Fubini's Theorem and the Magic of Convolution

Geoff Vooys

April 7, 2015

Algebraic Preliminaries

Definition; [Conway], Definition II.3.1

Let X be a set and let $A \subseteq \mathcal{P}(X)$. Then A is said to be a σ-algebra if the following hold:

1. $\varnothing, X \in A$.
2. If the sets $S, T \in A$ then $T \setminus S \in A$.
3. If the collection
$$\mathscr{E} = \{E_k \mid k \in \mathbb{N}\} \subseteq A,$$
then the union
$$E := \bigcup_{k \in \mathbb{N}} E_k \in A.$$

Note that after applying De Morgan's Law to condition (3), it follows that if
$$\mathscr{E} := \{E_k \mid k \in \mathbb{N}\} \subseteq A,$$
then
$$\bigcap_{k \in \mathbb{N}} E_k \in A.$$

Borelling Towards Measure Theory

Constructing σ-algebras

Let $\mathcal{S}$ be a (possibly empty) collection of subsets of an arbitrary set X. Then the σ-algebra generated by $\mathcal{S}$ is defined to be the set $\mathfrak{A}$ given by, for A a σ-algebra over X,

$$\mathfrak{A} := \bigcap_{\mathcal{S} \subseteq A} A.$$

Definition; [Conway], Definition II.3.3

Let X be a metric space equipped with its metric topology. Then the σ-algebra of Borel sets is the σ-algebra generated by the collection $\mathscr{O}$ of open subsets of X. We will call this σ-algebra $\mathfrak{B}_X$ because reasons (namely, it's frak-ing awesome).

Definition

A set $B \subseteq X$ is said to be Borel if $B \in \mathfrak{B}_X$.

A Measure of Progress

UNIVERSITY OF CALGARY

Definition

Define $\hat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Then if X is a set and A is a σ-algebra over X, we say that a function $f : X \to \hat{\mathbb{R}}$ is A-measurable if and only if $f^{-1}(B) \in A$ for every Borel set $B \subseteq \hat{\mathbb{R}}$. If X is a metric space uder its metric topology and $A = \mathfrak{B}_X$, then the class of $\mathfrak{B}_X$-measurable functions are called Borel functions.

Proposition; [Conway], Prop. II.3.8

Let us select a branch cut $\mathcal{B} = (-\infty, \infty]$ or $\mathcal{C} = [-\infty, \infty)$ of $\hat{\mathbb{R}}$. Then if A is a σ-algebra over a set X and $f, g : X \to \hat{\mathbb{R}}$ are A-measurable functions taking values in exactly one of the branch cuts $\mathcal{B}$ or $\mathcal{C}$, then the functions $f + g$, fg, and rf are all A-measurable for every $r \in \mathbb{R}$.

Measuring Up

UNIVERSITY OF

Definition; [Conway], Definition II.4.1

Let X be a set and A a σ-algebra over X. Then a measure is a function $\mu : X \to [0, \infty]$ satisfying the following conditions:

1. $\mu(\varnothing) = 0$.
2. if $\mathcal{E} := \{E_k \mid k \in \mathbb{N}\} \subseteq A$, then
$$\mu\left(\bigcup_{k\in\mathbb{N}} E_k\right) \leq \sum_{k\in\mathbb{N}} \mu(E_k).$$
3. If $\mathcal{E} := \{E_k \mid k \in \mathbb{N}\} \subseteq A$ is a collection of pairwise disjoint sets, then
$$\mu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \mu(E_k).$$

The triple (X, A, μ) is called a measure space, and if $\mu(X) < \infty$ the triple (X, A, μ) is called a finite measure space.

Examples, Part One: The Legbesgue Measure

The Lebesgue Measure

We construct the Lebesgue measure as follows. Begin by letting $a < b \in \mathbb{R}$ and setting $X := [a, b]$. Then for every open set $G \subset X \subset \mathbb{R}$ define

$$\lambda^*(G) := \sup_{f \in \mathcal{C}(X)} \left\{ \int_a^b f(t)\,\mathrm{d}t \mid 0 \leq f \leq \chi_G \right\}.$$

Now, for any set $S \subseteq X$ define the quantity

$$\lambda^*(E) := \inf_{E \subseteq G} \{\lambda^*(G) \mid G \text{ open}\}.$$

We construct the σ-algebra A by giving conditions upon the sets that A may contain. We say that a set $S \subseteq X$ belongs to A if for every real number $\varepsilon > 0$ there is a compact set $C \subseteq S$ and an open set $O \supseteq S$ such that

$$\lambda^*(O \setminus C) < \varepsilon.$$

Examples, Part Deux: Count Lebesgue

The Lebesgue Measure, Cont.

We can verify that A, defined in this way, indeed is a σ-algebra. Define the function $\lambda : A \to [0, \infty]$ by

$$S \mapsto \lambda^*(S).$$

Then λ is a measure on A and the resulting measure space is called the Lebesgue Measure on X.

The Counting Measure

Let X be an arbitrary set and define

$$\mu : \mathcal{P}(X) \to [0, \infty]$$

by, if $|S| \geq \aleph_0$, setting $\mu(S) := \infty$ and defining $\mu(S) := |S|$ if S is finite. Then $(X, \mathcal{P}(X), \mu)$ is a measure space. We call μ the Counting Measure on X.

Examples, Part Three: Radon Measures

Radon Measures

Let X be a compact metric space and let $L : \mathcal{C}(X) \to \mathbb{R}$ be a positive linear functional (i.e., $L(f) \geq 0$ whenever $f(x) \geq 0$) and define, for all open sets $O \subseteq X$,

$$\mu^*(O) := \sup_{f \in \mathcal{C}(x)} \{L(f) \mid 0 \leq f \leq \chi_O\}.$$

The definition for μ^* on a nonopen set is the same is in the Lebesgue measure, just now with respect to the linear functional L; analogously, we define a set $S \subseteq X$ to be measurable in the exact same way as the Lebesgue measure, just this time that we insist on whenever $\varepsilon > 0$, there exist K compact and O open with $K \subseteq S \subseteq O$,

$$\mu^*(O \setminus K) < \varepsilon.$$

Define A to be the collection of all such sets $S \subseteq X$. Then $\mu : A \to [0, \infty]$ by $S \mapsto \mu^*(S)$ is a measure and the resulting measure space (X, A, μ) is a finite measure space called the Radon measure space associated to the linear functional L.

Properties of Measures

Proposition; [Conway], Prop. II.4.3

Let (X, A, μ) be a measure space. Then the following conditions hold:

1. If $E \in A$ is a fixed set, then define $A_E := \{E \cap S \mid S \in A\}$ and $\mu_E(S) := \mu(E \cap S)$. Then (E, A_E, μ_E) is a measure space.
2. If $t \in \mathbb{R}_+$ then $(X, A, t\mu)$ is a measure space.
3. If $(S_n) \subseteq A$ is a sequence such that we have the ascending chain
$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots$$
and
$$S = \bigcup_{n \in \mathbb{N}} S_n,$$
then $\mu(S_n) \to \mu(S)$.
4. If $(T_n) \subseteq A$ is a sequence of sets in A such that we get the descending chain $S_n \supseteq S_{n+1}$ such that $\mu(S_n) \in \mathbb{R}_+$ for all $n \in \mathbb{N}$ and if $S = \cap_{n \in \mathbb{N}} S_n$, then $\mu(S_n) \to \mu(S)$.

A Proprietary Proof

Proof

(1): By construction $\varnothing \in A_E$ and so $\mu_E(\varnothing) = \mu(\varnothing \cap E) = \mu(\varnothing) = 0$. Now let $\mathcal{S} := \{S_k \mid k \in \mathbb{N}, S_k \in A_E\} \subseteq A_E$. Then we find that

$$\mu_E\left(\bigcup_{k\in\mathbb{N}} S_k\right) = \mu\left(\left(\bigcup_{k\in\mathbb{N}} S_k\right) \cap E\right) = \mu\left(\bigcup_{k\in\mathbb{N}} (S_k \cap E)\right)$$
$$\leq \sum_{k\in\mathbb{N}} \mu(S_k \cap E) = \sum_{k\in\mathbb{N}} \mu_E(S_k).$$

Finally, we compute for
$\mathcal{S} := \{S_k \mid S_k \cap S_m = \varnothing, k, m \in \mathbb{N}, k \neq m, S_k \in A_E\} \subseteq A_E$,

$$\mu_E\left(\bigcup_{k\in\mathbb{N}} S_k\right) = \mu\left(\left(\bigcup_{k\in\mathbb{N}} S_k\right) \cap E\right) = \mu\left(\bigcup_{k\in\mathbb{N}} (S_k \cap E)\right)$$
$$= \sum_{k\in\mathbb{N}} \mu(S_k \cap E) = \sum_{k\in\mathbb{N}} \mu_E(S_k).$$

This shows that (E, A_E, μ_E) is a measure space.

Continuing with Proprietary Information

Proof, Cont.

(3) Let $\{S_k \mid S_k \subseteq S_{k+1}\} \subseteq A$ be an ascending chain of sets in A and set $S := \cup_{k\in\mathbb{N}} S_k$. Then set $T_0 := S_0$ and construct a new sequence, for all positive $n \in \mathbb{N}$, $T_n := S_n \setminus S_{n-1}$. Then the sequence (T_k) is a sequence of mutually disjoint sets with $\bigcup_{k\in\mathbb{N}} T_k = S = \bigcup_{k\in\mathbb{N}} S_k$. This then yields that

$$\mu(S) = \mu\left(\bigcup_{k\in\mathbb{N}} T_k\right) = \sum_{k\in\mathbb{N}} \mu(T_k).$$

Now, for any finite $n \in \mathbb{N}$ we find that

$$\sum_{k=0}^{n} \mu(T_k) = \mu\left(\bigcup_{k=0}^{n} T_k\right) = \mu(S_n).$$

Thusly as $n \to \infty$, $\mu(S_n) \to \mu(S)$. This proves part (3).

Proof of Part (4)

UNIVERSITY OF CALGARY

Proof, Cont.

(4): Let $(S_k)_{k\in\mathbb{N}}$ be a sequence of sets with finite measure in A such that $S_k \supseteq S_{k+n}$ for all $n \in \mathbb{N}^\times$; furthermore, set $S := \bigcup_{k\in\mathbb{N}} S_k$. Then define $C := S_0$ and set $T_m = C \setminus S_m$. Then (T_m) is a sequence that forms an ascending chain in A with infinite union $T = C \setminus S$. Because $\mu(C) < \infty$ it follows from part (3) that

$$\mu(C) - \mu(S) = \mu(T) = \lim_{n\to\infty} \mu(T_n) = \mu(C) - \lim_{n\to\infty} \mu(S_n).$$

From here it follows after killing the $\mu(C)$ term that $\mu(S_n) \to \mu(S)$ and we are done. $\square$

A Beginning of Integrating over a Measure

Definition; [Conway], Definition II.4.10

Let X be a set and A a σ-algebra. Then a function $s : X \to \mathbb{R}$ is said to be a Simple Measurable Function if s is A-measurable and s takes on only a finite number of distinct values in its image. Equivalently, s is a simple measurable function if there exist a collection
$\mathcal{S} := \{S_0, \cdots, S_n \mid n \in \mathbb{N}, S_n \cap S_m = \varnothing, m \neq n\}$ of sets $S_k \in A$ such that

$$s = \sum_{k=0}^{n} a_k \chi_{E_k}.$$

If each of the sets $S_k \in \mathcal{S}$ has finite measure we say that s is integrable and define

$$\int_X s \, \mathrm{d}\mu := \sum_{k=0}^{n} a_k \mu(S_k).$$

$L^1_s(\mu)$ (Yet another L^1 to worry about)

UNIVERSITY OF

Definition

Simple L^1 space over μ is defined as a set as

$$L^1_s(\mu) := \{s : X \to \mathbb{R} \text{ simple measurable} \mid s \text{ integrable}\}.$$

Proposition

Define addition and $\mathbb{R}$-scalar multiplication on $L^1_s(\mu)$ pointwise. Then $L^1_s(\mu) \in \mathrm{ob}(\mathbb{R}\text{-}\mathbf{Vect})$ and furthermore, for any $s, t \in L^1_s(\mu)$ and $a \in \mathbb{R}$, we have that $\int (s + at)\,\mathrm{d}\mu = \int s\,\mathrm{d}\mu + a \int t\,\mathrm{d}\mu$. It also follows that if $s \geq 0$ everywhere, $\int s\,\mathrm{d}\mu \geq 0$.

Propositon

$L^1_s(\mu)$ is a semi-normed space with the semi-norm $\|\cdot\|$ given by

$$\|s\| := \int |s|\,\mathrm{d}\mu.$$

A Proper Semi-norm and a Quotient Space Fixing This

UNIVERSITY OF CALGARY

Claim

The norm $\|\cdot\|$ on $L^1_s(\mu)$ is a proper semi-norm.
Proof:
Let $S \in A$ with $S \neq \varnothing$ but $\mu(S) = 0$. Then we find that $\|\chi_S\| = 0$ but by construction $\chi_S \neq 0$. This shows that $\|\cdot\|$ is a proper seminorm.

Fixing This Problem

We fix the problem of $\|\cdot\|$ by finding a quotient space of $L^1_s(\mu)$ for which $\|\cdot\|$ is a proper norm; luckily for us, this is done very naturally! Form a relation $\sim$ on $L^1_s(\mu)$ by defining $s \sim t$ if and only if the set

$$\mathscr{D} := \{x \in X \mid s(x) \neq t(x)\}$$

has measure $\mu(\mathscr{D}) = 0$. This then allows us to form a subspace N of $L^1_s(\mu)$ defined by $N := \{u \in L^1_s(\mu) \mid \|u\| = 0\}$. From here on we identify $L^1_s(\mu)$ as the quotient space $L^1_s(\mu)/N$. Perversely we use the same name for both spaces.

There is Zero Measurement Here

UNIVERSITY OF CALGARY

Definition; [Conway], Definition II.4.13

Two A-measurable functions f and g on a set X agree almost everywhere with respect to μ if $\mu(\{x \mid f(x) \neq g(x)\}) = 0$.

Definition

Let M_+ denote the set of nonnegative A-mesuarble functions on A and let $L^1_s(\mu)_+$ denote the set of nonnegative $s \in L^1_s(\mu)$. Then for every $f \in M_+$ we define

$$\int_X f \,\mathrm{d}\mu := \sup_{s \in L^1_s(\mu)_+} \left\{ \int_X s \,\mathrm{d}\mu \mid s \leq f \text{ almost everywhere} \right\}$$

$$= \sup_{s \in L^1_s(\mu)_+} \left\{ \int_X f \,\mathrm{d}\mu \mid s \leq f \right\}.$$

Yet Another Sighting of the Monotone Convergence Theorem

UNIVERSITY OF CALGARY

Lemma

Let $s \in L^1_s(\mu)_+$ and define, for all $T \in A$, $\nu(T) := \int \chi_T s \, \mathrm{d}\mu$. Then ν is a measure on (X, A).

Monotone Convergence Theorem; [Conway], Theorem II.4.16

Let (f_n) be a sequence in M_+ such that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ almost everywhere with respect to μ. Then if the $f(x) = \lim_{n\to\infty} f_n(x)$ almost everywhere with respect to μ, we have that $f \in M_+$ and

$$\int f_n \, \mathrm{d}\mu \to \int f \, \mathrm{d}\mu.$$

Proof of the MCT

Proof

Define a sequence of sets $(S_n) \subseteq A$ by giving $S_n := \{x \mid f_n(x) > f_{n+1}(x)\}$. Then $\mu(S_n) = 0$. Thusly it follows that if $S = \cup_{n\in\mathbb{N}} S_n$ we have $\mu(S) = \mu(\cup_{n\in\mathbb{N}} S_n) \leq \sum_{n\in\mathbb{N}} \mu(S_n) = 0$. Redefine each f_n by taking $f_n(x) = 0$ for all $x \in S$ and note that this new $f_n \in M_+$. This new sequence of (f_n) then satisfies $f_n(x) \leq f_{n+1}(x)$ for every $x \in X$. In a similar process, we again redefine each f_n on a set of measure 0 in A in order to derive that $\lim_{n\to\infty} f_n = f$ exists for every $x \in X$.
It then follows immediately from the monotone behavior of (f_n) that the limit f satisfies $f \in M_+$. Furthermore, the sequence $\left(\int f_n \, \mathrm{d}\mu\right)$ is an increasing sequence of extended reals, which implies that the limit

$$\alpha = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$$

exists, although it may be true that $\alpha = \infty$. Because f is the limiting function of (f_n), it follows that $f_n \leq f$ for each $n \in \mathbb{N}$, and soit follows that $\alpha \leq \int f \, \mathrm{d}\mu$. In order to complete the proof we need to show that $\alpha \geq \int f \, \mathrm{d}\mu$. To do this we let $s \in L^1_s(\mu)_+$ such that $s \leq f$. Let $a \in (0,1)$ and define the set $T_n := \{x \mid as(x) \leq f_n(x)\}$.

Proof of the MCT, Cont.

UNIVERSITY OF CALGARY

Proof, Cont.

Each such $T_n \in A$ and because (f_n) is an increasing sequence with $f_n \to f$ and $\cup_{n \in \mathbb{N}} T_n = X$, we get that $T_n \subseteq T_{n+1}$ for all $n \in \mathbb{N}$. However, we also have that

$$\int f_n \,\mathrm{d}\mu \geq \int_{T_n} f_n \,\mathrm{d}\mu \geq a \int_{T_n} s \,\mathrm{d}\mu.$$

Since the map $\nu : A \to [0, \infty]$ defined by $C \mapsto a \int_C s \,\mathrm{d}\mu$ is a measure on (X, A), the proposition describing various properties of a measure implies that $a \int_{T_n} s \,\mathrm{d}\mu \to a \int s \,\mathrm{d}\mu$. This yields that $\alpha \geq a \int s \,\mathrm{d}\mu$ for all $a \in (0, 1)$. Allowing $a \to 1$ then gives that $\alpha \geq \int s \,\mathrm{d}\mu$ for all nonnegative $s \in L^1_s(\mu)_+$ with $s \leq f$. It then follows from definition that $\alpha \geq \int f \,\mathrm{d}\mu$, which completes the proof. $\square$

Viewing M_+ as Convergent Sequences in $L^1_s(\mu)$

UNIVERSITY OF

Proposition

Let $f \in M_+$. Then there is a seuqnece (s_n) of nonnegative measurabble simple functions such that for every $x \in X$ we have that $(u_n(x))$ is increasing and converges to $f(x)$. Additionally, if f is integrable, then the sequence (s_n) can be chosen with $s_n \in L^1_s(\mu)$.

Proof

Consider the interval, for each positive integer n, $[0, n]$ and divide it into 2^n subintervals. Now consider for each $1 \leq k \leq n2^n$ for $k \in \mathbb{N}^\times$ the set $S_{nk} := \{x \mid (k-1)/2^n \leq f(x) \leq k/2^n\}$ and define

$$s_n := \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{S_{nk}}.$$

Since $S_{nk} = S_{n+1,2k-1} \cup S_{n+1,2k}$, it follows that $s_n \leq s_{n+1}$ for each $x \in X$. We claim that $(s_n(x)) \to f(x)$.

Proof, Cont.

To see that $(s_n(x)) \to f(x)$ note first that the dyadic rationals are dense in $\mathbb{R}$. As such, we may give two nonnegative dyadic rationals a and b such that

$$a \leq f(x) \leq b;$$

in particular, let $\varepsilon > 0$ be given and fix an $x \in X$ and an $n \in \mathbb{N}$ such that, for some given $k \in \mathbb{N}^{\times}$, we have

$$\frac{k-1}{2^n} \leq f(x) \leq \frac{k}{2^n}$$

Then we find that $s_n(x) = (k-1)/2^n \leq f(x)$ and so

$$|f(x) - s_n(x)| < \max\{\varepsilon, s_{n+1}(x)\} \in [s_n(x), f(x)]$$

which implies that

$$|f(x) - s_{n+1}(x)| < \varepsilon.$$

This shows that $(s_n(x)) \to f(x)$. To show that we can take $(s_n) \in L_s^1(\mu)_+$ if f is integrable, note that f integrable implies $\mu(E_{nk}) \in \mathbb{R}$ for every choice of n and k. This shows that each $s_n \in L_s^1(\mu)$. $\square$

Properties of M_+

UNIVERSITY OF CALGARY

Proposition

Let $f, g \in M_+$ and let $a \in \mathbb{R}_{\geq}$ and let $L^1(\mu) := \overline{(L_s^1(\mu), \|\cdot\|)}$. Then the following hold:

1. $\int (f + ag)\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu + a\int g\,\mathrm{d}\mu$.
2. $f \geq g$ and both f and g are integrable, then $\int (f - g)\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu - \int g\,\mathrm{d}\mu$.
3. Each integrable function $f \in M_+$ corresponds to an element $L^1(\mu)$.

Comments

The proof of the conditions in the above proposition are fairly routine and grindy, save for two aspects: when we prove condition (3) we have to cosntruct a sequence $(s_n) \to f$ with $(s_n) \subseteq L_s^1(\mu)$, show that (s_n) is Cauchy, and finally show that $f \in L^1(\mu)$. This is really where the meat of the proof lies, so we will do this on the next slide.

Cauchy Sequences of $L_s^1(\mu)$ to $f \in M_+$ Integrable

Sketching Why (s_n) is Cauchy

Let $(s_n) \to f \in M_+$ with $(s_n) \subseteq L_s^1(\mu)$. Then by the MCT and a previous proposition, we know that $\left(\int f_n \, \mathrm{d}\mu\right) \to \int f \, \mathrm{d}\mu$. Let $\varepsilon > 0$ and choose an $N \in \mathbb{N}$ such that for all $n \geq N$

$$0 \leq \int f \, \mathrm{d}\mu - \int s_n \, \mathrm{d}\mu < \frac{\varepsilon}{2}.$$

Then if $m \geq n \geq N$ we have that

$$\begin{aligned} \|s_m - s_n\| = \int (s_m - s_n) \, \mathrm{d}\mu &= \int s_m \, \mathrm{d}\mu - \int s_n \, \mathrm{d}\mu \\ &= \int s_m \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu + \int f \, \mathrm{d}\mu - \int s_n \, \mathrm{d}\mu \\ &\leq \left| \int s_m \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu \right| + \left| \int f \, \mathrm{d}\mu - \int s_n \, \mathrm{d}\mu \right| < \varepsilon. \end{aligned}$$

The final proof is to show that f does not depend on the choice of Cauchy seuqence that converges to it. This is a standard argument, however, and will be omitted.

Finally, Integration

Definition

Let $f \in M_+$ and define $f_+ := \max\{f(x), 0\}$ and $f_- := \min\{-f(x), 0\}$. Then $f_+, f_- \in M_+$ and $f = f_+ - f_-$. Similarly, $|f| = f_+ + f_-$.

Definition; [Conway], Definition II.4.20

A function $f : X \to \hat{R}$ is μ-integrable if f is measurable and both f_+ and f_- are integrable. In particular, we define the integral of f by

$$\int f \,\mathrm{d}\mu = \int f_+ \,\mathrm{d}\mu - \int f_- \,\mathrm{d}\mu.$$

Integrable Functions and $L^1(\mu)$

UNIVERSITY OF CALGARY

Proposition

If f is an integrable function, then f corresponds to an element in $L^1(\mu)$. Furthermore,

$$\|f\| = \int |f| \, \mathrm{d}\mu = \int f_+ \, \mathrm{d}\mu + \int f_- \, \mathrm{d}\mu$$

defines a norm on $L^1(\mu)$.

Theorem; [Conway], Proposition II.4.22

1. The space $I_\mu(X)$ of μ-integrable functions on (X, A) is a real vector space and the corresponding μ-integral acts as a linear functional on $I_\mu(X)$.
2. If f, g are integrable functions with $f \leq g$, then $\int f \, \mathrm{d}\mu \leq \int g \, \mathrm{d}\mu$.
3. If f is an integrable function, then $\mu(\{x \mid |f(x)| = \infty\}) = 0$.

Variation of a Measure

Complex Measurements

We now allowing our measures to be complex. Intuitively, if X is a set and A is a σ-algbra over X, we say that a function $\mu : A \to \mathbb{C}$ is a measure on X if we may write $\mu(S) = (\mu_1 - \mu_2)(S) + i(\mu_3 - \mu_4)(S)$ and all of the μ_i are real-valued signed measures (measures into $\hat{\mathbb{R}}$). We define, for a set X and σ-algebra A over X, a function $f : X \to \mathbb{C}$ to be a measureable function if, for every open set $D \subseteq \mathbb{C}$, the set $f^{-1}(D) \in A$.

Variations in Measurement

Let μ be a complex measure on (X, A). Let $S \in A$ be arbitrary and define

$$|\mu|(S) := \sup \left\{ \sum_{k=0}^{n} |\mu(P_k)| \;\middle|\; \mathcal{P} = \{P_k\}_{k=0}^{n}, n \in \mathbb{N}, \mathcal{P} \text{ partitions } S, P_k \in A \right\}.$$

Regular Borel Measures, as Opposed to Abnomral Borel Measures

Regular Borel Measures; Definition

Let X be a locally compact metric space equipped with its metric topology and let A be a σ-algebra over X such that $\mathfrak{B}_X \subseteq A$. We then say that a measure μ on X is a regular Borel measure on X if $\mu : A \to \hat{\mathbb{C}}$, $\mu : A \to (-\infty, \infty]$, or if $\mu : A \to [-\infty, \infty)$ such that the following three conditions hold:

1. $|\mu|(K) < \infty$ for every compact $K \subseteq X$.
2. For every set $S \in A$,
$$|\mu|(S) = \inf_{G \text{ open}} \left\{ |\mu(G)| \;\middle|\; S \subseteq G \right\}.$$
3. For every set $S \in A$,
$$|\mu|(S) = \sup_{K \text{ compact}} \left\{ |\mu(K)| \;\middle|\; K \subseteq S \right\}.$$

A Haar-y Situation

Haar Measures; cf. [Farb], Exercise I.27

Let G be a topological group and let A be a σ-algebra such that $\mathfrak{B}_G \subseteq A$. Then a measure $\mu : A \to [0, \infty]$ is said to be a left Haar Measure on G if μ is a regular Borel measure and, for all $g \in G$ and every $S \in A$, $\mu(gS) = \mu(S)$. In the spirit of every noncommutative theory ever, right Haar measures are defined similarly.
A Haar measure that is both simultaneously a left and right Haar measure is called bi-invariant. This is satisfied for free on all Abelian groups with a Haar measure.

An Example of a Haar Measure

Let G be a finite topological group and let $\mathfrak{B}_G$ be the Borel algebra on G. Then, after normalizing the counting measure $\mu : \mathfrak{B}_G \to [0, \infty]$ by defining

$$\mu^*(S) := \frac{|S|}{|G|},$$

it follows that μ^* defines a new measure $\mu^* : \mathfrak{B}_G \to [0, 1]$. This is a Haar measure on G.

Properties of Haar Measures

UNIVERSITY OF CALGARY

Proposition

Let G be a locally compact topological group and let μ be a regular Borel measure on G that is finite on all compact $K \subseteq G$. Then the following hold:

1. If μ is a left Haar measure on G and $\varphi \in \mathrm{Aut}_{\mathbf{TopGrp}}(G)$, then $\mu\varphi$ is a left Haar measure on G.
2. If μ is a left Haar measure on G, then μ is positive on all nonempty open subsets of G and for all $f \in \mathcal{C}_c^+(G)$

$$\int_G f \, \mathrm{d}\mu > 0.$$

3. If μ is a left Haar measure on G then $\mu(G) < \infty$ if and only if G is compact.

Proof of Property (1)

Proof

(1): We will show that if $\varphi \in \mathrm{Aut}_{\mathbf{TopGrp}}(G)$ and μ is a left Haar measure on G, then $\mu\varphi$ is a left Haar measure. WOLOG take our σ-algebra to $\mathfrak{B}_G$, as this is really where the work is being done. To begin, note that since φ is a homeomorphism, a set $U \subseteq G$ is open if and only if $\varphi(U)$ is open, C is closed if and only if $\varphi(C)$ is closed, $A \in \mathfrak{B}_G$ if and only if $\varphi(A) \in \mathfrak{B}_G$, and K is compact in G if and only if $\varphi(K)$ is compact in G. Thusly we find that

$$|\mu|(K) < \infty \iff |\mu|(\varphi(K)) < \infty,$$

which is satisfied because μ is a Haar measure and thusly a regular Borel measure. Similarly because for any set $S \in \mathfrak{B}_G$ $|\mu|(S) = \inf_{U \text{ open}}\{|\mu|(U) \mid S \subseteq U\}$ it follows that

$$|\mu\varphi|(S) = |\mu|(\varphi(S)) = \inf_{U \text{ open}}\{|\mu|(U) \mid \varphi(S) \subseteq U\}$$

$$= \inf_{\varphi(V) \text{ open}}\{|\mu|(\varphi(V)) \mid \varphi(S) \subseteq \varphi(V)\} = \inf_{\varphi(U) \text{ open}}\{|\mu\varphi|(V) \mid \varphi(S) \subseteq \varphi(V)\}$$

for some unique open V that maps to U under φ.

The Riesz Representation Theorem (the Saddest Representation Theory)

UNIVERSITY OF CALGARY

Proof, Cont.

The final condition for $\mu\varphi$ being a regular Borel measure is proved similarly. Lastly, we show that $\mu\varphi$ is left G-invariant. To see this, let $g \in G$ and $A \in \mathfrak{B}_G$. Then

$$\mu\varphi(gA) = \mu(\varphi(gA)) = \mu(\varphi(g)\varphi(A)) = \mu(hB) = \mu(B) = \mu\varphi(A)$$

for the unique $B = \varphi(A)$ and $h = \varphi(g)$.

Riesz Representation Theorem; [Conway], Theorem IV.4.1

Let X be a locally compact, σ-compact metric space and let $L : \mathcal{C}_0(X) \to \mathbb{C}$ be a bounded linear functional, where $\mathcal{C}_0(X)$ is the set of continuous functions that vanish at ∞. Then there exists a unique finite regular Borel Measure μ such that $L(f) = \int f \,\mathrm{d}\mu$ for all $f \in \mathcal{C}_0(X)$; moreover, $\|L\| = \|\mu\|$.

Two Maps and the Power of Tensors

Definition

Let X and Y be given sets and let $f : X \prod Y \to \mathbb{C}$. Then for each $x \in X$ and $y \in Y$, we define the functions $\iota_x : Y \to \mathbb{C}$ and $\iota_y : X \to \mathbb{C}$ by $\iota_x(y_1) := f(x, y_1)$ and $\iota_y(x_1) := f(x_1, y)$.

Definition

Let (X, d) and (Y, ∂) be metric spaces. Then define the metric δ on $X \prod Y$ by

$$\delta((x, y), (a, b)) := d(x, a) + \partial(y, b).$$

Note that the metric δ is a choice of convenience; however, we only want to use a metric up to the equivalence of δ, and the use of δ will make our lives easier, so we may as well use it. Now, for any two functions $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$, define the tensor product of f and g by

$$f \otimes g : X \prod Y \to \mathbb{C}, \qquad (f \otimes g)(x, y) := f(x)g(y).$$

Note that $\iota_x(f \otimes g) = g$ and $\iota_y(f \otimes g) = f$. In particular, if X and Y are σ-compact, $f \in \mathcal{C}_0(X)$, and $g \in \mathcal{C}_0(Y)$, then $f \otimes g \in \mathcal{C}_0\left(X \prod Y\right)$.

More on Tensors and Product Measures

UNIVERSITY OF CALGARY

Theorem; [Conway], Proposition IV.7.3

Let X and Y be σ-compact locally compact metric spaces and let $\mathfrak{T}_0(X, Y)$ be the linear span of $f \otimes g$ for all $f \in \mathcal{C}_0(X)$ and $g \in \mathcal{C}_0(Y)$ (i.e., $\mathfrak{T}_0(X, Y)$ is the tensor algebra on $\mathcal{C}_0(X) \otimes_{\mathbb{C}} \mathcal{C}_0(Y)$). Then $\mathfrak{T}_0(X, Y)$ is dense in $\mathcal{C}_0(X, Y)$.

Lemma; [Conway], Lemma IV.7.5

There exists a regular Borel measure on $X \prod Y$ such that for every continuous function f on $X \prod Y$ we have

$$\int f(x,y)\,\mathrm{d}(x,y) = \int \left(\int f(x,y)\,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y)$$

$$= \int \left(\int f(x,y)\,\mathrm{d}\nu(y) \right) \mathrm{d}\mu(x).$$

A Sketch of the Proof of the Lemma

Proof

Define $L : \mathcal{C}_0\left(X \prod Y\right) \to \mathbb{C}$ by

$$L(f) := \int \left(\int f(x,y)\,\mathrm{d}\mu(x) \right)\,\mathrm{d}\nu(y).$$

The trick now is to show that the function $y \mapsto \int f(x,y)\mathrm{d}\mu(x)$ is ν-integrable. To do this it is sufficient to show that the map just described is continuous in Y; in particular, if $(y_n) \to y$ then $f(x, y_n) \to f(x,y)$ for every $x \in X$ and $|f(x,y)|$ is dominated by $\|f\|$. Thusly, by the DCT (see Erik's first presentation for the DCT), it follows that $\int f(x,y_n)\,\mathrm{d}\mu(x) \to \int f(x,y)\,\mathrm{d}\mu(x)$, which does the job. Thusly our the outer integral used in our definition of $L(f)$ actually makes sense.

By construction, it follows that $L(f)$ is a positive linear functional with $|L(f)| \leq \|\mu\|\|\nu\|\|f\|$, which gives us the existence of a finite positive Radon measure η on $X \prod Y$ by the Riesz Representation Theorem such that, for all $f \in \mathcal{C}_0\left(X \prod Y\right)$,

$$\int f(x,y)\,\mathrm{d}\eta(x,y) = \int \left(\int f(x,y)\,\mathrm{d}\mu(x) \right)\,\mathrm{d}\nu(y).$$

Yet Another Proof Continuation

Proof, Cont

We can show similarly that there is a finite positive Radon measure τ on $X \prod Y$ that gives the integral for all $f \in \mathcal{C}_0 \left(X \prod Y \right)$

$$\int f(x,y) \, \mathrm{d}\tau(x,y) = \int \left(\int f(x,y) \, \mathrm{d}\nu(y) \right) \mathrm{d}\mu(x).$$

Now for every $a \in \mathcal{C}_0(X)$ and $b \in \mathcal{C}_0(Y)$ we have that

$$\int (a \otimes b) \, \mathrm{d}\eta = \int a \, \mathrm{d}\mu \int b \, \mathrm{d}\nu = \int (a \otimes b) \, \mathrm{d}\tau.$$

Thusly integration with respect to η is exactly eqiuvalent to integration with respect to τ on $\mathfrak{T}_0 \left(X \prod Y \right)$. By the density of this algebra in $\mathcal{C}_0 \left(X \prod Y \right)$ it will follow that integration in η is the same as integration in τ. This completes the sketch.

The Final Two Lemmas for Fubini's Theorem

Lemma; [Conway], Lemma IV.7.7

Let $S \subseteq X \prod Y$ be either an open or compact subset. Then $y \mapsto \mu\iota_y(S)$ is ν-measurable, $x \mapsto \nu\iota_x(S)$ is μ-measurable, and

$$\eta(S) = \int \mu\iota_y(S)\,\mathrm{d}\nu(y) = \int \nu\iota_x(S)\,\mathrm{d}\mu(x).$$

Lemma

Let $S \subseteq X \prod Y$ such that $\eta(S) = 0$. Then $\iota_y(S) \in \mathfrak{B}_X$ and $\mu\iota_y(S) = 0$ almost everywhere wrt ν, while $\iota_x(S) \in \mathfrak{B}_Y$ and $\nu\iota_x(S) = 0$ almost everywhere wrt μ. Thusly, if $S \in \mathfrak{B}_{X \prod Y}$, then $\iota_y(S) \in \mathfrak{B}_X$ almost everywhere wrt ν and $\iota_x(S) \in \mathfrak{B}_Y$ almost everywhere wrt μ.

The Measure Theoretic Version of Fubini for LC Spaces

UNIVERSITY OF CALGARY

Fubini's Theorem; [Conway], Theorem IV.7.1 (cf. [Deitmar], Theorem VIII.2.1)

Let X and Y be σ-compact locally compact metric spaces equiped with their metric topologies. If μ and ν are finite positive regular Borel measures on X and Y, repsectively (i.e., $\mu : A \to \mathbb{R}_{\geq 0}$ and $\nu : B \to \mathbb{R}_{\geq 0}$ are regular Borel measures, $\mathfrak{B}_X \subseteq A$, and $\mathfrak{B}_Y \subseteq B$) then there exists a unique finite positive regular Borel measure $\eta := \mu \times \nu$ that satisfies the following properties:

1. For all Borel sets $S \in A$ and $T \in B$, $(\mu \times \nu)(S \times T) = \mu(S)\nu(T)$.
2. If $f \in L^1(\mu \times \nu)$, then $\iota_x(f) \in L^1(\nu)$ for almost every $x \in X$ with respect to μ and $\iota_y(f) \in L^1(f)$ for almost every $y \in Y$ with respect to ν. Furthermore,

$$\int_{X \times Y} f(x,y)\,\mathrm{d}(\mu \times \nu)(x,y)$$
$$= \int_Y \left(\int_X f(x,y)\,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y) = \int_X \left(\int_Y f(x,y)\,\mathrm{d}\nu(y) \right) \mathrm{d}\mu(x).$$

Proof of Fubini's Theorem

UNIVERSITY OF CALGARY

Proof; [Conway], p.138,139

We have already seen that the product measure $\eta = \mu \prod \nu$ exists. We now pretend that orderings are a natural phenomenon and prove condition (2) of Fubini's Theorem. We complete the proof now by running the standard machinery of measure theory. Let S be some η-measurable set and assume that $f = \chi_S$. Then by the first of the final two lemmas, condition (2) follows whenever S is open or compact. In general, let (K_n) be a sequence of ascending compact sets contained in S with $\eta(K_n) \to \eta(S)$. Then it follows that, up to a set of η-measure zero, $\chi_{K_n} \to \chi_S$. Now write $K = \cup_{n \in \mathbb{N}} K_n$. Then $\eta(S \setminus K) = 0$ and so the second of the final two lemmas implies that there exists a set $A \subseteq Y$ with $\nu(A) = 0$ such that $\mu\iota_y(A) = 0$ whenever $y \notin A$. Thusly we find that the sequence $(\iota_y \chi_{K_n}) \to (\iota_y \chi_S)$ almost everywhere wrt μ whenever $y \notin A$. Then, by the MCT, the map $y \mapsto \mu\iota_y(S)$ is ν-measurable and $\eta(K_n) = \int \mu\iota_y(K_n)\,\mathrm{d}\nu(y) \to \int \mu\iota_y(S)\,\mathrm{d}\nu(y)$. Since $\eta(K_n) \to \eta(S)$, we conclude that $\eta(S) = \int \mu\iota_y(S)\,\mathrm{d}\nu(y)$ for each $y \notin A$. A symmetric argument shows that $\eta(S) = \int \nu\iota_x(S)\,\mathrm{d}\mu(x)$ almost everywhere wrt ν.

Proof of Fubini, Part Two

Proof, Cont.

From what we have just shown, it follows easily through the $\mathbb{C}$-linearity of the integral that condition (2) of Fubini holds for positive, simple η-measurable functions. Let f be an arbitrary positive measurable function. Now we use the density of $L^1_s(\eta)$ to produce a sequence of functions $(s_n) \subseteq L^1_s(\eta)$ such that $(s_n) \to f$ almost everywhere wrt η and $\left(\int s_n \,\mathrm{d}\eta\right) \to \int f \,\mathrm{d}\eta$. Take $S \subseteq X \prod Y$ to be a set with $\eta(S) = 0$ such that $s_n(x, y) \to f(x, y)$ for all $(x, y) \notin S$. Then, again by the second of the two final lemmas, there is a set $A \subseteq Y$ such that $\nu(A) = 0$ and $\mu\iota_y(S) = 0$ whenever $y \notin A$. As such it follows that for all $y \notin A$ and all $x \notin \iota_y(S)$ that $(\iota_y(s_n)) \to \iota_y(f)$ and hence $\int \iota_y(s_n) \,\mathrm{d}\mu \to \int \iota_y(f) \,\mathrm{d}\mu$. Thusly the map $y \mapsto \int \iota_y(f) \,\mathrm{d}\mu$ is integrable. A routine application of limits now gives that

$$\int f \,\mathrm{d}\eta = \imath_{n\to\infty} \int (\iota_y(s_n) \,\mathrm{d}\mu) \,\mathrm{d}\nu = \int \left(\int \iota_y(f) \,\mathrm{d}\mu \right) \mathrm{d}\nu.$$

The conclusion for the other variable is given by running the same argument over x. Now, since every function in $L^1(\eta)$ is the difference of two nonnegative functions in $L^1(\eta)$, we have shown condition (2) of Fubini! To complete the proof take $f = \chi_{S \prod T}$ for $S \in \mathfrak{B}_X$ and $T \in \mathfrak{B}_Y$.

A *Character*ization of Fourier Transforms

Let A be an LCA group and fix a Haar measure μ on A in order to obtain the Haar integral

$$I(f) := \int_A f \,\mathrm{d}\mu.$$

Let $\hat{A}$ be the dual group, i.e., the group of all characters

$$\chi : A \to \mathbb{S}^1 \subseteq \mathbb{C}.$$

Then, for every $f \in L^1_{bc}(\mu)$ define the Fourier Transform $\hat{f} : \hat{A} \to \mathbb{C}$ by

$$\hat{f}(\chi) := \int_A f(a)\overline{\chi(a)}\,\mathrm{d}\mu(a).$$

A Convolved Theorem

UNIVERSITY OF CALGARY

Theorem; [Deitmar], Theorem VIII.3.1

Let A be an LCA group with Haar measure μ. Then, for $f, g \in L^1_{bc}(\mu)$ the function

$$(f * g)(x) := \int f(xy^{-1})g(y)\,\mathrm{d}\mu(y)$$

exists for every $x \in A$ and defines a function $f * g \in L^1_{bc}(\mu)$. For the Fourier transform (i.e., we add a bunch of 4s and make things ``Fourier'') we have

$$\widehat{f * g}(\chi) = \hat{f}(\chi)\hat{g}(\chi).$$

Proof of the Convolution

Proof of Existence of the Integral and the Beginning of Continuity

Since $f \in L^1_{bc}(\mu)$, assume that $|f(x)| \le C$ for some $C \in \mathbb{R}_+$. Then we find that $\left|\int f(xy^{-1})g(y)\,\mathrm{d}\mu(y)\right| \le \int |f(xy^{-1})g(y)|\,\mathrm{d}\mu(y) \le C\int |g(y)\,\mathrm{d}\mu| = C\|g\|_1$.
This shows that the integral exists and that $f * g$ is bounded.
We now show continuity. Since f, g are both bounded, assume that $|f(x)|, |g(x)| \le M \in \mathbb{R}_+$ for all $x \in A$ and assume that $g \neq 0$. Then for each $\varepsilon > 0$ there is a function $\varphi \in \mathcal{C}_c(A)_+$ such that $\varphi \le |g|$ and

$$\int (|g(y)| - \varphi(y))\,\mathrm{d}\mu(y) < \frac{\varepsilon}{4M}.$$

On any compact set $K \subseteq A$ the function f is uniformly continuous, so there is a neighborhood N of 1_A such that for every $x \in x_0N$ we have $y \in \mathrm{supp}(\varphi)$ implies that

$$|f(xy^{-1}) - f(x_0y^{-1})| \le \frac{\varepsilon}{2\|g\|_1}.$$

Proof of Continuity

UNIVERSITY OF CALGARY

Proof of Continuity, Cont.

It then follows that for each $x \in x_0 N$ we have

$$\int |f(xy^{-1}) - f(x_0 y^{-1})|\varphi(y)\,\mathrm{d}\mu(y) \leq \frac{\varepsilon}{2\|g\|_1}\int \varphi\,\mathrm{d}\mu \leq \frac{\varepsilon}{2}.$$

However, we also have the estimate that

$$\int |f(xy^{-1}) - f(x_0 y^{-1})|(|g(y)| - \varphi(y))\,\mathrm{d}\mu(y) \leq 2M\int (|g| - \varphi)\,\mathrm{d}\mu < \frac{\varepsilon}{2}.$$

Finishing Continuity and Boundedness of $f * g$

Proof, Cont.

Then, over $x_0 N$ we have

$$|(f*g)(x) - (f*g)(x_0)| = \left| \int (f(xy^{-1}) - f(xy^{-1}))g(y)\,\mathrm{d}\mu(y) \right|$$
$$\leq \int |f(xy^{-1}) - f(x_0 y^{-1})||g(y)|\,\mathrm{d}\mu(y)$$
$$= \int |f(xy^{-1}) - f(x_0 y^{-1})|(|g(y)| - \varphi(y) + \varphi(y))\,\mathrm{d}\mu(y) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows continuity. To see the boundedness of $f * g$ in the L^1-norm, we use Fubini's Theorem and the fact that μ is a Haar measure on A.

$$\|f*g\|_1 = \int |(f*g)(x)|\,\mathrm{d}\mu(x) = \int \left| \int f(xy^{-1})g(y)\,\mathrm{d}\mu(y) \right| \mathrm{d}\mu(x)$$
$$\leq \int \int |f(xy^{-1})\,\mathrm{d}\mu(y)|\,\mathrm{d}\mu(x) = \int \int |f(xy^{-1})f(y)|\,\mathrm{d}\mu(x)\mathrm{d}\mu(y)$$
$$= \int |f|\,\mathrm{d}\mu \int |g|\,\mathrm{d}\mu = \|f\|_1 \|g\|_1 < \infty.$$

The Transform Formula

UNIVERSITY OF CALGARY

Proof of the Transformula

We prove the transformula by just straight up plugging and chugging. Explicitly, we compute for $f, g \in L^1_{bc}(\mu)$

$$\widehat{f * g}(\chi) = \int (f * g)(x)\overline{\chi(x)}\,\mathrm{d}\mu(x) = \int\int f(xy^{-1})g(y)\overline{\chi(x)}\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)$$

$$= \int g(y)\overline{\chi(y)} f(xy^{-1})\overline{\chi(xy^{-1})}\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)$$

$$= \int g(y)\overline{\chi(y)}\,\mathrm{d}\mu(y) \int f(x)\overline{\chi(x)}\,\mathrm{d}\mu(x) = \hat{g}(\chi)\hat{f}(\chi)$$

This proves the theorem and provides us with a nice introduction of how to manipulate the Haar measure on an LCA group. We now move on to provide a somewhat alternative view to a classical theorem of Algebra/Representation Theory.

A Mash-Up With Maschke's Theorem

UNIVERSITY OF CALGARY

Maschke's Theorem Version One; [Serre], Proposition 6.10

Let K be an algebraically closed field of characteristic zero and let G be a finite group. Then $K[G]$ is semisimple in the following way: let $\rho_i : G \to \mathrm{GL}(W_i)$ be the distinct irreducible *rho*presentations of G over K-vector spaces W_i (up to isomorphism) and set $n_i = \dim(W_i)$ so that $\mathrm{End}_{\mathbf{K-Vect}}(W_i) \cong \mathrm{Mat}_{n_i}(K)$. Then each map ρ_i extends by linearity to a homomorphism of K-aglebras $\hat{\rho}_i : K[G] \to \mathrm{End}_{\mathbf{K-Vect}}(W_i)$. Combining these as a family gives an isomorphism of K-aglebras $\hat{\rho} : K[G] \to \prod_{i=1}^{m} \mathrm{Mat}_{n_i}(K)$; the content of Maschke's Theorem is that $\hat{\rho}$ is an isomorphism.

Maschke's Theorem Version Two; [Farb], Exercise 1.27

Let K be a field and let G be a finite group such that char$(K) \nmid |G|$. Then $K[G]$ is a semisimple project.

A New Perspective

A New Perspective; cf. [Farb], Exercise I.27

The isomorphism $\hat{\rho}$ described on the previous slide tells us that whenever we have a short exact sequence $0 \to B \to A \to A/B \to 0$ of $K[G]$ modules, the sequence splits. This is surprising! While we know that it certainly splits when we treat each module as a K-vector space, i.e., there is a section $s : A/B \to A$, it is not clear that it splits over $K[G]$. However, if we define a map $S : A/B \to A$ of $K[G]$-modules by $S(x) := \sum_{g \in G} gs(g^{-1}x)$, the map S turns out to be a section on A/B! Now, we ask, what does this have to do with Harmonic Analysis? Well, since G is finite, treat it as a discrete topological group acting on K-vector spaces and give G its Haar measure (because G is finite it has only one: the normalized counting measure). Then sums are transformed into integrals and the splitting formula now may be written as

$$\frac{1}{|G|} S(x) = \frac{1}{|G|} \sum_{g \in G} gs(g^{-1}x) = \int_G gs(g^{-1}x) \, \mathrm{d}\mu(g).$$

This is just the convolution of the identity with s and so, quite surprisingly, the above splitting is actually a Fourier Inversion Formula!

References

UNIVERSITY OF CALGARY

References Cited

Conway, John B.. *A Course in Abstract Analysis*. 1st Ed. USA: AMS. 2012. Print. GSM 141.

Deitmar, Anton. *A First Course in Harmonic Analysis*. 2nd Ed. New York NY: Springer-Verlag. 2005. Print. Universitext.

Farb, Benson, and R. Keith Dennis. *Noncommutative Algebra*. 1st Ed. New York NY: Springer-Verlag. 1993. Print. GTM 144.

Serre, Jean-Pierre. *Linear Representations of Finite Groups*. 2nd Ed. New York, NY: Springer-Verlag. 1977. Print. GTM 42.