Introduction to Filters

Scope and Background Reading

This session is Based on the section **Filtering** from Chapter 5 of f Benoit Boulet, Fundamentals of Signals and Systems (http://site.ebrary.com/lib/swansea/docDetail.action?docID=10228195) from the **Recommended Reading List**.

This material is an introduction to analogue filters. You will find much more in-depth coverage on Pages 11-1—1-48 of Karris.

Agenda

- Frequency Selective Filters
- Ideal low-pass filter
- Butterworth low-pass filter
- High-pass filter
- Bandpass filter

Introduction

- Filter design is an important application of the Fourier transform
- Filtering is a rich topic often taught in graduate courses so we give only an introduction.
- Our introduction *will* illustrate the usefulness of the frequency domain viewpoint.
- We will explore how filters can shape the spectrum of a signal.

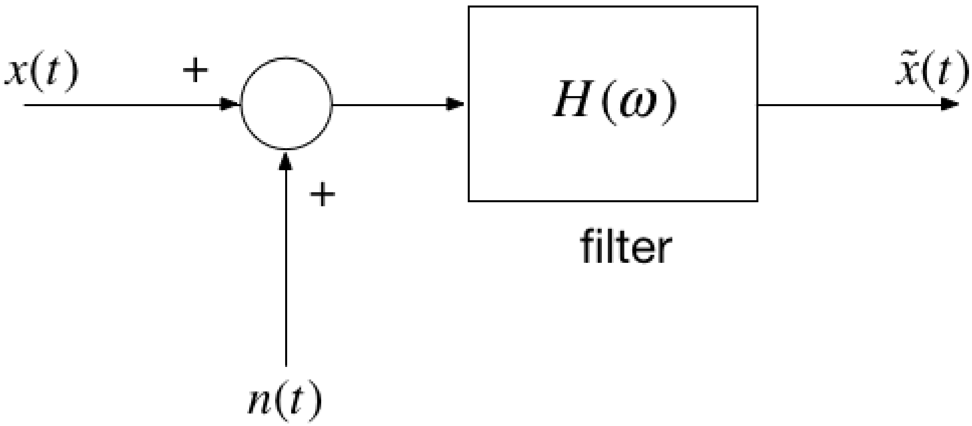
Other applications of the Fourier transform are sampling theory (introduced next week) and modulation.

Frequency Selective Filters

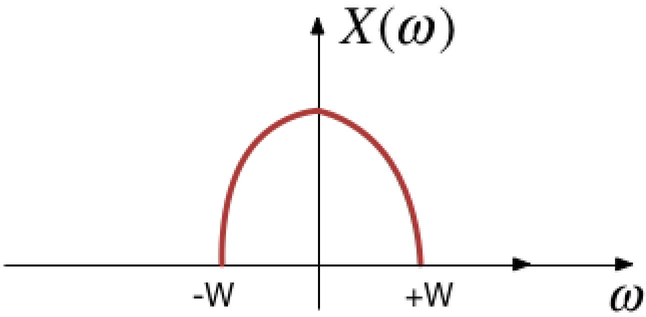
An ideal frequency-selective filter is a system that let's the frequency components of a signal through undistorted while components at other components are completely cut off.

- The range of frequencies which are let through belong to the **pass Band**
- The range of frequencies which are cut-off by the filter are called the **stopband**
- A typical scenario where filtering is needed is when noise $n(t)$ is added to a signal $x(t)$ but that signal has most of its energy outside the bandwidth of a signal.

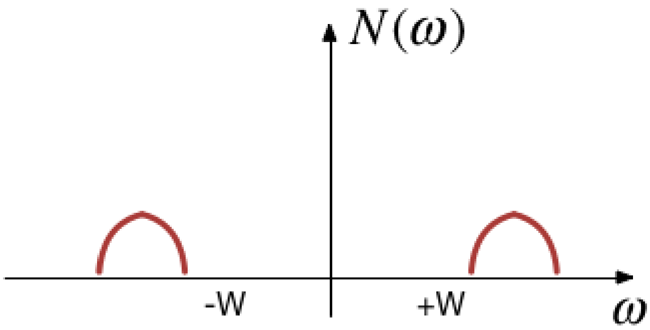
Typical filtering problem



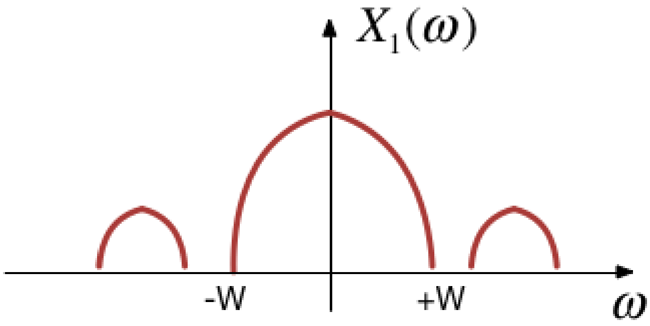
Signal



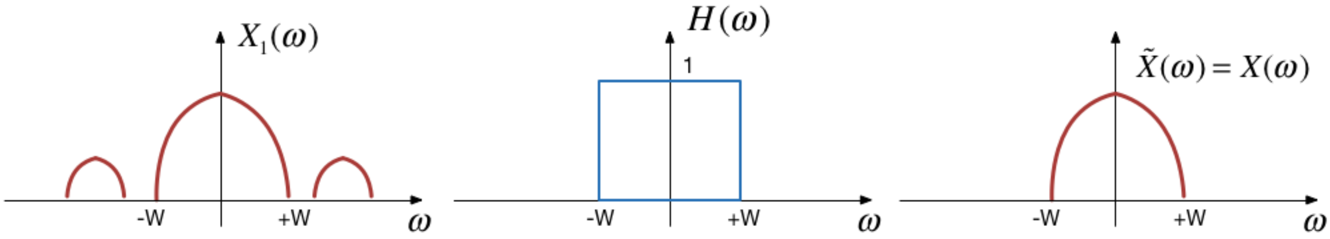
Out-of Bandwidth Noise



Signal plus Noise



Filtering



Motivating example

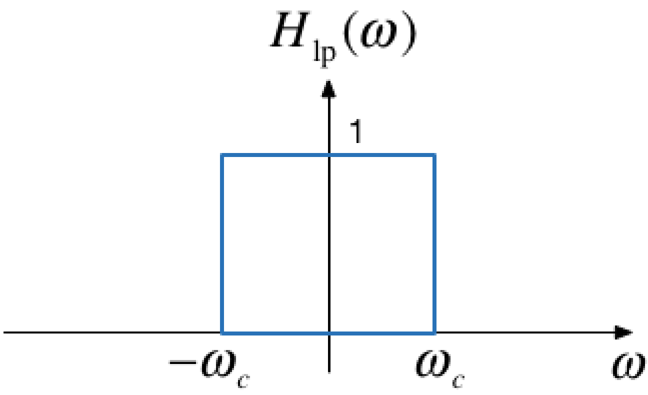
See the notes in the OneNote Class Room notebook (https://swanseauniversity-my.sharepoint.com/personal/c_p_jobling_swansea_ac_uk/_layouts/15/WopiFrame.aspx?sourcedoc={26f94375-62db-439d-bbcb-5fe8cf2fa0fb}&action=edit&wd=target%28_Content%20Library%2FLessons%2FLesson%2016%2Eone%7C7 0238-6A44-8FDF-0184D3855DB0%2FBefore%20Class%7CE5AD343A-E348-0141-8096-60E0CA201E57%2F%29onenote%3Ahttps%3A%2F%2Fswanseauniversity-my%2Esharepoint%2Ecom%2Fpersonal%2Fc_p_jobling_swansea_ac_uk%2FDocuments%2FClass%20Note 247%20Signals%20and%20Systems%20%282015-2016) or on Blackboard.

Ideal Low-Pass Filter

An ideal low pass filter cuts-off frequencies higher than its *cutoff frequency*, ω_c.

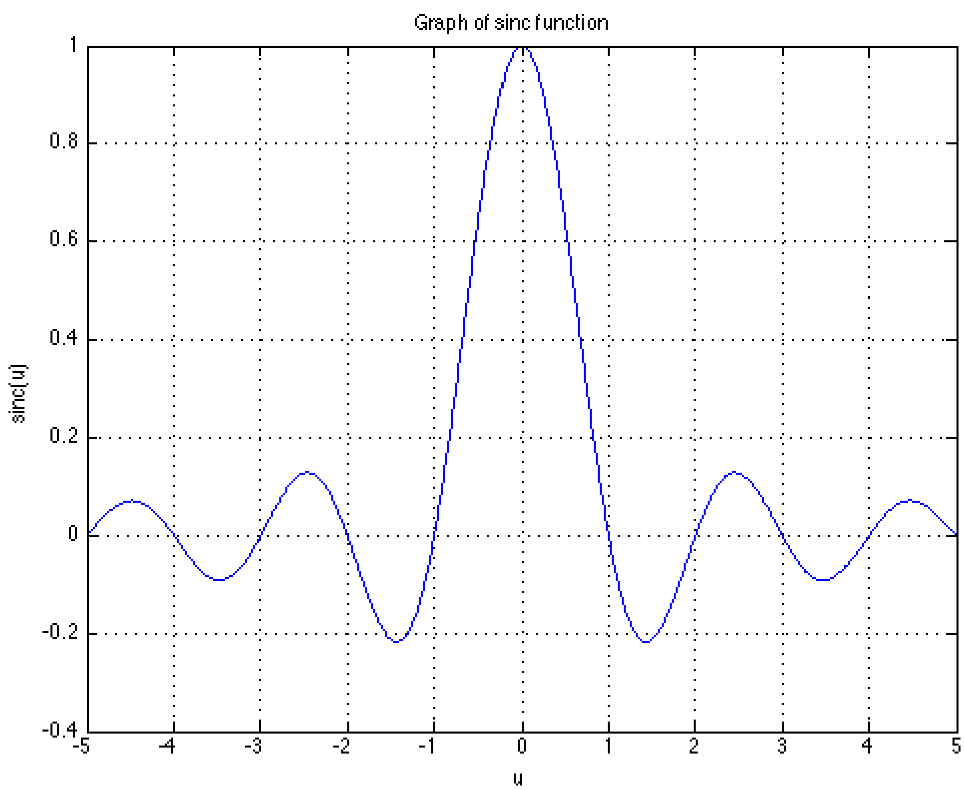
$$H_{\mathrm{lp}}(\omega) = \begin{cases} 1, & |\omega| \; < \omega_c \\ 0, & |\omega| \; \geq \omega_c \end{cases}$$

Frequency response



Impulse response

$$h_{lp}(t) = \frac{\omega_c}{\pi}\mathrm{sinc}\left(\frac{\omega_c}{\pi}t\right)$$



Filtering is Convolution

The output of an LTI system with impulse response

$$h(t) \Leftrightarrow H(\omega)$$

subject to an input signal

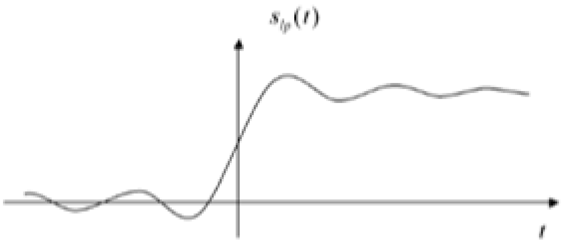
$$x(t) \Leftrightarrow X(\omega)$$

is given by

$$y(t) = h(t) * x(t) \Leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

Issues with the "ideal" filter

This is the step response:



(reproduced from Boulet Fig. 5.23 p. 205)

Ripples in the impulse resonse would be undesireable, and because the impulse response is non-causal it cannot actually be implemented.

Butterworth low-pass filter

N-th Order Butterworth Filter

$$|H_B(\omega)| = \frac{1}{\left(1 + \left(\frac{\omega}{\omega_c}\right)^{2N}\right)^{\frac{1}{2}}}$$

Remarks

- DC gain is $|H_B(j0)| = 1$
- Attenuation at the cut-off frequency is $|H_B(j\omega_c)| = 1/\sqrt{2}$ for any N

More about the Butterworth filter: Wikipedia Article (http://en.wikipedia.org/wiki/Butterworth_filter)

Example 5: Second-order BW Filter

The second-order butterworth Filter is defined by is Charecteristic Equation (CE):

$$p(s) = s^2 + \omega_c\sqrt{2}s + \omega_c^2 = 0^*$$

Calculate the roots of $p(s)$ (the poles of the filter transfer function) in both Cartesian and polar form.

Note: This has the same characteristic as a control system with damping ratio $\zeta = 1/\sqrt{2}$ and $\omega_n = \omega_c$!

Solution

Example 6

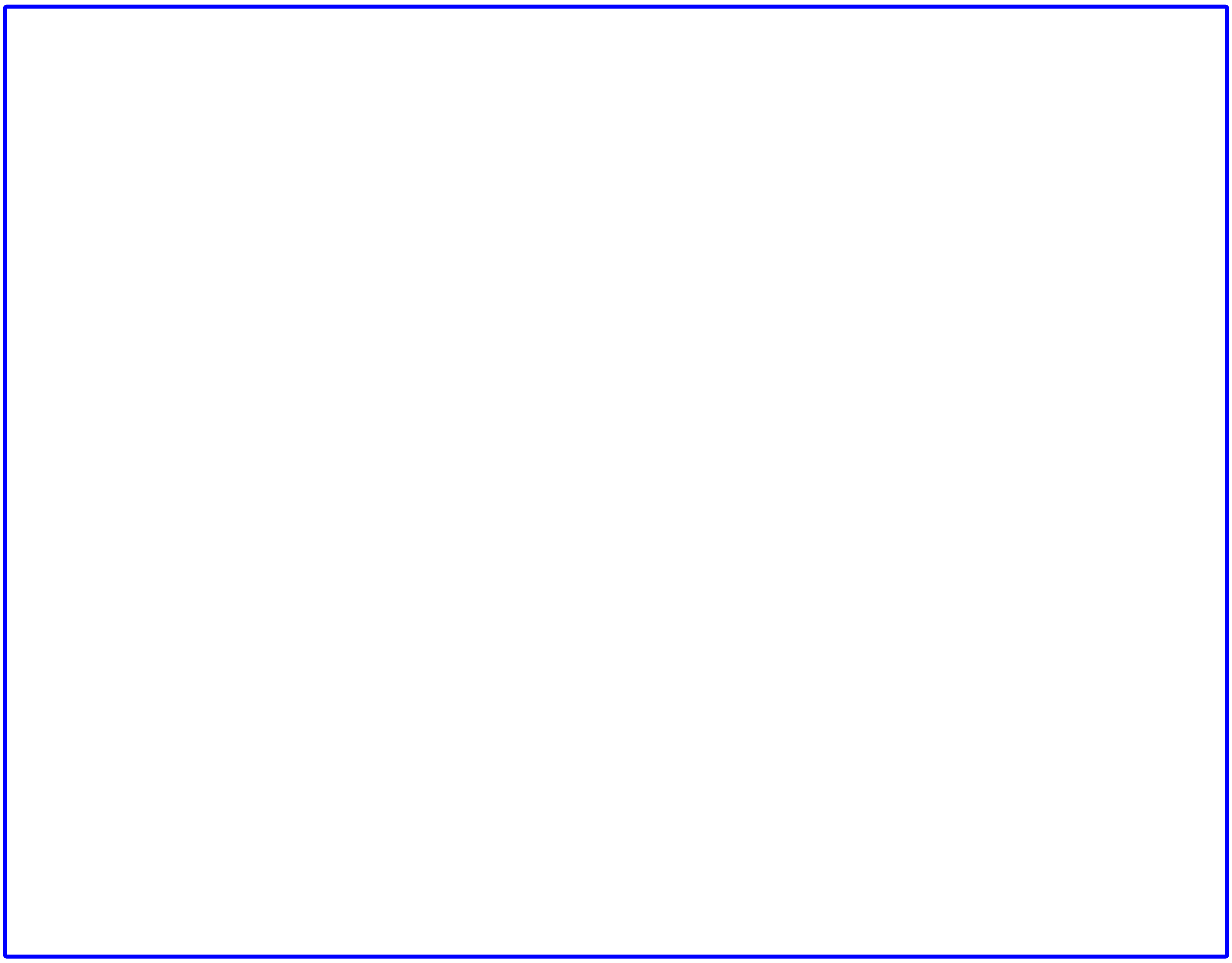
Derive the differential equation relating the input $x(t)$ to output $y(t)$ of the 2nd-Order Butterworth Low-Pass Filter with cutoff frequency ω_c.

Solution

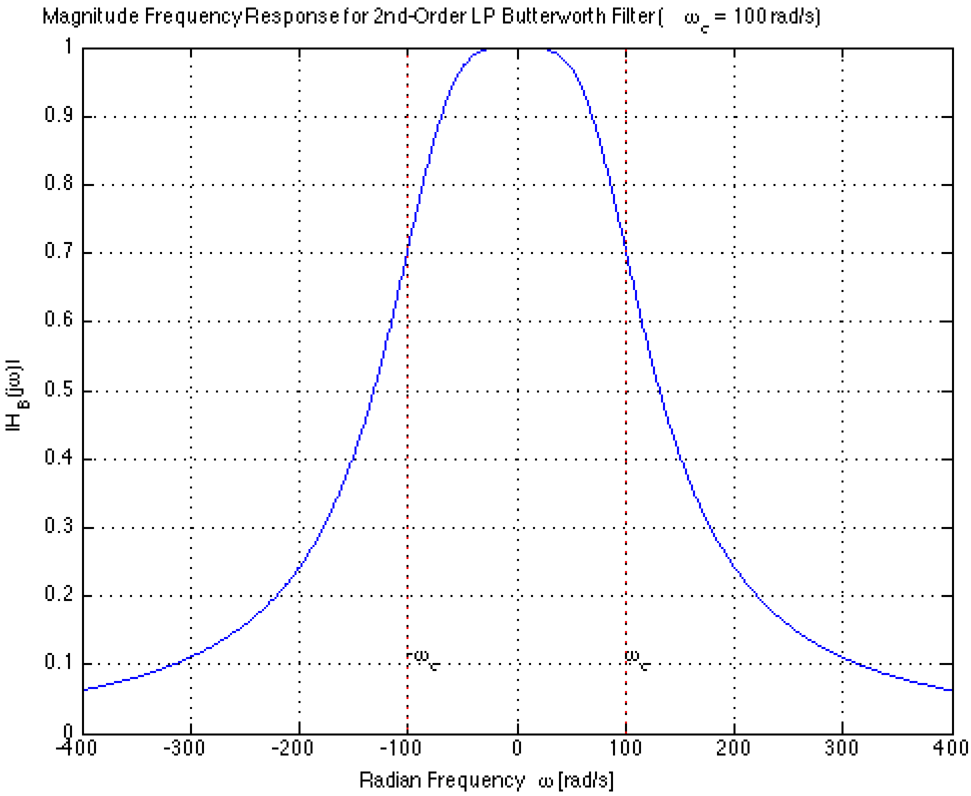
Example 7

Determine the frequency response $H_B(\omega) = Y(\omega)/X(\omega)$

Solution



Magnitude of frequency response of a 2nd-order Butterworth Filter



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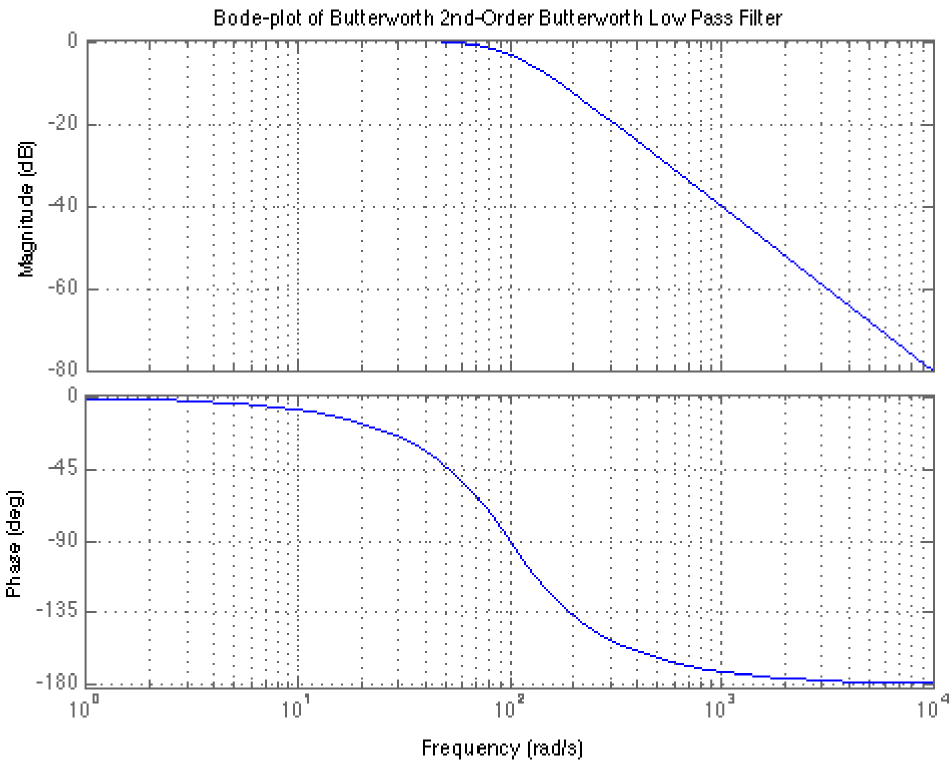
Bode-plot of a 2nd-order Butterworth Filter



Matlab:

```
wc = 100;
H = tf(wc^2,[1, wc*sqrt(2), wc^2])
bode(H)
```

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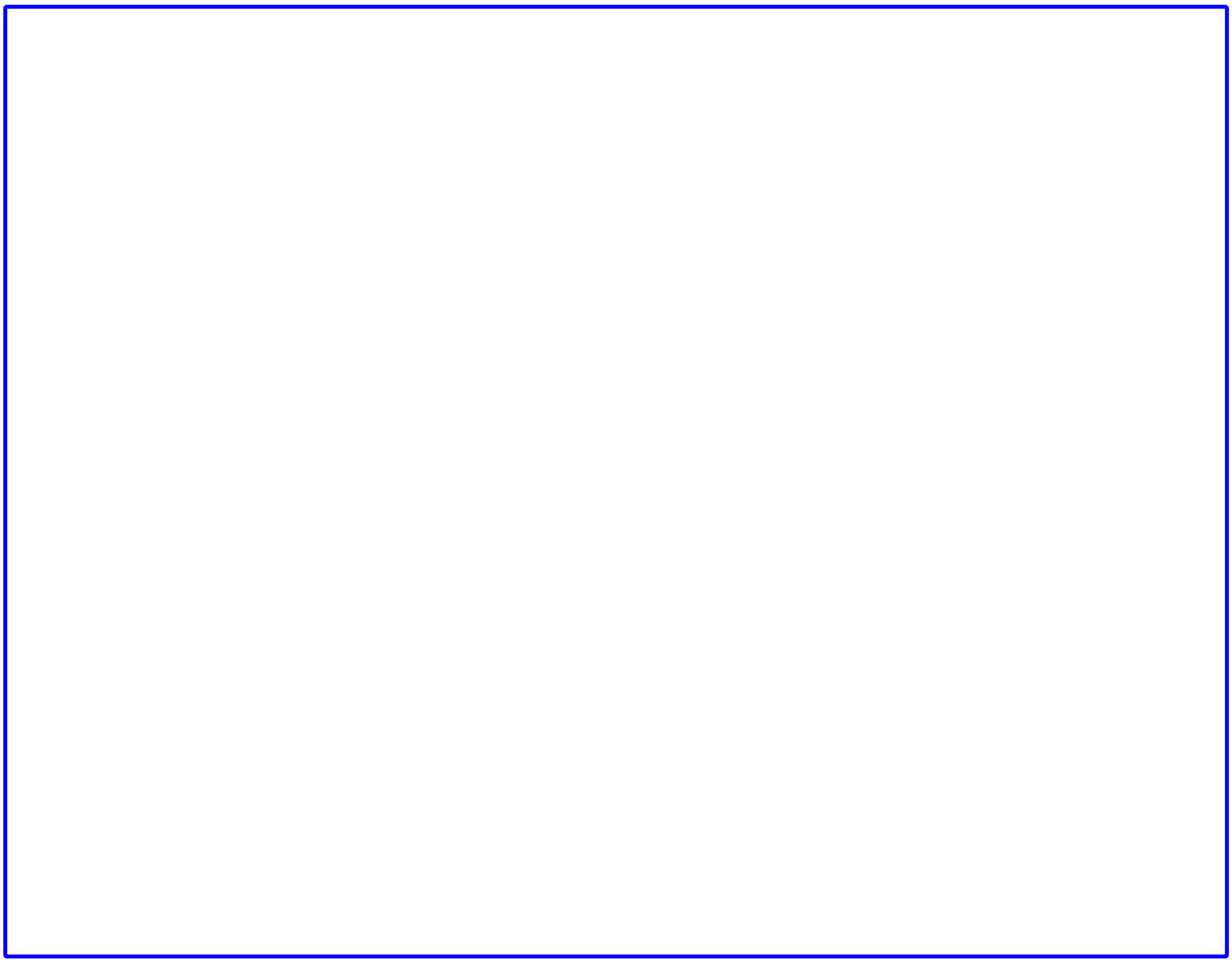
Example 8

Determine the impulse response of the butterworth filter.

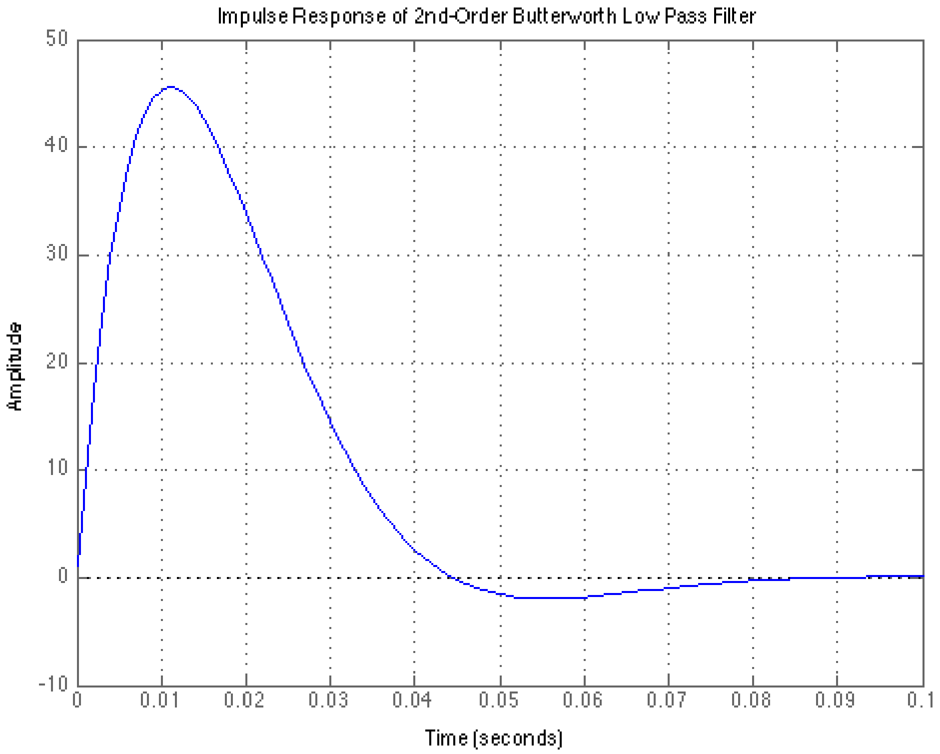
You will find this Fourier transform pair useful:

$$e^{-at}\sin\omega_0 t\; u_0(t) \Leftrightarrow \frac{\omega_0}{(j\omega + a)^2 + \omega_0^2}$$

Solution



Impulse response



Matlab:

```
impulse(H)
```

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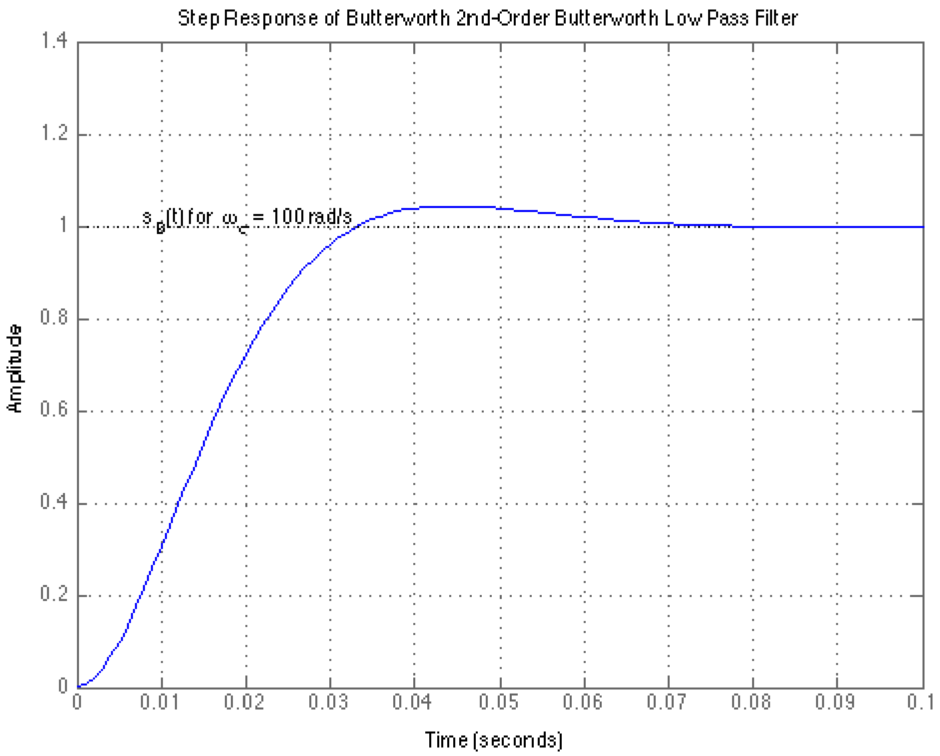
Step response of of a 2nd-order Butterworth Filter



Matlab:

```
step(H)
```

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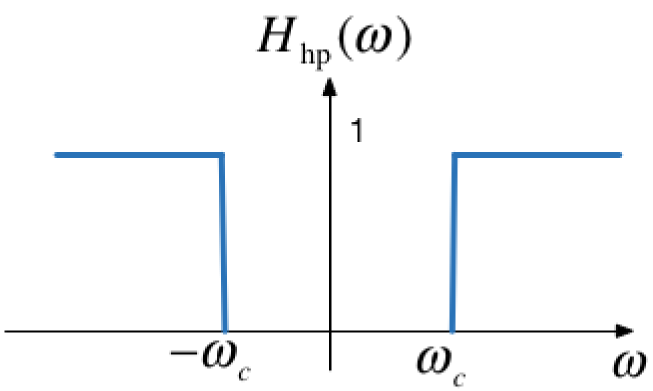
High-pass filter

An ideal highpass filter cuts-off frequencies lower than its *cutoff frequency*, ω_c.

$$H_{\text{hp}}(\omega) = \begin{cases} 0, & |\omega| \leq \omega_c \\ 1, & |\omega| > \omega_c \end{cases}$$

Frequency response



Responses

Frequency response

$$H_{\mathrm{hp}}(\omega) = 1 - H_{\mathrm{lp}}(\omega)$$

Impulse response

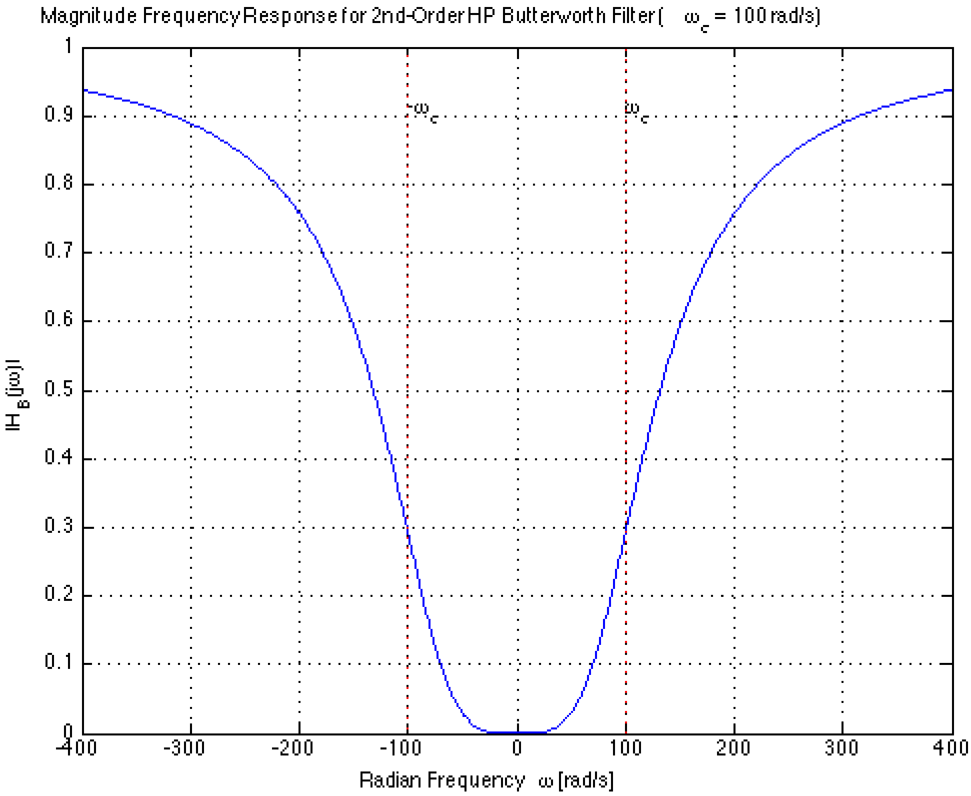
$$h_{\mathrm{hp}}(t) = \delta(t) - h_{\mathrm{lp}}(t)$$

Example 9

Determine the frequency response and impulse response of a 2nd-order butterworth highpass filter

Solution

Magnitude of frequency response of a 2nd-order Butterworth High-Pass Filter



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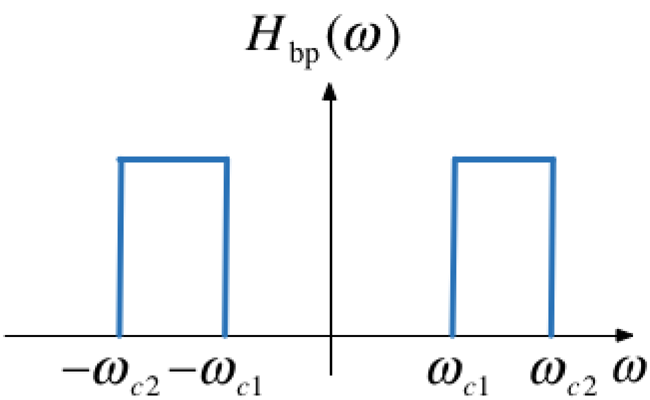
Band-pass filter

An ideal bandpass filter cuts-off frequencies lower than its first *cutoff frequency* ω_{c1}, and higher than its second *cutoff frequency* ω_{c2}.

$$H_{\mathrm{bp}}(\omega) = \begin{cases} 1, & \omega_{c1} < |\omega| < \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$



Bandpass filter design

A bandpass filter can be obtained by multiplying the frequency responses of a lowpass filter by a highpass filter.

$$H_{\mathrm{bp}}(\omega) = H_{\mathrm{hp}}(\omega)H_{\mathrm{lp}}(\omega)$$

- The highpass filter should have cut-off frequency of ω_{c1}

- The lowpass filter should have cut-off frequency of ω_{c2}

Summary

- Frequency Selective Filters
- Ideal low-pass filter
- Butterworth low-pass filter
- High-pass filter
- Bandpass filter

Next Lesson – sampling theory