$FU \xrightarrow{e} \prod_{f\in S} P(\operatorname{Dom} f) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{\substack{f\in S, g\in \operatorname{Mor}\mathscr{C} \\ \operatorname{Codom} g = \operatorname{Dom} f}} F(\operatorname{Dom} g)$$

is an equalizer as well. This proves that F is a J-sheaf and concludes the proof of the theorem. $\square$

Remark 1.32. Note that a subtle consequence (or subtle elucidation) of Theorem 1.30 above is the fact that different topologies can generate the same sheaves. This shows, in fact, that if τ and τ' are pretopologies that generate the same topology J, then they generate the same sheaves, *and* how you would go about checking it!

1.2 First Properties of the Category of Sheaves

Now that we have made the acquaintance of sheaves on a site $(\mathscr{C}, J)$, we would like to study some of the basic properties of the category of J-sheaves. This will be especially important as we begin to study stacks, so becoming familiar with sheaf categories now will be to our advantage later.

We will begin our examination of sheaves by first defining the category of J-sheaves and then showing that the category of sheaves has all **Set**-indexed limits. From there we will discuss the associated sheaf functor and use this to show that it allows us to conclude that the category of sheaves admits all colimits. Afterwards, we will discuss the Day Reflection Theorem (cf. Theorem 1.49 below) and use this, together with the Associated Sheaf Functor, to prove that the category of J-sheaves is Cartesian Closed.

Definition 1.33. Let $(\mathscr{C}, J)$ be a site. A morphism between J-sheaves F and G is then a natural transformation $\alpha : F \to G$ in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, and the category of all J-sheaves, $\mathbf{Shv}(\mathscr{C}, J)$, is the full subcategory of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ comprised of J-sheaves.

Proposition 1.34. *All* **Set***-indexed limits exist in* $\mathbf{Shv}(\mathscr{C}, J)$.

Proof. Begin by letting I be a category for which $\operatorname{Ob} I$ is a set and consider a family $\{P_i \mid i \in I\}$ of J-sheaves. Now let P be the limit

$$P := \varprojlim_{i\in I} P_i$$

of some diagram $D : I \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$; note that the limit P exists because $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a topos and hence complete. Now consider that for all $i \in I$, for all $U \in \operatorname{Ob}\mathscr{C}$, and for all $S \in J(U)$, we have that the diagram

$$P(U) \xrightarrow{e} \prod_{f\in S} P_i(\operatorname{Dom} f) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{\substack{f\in S, g\in \operatorname{Mor}\mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P_i(\operatorname{Dom} g).$$

is an equalizer. Taking the limit allows us to produce a diagram

$$\prod_{f\in S} P(\operatorname{Dom} f) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{\substack{f\in S, g\in \operatorname{Mor}\mathscr{C} \\ \operatorname{Dom} f = \operatorname{Codom} g}} P(\operatorname{Dom} g).$$

which commutes because it commutes for each $P_i(U)$; now, because P is a limit, because an equalizer is a limit, and because limits commute with each other, it then follows that the above diagram is an equalizer diagram and so P is a J-sheaf. $\square$

From here we will move on to discuss the Associated Sheaf Functor. To do this, we will go use Grothendieck's Double Plus construction, which has the benefit of while being technical, making it immediate as to why the functor is flat (or, equivalently, left exact), as well as giving natural ideas for how to get the theory working in other toposes. To go about this path, we need one definition before we will proceed.

Definition 1.35. Let $(\mathscr{C}, J)$ be a site. We then say that a presheaf P on $\mathscr{C}$ is a *separated presheaf* if for each object $U \in \mathscr{C}$ and for each sieve $S \in J(U)$, given a natural transformation $\alpha : S \to P$, there is *at most one* natural transformation $\beta : \mathscr{C}(-, U) \to P$ making the diagram

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d] & P \\ \mathscr{C}(-,U) \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd}$$

commute.

We will proceed now by defining a $(-)^+$ functor on $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ which will construct a separated presheaf out of a presheaf; while this does not get us to sheaves directly, this plus functor will get us halfway there. Additionally, it is here that we will be able to prove the flatness of the Associated Sheaf Functor as a simple corollary of the flatness of the plus functor.

Definition 1.36. If $(\mathscr{C}, J)$ is a site and P is a presheaf on $\mathscr{C}$, define a presheaf P^+ via the equation

$$P^+(U) := \varinjlim_{S \in J(U)} [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](S, P),$$

where $J(U)$ is regarded as a poset with respect to inclusion of sieves. If $\varphi \in \mathscr{C}(V, U)$, define $P^+(\varphi)$ by taking the colimit over the induced function $\varphi^* : J(U) \to J(V)$ which sends a sieve $S \in J(U)$ to $\varphi^*(S) \ni J(V)$.

Remark 1.37. The colimit defining P^+ is a filtered colimit: If $x = \{\alpha(f) \mid f \in S\}$ and $y = \{\beta(g) \mid g \in R\}$, for sieves $R, S \in \alpha$ and for natural transformations $\alpha : S \to P$ and $\beta : R \to P$, then $x \simeq y$ in $P^+(U)$ if and only if there exists a refinement $T \subseteq R \cap S$ such that $T \in J(U)$ and for all $\varphi \in T$, $\alpha(\varphi) = \beta(\varphi)$; that is to say, there is a sieve T on U such that α and β are "eventually equal" on T. Moreover, if φ is any natural transformation of presheaves $P \to Q$, then the post-composition morphism

$$[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](\mathrm{id}_S, \varphi) : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](S, P) \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](S, Q),$$

for all $S \in J(U)$ and for all $U \in \mathrm{Ob}\,\mathscr{C}$, then gives a morphism $\varphi^+ : P^+ \to Q^+$, and this assignment is easily shown to be compatible with the colimit operation, i.e., for all objects $U \in \mathrm{Ob}\,\mathscr{C}$, there is a corresponding morphism

$$\varinjlim_{S \in J(U)} \mathscr{C}(\mathrm{id}_S, \varphi) = \varphi_U^+ : P^+(U) \to Q^+(U),$$

which is natural in U. Taking these observations together shows that the $(-)^+$ assignment determines a functor, yielding the content of the proposition below.

Proposition 1.38. *The assignment* $(-)^+ : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ *is a functor.*

In order to proceed with studying the Plus functor, we would like to show that there is a natural transformation $\eta : \mathrm{id}_{[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]} \to (-)^+$ which plays well with J-sheaves. To show how to define η, let $P \in \mathrm{Ob}\,[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ be a presheaf and let $U \in \mathrm{Ob}\,\mathscr{C}$. Then define $\eta_U : PU \to P^+U$ by

$$\eta_U(x) = \{P(f)(x) \mid f \in \mathrm{Codom}\, U\}.$$

Note that this represents an equivalence class in $P^+(U)$, and if $\varphi \in \mathscr{C}(V, U)$ for all $x \in PU$,

$$P^+(\varphi)(\eta_U(x)) = P^+(\varphi)(\{P(f)(x) \mid f \in \mathrm{Codom}\, U\}).$$

However, since $P^+(\varphi)$ is induced by sending sieves S in $J(U)$ to $\varphi^*(S) \in J(V)$, we find that $P(\varphi)^+$ acts on the sieve $\operatorname{Codom} U$ by

$$\varphi^*(\operatorname{Codom} U) = \{g \in \operatorname{Mor} \mathscr{C} \mid \varphi \circ g \in \operatorname{Codom} U\} = \{g \in \operatorname{Mor} \mathscr{C} \mid \operatorname{Codom} g = V\} = \operatorname{Codom} V$$

so we calculate that

$$\begin{aligned} P^+(\varphi)(\eta_U(x)) = P^+(\varphi)(\{P(f)(x) \mid f \in \operatorname{Codom} U\} &= \{P(\varphi \circ g)(x) \mid g \in \operatorname{Codom} V\} \\ &= \{P(g)\big(P(\varphi)(x)\big) \mid g \in \operatorname{Codom} V\} = \eta_V(P(\varphi)(x)) \end{aligned}$$

which shows that the diagram

$$\begin{tikzcd} PU \arrow[r, "\eta_U"] \arrow[d, "P(\varphi)"'] & P^+U \arrow[d, "P^+\varphi"] \\ PV \arrow[r, "\eta_V"'] & P^+V \end{tikzcd}$$

commutes. This proof yields the lemma below that η is a natural transformation, which in turn helps us get our study of the Double Plus Functor running.

Lemma 1.39. *The family of functions $\eta_U : PU \to P^+U$, for all $U \in \operatorname{Ob} \mathscr{C}$, defined by*

$$\eta_U(x) := \{P(f)(x) \mid f \in \operatorname{Codom} U\}$$

determines a natural transformation $\eta : \mathrm{id}_{[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]} \to (-)^+$.

We will now use a structural lemma about sheaves and how they behave with respect to the natural transformation η. It turns out that we can determine whether or not a presheaf is both separated *or* a J-sheaf by studying whether or not the map $\eta : P \to P^+$ is monic or an isomorphism.

Lemma 1.40. *Let $(\mathscr{C}, J)$ be a site and let P be a presheaf on $\mathscr{C}$. Then:*

1. *P is separated if and only if $\eta : P \to P^+$ is monic;*
2. *P is a J-sheaf if and only if $\eta : P \to P^+$ is an isomorphism.*

Proof. (1): Begin by observing that if $x, y \in P(U)$, for some $U \in \operatorname{Ob} \mathscr{C}$, then $\eta(x) = \eta(y)$ implies that there is a covering sieve $S \in J(U)$ such that for all $f \in S$, $P(f)(x) = P(f)(y)$. We can then conclude that $x = y$ if and only if there is at most one natural transformation $\beta : \mathscr{C}(-, U) \to P$ extending the diagram below along the dotted arm:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & P \\ \mathscr{C}(-,U) \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd}$$

Finally, the claim for (2) follows by using the surjecivity of η to give the existence of a lift, while the fact that it is monic implies that there is at most one such lift. $\square$

Lemma 1.41. *The functor $(-)^+$ is flat.*

Proof. Since $(-)^+$ is determined pointwise by a filtered colimit in **Set** and filtered colimits in set commute with finite limits, it follows immediately that $(-)^+$ is flat because limits in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ are computed pointwise in **Set**. $\square$

Corollary 1.42. *The functor $(-)^{++} := (-)^+ \circ (-)^+$ is flat.*

Lemma 1.43. *If F is a J-sheaf and if $P \in \operatorname{Ob}[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, then if $\varphi \in [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](P, F)$ there exists a unique $\varphi^\sharp \in [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](P^+, F)$ making the diagram*

$$\begin{tikzcd} P \arrow[r, "\eta_P"] \arrow[dr, "\varphi"'] & P^+ \arrow[d, dashed, "\exists! \varphi^\sharp"] \\ & F \end{tikzcd}$$

commute.

Proof. We will begin by exploring how P^+ acts on its elements in order to determine necessary conditions on the map $\varphi^\sharp$. Begin by noting that if $x \in P^+U$, then we can find a collection of elements $\{x_f \mid f \in S\}$ induced by elements of a natural transformation from some covering sieve $S \in J(U)$ to P which represent x. i.e.,

$$x = [\{x_f \mid f \in S\}].$$

Now let $\rho : V \to U$ be a morphism in S and recall that

$$\eta_V(x_\rho) = \{P(k)(x_\rho) \mid k \in \operatorname{Codom} V\}$$

and that $P^+(\rho)$ acts on x by

$$P^+(\rho)(x) = P^+(\rho)(\{x_f \mid f \in S\}) = \{x_{\rho \circ g} \mid g \in \rho^*(S)\}.$$

Moreover, by the naturality of η we have that

$$\eta_V(x_\rho) = P^+(\rho)(\{x_f \mid f \in S\}).$$

Thus, if $\varphi^\sharp$ were to exist, it would be uniquely determined by the fact that $\varphi^\sharp$ must map the equivalence class of $\{x_f \mid f \in S\}$ to the unique $y \in FU$ satisfying the equations

$$F(\rho)(y) = F(\rho)\Big(\varphi^\sharp(\{x_f \mid f \in S\})\Big) = \varphi^\sharp\left(\Big[P^+(\rho)(\{x_f \mid f \in S\})\Big]\right) = \varphi^\sharp(\eta_V(x_\rho)) = \varphi(x_\rho)$$

for all $\rho \in S$. This implies that y is a unique lift of the transformation $S \to F$ induced by the factorization

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & P \arrow[d, "\varphi"] \\ & F \end{tikzcd}$$

of natural transformations; however, this unique lift exists because F is a J-sheaf and because the family $\{\varphi(x_\rho) \mid \rho \in S\}$ is induced from the factorization above. ☐

Unfortunately, this lemma does not prove that $(-)^+$ is a left adjoint because it *only* provides a factorization for J-sheaves, not for all presheaves. However, if we knew that $(-)^+$ was a sheaf, we'd be done because this says that the plus functor would be left adjoint to the inclusion $\iota : \mathbf{Shv}(\mathscr{C}, J) \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$. However, all is not lost! We will show below that P^+ is separated, and then show that if P is separated P^{++} is an isomorphism. This, together with the fact that $\eta|_{\mathbf{Shv}(\mathscr{C},J)} \cong \mathrm{id}_{\mathbf{Shv}(\mathscr{C},J)}$, will allow us to conclude that the Double Plus Functor is left adjoint to the inclusion of $\mathbf{Shv}(\mathscr{C}, J)$ into the presheaf topos $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.

Lemma 1.44. *If P is a presheaf on a site $(\mathscr{C}, J)$, then P^+ is separated.*

Proof. We begin by observing that in order to show that P^+ is separated, it suffices to show that if $x, y \in P^+U$ for some $U \in \operatorname{Ob}\mathscr{C}$ such that there exists a sieve $Q \in J(U)$ with the property that $P^+(h)(x) = P^+(h)(y)$ for all $h \in Q$, then $x = y$. As such, find sieves $S, R \in J(U)$ and natural transformations, say $\alpha : S \to P^+$,

$\beta : R \to P^+$, which represent x and y in P^+U, respectively. Then we have the equalities, if we set $x_f := \alpha_{\mathrm{Dom}\, f}(f)$ and $y_g := \beta_{\mathrm{Dom}\, g}(g)$ for all $f \in S, g \in R$, then

$$x = [\{x_f \mid f \in S\}]$$

and

$$y = [\{y_g \mid g \in R\}] .$$

Let $\rho \in Q$ be given and write $\rho : V \to U$. Then since we have that $P^+(\rho)(x) = P^+(\rho)(y)$, there exists a cover $T_\rho \subseteq \rho^*(S) \cap \rho^*(R)$ with $T_\rho \subseteq J(V)$ such that $x_{\rho\circ f'} = y_{\rho\circ f'}$ for all $f' \in T_\rho$. However, by the transitivity axiom, the family

$$T := \{\rho \circ f \mid \rho \in Q, f \in T_\rho\}$$

is a J-covering sieve of U with $T \subseteq R \cap S$. Moreover, for all $\varphi \in T$ we have that $P^+(\varphi)(x)0 = P^+(\varphi)(y)$ so it follows that $x = y$ in P^+U. This shows that P^+ is separated. □

Lemma 1.45. *If P is a separated presheaf on a site $(\mathscr{C}, J)$, then P^+ is a J-sheaf.*

Proof. Let $\alpha : S \to P^+$ be a natural transformation for S a J-covering sieve on U. For each $f \in S$, define

$$x_f := \alpha_{\mathrm{Dom}\, f}(f) \in P^+ \,\mathrm{Dom}\, f$$

and note that since α is a natural transformation, the family $(x_f)_{f\in S}$ satisfies the equation

$$\begin{aligned} q(x_f)_{f\in S} &= \big(P^+(g)(x_f)\big)_{f\in S, g\in \mathrm{Mor}\,\mathscr{C}, \mathrm{Codom}\, g = \mathrm{Dom}\, f} = \big((P^+(g)(\alpha_{\mathrm{Dom}\, f}(f))\big)_{f\in S, g\in \mathrm{Mor}\,\mathscr{C}, \mathrm{Codom}\, g = \mathrm{Dom}\, f} \\ &= \big(\alpha_{\mathrm{Dom}\, g}(f \circ g)\big)_{f\in S, g\in \mathrm{Mor}\,\mathscr{C}, \mathrm{Codom}\, g = \mathrm{Dom}\, f} = \big(x_{f\circ g}\big)_{f\in S, \mathrm{Mor}\,\mathscr{C}, \mathrm{Codom}\, g = \mathrm{Dom}\, f} \\ &= p(x_f)_{f\in S} \end{aligned}$$

where p and q are defined as in the sheaf diagram

$$\prod_{f\in S} P^+(\mathrm{Dom}\, f) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{\substack{f\in S, g\in \mathrm{Mor}\,\mathscr{C} \\ \mathrm{Codom}\, g = \mathrm{Dom}\, f}} P^+ \,\mathrm{Dom}\, g$$

We now need to show how to construct a single element $y \in P^+U$ that has $P(f)(y) = x_f$ for all $f \in S$. Before doing this, however, we need to consider how to represent the naturality condition on the $(x_f)_{f\in S}$ in the sets $P^+ \,\mathrm{Dom}\, g$. To do this, note that because $x_f \in P^+ \,\mathrm{Dom}\, f$ there exists a J-covering sieve $S_f \in J(\mathrm{Dom}\, f)$ such that x_f is represented by the set

$$x_f = [\{(x_f)_g \mid g \in S_f\}] = [\{(\alpha_{\mathrm{Dom}\, f}(f))_g \mid g \in S_f\}]$$

where we write $(x_f)_g := (\alpha_{\mathrm{Dom}\, f}(f))_g$. Fix now some $\rho : \mathrm{Dom}\, \rho \to \mathrm{Dom}\, f$ in $\mathscr{C}$. Then because

$$P^+(\rho)(\alpha(f)) = \alpha(f \circ \rho)$$

it follows that in $P^+(\mathrm{Dom}\, \rho)$ there is a similarity between the sets

$$\{(\alpha(f))_{\rho\circ g} \mid g \in \rho^*(S_f)\} \simeq \{(\alpha(f \circ \rho))_g \mid g \in S_{f\circ\rho}\},$$

i.e., the sets represent the same equivalence class in the colimit. So there exists a sieve $T_{f,\rho} \subseteq \rho^*(S_f) \cap S_{f\circ\rho}$ such that for all $k \in T_{f,\rho}$ we have that

$$(\alpha(f))_{\rho\circ k} = (\alpha(f \circ \rho))_k.$$

We will use the above representations to construct an element y in P^+U equalizing p and q. Define the sieve R by the equation

$$R := \{f \circ g \mid f \in S, g \in S_f\}$$

and note that since $S \in J(U)$ and, for all $f \in S$, $S_f \in J(\operatorname{Dom} f)$, it follows from the transitivity axiom for topologies that $R \in J(U)$ as well. Now define $Y \in P^+U$ via

$$y_{f\circ g} := (\alpha(f))_g.$$

To see that y is a well-defined element of P^+U, we must show that it does not depend on the choice of factorization for $f \circ g$. To this end, assume that there exists $h \in S$ and $k \in S_h$ such that $f \circ g = h \circ k$. Then we derive that if $\rho \in T_{f,g} \cap T_{h,k}$, we have that

$$P^+(\rho)\big((\alpha(f))_g\big) = ((\alpha(f))_{g\circ\rho} = (\alpha(f \circ g))_\rho = (\alpha(h \circ k))_\rho = (\alpha(h))_{k\circ\rho} = P^+(\rho)\big((\alpha(h))_k\big).$$

Because $T_{f,g} \cap T_{h,k}$ is a J-covering sieve and P is a separated sheaf, it follows then $(\alpha(f))_g = (\alpha(h))_k$. Defining the natural transformation $\beta : R \to P$ via the equation $\beta(\rho) := y_\rho$ for all $\rho \in R$ then yields that the set

$$y := [\{y_\rho \mid \rho \in R] \in P^+U.$$

Then we compute that for all $f \in S$,

$$P^+(f)(y) = \alpha(f)$$

by construction, so

$$e(y) = \langle P^+(f)\rangle_{f\in S}(y) = (P^+(f)(y))_{f\in S} = (\alpha(f))_{f\in S}$$

and by design $p \circ e = q \circ e$. Moreover, since P is separated, it follows that y is the unique such element that maps through e to α; it then follows that

$$P^+U \xrightarrow{e} \prod_{f\in S} P(\operatorname{Dom} f) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{\substack{f\in S, g\in \operatorname{Mor}\mathscr{C} \\ \operatorname{Codom} g = \operatorname{Dom} f}} P^+(\operatorname{Dom} g)$$

is an equalizer diagram, which proves that P^+ is a J-sheaf. □

Corollary 1.46. *The functor $(-)^{++} : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ takes values in $\mathbf{Shv}(\mathscr{C}, J)$, i.e., $(-)^{++}$ factors as:*

$$\begin{tikzcd} {[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]} \arrow[rr, "(-)^{++}"] \arrow[dr, "(-)^{++}"'] & & {[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]} \\ & \mathbf{Shv}(\mathscr{C}, J) \arrow[ur, "\iota"'] & \end{tikzcd}$$

Corollary 1.47. *The functor $(-)^{++} : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Shv}(\mathscr{C}, J)$ is flat and left adjoint to the inclusion $\iota : \mathbf{Shv}(\mathscr{C}, J) \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.*

Proof. The flatness of $(-)^{++}$ follows from the fact that $(-)^{+}$ is flat, while the fact that $(-)^{++}$ is left adjoint to ι follows from the fact that there is a universal property for $\eta' : \mathrm{id}_{[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]} \to \iota \circ (-)^{++}$ given by the equation

$$\eta'_P := \eta_{P^+} \circ \eta_P$$

and then using the induced universal property for η'. □

Corollary 1.48. *The counit of adjuction $\varepsilon : (-)^{++} \circ \iota \to \mathrm{id}_{\mathbf{Shv}(\mathscr{C},J)}$ is an isomorphism.*

Proof. This is immediate from the fact that ι is a fully faithful right adjoint. □

Finally, to complete this section and our introductory study of $\mathbf{Shv}(\mathscr{C}, J)$, we need to discuss the Day Reflection Theorem. It is an intuitive theorem on symmetric monoidal categories, but gives a nice way of showing that $\mathbf{Shv}(\mathscr{C}, J)$ is Cartesian Closed by relying only on the Cartesian Monoidal structure of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ (which is guaranteed by the fact that $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a topos).

Theorem 1.49 (Day Reflection Theorem; cf. [1])**.** *Let $R : \mathscr{C} \to \mathscr{D}$ be a fully faithful right adjoint with left adjoint $L : \mathscr{D} \to \mathscr{C}$. Then if $(\mathscr{D}, \otimes, I)$ is a closed symmetric monoidal structure on $\mathscr{D}$ and let*

$$(\eta, \varepsilon) : (-) \otimes Y \dashv [Y, -] : \mathscr{D} \to \mathscr{D}$$

describe the tensor/internal hom adjunction in $\mathscr{D}$. Then, for all $U \in \operatorname{Ob}\mathscr{C}$ and all $V \in \operatorname{Ob}\mathscr{D}$, if any of the following natural transformations are invertible, then they all are:

1. $\eta_{[V,RU]} : [V, RU] \to (R \circ L)[V, RU]$;
2. $[\eta_V, \mathrm{id}_{RU}] : [(R \circ L)V, RU] \to [V, RU]$;
3. $L(\eta_V \otimes \mathrm{id}_{V'}) : L(V \times V') \to L((R \circ L)V \otimes V')$;
4. $L(\eta_V \otimes \eta_{V'}) : L(V \otimes V') \to L((R \circ L)V \otimes (R \circ L)V')$.

In particular, if $\mathscr{D}$ is Cartesian Closed and if L preserves products, then $\mathscr{C}$ is an exponential ideal of $\mathscr{D}$.

Corollary 1.50. *If $\mathscr{C}$ is a reflexive subcategory of $\mathscr{D}$ with reflector $L : \mathscr{D} \to \mathscr{C}$, if $\mathscr{D}$ is Cartesian Closed, and if L preserves products, then $\mathscr{C}$ is Cartesian Closed.*

Corollary 1.51. *The category $\mathbf{Shv}(\mathscr{C}, J)$ is Cartesian Closed.*

Proof. Since $\mathbf{Shv}(\mathscr{C}, J)$ is a reflective subcategory of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ with flat reflector $(-)^{++}$, and since $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a topos (and hence Cartesian Closed), $\mathbf{Shv}(\mathscr{C}, J)$ is Cartesian Closed. □

Proposition 1.52. *If Ω is the subobject classifier in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ then $(\Omega)^{++}$ is a subobject classifier in $\mathbf{Shv}(\mathscr{C}, J)$.*

Proof. This follows from Remark B2.2.7 of [4]. More explicitly, the subobject classifier Ω in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ may be described on objects by observing that the Yoneda Lemma forces $\Omega(U)$ to have its elements determined by considering morphisms, for each $U \in \operatorname{Ob}\mathscr{C}$ and $\Omega_{\mathbf{Set}} = \{0, 1\}$ the subobject classifier in $\mathbf{Set}$,

$$\Omega(U) \cong [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}](\mathscr{C}(-, U), \Omega)$$

and hence are locally given by considering that these must classify subfunctors of $\mathscr{C}(-, U)$ for each U. Thus we define

$$\Omega(U) := \{S \in [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \mid S \,\text{is a subfunctor of}\, \mathscr{C}(-, U)\}.$$

Moreover, for each $\rho \in \mathscr{C}(V, U)$ we get a morphism $\Omega(\rho) : \Omega(U) \to \Omega(V)$ by defining

$$\Omega(\rho)(S) = \rho^*(S)$$

for each sieve $S \in \Omega(U)$. This makes Ω into a functor $\Omega \in [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, while the map true : $\top \to \Omega$ is given on each $U \in \mathscr{C}$ as

$$\mathrm{true}_U : \top(U) \to \Omega(U)$$

by

$$\mathrm{true}_U(*) = \mathscr{C}(-, U) = \operatorname{Codom} U.$$

Take $\mu : P \to Q$ to be any monic in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ and define $\chi_\mu : Q \to \Omega$ by saying, for each $U \in \operatorname{Ob}\mathscr{C}$ and for each $x \in QU$,

$$(\chi_\mu)_U(x) := \{f \in \operatorname{Mor}\mathscr{C} \mid \operatorname{Codom} f = U, Q(f)(x) \in P(\operatorname{Dom} f)\}.$$

Then χ_μ determines a natural transformation and the diagram

$$\begin{tikzcd} P \arrow[r, dashed, "\exists!"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\lrcorner", very near start] & \top \arrow[d, "\mathrm{true}"] \\ Q \arrow[r, "\chi_\mu"'] & \Omega \end{tikzcd}$$

is easily seen to be a pullback. Moreover, χ_μ is the unique such morphism to Ω making this a pullback, so Ω is indeed the subobject classifier in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.

Finally, it can be readily checked that for all $U \in \operatorname{Ob} \mathscr{C}$,

$$\Omega^{++}(U) = \{S \in \Omega(U) \mid S \in J(U)\}$$

so $\Omega^{++}(U)$ contains all J-covering sieves on U. Moreover, since $(-)^{++}$ is flat, $\top^{++} = \top$ so we can check that

$$\mathrm{true}_U^{++}(*) = \operatorname{Codom} U = \mathscr{C}(-, U),$$

which is always a J-covering sieve. Then for any J-sheaves F and G with a monic $\mu : F \to G$, define $\chi_\mu : G \to \Omega^{++}$ by, for all $U \in \operatorname{Ob} \mathscr{C}$ and for all $x \in GU$,

$$(\chi_\mu)_U(x) := \{f \in \operatorname{Mor} \mathscr{C} \mid \operatorname{Codom} f = U, G(f)(x) \in F(\operatorname{Dom} f)\}.$$

It then follows more or less immediately that

$$\begin{tikzcd} F \arrow[r, dashed, "\exists!"] \arrow[d, "\mu"'] & \top \arrow[d, "\mathrm{true}^{++}"] \\ G \arrow[r, "\chi_\mu"'] & \Omega^{++} \end{tikzcd}$$

is a pullback in $\mathbf{Shv}(\mathscr{C}, J)$ with χ_μ the unique natural transformation making the diagram a pullback. This implies that Ω^{++} is the subobject classifier in $\mathbf{Shv}(\mathscr{C}, J)$ and proves the proposition. □

Corollary 1.53. *If $(\mathscr{C}, J)$ is a site then $\mathbf{Shv}(\mathscr{C}, J)$ is a topos.*

Proof. The functors ι and $(-)^{++}$ both create limits and colimits in $\mathbf{Shv}(\mathscr{C}, J)$, so $\mathbf{Shv}(\mathscr{C}, J)$ is **Set**-complete and **Set**-cocomplete. Moreover, Corollary 1.51 shows that $\mathbf{Shv}(\mathscr{C}, J)$ is Cartesian closed, while Proposition 1.52 shows that $\mathbf{Shv}(\mathscr{C}, J)$ has a subobject classifier. Thus $\mathbf{Shv}(\mathscr{C}, J)$ is a complete and cocomplete Cartesian Closed category with a subobject classifier, and hence an elementary topos. □

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