The Impulse Response and Convolution (Part 1)

Scope and Background Reading

This session is an introduction to the impulse response of a system and time convolution. Together, these can be used to determine a Linear Time Invariant (LTI) system's time response to any signal.

As we shall see, in the determination of a system's response to a signal input, time convolution involves integration by parts and is a tricky operation. But time convolution becomes multiplication in the Laplace Transform domain, and is much easier to apply.

The material in this presentation and notes is based on Chapter 6 of Steven T. Karris, Signals and Systems: with Matlab Computation and Simulink Modelling, 5th Edition. (http://site.ebrary.com/lib/swansea/docDetail.action?docID=10547416) and builds on the time response of a state-space model that was developed in the previous session (http://nbviewer.ipython.org/github/cpjobling/EG-247-Resources/blob/master/week4/state_space.ipynb).

Agenda

The material to be presented will need two sessions.

Today

- The Impulse Response of a System in Time Domain
- Even and Odd Functions of Time

Next Session

- Time Convolution
- Graphical Evaluation of the Convolution Integral
- System Response by Convolution
- System Response by Laplace

The Impulse Response of a System in Time Domain

In the last session we showed that if the state-space model of a SISO system was:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + du$$

the state response would be

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$

which for our later convenience can be rewritten

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + e^{\mathbf{At}}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d\tau$$

Impulse response (1)

If we assume zero initial conditions $\mathbf{x_0} = \mathbf{0}$ and $u(t) = \delta(t)$ (Matlab `dirac`), then the state-reponse to an impulse input is:

$$\mathbf{x}(t) = e^{\mathbf{At}}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\delta(\tau)d\tau$$

Using the *sifting property* of the delta function

$$\int_{-\infty}^{\infty} f(t)\delta(\tau)d\tau = f(0)$$

then

$$\mathbf{x}(t) = e^{\mathbf{At}}\mathbf{B}$$

so, the impulse response is:

$$y(t) = \mathbf{C}e^{\mathbf{At}}\mathbf{B} + d\delta(t)$$

Impulse response (2)

In most systems that you will encounter on this course the scalar quantity $d = 0$ so the impulse response, which we donate as $h(t)$, is

$$h(t) = \mathbf{C}e^{\mathbf{At}}\mathbf{B}u_0(t)$$

where the unit step function $u_0(t)$ has been included to indicate that the impulse response is only defined for $t > 0$.

Note

In the text book, Karris presents the impulse response as

$$\mathbf{x}(t) = \mathbf{h}(t) = e^{\mathbf{A}t}\mathbf{B}u_0(t)$$

but this is the impulse response of the state variables and is a vector quantity.

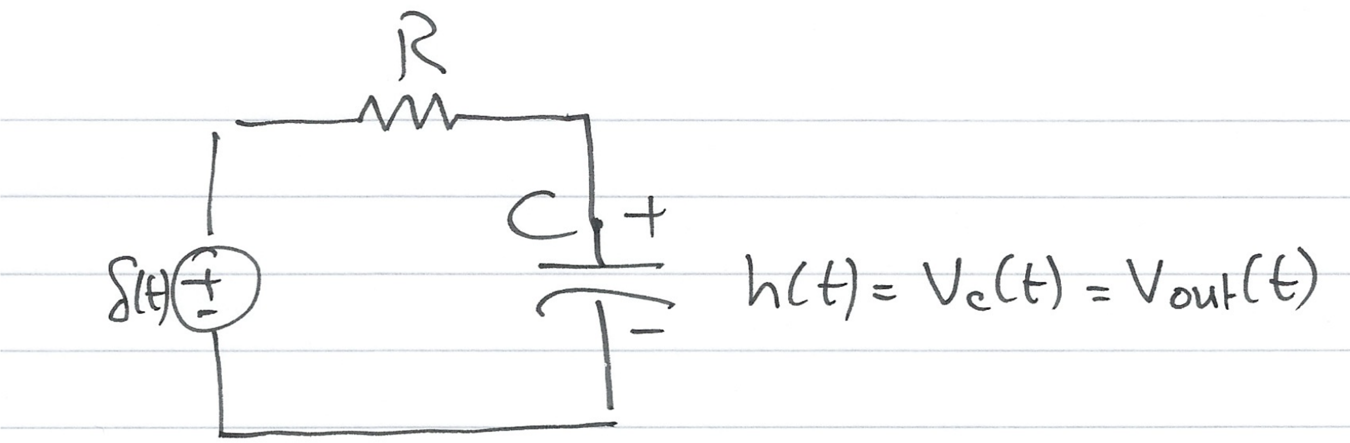
You need to introduce the output equation to find the actual scalar impulse response $h(t)$ which for a SISO system.

Karris gets away with this in his book because he uses voltages and currents as *physical state variables* and the coefficient of the corresponding $\mathbf{C}$ matrix will be unity.

In general, we cannot assume that this will be true so I prefer to be a little more careful in my presentation.

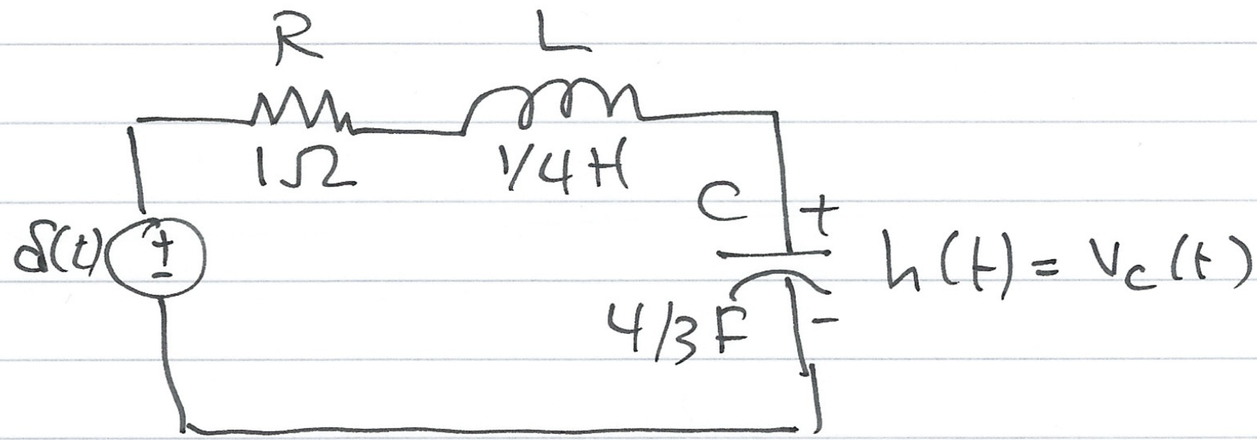
Example 1

Compute the impulse response of the series RC circuit shown below in terms of the constants R and C, where the response is considered to be the voltage across the capicitor, and $v_c(0^-) = 0$. Then, compute the current through the capacitor.



Example 2

In the RLC circuit shown below, compute the impulse response $h(t) = v_c(t)$ given that the initial conditions are zero, that is $i_L(0^-) = 0$ and $V_c(0^-) = 0$.



Solution

We tackled this problem as Example 6 in the previous session and found that:

$$\mathbf{B} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} + \frac{3}{8}e^{-3t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} \end{bmatrix}$$

so the impulse response of the state variables is:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{B}u_0(t) = \begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} + \frac{3}{8}e^{-3t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} u_0(t) = \begin{bmatrix} -2e^{-t} + 6e^{-3t} \\ \frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix} u_0(t)$$

Impulse response

In Example 6 in the previous session, we defined $x_1 = i_L$ and $x_2 = v_c$ so if we want the capacitor voltage to be the circuit output, the output vector $\mathbf{C}$ will be

$$\mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

so

$$\begin{aligned} h(t) = y(t) = v_c(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{B}u_0(t) \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -2e^{-t} + 6e^{-3t} \\ \frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix} u_0(t) \\ &= \left(\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t}\right) u_0(t) = \frac{3}{2}\left(e^{-t} + e^{-3t}\right) u_0(t) \end{aligned}$$

Even and Odd Functions of Time

(**This should be revision!**)

We need to be reminded of *even* and *odd* functions so that we can develop the idea of time convolution which is a means of determining the time response of any system for which we know its *impulse response* to any signal.

The development requires us to find out if the Dirac delta function ($\delta(t)$) is an *even* or an *odd* function of time.

Even Functions of Time

A function $f(t)$ is said to be an *even function* of time if the following relation holds

$$f(-t) = f(t)$$

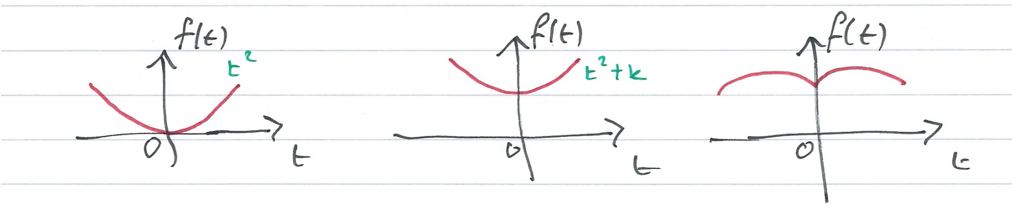
that is, if we relace t with $-t$ the function $f(t)$ does not change.

Polynomials with even exponents only, and with or without constants, are even functions. For example:

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots$$

is even.

Other Examples of Even Functions



Odd Functions of Time

A function $f(t)$ is said to be an *odd function* of time if the following relation holds

$$-f(-t) = f(t)$$

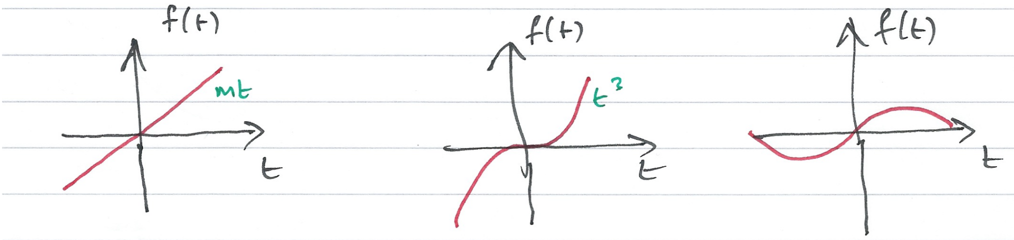
that is, if we relace t with $-t$, we obtain the negative of the function $f(t)$.

Polynomials with odd exponents only, and no constants, are odd functions. For example:

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots$$

is odd.

Other Examples of Odd Functions



Observations

- For odd functions $f(0) = 0$.
- If $f(0) = 0$ we should not conclude that $f(t)$ is an odd function. *c.f.* $f(t) = t^2$ is even, not odd.
- The product of *two even* or *two odd* functions is an even function.
- The product of an even and an odd function, is an odd function.

In the following $f_e(t)$ will donate an even function and $f_o(t)$ an odd function.

Time integrals of even and odd functions

For an even function $f_e(t)$

$$\int_{-T}^{T} f_e(t)dt = 2\int_{0}^{T} f_e(t)dt$$

For an odd function $f_o(t)$

$$\int_{-T}^{T} f_o(t)dt = 0$$

Even/Odd Representation of an Arbitrary Function

A function $f(t)$ that is neither even nor odd can be represented as an even function by use of:

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$$

or as an odd function by use of:

$$f_o(t) = \frac{1}{2}[f(t) - f(-t)]$$

Adding these together, an abitrary signal can be represented as

$$f(t) = f_e(t) + f_o(t)$$

That is, any function of time can be expressed as the sum of an even and an odd function.

Example 3

Is the Dirac delta $\delta(t)$ an even or an odd function of time?

Solution

Let $f(t)$ be an arbitrary function of time that is continous at $t = t_0$. Then by the sifting property of the delta function

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

and for $t_0 = 0$

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

Also for an even function $f_e(t)$

$$\int_{-\infty}^{\infty} f_e(t)\delta(t)dt = f_e(0)$$

and for an odd function $f_o(t)$

$$\int_{-\infty}^{\infty} f_o(t)\delta(t)dt = f_o(0)$$

Even or odd?

An odd function $f_o(t)$ evaluated at $t = 0$ is zero, that is $f_o(0) = 0$.

Hence

$$\int_{-\infty}^{\infty} f_o(t)\delta(t)dt = f_o(0) = 0$$

Hence the product $f_o(t)\delta(t)$ is odd function of t.

Since $f_o(t)$ is odd, $\delta(t)$ must be even because only an *even* function multiplied by an *odd* function can result in an *odd* function.

(Even times even or odd times odd produces an even function. See earlier slide)

Next Time

We will conclude our discussion of *Time Convolution* by presenting:

- Time Convolution
- Graphical Evaluation of the Convolution Integral
- System Response by Convolution
- System Response by Laplace