The Impulse Response and Convolution (Part 2)

Scope and Background Reading

This session continues our introduction to time convolution.

As we shall see, in the determination of a system's response to a signal input, time convolution involves integration by parts and is a tricky operation. But time convolution becomes multiplication in the Laplace Transform domain, and is much easier to apply.

The material in this presentation and notes is based on Chapter 6 of Steven T. Karris, Signals and Systems: with Matlab Computation and Simulink Modelling, 5th Edition. (http://site.ebrary.com/lib/swansea/docDetail.action?docID=10547416) and builds on the time response of a state-space model that was developed in the previous session (http://nbviewer.ipython.org/github/cpjobling/EG-247-Resources/blob/master/week4/state_space.ipynb).

Agenda

The material to be presented will need two sessions.

Last Session

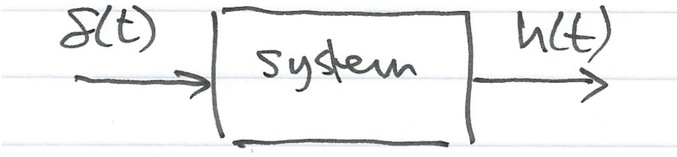
- The Impulse Response of a System in Time Domain
- Even and Odd Functions of Time

This Session

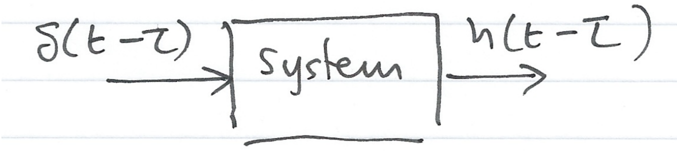
- Time Convolution
- Graphical Evaluation of the Convolution Integral
- System Response by Convolution
- System Response by Laplace

Time Convolution

Consider a system whose input is the Dirac delta ($\delta(t)$), and its output is the impulse response $h(t)$. We can represent the inpt-output relationship as a block diagram

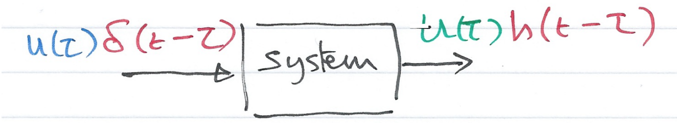


In general



Add an arbitrary input

Let $u(t)$ be any input whose value at $t = \tau$ is $u(\tau)$, Then because of the sampling property of the delta function



(output is $u(\tau)h(t - \tau)$)

Integrate both sides

Integrating both sides over all values of τ ($-\infty < \tau < \infty$) and making use of the fact that the delta function is even, i.e.

$$\delta(t-\tau) = \delta(\tau - t)$$

we have:

Since $\delta(t-\tau) = \delta(\tau-t)$ because $\delta(t)$ is even.

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\,d\tau \\ &\int_{-\infty}^{\infty} u(\tau)\delta(\tau-t)\,d\tau \end{aligned}\right\} \longrightarrow \boxed{\text{System}} \longrightarrow \left\{\begin{aligned} &\int_{-\infty}^{\infty} u(\tau)h(t-\tau)\,d\tau \\ &\int_{-\infty}^{\infty} u(t-\tau)h(\tau)\,d\tau \end{aligned}\right.$$

Use the sifting property of delta

The second integral on the left side reduces to $u(t)$

$$u(t) \longrightarrow \boxed{\text{System}} \longrightarrow \left\{\begin{aligned} &\int_{-\infty}^{\infty} u(\tau)h(t-\tau)\,d\tau \\ &\int_{-\infty}^{\infty} u(t-\tau)h(\tau)\,d\tau \end{aligned}\right.$$

The Convolution Integral

The integral

$$\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

or

$$\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

is known as the *convolution integral*; it states that if we know the impulse response of a system, we can compute its time response to any input by using either of the integrals.

The convolution integral is usually written $u(t) * h(t)$ or $h(t) * u(t)$ where the asterisk ($*$) denotes convolution.

Convolution and State-Space Models

In the previous session, we found that the impulse response of a SISO system (with $d = 0$) was

$$h(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}$$

Therefore, if we know $h(t)$, we can use the convolution integral to compute the response $y(t)$ to any input $u(t)$ using the relation

$$h(t) = \int_{-\infty}^{\infty} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$
$$h(t) = \mathbf{C}e^{\mathbf{A}t}\int_{-\infty}^{\infty} e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d\tau$$

Graphical Evaluation of the Convolution Integral

The convolution integral is most conveniently evaluated by a graphical evaluation. The text book gives three examples (6.4-6.6) which we will demonstrate using a graphical visualization tool (http://www.mathworks.co.uk/matlabcentral/fileexchange/25199-graphical-demonstration-of-convolution) developed by Teja Muppirala of the Mathworks.

The tool: convolutiondemo.m (matlab/convolutiondemo.m) (see license.txt (matlab/license.txt)).

Convolution by Graphical Method - Summary of Steps

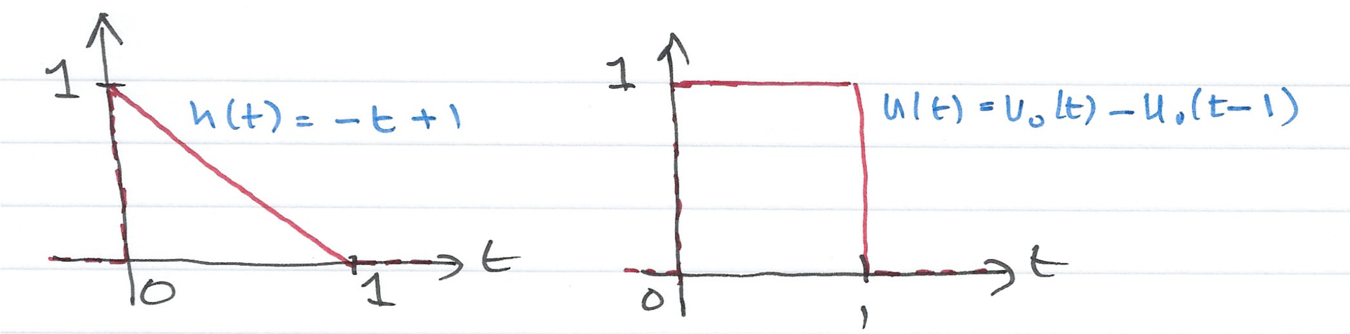
For simplicity, we give the rules for $u(t)$, but the procedure is the same if we reflect and slide $h(t)$

1. Substitute $u(t)$ with $u(\tau)$ – this is a simple change of variable. It doesn't change the definition of $u(t)$.
2. Reflect $u(\tau)$ about the vertical axis to form $u(-\tau)$
3. Slide $u(-\tau)$ to the right a distance t to obtain $u(t-\tau)$
4. Multiply the two signals to obtain the product $u(t-\tau)h(\tau)$
5. Integrate the product over all t from $-\infty$ to ∞.

Example 1

(This is example 6.4 in the textbook)

The signals $h(t)$ and $u(t)$ are shown below. Compute $h(t) * u(t)$ using the graphical technique.

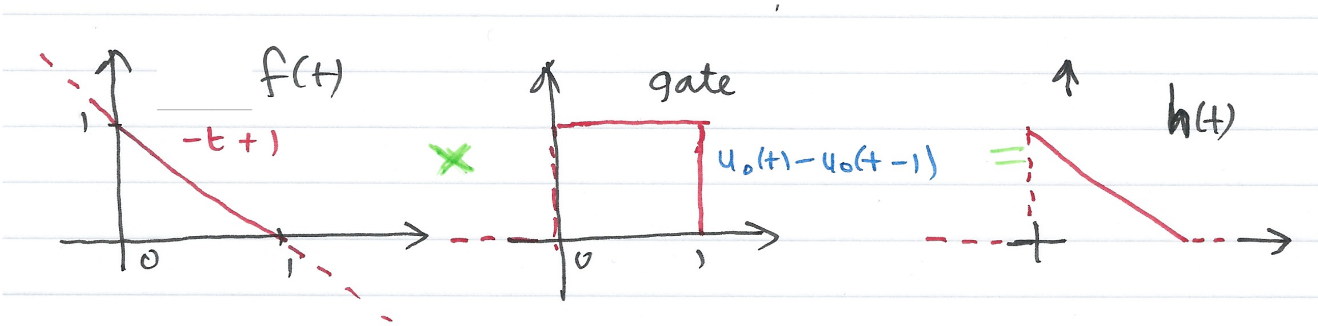


Prepare for convolutiondemo

To prepare this problem for evaluation in the `convolutiondemo` tool, we need to determine the Laplace Transforms of $h(t)$ and $u(t)$.

h(t)

The signal $h(t)$ is the straight line $f(t) = -t + 1$ but this is defined only between $t = 0$ and $t = 1$. We thus need to gate the function by multiplying it by $u_0(t) - u_0(t-1)$ as illustrated below:



Thus

$h(t) = (-t+1)(u_0(t) - u_0(t-1)) = (-t+1)u_0(t) - (-(t-1)u_0(t-1)) = -tu_0(t) + u_0(t) + (t -$

$$-\frac{1}{s^2} + \frac{1}{s} + \frac{e^{-s}}{s^2}$$

$$H(s) = \frac{s^2 + e^{-s} - 1}{s^2}$$

u(t)

The input $u(t)$ is the gating function:

$$u(t) = u_0(t) - u_0(t-1)$$

so

$$U(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1 - e^{-s}}{s}$$

convolutiondemo settings

- Let `g = (1 - exp(-s))/s`
- Let `h = (s + exp(-s) - 1)/s^2`
- Set range $-2 < \tau < -2$

Summary of result

1. For $t < 0$: $u(t-\tau)h(\tau) = 0$
2. For $t = 0$: $u(t-\tau) = u(-\tau)$ and $u(-\tau)h(\tau) = 0$
3. For $0 < t \leq 1$: $h * u = \int_0^t (1)(-\tau+1)d\tau = \tau - \tau^2/2\Big|_0^t = t - t^2/2$
4. For $1 < t \leq 2$: $h * u = \int_{t-1}^1 (-\tau+1)d\tau = \tau - \tau^2/2\Big|_{t-1}^1 = t^2/2 - 2t + 2$
5. For $2 \leq t$: $u(t-\tau)h(\tau) = 0$

Example 2

This is example 6.5 from the text book.

$$h(t) = e^{-t}$$
$$u(t) = u_0(t) - u_0(t-1)$$

Answer 2

$$y(t) = \begin{cases} 0 : t \leq 0 \\ 1 - e^{-t} : \ 0 < t \leq 1 \\ e^{-t}(e-1) : \ 1 < t \leq 2 \\ 0 : \ 2 \leq t \end{cases}$$

Example 3

This is example 6.6 from the text book.

$$h(t) = 2(u_0(t) - u_0(t-1))$$
$$u(t) = u_0(t) - u_0(t-2)$$

Answer 3

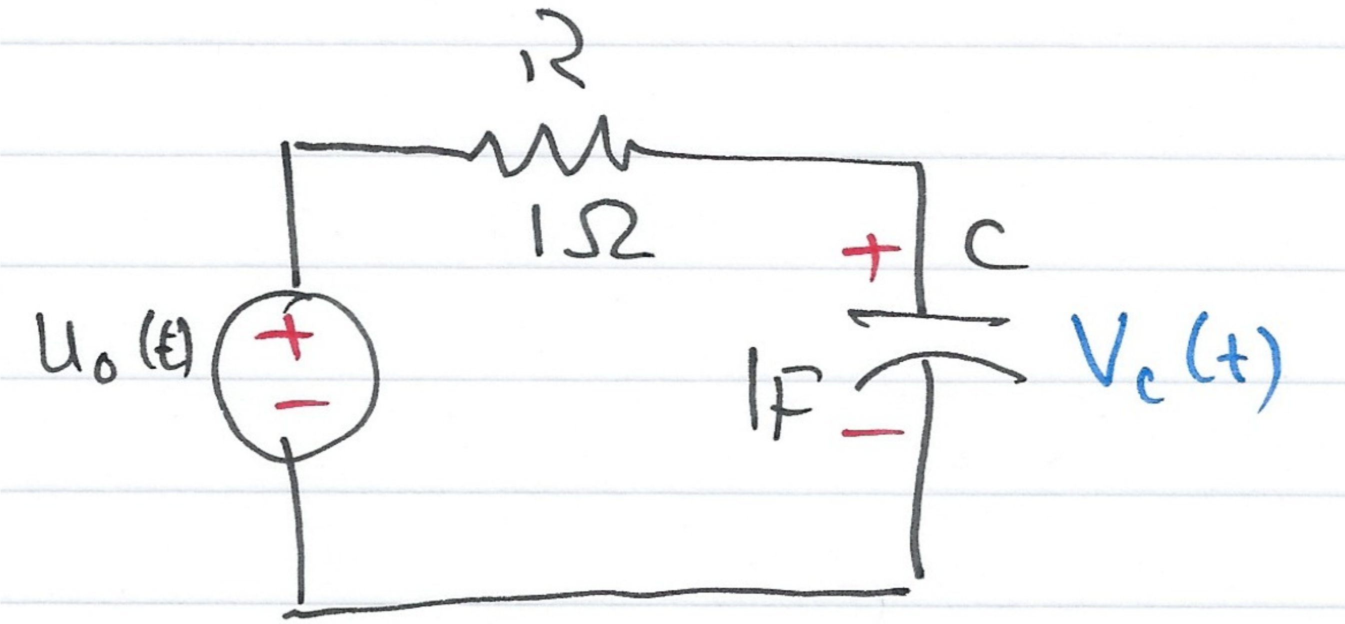
$$y(t) = \begin{cases} 0 : t \leq 0 \\ 2t : \; 0 < t \leq 1 \\ 2 : \; 1 < t \leq 2 \\ -2t + 6 : \; 2 < t \leq 3 \\ 0 : \; 3 \leq t \end{cases}$$

System Response by Convolution

Example 4

This is example 6.7 from the textbook.

For the circuit shown below, use the convolution integral to find the capacitor voltage when the input is the unit step function $u_0(t)$ and $v_c(0^-) = 0$



Solution 4

$$h(t) = \frac{1}{RC} e^{-t/RC} u_0(t)$$

which when $C = 1\ \mathrm{F}$ and $R = 1\ \Omega$ reduces to

$$h(t) = e^{-t} u_0(t)$$

It is relatively straight forward to show that

$$y(t) = \left(1 - e^{-t}\right) u_0(t)$$

System Response by Laplace

In the discussion of Laplace, we stated that

$$\mathcal{L}\left\{f(t) * g(t)\right\} = F(s)G(s)$$

We can use this property to make the solution of convolution problems even simpler.

Example 5

Solve Example 4 using Laplace.

Solution 5

$$h(t) = e^{-t}u_0(t) \Leftrightarrow H(s) = \frac{1}{s+1}$$

$$u(t) = u_0(t) \Leftrightarrow U(s) = \frac{1}{s}$$

$$y(t) = h(t) * u(t) \Leftrightarrow Y(s) = H(s)U(s) = \left(\frac{1}{s}\right) \times \left(\frac{1}{s+1}\right)$$

By PFE

$$Y(s) = \frac{r_1}{s} + \frac{r_2}{s+1}$$

The residues are $r_1 = 1$, $r_2 = -1$, so

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} \Leftrightarrow y(t) = \left(1 - e^{-t}\right) u_0(t)$$

Impulse Response and Transfer Functions

A consequence of Laplace is that the transform of the impulse response of a transfer function $G(s)$ is given by the transfer function itself.

$$y(t) = g(t) * \delta(t) \Leftrightarrow Y(s) = G(s).1 = G(s)$$

Thus the Laplace transform of any system subject to an input $u(t)$ is simply

$$Y(s) = G(s)U(s)$$

and

$$y(t) = \mathcal{L}^{-1}\left\{G(s)U(s)\right\}$$

Using tables, solution of a convolution problem by Laplace is usually simpler than using convolution directly. And if the system is particularly complex we can always fall back on the State-Space solution:

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d\tau$$