

# The Inverse Laplace Transform

## Definition

The formal definition of the Inverse Laplace Transform is

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} f(t)e^{st} ds$$

but this is difficult to use in practice because it requires contour integration using complex variable theory.

For most engineering problems we can instead refer to **Tables of Properties** and **Common Transform Pairs** to look up the *Inverse Laplace Transform*

(Or, if we are not taking an exam, we can use a computer or mobile device.)

## Partial Fraction Expansion

Quite often the Laplace Transform we start off with is a *rational polynomial* in  $s$ .

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_1 s + a_0}$$

The coefficients  $a_k$  and  $b_k$  are real for  $k = 1, 2, 3, \dots$

## Proper and Improper Rational Functions

- If  $m < n$  proper rational function.
- If  $m \geq n$   $F(s)$  is said to be an *improper rational function*

(Think proper fractions (<http://www.mathsisfun.com/proper-fractions.html>) and improper fractions (<http://www.mathsisfun.com/improper-fractions.html>)).

## Zeros

- The *roots* of the numerator polynomial  $N(s)$  are found by setting  $N(s) = 0$
- When  $s$  equals one of the  $m$  roots of  $N(s)$  then  $F(s)$  will be zero.
- Thus the roots of  $N(s)$  are the **zeros** of  $F(s)$ .

## Poles

- The *roots* (zeros) of the denominator polynomial are found by setting  $D(s) = 0$ .
- When  $s$  equals one of the  $n$  roots of  $D(s)$  then  $F(s)$  will be infinite  $F(s_r) = N(s_r)/0 = \infty$ .
- These are called the **poles** of  $F(s)$ .

(Imagine telegraph poles planted at the points on the  $s$ -plane where  $D(s)$  is zero.)

## A Further Simplifying Assumption

If  $F(s)$  is proper then it is conventional to make the coefficient  $s_n$  unity thus:

$$F(s) = \frac{N(s)}{D(s)} = \frac{1/a_n (b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \dots + b_1 s + b_0)}{s^n + \frac{a_{n-1}}{a_n} s^{n-1} + \frac{a_{n-2}}{a_n} s^{n-2} + \dots + \frac{a_1}{a_n} s + \frac{a_0}{a_n}}$$

(I know it doesn't look simpler, but remember that the  $a$ s and  $b$ s are numbers in practice!)

## Inverse Laplace Transform by Partial Fraction Expansion (PFE)

The poles of  $F(s)$  can be real and distinct, real and repeated, complex conjugate pairs, or a combination.

### Defining the problem

The nature of the poles governs the best way to tackle the PFE that leads to the solution of the Inverse Laplace Transform. Thus, we need to structure our presentation to cover one of the following cases:

- The case where  $F(s)$  has distinct poles
- The case where  $F(s)$  has complex poles
- The case where  $F(s)$  has repeated poles
- The case where  $F(s)$  is an improper rational polynomial

We will examine each case by means of a worked example. Please refer to Chapter 3 of Karris for full details.

### The case of the distinct poles

If the poles  $p_1, p_2, p_3, \dots, p_n$  are *distinct* we can factor the denominator of  $F(s)$  in the form

$$F(s) = \frac{N(s)}{(s - p_1)(s - p_2)(s - p_3) \dots (s - p_n)}$$

Next, using *partial fraction expansion*

$$F(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \frac{r_3}{s - p_3} + \dots + \frac{r_n}{s - p_n}$$

To evaluate the *residue*  $r_k$ , we multiply both sides by  $(s - p_k)$  then let  $s \rightarrow p_k$

$$r_k = \lim_{s \rightarrow p_k} (s - p_k) F(s) = (s - p_k) F(s) \big|_{s=p_k}$$

## Example 1

Use the PFE method to simplify  $F_1(s)$  below and find the time domain function  $f_1(t)$  corresponding to  $F_1(s)$

$$F_1(s) = \frac{2s + 5}{s^2 + 5s + 6}$$

(Quick solution: [Wolfram|Alpha](http://www.wolframalpha.com/input/?i=inverse+laplace+transform+%282s+%2B+5%29%2F%28s%5E2+%2B+5s+%2B+6%29%29) (<http://www.wolframalpha.com/input/?i=inverse+laplace+transform+%282s+%2B+5%29%2F%28s%5E2+%2B+5s+%2B+6%29%29>))

## Matlab Solution - Numerical

In [ ]:

```
# Set up in iPython
from pymatbridge import Matlab
mlab = Matlab()
mlab.start()
import pymatbridge as pymat
ip = get_ipython()
pymat.load_ipython_extension(ip)
```

In [ ]:

```
%%matlab
Ns = [2, 5]; Ds = [1, 5, 6];
[r,p,k] = residue(Ns, Ds)
```

Interpreted as:

$$F_1(s) = \frac{1}{s+3} + \frac{1}{s+2}$$

which because of the linearity property of the Laplace Transform and using tables results in the *Inverse Laplace Transform*

$$f_1(t) = e^{-3t} + e^{-2t}$$

## Matlab solution - symbolic

In [ ]:

```
%%matlab
syms s t;
Fs = (2*s + 5)/(s^2 + 5*s + 6);
ft = ilaplace(Fs);
pretty(ft)
```

## Example 2

Determine the Inverse Laplace Transform of

$$F_2(s) = \frac{3s^2 + 2s + 5}{s^3 + 9s^2 + 23s + 15}$$

(Quick solution: [Wolfram|Alpha](http://www.wolframalpha.com/input/?i=inverse+laplace+transform+%283s%5E2+%2B+2s+%2B+5%29%2F%28s%5E3+%2B+9s%5E2+%2B+23s+%2B+15%29) (<http://www.wolframalpha.com/input/?i=inverse+laplace+transform+%283s%5E2+%2B+2s+%2B+5%29%2F%28s%5E3+%2B+9s%5E2+%2B+23s+%2B+15%29>))

## Solution 2

Because the denominator of  $F_2(s)$  is a cubic, it will be difficult to factorise without computer assistance so we use Matlab to factorise  $D(s)$

In [ ]:

```
%%matlab
syms s;
factor(s^3 + 9*s^2 + 23*s + 15)
```

In an exam you'd be given the factors

We can now use the previous technique to find the solution which according to Matlab should be

$$f_1(t) = \frac{3}{4}e^{-t} - \frac{13}{2}e^{-3t} + \frac{35}{4}e^{-5t}$$

## The case of the complex poles

Quite often the poles of  $F(s)$  are complex and because the complex poles occur as complex conjugate pairs, the number of complex poles is even. Thus if  $p_k$  is a complex root of  $D(s)$  then its complex conjugate  $p_k^*$  is also a root of  $D(s)$ .

You can still use the PFE with complex poles, as demonstrated in Pages 3-5—3-7 in the textbook. However it is easier to use the fact that complex poles will appear as quadratic factors of the form  $s^2 + as + b$  and then call on the two transforms in the PFE

$$\frac{s}{(s-a)^2 + \omega^2} \Leftrightarrow e^{at} \sin \omega t$$

$$\frac{s+a}{(s-a)^2 + \omega^2} \Leftrightarrow e^{at} \cos \omega t$$



## Example 4

Find the inverse Laplace Transform of

$$F_4(s) = \frac{s+3}{(s+2)(s+1)^2}$$

Note that the transform

$$te^{at} \Leftrightarrow \frac{1}{(s-a)^2}$$

will be useful.

(Quick solution: [Wolfram|Alpha \(http://www.wolframalpha.com/input/?i=inverse+laplace+transform+%28s%2B3%29%2F%28%28s%2B2%29%28s+%2B+1%29%5E2%29\)](http://www.wolframalpha.com/input/?i=inverse+laplace+transform+%28s%2B3%29%2F%28%28s%2B2%29%28s+%2B+1%29%5E2%29))

## Solution

We will leave the solution that makes use of the residue of repeated poles formula for you to study from the text book. In class we will illustrate the slightly simpler approach also presented in the text.

For exam preparation, I would recommend that you use whatever method you find most comfortable.

## The case of the improper rational polynomial

If  $F(s)$  is an improper rational polynomial, that is  $m \geq n$ , we must first divide the numerator  $N(s)$  by the denominator  $D(s)$  to derive an expression of the form

$$F(s) = k_0 + k_1 s + k_2 s^2 + \dots + k_{m-n} s^{m-n} + \frac{N(s)}{D(s)}$$

and then  $N(s)/D(s)$  will be a proper rational polynomial.

## Example 5 - and some new transform pairs.

$$F_6(s) = \frac{s^2 + 2s + 2}{s + 1}$$

(Quick solution: [Wolfram|Alpha \(http://www.wolframalpha.com/input/?i=inverse+laplace+transform+%28s%5E2+%2B+2s+%2B+2%29%2F%28s+%2B+1%29\)](http://www.wolframalpha.com/input/?i=inverse+laplace+transform+%28s%5E2+%2B+2s+%2B+2%29%2F%28s+%2B+1%29))

Dividing  $s^2 + 2s + 2$  by  $s + 1$  gives

$$F_6(s) = s + 1 + \frac{1}{s + 1}$$

$$\frac{1}{s + 1} \Leftrightarrow e^{-t}$$

$$1 \Leftrightarrow \delta(t)$$

$$s \Leftrightarrow ?$$

## What function of $t$ has Laplace transform $s$ ?

Recall from Session 2:

$$\frac{d}{dt}u_0(t) = u'_0(t) = \delta(t)$$

and

$$\frac{d^2}{dt^2}u_0(t) = u''_0(t) = \delta'(t)$$

Also, by the time differentiation property

$$u''_0(t) = \delta'(t) \Leftrightarrow s^2 \mathcal{L}u_0(t) - su_0(0) - \frac{d}{dt}u_0(t)\Big|_{t=0} = s^2 \frac{1}{s} = s$$

## New Transform Pairs

$$s \Leftrightarrow \delta'(t)$$

$$\frac{d^n}{dt^n}\delta(t) \Leftrightarrow s^n$$

$$f_6(t) = e^{-t} + \delta(t) + \delta'(t)$$

## Matlab verification

In [ ]:

```
%%matlab
Ns = [1, 2, 2]; Ds = [1 1];
[r, p, k] = residue(Ns, Ds)
```

In [ ]:

```
%%matlab
syms s;
F6 = (s^2 + 2*s + 2)/(s + 1);
f6 = ilaplace(F6)
```

## Homework

Do the end of the chapter exercises (Section 3.67) from the textbook. Don't look at the answers until you have attempted the problems.

## Lab Work

In the lab, on Monday, we will explore the tools provided by Matlab for taking Laplace transforms, representing polynomials, finding roots and factorizing polynomials and solution of inverse laplace transform problems.