Chapter 12

Topics in Two-Dimensional Analytic Geometry

In this chapter we look at topics in analytic geometry so we can use our calculus in many new settings. Most of the discussion will involve developing these settings, but once developed we will have some immediate input from our calculus.

Analytic geometry is the name we give to the combining of geometry with theories of equations. For instance, we place xy-coordinate axes in a plane, renaming the plane Euclidean or Cartesian 2-space, and immediately we can associate every line with an equations $Ax + By = C$ for some $A, B, C \in \mathbb{R}$. Thus we can associate geometric figures with equations involving x and y, and vice-versa, which opens up the possibility of algebraic and calculus-based analysis of geometric figures, and geometric analysis of the equations. It is nearly impossible to exaggerate the importance of these connections between equations and geometric objects.

We will not always use x and y, also known as *rectangular coordinates*, to describe points and equations. In particular, we will be interested in *polar coordinates,* which have many uses as well. We will also generalize the idea of a *position* (x, y) to the idea of a *vector* $\overrightarrow{PQ}$.

In the next chapter, we will extend these ideas into three-dimensional space, with rectangular coordinates (x, y, z), as well as cylindrical and spherical coordinates. We will examine lines and planes in space, surfaces in space, three-dimensional vectors and their applications.

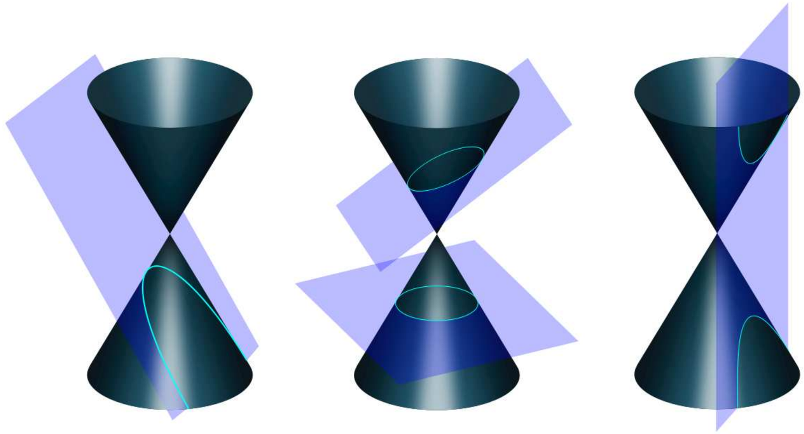


Figure 12.1: Congruent infinite right-circular cones (with the same axes of symmetry and vertices), with intersecting planes. The plane will intersect the cones in either a single point, a line, two lines or a *conic section:* i.e., a parabola, a circle, an ellipse, or a hyperbola. Graphic is from Wikimedia Commons.

12.1 Conic Sections

In short, conic sections are plane curves which are either circles, ellipses, parabolas or hyperbolas. By their natures, there are multiple ways of approaching these curves. We will only consider those with vertical or horizontal lines of symmetry.[1] We will first look at the actually definitions of the conic sections, but these are only occasionally necessary in applications, particularly in optics and astronomy, so we will then look at how to draw their graphs based upon their equations without referencing their definitions. Finally we will see how the definitions give rise to the equations, which will help us to write the equations given different types of data for the curves, including the location of foci.

12.1.1 Conic Sections Defined Geometrically

A *conic section* is an intersection of a single plane with two stacked, congruent, infinite right circular cones. This is illustrated in Figure 12.1. From there we can see that there are several ways these geometric figures can intersect. Not included in the Figure 12.1 above are other possibilities, which are not of interest here except as "degenerate cases," such as a single point, a single line or two lines meeting at the vertices of the cones (left to the imagination of the reader). The more interesting cases we deal with here are the parabola, ellipse (including the circle), and the hyperbola. When one uses the term "conic section," it is usually in reference to these "more interesting" cases.

However, there are other definitions of these figures, based upon distances. The circle is the easiest: the set of all points in a plane which are the same, fixed distance, or *radius* from a fixed point, called the *center*, also in the plane. If the center is (h, k) and the radius is r, this definition quickly becomes (from the Pythagorean Theorem)

$$\sqrt{(x-h)^2+(y-k)^2}=r,$$

[1]A dedicated analytic geometry or a linear algebra course can be appropriate settings to consider equations of *rotated* conic sections with non-vertical or non-horizontal lines of symmetry.

or the more common form, which is equivalent since $r > 0$:

$$(x-h)^2 + (y-k)^2 = r^2. \tag{12.1}$$

For $r = 0$ our "circle" would be a single point, which is also a possible intersection of a plane with our cones above. Thus one can debate whether or not to consider a single point to somehow be a circle.

The ellipse is defined somewhat similarly, except that for the ellipse we take two fixed points, or foci (singular is "focus") in the plane, and find all points whose distances sum to some fixed total distance. A special case of this can be seen in Figure 12.6, page 822. It will take some effort to derive the form of an equation for an ellipse from this definition, and we will do so later, in Subsection 12.1.3.

Next in complexity is the parabola, which is defined by all the points in the plane whose distance to a fixed point, or focus in the plane, is the same as the distance to a fixed line, or directrix lying in the plane but not containing the focus. This is illustrated in Figure 12.5, page 821. The derivation of an equation for a parabola is somewhat simpler, and will also be included later.

Finally, there is the hyperbola, which is defined by all points in a plane whose distances to two fixed points, or foci in the plane, differ (in absolute value) by a constant. This is illustrated in Figure 12.7, page 824.

While the definitions are interesting and illustrative, we can do much with these figures without resorting to these definitions, or even the foci or (in the case of the parabola) directrixes (also called directrices). This is because we can get much general information regarding the positions and shapes of the figures from simplified equations in which a focus or directrix does not appear directly. For completeness we will return to these definitions in Subsection 12.1.3, and use information from those derivations in later computations. While we can draw, say, a parabola without knowing its focus or directrix, if we are interested in optics or acoustics these things are more important. For most calculus applications they are not so much.

12.1.2 Simplest Equations of Conic Sections

Most algebra students learn quickly that the graph of $y = x^2$ is a parabola.[2] After plotting simple points, we see a line of symmetry $x = 0$, intersecting at a *vertex* $(0, 0)$. After a small amount of the graphical theory of functions, one usually then learns that any equation of the form

$$f(x) = a(x-h)^2 + k, \qquad a \neq 0 \tag{12.2}$$

will be similar, but with the vertex shifted to (h, k), the axis of symmetry now being $x = h$, and the parabola opening upward if $a > 0$, and downward if $a < 0$. Shortly after that, one is taught that any function of the form $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is also a parabola, and we can use

[2]However, many algebra students make the mistake of believing that any "U-shaped," or upside-down U-shaped curve is a parabola. Just as not every round "loop"is a circle, we should not expect every curve whose shape superficially resembles a parabola to actually *be* a parabola. The curve $y = \sin x$, for $x \in [0, \pi]$ is one such example, which resembles a parabola but is in fact not a piece of a parabola. Nor are either branches of a hyperbola "parabolic." Similarly, not every "oval" is an ellipse.

completing the square to find its form (12.2):[3]

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c \\ &= a\left(x - \frac{b}{2a}\right)^2 - a \cdot \frac{b^2}{4a^2} + c \\ &= a\left(x - \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right). \end{aligned}$$

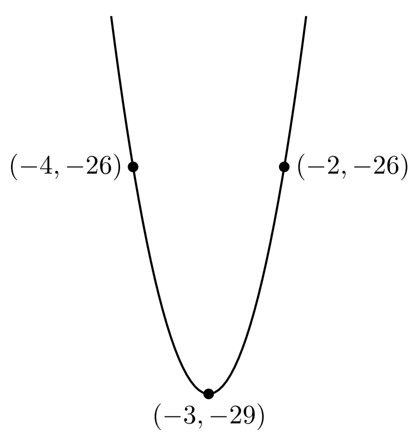
Thus $y = f(x)$ has the same shape as $y = ax^2$ except it has been moved horizontally by $h = \frac{b}{2a}$ and vertically by $k = c - \frac{b^2}{4a}$.

Example 12.1.1 *Find the vertex and graph the parabola* $y = 3x^2 + 18x - 2$.

<u>Solution</u>: *We do as above, completing the square after first factoring the second and first-degree terms collectively. Recall that when completing a square of the form* $x^2 + \beta x$, *we add and subtract* $(\beta/2)^2$, *to get* $x^2 + \beta x + \beta^2/4 - \beta^2/4 = \left(x + \frac{\beta}{2}\right)^2 - \beta^2/4$.

$$\begin{aligned} y = 3(x^2 + 6x) - 2 &= 3(\underbrace{x^2 + 6x + 9}_{\textit{perfect square}} - 9) - 2 \\ &= 3(x+3)^2 - 27 - 2 = 3(x+3)^2 - 29. \end{aligned}$$

Thus we are asked to plot $y = 3(x+3)^2 - 29$, *which has a vertex at* $(-3, -29)$. *It is common practice to plot the vertex and two symmetric points, such as* $x = h \pm 1$ *if it is convenient. Here that would be the points* $(-2, -26)$ *and* $(-4, -26)$. *Because of the location of the curve, we omit the axes here:*



Of course there are also those of the form

$$x = a(y-k)^2 + h, \qquad \text{or} \qquad x - h = a(y-k)^2.$$

[3]Since we have calculus here we can certainly use it to find that the one critical point where $f'(x) = 0$ is at $x = -b/2a$, and that this must be the *vertex* of the parabola.

These open horizontally, to the right (vertex on the left) if $a > 0$, and to the left if $a < 0$.

Now consider an equation which is claimed to represent an ellipse:

Example 12.1.2 *Consider the curve* $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$.

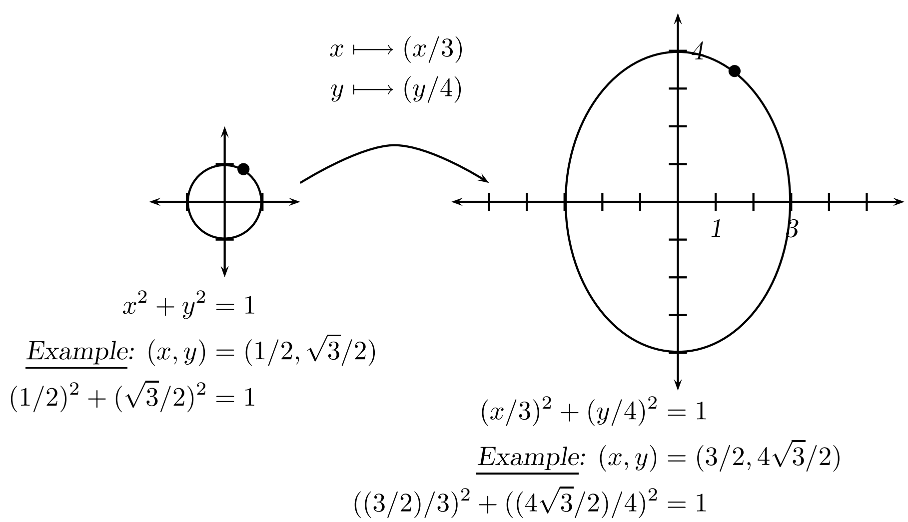
Note that this is a variation on the unit circle $x^2 + y^2 = 1$, *except for scaling in the horizontal and vertical directions. In fact the above equation can be rewritten*

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1.$$

If we look at ordered pairs (X, Y) *which satisfy* $x^2 + y^2 = 1$, *then for analogous points on the ellipse* $(x/3)^2 + (y/4)^2 = 1$ *we require* $(3X, 4Y)$*:*

$$(3X/3)^2 + (4Y/4)^2 = 1 \iff X^2 + Y^2 = 1.$$

This effectively stretches the graph by a factor of 3 in the horizontal direction, and by a factor of 4 in the vertical direction, giving us our ellipse. Below we give the graphs, equations, and one analogous point on each.



The example above, properly generalized, indicates a method of efficient plotting of ellipses. If we generalize this to have different centers (h, k) (not simply $(0, 0)$ as above), we get a form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \tag{12.3}$$

Here we assume $a, b > 0$. For any such equation, graphing the ellipse is straight-forward:

- Identify the center (h, k), and label it.
- From the center, move left and right by a to find the points along the horizontal axis.
- From the center, move up and down by b to find the points on the vertical axis.
- Graph the ellipse.

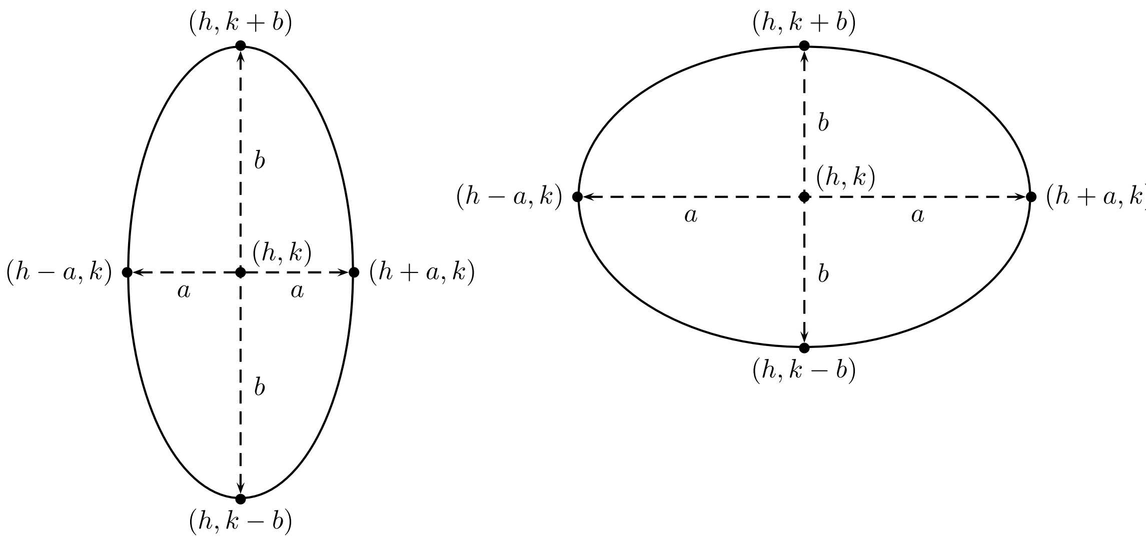
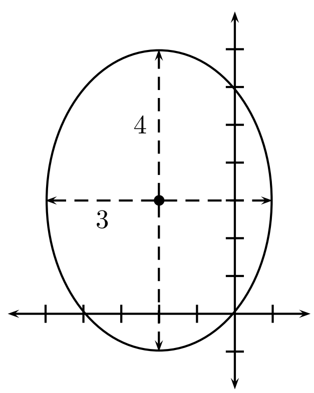


Figure 12.2: Ellipses with equations $(x-h)^2/a^2 + (y-k)^2/b^2 = 1$. The centers are (h, k) in both cases. For the first ellipse, $a < b$, so it has a vertical *major axis* of length $2a$, and horizontal *minor axis* of length $2b$. In the second ellipse, $a > b$ so the major axis is horizontal with length $2a$, and the minor axis is vertical with length $2b$.

See Figure 12.2. There we see two axes: a horizontal axis of length $2a$, and a vertical axis of length $2b$. The longer of these axes is called the *major axis*, and the shorter of these axes is called the *minor axis*.

Example 12.1.3 *Consider the ellipse* $\dfrac{(x+2)^2}{9} + \dfrac{(y-3)^2}{16} = 1$*. The center is* $(-2, 3)$*, and* $a = 3$*,* $b = 4$*, so four points we can plot immediately are* $(-2 \pm 3, 3)$*,* $(-2, 3 \pm 4)$*, i.e, points* $(-5, 3)$*,* $(1, 3)$*,* $(-2, -1)$ *and* $(-2, 7)$*:*



On occasion, some algebraic manipulations are required to get the form (12.3).

Example 12.1.4 *Find the center and four axis (major and minor) points of the ellipse*

$$4x^2 - 12x + 5y^2 + 20y = 0.$$

Solution: The usual process of completing the square, and the final steps to have the constant 1 on the right-hand side, are as follow:

$$\begin{aligned}
& 4x^2-12x+5y^2+20y=0\\
\iff\quad & 4(x^2-3x)+5(y^2+4y)=0\\
\iff\quad & 4\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)+5(y^2+4y+4-4)=0\\
\iff\quad & 4(x-3/2)^2-9+5(y+2)^2-20=0\\
\iff\quad & 4(x-3/2)^2+5(y+2)^2=29\\
\iff\quad & \frac{4}{29}(x-3/2)^2+\frac{5}{29}(y+2)^2=1\\
\iff\quad & \frac{(x-3/2)^2}{29/4}+\frac{(y+2)^2}{29/5}=1.
\end{aligned}$$

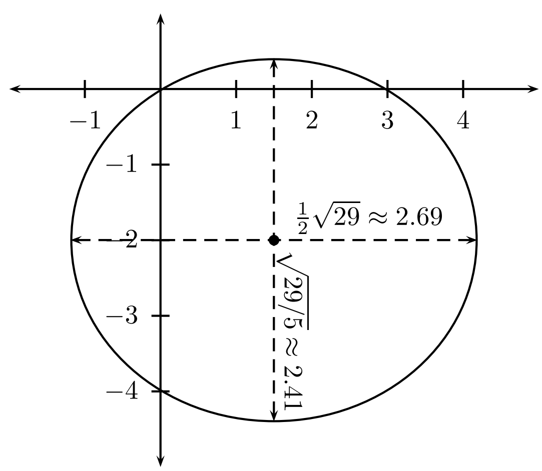
From this we see the center is $(3/2,-2)$, and the points on the axes are

$$\left(\frac{3}{2}\pm\frac{\sqrt{29}}{2},-2\right),\qquad \left(\frac{3}{2},-2\pm\sqrt{\frac{29}{5}}\right).$$

More useful for graphing perhaps are decimal approximations of these points:

$$(4.19,-2),(-1.19,-2),(1.5,.41),(1.5,-4.4).$$

A rough sketch of this ellipse is then relatively easy, either by plotting the four points above, or by using the distances $a=\frac{1}{2}\sqrt{29}\approx 2.69$, and $b=\sqrt{29/5}\approx 2.41$. Since these are so similar in length, the ellipse is closer to "circular" than the previous examples.



For the hyperbola, we start with the simplest examples, namely $x^2-y^2=1$ and $y^2-x^2=1$. These are illustrated in Figure 12.3 Let us take these in turn, though we note that there is a clear analogy between the two curves, as x and y basically exchange roles.

- $x^2-y^2=1$: For this curve, we first note that $x^2=1-y^2\in[1,\infty$, which requires $|x|\in[1,\infty)$, i.e., $x\in(-\infty,-1]\cup[1,\infty)$. Therefore x is somewhat limited in possible values, while y is not. In fact the "vertices" of the hyperbola will occur where $x=\pm1$,

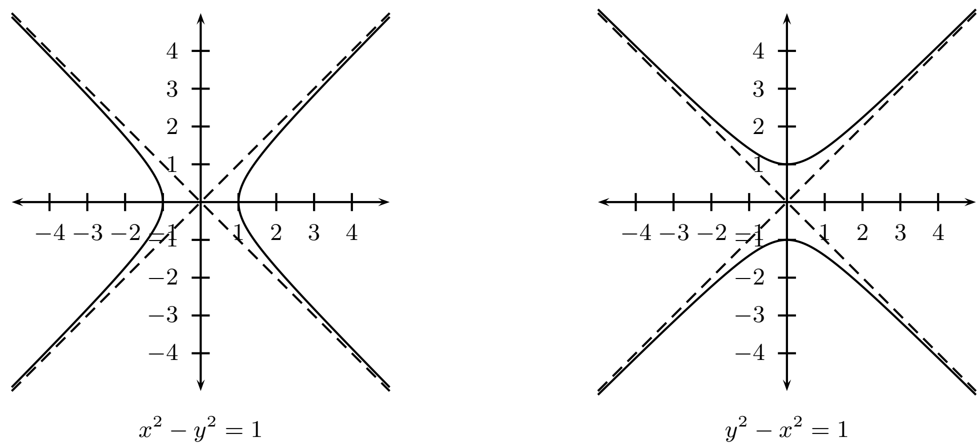


Figure 12.3: Illustrations of the simplest hyperbolas considered here: $x^2 - y^2 = 1$ and $y^2 - x^2 = 1$. In both cases, for large x we have $y \approx \pm x$, giving us two linear asymptotes.

and $y = 0$. From the equation, the graph is obviously symmetric with respect to both axes, since (a, b) on the graph means all possible combinations of $(\pm a, \pm b)$ will also be on the graph. The line of symmetry passing through both vertices is called the *axis* of the hyperbola, which in this case is the x-axis. Finally, for large x we have

$$y = \pm\sqrt{x^2+1} \approx \pm\sqrt{x^2} = \pm|x| = \pm x,$$

giving us the two asymptotic lines $y = x$ and $y = -x$.

- $y^2 - x^2 = 1$: For this curve, we require $y \in (-\infty, -1] \cup [1, \infty)$, but there is no restriction on x. The vertices occur where $y = \pm 1$, and $x = 0$, and the graph is symmetric with respect to both axes, the y-axis being the *axis* of this particular hyperbola. Finally, for large x we have

$$y = \pm\sqrt{x^2+1} \approx \pm\sqrt{x^2} = \pm|x| = \pm x,$$

again giving us asymptotic lines $y = x$ and $y = -x$.

Note that both hyperbolas have a natural "center" at $(0, 0)$, which is both where the asymptotes intersect each other, and a point with respect to which the parabola is symmetric (in the "inverting lens" fashion).

We can now find our general equation of the hyperbola centered at (h, k):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \tag{12.4}$$

$$\frac{(y-h)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1. \tag{12.5}$$

For the case (12.4), the vertices are at $(h \pm a, k)$, and the axis is the line $y = k$. For case (12.5), the vertices are at $(h, k \pm b)$, and the axis is the line $x = h$. In both cases, we have center (h, k), asymptotes through (h, k) with slopes $m = \pm\frac{b}{a}$:

$$y - k \approx \pm\frac{b}{a}(x - h). \tag{12.6}$$

In fact, when graphing these it is simpler to use the point (h, k) and the slopes $\pm b/a$, rather than using (12.6).

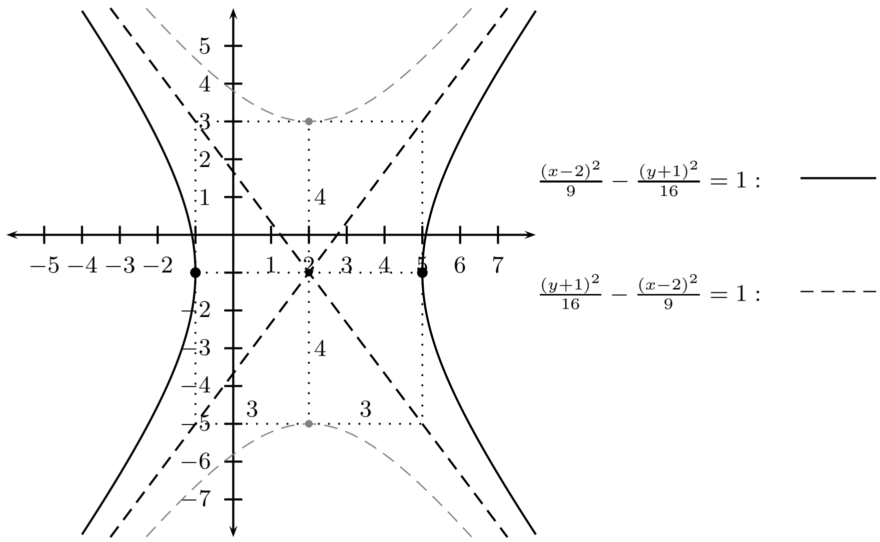


Figure 12.4: Illustration of graphing technique for two hyperbolas, with the same center (h, k), and same values for a and b. They will have the same asymptotes, and we can use the same "box" to guide our graphs for both. See Example 12.1.5

Example 12.1.5 *Graph the hyperbola* $\dfrac{(x-2)^2}{9} - \dfrac{(y+1)^2}{16} = 1.$

Solution: The equation puts restrictions on x, and the vertices will occur where $(x-2)^2/9 = 1$, i.e., where $x - 2 = \pm 3$, while $(y+1)^2/16 = 0$, so the vertices are at at $(2 \pm 3, -1)$. In fact, a quick method of graphing this is to

1. *first, make note of the center $(2, -1)$;*
2. *next, move $a = 3$ units horizontally both left and right to find the vertices,*
3. *next, draw the asymptotes as lines through the center $(2, -1)$ and with slopes $\pm b/a = \pm 4/3$, and*
4. *finally, draw a hyperbola through the vertices found above, approaching the asymptotes (usually rather quickly).*
5. *Alternatively, draw the (dotted) box below, and use it to find the vertices and asymptotes to draw the curve, as explained below.*

   See Figure 12.4 for the graph illustrating the technique. The extra "dotted" line segments are optional, and represent

   (a) *points obtained by moving from the center $(h, k) = (2, -1)$ horizontally by $\pm a = \pm 3$, and vertically by $\pm b = \pm 4$, and will include the vertices, and*

   (b) *the "corners" of a box through which the lines passing through the center and have slopes $\pm b/a = \pm 4/3$. These help to draw the asymptotes.*

   The lengths of each dotted segment are $a = 3$ for horizontal segments, and $b = 4$ for each vertical segment.

It should be pointed out that the related hyperbola

$$\frac{(y+1)^2}{16} - \frac{(x-2)^2}{9} = 1$$

will have the same asymptotes, and the same "box" as above, though the vertices will be at $(h, k \pm b) = (-2, 3), (-2, -5)$. The graph of this hyperbola is superimposed on the graph for the hyperbola in Example 12.1.5, but in dashed, gray lines.

If the equation of the hyperbola is not given in the standard forms (12.4) or (12.5), we may need to manipulate the given equation to achieve such form.

Example 12.1.6 *Graph the equation $2x^2 - 6x = 3y^2 - 18y$.*

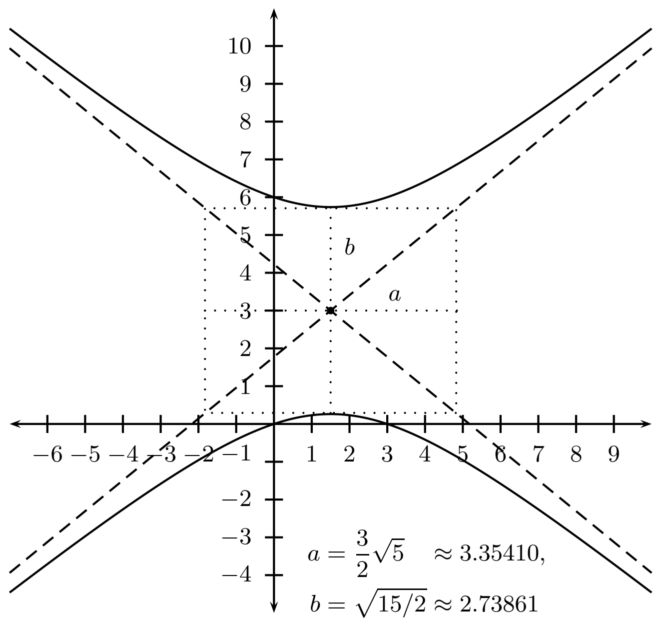
Solution: We will complete both squares separately, and see what is left outside the squares to decide which form the standard equation should have.

$$\begin{aligned}
& & 2(x^2-3x) &= 3(y^2-6y) \\
&\iff & 2\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right) &= 3(y^2-6y+9-9) \\
&\iff & 2\left(x-\frac{3}{2}\right)^2-\frac{9}{2} &= 3(y-3)^2-27 \\
&\iff & 27-\frac{9}{2} &= 3(y-3)^2-2\left(x-\frac{3}{2}\right)^2 \\
&\iff & \frac{45}{2} &= 3(y-3)^2-2\left(x-\frac{3}{2}\right)^2 \\
&\iff & 1 &= \frac{3(y-3)^2}{45/2}-\frac{2\left(x-\frac{3}{2}\right)^2}{45/2} \\
&\iff & 1 &= \frac{(y-3)^2}{15/2}-\frac{\left(x-\frac{3}{2}\right)^2}{45/4}.
\end{aligned}$$

If we wish to make the form more obvious, we could write

$$\frac{(y-3)^2}{\left(\sqrt{\frac{15}{2}}\right)^2} - \frac{\left(x-\frac{3}{2}\right)^2}{\left(\frac{3\sqrt{5}}{2}\right)^2} = 1.$$

Here, $a = \frac{3}{2}\sqrt{5} \approx 3.3541$, *and* $b = \sqrt{15/2} \approx 2.73861$, *which will help us to graph the hyperbola reasonably. Note that the slopes of the asymptotes are* $\pm b/a = \pm\sqrt{\frac{5\cdot 3}{2}}\cdot\frac{2}{3\sqrt{5}} = \sqrt{2/3} \approx \pm 0.81650$. *Using the center* $(3/2, 3)$ *we can graph the hyperbola, though if we sketch it by hand we can use* $\pm b/a \approx \pm 4/5$, *for instance (though the graph below is computer-generated).*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$b = \sqrt{15/2} \approx 2.73861$

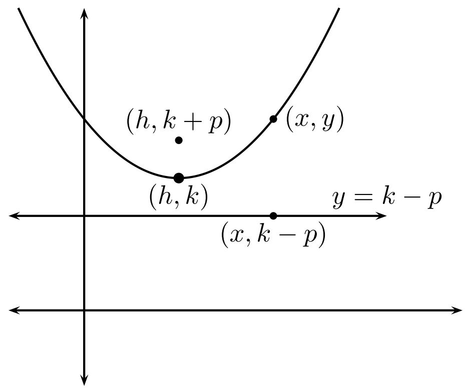


Figure 12.5: Illustration of a parabola opening vertically, with vertex at (h, k), and p being the directed distance from the vertex to the focus. The directrix is then a displacement of $-p$ from the vertex, opposite the focus. Any point (x, y) on the parabola will be the same distance from the focus $(h, k+p)$ and the directrix $y = k - p$. The case illustrated here has $p > 0$. If $p < 0$ then the parabola would open downward.

12.1.3 Equations of Conic Sections from Definitions

Consider the definition of a parabola: the set of all points which are the same distance from a fixed point (the focus), and a fixed line (the directrix). The vertex will be the point on the parabola which is the shortest distance from both of these. We will only consider parabolas with vertical or horizontal axes (lines) of symmetry, which will pass through both the vertex and the focus (and will be perpendicular to the directrix). Our derivation will be for a vertically opening parabola (with a vertical axis of symmetry), from which the horizontally opening parabolas' equations will follow.

We will take (h, k) to be the vertex, and p to be the displacement, or "directed distance," from the vertex to the focus. (See Figure 12.5.) Thus $-p$ is the displacement between the vertex and the directrix.

Now consider any point (x, y) on the parabola. Then the distance to the focus being equal to the distance to the directrix will give us

$$\sqrt{(x-h)^2 + [y-(k+p)]^2} = |y-(k-p)|.$$

We now square both sides, expand and cancel, though for reasons that will become clearer later, we will not expand the $(x-h)^2$ term. Note $|y-(k-p)|^2 = [y-(k-p)]^2$.

$$\begin{aligned}
& (x-h)^2 + [y-(k+p)]^2 = [y-(k-p)]^2 \\
\iff & (x-h)^2 + \cancel{y^2} - 2(k+p)y + (k+p)^2 = \cancel{y^2} - 2(k-p)y + (k-p)^2 \\
\iff & (x-h)^2 - \cancel{2ky} - 2py + \cancel{k^2} + 2kp + \cancel{p^2} = -\cancel{2ky} + 2py + \cancel{k^2} - 2kp + \cancel{p^2} \\
\iff & (x-h)^2 = 4py - 4pk \\
\iff & (x-h)^2 = 4p(y-k).
\end{aligned}$$

With this, and the analogous result for horizontall-opening parabolas, also with vertex at (h, k), we have the following:

- A vertical-axis parabola with vertex at (h, k) and the focus at a vertical displacement p

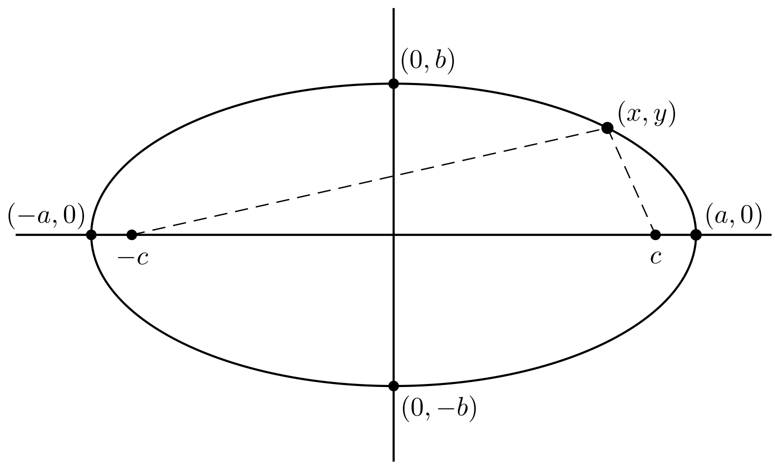


Figure 12.6: A special case of an ellipse, with foci at $(-c, 0)$ and $(c, 0)$. When we add the distances from any point (x, y) on the ellipse to the foci, the sum will be constant.

from the vertex will have equation

$$\begin{aligned} (x-h)^2 &= 4p(y-k), \\ \underline{\text{focus}} \ &= (h, k+p), \\ \underline{\text{directrix}}\text{:} \ & y = k-p. \end{aligned} \tag{12.7}$$

- A horizontal-axis parabola with vertex at (h, k) and the focus at a horizontal displacement p from the vertex will have equation

$$\begin{aligned} (y-k)^2 &= 4p(x-h), \\ \underline{\text{focus}} \ &= (h+p, k), \\ \underline{\text{directrix}}\text{:} \ & x = k-p. \end{aligned} \tag{12.8}$$

For the ellipse, we choose some distance d which is the sum of the distances from a point (x, y) to the foci, which we will assume for simplicity are located at $(\pm c, 0)$. (We can easilly adjust for more complicated cases later.) If we let d be the sum of distances from (x, y) to the two foci, we get

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = d.$$

One needs to square both sides to remove the radicals, but this can not be done in one step. It is usually algebraically simpler to have one of the radicals alone on one side before squaring.

Proceeding as before, we have

$$\begin{aligned}
&& \sqrt{(x+c)^2+y^2} &= d-\sqrt{(x-c)^2+y^2}\\
\Longrightarrow && \cancel{x^2}+2cx+\cancel{c^2}+\cancel{y^2} &= d^2-2d\sqrt{(x-c)^2+y^2}+\cancel{x^2}-2cx+\cancel{c^2}+\cancel{y^2}\\
\Longleftrightarrow && 2d\sqrt{(x-c)^2+y^2} &= d^2-4cx\\
\Longrightarrow && 4d^2[x^2-2cx+c^2+y^2] &= d^4-8cd^2x+16c^2x^2\\
\Longleftrightarrow && 4d^2x^2-8\cancel{c}d^2x+4c^2d^2+4d^2y^2 &= d^4-8\cancel{c}d^2x+16c^2x^2\\
\Longleftrightarrow && (4d^2-16c^2)x^2+4d^2y^2 &= d^4-4c^2d^2\\
\Longleftrightarrow && 4(d^2-4c^2)x^2+4d^2y^2 &= d^2(d^2-4c^2)\\
\Longleftrightarrow && \frac{4}{d^2}x^2+\frac{4}{d^2-4c^2}y^2 &= 1\\
\Longleftrightarrow && \frac{x^2}{d^2/4}+\frac{y^2}{(d^2-4c^2)/4} &= 1.
\end{aligned}$$

Now if we let $a^2 = d^2/4$ and $b^2 = (d^2-4c^2)/4$, we see that

$$a^2-b^2 = \frac{d^2}{4}-\frac{d^2-4c^2}{4} = \frac{4c^2}{4} = c^2.$$

Thus we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \tag{12.9}$$

where

$$a^2-b^2 = c^2, \tag{12.10}$$

If one is interested, we also find d since $d^2 = a^2/4$, though for an ellipse with the foci on a vertical axis we would replace a with b. In fact, the longer (major) axis will be in the direction of the variable x if $a > b$, and y if $b > a$. However, a perhaps more easily recalled method for finding d might be to find the foci and one point on the ellipse, and simply compute the sum of distances to the foci from that point. Because d does not explicitly appear in the equation of an ellipse, it is usually not included in the discussion, beyond the derivation.

By considering vertical and horizontal translations of the equation above, and considering cases where the major axis is vertical, we conclude that the general equation for an ellipse centered at (h,k), is given by

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2} = 1, \tag{12.11}$$

where

- if $a > b$ the major axis (containing the foci) is horizontal, with endpoints $(h \pm a)$, with foci $(h \pm c, k)$, $c^2 = a^2-b^2$, and minor axis endpoints $(h, k \pm b)$;
- if $b > a$, the major axis (containing the foci) is vertical, with endpoints $(h, k \pm b)$, with foci $(h, k \pm c)$, $c^2 = b^2-a^2$, and minor axis with endpoints $(h \pm a, k)$.
- In all cases, the center is at (h,k), axes have endpoints $(h \pm a, k)$ and $(h, k \pm b)$, the foci are a distance c (i.e., a displacement $\pm c$) from the center (h,k) along the major (longer) axis, where
$$c^2 = |a^2-b^2|. \tag{12.12}$$

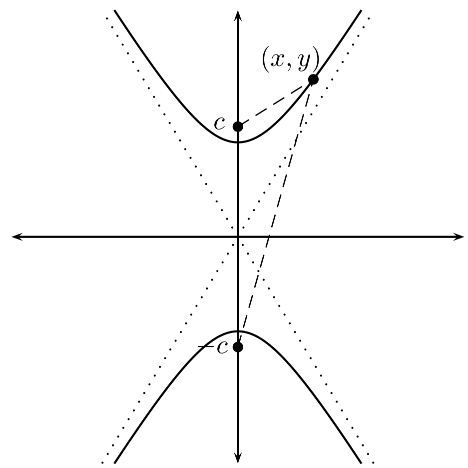


Figure 12.7: A special case of the hyperbola, with foci at $(0, -c)$ and $(0, c)$. The distances from any point (x, y) on the hyperbola to the foci will differ by a constant (in the sense that the abolute value of the differences will be constant). The large-x and large-y behavior of the hyperbola is asymptotically linear (see dotted lines).

For the hyperbola, we will first look at the case where the foci are located at $(0, \pm c)$, as in Figure 12.7. For any point (x, y) on the hyperbola, this translates to

$$\left|\sqrt{x^2 + (y-c)^2} - \sqrt{x^2 + (y+c)^2}\right| = d.$$

We begin with the case that the first distance is greater, and leave the other case as an exercise. (Note that for this case, $y < 0$.)

$$\begin{aligned}
& \sqrt{x^2 + (y-c)^2} - \sqrt{x^2 + (y+c)^2} = d \\
\iff & \sqrt{x^2 + (y-c)^2} = d + \sqrt{x^2 + (y+c)^2} \\
\implies & \cancel{x^2} + \cancel{y^2} - 2cy + \cancel{c^2} = d^2 + 2d\sqrt{x^2 + (y+c)^2} + \cancel{x^2} + \cancel{y^2} + 2cy + \cancel{c^2} \\
\iff & -4cy - d^2 = 2d\sqrt{x^2 + (y+c)^2} \\
\implies & 16c^2y^2 + 8cd^2y + d^4 = 4d^2(x^2 + y^2 + 2cy + c^2) \\
\iff & 16c^2y^2 + 8\cancel{c}d^2y + d^4 = 4d^2x^2 + 4d^2y^2 + 8\cancel{c}d^2y + 4c^2d^2 \\
\iff & (16c^2 - 4d^2)y^2 - 4d^2x^2 = 4c^2d^2 - d^4 \\
\iff & 4(4c^2 - d^2)y^2 - 4d^2x^2 = d^2(4c^2 - d^2) \\
\iff & \frac{4}{d^2}y^2 - \frac{4}{4c^2 - d^2}x^2 = 1 \\
\iff & \frac{y^2}{d^2/4} - \frac{x^2}{(4c^2 - d^2)/4} = 1
\end{aligned}$$

Letting $b^2 = d^2/4$ and $a^2 = (4c^2 - d^2)/4$, this becomes

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1. \tag{12.13}$$

Note that $b^2 + a^2 = 4c^2/4$, i.e.,

$$a^2 + b^2 = c^2. \tag{12.14}$$

Note that if we solve for y in (12.13), we have

$$\begin{aligned} \frac{y^2}{b^2} &= 1 + \frac{x^2}{a^2} \\ \Longrightarrow \quad y^2 &= b^2 + \frac{b^2}{a^2}x^2 \\ \Longleftrightarrow \quad y &= \pm\frac{b}{a}\sqrt{a^2 + x^2} \approx \pm\frac{b}{a}\sqrt{x^2}, \end{aligned} \tag{12.15}$$

this last line being for large x. Recall $\sqrt{x^2} = |x| = \pm x$, depending upon whether $x \geq 0$ or $x < 0$. Thus,

$$x \text{ large} \quad \Longrightarrow \quad y \approx \pm\frac{b}{a}\,x. \tag{12.16}$$

We see that the large-x behavior is for $y \approx \pm\frac{b}{a}x$, which gives two linear asymptotes: $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$. It is also notable that $\frac{y^2}{b^2} \geq 1$ gives us $y^2 \geq b^2$, requiring $y \geq b$ or $y \leq -b$, i.e., $y \in (-\infty, -b] \cup [b, \infty)$, so y is somewhat limited in possible values, where x is not, as we can see within the equations (12.15). The *vertices* of the hyperbola are at $(0, \pm b)$. Through those vertices is the *axis*. Of course there are two lines of symmetry, the axis and the line through the *center* (0,0) and perpendicular to the axis.

Looking at translations of our simplified model, we get a general form with translated features. The equation and the hyperbola's features will be as follow:

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1 \tag{12.17}$$

- centered at (h, k),
- vertical axis line $x = h$
- asymptotes $y - k = \pm\frac{b}{a}(x - h)$, i.e., containing (h, k) with slopes $\pm\frac{b}{a}$,
- vertices at $(h, k \pm b)$,
- foci at vertical distances c from the center (h, k), i.e., at $(h, k \pm c)$, where $c^2 = a^2 + b^2$

If instead we have a horizontal axis, we have:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{12.18}$$

- centered at (h, k),
- horizontal axis line $y = k$
- asymptotes $y - k = \pm\frac{b}{a}(x - h)$, i.e., containing (h, k) with slopes $\pm\frac{b}{a}$,
- vertices at $(h \pm a, k)$,
- foci at vertical distances c from the center (h, k), i.e., at $(h \pm c, k)$, where $c^2 = a^2 + b^2$

12.1.4 Eccentricity

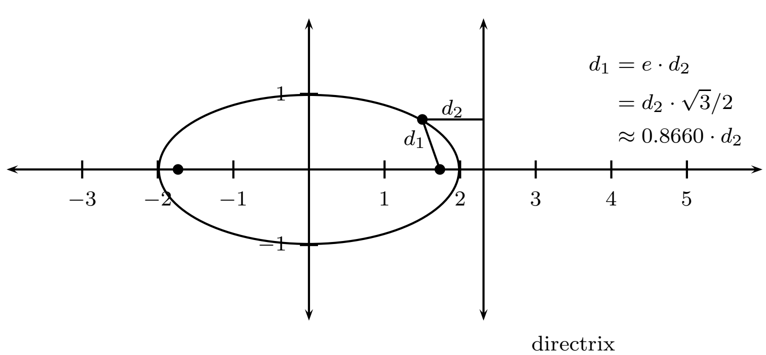
Physicists often discuss a concept called *eccentricity*, associated with conic sections or, more loosely, how much a curved path deviates from a circle (which then has zero eccentricity). For most of our conic sections, it is the ratio of the distance from the "center" to a focus, divided by the distance from the center to a vertex. For the ellipse, this would be $e = c/a$ if $a > b$, or $e = c/b$ if $b > a$, so we can simply write $e = c/\max\{a, b\}$. Since $c^2 = |a^2 - b^2|$, we have $c < \max\{a, b\}$, $0 < e < 1$ for an ellipse. A circle is like an ellipse with both foci at the center ($c = 0$), so $e = 0$. For a hyperbola, $c^2 = a^2 + b^2$ so $e = c/\max\{a, b\} > 1$.

For the parabola, it would seem strange to discuss a "center." For the other conic sections, it seems that "center" could be the one point through which the conic section is symmetric. Recall that "symmetric with respect to a point" means that, were that point a "lens," any part of the figure on one "side" of the point ("lens") has a corresponding part which is the optic inversion of the first part through that point ("lens"). However the parabola has no such symmetry with respect to a point. (For the others, it is at their center (h, k).) The solution is to instead define a "directrix" for each figure, and not just for the parabola, and then the eccentricity is the ratio of the distance from any point on the figure to the focus versus the distance to the directrix.[4] From the definition of the parabola—all points which are the same distance to the focus as the directrix—we trivially have $e = 1$.

Exercises

1. Compute the general formula (12.6) for the asymptotes of a hyperbola for both (12.4) and (12.5).

[4] We will not pursue this in depth here, but mention one case here briefly. For an ellipse, the directrix is a line such that the ellipse can be defined as all points (x, y) for which the distance d_1 to the focus is proportional to the perpendicular distance d_2 to the directrix. That proportionality constant is then $e < 1$. For the ellipse in Figure 12.6, a directrix would be $x = a^2/c$, which is to the right of the ellipse since $a^2/c = a \cdot \frac{a}{c} > a$. For the ellipse below, $a = 2$, $b = 1$, so $c = \sqrt{a^2 - b^2} = \sqrt{3}$, and then $e = c/a = \sqrt{3}/2 < 1$.



12.2 Parametric Curves

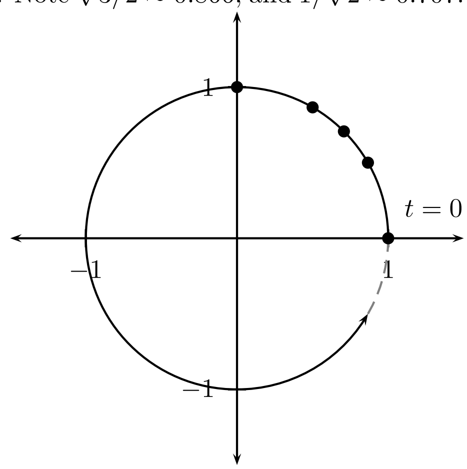
The term **parametric curve** comes from the idea that both the x-coordinate and the y-coordinate of the curve will be functions of a third variable, called the **parameter**. This allows for many more types of curves than those given functionally by $y = f(x)$, and even includes some that would be difficult to give implicitly.

An example of a parametric curve in the plane can be

$$x = \cos t,$$
$$y = \sin t.$$

Thus we follow the x-coordinate as t varies, and separately (if we like) the y-coordinate as t varies. When we graph this for the first time, we might make a chart and graph the points that occur, and attempt to deduce the shape of the graph. Choosing t-values with known sines and cosines, we might produce the following table and graph. Note $\sqrt{3}/2 \approx 0.866$, and $1/\sqrt{2} \approx 0.707$.

t	$x = \cos t$	$y = \sin t$
0	1	0
$\pi/6$	$\sqrt{3}/2$	$1/2$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	1
$2\pi/3$	$\sqrt{3}/2$	$-1/2$
$3\pi/4$	$1/\sqrt{2}$	$-1/\sqrt{2}$
$5\pi/6$	$1/2$	$-\sqrt{3}/2$
π	-1	0
$7\pi/6$	$-\sqrt{3}/2$	$-1/2$
$5\pi/4$	$-1/\sqrt{2}$	$-1/\sqrt{2}$
$4\pi/3$	$-1/2$	$-\sqrt{3}/2$
$3\pi/2$	0	-1
$5\pi/3$	$1/2$	$-\sqrt{3}/2$
$7\pi/4$	$1/\sqrt{2}$	$-1/\sqrt{2}$
$11\pi/6$	$\sqrt{3}/2$	$-1/2$



12.3 Polar Coordinates

12.4 Calculus in Polar Coordinates

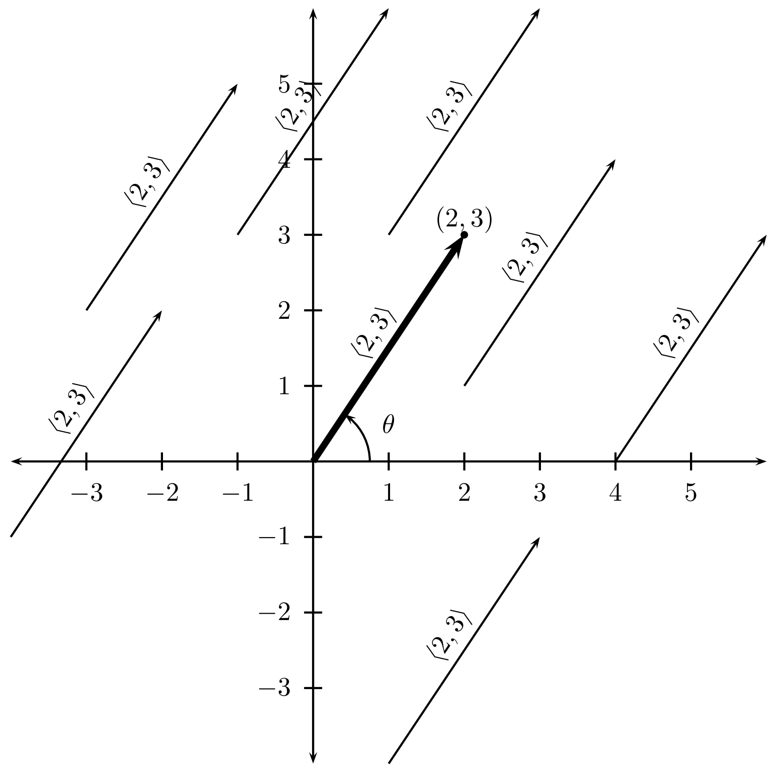


Figure 12.8: For each vector illustrated above, the displacement from the "tail" to the "head" is +2 horizontally, and +3 vertically. Each therefore represent the same (displacement) vector, namely $\langle 2,3\rangle$. If we plot the vector $\langle 2,3\rangle$ in *standard position*, i.e., with its tail at the origin $(0,0)$, we see the head is indeed at $(2,3)$. The length is $\|\,\langle 2,3\rangle\,\| = \sqrt{13}$, and the vector points in the direction $\theta = \tan^{-1}\frac{3}{2}$, since it is in the first quadrant.

12.5 Vectors in $\mathbb{R}^2$; Scalar (Dot) Products

In this section we will look at vectors "in the plane," particularly the familiar xy-plane, which is a two-dimensional space.[5] Later in the text we will examine vectors in space, also sometimes known as xyz-space, which is a three-dimensional space.

The classical definition of a vector is *a quantity with both magnitude and direction.* We will see that there is an intentional ambiguity in this definition.

Another common defintion of a vector is *a directed line segment.* This is even more problematic, as it removes exactly the ambiguity that we will see later is crucial to the usefulness of vectors. Indeed, describing a vector as a directed line segment is a bit like describing an angle as the union of two rays with the same origin. In the former you lose that the crucial elements are the length and direction (not the geometric position of the vector), while with angles we find it useful to think of them as rotations of well-defined amounts (and in particular directions) regardless of the pivot points.

Both definitions are somewhat geometric. As with derivatives and definite integrals, there is a geometric context in which vectors are easily visualized. However, there are other quanti-

[5] A two-dimensional space is one in which we require two variable to describe a point's position.

ties which are "vector quantities," and there is a purely algebraic definition of a vector which accommodates these as well.

To introduce the actual concept of vector, we will use one such example of a vector quantity, which closely mirrors the geometric intuitions. That is the concept of a vector as a *displacement*. As before, a **displacement** can be defined to be a net change in position. Suppose for instance we move from $(0, 0)$ to $(2, 3)$. This would be a change of $+2$ in the horizontal position, and $+3$ in the vertical.

Now suppose instead we move from $(-4, -1)$ to $(-2, 2)$. This would also be a change of $+2$ in the horizontal and $+3$ in the vertical directions, repsectively. Both motions represent the same net displacement, which we signify by the **vector** $\langle 2, 3\rangle$. It does not matter what is our initial point, as long as our final point is right 2 and up 3 from our initial point. In all cases it is represented by the same vector $\langle 2, 3\rangle$. See Figure 12.8 at the beginning of this secction.

While it is important that we realize that each of these displacements—of $+2$ in the horizontal and $+3$ in the vertical—is considered to be *the same net displacement* and therefore *the same vector*, for many purposes it is best to define a **standard position** for vectors, namely that the tail is fixed to the origin $(0, 0)$. Then the head of the vector $\langle 2, 3\rangle$ would lie at $(2, 3)$.[6]

The standard position of a vector allows us to easily explain these concepts of magnitude and direction. The magnitude is a measure of the vector's size, and the direction is, of course, the direction it points. The length of the vector is given by the notation $\|\langle 2, 3\rangle\|$, and the Pythagorean Theorem gives it to us immediately: $\|\langle 2, 3\rangle\| = \sqrt{2^2 + 3^2} = \sqrt{13}$. More generally,

$$\|\langle a, b\rangle\| = \sqrt{a^2 + b^2}. \tag{12.19}$$

The length of the vector is also known as its **magnitude**, modulus, and sometimes called its absolute value.[7] The direction θ in which the vector points is measured off of the positive x-axis, just as is an angle in standard position. In general,

$$\tan\theta = \frac{a}{b}, , \tag{12.20}$$

and does not have to be in any particular range. Some texts will have $0 \le \theta < 360°$, while others will use $-180° < \theta \le 180°$, but all that is required is that we allow the angles available to describe all possible directions in which a vector can point. Note that we usually decline to define a direction for the **zero vector** $\langle 0, 0\rangle$, though it clearly has length $\|\langle 0, 0\rangle\| = \sqrt{0^2 + 0^2} = 0$. Note also that at the moment we are interested in the geometry of these vectors, not the calculus, so using radian measure for θ is not yet necessary. Some texts defined the *argument* of the vector, $\arg\langle a, b\rangle$ to be some particular angle θ, but we will usually just describe (albeit more verbosely) the angle in question.

When looking at vectors in the plane, it is only necessary to specify the two coordinates a and b of the endpoints (a, b) where the head of vector $\langle a, b\rangle$ points when $\langle a, b\rangle$ is in standard position. Because $a, b \in \mathbb{R}$, we often signify the set of all such possible vectors as

$$\mathbb{R}^2 = \left\{ \langle a, b\rangle \,\middle|\, a, b \in \mathbb{R} \right\}.$$

Because it takes exactly two numbers to specify to identify the vector, $\mathbb{R}^2$ is called a *two-dimensional space*.

[6]This is similar to the "standard position" of an angle θ, which allows much of the analysis—especially of the trigonometric kind—of such objects.

[7]When the length of $\langle a, b\rangle$ is called its it absolute value, the notation usually reflects this as well: $|\langle a, b\rangle| = \sqrt{a^2 + b^2}$.

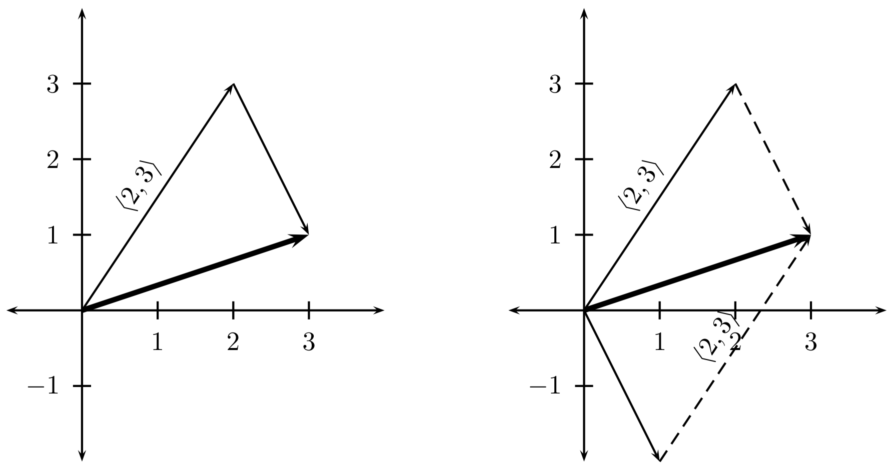


Figure 12.9: Two traditional ways to geometrically describe vector addition. Here we illustrate how these methods predict $\langle 2,3\rangle + \langle 1,-2\rangle = \langle 3,-1\rangle$.

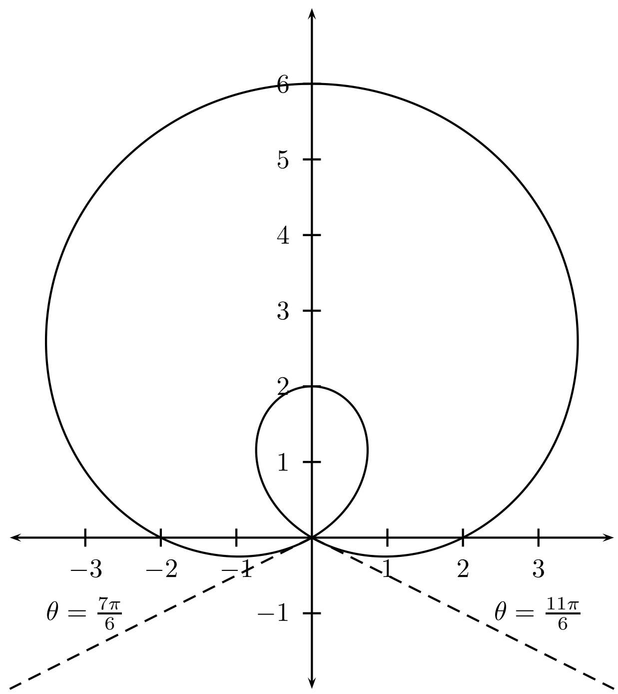
One of the aspects that makes vectors interesting is how they "add" and similarly combine. It is quite intuitive when put simply, and quite interesting when viewed geometrically. When we add two vectors in $\mathbb{R}^2$, we do so as follows:

$$\langle a_1, b_1\rangle + \langle a_2, b_2\rangle = \langle a_1 + a_2, b_1 + b_2\rangle.$$

The new vector is then called the *resultant* vector, representing the net displacement. As we see from the equation above, to find the net displacement—when we first displace by $\langle a_1, b_1\rangle$, followed by another displacement of $\langle a_2, b_2\rangle$— we look at the total horizontal displacement a_1+a_2 and the total vertical displacement $b_1 + b_2$ to form our new vector. Algebraically this is very simple, but at times we will be interested in a geometric perspective. Geometric vector addition is traditionally described in the following two ways:

Tail-to-Head: where the origin (tail) of the second vector is placed at the head of the first;

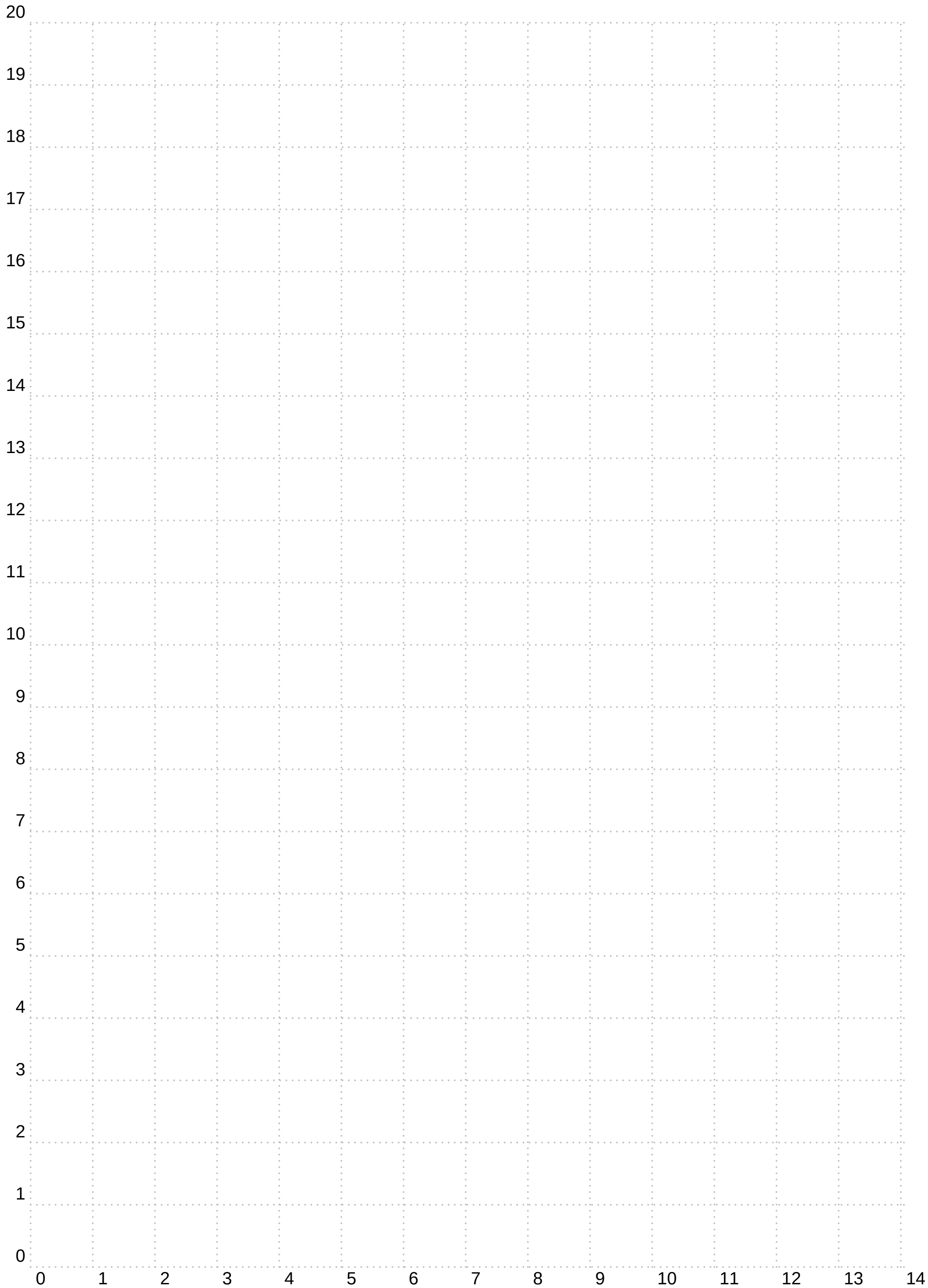
Parallelogram Rule: where we form a parallelogram with the two vectors, in standard position, forming one corner, and the resultant vector coming from that corner to the diagonally opposed corner.


6
5
4
3
2
1
−1
−3
−2
−1
1
2
3
$\theta = \frac{7\pi}{6}$
$\theta = \frac{11\pi}{6}$

12.6 Three-Dimensional Space

12.7 Vectors in $\mathbb{R}^3$; Vector (Cross) Products

12.8 Lines and Planes in $\mathbb{R}^3$


20
19
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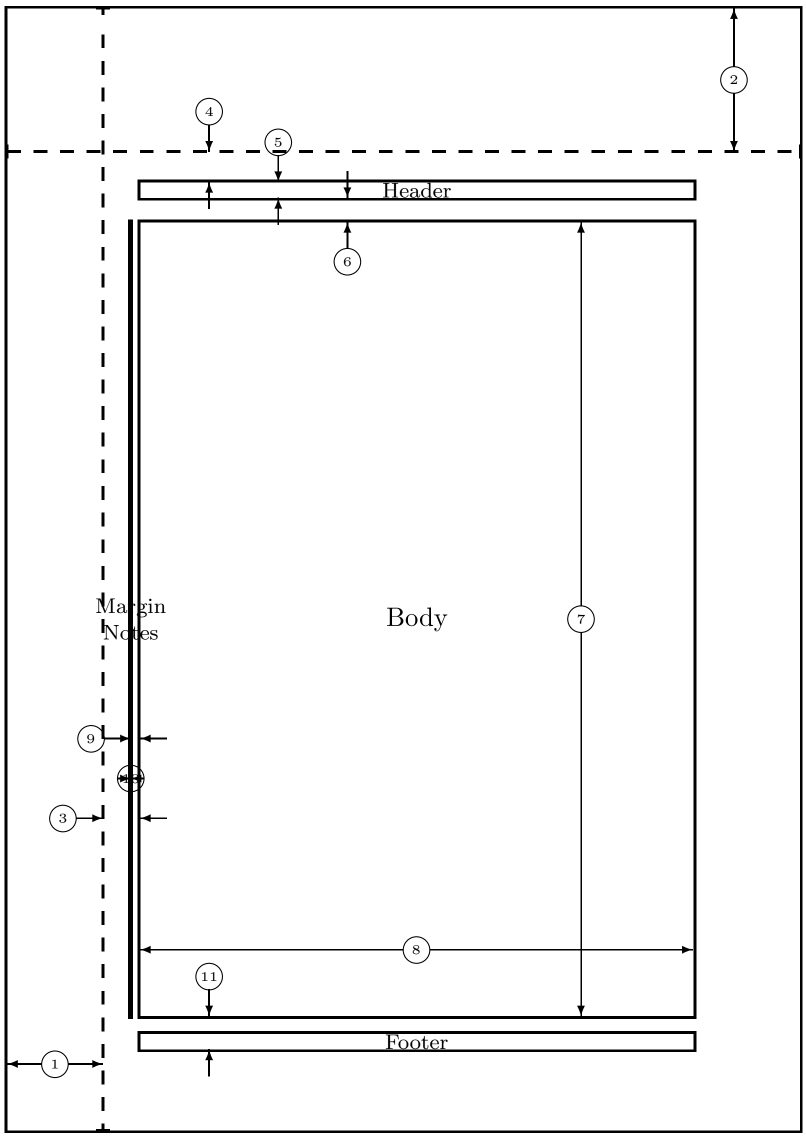
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14



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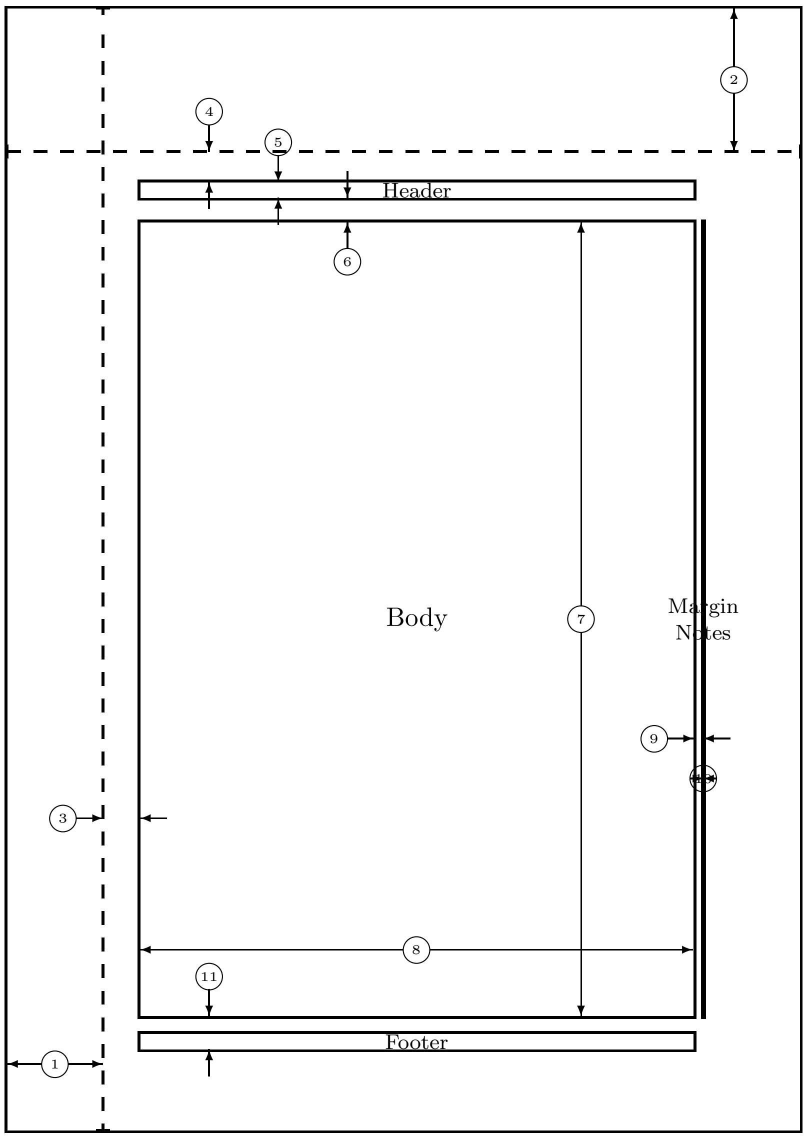

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