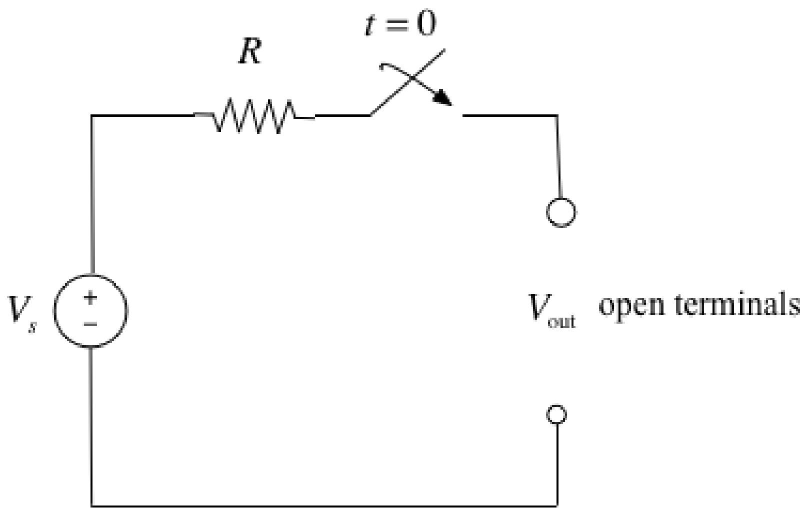
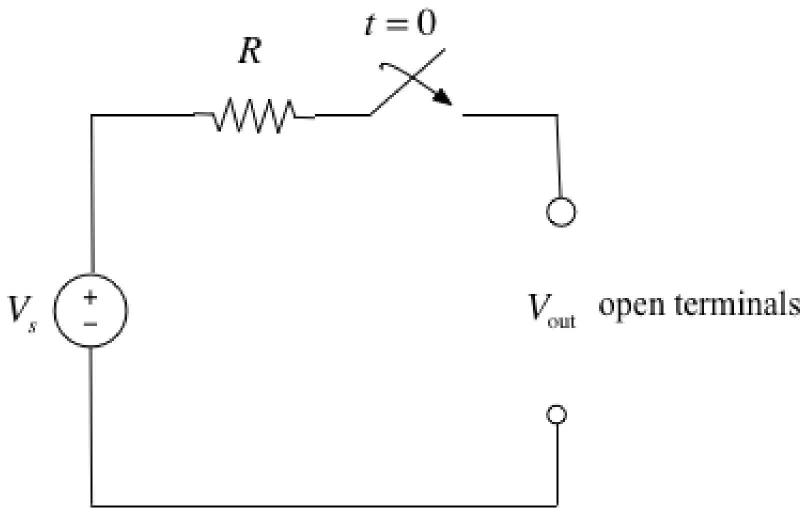
Elementary Signals

Consider this circuit:



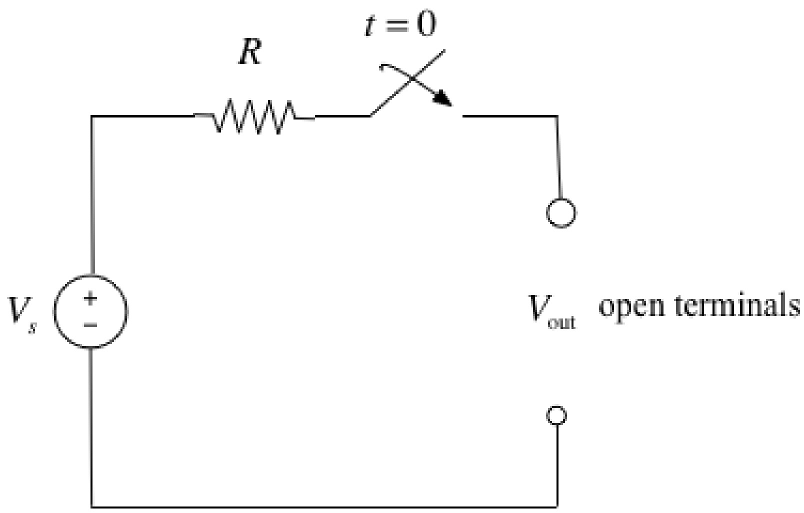
What happens **before** $t = 0$?

1. $v_{\text{out}} = \text{undefined}$
2. $v_{\text{out}} = 0$
3. $v_{\text{out}} = V_s$
4. $v_{\text{out}} = 1/2$
5. $v_{\text{out}} = \infty$



What happens **after** $t = 0$?

1. $v_{\text{out}} = \text{undefined}$
2. $v_{\text{out}} = 0$
3. $v_{\text{out}} = V_s$
4. $v_{\text{out}} = 1/2$
5. $v_{\text{out}} = \infty$



What happens **at** $t = 0$?

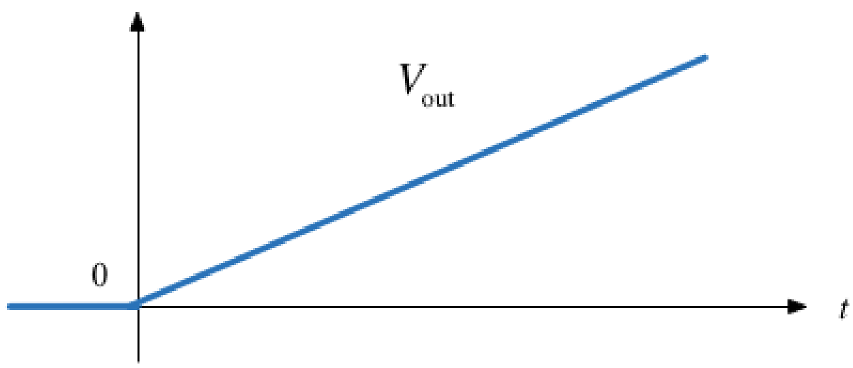
1. $v_{\text{out}} = \text{undefined}$
2. $v_{\text{out}} = 0$
3. $v_{\text{out}} = V_s$
4. $v_{\text{out}} = 1/2$
5. $v_{\text{out}} = \infty$

What does the response of V_{out} look like?

1:

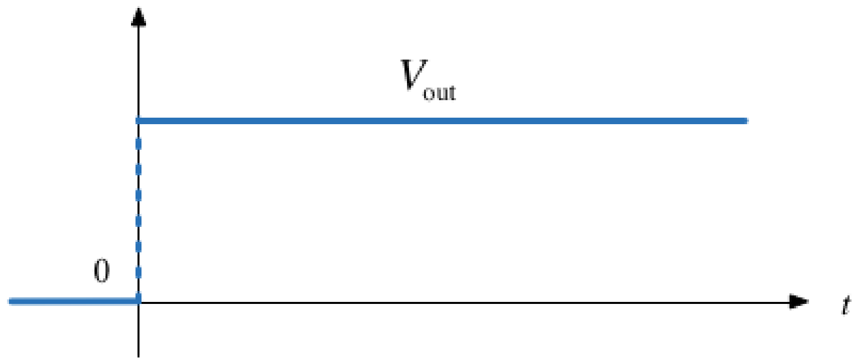
What does the response of V_{out} look like

2:



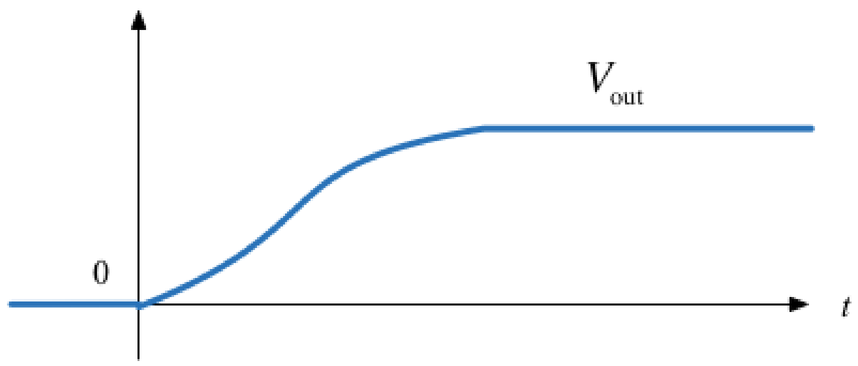
What does the response of V_{out} look like

3:



What does the response of V_{out} look like?

4:



Answers

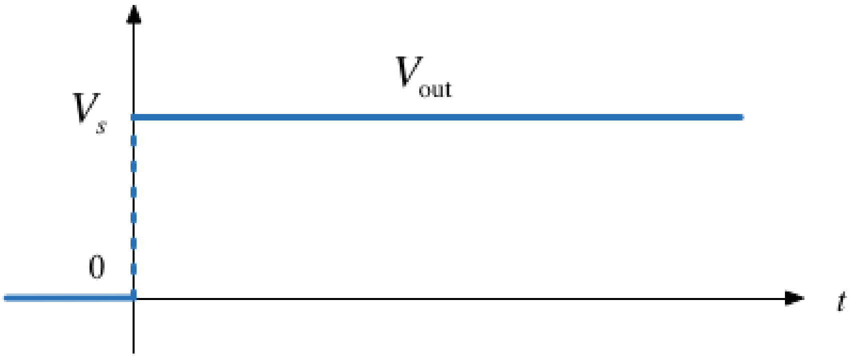
Mathematically

1. $v_{\text{out}} = 0$ when $-\infty < t < 0$ (answer 2)
2. $v_{\text{out}} = V_s$ when $0 < t < \infty$ (answer 3)
3. $v_{\text{out}} = \text{undefined}$ when $t = 0$ (answer 1)

V_{out} jumps from 0 to V_s instantanously when the switch is closed. We call this a discontinuous signal!

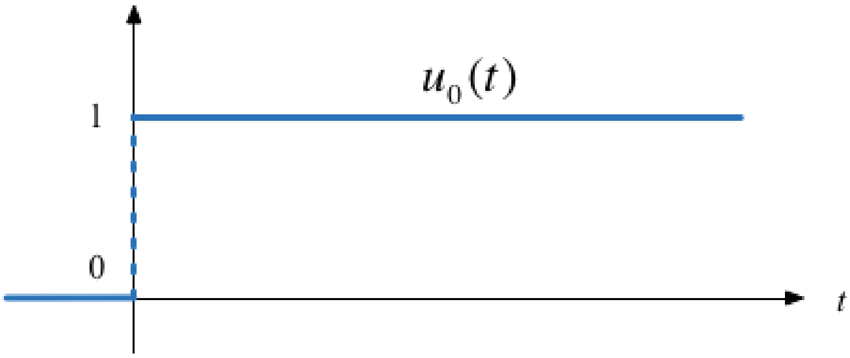
The correct image is:

3:



Unit Step Function

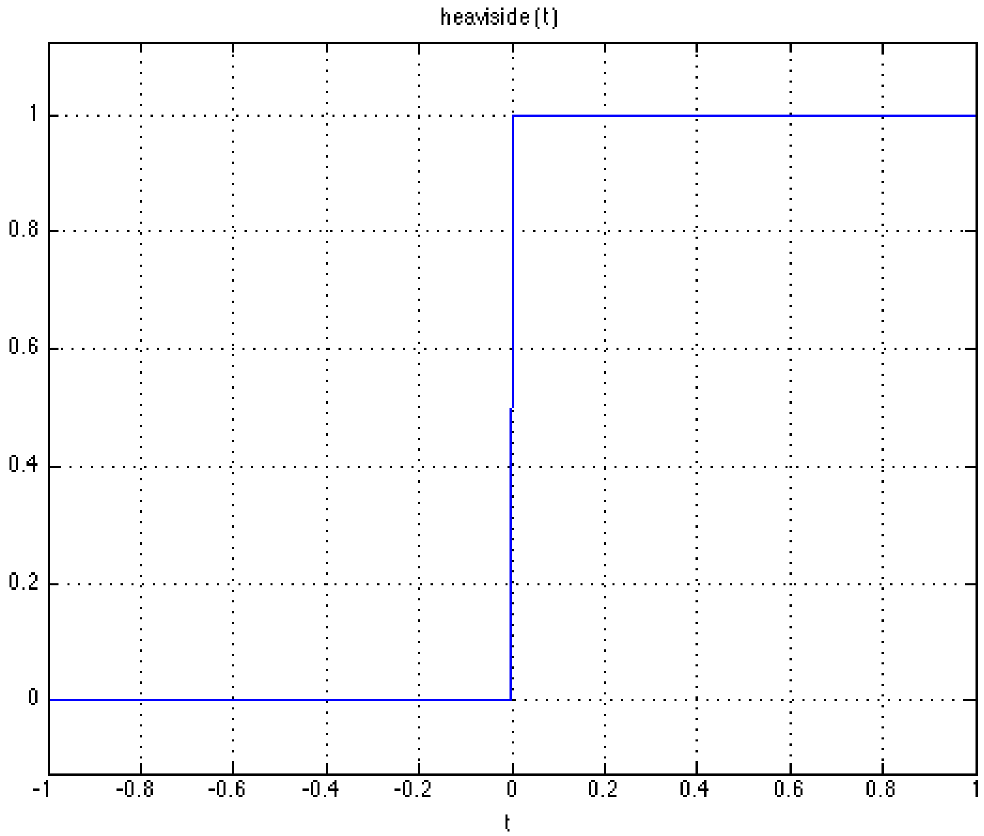
$$u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



In Matlab

In Matlab, we use the `heaviside` function (Named after Oliver Heaviside (http://en.wikipedia.org/wiki/Oliver_Heaviside)).

```
syms t
ezplot(heaviside(t),[-1,1])
```



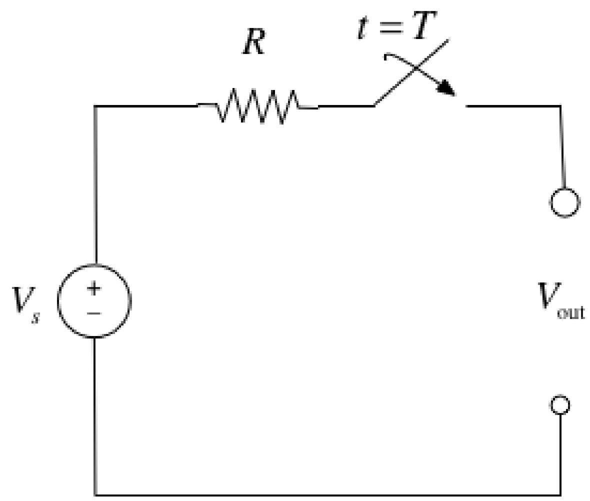
See: heaviside_function.m (matlab/heaviside_function.m)

Note that, so it can be plotted, Matlab defines the *heaviside function* slightly differently from the mathematically ideal unit step:

$$\mathrm{heaviside}(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$

Circuit Revisited

Consider the network shown below, where the switch is closed at time $t = T$.



Express the output voltage v_{out} as a function of the unit step function, and sketch the appropriate waveform.

Simple Signal Operations

Amplitude Scaling

Sketch $Au_0(t)$ and $-Au_0(t)$

Time Reversal

Sketch $u_0(-t)$

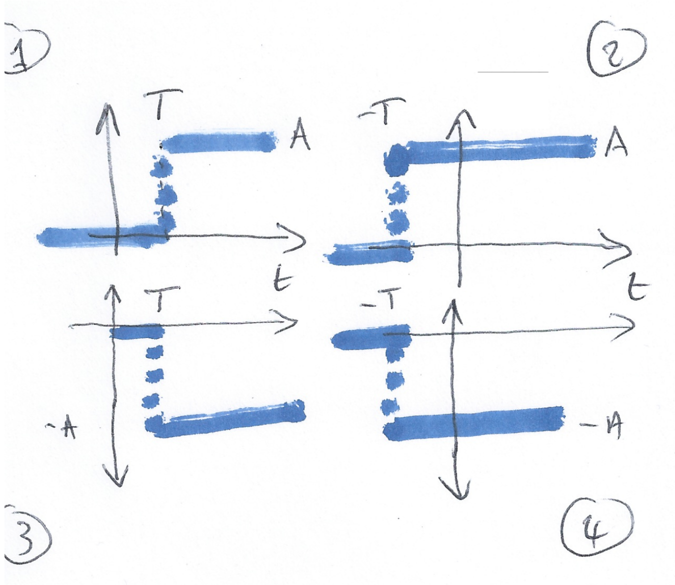
Time Delay and Advance

Sketch $u_0(t-T)$ and $u_0(t+T)$

Examples

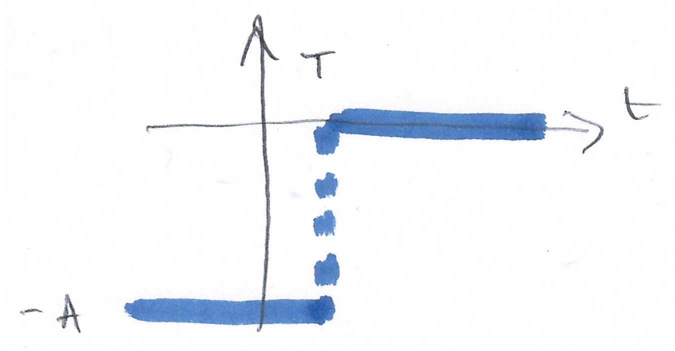
Example 1

Which of these signals represents $-Au_0(t+T)$?



Example 2

What is represented by

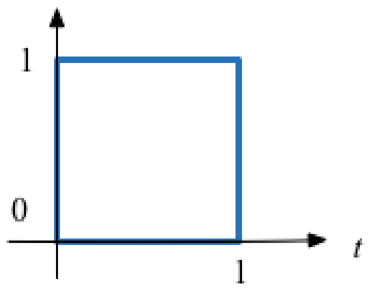


1. $-Au_0(t-T)$
2. $-Au_0(-t+T)$
3. $-Au_0(-t-T)$
4. $-Au_0(t-T)$

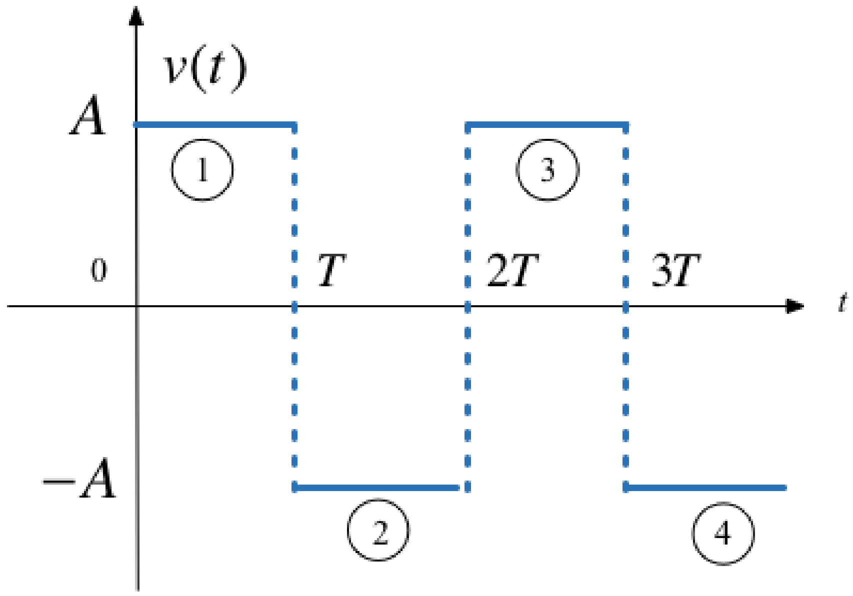
Synthesis of Signals from Unit Step

Unit step functions can be used to represent other time-varying functions such as rectangular pulses, square waves and triangular pulses.

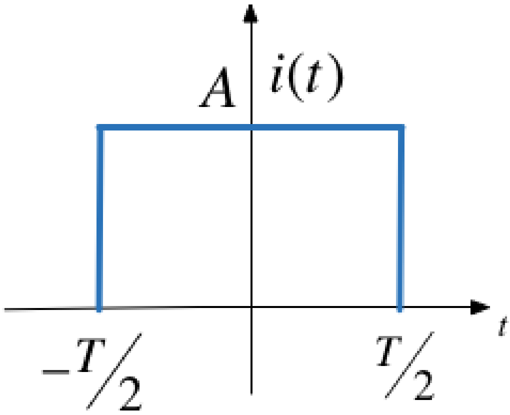
Synthesize Rectangular Pulse



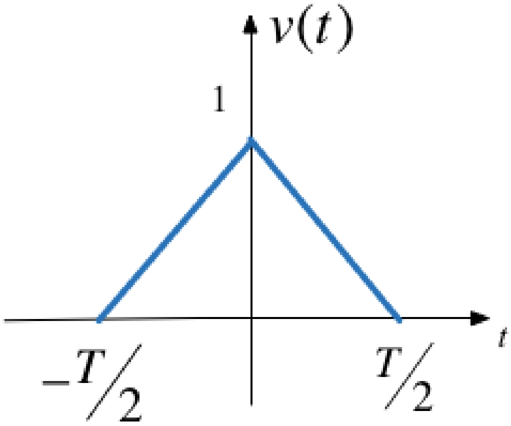
Synthesize Square Wave



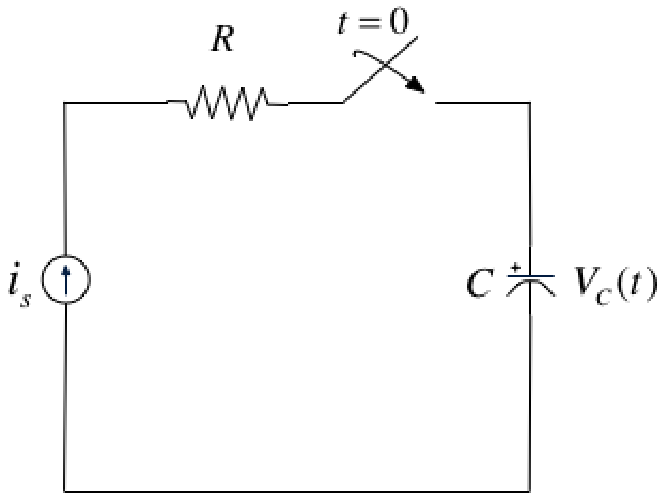
Synthesize Symmetric Rectangular Pulse



Synthesize Symmetric Triangular Pulse



The Ramp Function



In the circuit shown above i_s is a constant current source and the switch is closed at time $t = 0$. Show that the voltage across the capacitor can be represented as

$$v_C(t) = \frac{i_s}{C} t u_0(t)$$

and sketch the wave form.

The unit ramp function is defined as

$$u_1(t) = \int_{-\infty}^{t} u_0(\tau) d\tau$$

so

$$u_1(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

and

$$u_0(t) = \frac{d}{dt} u_1(t)$$

Note

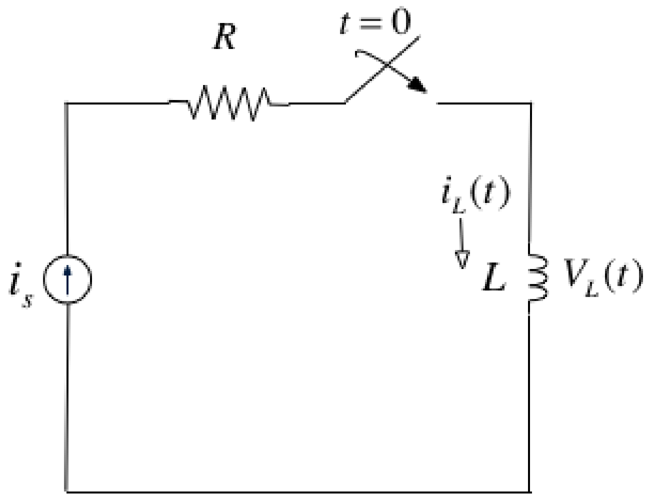
Higher order functions of t can be generated by the repeated integration of the unit step function.

For future reference, you should determine $u_2(t)$, $u_3(t)$ and $u_n(t)$ for yourself and make a note of the general rule:

$$u_{n-1} = \frac{1}{n} \frac{d}{dt} u_n(t)$$

Details are given in equations 1.26—1.29 in the textbook.

The Dirac Delta Function



In the circuit shown above, the switch is closed at time $t = 0$ and $i_L(t) = 0$ for $t < 0$. Express the inductor current $i_L(t)$ in terms of the unit step function and hence derive an expression for $v_L(t)$.

Notes

To solve this problem we need to invent a function that represents the derivative of the unit step function. This function is called $\delta(t)$ or the *dirac delta* function (named after Paul Dirac (http://en.wikipedia.org/wiki/Paul_Dirac)).

The delta function

The unit impulse or the delta function, denoted as $\delta(t)$, is the derivative of the unit step.

This function is tricky because $u_0(t)$ is discontinuous at $t = 0$ but it must have the properties

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u_0(t)$$

and

$\delta(t) = 0$ for all $t \neq 0$.

Sketch of the delta function

Important properties of the delta function

Sampling Property

The *sampling property* of the delta function states that

$$f(t)\delta(t-a) = f(a)\delta(t-a)$$

or, when $a = 0$,

$$f(t)\delta(t) = f(0)\delta(t)$$

Multiplication of any function $f(t)$ by the delta function $\delta(t)$ results in sampling the function at the time instants for which the delta function is not zero.

The study of descrete-time (sampled) systems is based on this property.

You should work through the proof for youself.

Sifting Property

The *sifting property* of the delta function states that

$$\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)$$

That is, if multiply any function $f(t)$ by $\delta(t-\alpha)$, and integrate from $-\infty$ to $+\infty$, we will get the value of $f(t)$ evaluated at $t=\alpha$.

You should also work through the proof for yourself.

Higher Order Delta Fuctions

the n*th*-order *delta function* is defined as the n*th* derivative of $u_0(t)$, that is

$$\delta^n(t) = \frac{d^n}{dt^n}[u_0(t)]$$

The function $\delta'(t)$ is called the *doublet*, $\delta''(t)$ is called the *triplet* and so on.

By a procedure similar to the derivation of the sampling property we can show that

$$f(t)\delta'(t-a) = f(a)\delta'(t-a) - f'(t)\delta(t-a)$$

Also, derivation of the sifting property can be extended to show that

$$\int_{-\infty}^{infty} f(t)\delta^n(t-\alpha)dt = (-1)^n \frac{d^n}{dt^n}[f(t)]\bigg|_{t=\alpha}$$

Examples

Example 3

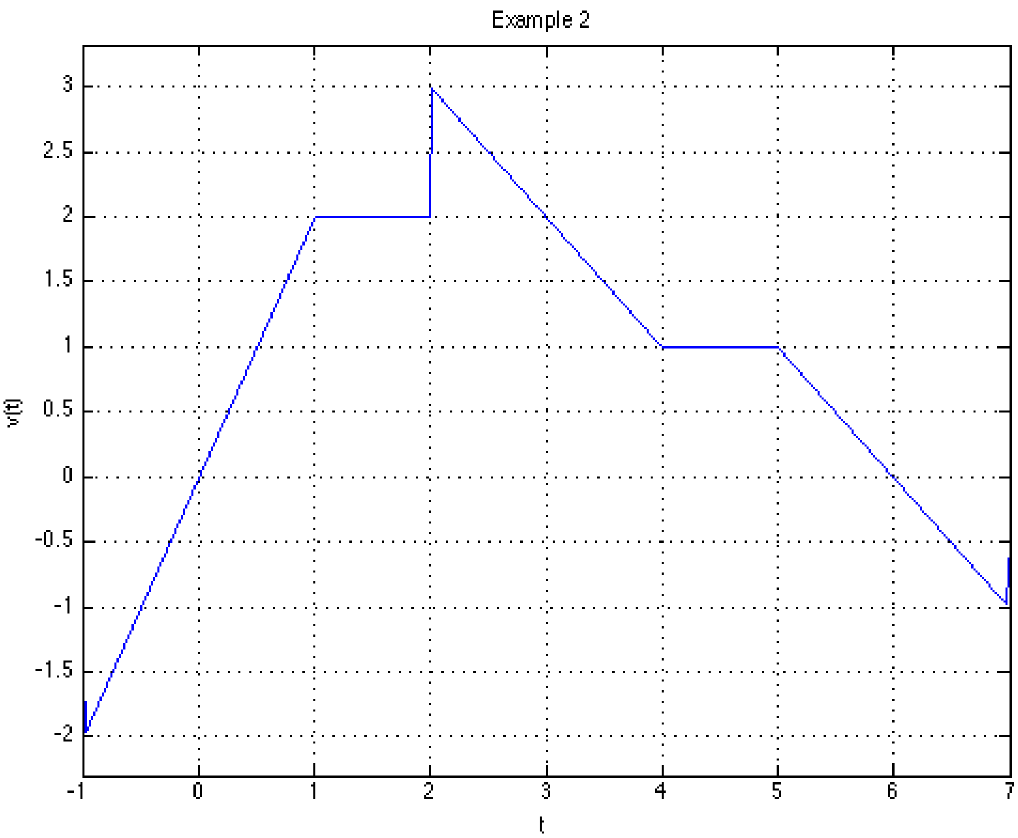
Evaluate the following expressions

1. $$3t^4\delta(t-1)$$

2. $$\int_{-\infty}^{\infty} t\delta(t-2)dt$$

3. $$t^2\delta'(t-3)$$

Example 4



1. Express the voltage waveform $v(t)$ shown above aa sum of unit step functions for the time interval $-1 < t < 7$ s

2. Using the result of part (1), compute the derivative of $v(t)$ and sketch its waveform.

Lab Work

In the lab, a week on Thursday, we will solve Example 2 using Matlab/Simulink following the procedure given in Pages between pages 1-17 and 1-22 of the textbook. We will also explore the `heaviside` and `dirac` functions.