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Analytical Solutions in the Darboux Problem and the Bortz Equation and the Approach to Orientation Algorithms Based on Them

A.V. Molodenkov[a,*], Ya.G. Sapunkov[a,*], T.V. Molodenkova[b,], and S.E. Perelyaev[c,***]**

[a] *Institute of Precision Mechanics and Control, Russian Academy of Sciences, Saratov, Russia*
[b] *Gagarin Saratov State Technical University, Saratov, Russia*
[c] *Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russia*
**e-mail: molalexei@yandex.ru*
***e-mail: moltw@yandex.ru*
****e-mail: sergey-perelyaev@mail.ru*

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Abstract—We consider the problem of determining the angular position of a rigid body in space from its known angular velocity and initial position (the Darboux problem) in the quaternion setting. Based on the exact solution of the Bortz approximate differential equation with respect to the orientation vector of the rigid body, we analytically solve the problem to determine the quaternion of the orientation of the rigid body for each arbitrary angular velocity and small rotation angle of the rigid body. Based on this solution, we propose an approach to design a new algorithm to compute the orientation of moving objects by means of strapdown inertial navigation systems.

Keywords: analytical solution, orientation, arbitrary angular velocity, rigid body, strapdown INS, quaternion, algorithm

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INTRODUCTION

In the operation of various strapdown inertial navigation systems (SINS), the orientation vector of a rigid body (object) with respect to an inertial space is periodically computed by means of the approximate resolving of the Bortz approximate linear differential equation (in the theory and practice of SINS design in superfast cycles of algorithms, the nonlinear term of the Bortz equation is disregarded under small rotation angles, see [1–3]). For the Bortz equation, the input value is the vector of the angular velocity of the rigid body. Note that the total nonlinear Bortz equation with respect to the orientation vector of the rigid body is an analog of the quaternion linear equation with respect to the orientation quaternion of the rigid body; the vector and the quaternion are linked between each other by means of known relations. The Bortz approximate linear vector differential equation is solved by various numerical methods. For example, it can be solved by the Picard method; since, in practice, the second iteration of this method can be treated as the final one. The corresponding term of the iteration relation of the Picard method is called the noncommutative rotation vector, or coning. For certain motions of a rigid body, this term essentially affects
1 the method error. The investigation of noncommutative rotations (conings) as a kind of mechanical motion of bodies and the separation of numerical algorithms to detect the orientation of a rigid body (SINS) into the superfast, fast, and slow computing cycles are aimed to compensate the influence of this phenomenon (see [2–5]). However, for a new vector of the angular velocity, obtained in the problem of the detection of the orientation of the rigid body (SINS) by means of an arbitrary original vector of the angular velocity and one-to-one changes of variables in motion equations for the rigid body, the Bortz approximate equation admits a precise analytic solution. Let us show this.

We set the following Darboux problem: using the (arbitrary) angular velocity vector $\boldsymbol{\omega}(t)$ and initial angular location of a rigid body in space, find its orientation quaternion $\boldsymbol{\Lambda}(t)$ by means of the quaternion kinematic equation. Further, we change variables as follows: $\boldsymbol{\Lambda} \to \mathbf{U}$, where $\mathbf{U}$ is the orientation quaternion of an introduced coordinate system such that the inverse passage $\mathbf{U} \to \boldsymbol{\Lambda}$ is possible). These changes, being rotation transformations, reduce the original problem to find the orientation of a rigid body (the quaternion $\boldsymbol{\Lambda}$) with an arbitrary variable vector $\boldsymbol{\omega}(t)$ of the angular velocity to the problem, where the vector $\mathbf{w}(t)$ of the angular velocity of the introduced coordinate system rotates around an axis of the coordinate system though its modulus is, in general, variable; this motion is a generalized conical precession and it is well coordinated with the following Poinsot concept: each rotation of a rigid body around a fixed point can be represented as a conical motion. It is still a hard problem to find an analytic solution of the quaternion differential equation with respect to the new unknown quaternion $\mathbf{U}$. However, the equation with the angular velocity vector $\mathbf{w}(t)/2$ (instead of $\mathbf{w}(t)$) can be solved in closed form. Note that the homogeneous Poincaré vector differential equation is isomorphic to the quaternion differential equation.

To the obtained problem with the angular velocity vector $\mathbf{w}(t)$ and an unknown orientation quaternion $\mathbf{U}$, we put in correspondence the total Bortz equation with respect to the unknown orientation vector $\boldsymbol{\varphi}$. The approximate linear Bortz equation, which is a homogeneous vector differential equation such that its homogeneous part is equivalent to the Poisson equation with the vector coefficient $\mathbf{w}(t)/2$, becomes analytically solvable; its solution $\boldsymbol{\varphi}^*$ is found in analytical functions and quadratures by means of the Lagrange method.

In this paper, the precise solution of the Bortz approximate equation obtained is used to analytically solve the problem of finding the orientation quaternion of a rigid body under the assumption the angular velocity vector of the rigid body is arbitrary and its rotation angle is small. Based on this solution, we propose the following approach for constructing a new algorithm to compute the inertial orientation of SINS:

(1) Using the given components of the vector $\boldsymbol{\omega}(t)$ of the angular velocity of the rigid body and one-to-one changes of variables, we compute the vector $\mathbf{w}(t)$ of a new introduced coordinate system for each time.

(2) Using the new angular velocity vector $\mathbf{w}(t)$ and the initial location of the rigid body, we find the precise solution $\boldsymbol{\varphi}^*$ of the Bortz approximate equation with the zero initial-value condition by means of quadratures.

(3) Using the orientation vector, we find the value of orientation quaternion of the rigid body (SINS) according to the $\boldsymbol{\varphi}^* \approx \boldsymbol{\varphi} \Leftrightarrow \mathbf{U} \rightarrow \boldsymbol{\Lambda}$ scheme.

Note that, at each step of the constructing of the SINS orientation algorithm, the change of variables takes into account the previous step of the algorithm in order to nullify the initial value of the orientation vector of the rigid body.

Since the proposed algorithm of the analytical resolving of the Bortz linear approximate equation is precise, it follows that it is regular for all angular motions of the rigid body.

This paper continues the investigations originated in [6, 7].

1. ORIENTATION DETECTION FOR RIGID BODIES (SINS): QUATERNION PROBLEM SETTING

Consider the following Cauchy problem for the quaternion kinematic equation (see [8]) with an arbitrary given vector function $\boldsymbol{\omega}(t)$ of the angular velocity:

$$2\dot{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda} \circ \boldsymbol{\omega}(t), \tag{1.1}$$

$$\boldsymbol{\Lambda}(t_0) = \boldsymbol{\Lambda}_0, \tag{1.2}$$

where $\boldsymbol{\Lambda}(t) = \lambda_0(t) + \lambda_1(t)i_1 + \lambda_2(t)i_2 + \lambda_3(t)i_3$ is the quaternion describing the location of the rigid body in the inertial space, $\boldsymbol{\omega}(t) = \omega_1(t)\mathbf{i}_1 + \omega_2(t)\mathbf{i}_2 + \omega_3(t)\mathbf{i}_3$ is the angular velocity vector of the rigid body, given by its projection on the axes of the coordinate system related to the rigid body, i_1, i_2, and i_3 are unit vectors are the unit vectors of the hypercomplex space (Hamilton imaginary units) that can be identified with the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ of the three-dimensional vector space, the symbol $\circ$ denotes the quaternion product, and $\boldsymbol{\Lambda}_0$ is the initial value of the quaternion $\boldsymbol{\Lambda}(t)$ for $t = t_0$, $t \in [t_0, \infty)$ (t_0 can be assigned to be equal to zero). It is required to find the quaternion $\boldsymbol{\Lambda}(t)$.

This problem is called the Darboux problem.

The following matrix form of the problem, using, e.g., $\mathbf{n}$-type quaternion matrices (see [9]), is equivalent to its quaternion form:

$$\mathbf{n}(\boldsymbol{\Lambda}) = \begin{bmatrix} \lambda_0 & -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_1 & \lambda_0 & \lambda_3 & -\lambda_2 \\ \lambda_2 & -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_3 & \lambda_2 & -\lambda_1 & \lambda_0 \end{bmatrix}, \quad \mathbf{n}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}, \tag{1.3}$$

$$\mathbf{2n}(\mathbf{\Lambda})^{\bullet} = \mathbf{n}(\boldsymbol{\omega}(t))\mathbf{n}(\mathbf{\Lambda}), \tag{1.4}$$

$$\mathbf{n}(\mathbf{\Lambda}(t_0)) = \mathbf{n}(\mathbf{\Lambda}_0). \tag{1.5}$$

Several approaches to solving this problem are known: using change of variables, one can reduce the original equations to a nonlinear first-order differential equation of the Riccati type (the Darboux approach from [10]) or to a linear second-order differential equation (see [9]) with variable coefficients with respect to a complex desired function; one can identify the Darboux problem with the problem to find a vector function via known modules of its derivatives (see [11]), reducing the Darboux problem to a linear third-order differential equation with variable coefficients with respect to a real desired function. However, an analytic solution of the Darboux problem in closed form for an arbitrary vector of the angular velocity was not found for all approaches. Only a few special cases admitting the construction of an exact solution of this problem are found (see [6, 8, 9, 11–17]).

However, as shown in [18], each first-order system of linear differential equations with a skew-symmetric matrix of coefficients is reducible in the Lyapunov sense, i.e., there exists a change of variables (the Lyapunov transformation) reducing the specified system to a system with constant coefficients. The system of Eqs. (1.4) of the Darboux problem has a skew-symmetric matrix of coefficients and, therefore, the problem is reducible in the Lyapunov sense. Hence, the search of a closed form of the solution of the Darboux problem in the general case of the given angular velocity of the rigid body is not a hopeless task. On the other hand, since no exact solutions of the Darboux problem are known so far, constructing new highly efficient operation algorithms for SINS, implementing real-time integration of differential equations (describing the orientation with respect to the data from SINS sensors) by the onboard calculator, is still relevant.

In the present paper, approaches related to the search for the solution of the Darboux problem in closed form (see [6]) are used to propose a new orientation-detection algorithm for rigid bodies (SINS).

2. ORIENTATION DETECTION PROBLEM: THE BORTZ EQUATION FORM

Another setting of the problem to find the orientation vector $\boldsymbol{\varphi}(t)$ of a rigid body with respect to an inertial space is the exact Bortz equation

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\varphi}\times\boldsymbol{\omega} + \frac{1}{\varphi}\left(1 - \frac{\varphi\sin\varphi}{2(1-\cos\varphi)}\right)\boldsymbol{\varphi}\times(\boldsymbol{\varphi}\times\boldsymbol{\omega}), \tag{2.1}$$

where $\times$ denotes the vector product (see [1, 2]). In Eq. (2.1), the input data is the angular velocity vector $\boldsymbol{\omega}$. Note that the Bortz nonlinear equation for the orientation vector $\boldsymbol{\varphi}$ of the rigid body, given by (2.1), is an analog of the quaternion linear equation given by (1.1): the vector $\boldsymbol{\varphi}$ and the quaternion $\mathbf{\Lambda}$ are linked by the relations

$$\begin{gathered}\boldsymbol{\varphi} = \varphi\mathbf{e}, \quad \mathbf{e} = e_1\mathbf{i}_1 + e_2\mathbf{i}_2 + e_3\mathbf{i}_3, \quad |\mathbf{e}| = (e_1^2 + e_2^2 + e_3^2)^{1/2} = 1,\\ \lambda_0 = \cos(\varphi/2), \quad \lambda_j = \sin(\varphi/2)e_j, \quad j = 1,2,3,\end{gathered} \tag{2.2}$$

where φ is the orientation angle of the rigid body, while $\mathbf{e}$ is the Eulerian rotation axis. In practice, constructing SINS-orientation algorithms by means of the numerical solution of Eq. (2.1) on the time segment defined by the inequality $t_{m-1} \le t < t_m$, one disregards the third term of this equations for small values of the angle φ (this is a value of order φ^2). If the simplified (approximate) differential equation

$$\dot{\boldsymbol{\varphi}}^* = \boldsymbol{\omega} + \boldsymbol{\varphi}^*\times\boldsymbol{\omega}/2 \tag{2.3}$$

is solved by the Picard iteration method, then its second iteration is assigned to be final (see [2]):

$$\begin{gathered}\boldsymbol{\varphi}_m^* = \int\limits_{t_{m-1}}^{t_m}(\boldsymbol{\omega}(t) + \boldsymbol{\alpha}(t)\times\boldsymbol{\omega}(t)/2)dt = \boldsymbol{\alpha}_m + \boldsymbol{\beta}_m,\\ \boldsymbol{\alpha}(t) = \int\limits_{t_{m-1}}^{t}\boldsymbol{\omega}(\tau)d\tau, \quad \boldsymbol{\alpha}_m = \boldsymbol{\alpha}(t_m), \quad \boldsymbol{\beta}(t) = \int\limits_{t_{m-1}}^{t}\boldsymbol{\alpha}(\tau)\times\boldsymbol{\omega}(\tau)d\tau/2,\\ \boldsymbol{\beta}_m = \boldsymbol{\beta}(t_m),\end{gathered} \tag{2.4}$$

where the vector $\boldsymbol{\beta}$ is called the vector of the noncommutative rotation or the coning. Under certain motions of the rigid body, the term $\boldsymbol{\beta}_m$ of relation (2.4) substantially affects the method error. The inves-

1 tigation of noncommutative rotations (conings) as a kind of mechanical motion of bodies and the separation of numerical algorithms to detect the orientation of a rigid body (SINS) into the superfast, fast, and slow computing cycles is intended to compensate the influence of this phenomenon. Taking this into consideration, we note that algorithms of the superfast cycle are intended to integrate fast absolute angular motions of the object, using auxiliary variables (e.g., the orientation vector or the vector of the final rotation). The fast-cycle algorithm implements the computing of the classical quaternion of the object rotation on the fast-cycle step in the inertial coordinate system. The slow-cycle algorithm is used to compute the classical quaternion of the object orientation in the normal geographical system of coordinates and airplane angles (see [19]).

However, for a new vector $\mathbf{w}(t)$ of the angular velocity, obtained in the problem of the detection of the orientation of the rigid body (SINS) by means of an arbitrary original vector $\boldsymbol{\omega}(t)$ of the angular velocity and one-to-one changes of variables in motion equations for the rigid body, the Bortz approximate equation admits a precise analytic solution. Let us show this.

3. CHANGES OF VARIABLES AND RELATED EFFECTS

In problem (1.1) and (1.2) the one-to-one change of variables of the Lyapunov-transformation type can be represented by the scheme $\Lambda \to \mathrm{U}$, where $\mathrm{U}(t)$ is the orientation quaternion of an introduced coordinate system (a new variable), the quaternion $\mathrm{V}(t)$ is the given transitional operator, and K is an arbitrary constant quaternion:

$$\boldsymbol{\Lambda}(t) = \mathbf{U}(t) \circ \mathbf{K} \circ \mathbf{V}(t), \quad \|\mathbf{K}\| = \|\mathbf{V}\| = 1, \tag{3.1}$$

$$\mathbf{V}(t) = (-\mathbf{i}_1 \sin \mathrm{N}(t) + \mathbf{i}_2 \cos \mathrm{N}(t)) \circ \exp(\mathbf{i}_3 \mathrm{N}(t)/2) \circ \exp(\mathbf{i}_1 \Omega_1(t)/2), \tag{3.2}$$

$$\mathrm{N}(t) = \int_0^t \nu(\tau)\, d\tau, \quad \Omega_1(t) = \int_0^t \omega_1(\tau)\, d\tau, \tag{3.3}$$

and

$$\nu(t) = \omega_2(t) \sin \Omega_1(t) + \omega_3(t) \cos \Omega_1(t). \tag{3.4}$$

Here $\|g\|$ denotes the quaternion norm ($\|\mathbf{K}\| = k_0{}^2 + k_1{}^2 + k_2{}^2 + k_3{}^2$) and $\exp(g)$ denotes the quaternion exponent

$$\exp(\mathbf{Z}) = \exp(z_0)(\cos(|\mathbf{z}_v|) + \sin(|\mathbf{z}_v|)\, \mathbf{z}_v/|\mathbf{z}_v|), \tag{3.5}$$

where z_0, and $\mathbf{z}_v = z_1\mathbf{i}_1 + z_2\mathbf{i}_2 + z_3\mathbf{i}_3$ are the scalar part and vector part (respectively) of the quaternion $\mathbf{Z}$ and the vector part of the quaternion $\mathbf{Z}(t)$ has a constant direction (see [8]).

Then, the original problem given by (1.1)–(1.2) passes into the following problem with the new vector $\mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}}$ of the angular velocity:

$$2\dot{\mathbf{U}} = \mathbf{K} \circ \mathbf{V} \circ (\boldsymbol{\omega}(t) - 2\dot{\mathbf{V}}) \circ \tilde{\mathbf{V}} \circ \tilde{\mathbf{K}} = \mathbf{U} \circ \mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}}, \tag{3.6}$$

$$\mathbf{w}(t) = \mu(t)(-\mathbf{i}_1 \sin \mathrm{N}(t) + \mathbf{i}_2 \cos \mathrm{N}(t)) - 2\mathbf{i}_3 \nu(t), \tag{3.7}$$

$$\mu(t) = \omega_2(t) \cos \Omega_1(t) - \omega_3(t) \sin \Omega_1(t), \tag{3.8}$$

and

$$\mathbf{U}(0) = \boldsymbol{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \tilde{\mathbf{K}}, \tag{3.9}$$

where $\mathrm{N}(t)$, $\Omega_1(t)$, and $\nu(t)$ are determined by relations (3.3) and (3.4), the tilde denotes the quaternion conjugation, and the orthogonal transformation contained in the right-hand side of the quaternion differential equation given by (3.6) takes the following form:

$$\begin{aligned}
\mathbf{i}_1 &: w_1(k_0^2 + k_1^2 - k_2^2 - k_3^2) + 2w_2(k_1k_2 - k_0k_3) + 2w_3(k_1k_3 + k_0k_2), \\
\mathbf{i}_2 &: 2w_1(k_1k_2 + k_0k_3) + w_2(k_0^2 + k_2^2 - k_1^2 - k_3^2) + 2w_3(k_2k_3 - k_0k_1), \\
\mathbf{i}_3 &: 2w_1(k_1k_3 - k_0k_2) + 2w_2(k_2k_3 + k_0k_1) + w_3(k_0^2 + k_3^2 - k_1^2 - k_2^2).
\end{aligned} \tag{3.10}$$

(on the example of the angular velocity vector $\mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}}$ in the component-wise form).

The vector coefficient in Eq. (3.6) is still interpreted as the angular velocity vector of a coordinate system. However, unlike the arbitrary variable vector $\omega(t)$ in Eq. (1.1), the angular velocity vector $\mathbf{w}(t)$ given by (3.7) rotates in the plane $(\mathbf{i}_1, \mathbf{i}_2)$ around the axis $\mathbf{i}_3$ (this motion is a conical precession) though its modulus is, in general, variable. The introduced arbitrary constant quaternion $\mathbf{K}$ generalizes this motion: it becomes a generalized conical precession and is well coordinated with the known Poinsot concept that each rotation of a rigid body around a fixed point can be represented as a conical motion.

Note that there is one-to-one correspondence between problem (3.6)–(3.9) and the original problem to find the orientation of a rigid body via its known angular velocity and initial angular location in space, given by (1.1)–(1.2).

It is still a hard problem to find an analytic solution of the quaternion differential equation given by (3.6). However, the equation differs from this only by the coefficient 1/2 on the right-hand side (i.e., the angular velocity vector is $\mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}}/2$ instead of $\mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}}$). Note that the homogeneous Poincaré vector differential equation is isomorphic to the quaternion differential equation, i.e.,

$$2\dot{\mathbf{\Psi}} = \mathbf{\Psi} \circ \mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}}/2, \tag{3.11}$$

$$\mathbf{\Psi}(0) = \mathbf{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \tilde{\mathbf{K}} \tag{3.12}$$

can be solved in closed form. This fact is established heuristically and has a mathematical nature. Note that a number of pradoxical results not interpreted by physics or mechanics clearly are related to the Darboux problem reduced to form (3.6)–(3.9) (see [6]). Assign

$$\mathbf{K} = \mathbf{\Lambda}_0 \circ (-\mathbf{i}_2). \tag{3.13}$$

Then the initial-value conditions (3.9), (3.12) take the unit form $\mathbf{U}(0) = \mathbf{\Psi}(0) = 1$. Note that this operation with the quaternion $\mathbf{K}$ is important for the constructing of the SINS-orientation algorithm below. In this case, the exact solution of the Cauchy problem given by (3.11)–(3.13) is represented as follows:

$$\mathbf{\Psi} = \mathbf{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \mathbf{\Phi}(t) \circ \mathbf{i}_2 \circ \tilde{\mathbf{\Lambda}}_0, \tag{3.14}$$

$$\mathbf{\Phi}(t) = \exp(\mathbf{i}_2 \mathrm{M}(t)/4) \circ \exp(-\mathbf{i}_3 \mathrm{N}(t)/2), \quad \mathrm{M}(t) = \int_0^t \mu(\tau)\, d\tau, \tag{3.15}$$

where the function $\mu(t)$ is defined by means of relations (3.3) and (3.8) via known components of the vector $\omega(t)$ of the angular velocity of the rigid body.

To verify that a solution of problem (3.11)–(3.13) is obtained, differentiate expression (3.14), taking into account (3.3), (3.7), (3.8), and (3.15) and using expression (3.5) of the quaternion exponent and relations of type (3.10) for the orthogonal transformation:

$$\begin{gathered}\dot{\mathbf{\Psi}}(t) = \mathbf{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \mathbf{\Phi}(t) \circ (\mu(t) \exp(\mathbf{i}_3 \mathrm{N}(t)/2) \circ \mathbf{i}_2 \circ \exp(-\mathbf{i}_3 \mathrm{N}(t)/2)/4 \\ - \mathbf{i}_3 \nu(t)/2) \circ \mathbf{i}_2 \circ \tilde{\mathbf{\Lambda}}_0 = \mathbf{\Psi}(t) \circ \mathbf{\Lambda}_0 \circ (-\mathbf{i}_2) \\ \circ (\mu(t)(-\mathbf{i}_1 \sin \mathrm{N}(t) + \mathbf{i}_2 \cos \mathrm{N}(t)) - 2\mathbf{i}_3 \nu(t)) \circ \mathbf{i}_2 \circ \tilde{\mathbf{\Lambda}}_0/4\end{gathered}$$

or, which is the same,

$$2\dot{\mathbf{\Psi}} = \mathbf{\Psi} \circ \mathbf{\Lambda}_0 \circ (-\mathbf{i}_2) \circ \mathbf{w}(t) \circ \mathbf{i}_2 \circ \tilde{\mathbf{\Lambda}}_0/2,$$

where $\mathbf{\Psi}(0) = 1$, which coincides with expressions (3.11) and (3.12) provided that condition (3.13) is satisfied.

Note that the quaternion differential equation given by (3.11) is equivalent to the Poisson vector differential equation

$$d(\tilde{\mathbf{\Psi}}(t) \circ \mathbf{c} \circ \mathbf{\Psi}(t))/dt = (\tilde{\mathbf{\Psi}}(t) \circ \mathbf{c} \circ \mathbf{\Psi}(t)) \times (\mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}})/2, \tag{3.15}$$

where c is an arbitrary vector constant (see [8]); its exact solution is constructed by means of relations (3.14) and (3.15). Below, this is used to obtain an explicit analytic solution of the approximate Bortz equation.

4. PRECISE SOLUTIONS OF THE APPROXIMATE BORTZ EQUATION AND RELATED ORIENTATION-DETECTION SINS ALGORITHMS

Based on relations of type (2.2), to the reduced problem of the detection of the orientation quaternion U(t), given by (3.6)–(3.8), with the single initial-value condition given by (3.9), we associate the problem with the nonlinear vector Bortz equation given by (2.1) and zero initial-value condition (in this case, the correctness of the passage from the original problem given by (1.1) and (1.2) to Eq. (2.1) through problem (3.6)–(3.9) is not violated). Then, according to the SINS theory and practice, instead of the total Bortz equation corresponding to the quaternion vector given by (3.6), we consider the simplified Bortz equation with respect to the orientation vector $\boldsymbol{\varphi}^*(t)$, given by (2.3), and the zero initial-value condition:

$$\dot{\boldsymbol{\varphi}}^* = \mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}} + \boldsymbol{\varphi}^* \times (\mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}})/2, \tag{4.1}$$

$$\boldsymbol{\varphi}^*(0) = 0. \tag{4.2}$$

Note that the homogeneous part of the vector linear differential equation (4.1) is equivalent to the solvable system given by (3.9) and presented as the Poisson vector differential equation given by (3.15). Following the Lagrange method of resolving linear heterogeneous differential systems of equations, from (3.14) and (3.15), we conclude that the exact solution of the approximate Bortz equation given by (4.1) with the initial-value condition given by (4.2) has the form

$$\boldsymbol{\varphi}^* = \mathbf{K} \circ \tilde{\boldsymbol{\Phi}}(t) \circ \int_0^t \boldsymbol{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\boldsymbol{\Phi}}(\tau) d\tau \circ \boldsymbol{\Phi}(t) \circ \tilde{\mathbf{K}}. \tag{4.3}$$

To verify the correctness of the solution of Eq. (4.1), differentiate expression (4.3):

$$\begin{aligned}
\dot{\boldsymbol{\varphi}}^* &= \mathbf{K} \circ (\tilde{\boldsymbol{\Phi}}(t) \circ \int_0^t \boldsymbol{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\boldsymbol{\Phi}}(\tau) d\tau \circ \boldsymbol{\Phi}(t) \circ \mathbf{w}(t) \\
&- \mathbf{w}(t) \circ \tilde{\boldsymbol{\Phi}}(t) \circ \int_0^t \boldsymbol{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\boldsymbol{\Phi}}(\tau) d\tau \circ \boldsymbol{\Phi}(t)) \circ \tilde{\mathbf{K}}/4 + \mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}} \\
&= \mathbf{K} \circ (\tilde{\boldsymbol{\Phi}}(t) \circ \int_0^t \boldsymbol{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\boldsymbol{\Phi}}(\tau) d\tau \circ \boldsymbol{\Phi}(t) \circ \tilde{\mathbf{K}} \circ \mathbf{K} \circ \mathbf{w}(t) \\
&- \mathbf{w}(t) \circ \tilde{\mathbf{K}} \circ \mathbf{K} \circ \tilde{\boldsymbol{\Phi}}(t) \circ \int_0^t \boldsymbol{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\boldsymbol{\Phi}}(\tau) d\tau \circ \boldsymbol{\Phi}(t)) \circ \tilde{\mathbf{K}}/4 \\
&+ \mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}} = \boldsymbol{\varphi}^* \times (\mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}})/2 + \mathbf{K} \circ \mathbf{w}(t) \circ \tilde{\mathbf{K}}.
\end{aligned}$$

Thus, using (2.3) and taking into account changes of variables, provided in Sec. 3, we completely solve in quadratures problem (1.1), (1.2), (2.1), (2.2) of the orientation detection for the rigid body. The analytic algorithm of the orientation detection for the rigid body (SINS) is as follows.

(1) According to the given components of the vector $\boldsymbol{\omega}(t)$ of the angular velocity of the rigid body, compute the functions $\mu(t)$ and $\nu(t)$ as follows:

$$\begin{aligned}
\Omega_1(t) &= \int_0^t \omega_1(\tau)\, d\tau, \\
\mu(t) &= \omega_2(t) \cos \Omega_1(t) - \omega_3(t) \sin \Omega_1(t), \\
\nu(t) &= \omega_2(t) \sin \Omega_1(t) + \omega_3(t) \cos \Omega_1(t).
\end{aligned} \tag{4.4}$$

(2) Using the computed functions $\mu(t)$, and $\nu(t)$, find the vector $\mathbf{w}(t)$ as follows:

$$\begin{aligned}
\mathrm{N}(t) &= \int_0^t \nu(\tau)\, d\tau, \\
\mathbf{w}(t) &= \mu(t)(-\mathbf{i}_1 \sin \mathrm{N}(t) + \mathbf{i}_2 \cos \mathrm{N}(t)) - 2\mathbf{i}_3 \nu(t).
\end{aligned} \tag{4.5}$$

(3) Using the vector $\mathbf{w}(t)$ and the initial location of the rigid body $\mathbf{\Lambda}_0$, compute the value of the orientation vector $\boldsymbol{\varphi}^*$ for the rigid body as follows:

$$\mathrm{M}(t) = \int_0^t \mu(\tau)\,d\tau,$$
$$\mathbf{\Phi}(t) = \exp(\mathbf{i}_2\mathrm{M}(t)/4) \circ \exp(-\mathbf{i}_3\mathrm{N}(t)/2), \tag{4.6}$$

and

$$\boldsymbol{\varphi}^* = \mathbf{K} \circ \tilde{\mathbf{\Phi}}(t) \circ \int_0^t \mathbf{\Phi}(\tau) \circ \mathbf{w}(\tau) \circ \tilde{\mathbf{\Phi}}(\tau) d\tau \circ \mathbf{\Phi}(t) \circ \tilde{\mathbf{K}}, \quad \mathbf{K} = \mathbf{\Lambda}_0 \circ (-\mathbf{i}_2). \tag{4.7}$$

(4) Using the orientation vector $\boldsymbol{\varphi}^*$, find the components of the quaternion $\mathbf{U}$ as follows:

$$u_0 = \cos(\varphi/2), \quad u_j = \sin(\varphi/2)e_j, \quad j = 1,2,3,$$
$$\varphi = |\boldsymbol{\varphi}^*|, \quad \mathbf{e} = \boldsymbol{\varphi}^*/\varphi, \quad \varphi(t) \neq 0, \quad \forall t. \tag{4.8}$$

(5) Find the approximate value $\mathbf{\Lambda}^{\text{approx}}$ of the orientation quaternion for the rigid body (SINS) as follows:

$$\mathbf{\Lambda}^{\text{approx}} = \mathbf{U}(t) \circ \mathbf{K} \circ (-\mathbf{i}_1 \sin \mathrm{N}(t) + \mathbf{i}_2 \cos \mathrm{N}(t)) \circ \exp(\mathbf{i}_3\mathrm{N}(t)/2) \circ \exp(\mathbf{i}_1\Omega_1(t)/2). \tag{4.9}$$

Implementing the SINS orientation algorithm with time digitization at each subsequent step m of the algorithm, one has to select the quaternion $\mathbf{K}$ as follows: $\mathbf{K}_m = \mathbf{\Lambda}_{m-1} \circ (-\mathbf{i}_2)$. Then, the initial value with respect to variable $\boldsymbol{\varphi}^*$ is equal to zero each time. The data about the angular-motion trajectory of the rigid body (object) according to a relation of type (4.9) is accumulated by means of the quaternion $\mathbf{K}$.

CONCLUSIONS

The proposed quaternion algorithm to find the orientation of a rigid body (object using SINS), based on the analytic solution of the approximate Bortz linear equation, is regular for all angular motions of the rigid body because the solution is exact. Unlike orientation-detection algorithms using approximate numerical solutions of the truncated Bortz equation and reading the data about the angular velocity of the object out of SINS sensors intermediately (see [1–5]), the substance of the proposed approach is as follows: once the said data is preliminarily transformed according to relations (4.6) and (4.7), the quaternion $\mathbf{\Phi}(t)$ used as the basis of constructing the solution of the problem is presented in elementary functions and quadratures according to relations (3.5) and (4.6).

Other equations are used in the SINS theory and practice as well: they are the nonlinear differential equation

$$\dot{\mathbf{x}} = \boldsymbol{\omega}(t) + \mathbf{x} \times \boldsymbol{\omega}(t)/2 + (\mathbf{x}, \boldsymbol{\omega}(t))\mathbf{x}/4$$

for the vector $\mathbf{x}$(t) of the final rotation of the rigid body, which is an analog of the linear quaternion equation given by (1.1) (see [8]), and the quaternion Riccati equation

$$\dot{\mathbf{y}} = \mathbf{a}(t) + 2\mathbf{y} \times \mathbf{a}(t) - \mathbf{y} \circ \mathbf{a}(t) \circ \mathbf{y}, \quad \mathbf{a}(t) = \boldsymbol{\omega}(t)/4$$

with respect to the quaternion $\mathbf{y}(t)$ with the scalar part $y_0 = 0$, where $(\cdot, \cdot)$ denotes the scalar product of vectors, the components of the quaternion $\mathbf{\Lambda}(t)$ and the final rotation vector $\mathbf{x}(t)$ of the rigid body are such that

$$\mathbf{x} = 2\boldsymbol{\lambda}_v/\lambda_0 = 2\tan(\phi/2)\mathbf{e}, \quad \boldsymbol{\lambda}_v = \lambda_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3,$$
$$\mathbf{e} = \boldsymbol{\lambda}_v/\sqrt{\|\boldsymbol{\lambda}_v\|}, \quad \cos\varphi = \lambda_0, \quad \sin\varphi = \sqrt{\|\boldsymbol{\lambda}_v\|}, \quad 0 \le \varphi < \pi,$$

and the quaternion $\mathbf{y}(t)$ with the zero scalar part and the orientation quaternion $\mathbf{\Lambda}(t)$ are related by a one-to-one correspondence (see [9, 20]).

For the linear parts

$$\dot{\mathbf{x}} = \boldsymbol{\omega}(t) + \mathbf{x} \times \boldsymbol{\omega}(t)/2$$

and

$$\dot{\mathbf{y}} = (\boldsymbol{\omega}(t)/2 + \mathbf{y} \times \boldsymbol{\omega}(t))/2$$

of these differential equations, one can argue in the same way as for the Bortz equation above (see [21] as well).

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