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Engineering Notes

Solution of Approximate Equation for Modified Rodrigues Vector and Attitude Algorithm Design

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I. Introduction

DURING operation of strapdown inertial navigation systems (SINS), a vector responsible for the orientation of a rigid body (object) in space can be periodically calculated by the method of approximate solution of the kinematic equation with respect to the vector of modified Rodrigues parameters (MRPs) or, in other words, the kinematic equation written in three-dimensional skew-symmetric operators [1] (in the theory and practice of SINS construction, in ultrarapid cycles of algorithms for small angles of rotation, the nonlinear terms in this equation are neglected). The angular velocity vector of a rigid body is the input quantity in the equation. Note that the complete nonlinear equation for the MRPs orientation vector of a rigid body is an analog of the full quaternion linear equation [1,2]; the vector and the quaternion of a rigid body orientation are linked by known relations. The approximate linear differential equation in skew-symmetric operators is solved by various numerical methods, for example, by Picard's method, and then, the second iteration of this method in the practice of SINS can be taken for the final one. This term in the iteration formula of Picard's method is called a noncommutative rotation vector, or "coning." For certain motions of a rigid body, this term makes a significant contribution to the error of the method. The study of noncommutative rotation (or term of the "coning" type) as a kind of mechanical motion of bodies, separation of numerical algorithms for determining the orientation of a rigid body (SINS) for rapid and slow counting cycles are aimed at compensation for the effect of this phenomenon [3–6]. Meanwhile, for some new angular velocity vector, which is obtained in determining the orientation of a rigid body (SINS), based on the initial arbitrary angular velocity vector in unambiguous interchanges of variables in the motion equations for a rigid body, the approximate differential equation in skew-symmetric operators admits of an exact analytic solution. We will show this.

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The problem is to define the quaternion of orientation $\boldsymbol{\Lambda}$ of a rigid body with respect to an arbitrary given angular velocity vector $\boldsymbol{\omega}(t)$ and the initial angular position of a rigid body in space based on the quaternion kinematic equation [1,2] known as the Darboux problem (e.g., [7]). Further, we make changes of variables (without violating generality of the problem) according to the scheme $\boldsymbol{\Lambda} \to \boldsymbol{U}$, where $\boldsymbol{U}$ is the quaternion of the orientation of some introduced coordinate system; it is always possible to reverse the transition $\boldsymbol{U} \to \boldsymbol{\Lambda}$. These changes have the character of rotation transformation and reduce the initial problem of determining the orientation of a rigid body (quaternion $\boldsymbol{\Lambda}$) with an arbitrary variable angular velocity vector $\boldsymbol{\omega}(t)$ to the problem where the angular velocity vector $\boldsymbol{w}(t)$ of the introduced coordinate system, remaining generally variable in absolute value, performs a definite motion: rotates around one of the axes of the coordinate system. This motion is generalized conical precession, which agrees well with the known Poinsot's concept that any rigid body rotation about a fixed point can be represented as a conical motion. At the same time, the generality of the original problem is not violated. Finding an analytical solution of the quaternion differential equation obtained with respect to the new unknown quaternion $\boldsymbol{U}$ is still a difficult problem. However, the equation differing from this only by the coefficient "1/2" in the right-hand side [i.e., with the angular velocity vector $\boldsymbol{w}(t)/2$] is solved in closed form. Moreover, we note that the quaternion differential equation is isomorphic to the homogeneous vector differential equation of Poisson.

The resulting problem with the angular velocity vector $\boldsymbol{w}(t)$ and the unknown quaternion of orientation $\boldsymbol{U}$ is associated with the MRPs complete nonlinear vector differential equation (the equation in skew-symmetric operators) with respect to the unknown orientation vector $\boldsymbol{\theta}$. The approximate linear equation in skew-symmetric operators, which mathematically is an inhomogeneous differential vector equation where homogeneous part is equivalent to the Poisson equation with the vector coefficient $\boldsymbol{w}(t)/2$, becomes analytically solvable, and its solution $\boldsymbol{\theta}^*$ is obtained in elementary functions and quadratures by the Lagrange method.

The exact solution of the approximate linear equation in skew-symmetric operators made it possible to solve the problem of determining the quaternion of orientation of a rigid body for an arbitrary angular velocity and small angle of rotation of a rigid body with the help of quadratures (the truncated Darboux problem is actually solved). Proceeding from this solution, the following approach to the design of a new algorithm for computation of SINS orientation is proposed: 1) by the set components of the angular velocity of a rigid body $\omega_i(t)$, $i = 1, 2, 3$ on the basis of unambiguous interchanges of the variables at each time point, a new angular velocity $\boldsymbol{w}(t)$ of some new coordinate system is calculated; 2) using the new angular velocity and the initial position of a rigid body, we find the exact solution $\boldsymbol{\theta}^*$ of the approximate linear equation in skew-symmetric operators with a zero initial condition with the help of quadratures; 3) the value of the quaternion orientation of a rigid body (SINS) is determined by the vector $\boldsymbol{\theta}^*$ according to the scheme $\boldsymbol{\theta}^* \approx \boldsymbol{\theta} \Leftrightarrow \boldsymbol{U} \to \boldsymbol{\Lambda}$.

During construction of the algorithm for SINS orientation at each subsequent step, the change of the variables takes into account the previous step of the algorithm in such a way that each time the initial value of the unknown MRPs vector will be zero. Because the proposed algorithm for the analytical solution of the approximate linear equation written in skew-symmetric operators is exact, it is regular for any angular motion of a rigid body.

Previously, the authors constructed analytical solutions to the Bortz approximate equation for the orientation vector of a rigid body and for the approximate equation for the finite rotation vector of a rigid body, and quaternion orientation algorithms of SINS based on these solutions [8,9].

II. Statement of the Problem of Determining Orientation of a Rigid Body (SINS)

Consider the Cauchy problem for quaternion kinematic equation [1,2] with arbitrary given angular velocity vector-function $\boldsymbol{\omega}(t)$, written in the following form:

$$2\dot{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda} \circ \boldsymbol{\omega}(t) \tag{1}$$

$$\boldsymbol{\Lambda}(t_0) = \boldsymbol{\Lambda}_0 \tag{2}$$

Here $\boldsymbol{\Lambda}(t) = \lambda_0(t) + \lambda_1(t)i_1 + \lambda_2(t)i_2 + \lambda_3(t)i_3$ is a quaternion describing the position of a rigid body in an inertial space; $\boldsymbol{\omega}(t) = \omega_1(t)\boldsymbol{i}_1 + \omega_2(t)\boldsymbol{i}_2 + \omega_3(t)\boldsymbol{i}_3$ is the angular velocity vector of the rigid body specified by its projections onto body-fixed coordinate axes; i_1, i_2, i_3 are the units of the hypercomplex space (imaginary Hamiltonian units), which can be identified with the vectors of a three-dimensional vector space $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$; the symbol "$\circ$" stands for the quaternion product; and $\boldsymbol{\Lambda}_0$ is the initial value of the quaternion $\boldsymbol{\Lambda}(t)$ at $t = t_0$, $t \in [t_0, \infty)$ (t_0 set equal to 0). The problem is to find the quaternion $\boldsymbol{\Lambda}(t)$.

The problem of finding the quaternion $\boldsymbol{\Lambda}(t)$ through the angular velocity vector $\boldsymbol{\omega}(t)$ using quadratures will be called the Darboux problem after the scientist who first dealt with it in the general formulation [7]. It should be noted that in his papers in the *Bulletin des Sciences Mathematiques* G. Darboux reduced a system of differential equations of type (1) to an ordinary Riccati equation and tried to solve it in the general case.

We can also formulate the problem of determining the MRPs vector $\boldsymbol{\theta}(t)$, responsible for the orientation of a rigid body (object), by solving the exact vector kinematic equation [1]

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}(1 - \boldsymbol{\theta}^2)/4 + \boldsymbol{\theta} \times \boldsymbol{\omega}/2 + (\boldsymbol{\theta} \cdot \boldsymbol{\omega})\boldsymbol{\theta}/2, \quad \boldsymbol{\theta}^2 = \theta_1^2 + \theta_2^2 + \theta_3^2 \tag{3}$$

where "$\times$" and "$\cdot$" mean the vector and the scalar products. In Eq. (3) the input quantity is the angular velocity vector $\boldsymbol{\omega}(t)$. Note that the nonlinear equation (3) for the MRPs vector $\boldsymbol{\theta}$ is an analog of the quaternion linear equation (1); vector $\boldsymbol{\theta}$ and quaternion $\boldsymbol{\Lambda}$ are connected by the relations [1]

$$\begin{aligned} &\boldsymbol{\theta} = \boldsymbol{e}\,\mathrm{tg}(\varphi/4), \qquad \boldsymbol{e} = e_1\boldsymbol{i}_1 + e_2\boldsymbol{i}_2 + e_3\boldsymbol{i}_3 \\ &|\boldsymbol{e}| = \left(e_1^2 + e_2^2 + e_3^2\right)^{1/2} = 1 \\ &\lambda_0 = \cos(\varphi/2), \quad \lambda_j = \sin(\varphi/2)e_j, \quad j = 1,2,3 \end{aligned} \tag{4}$$

where φ is the angle of rotation of a rigid body, and $\boldsymbol{e}$ is the unit vector of the Euler axis of rotation.

It should be noted that with respect to the vector $\boldsymbol{y} = -\boldsymbol{e}\,\mathrm{tg}(\varphi/4)$ which differs from the vector θ by a sign and is interpreted as a quaternion with a zero scalar part, an alternative notation of the kinematic equation is permissible [10,11]

$$\dot{\boldsymbol{y}} = -\boldsymbol{\omega}/4 + \boldsymbol{y} \times \boldsymbol{\omega}/2 + \boldsymbol{y} \circ \boldsymbol{\omega} \circ \boldsymbol{y}/4 \tag{5}$$

The nonlinear equation (5) for a quaternion with zero scalar part $\boldsymbol{y}$ (kinematic equation in four-dimensional skew-symmetric operators) is also an analog of the linear quaternion equation (1). As noted in [10], the quaternion kinematic equation in four-dimensional skew-symmetric operators (5) is qualitatively different from the vector kinematic equation in three-dimensional skew-symmetric operators (3). The properties of three-dimensional and four-dimensional skew-symmetric matrices differ qualitatively: three-dimensional skew-symmetric matrices are singular (their determinants are equal to zero), whereas four-dimensional skew-symmetric matrices are not singular (their determinants are always nonzero). In addition, whereas a polynomial of any degree of a three-dimensional skew-symmetric matrix reduces to a second-degree polynomial, a polynomial of any degree of a four-dimensional skew-symmetric matrix reduces to a first-degree polynomial. The latter circumstance makes using matrix kinematic equation in four-dimensional skew-symmetric operators to construct algorithms for determining the orientation of moving objects using SINS more efficient than vector kinematic equation (3) and its matrix analog in three-dimensional skew-matrix operators. At the same time the linear parts of equations (3) and (5) practically coincide. In this regard, further reasoning is valid for both Eqs. (3) and (5).

In the practice of constructing SINS orientation algorithms by numerical solution of equation type of Eq. (3) on a time interval $t_{m-1} \le t < t_m$, the nonlinear members in this equation are neglected for small angles of rotation (it has the magnitude of the second order). If the derived approximate (truncated) differential equation

$$\dot{\boldsymbol{\theta}}^* = \boldsymbol{\omega}/4 + \boldsymbol{\theta}^* \times \boldsymbol{\omega}/2 \tag{6}$$

[problem of determining orientation of a rigid body with Eq. (6) can be called the truncated Darboux problem] is solved by Picard's iterative method, then the second iteration of this method is taken for the final one in these types of tasks [3–5]:

$$\begin{aligned} \boldsymbol{\theta}_m^* &= \int_{t_{m-1}}^{t_m} (\boldsymbol{\omega}(t)/4 + \boldsymbol{\alpha}(t) \times \boldsymbol{\omega}(t)/2)\,\mathrm{d}t = \boldsymbol{\alpha}_m + \boldsymbol{\beta}_m \\ \boldsymbol{\alpha}(t) &= \int_{t_{m-1}}^{t} \boldsymbol{\omega}(\tau)\,\mathrm{d}\tau/4, \quad \boldsymbol{\alpha}_m = \boldsymbol{\alpha}(t_m), \\ \boldsymbol{\beta}(t) &= \int_{t_{m-1}}^{t} \boldsymbol{\alpha}(\tau) \times \boldsymbol{\omega}(\tau)\,\mathrm{d}\tau/2, \quad \boldsymbol{\beta}_m = \boldsymbol{\beta}(t_m) \end{aligned}$$

where vector $\boldsymbol{\beta}$ is called a noncommutative rotation vector, or "coning." For certain motions of a rigid body, this term makes a significant contribution to the error of the method. The investigation of noncommutative rotation (or "coning") as a kind of mechanical motion of bodies and the separation of numerical algorithms, which calculate the orientation of a rigid body (SINS), into the ultrarapid, rapid, and slow computing cycles, are aimed at compensation for the effect of this phenomenon. Taking this into consideration, we note that algorithms of the rapid cycle are intended to integrate fast absolute angular motions of the object, using auxiliary variables (e.g., the orientation vector or the vector of finite rotation). The rapid-cycle algorithm implements the computing of the classical quaternion of the object rotation on the rapid-cycle step in the inertial coordinate system. The slow-cycle algorithm is used to compute the classical quaternion of the object orientation in the normal geographical system of coordinates and nautical angles [6].

Meanwhile, for some new angular velocity vector $\boldsymbol{w}(t)$, which is obtained in the process of determining the orientation of a rigid body (SINS), based on the initial arbitrary angular velocity vector $\boldsymbol{\omega}(t)$ in unambiguous replacements of variables in the motion equations for a rigid body, the approximate kinematic equation (6) admits of an exact analytic solution (i.e., the truncated Darboux problem becomes solvable), which will be shown in what follows.

III. Variables Replacements and Related Effects

Let us write unambiguous replacements of variables (known in the theory of linear differential systems as the Lyapunov transformation) in problems (1) and (2) according to the scheme $\boldsymbol{\Lambda} \to \boldsymbol{U}$, where $\boldsymbol{U}(t)$ is the quaternion of orientation of some introduced coordinate system (new variable), quaternion $\boldsymbol{V}(t)$ is the introduced transition operator, and $\boldsymbol{K}$ is an arbitrary constant quaternion [12]:

$$\boldsymbol{\Lambda}(t) = \boldsymbol{U}(t) \circ \boldsymbol{K} \circ \boldsymbol{V}(t), \quad \|\boldsymbol{K}\| = \|\boldsymbol{V}\| = 1 \tag{7}$$

$$\boldsymbol{V}(t) = (-\boldsymbol{i}_1 \sin N(t) + \boldsymbol{i}_2 \cos N(t)) \circ \exp(\boldsymbol{i}_3 N(t)/2) \circ \exp(\boldsymbol{i}_1 \Omega_1(t)/2) \tag{8}$$

$$N(t) = \int_0^t \nu(\tau)\,\mathrm{d}\tau, \qquad \Omega_1(t) = \int_0^t \omega_1(\tau)\,\mathrm{d}\tau \tag{9}$$

$$\nu(t) = \omega_2(t)\sin\Omega_1(t) + \omega_3(t)\cos\Omega_1(t) \tag{10}$$

Here "$\|\cdot\|$;" denotes the quaternion norm ($\|\boldsymbol{K}\| = k_0^2 + k_1^2 + k_2^2 + k_3^2$) and "exp(.)" denotes the quaternion exponent

$$\exp(\boldsymbol{Z}) = \exp(z_0)(\cos(|z_v|) + \sin(|z_v|)z_v/|z_v|) \tag{11}$$

where $z_0, \boldsymbol{z}_v = z_1\boldsymbol{i}_1 + z_2\boldsymbol{i}_2 + z_3\boldsymbol{i}_3$ are the scalar part and vector part (respectively) of the quaternion $\boldsymbol{Z}$; note that the vector part of the quaternion $\boldsymbol{Z}(t)$ has a constant direction [2].

Then, the original problem given by Eqs. (1) and (2) transforms into the following problem with the new angular velocity vector $\boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}$:

$$2\dot{\boldsymbol{U}} = \boldsymbol{K} \circ \boldsymbol{V} \circ (\boldsymbol{\omega}(t) - 2\dot{\boldsymbol{V}}) \circ \tilde{\boldsymbol{V}} \circ \tilde{\boldsymbol{K}} = \boldsymbol{U} \circ \boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}} \tag{12}$$

$$\boldsymbol{w}(t) = \mu(t)(-\boldsymbol{i}_1 \sin N(t) + \boldsymbol{i}_2 \cos N(t)) - 2\boldsymbol{i}_3 v(t) \tag{13}$$

$$\mu(t) = \omega_2(t)\cos\Omega_1(t) - \omega_3(t)\sin\Omega_1(t) \tag{14}$$

$$\boldsymbol{U}(0) = \boldsymbol{\Lambda}_0 \circ (-\boldsymbol{i}_2) \circ \tilde{\boldsymbol{K}} \tag{15}$$

where $N(t)$, $\Omega_1(t)$, and $\nu(t)$ are determined by relations (9) and (10); the tilde denotes the quaternion conjugation; and the orthogonal transformation, contained in the right-hand side of the quaternion differential equation given by Eq. (12), takes the following form:

$$\begin{aligned}
&\boldsymbol{i}_1\colon w_1(k_0^2 + k_1^2 - k_2^2 - k_3^2) + 2w_2(k_1k_2 - k_0k_3) + 2w_3(k_1k_3 + k_0k_2)\\
&\boldsymbol{i}_2\colon 2w_1(k_1k_2 + k_0k_3) + w_2(k_0^2 + k_2^2 - k_1^2 - k_3^2) + 2w_3(k_2k_3 - k_0k_1)\\
&\boldsymbol{i}_3\colon 2w_1(k_1k_3 - k_0k_2) + 2w_2(k_2k_3 + k_0k_1) + w_3(k_0^2 + k_3^2 - k_1^2 - k_2^2)
\end{aligned} \tag{16}$$

(these are the components of the angular velocity vector $\boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}$).

The quaternion differential equation (12) with the new angular velocity $\boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}$ can be matched with the full nonlinear differential equation for the MRPs vector $\boldsymbol{\theta}$ type of (3):

$$\begin{aligned}
\dot{\boldsymbol{\theta}} = \boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}(1 - \boldsymbol{\theta}^2)/4 + \boldsymbol{\theta} \times \left(\boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}\right)/2\\
+ \left(\boldsymbol{\theta} \cdot \left(\boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}\right)\right)\boldsymbol{\theta}/4
\end{aligned} \tag{17}$$

The vector coefficient in Eq. (12) is still interpreted as the angular velocity vector of a coordinate system $\boldsymbol{U}$. However, unlike the arbitrary variable vector $\boldsymbol{\omega}(t)$ in Eq. (1), the angular velocity vector given by Eq. (13) rotates in a plane around the axis $\boldsymbol{i}_3$ (this motion is a conical precession) though its modulus is, in general, variable. The introduced arbitrary constant quaternion $\boldsymbol{K}$ generalizes this motion: it becomes a generalized conical precession and is well coordinated with the known Poinsot concept that any rotation of a rigid body around a fixed point can be represented as a generalized conical motion.

Note that there is one-to-one correspondence between problems (12–15) or (17) and the original Darboux problem to find the orientation of a rigid body from its known angular velocity and initial angular location in space, given by Eq. (1), (2), or (3). It is still a hard problem to find an analytic solution of the quaternion differential equation given by Eq. (12).

However, the equation that differs from this one only by the coefficient "1/2" on the right side (i.e., with the angular velocity vector $\boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}/2$)

$$2\dot{\boldsymbol{\Psi}} = \boldsymbol{\Psi} \circ \boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}/2 \tag{18}$$

$$\boldsymbol{\Psi}(0) = \boldsymbol{\Lambda}_0 \circ (-\boldsymbol{i}_2) \circ \tilde{\boldsymbol{K}} \tag{19}$$

is solved in a closed form. This fact is established heuristically and has a mathematical nature. It should be noted that a number of paradoxical results not interpreted by physics or mechanics clearly are related to the Darboux problem reduced to form (12–15) [12]. Let us choose quaternion in the form

$$\boldsymbol{K} = \boldsymbol{\Lambda}_0 \circ (-\boldsymbol{i}_2) \tag{20}$$

so that the initial conditions (15) and (19) become unit $\boldsymbol{U}(0) = \boldsymbol{\Psi}(0) = 1$. Note that this technique with quaternion $\boldsymbol{K}$ is important in the subsequent construction of the algorithm of SINS orientation. The solution of the Cauchy problem (18–20) can be written as follows:

$$\boldsymbol{\Psi} = \boldsymbol{\Lambda}_0 \circ (-\boldsymbol{i}_2)\boldsymbol{\Phi}(t) \circ \boldsymbol{i}_2 \circ \tilde{\boldsymbol{\Lambda}}_0 \tag{21}$$

$$\boldsymbol{\Phi}(t) = \exp(\boldsymbol{i}_2 M(t)/4) \circ \exp(-\boldsymbol{i}_3 N(t)/2), \quad M(t) = \int_0^t \mu(\tau)\,\mathrm{d}\tau \tag{22}$$

where the function $\mu(t)$ is defined by means of relations (9) and (14) via known components of the vector $\boldsymbol{\omega}(t)$ of the angular velocity of a rigid body.

We can check the correctness of the obtained solution of the problem (18–20) by differentiating the expression (21) taking into account Eqs. (9), (13), (14), and (22), and using expression (11) of the quaternion exponent and relations of type (16) for the orthogonal transformation:

$$\begin{aligned}
\dot{\boldsymbol{\Psi}}(t) &= \boldsymbol{\Lambda}_0 \circ (-\boldsymbol{i}_2) \circ \boldsymbol{\Phi}(t) \circ (\mu(t)\exp(\boldsymbol{i}_3 N(t)/2) \circ \boldsymbol{i}_2\\
&\quad \circ \exp(-\boldsymbol{i}_3 N(t)/2)/4 - \boldsymbol{i}_3 \nu(t)/2) \circ \boldsymbol{i}_2 \circ \tilde{\boldsymbol{\Lambda}}_0\\
&= \boldsymbol{\Psi}(t) \circ \boldsymbol{\Lambda}_0 \circ (-\boldsymbol{i}_2) \circ (\mu(t)(-\boldsymbol{i}_1 \sin N(t) + \boldsymbol{i}_2 \cos N(t))\\
&\quad - 2\boldsymbol{i}_3 v(t)) \circ \boldsymbol{i}_2 \circ \tilde{\boldsymbol{\Lambda}}_0/4
\end{aligned}$$

or, which is the same,

$$2\dot{\boldsymbol{\Psi}} = \boldsymbol{\Psi} \circ \boldsymbol{\Lambda}_0 \circ (-\boldsymbol{i}_2) \circ \boldsymbol{w}(t) \circ \boldsymbol{i}_2 \circ \tilde{\boldsymbol{\Lambda}}_0/2$$

where $\boldsymbol{\Psi}(0) = 1$, which coincides with expressions (18) and (19) provided that condition (20) is satisfied.

Note that the quaternion differential equation given by Eq. (18) is equivalent to the Poisson vector differential equation [2]

$$d\left(\tilde{\boldsymbol{\Psi}}(t) \circ \boldsymbol{c} \circ \boldsymbol{\Psi}(t)\right)/dt = \left(\tilde{\boldsymbol{\Psi}}(t) \circ \boldsymbol{c} \circ \boldsymbol{\Psi}(t)\right) \times \left(\boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}\right)/2 \tag{23}$$

where $\boldsymbol{c}$ is an arbitrary vector constant; its exact solution is constructed by means of relations (21) and (22). Below, this fact is used to obtain an explicit analytic solution of the approximate kinematic equation with respect to the MRPs vector of type (6) with the angular velocity $\boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}$.

IV. Exact Solution of the Vector Approximate Kinematic Equation and Quaternion Attitude Algorithm

Using expressions of type (4), we associate the reduced quaternion problem of determining orientation (12–15) with approximate differential equation obtained from Eq. (17):

$$\dot{\boldsymbol{\theta}}^* = \boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}/4 + \boldsymbol{\theta}^* \times \left(\boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}\right)/2 \tag{24}$$

$$\boldsymbol{\theta}^*(0) = 0 \tag{25}$$

We note that the homogeneous part of the linear differential equation (24) is equivalent to the solvable system (18) written in the form of a vector differential Poisson equation (23). From the Lagrange method of solving linear inhomogeneous differential systems of equations, the exact solution of the approximate Eq. (24) with the initial condition (25) will have the form on the basis of Eqs. (21) and (22):

$$\boldsymbol{\theta}^* = \boldsymbol{K} \circ \tilde{\boldsymbol{\Phi}}(t) \circ \int_0^t \boldsymbol{\Phi}(\boldsymbol{\tau}) \circ \boldsymbol{w}(\boldsymbol{\tau}) \circ \tilde{\boldsymbol{\Phi}}(\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau} \circ \boldsymbol{\Phi}(t) \circ \tilde{\boldsymbol{K}}/4 \quad (26)$$

We check the correctness of the obtained solution of the Eq. (24) by differentiating the expression (26):

$$\begin{aligned}\dot{\boldsymbol{\theta}}^* &= \boldsymbol{K} \circ \left(\tilde{\boldsymbol{\Phi}}(t) \circ \int_0^t \boldsymbol{\Phi}(\tau) \circ \boldsymbol{w}(\tau) \circ \tilde{\boldsymbol{\Phi}}(\tau)\,\mathrm{d}\tau \circ \boldsymbol{\Phi}(t) \circ \boldsymbol{w}(t) - \boldsymbol{w}(t)\right.\\ &\quad \left.\circ\, \tilde{\boldsymbol{\Phi}}(t) \circ \int_0^t \boldsymbol{\Phi}(\tau) \circ \boldsymbol{w}(\tau) \circ \tilde{\boldsymbol{\Phi}}(\tau)\,\mathrm{d}\tau \circ \boldsymbol{\Phi}(t)\right) \circ \boldsymbol{K}/16\\ &\quad + \boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}/4 = \boldsymbol{K} \circ \left(\tilde{\boldsymbol{\Phi}}(t) \circ \int_0^t \boldsymbol{\Phi}(\tau) \circ \boldsymbol{w}(\tau) \circ \tilde{\boldsymbol{\Phi}}(\tau)\,\mathrm{d}\tau\right.\\ &\quad \circ\, \boldsymbol{\Phi}(t) \circ \tilde{\boldsymbol{K}} \circ \boldsymbol{K} \circ \boldsymbol{w}(t) - \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}} \circ \boldsymbol{K} \circ \tilde{\boldsymbol{\Phi}}(t)\\ &\quad \left.\circ \int_0^t \boldsymbol{\Phi}(\tau) \circ \boldsymbol{w}(\tau) \circ \tilde{\boldsymbol{\Phi}}(\tau)\,\mathrm{d}\tau \circ \boldsymbol{\Phi}(t)\right) \circ \tilde{\boldsymbol{K}}/16\\ &\quad + \boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}/4 = \boldsymbol{\theta}^* \times (\boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}})/2 + \boldsymbol{K} \circ \boldsymbol{w}(t) \circ \tilde{\boldsymbol{K}}/4\end{aligned}$$

Thus, the problem of determining the orientation of a rigid body [Eqs. (1–3)] on the basis of Eq. (6) for the case of small rotation angles is completely solved in elementary functions with the help of quadratures. We give the analytical algorithm for determining orientation of a rigid body (SINS) for the case of arbitrary angles of rotation:

1) Using the components of angular velocity vector $\boldsymbol{\omega}(t)$ of a rigid body, functions $\mu(t), \nu(t)$ are calculated at each moment of time t by the formulas

$$\begin{aligned}\Omega_1(t) &= \int_0^t \omega_1(\tau)\,\mathrm{d}\tau\\ \mu(t) &= \omega_2(t)\cos\Omega_1(t) - \omega_3(t)\sin\Omega_1(t)\\ \nu(t) &= \omega_2(t)\sin\Omega_1(t) + \omega_3(t)\cos\Omega_1(t)\end{aligned} \quad (27)$$

2) Vector $\boldsymbol{w}(t)$ is determined by the calculated $\mu(t), \nu(t)$:

$$\begin{aligned}N(t) &= \int_0^t \nu(\tau)\,\mathrm{d}\tau\\ \boldsymbol{w}(t) &= \mu(t)(-\boldsymbol{i}_1\sin N(t) + \boldsymbol{i}_2\cos N(t)) - 2\boldsymbol{i}_3 v(t)\end{aligned} \quad (28)$$

3) Approximate value $\boldsymbol{\theta}^*$ of the MRPs orientation vector of a rigid body θ is calculated using vector $\boldsymbol{w}(t)$ and the initial position of a rigid body $\boldsymbol{\Lambda}_0$:

$$M(t) = \int_0^t \mu(\tau)\,\mathrm{d}\tau, \quad \boldsymbol{\Phi}(t) = \exp(\boldsymbol{i}_2 M(t)/4) \circ \exp(-\boldsymbol{i}_3 N(t)/2) \quad (29)$$

$$\begin{aligned}\boldsymbol{\theta}^* &= \boldsymbol{K} \circ \tilde{\boldsymbol{\Phi}}(t) \circ \int_0^t \boldsymbol{\Phi}(\tau) \circ \boldsymbol{w}(\tau) \circ \tilde{\boldsymbol{\Phi}}(\tau)\,\mathrm{d}\tau \circ \boldsymbol{\Phi}(t) \circ \tilde{\boldsymbol{K}}/4,\\ \boldsymbol{K} &= \boldsymbol{\Lambda}_0 \circ (-\boldsymbol{i}_2)\end{aligned} \quad (30)$$

4) Components of quaternion $\boldsymbol{U}$ are determined from the vector $\boldsymbol{\theta}^*$ on the basis of formulas of the type (4):

$$\begin{aligned}&\boldsymbol{e} = \boldsymbol{\theta}^*/|\boldsymbol{\theta}^*|, \qquad \varphi = 4\mathrm{arctg}|\boldsymbol{\theta}^*|\\ &u_0 = \cos(\varphi/2), \quad u_j = \sin(\varphi/2)e_j, \quad j = 1, 2, 3\end{aligned} \quad (31)$$

5) Approximate value of quaternion of a rigid body (SINS) orientation $\boldsymbol{\Lambda}^{\mathrm{approx}}$ is obtained:

$$\begin{aligned}\boldsymbol{\Lambda}^{\mathrm{approx}} &= \boldsymbol{U}(t) \circ \boldsymbol{K} \circ (-\boldsymbol{i}_1\sin N(t) + \boldsymbol{i}_2\cos N(t))\\ &\quad \circ \exp(\boldsymbol{i}_3 N(t)/2) \circ \exp(\boldsymbol{i}_1\Omega_1(t)/2)\end{aligned} \quad (32)$$

When implementing the SINS orientation algorithm, at each subsequent step m of algorithm, quaternion $\boldsymbol{K}$ should be selected in the form $\boldsymbol{K}_m = \boldsymbol{\Lambda}_{m-1} \circ (-\boldsymbol{i}_2)$. Then the initial value of variable $\boldsymbol{\theta}^*$ will be zero each time. The data about the angular motion trajectory of the rigid body (object) according to a relation of type (32) is accumulated by means of the quaternion $\boldsymbol{K}$.

V. Conclusions

The truncated Darboux problem is analytically solved. The proposed quaternion algorithm to find the orientation of a rigid body (object using SINS), based on the analytic solution of the approximate kinematic equation for the modified Rodrigues vector, is regular for any angular motion of the rigid body because this solution is exact. In contrast to the known algorithms for determining the orientation of an object, which use the approximate numerical solutions of the truncated Bortz equation for the vector of orientation of a rigid body and read information about the angular velocity of an object directly from sensors of SINS, the essence of the approach proposed in this paper is that by transforming mentioned information using formulas (27) and (28), the truncated kinematic equation for the modified Rodrigues vector $\boldsymbol{\theta}^*$ becomes clearly solvable by formulas (29) and (30). The quaternion, on which the solution of the problem is based, is written in elementary functions and quadratures by formulas (28–32).

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