Discrete-Time System Models

Scope and Background Reading

This we will explore digital systems and learn more about the z-transfer function model.

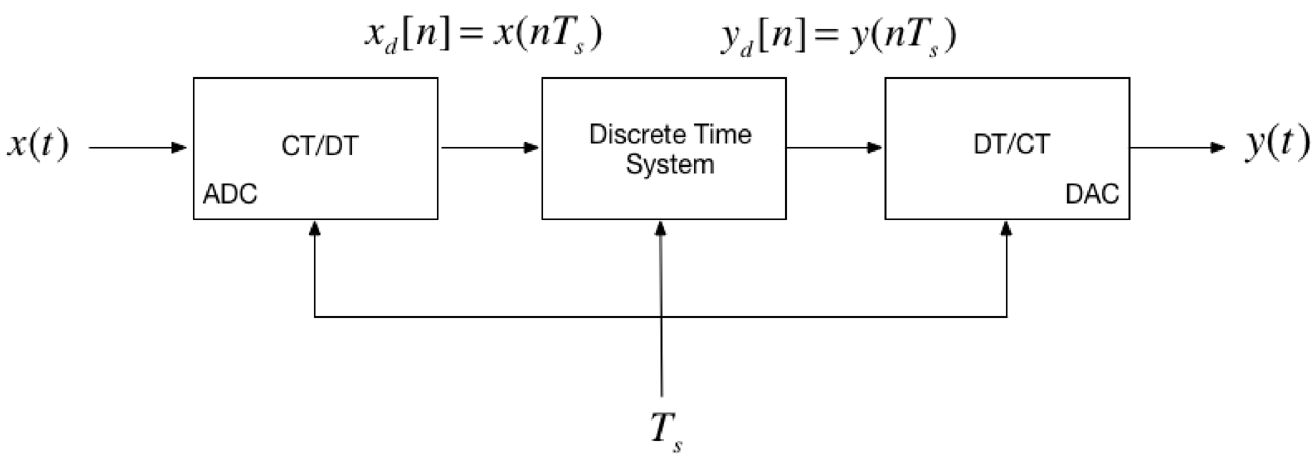
The material in this presentation and notes is based on Chapter 9 (Starting at Section 9.7) of Steven T. Karris, Signals and Systems: with Matlab Computation and Simulink Modelling, 5th Edition. (http://site.ebrary.com/lib/swansea/docDetail.action?docID=10547416) from the **Required Reading List**. I have skipped the section on digital state-space models.

Agenda

- Discrete Time Systems
- Transfer Functions in the Z-Domain
- Modelling digital systems in Matlab/Simulink
- Continuous System Equivalents
- Example: Digital Butterworth Filter

Discrete Time Systems

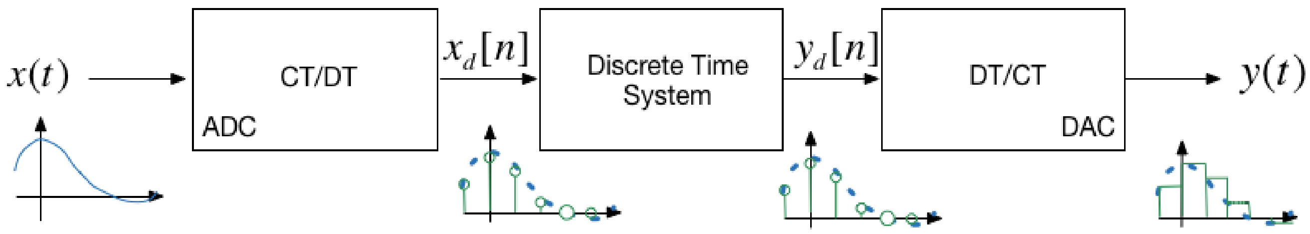
In the lecture that introduced the z-transform we talked about the representation of a discrete-time (DT) system by the model shown below:



In this session, we want to explore the contents of the central block.

DT System as a Sequence Processor

- As noted in the previous slide, the discrete time system takes as an input the sequence $x_d[n]$[1].
- It produces another sequence $y_d[n]$ by *processing* the input sequence in some way.
- The output sequence is converted into an analogue signal $y(t)$ by a digital to analogue converter.



What is the nature of the DTS?

- The discrete time system (DTS) is a block that converts a sequence $x_d[n]$ into another sequence $y_d[n]$
- The transformation will be a *difference equation* $h[n]$

- By analogy with CT systems, $h[n]$ is the impulse response of the DTS, and $y[n]$ can be obtained by *convolving* $h[n]$ with $x_d[n]$ so:
$$y_d[n] = h[n] * x_d[n]$$

- Taking the z-transform of $h[n]$ we get $H(z)$, and from the transform properties, convolution of the signal $x_d[n]$ by system $h[n]$ will be *multiplication* of the z-transforms:
$$Y_d(z) = H(z)X_d(z)$$

- So, what does $h[n]$ and therefore $H(z)$ look like?

Transfer Functions in the z-Domain

Let us assume that the sequence transformation is a *difference equation* of the form[2]:
$$\begin{aligned} y[n] + a_1 y[n-1] + a_2 y[n-2] + \cdots + a_k y[n-k] \\ = b_0 x[n] + b_1 u[n-1] + b_2 u[n-2] + \cdots + b_k u[n-k] \end{aligned}$$

Take Z-Transform of both sides

From the z-transform properties
$$f[n-m] \Leftrightarrow z^{-m} F(z)$$

so....

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_k z^{-k} Y(z) = \ldots$$

$$b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \cdots + b_k z^{-k} U(z)$$

Gather terms

$$\left(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_k z^{-k}\right) Y(z) =$$
$$\left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_k z^{-k}\right) U(z)$$

from which ...

$$Y(z) = \left(\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_k z^{-k}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_k z^{-k}}\right) U(z)$$

Define transfer function

We define the *discrete time transfer function* $H(z) := Y(z)/U(z)$ so...

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_k z^{-k}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_k z^{-k}}$$

... or more conventionally[3]:

$$H(z) = \frac{b_0 z^k + b_1 z^{k-1} + b_2 z^{k-2} + \cdots b_{k-1} z + b_k}{z^k + a_1 z^{k-1} + a_2 z^{k-2} + \cdots a_{k-1} z + a_k}$$

DT impulse response

The *discrete-time impulse reponse* $h[n]$ is the response of the DT system to the input $x[n] = \delta[n]$

Last week we showed that $\mathcal{Z}\{\delta[n]\}$ was defined by the transform pair

$$\delta[n] \Leftrightarrow ?$$

$$\delta[n] \Leftrightarrow 1$$

so

$$h[n] = \ldots$$

$$h[n] = \mathcal{Z}^{-1}\{H(z).1\} = \mathcal{Z}^{-1}\{H(z)\}$$

Example 1

Karris Example 9.10:

The difference equation describing the input-output relationship of a DT system with zero initial conditions, is:

$$y[n] - 0.5y[n-1] + 0.125y[n-2] = x[n] + x[n-1]$$

Compute:

1. The transfer function $H(z)$
2. The DT impulse response $h[n]$
3. The response $y[n]$ when the input $x[n]$ is the DT unit step $u_0[n]$

Karris Example 9.10:

The difference equation describing the input-output relationship of a DT system with zero initial conditions, is:

$$y[n] - 0.5y[n-1] + 0.125y[n-2] = x[n] + x[n-1]$$

Compute:

1. The transfer function $H(z)$
2. The DT impulse response $h[n]$
3. The response $y[n]$ when the input $x[n]$ is the DT unit step $u_0[n]$

1. The transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \ldots ?$$

1. Solution

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2 + z}{z^2 - 0.5z + 0.125}$$

2. The DT impulse response

Start with:

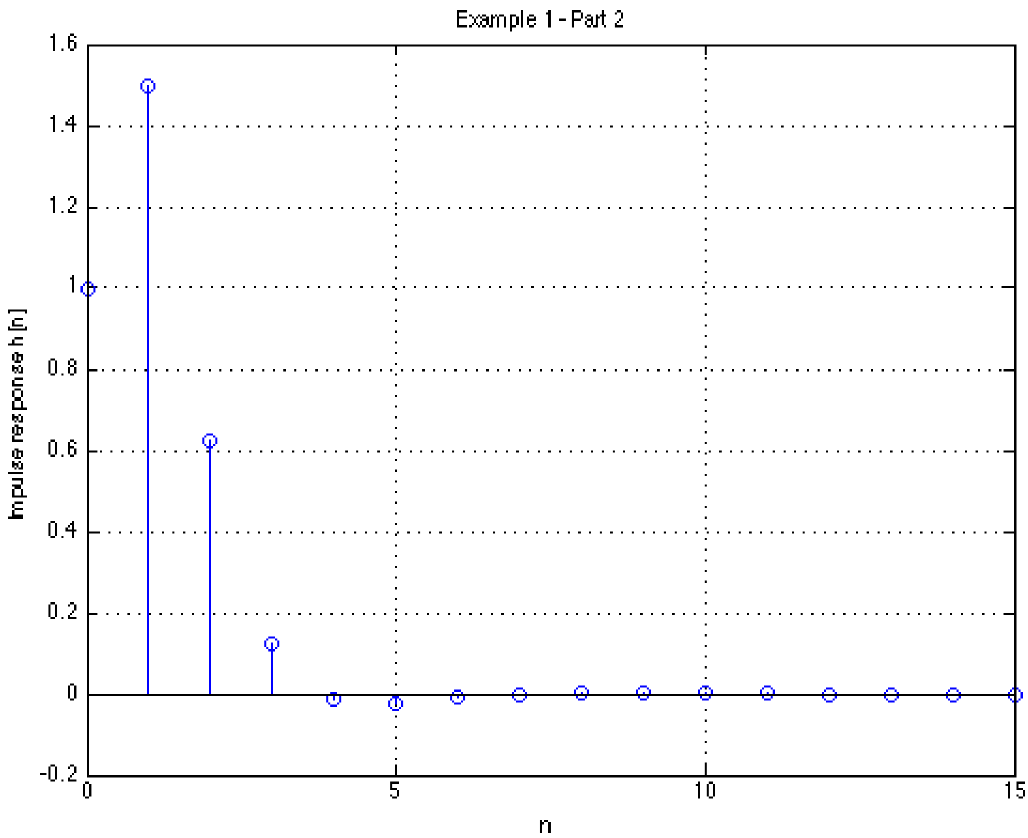
$$\frac{H(z)}{z} = \frac{z - 1}{z^2 + 0.5z + 0.125}$$

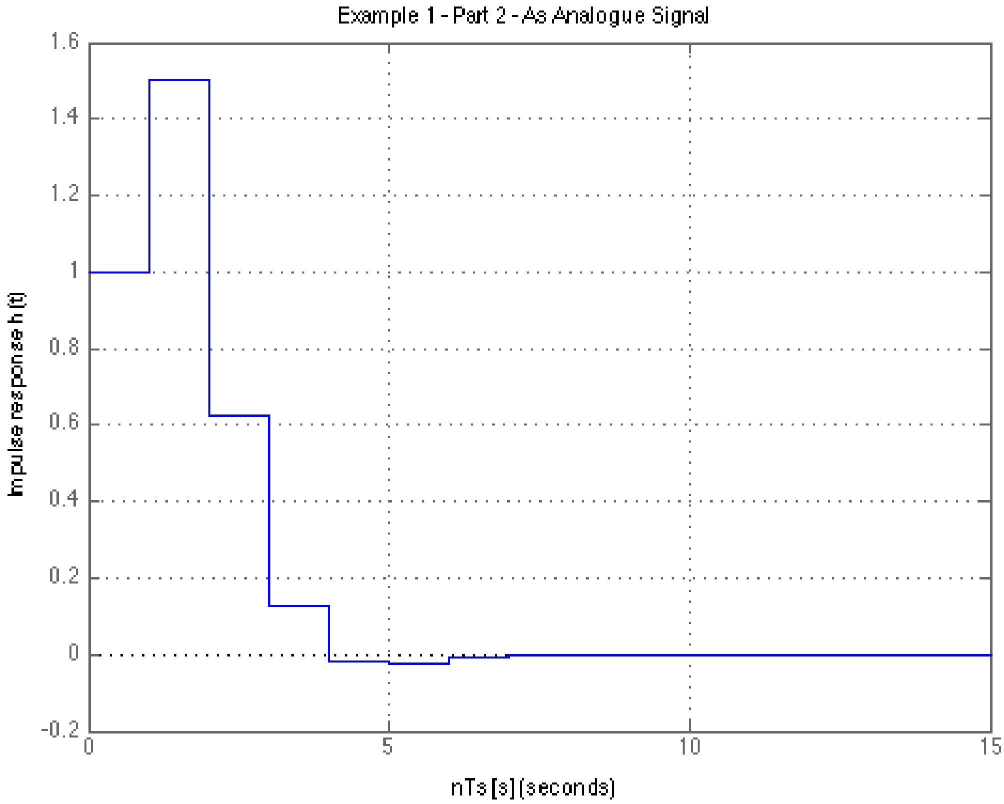
2. Solution

$$h[n] = \left(\frac{\sqrt{2}}{4}\right)^n \left(\cos\left(\frac{n\pi}{4}\right) + 5\sin\left(\frac{n\pi}{4}\right)\right)$$

Matlab Solution

See dtm_ex1_2.m (matlab/dtm_ex1_2.m):





3. The DT step response

$$Y(z) = H(z)X(z)$$

$$u_0[n] \Leftrightarrow \frac{z}{z-1}$$

$$\begin{aligned} Y(z) = H(z)U_0(z) &= \frac{z^2+z}{z^2+0.5z+0.125} \cdot \frac{z}{z-1} \\ &= \frac{z(z^2+z)}{(z^2+0.5z+0.125)(z-1)} \end{aligned}$$

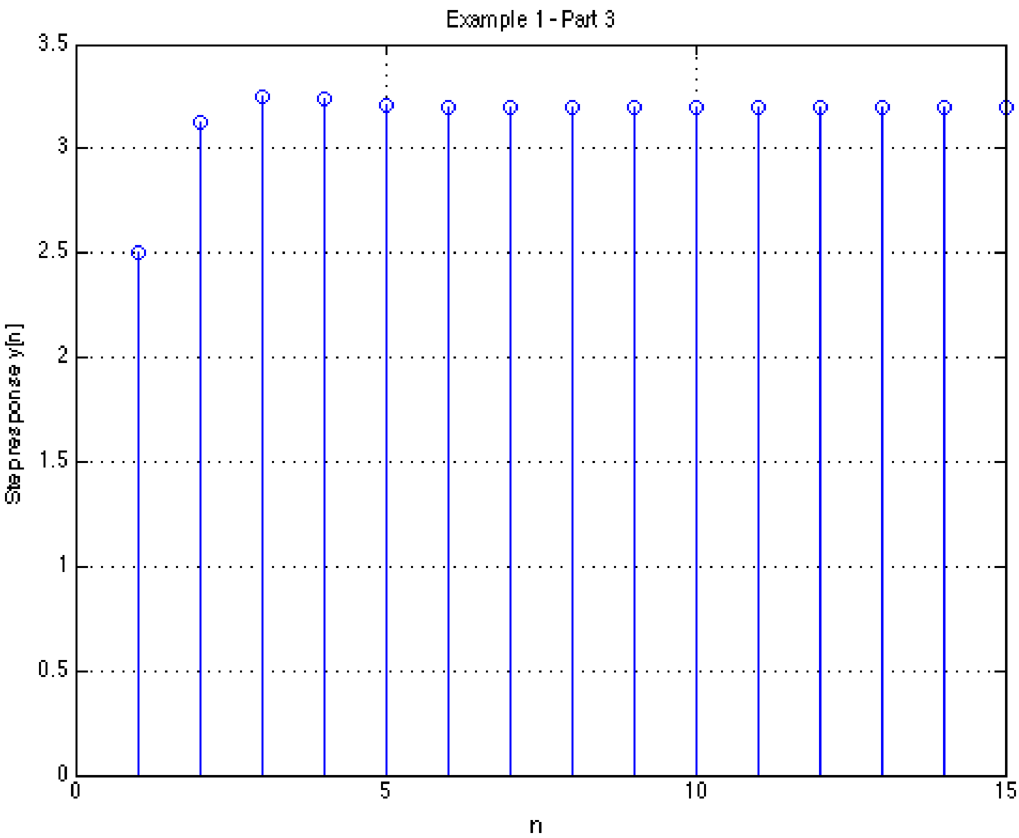
$$\frac{Y(z)}{z} = \frac{z^2 + z}{(z^2 + 0.5z + 0.125)(z - 1)}$$

3. Solution

$$y[n] = \left(3.2 - \left(\frac{\sqrt{2}}{4}\right)^n \left(2.2\cos\left(\frac{n\pi}{4}\right) + 0.6\sin\left(\frac{n\pi}{4}\right)\right)\right) u_0[n]$$

Matlab Solution

See dtm_ex1_3.m (matlab/dtm_ex1_3.m):

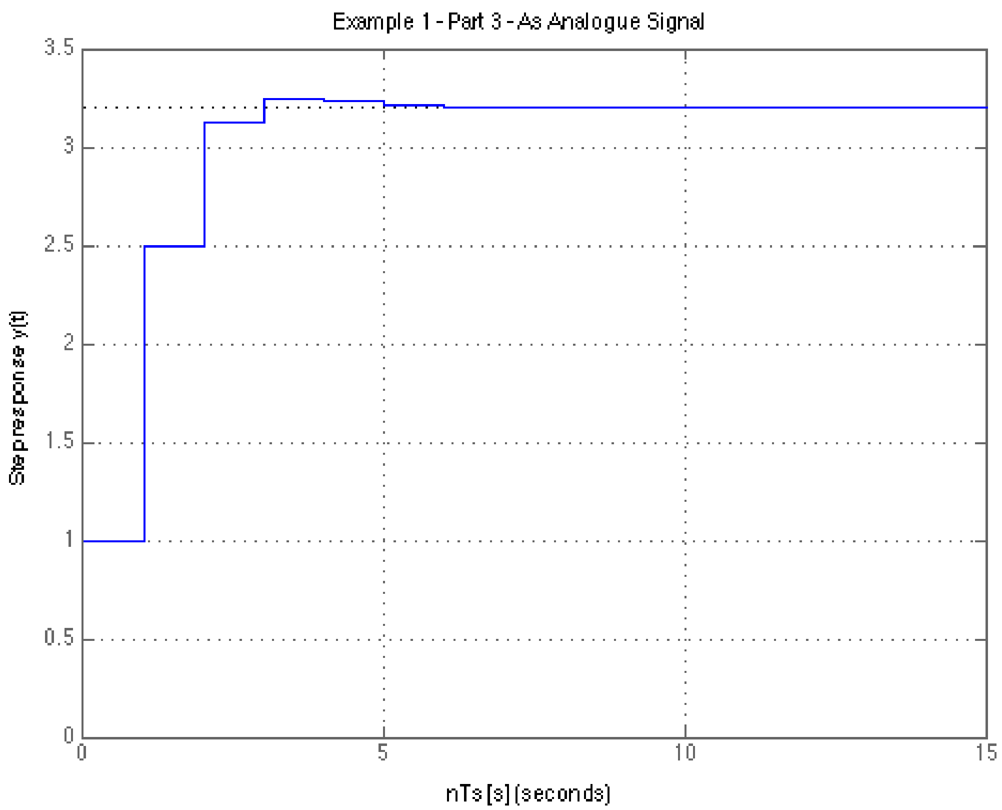


Modelling DT systems in Matlab and Simulink

Matlab

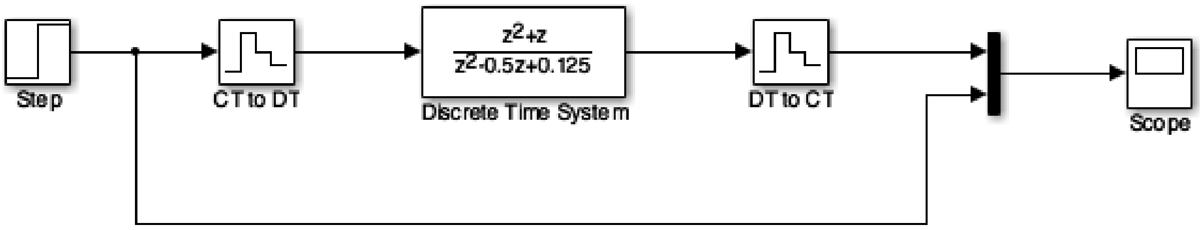
Code extracted from dtm_ex1_3.m (matlab/dtm_ex1_3.m):

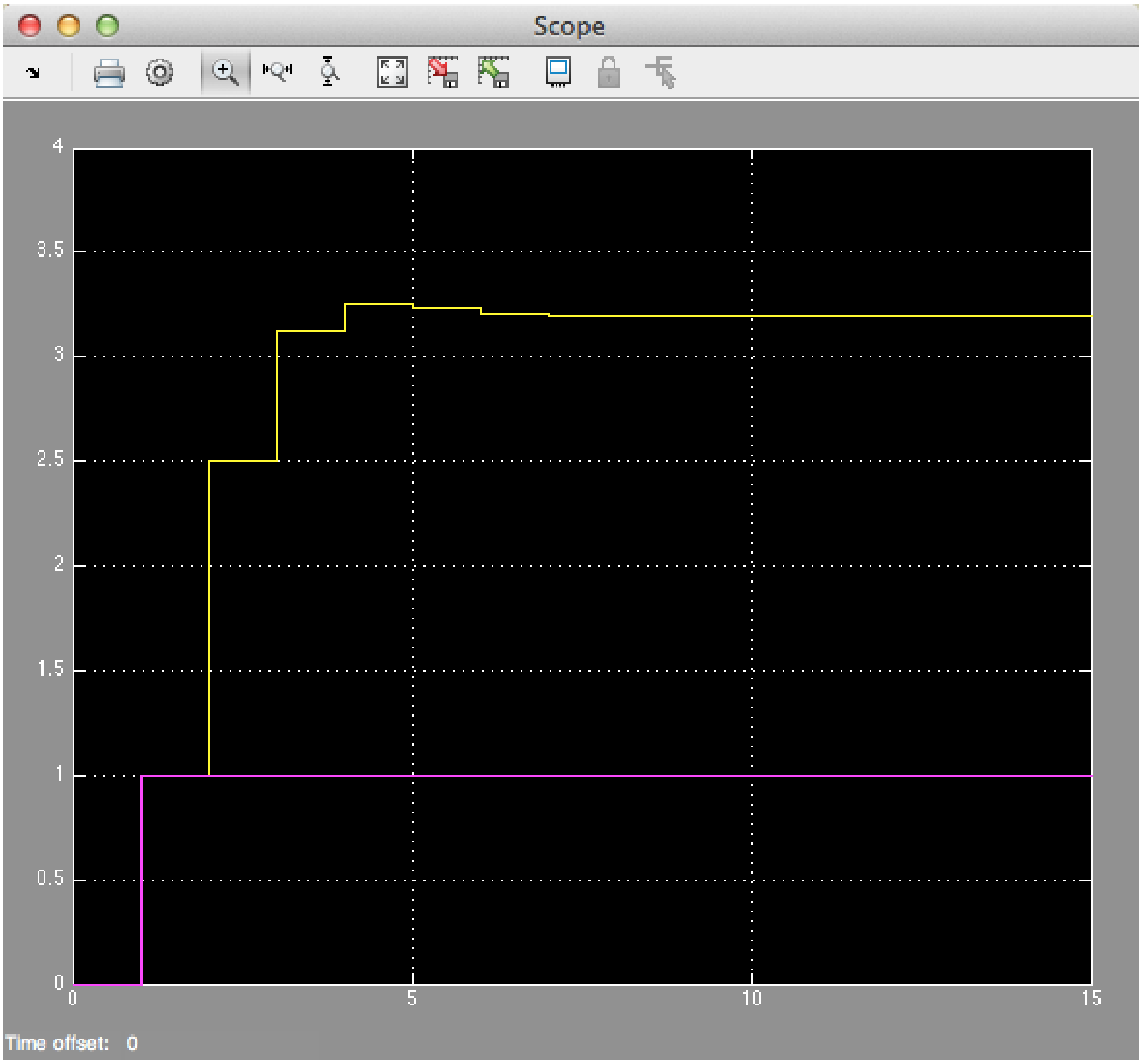
```
Ts = 1;
z = tf('z', Ts)
Hz = (z^2 + z)/(z^2 - 0.5 * z + 0.125)
step(Hz)
grid
title('Example 1 - Part 3 - As Analogue Signal')
xlabel('nTs [s]')
ylabel('Step response y(t)')
axis([0,15,0,3.5])
```



Simulink Model

See dtm.slx (matlab/dtm.slx):





Converting Continuous Time Systems to Discrete Time Systems

- In analogue electronics, to implement a filter we would need to resort to op-amp circuits with resistors, capacitors and inductors acting as energy dissipation, storage and release devices.
- In modern digital electronics, it is often more convenient to take the original transfer function $H(s)$ and produce an equivalent $H(z)$.
- We can then determine a *difference equation* that will respresent $h[n]$ and implement this as *computer algorithm*.
- Simple storage of past values in memory becomes the repository of past state rather than the integrators and derivative circuits that are needed in the analogue world.
- To achieve this, all we need is to be able to do is to *sample* and *process* the signals quickly enough to avoid violating Nyquist-Shannon's sampling theorem.

Continuous System Equivalents

- There is no digital system that uniquely represents a continuous system
- This is because as we are sampling, we only have knowledge of signals being processed at the sampling instants, and need to *reconstruct* the inter-sample behaviour.
- In practice, only a small number of transormations are used.
- The derivation of these is beyond the scope of this module, but we'll mention the ones that Matlab provides in a function called `c2d`

Matlab c2d function

This is what the help function says:

```
>> help c2d
SYSD = c2d(SYSC,TS,METHOD) computes a discrete-time model SYSD with
    sampling time TS that approximates the continuous-time model SYS
C.
    The string METHOD selects the discretization method among the fo
llowing:
       'zoh'       Zero-order hold on the inputs
       'foh'       Linear interpolation of inputs
       'impulse'   Impulse-invariant discretization
       'tustin'    Bilinear (Tustin) approximation.
       'matched'   Matched pole-zero method (for SISO systems only).
    The default is 'zoh' when METHOD is omitted. The sampling time T
S should
    be specified in the time units of SYSC (see "TimeUnit" propert
y).
...
```

Example 2

- Design a 2nd-order butterworth anti-aliasing filter with transfer function $H(s)$ for use in sampling music.
- The cut-off frequency $\omega_c = 20$ kHz and the filter should have an attenuation of at least -80 dB in the stop band.
- Choose a suitable sampling frequency for the audio signal and give the transfer function $H(z)$ and an algorithm to implement $h[n]$

Solution

See digi_butter.m (matlab/digi_butter.m):

$$\omega_c = 2\pi f_c = 2 \times \pi \times 20 \times 10^3 = 125.6637 \times 10^3 \text{ rad/s}$$

From the lecture on filters, we know the 2nd-order butterworth filter has transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_c^2}{s^2 + \omega_c \sqrt{2}\, s + \omega_c^2}$$

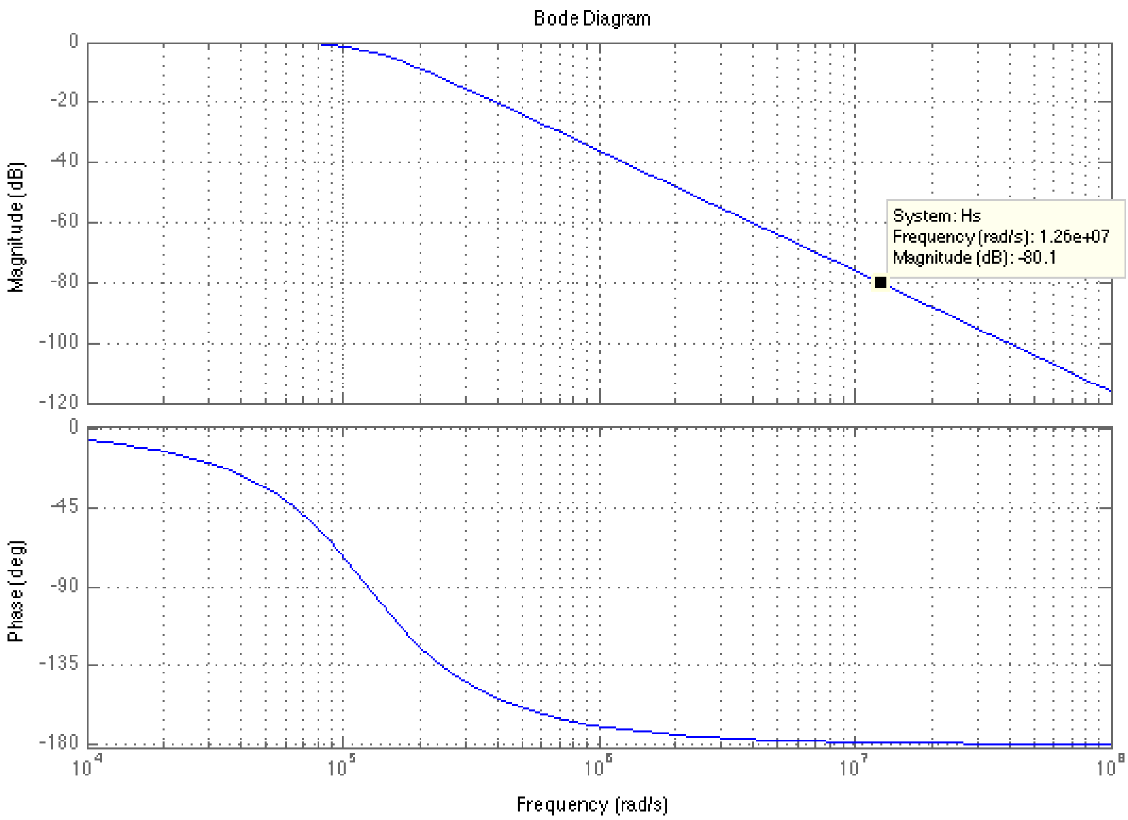
Substituting for $\omega_c = 125.6637 \times 10^3$ this is ...?

$$H(s) = \frac{15.79 \times 10^9}{s^2 + 177.7 \times 10^3 s + 15.79 \times 10^9}$$

Bode plot

Matlab:

```
wc = 2*pi*20e3;
Hs = tf(wc^2,[1 wc*sqrt(2), wc^2]);
bode(Hs,{1e4,1e8})
grid
```



Sampling Frequency

From the bode diagram, the frequency at which $|H(j\omega)|$ is -80 dB is approx 12.6×10^6 rad/s.

To avoid aliasing, we should choose a sampling frequency twice this = ?

So sampling frequency $\omega_s = 2 \times 12.6 \times 10^6 = 25.2 \times 10^6$ rad/s.

Sampling frequency in Hz f_s = ?

$$f_s = \omega_s/(2\pi) = 25.2 \times 10^6/(2 \times \pi) = 40.1 \text{ Mhz}$$

Sampling time $T_s = ?$

$$T_s = 1/f_s \approx 0.25 \ \mu\text{s}$$

Digital Butterworth

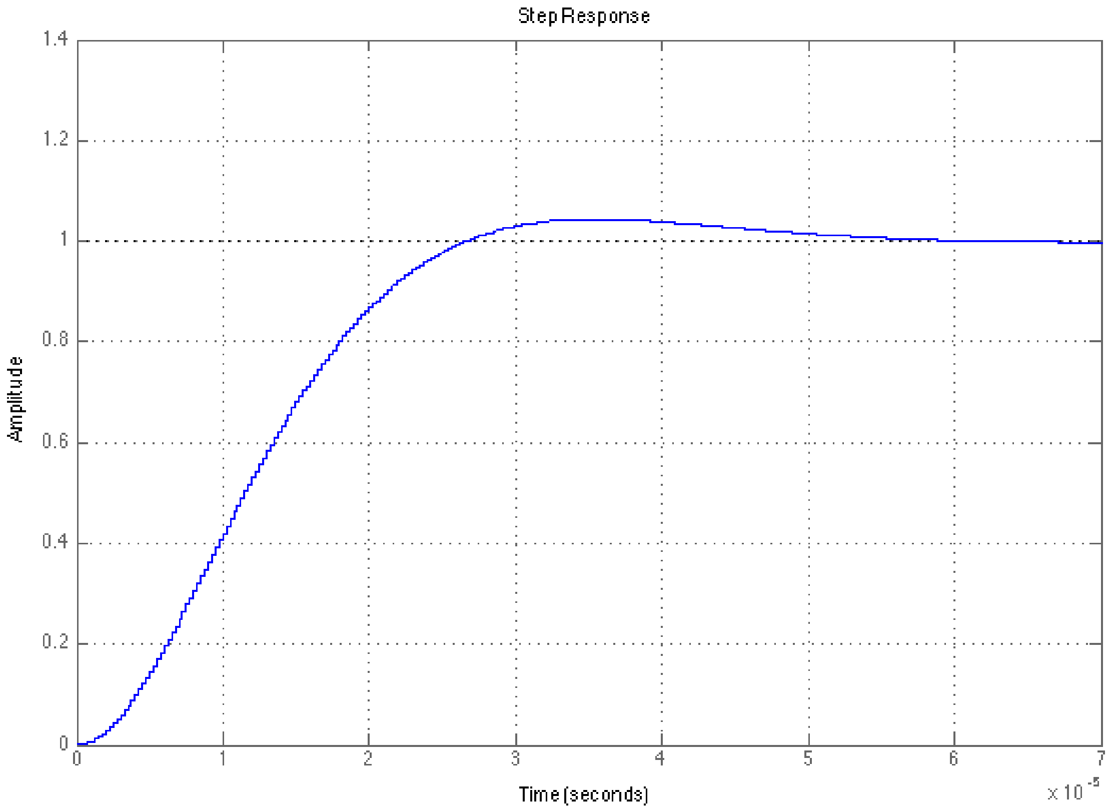
```
>> Ts = 0.25e-6;
>> Hz = c2d(Hs, Ts) % zero-order-hold equivalent

Hz =

  0.0004836 z + 0.0004765
  -----------------------
  z^2 - 1.956 z + 0.9567

Sample time: 2.4933e-07 seconds
Discrete-time transfer function.
```

Step response



Algorithm

From previous result:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{486.6 \times 10^{-6} z + 476.5 \times 10^{-6}}{z^2 - 1.956z + 0.9567}$$

Dividing top and bottom by z^2 ...

$$H(z) = \frac{Y(z)}{U(z)} = \frac{486.6 \times 10^{-6} z^{-1} + 476.5 \times 10^{-6} z^{-2}}{1 - 1.956z^{-1} + 0.9567z^{-2}}$$

expanding out ...

$$\begin{gathered} Y(z) - 1.956z^{-1}Y(z) + 0.9567z^{-2}Y(z) = \\ 486.6 \times 10^{-6} z^{-1} U(z) + 476.5 \times 10^{-6} z^{-2} U(z) \end{gathered}$$

Inverse z-transform gives ...

Algorithm ... continued

$$\begin{gathered} y[n] - 1.956y[n-1] + 0.9567y[n-2] = \\ 486.6 \times 10^{-6} u[n-1] + 476.5 \times 10^{-6} u[n-2] \end{gathered}$$

in algorithmic form (compute $y[n]$ from past values of u and y) ...

$$\begin{aligned} y[n] = 1.956y[n-1] - 0.9567y[n-2] + 486.6 \times 10^{-6} u[n-1] + \ldots \\ 476.5 \times 10^{-6} u[n-2] \end{aligned}$$

Now convert to code

Convert to code

To implement:

$$y[n] = 1.956y[n-1] - 0.9567y[n-2] + 486.6 \times 10^{-6} u[n-1] + 476.5 \times 10^{-6} u[n-2]$$

```
/* Initialize */
ynm1 = 0; ynm2 = 0; unm1 = 0; unm2 = 0;
while (true) {
    un = read_adc;
    yn = 1.956*ynm1 - 0.9567*ynm2 + 486.6e-6*unm1 + 476.5e-6*unm2;
    write_dac(yn);
    /* store past values */
    ynm2 = ynm1; ynm1 = yn;
    unm2 = unm1; unm1 = un;
}
```

Comments

PC soundcards can sample audio at 44.1 kHz so this implies that the anti-aliasing filter is much sharper than this one as $f_s/2 = 22.05$ kHz.

You might wish to find out what order butterworth filter would be needed to have $f_c = 20$ kHz and f_{stop} of 22.05 kHz.

Summary

- Discrete Time Systems
- Transfer Functions in the Z-Domain
- Modelling digital systems in Matlab/Simulink
- Continuous System Equivalents
- Example: Digital Butterworth Filter

The End?

- This concludes this module.
- There is some material that I have not covered, most notably **Discrete Fourier Transform**.
- This is covered in Karris Chapter 10 and Boulet. It will not be examined.
- There is a significant amount of additional information about **Filter Design** (including the use of Matlab for this) in Chapter 11 of Karris.