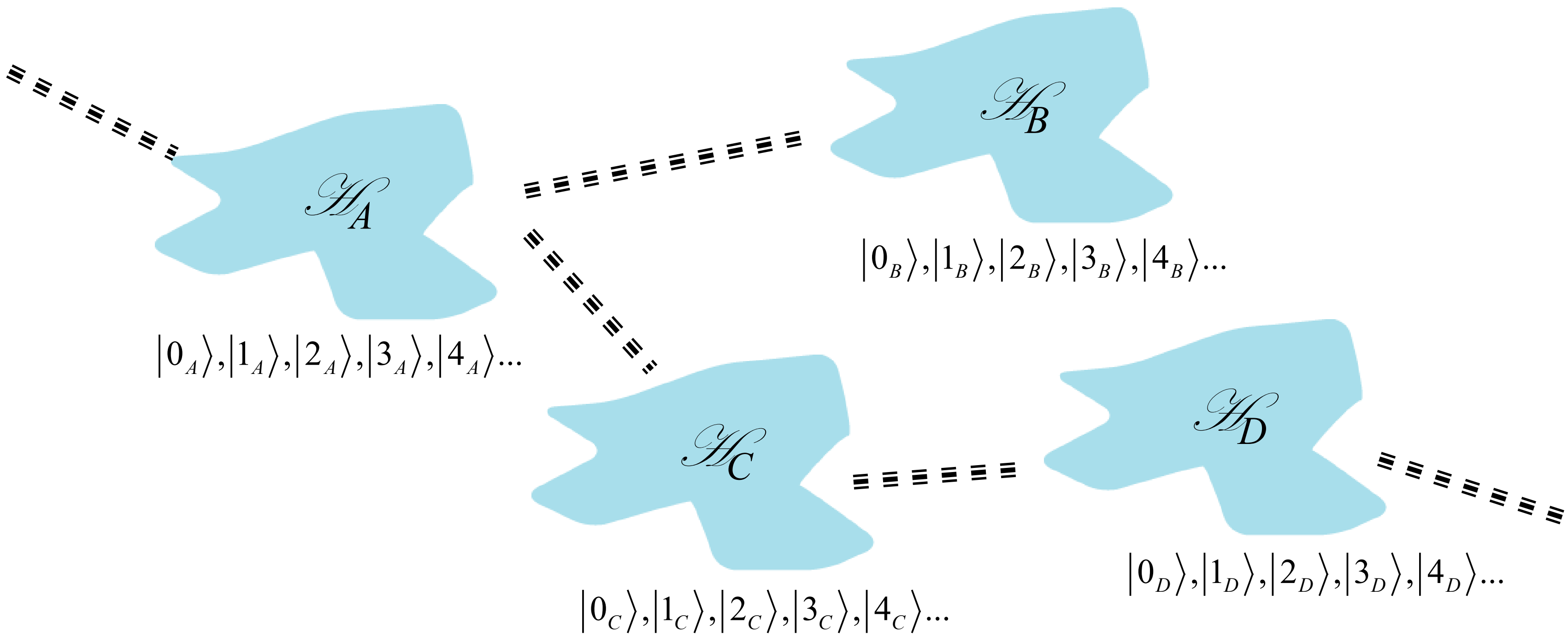
Understanding Entanglement

- Entanglement and separability
- Schmidt decomposition.
- Acín decomposition.
- The reduced density matrix.
  - Von Neumann-Shannon Entropy
- Tangle and Concurrence entanglement measures
- Local unitary operations.
- Transformations between entangled states.
  - Nielson
  - Jonathan and Plenio
- Entanglement concentration.
- Experimental entanglement.

N – particle state – shared between systems *A,B,C,D,….*



$$\boxed{|\psi_N\rangle = \sum_{i,j,k,l,\cdots} a_{i,j,k,l,\cdots} |i_A\rangle |j_B\rangle |k_C\rangle |l_D\rangle \cdots}$$

- For such a system we look at each subset of particles and ask if its state is separable from the state of all other particles.
  - If no subset is separable then the state is **fully entangled**.
  - If some subsets are separable then the state is **partially separable** and therefore **partially entangled.**
  - If all subsets are separable then the state is **separable**.

Example: 2 qubit states

- Determine if the state

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{23}}\left(|00\rangle + 3i|01\rangle - 3i|10\rangle + 2|11\rangle\right)}$$

is entangled.

- A general two-qubit state has the form

$$|\psi\rangle = \alpha_1|0\rangle|0\rangle + \alpha_2|0\rangle|1\rangle + \alpha_3|1\rangle|0\rangle + \alpha_4|1\rangle|1\rangle. \qquad (1)$$

- It is separable and therefore not entangled if it can be written as

$$|\psi\rangle = \left(a_1|0\rangle + b_1|1\rangle\right)\left(a_2|0\rangle + b_2|1\rangle\right),$$
$$= a_1a_2|0\rangle|0\rangle + a_1b_2|0\rangle|1\rangle + b_1a_2|1\rangle|0\rangle + b_1b_2|1\rangle|1\rangle. \quad (2)$$

- Comparing eqn (1) with eqn (2) yields 4 identities:

$$\alpha_1 = a_1a_2 = \frac{1}{\sqrt{23}} \qquad (3)$$

$$\alpha_2 = a_1b_2 = \frac{3i}{\sqrt{23}} \qquad (4)$$

$$\alpha_3 = b_1a_2 = \frac{-3i}{\sqrt{23}} \qquad (5)$$

$$\alpha_4 = b_1b_2 = \frac{2}{\sqrt{23}} \qquad (6)$$

- Dividing identity (3) by identity (4) and dividing identity (5) by identiy (6) we have

$$\frac{\alpha_1}{\alpha_2} = \frac{a_2}{b_2} = \frac{1}{3i} \qquad (7)$$

$$\frac{\alpha_3}{\alpha_4} = \frac{a_2}{b_2} = \frac{-3i}{2} \qquad (8)$$

- Since equations (7) and (8) are incompatible the state $|\psi\rangle$ cannot be separable and must therefore be entangled.

- Determine if the state

$$\boxed{|\psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)}$$

is entangled.

- For this state

$$\frac{1}{2} = a_1a_2,\ \frac{1}{2} = a_1b_2,\ \frac{1}{2} = b_1a_2,\ \frac{1}{2} = b_1b_2.$$

- These equations are compatible and give $1/\sqrt{2} = a_1 = a_2 = b_1 = b_2$ so that

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right).$$

is separable.

Note 1: $|ij\rangle = |i_2 j_1\rangle = |i_2\rangle|j_1\rangle = |j_1\rangle|i_2\rangle$, where numbers label particles. Note 2: $|i_2\rangle|j_1\rangle \neq |i_1\rangle|j_2\rangle$.

Example: 3 qubit state

- Determine if the state

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)}$$

is entangled.

- A general three-qubit state has the form

$$|\psi\rangle = \alpha_1|000\rangle + \alpha_2|001\rangle + \alpha_3|010\rangle + \alpha_4|011\rangle$$
$$+\alpha_5|100\rangle + \alpha_6|101\rangle + \alpha_7|110\rangle + \alpha_8|111\rangle \qquad (1)$$

- If $|\psi\rangle$ is fully separable $[111]$ it can be written as

$$|\psi\rangle = \left(a_1|0\rangle + b_1|1\rangle\right)\left(a_2|0\rangle + b_2|1\rangle\right)\left(a_3|0\rangle + b_3|1\rangle\right),$$
$$= a_1a_2a_3|0\rangle|0\rangle|0\rangle + a_1a_2b_3|0\rangle|0\rangle|1\rangle + a_1b_2a_3|0\rangle|1\rangle|0\rangle$$
$$+ a_1b_2b_3|0\rangle|1\rangle|1\rangle + b_1a_2a_3|1\rangle|0\rangle|0\rangle + b_1a_2b_3|1\rangle|0\rangle|1\rangle$$
$$+ b_1b_2a_3|1\rangle|1\rangle|0\rangle + b_1b_2b_3|1\rangle|1\rangle|1\rangle. \qquad (2)$$

Comparing (1) and (2) yields $2^3 = 8$ identities:

$$\frac{1}{\sqrt{2}} = a_1a_2a_3 = b_1b_2b_3 \qquad (3)$$

$$0 = a_1a_2b_3 = a_1b_2a_3 = a_1b_2b_3 = b_1a_2a_3 = b_1a_2b_3 = b_1b_2a_3 \qquad (4)$$

The identities (3) and (4) are incompatible.

Identities (3) require

$a_1 \neq 0, a_2 \neq 0, a_3 \neq 0, b_1 \neq 0, b_2 \neq 0, b_3 \neq 0.$

Identities (4) require some of the coefficients a_i, b_i to be zero.

$|\psi\rangle$ and is therefore not fully separable.

- If $|\psi\rangle$ is partially separable $[12]$ for particle permutation 123, it can be written as

$$|\psi\rangle = \left(a_1|0\rangle + b_1|1\rangle\right)\left(\beta_1|00\rangle + \beta_2|01\rangle + \beta_3|10\rangle + \beta_4|11\rangle\right),$$
$$= a_1\beta_1|0\rangle|00\rangle + a_1\beta_2|0\rangle|01\rangle + a_1\beta_3|0\rangle|10\rangle$$
$$+ a_1\beta_4|0\rangle|11\rangle + b_1\beta_1|1\rangle|00\rangle + b_1\beta_2|1\rangle|01\rangle$$
$$+ b_1\beta_3|1\rangle|10\rangle + b_1\beta_4|1\rangle|11\rangle. \qquad (5)$$

Comparing (1) and (5) yields 8 identities:

$$\frac{1}{\sqrt{2}} = a_1\beta_1 = b_1\beta_4 \qquad (6)$$

$$0 = a_1\beta_2 = a_1\beta_3 = a_1\beta_4 = b_1\beta_1 = b_1\beta_2 = b_1\beta_3 \qquad (7)$$

The identities (6) and (7) are incompatible. Identities (6) require $a_1 \neq 0, \beta_1 \neq 0, b_1 \neq 0, \beta_4 \neq 0.$ Identities (7) require either either, a_1 or β_4, or, b_1 or β_1, be zero.

- The state is therefore not $[12]$ separable for particle permutation 123. It can be shown that the same is also true for particle permutations 213 and 312. $|\psi\rangle$ and is therefore not partially separable.
- Since $|\psi\rangle$ is not fully separable and it is not partially separable, it must be fully entangled.
- Test your understanding using

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{8}}\left(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\right).}$$

Note 1: $|ijk\rangle = |i_3 j_2 k_1\rangle = |i_3\rangle|j_2\rangle|k_1\rangle = |k_1\rangle|j_2\rangle|i_3\rangle$ etc. Note 2: $|i_3\rangle|j_2\rangle|k_1\rangle \neq |i_3\rangle|j_1\rangle|k_2\rangle$ etc.

Schmidt Decomposition: bipartite separability.

- A general bipartite state has the form:

$$\boxed{|\psi_{A,B}\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N} a_{i,j}|i_A\rangle|j_B\rangle,}$$

$a:\ M\times N$ matrix with $\boxed{M \geq N}$ w.l.o.g.

where $|i_A\rangle$ and $|j_B\rangle$ are basis states for systems A and B.

- We will show it can be written in the form:

$$\boxed{|\psi_{A,B}\rangle = \sum_{k=1}^{N}\sqrt{\lambda_k}\,|k'_A\rangle|k'_B\rangle,}$$

$\sqrt{\lambda_k}$ - Schmidt Coefficients.
$|k'_A\rangle, |k'_B\rangle$ - Schmidt Basis States.

where $|k'_A\rangle$ and $|k'_B\rangle$ are are linear combinations of $|i_A\rangle$ and $|j_B\rangle$ respectively.

- We do this by decomposing the matrix a as:

$$\boxed{a = UdV^{\dagger},}$$

$U: M\times N$, matrix
$d: N\times N$, diagonal
$V^{\dagger}: N\times N$, unitary

where V is a unitary matrix and d is a diagonal matrix with positive or zero entries.

- This type of decomposition is known as ***singular-value decomposition***.

Schmidt Decomposition: bipartite separability.

- Proof : $\boxed{a = UdV^{\dagger}}$ $\boxed{a:\ M \times N \text{ matrix}}$

- We define a Hermitian $N \times N$ matrix H such that :

$$H = a^{\dagger} a, \qquad \boxed{H : N \times N \text{ matrix}}$$

$$H^{\dagger} = a^{\dagger} a$$

$$\Rightarrow \quad H = H^{\dagger}$$

- Eigenvalues and Eigenvectors of H:

$$\boxed{H v_i = \lambda_i v_i} \qquad \boxed{\begin{array}{l} i = 1, \ldots, N \\ \lambda_i :\ N \text{ eigenvalues,} \\ v_i :\ N \text{ eigenvectors.} \end{array}}$$

- The eigenvalues of H are real and either positive or zero:

$$\left|a v_i\right|^2 = (a v_i)^{\dagger} . a v_i \geq 0$$

$$= v_i^{\dagger} a^{\dagger} a v_i \geq 0$$

$$= v_i^{\dagger} \lambda_i v_i \geq 0$$

$$= \lambda_i \geq 0$$

- We now define a set of vectors u_i : $\boxed{u_i = \dfrac{a v_i}{\sqrt{\lambda_i}}}$

Note 1: if $\lambda_i = 0$ the vector u_i does not need to be defined since it only appears in the decomposition of a multiplied by $\lambda_i = 0$.
Note 2: the vectors u_i are of length M.

Schmidt Decomposition: bipartite separability.

- Writing v_i and u_i as column vectors we can define the matrices V and U:

$$V = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ v_1 & v_2 & \cdot & \cdot & v_N \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad U = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_1 & u_2 & \cdot & \cdot & u_N \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

- Using $\sqrt{\lambda_i}$ as elements we can define a diagonal matrix d:

$$d = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\lambda_N} \end{pmatrix}$$

- $\boxed{\begin{array}{l} U : M \times N, \text{matrix} \\ d : N \times N, \text{diagonal} \\ V^\dagger : N \times N, \text{unitary} \end{array}}$

- Using V, U and d, the results from the previous slides can be written in matrix form:

$$\boxed{Hv_i = \lambda_i v_i \quad \Rightarrow \quad HV = Vd^2}$$

$$\boxed{u_i = \frac{av_i}{\sqrt{\lambda_i}} \quad \Rightarrow \quad U = aVd^{-1}} \quad *$$

Note: if $\lambda_i = 0$ we set, $d_i^{-1} = 0$. The vector u_i will not appear in the decomposition of a and therefore can be left empty in U or set to any length M vector, usually $u_i = 0$.

- Inverting the relationship *, we find: $\boxed{a = UdV^{-1}.}$

- Since H is Hermitian, V is unitary, $V^{-1} = V^\dagger$, hence:

$$\boxed{a = UdV^\dagger.}$$

Schmidt Decomposition: bipartite separability.

- Finally, putting everything together : $\boxed{a = UdV^{\dagger}}$

$$\boxed{\begin{aligned}
\left|\psi_{A,B}\right\rangle &= \sum_{i=1}^{M}\sum_{j=1}^{N} a_{i,j}\left|i_A\right\rangle\left|j_B\right\rangle \qquad\qquad M \geq N \\
&= \sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{N} U_{i,k} d_k V^{\dagger}_{k,j}\left|i_A\right\rangle\left|j_B\right\rangle \\
&= \sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{N} U_{i,k} d_k V^{*}_{j,k}\left|i_A\right\rangle\left|j_B\right\rangle \\
&= \sum_{k=1}^{N} d_k \left(\sum_{i=1}^{M} U_{i,k}\left|i_A\right\rangle\right)\left(\sum_{j=1}^{N} V^{*}_{j,k}\left|j_B\right\rangle\right) \\
&= \sum_{k=1}^{N} d_k \left|k'_A\right\rangle\left|k'_B\right\rangle \\
&= \sum_{k=1}^{N} \sqrt{\lambda_k}\left|k'_A\right\rangle\left|k'_B\right\rangle
\end{aligned}}$$

- We use these definitions:

$$\boxed{\begin{aligned}
\left|k'_A\right\rangle &= \sum_{i=1}^{M} U_{i,k}\left|i_A\right\rangle \\
&= \sum_{i=1}^{M} \left(u_k\right)_i \left|i_A\right\rangle \\
\left|k'_B\right\rangle &= \sum_{j=1}^{N} V^{*}_{j,k}\left|j_B\right\rangle \\
&= \sum_{j=1}^{N} \left(v^{*}_k\right)_j \left|j_B\right\rangle
\end{aligned}}$$

$$V = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ v_1 & v_2 & \cdot & \cdot & v_N \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \Rightarrow V_{i,k} = \left(v_k\right)_i$$

$$U = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_1 & u_2 & \cdot & \cdot & u_N \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \Rightarrow U_{i,k} = \left(u_k\right)_i$$

Schmidt Decomposition: bipartite separability.

- In summary:

$$\left|\psi_{A,B}\right\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N} a_{i,j}\left|i_A\right\rangle\left|j_B\right\rangle \qquad M \geq N$$

$$\rightarrow$$

$$\left|\psi_{A,B}\right\rangle = \sum_{k=1}^{N}\sqrt{\lambda_k}\left|k'_A\right\rangle\left|k'_B\right\rangle$$

$$a^{\dagger}av_i = \lambda_i v_i$$

$$u_i = \frac{av_i}{\sqrt{\lambda_i}}, \qquad u_i = arbitrary \text{ if } \lambda_i = 0$$

$$\left|k'_A\right\rangle = \sum_{i=0}^{M-1}\left(u_k\right)_i\left|i_A\right\rangle$$

$$\left|k'_B\right\rangle = \sum_{j=0}^{N-1}\left(v_k^*\right)_j\left|j_B\right\rangle$$

- $\sqrt{\lambda_k}$ - Schmidt Coefficients.
  $\left|k'_A\right\rangle, \left|k'_B\right\rangle$ - Schmidt Basis States.

- The ***Schmidt number*** is the number of **non-zero** eigenvalues λ_k.

- The state is ***entangled*** if more than one eigenvalue is non-zero.

- If one of the eigenvalues is nearly equal to unity and the others are all near zero, the state is said to be ***weakly entangled***.

Schmidt Decomposition: Example I

- Consider the state: $\left|\psi_{A,B}\right\rangle = \frac{1}{2}\left(\left|\uparrow_A\right\rangle\left|\uparrow_B\right\rangle + \left|\uparrow_A\right\rangle\left|\downarrow_B\right\rangle + \left|\downarrow_A\right\rangle\left|\uparrow_B\right\rangle - \left|\downarrow_A\right\rangle\left|\downarrow_B\right\rangle\right).$

- For this state:

$$a = a^\dagger = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

$$H = a^\dagger a = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

- Eigenvectors and eigenvalues:

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \lambda_1 = \frac{1}{2}.$$

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \lambda_2 = \frac{1}{2}.$$

Note: because H is diagonal, we could have used **any pair** of orthogonal vectors for v_1 and v_2.

Schmidt Decomposition: Example I

- Find u from v :

$$\lambda_1=\frac{1}{2},\ v_1=\begin{pmatrix}1\\0\end{pmatrix},\ u_1=\frac{av_1}{\sqrt{\lambda_1}}=\frac{1}{\sqrt{1/2}}\begin{pmatrix}1/2 & 1/2\\1/2 & -1/2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},$$

$$\lambda_2=\frac{1}{2},\ v_2=\begin{pmatrix}0\\1\end{pmatrix},\ u_2=\frac{av_2}{\sqrt{\lambda_2}}=\frac{1}{\sqrt{1/2}}\begin{pmatrix}1/2 & 1/2\\1/2 & -1/2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

- Schmidt vectors:

$$\left|1'_A\right\rangle=(u_1)_1\left|\uparrow_A\right\rangle+(u_1)_2\left|\downarrow_A\right\rangle=\frac{1}{\sqrt{2}}\left|\uparrow_A\right\rangle+\frac{1}{\sqrt{2}}\left|\downarrow_A\right\rangle,$$

$$\left|2'_A\right\rangle=(u_2)_1\left|\uparrow_A\right\rangle+(u_2)_2\left|\downarrow_A\right\rangle=\frac{1}{\sqrt{2}}\left|\uparrow_A\right\rangle-\frac{1}{\sqrt{2}}\left|\downarrow_A\right\rangle,$$

$$\left|1'_B\right\rangle=\left(v_1^*\right)_1\left|\uparrow_B\right\rangle+\left(v_1^*\right)_2\left|\downarrow_B\right\rangle=\left|\uparrow_B\right\rangle,$$

$$\left|2'_B\right\rangle=\left(v_2^*\right)_1\left|\uparrow_B\right\rangle+\left(v_2^*\right)_2\left|\downarrow_B\right\rangle=\left|\downarrow_B\right\rangle.$$

$$\left|k'_A\right\rangle=\sum_{i=1}^{2}(u_k)_i\left|i_A\right\rangle$$

$$\left|k'_B\right\rangle=\sum_{j=1}^{2}\left(v_k^*\right)_j\left|j_B\right\rangle$$

- The Schmidt form is:

$$\begin{aligned}\left|\psi_{A,B}\right\rangle&=\sum_{k=1}^{2}\sqrt{\lambda_k}\left|k'_A\right\rangle\left|k'_B\right\rangle,\\&=\frac{1}{\sqrt{2}}\left|1'_A\right\rangle\left|1'_B\right\rangle+\frac{1}{\sqrt{2}}\left|2'_A\right\rangle\left|2'_B\right\rangle,\\&=\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\left|\uparrow_A\right\rangle+\left|\downarrow_A\right\rangle\right)\left|\uparrow_B\right\rangle+\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\left|\uparrow_A\right\rangle-\left|\downarrow_A\right\rangle\right)\left|\downarrow_B\right\rangle.\end{aligned}$$

Schmidt Decomposition: Example II

$$|\psi_{AB}\rangle = (0.1+0.4i)|0_A\rangle|0_B\rangle + 0.2|1_A\rangle|0_B\rangle + (0.4+0.5i)|2_A\rangle|0_B\rangle + 0.2|0_A\rangle|1_B\rangle + (0.4+0.1i)|1_A\rangle|1_B\rangle + 0.41231|2_A\rangle|1_B\rangle$$

$$a = \begin{pmatrix} 0.1+0.4i & 0.2 \\ 0.2 & 0.4+0.1i \\ 0.4+0.5i & 0.41231 \end{pmatrix}, \; a^\dagger = \begin{pmatrix} 0.1-0.4i & 0.2 & 0.4-0.5i \\ 0.2 & 0.4-0.1i & 0.41231 \end{pmatrix}$$

$$\boxed{\begin{aligned} M &= 3, \\ N &= 2. \end{aligned}}$$

$$\boxed{0.41231\ldots = \sqrt{0.17}}$$

$$H = a^\dagger a = \begin{pmatrix} 0.62 & 0.2649..-0.2661..i \\ 0.2649..+0.2661..i & 0.38 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.5697..+0.5723..i & 0.4160..+0.4179..i \\ 0.5897.. & -0.8075.. \end{pmatrix}, \quad V^\dagger = V^{-1} = \begin{pmatrix} 0.5697..-0.5723..i & 0.5897.. \\ 0.4160..-0.4179..i & -0.8075.. \end{pmatrix}$$

$$d = \begin{pmatrix} 0.9456.. & 0 \\ 0 & 0.3253.. \end{pmatrix}$$

$$U = aVd^{-1} = \begin{pmatrix} -0.0571.. + 0.3015..i & -0.8828.. + 0.6402..i \\ 0.3699.. + 0.1834..i & -0.7374.. + 0.0087..i \\ 0.1954.. + 0.5433..i & -1.1547.. + 1.1537..i \end{pmatrix}$$

$$|0'_A\rangle = \sum_{i=0}^{M-1} (u_0)_i |i_A\rangle = (-0.0571.. + 0.3015..i)|0_A\rangle + (0.3699.. + 0.1834..i)|1_A\rangle + (0.1954.. + 0.5433..i)|2_A\rangle$$

$$|1'_A\rangle = \sum_{i=0}^{M-1} (u_1)_i |i_A\rangle = (-0.8828.. + 0.6402..i)|0_A\rangle + (-0.7374.. + 0.0087..i)|1_A\rangle + (-1.1547.. + 1.1537..i)|2_A\rangle$$

$$|0'_B\rangle = \sum_{j=0}^{N-1} (v_0^*)_j |j_B\rangle = (0.5697..-0.5723..i)|0_B\rangle + 0.5897..|1_B\rangle$$

$$|1'_B\rangle = \sum_{j=0}^{N-1} (v_1^*)_j |j_B\rangle = (0.4160..-0.4179..i)|0_B\rangle - 0.8075..|1_B\rangle$$

$$|\psi_{AB}\rangle = 0.9456..|0'_A\rangle|0'_B\rangle + 0.3253..|1'_A\rangle|1'_B\rangle$$

Schmidt Decomposition: Example III

$$\left|\psi_{A,B}\right\rangle = \alpha\left|\uparrow_A\right\rangle\left|\uparrow_B\right\rangle + \beta\left|\uparrow_A\right\rangle\left|\downarrow_B\right\rangle + \gamma\left|\downarrow_A\right\rangle\left|\uparrow_B\right\rangle + \delta\left|\downarrow_A\right\rangle\left|\downarrow_B\right\rangle$$

- Hence:

$$H = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} |\alpha|^2 + |\gamma|^2 & \alpha^*\beta + \gamma^*\delta \\ \alpha\beta^* + \gamma\delta^* & |\beta|^2 + |\delta|^2 \end{pmatrix}$$

- Schmidt vectors:

$$\left|0_A\right\rangle = a\left|\uparrow_A\right\rangle + b^*\left|\downarrow_A\right\rangle, \qquad \left|1_A\right\rangle = b\left|\uparrow_A\right\rangle - a^*\left|\downarrow_A\right\rangle$$

$$\left|0_B\right\rangle = c\left|\uparrow_B\right\rangle + d^*\left|\downarrow_B\right\rangle, \qquad \left|1_B\right\rangle = d\left|\uparrow_B\right\rangle - c^*\left|\downarrow_B\right\rangle$$

- Schmidt coefficients: $\lambda_1 = \cos^2\theta \quad \lambda_2 = \sin^2\theta$

- The Schmidt form is:

$$\left|\psi_{A,B}\right\rangle = \cos\theta\left|0_A\right\rangle\left|0_B\right\rangle + \sin\theta\left|1_A\right\rangle\left|1_B\right\rangle \qquad 0 \le \theta \le \pi/4$$

$\theta = 0$ $\quad \left|\psi_{A,B}\right\rangle = \left|0_A\right\rangle\left|0_B\right\rangle$ Separable.

$\theta < \frac{\pi}{4}$ $\quad \left|\psi_{A,B}\right\rangle = \frac{1}{\sqrt{1+\varepsilon^2}}\left(\left|0_A\right\rangle\left|0_B\right\rangle + \varepsilon\left|1_A\right\rangle\left|1_B\right\rangle\right)$ Weak entanglement.

$\theta = \frac{\pi}{4}$ $\quad \left|\psi_{A,B}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_A\right\rangle\left|0_B\right\rangle + \left|1_A\right\rangle\left|1_B\right\rangle\right)$ Maximally entangled.

Schmidt Decomposition: Example IV

Schmidt decomposition can be used to check for separability of many-particle states. For a given partition of the particle number $[n_1, n_2, \cdots]$, it is necessary to test for each particle permutation.

For example, consider the state:

$$\left|\Psi_4\right\rangle = \frac{1}{2}\left|0_1 0_2 1_3 1_4\right\rangle - \frac{1}{2}\left|0_1 1_2 0_3 1_4\right\rangle - \frac{1}{2}\left|1_1 0_2 1_3 0_4\right\rangle + \frac{1}{2}\left|1_1 1_2 0_3 0_4\right\rangle.$$

Test for [1,3] separability:

$$\begin{aligned}\left|\Psi_4\right\rangle &= \frac{1}{2}\left|0_1 0_2 1_3 1_4\right\rangle - \frac{1}{2}\left|0_1 1_2 0_3 1_4\right\rangle - \frac{1}{2}\left|1_1 0_2 1_3 0_4\right\rangle + \frac{1}{2}\left|1_1 1_2 0_3 0_4\right\rangle,\\ &= \frac{1}{2}\left|03\right\rangle - \frac{1}{2}\left|05\right\rangle - \frac{1}{2}\left|12\right\rangle + \frac{1}{2}\left|14\right\rangle,\\ &= -\frac{1}{2}\left|12\right\rangle + \frac{1}{2}\left|03\right\rangle + \frac{1}{2}\left|14\right\rangle - \frac{1}{2}\left|05\right\rangle.\end{aligned}$$

$$a = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \\ 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad H = a^\dagger a = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

$SVD \;\; d = \left(1/\sqrt{2}, 1/\sqrt{2}\right) \;\Rightarrow$ not [1,3] separable for permutation 1,2,3,4.

$$\begin{aligned}\left|\Psi_4\right\rangle &= \frac{1}{2}\left|0_2 0_1 1_3 1_4\right\rangle - \frac{1}{2}\left|1_2 0_1 0_3 1_4\right\rangle - \frac{1}{2}\left|0_2 1_1 1_3 0_4\right\rangle + \frac{1}{2}\left|1_2 1_1 0_3 0_4\right\rangle,\\ &= \frac{1}{2}\left|03\right\rangle - \frac{1}{2}\left|11\right\rangle - \frac{1}{2}\left|06\right\rangle + \frac{1}{2}\left|14\right\rangle.\end{aligned}$$

$$a = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \\ 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad H = a^\dagger a = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

$SVD \;\; d = \left(1/\sqrt{2}, 1/\sqrt{2}\right) \;\Rightarrow$ not [1,3] separable for permutation 2,1,3,4.

$$\begin{aligned}\left|\Psi_4\right\rangle &= \frac{1}{2}\left|1_3 0_1 0_2 1_4\right\rangle - \frac{1}{2}\left|0_3 0_1 1_2 1_4\right\rangle - \frac{1}{2}\left|1_3 1_1 0_2 0_4\right\rangle + \frac{1}{2}\left|0_3 1_1 1_2 0_4\right\rangle,\\ &= \frac{1}{2}\left|11\right\rangle - \frac{1}{2}\left|03\right\rangle - \frac{1}{2}\left|14\right\rangle + \frac{1}{2}\left|06\right\rangle.\end{aligned}$$

$$a = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \\ 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}, \quad H = a^\dagger a = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

$SVD \;\; d = \left(1/\sqrt{2}, 1/\sqrt{2}\right) \;\Rightarrow$ not [1,3] separable for permutation 3,1,2,4.

$$\begin{aligned}\left|\Psi_4\right\rangle &= \frac{1}{2}\left|1_4 0_1 0_2 1_3\right\rangle - \frac{1}{2}\left|1_4 0_1 1_2 0_3\right\rangle - \frac{1}{2}\left|0_4 1_1 0_2 1_3\right\rangle + \frac{1}{2}\left|0_4 1_1 1_2 0_3\right\rangle,\\ &= \frac{1}{2}\left|11\right\rangle - \frac{1}{2}\left|12\right\rangle - \frac{1}{2}\left|05\right\rangle + \frac{1}{2}\left|06\right\rangle.\end{aligned}$$

$$a = \begin{pmatrix} 0 & 1/2 \\ 0 & -1/2 \\ -1/2 & 0 \\ 1/2 & 0 \end{pmatrix}, \quad H = a^\dagger a = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

$SVD \;\; d = \left(1/\sqrt{2}, 1/\sqrt{2}\right) \;\Rightarrow$ not [1,3] separable for permutation 4,1,2,3.

Test for [2,2] seperability:

$$\begin{aligned}\left|\Psi_4\right\rangle &= \frac{1}{2}\left|0_1 0_2 1_3 1_4\right\rangle - \frac{1}{2}\left|0_1 1_2 0_3 1_4\right\rangle - \frac{1}{2}\left|1_1 0_2 1_3 0_4\right\rangle + \frac{1}{2}\left|1_1 1_2 0_3 0_4\right\rangle,\\ &= \frac{1}{2}\left|03\right\rangle - \frac{1}{2}\left|11\right\rangle - \frac{1}{2}\left|22\right\rangle + \frac{1}{2}\left|30\right\rangle.\end{aligned}$$

$$a = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad H = a^\dagger a = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}.$$

$SVD \;\; d = (1/2, 1/2, 1/2, 1/2) \;\Rightarrow$ not [2,2] separable for permutation 1,2,3,4.

$$\begin{aligned}\left|\Psi_4\right\rangle &= \frac{1}{2}\left|0_1 1_3 0_2 1_4\right\rangle - \frac{1}{2}\left|0_1 0_3 1_2 1_4\right\rangle - \frac{1}{2}\left|1_1 1_3 0_2 0_4\right\rangle + \frac{1}{2}\left|1_1 0_3 1_2 0_4\right\rangle,\\ &= \frac{1}{2}\left|11\right\rangle - \frac{1}{2}\left|03\right\rangle - \frac{1}{2}\left|30\right\rangle + \frac{1}{2}\left|22\right\rangle.\end{aligned}$$

$$a = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad H = a^\dagger a = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}.$$

$SVD \;\; d = (1/2, 1/2, 1/2) \;\Rightarrow$ not [2,2] separable for permutation 1,3,2,4.

$$\begin{aligned}\left|\Psi_4\right\rangle &= \frac{1}{2}\left|0_1 1_4 0_2 1_3\right\rangle - \frac{1}{2}\left|0_1 1_4 1_2 0_3\right\rangle - \frac{1}{2}\left|1_1 0_4 0_2 1_3\right\rangle + \frac{1}{2}\left|1_1 0_4 1_2 0_3\right\rangle,\\ &= \frac{1}{2}\left|11\right\rangle - \frac{1}{2}\left|12\right\rangle - \frac{1}{2}\left|21\right\rangle + \frac{1}{2}\left|22\right\rangle.\end{aligned}$$

$$a = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad H = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$SVD \;\; d = (1, 0) \;\Rightarrow$ [2,2] separable for permutation 1,4,2,3.

$$\boxed{0 = 000,\; 1 = 001,\; 2 = 010,\; 3 = 011,\; 4 = 100,\; 5 = 101,\; 6 = 110,\; 7 = 111.}$$

Schmidt Decomposition: Example V

Decompose the state:

$$|\psi\rangle = 1/\sqrt{69}\left(|0_1 0_2 0_3 0_4\rangle + |0_1 0_2 0_3 1_4\rangle + 3i|0_1 0_2 1_3 0_4\rangle + 3i|0_1 0_2 1_3 1_4\rangle + |0_1 1_2 0_3 0_4\rangle + 3i|0_1 1_2 1_3 0_4\rangle - 3i|1_1 0_2 0_3 0_4\rangle - 3i|1_1 0_2 0_3 1_4\rangle + 2|1_1 0_2 1_3 0_4\rangle + 2|1_1 0_2 1_3 1_4\rangle - 3i|1_1 1_2 0_3 0_4\rangle + 2|1_1 1_2 1_3 0_4\rangle\right).$$

Check [2,2] separability permutation 1,2,3,4:

$$|\psi\rangle = 1/\sqrt{69}\left(|0,0\rangle + |0,1\rangle + 3i|0,2\rangle + 3i|0,3\rangle + |1,0\rangle + 3i|1,2\rangle - 3i|2,0\rangle - 3i|2,1\rangle + 2|2,2\rangle + 2|2,3\rangle - 3i|3,0\rangle + 2|3,2\rangle\right).$$

$$a = \frac{1}{\sqrt{69}}\begin{pmatrix} 1 & 1 & 3i & 3i \\ 1 & 3i & 0 & 0 \\ -3i & -3i & 2 & 2 \\ -3i & 0 & 2 & 0 \end{pmatrix}, \quad H = a^\dagger a = \frac{1}{69}\begin{pmatrix} 20 & 10+3i & 15i & 9i \\ 10-3i & 19 & 9i & 9i \\ -15i & -9i & 17 & 13 \\ -9i & -9i & 13 & 13 \end{pmatrix}.$$

$SVD \;\; d = (7.0932, 3.4267, 2.5317, 0.7184)$

$\Rightarrow$ not [2,2] separable for permutation 1,2,3,4.

Check [2,2] separability permutation 1,3,2,4:

$$|\psi\rangle = 1/\sqrt{69}\left(|0_1 0_3 0_2 0_4\rangle + |0_1 0_3 0_2 1_4\rangle + |0_1 0_3 1_2 0_4\rangle + 3i|0_1 1_3 0_2 0_4\rangle + 3i|0_1 1_3 0_2 1_4\rangle + 3i|0_1 1_3 1_2 0_4\rangle - 3i|1_1 0_3 0_2 0_4\rangle - 3i|1_1 0_3 0_2 1_4\rangle - 3i|1_1 0_3 1_2 0_4\rangle + 2|1_1 1_3 0_2 0_4\rangle + 2|1_1 1_3 0_2 1_4\rangle + 2|1_1 1_3 1_2 0_4\rangle\right),$$

$$= 1/\sqrt{69}\left(|0,0\rangle + |0,1\rangle + |0,2\rangle + 3i|1,0\rangle + 3i|1,1\rangle + 3i|1,2\rangle - 3i|2,0\rangle - 3i|2,1\rangle - 3i|2,2\rangle + 2|3,0\rangle + 2|3,1\rangle + 2|3,2\rangle\right).$$

$$a = \frac{1}{\sqrt{69}}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 3i & 3i & 3i & 0 \\ -3i & -3i & -3i & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix}, \quad H = a^\dagger a = \frac{1}{69}\begin{pmatrix} 23 & 23 & 23 & 0 \\ 23 & 23 & 23 & 0 \\ 23 & 23 & 23 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$SVD \;\; d = (\sqrt{69}, 0, 0, 0)$

$\Rightarrow$ state is [2,2] separable for permutation 1,3,2,4.

Hence, we can write the [2,2] 1,3,2,4 permutation as:

$$|\psi\rangle = \left(|0_1\rangle|0_3\rangle, |0_1\rangle|1_3\rangle, |1_1\rangle|0_3\rangle, |1_1\rangle|1_3\rangle\right) U \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_3} & 0 \\ 0 & 0 & 0 & \sqrt{\lambda_4} \end{pmatrix} V^\dagger \begin{pmatrix} |0_2\rangle|0_4\rangle \\ |0_2\rangle|1_4\rangle \\ |1_2\rangle|0_4\rangle \\ |1_2\rangle|1_4\rangle \end{pmatrix},$$

$$= 1/\sqrt{69}\left(|0_1\rangle|0_3\rangle, |0_1\rangle|1_3\rangle, |1_1\rangle|0_3\rangle, |1_1\rangle|1_3\rangle\right) \begin{pmatrix} \frac{1}{\sqrt{23}} & -\frac{2}{\sqrt{5}} & -\frac{3i}{\sqrt{70}} & \frac{3i}{\sqrt{322}} \\ \frac{3i}{\sqrt{23}} & 0 & 0 & \sqrt{\frac{14}{23}} \\ -\frac{3i}{\sqrt{23}} & 0 & \sqrt{\frac{5}{14}} & \frac{9}{\sqrt{322}} \\ \frac{2}{\sqrt{23}} & \frac{1}{\sqrt{5}} & -3i\sqrt{\frac{2}{35}} & 3i\sqrt{\frac{2}{161}} \end{pmatrix}$$

$$\times \begin{pmatrix} \sqrt{69} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix} \begin{pmatrix} |0_2\rangle|0_4\rangle \\ |0_2\rangle|1_4\rangle \\ |1_2\rangle|0_4\rangle \\ |1_2\rangle|1_4\rangle \end{pmatrix},$$

$$= \frac{1}{\sqrt{69}}\left(|0_1\rangle|0_3\rangle + 3i|0_1\rangle|1_3\rangle - 3i|1_1\rangle|0_3\rangle + 2|1_1\rangle|1_3\rangle\right)\left(|0_2\rangle|0_4\rangle + |0_2\rangle|1_4\rangle + |1_2\rangle|0_4\rangle\right).$$

Neither of these states is further separable but they could be written in Schmidt form.

Acín decomposition of a three qubit state.

- A general three qubit state has the form :

$$\boxed{\begin{aligned}|\psi\rangle = a_{000}|0_A\rangle|0_B\rangle|0_C\rangle + a_{001}|0_A\rangle|0_B\rangle|1_C\rangle + a_{010}|0_A\rangle|1_B\rangle|0_C\rangle + a_{011}|0_A\rangle|1_B\rangle|1_C\rangle \\ + a_{100}|1_A\rangle|0_B\rangle|0_C\rangle + a_{101}|1_A\rangle|0_B\rangle|1_C\rangle + a_{110}|1_A\rangle|1_B\rangle|0_C\rangle + a_{111}|1_A\rangle|1_B\rangle|1_C\rangle,\end{aligned}} \quad (1)$$

- There is no general transformation of (1) to the form,

$$\boxed{|\psi_{A,B,C}\rangle \neq \sqrt{\lambda_0}\,|0'_A\rangle|0'_B\rangle|0'_C\rangle + \sqrt{\lambda_1}\,|1'_A\rangle|1'_B\rangle|1'_C\rangle,}$$

- however, there is a general transformation the form :

$$\boxed{|\psi\rangle = \lambda_1|0'_A\rangle|0'_B\rangle|0'_C\rangle + |1'_A\rangle\left(\lambda_2 e^{i\varphi}|0'_B\rangle|0'_C\rangle + \lambda_3|0'_B\rangle|1'_C\rangle + \lambda_4|1'_B\rangle|0'_C\rangle + \lambda_5|1'_B\rangle|1'_C\rangle\right).}$$

$$\boxed{\lambda_i \geq 0, \quad \sum_{i=1}^{5}\lambda_i^2 = 1.}$$

- Acin *et al* Phys. Rev. Lett. **85** 1560 (2000).

Acín decomposition of a three qubit state.

Proof: A general three-qubit state has 8 complex coefficients:

$$
\begin{aligned}
|\psi\rangle &= a_{000}|0_A\rangle|0_B\rangle|0_C\rangle + a_{001}|0_A\rangle|0_B\rangle|1_C\rangle + a_{010}|0_A\rangle|1_B\rangle|0_C\rangle + a_{011}|0_A\rangle|1_B\rangle|1_C\rangle \\
&\qquad + a_{100}|1_A\rangle|0_B\rangle|0_C\rangle + a_{101}|1_A\rangle|0_B\rangle|1_C\rangle + a_{110}|1_A\rangle|1_B\rangle|0_C\rangle + a_{111}|1_A\rangle|1_B\rangle|1_C\rangle, \\
&= |0_A\rangle\left(|0_B\rangle,|1_B\rangle\right)\begin{pmatrix} a_{000} & a_{001} \\ a_{010} & a_{011} \end{pmatrix}\begin{pmatrix} |0_C\rangle \\ |1_C\rangle \end{pmatrix} + |1_A\rangle\left(|0_B\rangle,|1_B\rangle\right)\begin{pmatrix} a_{100} & a_{101} \\ a_{110} & a_{111} \end{pmatrix}\begin{pmatrix} |0_C\rangle \\ |1_C\rangle \end{pmatrix}.
\end{aligned}
$$

We start by performing a unitary transformation $u = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}$ on the A qubit:

$$
\begin{aligned}
|\psi\rangle &= \left(u_{00}|0'_A\rangle + u_{01}|1'_A\rangle\right)\left(|0_B\rangle,|1_B\rangle\right)\begin{pmatrix} a_{000} & a_{001} \\ a_{010} & a_{011} \end{pmatrix}\begin{pmatrix} |0_C\rangle \\ |1_C\rangle \end{pmatrix} + \left(u_{10}|0'_A\rangle + u_{11}|1'_A\rangle\right)\left(|0_B\rangle,|1_B\rangle\right)\begin{pmatrix} a_{100} & a_{101} \\ a_{110} & a_{111} \end{pmatrix}\begin{pmatrix} |0_C\rangle \\ |1_C\rangle \end{pmatrix}, \\
&= |0'_A\rangle\left(|0_B\rangle,|1_B\rangle\right)\begin{pmatrix} a'_{000} & a'_{001} \\ a'_{010} & a'_{011} \end{pmatrix}\begin{pmatrix} |0_C\rangle \\ |1_C\rangle \end{pmatrix} + |1'_A\rangle\left(|0_B\rangle,|1_B\rangle\right)\begin{pmatrix} a'_{100} & a'_{101} \\ a'_{110} & a'_{111} \end{pmatrix}\begin{pmatrix} |0_C\rangle \\ |1_C\rangle \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{pmatrix} a'_{000} & a'_{001} \\ a'_{010} & a'_{011} \end{pmatrix} = \begin{pmatrix} a_{000}u_{00} + a_{100}u_{10} & a_{001}u_{00} + a_{101}u_{10} \\ a_{010}u_{00} + a_{110}u_{10} & a_{011}u_{00} + a_{111}u_{10} \end{pmatrix} = A'_0, \qquad \begin{pmatrix} a'_{100} & a'_{101} \\ a'_{110} & a'_{111} \end{pmatrix} = \begin{pmatrix} a_{000}u_{01} + a_{100}u_{11} & a_{001}u_{01} + a_{101}u_{11} \\ a_{010}u_{01} + a_{110}u_{11} & a_{011}u_{01} + a_{111}u_{11} \end{pmatrix} = A'_1.
$$

We then write A'_0 in Schmidt form $A'_0 = \begin{pmatrix} a'_{000} & a'_{001} \\ a'_{010} & a'_{011} \end{pmatrix} = UdV^\dagger = U\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}V^\dagger$ and $A'_1 = \begin{pmatrix} a'_{100} & a'_{101} \\ a'_{110} & a'_{111} \end{pmatrix} = U\begin{pmatrix} a''_{100} & a''_{101} \\ a''_{110} & a''_{111} \end{pmatrix}V^\dagger$,

so that:

$$
\begin{aligned}
|\psi\rangle &= |0'_A\rangle\left(|0_B\rangle,|1_B\rangle\right)U\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}V^\dagger\begin{pmatrix} |0_C\rangle \\ |1_C\rangle \end{pmatrix} + |1'_A\rangle\left(|0_B\rangle,|1_B\rangle\right)U\begin{pmatrix} a''_{100} & a''_{101} \\ a''_{110} & a''_{111} \end{pmatrix}V^\dagger\begin{pmatrix} |0_C\rangle \\ |1_C\rangle \end{pmatrix}, \\
&= |0'_A\rangle\left(|0'_B\rangle,|1'_B\rangle\right)\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}\begin{pmatrix} |0'_C\rangle \\ |1'_C\rangle \end{pmatrix} + |1'_A\rangle\left(|0'_B\rangle,|1'_B\rangle\right)\begin{pmatrix} a''_{100} & a''_{101} \\ a''_{110} & a''_{111} \end{pmatrix}\begin{pmatrix} |0'_C\rangle \\ |1'_C\rangle \end{pmatrix}.
\end{aligned}
$$

Acín decomposition of a three qubit state.

In order to reduce the number of basis states we choose the coefficients in

$$u = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix} = \begin{pmatrix} e^{i\varsigma}\cos\theta & \sin\theta \\ \sin\theta & -e^{-i\varsigma}\cos\theta \end{pmatrix}$$

so that $\eta_2 = 0.$

This gives

$$|\psi\rangle = \eta_1 |0'_A\rangle |0'_B\rangle |0'_C\rangle + |1'_A\rangle \left(|0'_B\rangle, |1'_C\rangle\right) \begin{pmatrix} a''_{100} & a''_{101} \\ a''_{110} & a''_{111} \end{pmatrix} \begin{pmatrix} |0'_B\rangle \\ |1'_C\rangle \end{pmatrix}.$$

We can find the parameters θ and ς by solving the equation $\det\left(A'_0\right) = \eta_1\eta_2 = 0$:

$$0 = \det\left(A'_0\right) = \left(a_{000}u_{00} + a_{100}u_{10}\right)\left(a_{011}u_{00} + a_{111}u_{10}\right) - \left(a_{001}u_{00} + a_{101}u_{10}\right)\left(a_{010}u_{00} + a_{110}u_{10}\right),$$

$$0 = \left(a_{000}a_{011} - a_{001}a_{010}\right)u_{00}^2 + \left(a_{100}a_{011} + a_{000}a_{111} - a_{101}a_{010} - a_{001}a_{110}\right)u_{10}u_{00} + \left(a_{100}a_{111} - a_{101}a_{110}\right)u_{10}^2,$$

$$0 = \left(a_{000}a_{011} - a_{001}a_{010}\right)\left(e^{i\varsigma}\cos\theta\right)^2 + \left(a_{100}a_{011} + a_{000}a_{111} - a_{101}a_{010} - a_{001}a_{110}\right)e^{i\varsigma}\cos\theta\sin\theta$$
$$+ \left(a_{100}a_{111} - a_{101}a_{110}\right)\left(\sin\theta\right)^2,$$

Dividing by $\left(e^{i\varsigma}\cos\theta\right)^2$ we have:

$$\boxed{0 = \left(a_{000}a_{011} - a_{001}a_{010}\right) + \left(a_{100}a_{011} + a_{000}a_{111} - a_{101}a_{010} - a_{001}a_{110}\right)t + \left(a_{100}a_{111} - a_{101}a_{110}\right)t^2,} \qquad (*)$$

where $t = e^{-i\varsigma}\tan\theta.$

Since $(*)$ is a quadratic, it will have two complex solutions for t. We can use either root for a solution. From $|t|$ we can find θ and from $\arg(t)$ we can find ς. u and A'_0 can then be calculated.

Acín decomposition of a three qubit state.

Writting $\eta_1 = \lambda_1, a''_{100} = \lambda_2 e^{i\varphi_2}, a''_{101} = \lambda_3 e^{i\varphi_3}, a''_{110} = \lambda_4 e^{i\varphi_4}, a''_{111} = \lambda_5 e^{i\varphi_5}$ we have:

$$|\psi\rangle = \lambda_1 |0'_A\rangle|0'_B\rangle|0'_C\rangle + \lambda_2 e^{i\varphi_2} |1'_A\rangle|0'_B\rangle|0'_C\rangle + \lambda_3 e^{i\varphi_3} |1'_A\rangle|0'_B\rangle|1'_C\rangle + \lambda_4 e^{i\varphi_4} |1'_A\rangle|1'_B\rangle|0'_C\rangle + \lambda_5 e^{i\varphi_5} |1'_A\rangle|1'_B\rangle|1'_C\rangle.$$

We can remove the three phases $\varphi_3, \varphi_4, \varphi_5$ if we perform a unitary transformation on each qubit of the form: $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\chi} \end{pmatrix}$.

Under these transformations, the wave function becomes:

$$|\psi\rangle = \lambda_1 |0'_A\rangle|0'_B\rangle|0'_C\rangle + \lambda_2 e^{i(\varphi_2+\chi_A)} |1''_A\rangle|0'_B\rangle|0'_C\rangle + \lambda_3 e^{i(\varphi_3+\chi_A+\chi_C)} |1''_A\rangle|0'_B\rangle|1''_C\rangle$$
$$+ \lambda_4 e^{i(\varphi_4+\chi_A+\chi_B)} |1''_A\rangle|1''_B\rangle|0'_C\rangle + \lambda_5 e^{i(\varphi_5+\chi_A+\chi_B+\chi_C)} |1''_A\rangle|1''_B\rangle|1''_C\rangle.$$

Choosing the phases so that, $\chi_A = -\varphi_3 - \varphi_4 + \varphi_5$, $\chi_B = -\varphi_5 + \varphi_3$, $\chi_C = \varphi_4 - \varphi_5$ then,
$-\varphi_3 = \chi_A + \chi_C$, $-\varphi_4 = \chi_A + \chi_B$, $-\varphi_5 = \chi_A + \chi_B + \chi_C$.

Finally, we find :

$$\boxed{|\psi\rangle = \lambda_1 |0'_A\rangle|0'_B\rangle|0'_C\rangle + \lambda_2 e^{i\varphi} |1''_A\rangle|0'_B\rangle|0'_C\rangle + \lambda_3 |1''_A\rangle|0'_B\rangle|1''_C\rangle + \lambda_4 |1''_A\rangle|1''_B\rangle|0'_C\rangle + \lambda_5 |1''_A\rangle|1''_B\rangle|1''_C\rangle,}$$

where $\varphi = \varphi_2 - \varphi_3 - \varphi_4 + \varphi_5$.

Acín decomposition of a three qubit state.

Consider the state:

$$
\begin{aligned}
|\psi\rangle = {} & \frac{1}{2}\sqrt{\frac{5}{2}}|0_A\rangle|0_B\rangle|0_C\rangle + \frac{1}{2}\sqrt{\frac{1}{10}}|0_A\rangle|0_B\rangle|1_C\rangle + \frac{1}{2}\sqrt{\frac{1}{10}}|0_A\rangle|1_B\rangle|0_C\rangle + \frac{1}{2}\sqrt{\frac{1}{10}}|0_A\rangle|1_B\rangle|1_C\rangle \\
& - \frac{3}{2}\sqrt{\frac{1}{10}}|1_A\rangle|0_B\rangle|0_C\rangle + \frac{1}{2}\sqrt{\frac{1}{10}}|1_A\rangle|0_B\rangle|1_C\rangle + \frac{1}{2}\sqrt{\frac{1}{10}}|1_A\rangle|1_B\rangle|0_C\rangle + \frac{1}{2}\sqrt{\frac{1}{10}}|1_A\rangle|1_B\rangle|1_C\rangle.
\end{aligned}
$$

This defines $A_0 = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{5}{2}} & \frac{1}{2}\sqrt{\frac{1}{10}} \\ \frac{1}{2}\sqrt{\frac{1}{10}} & \frac{1}{2}\sqrt{\frac{1}{10}} \end{pmatrix}$, $A_1 = \begin{pmatrix} -\frac{3}{2}\sqrt{\frac{1}{10}} & \frac{1}{2}\sqrt{\frac{1}{10}} \\ \frac{1}{2}\sqrt{\frac{1}{10}} & \frac{1}{2}\sqrt{\frac{1}{10}} \end{pmatrix}$ so that,

$$
\begin{aligned}
0 &= \left(a_{100}a_{111} - a_{101}a_{110}\right)t^2 + \left(a_{100}a_{011} + a_{000}a_{111} - a_{101}a_{010} - a_{001}a_{110}\right)t + \left(a_{000}a_{011} - a_{001}a_{010}\right) \\
&= -\frac{1}{10}t^2 + \frac{1}{10} = -\frac{1}{10}(t-1)(t+1).
\end{aligned}
$$

We will use $t = 1 = e^{-i\zeta}\tan\theta$ so that $\zeta = 0,\ \theta = \pi/4$.

This gives $u = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$ so that $A_0' = \frac{1}{2}\sqrt{\frac{1}{5}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_1' = \frac{2}{\sqrt{5}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Carrying out SVD on A_0' we find $U = \begin{pmatrix} 1/\sqrt{2} & \bullet \\ 1/\sqrt{2} & \bullet \end{pmatrix}$, $V = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ applying these transformations

we find:

$$
|\psi\rangle = \sqrt{\frac{1}{5}}|0'_A\rangle|0'_B\rangle|0'_C\rangle + \sqrt{\frac{1}{5}}|1'_A\rangle|0'_B\rangle|0'_C\rangle + \sqrt{\frac{1}{5}}|1'_A\rangle|0'_B\rangle|1'_C\rangle + \sqrt{\frac{1}{5}}|1'_A\rangle|1'_B\rangle|0'_C\rangle + \sqrt{\frac{1}{5}}|1'_A\rangle|1'_B\rangle|1'_C\rangle.
$$

Check this result yourself. Try further examples on dropbox: Acin_Example.nb.

The reduced density matrix

- There are many situations where we can only measure or manipulate or have any knowledge of **one** of a pair of entangled particles A, B.
- Our knowledge of such systems can be quantified using the **reduced density matrix.**
- If *B* is the unknown system the reduced density matrix is defined as:

$$\boxed{\hat{\rho}_A = trace_B\left(\left|\psi_{A,B}\right\rangle\left\langle\psi_{A,B}\right|\right),}$$

where $trace_B$ is a partial trace over the states of system *B*.

- A similar expression exists if A is the unknown system:

$$\boxed{\hat{\rho}_B = trace_A\left(\left|\psi_{A,B}\right\rangle\left\langle\psi_{A,B}\right|\right).}$$

A general density matrix $\hat{\rho}$ is:

Positive Semidefinite

$$z^\dagger \hat{\rho} z \geq 0$$

for any $z \in \mathbb{C}^n, z \neq 0.$

Hermitian

$$\hat{\rho}^\dagger = \hat{\rho}$$

and has

$$trace(\hat{\rho}) = 1.$$

The reduced density matrix

- In the Schmidt basis:

$$\hat{\rho}_{A,B} = |\psi_{A,B}\rangle\langle\psi_{A,B}|,$$
$$= \left(\sum_i \sqrt{\lambda_i}\,|i_A\rangle|i_B\rangle\right)\left(\sum_j \langle j_B|\langle j_A|\sqrt{\lambda_j}\right),$$
$$= \sum_{i,j}\sqrt{\lambda_i\lambda_j}\,|i_A\rangle|i_B\rangle\langle j_B|\langle j_A|.$$

Schmidt basis:

$$|\psi_{A,B}\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N} a_{i,j}|i_A\rangle|j_B\rangle \qquad M \geq N$$
$$= \sum_{k=1}^{N}\sqrt{\lambda_k}\,|k'_A\rangle|k'_B\rangle$$

- The reduced density matrix for particle *A*:

$$\hat{\rho}_A = trace_B(\hat{\rho}_{A,B})$$
$$= \sum_k\sum_{i,j}\sqrt{\lambda_i\lambda_j}\,|i_A\rangle\langle k_B|i_B\rangle\langle j_B|k_B\rangle\langle j_A|$$
$$= \sum_k\sum_{i,j}\sqrt{\lambda_i\lambda_j}\,|i_A\rangle\delta_{k,i}\delta_{j,k}\langle j_A|$$
$$= \sum_k \lambda_k |k_A\rangle\langle k_A|$$

In information theory, **entropy is a measure of the uncertainty in a random variable.**
It is the average unpredictability in a random variable, which is equivalent to its information content.

Note: these are mixed states. i.e a sum of pure states $|k\rangle\langle k|$. This means that each is a mix of both classical and quantum information.

- The reduced density matrix for particle *B*:

$$\boxed{\hat{\rho}_B = \sum_k \lambda_k |k_B\rangle\langle k_B|}$$

Von Neumann-Shannon entropy

- The **von Neuman-Shannon entropy** is defined by:

$$\boxed{S(\hat{\rho}) = -trace\left(\hat{\rho}\log_2(\hat{\rho})\right).}$$

- It is invariant under unitary trans-formations of the density matrix of the form: $\boxed{\hat{\rho} \to U\hat{\rho}U^{\dagger}.}$

In information theory, **entropy is a measure of the uncertainty in a random variable.** It is the average unpredictability in a random variable, which is equivalent to its information content.

- Proof:

$$\begin{aligned} S\left(U\hat{\rho}U^{\dagger}\right) &= -trace\left(U\hat{\rho}U^{\dagger}\log_2\left(U\hat{\rho}U^{\dagger}\right)\right), \\ &= -trace\left(U\hat{\rho}U^{\dagger}U\log_2(\hat{\rho})U^{\dagger}\right), \\ &= -trace\left(U\hat{\rho}\log_2(\hat{\rho})U^{\dagger}\right), \\ &= -trace\left(\hat{\rho}\log_2(\hat{\rho})\right), \\ &= S(\hat{\rho}). \end{aligned}$$

Von Neuman-Shannon entropy: Example I

$$\boxed{S(\rho) = -trace(\rho \log_2 \rho).}$$

- The pure state:

$$\rho = |\psi\rangle\langle\psi|.$$

- In the $|\psi\rangle$ basis the density matrix is:

$$\rho = 1$$

- Hence, $S(\rho) = -trace(1.\log_2 1)$

$$= 0$$

In information theory, **entropy is a measure of the uncertainty in a random variable.** It is the average unpredictability in a random variable, which is equivalent to its information content.

The von Neuman entropy of a pure state is zero.

Von Neumann-Shannon entropy of the reduced density matrix.

- Reduced density matrix (Mixed state): $\rho_R = \sum_i \lambda_i |i\rangle\langle i|.$

$$\rho_R = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & \lambda_N \end{pmatrix}, \quad \log_2(\rho_R) = \begin{pmatrix} \log_2 \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \log_2 \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & \log_2 \lambda_N \end{pmatrix},$$

$$\rho_R \log_2(\rho_R) = \begin{pmatrix} \lambda_1 \log_2 \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 \log_2 \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & \lambda_N \log_2 \lambda_N \end{pmatrix}.$$

- Entropy of entanglement: $\boxed{S(\rho_R) = -\sum_i \lambda_i \log_2 \lambda_i.}$

In information theory, **entropy is a measure of the uncertainty in a random variable.** It is the average unpredictability in a random variable, which is equivalent to its information content.

Entropy of entanglement is used as a quantitative measure of entanglement for two-particle systems.

Von Neumann-Shannon entropy: Example II

$$\boxed{\left|\psi_{A,B}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow_A\right\rangle\left|\downarrow_B\right\rangle - \left|\downarrow_A\right\rangle\left|\uparrow_B\right\rangle\right)}$$

In information theory, **entropy is a measure of the uncertainty in a random variable.** It is the average unpredictability in a random variable, which is equivalent to its information content.

- Density matrix: $\hat{\rho} = \left|\psi_{A,B}\right\rangle\left\langle\psi_{A,B}\right|$

$$= \frac{1}{2}\left(\left|\uparrow_A\right\rangle\left|\downarrow_B\right\rangle - \left|\downarrow_A\right\rangle\left|\uparrow_B\right\rangle\right)\left(\left\langle\uparrow_B\right|\left\langle\downarrow_A\right| - \left\langle\downarrow_B\right|\left\langle\uparrow_A\right|\right)$$

- Reduced density matrix:

$$\hat{\rho}_A = trace_B\left(\left|\psi_{A,B}\right\rangle\left\langle\psi_{A,B}\right|\right) = \left\langle\uparrow_B\middle|\psi_{A,B}\right\rangle\left\langle\psi_{A,B}\middle|\uparrow_B\right\rangle + \left\langle\downarrow_B\middle|\psi_{A,B}\right\rangle\left\langle\psi_{A,B}\middle|\downarrow_B\right\rangle$$

$$= \frac{1}{2}\left(\left|\uparrow_A\right\rangle\left\langle\uparrow_A\right| + \left|\downarrow_A\right\rangle\left\langle\downarrow_A\right|\right)$$

- Hence,

$$\rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 2^{-1} & 0 \\ 0 & 2^{-1} \end{pmatrix}$$

$$\Rightarrow \rho_A \log_2 \rho_A = \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$\Rightarrow \qquad S\left(\rho_A\right) = -trace_A\left(\rho_A \log_2 \rho_A\right) = \frac{1}{2} + \frac{1}{2} = 1$$

Entropy of entanglement for a maximally entangled state.

Tangle entanglement measure.

- For N qubits, the 'residual entangement' or 'tangle' is defined as

$$\boxed{\begin{aligned}\tau(\psi) = 2\Big|\sum a_{\alpha_1\ldots\alpha_n} a_{\beta_1\ldots\beta_n} a_{\gamma_1\ldots\gamma_n} a_{\delta_1\ldots\delta_n} \\ \times \varepsilon_{\alpha_1\beta_1}\varepsilon_{\alpha_2\beta_2}\ldots\varepsilon_{\alpha_{n-1}\beta_{n-1}}\varepsilon_{\alpha_{n-1}\beta_{n-1}}\varepsilon_{\gamma_1\delta_1}\varepsilon_{\gamma_2\delta_2}\ldots \\ \times\varepsilon_{\gamma_{n-1}\delta_{n-1}}\varepsilon_{\gamma_{n-1}\delta_{n-1}}\varepsilon_{\alpha_n\gamma_n}\varepsilon_{\beta_n\delta_n}\Big|,\end{aligned}}$$

where the a terms are the coefficients in the standard basis $|\psi\rangle = \sum_{i_1,\ldots,i_N} a_{i_1\ldots i_N}|i_1\ldots i_N\rangle$

and $\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

- It is defined so that :
  - (1.) $0 \le \tau(\psi) \le 1$,
  - (2.) $\tau(\psi) = 0$ for (fully) separable states,
  - (3.) $\tau(\psi) = 1$ for maximally extangled states,
  - (4.) $\tau(\psi)$ invariant under qubit permutations,
  - (5.) $\tau(\psi)$ invariant under local unitary operations.

- For 3 qubits, the sum expands to

$$\begin{aligned}\tau(\psi) \;=\; 4\big|&a_{011}^2 a_{100}^2 + a_{010}^2 a_{101}^2 + a_{001}^2 a_{110}^2 + a_{000}^2 a_{111}^2 \\ &-2a_{001}a_{011}a_{110}a_{100} + 4a_{001}a_{010}a_{111}a_{100} \\ &-2a_{000}a_{011}a_{111}a_{100} - 2a_{010}a_{011}a_{101}a_{100} \\ &-2a_{001}a_{010}a_{101}a_{110} + 4a_{000}a_{011}a_{101}a_{110} \\ &-2a_{000}a_{010}a_{101}a_{111} - 2a_{000}a_{001}a_{110}a_{111}\big|.\end{aligned}$$

- Examples

$$|\psi\rangle = |GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right),$$

$$\tau(\psi) = 1.$$

$$|\psi\rangle = |W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right),$$

$$\tau(\psi) = 0.$$ -despite this, the W state is **fully** entangled!

$$|\psi\rangle = \lambda_1|000\rangle + \lambda_2\exp(i\varphi)|100\rangle + \lambda_3|101\rangle + \lambda_4|110\rangle + \lambda_5|111\rangle,$$

$$\tau(\psi) = 4\lambda_1^2\lambda_5^2.$$

- For even numbers of qubits, the tangle is equal to the **concurrence**.

Wong and Christensen, Phys. Rev. A **63**, 044301 (2001).

Concurrence entanglement measure.

- The $2N$ – qubit concurrence is defined as

$$\boxed{C(\psi)=\left|\langle\psi|\tilde{\psi}\rangle\right|^2,}$$

where

$$|\tilde{\psi}\rangle=\varepsilon^{\otimes N}|\psi^*\rangle,$$

$$\varepsilon=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

- It is defined so that:
  - (1.) $0\le C(\psi)\le 1$,
  - (2.) $C(\psi)=0$ for (fully) separable states,
  - (3.) $C(\psi)=1$ for maximally extangled states,
  - (4.) $C(\psi)$ invariant under qubit permutations,
  - (5.) $C(\psi)$ invariant under local unitary operations.
- Not used for odd numbers of qubits.

- Example: 4 - qubit concurrence. For state

$$\psi=(a_{0000},a_{0001},a_{0010},a_{0011},\ a_{0100},a_{0101},a_{0110},a_{0111},\ a_{1000},a_{1001},a_{1010},a_{1011},\ a_{1100},a_{1101},a_{1110},a_{1111}).$$

we find $\tilde{\psi}$ by flipping spins and making basis states with an odd number of zeros negative:

$$\tilde{\psi}=(a_{1111},-a_{1110},-a_{1101},a_{1100},\ -a_{1011},a_{1010},a_{1001},-a_{1000},\ -a_{0111},a_{0110},a_{0101},-a_{0100},\ a_{0011},-a_{0010},-a_{0001},a_{0000}).$$

Hence,

$$\langle\tilde{\psi}|\psi\rangle=2(a_{0000}a_{1111}-a_{0001}a_{1110}-a_{0010}a_{1101}+a_{0011}a_{1100}-a_{0100}a_{1011}+a_{0101}a_{1010}+a_{0110}a_{1001}-a_{0111}a_{1000})$$

so that

$$\boxed{C(\psi)=4\left|a_{0000}a_{1111}-a_{0001}a_{1110}-a_{0010}a_{1101}+a_{0011}a_{1100}-a_{0100}a_{1011}+a_{0101}a_{1010}+a_{0110}a_{1001}-a_{0111}a_{1000}\right|^2.}$$

Concurrence of entangled and partially entangled states.

$$C(\psi)=4\left|a_{0000}a_{1111}-a_{0001}a_{1110}-a_{0010}a_{1101}+a_{0011}a_{1100}\right.$$
$$\left.-a_{0100}a_{1011}+a_{0101}a_{1010}+a_{0110}a_{1001}-a_{0111}a_{1000}\right|^2.$$

- Partially entangled state [22]:

$$|\psi_1\rangle=\frac{1}{\sqrt{69}}\big(|0000\rangle+|0001\rangle+3i|0010\rangle+3i|0011\rangle$$
$$+|0100\rangle+3i|0110\rangle-3i|1000\rangle$$
$$-3i|1001\rangle+2|1010\rangle+2|1011\rangle$$
$$-3i|1100\rangle+2|1110\rangle\big).$$

$$C(\psi)=4\frac{1}{69^2}\big|(1\times 0)-(1\times 2)-(3i\times 0)+(3i\times -3i)$$
$$-(1\times 2)+(0\times 2)+(3i\times -3i)-(0\times -3i)\big|^2,$$
$$=4\frac{1}{69^2}|9-2+9-2|^2,$$
$$=4\frac{1}{69^2}14^2$$
$$=0.16467\dots.$$

Note 1: $|ijkl\rangle=|i_4j_3k_2l_1\rangle=|i_4\rangle|j_3\rangle|k_2\rangle|l_1\rangle$, numbers label particles.

Note 2: careful with particle ordering e.g $|i_4\rangle|j_3\rangle|k_2\rangle|l_1\rangle\neq|i_4\rangle|j_3\rangle|k_1\rangle|l_2\rangle$.

- State related to $|\psi_1\rangle$ by a particle permutation:

$$|\psi_2\rangle=\frac{1}{\sqrt{69}}\big(|0000\rangle+|0001\rangle+|0010\rangle$$
$$+3i|0100\rangle+3i|0101\rangle+3i|0110\rangle$$
$$-3i|1000\rangle-3i|1001\rangle-3i|1010\rangle$$
$$+2|1100\rangle+2|1101\rangle+2|1110\rangle\big).$$

$$C(\psi)=4\frac{1}{69^2}\big|(1\times 0)-(1\times 2)-(1\times 2)+(0\times 2)$$
$$-(3i\times 0)+(3i\times -3i)+(3i\times -3i)-(0\times -3i)\big|^2,$$
$$=4\frac{1}{69^2}|9-2+9-2|^2,$$
$$=0.16467\dots.$$

- [1111], [112], and [13], separable states all have zero concurrence. This includes particle permutations.

$|\psi_1\rangle$ and $|\psi_2\rangle$ are related by the qubit permutaion (132):

$$|\psi_1\rangle=(132)|\psi_2\rangle.$$

For example:

$$\begin{array}{lcccccccccc} & 4 & 3 & 2 & 1 & & 4 & 3 & 2 & 1 \\ (132)| & 1 & 0 & 1 & 1\rangle & = & |1 & 1 & 0 & 1\rangle. \end{array}$$

Local unitary operations on bipartite states

- A **local** quantum operation of the form:

$$\left|i'_A\right\rangle = U_A\left|i_A\right\rangle.$$

$$U_A = \exp\left(-i\frac{\theta}{2}\hat{n}.\hat{\sigma}\right),$$

where

$\hat{n}$ – is a unit vector.

$\hat{\sigma} = \left(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\right)$ – is a vector of Pauli matrices.

leads to a change in the Schmidt decomposition of the form:

$$\left|\psi'_{A,B}\right\rangle = U_A\left|\psi_{A,B}\right\rangle = \sum_i \sqrt{\lambda_i}\left|i'_A\right\rangle\left|i_B\right\rangle.$$

$$\hat{\rho}_R = \sum_i \lambda_i \left|i\right\rangle\left\langle i\right|$$

- This implies:
  - Entropy of entanglement is unchanged.

$$S(\rho_R) = -\sum_i \lambda_i \log_2 \lambda_i.$$

  - Schmidt number is unchanged.
  - Any two states with the same Schmidt coefficients can be transformed into one another by local unitary operations only.

Local operations example: The Bell states.

$$\left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_A\right\rangle\left|1_B\right\rangle - \left|1_A\right\rangle\left|0_B\right\rangle\right), \quad \left|\phi^-\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_A\right\rangle\left|0_B\right\rangle - \left|1_A\right\rangle\left|1_B\right\rangle\right),$$

$$\left|\psi^+\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_A\right\rangle\left|1_B\right\rangle + \left|1_A\right\rangle\left|0_B\right\rangle\right), \quad \left|\phi^+\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_A\right\rangle\left|0_B\right\rangle + \left|1_A\right\rangle\left|1_B\right\rangle\right).$$

Note: memorise these states. They appear repeatedly throughout the course.

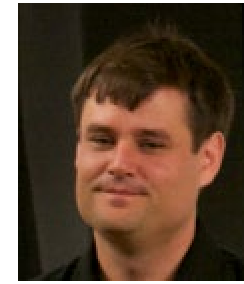
$\left|\psi^-\right\rangle$ can be transformed into any of the other Bell states with the following local operations acting on particle A alone:

$$U_{0,0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \left|0_A\right\rangle \to \left|0_A\right\rangle \qquad \left|1_A\right\rangle \to \left|1_A\right\rangle$$

$$U_{0,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \left|0_A\right\rangle \to \left|0_A\right\rangle \qquad \left|1_A\right\rangle \to -\left|1_A\right\rangle$$

$$U_{1,0} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \left|0_A\right\rangle \to -\left|1_A\right\rangle \qquad \left|1_A\right\rangle \to -\left|0_A\right\rangle$$

$$U_{1,1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \left|0_A\right\rangle \to \left|1_A\right\rangle \qquad \left|1_A\right\rangle \to -\left|0_A\right\rangle$$



Conditions for a Class of Entanglement Transformations

M. A. Nielsen*
Department of Physics, MC 12-33, California Institute of Technology, Pasadena, California 91125
(Received 2 December 1998)

Suppose Alice and Bob jointly possess a pure state, $|\psi\rangle$. Using local operations on their respective systems and classical communication it may be possible for Alice and Bob to *transform* $|\psi\rangle$ into another joint state $|\phi\rangle$. This Letter gives necessary and sufficient conditions for this process of *entanglement transformation* to be possible. These conditions reveal a partial ordering on the entangled states and connect quantum entanglement to the algebraic theory of *majorization*. As a consequence, we find that there exist essentially different types of entanglement for bipartite quantum systems.

The question "*What tasks may be accomplished using a given physical resource?*" is of fundamental importance in many areas of physics. In particular, the burgeoning field of quantum information [1,2] is much concerned with understanding transformations between different types of quantum information. A fundamental example is the problem of *entanglement transformation*: Suppose $|\psi\rangle$ is a pure state of some composite system AB; we refer to system A as Alice's system and to system B as Bob's system. Into what class of states $|\phi\rangle$ may $|\psi\rangle$ be *transformed,* assuming that Alice and Bob may use only local operations on their respective systems, and unlimited two-way classical communication?

Quantum information is physical information that is held in the "state" of a quantum system.

Conditions for a Class of Entanglement Transformations

M. A. Nielsen*
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Majorization :

suppose $x=(x_1,\dots x_d)$ and $y=(y_1,\dots y_d)$ are real d - dimensional vectors.

Then x is majorized by y (equivalently y majorizes x), written $x \prec y$, if for each k in the range $1,\dots,d$

$$\sum_{j=1}^{k} x_j^{\downarrow} \le \sum_{j=1}^{k} y_j^{\downarrow}$$

$e.g:\quad x \prec y$ for
$x=(0.4,0.3,0.3)$
$y=(0.6,0.2,0.2)$

with the equality holding when $k=d$, and where $\downarrow$ indicates that elements are to be taken in descending order, so, for example, $x_1^{\downarrow}$ will be the largest element in x.

Transformations between 2 qubit states.

- ***Nielsen's theorem*** : If Alice and Bob share the bipartite state

$$\boxed{|\psi_{AB}\rangle = \sum_{i=1}^{n} \sqrt{\lambda_i}\,|i_A\rangle|i_B\rangle,}$$

they ***can transform*** it into the bipartite state

$$\boxed{|\phi_{AB}\rangle = \sum_{i=1}^{n} \sqrt{\mu_i}\,|i_A\rangle|i_B\rangle,}$$

using arbitrary

local quantum operations

generalized measurements or transformations that either Alice or Bob can perform on their own part of the state.

and ***two-way classical communicaton***

two-way communication using classical bits over a classical channel. For example Alice and Bob could communicate by email.

if and ***only if*** $\lambda = \left(\lambda_1^{\downarrow}, \lambda_2^{\downarrow}, \ldots, \lambda_n^{\downarrow}\right)$ is majorized by $\mu = \left(\mu_1^{\downarrow}, \mu_2^{\downarrow}, \ldots, \mu_n^{\downarrow}\right)$.

- More succinctly, $\boxed{|\psi_{AB}\rangle \to |\phi_{AB}\rangle \quad \text{iff} \quad \lambda \prec \mu}$

$$\lambda \prec \mu \quad \text{iff}$$
$$\sum_{j=1}^{k} \lambda_j^{\downarrow} \le \sum_{j=1}^{k} \mu_j^{\downarrow} \qquad k = 1 \ldots n,$$
$$\lambda_1^{\downarrow} \ge \lambda_2^{\downarrow} \ge \cdots \ge \lambda_n^{\downarrow}$$
$$\mu_1^{\downarrow} \ge \mu_2^{\downarrow} \ge \cdots \ge \mu_n^{\downarrow}$$

Note, an arbitrary protocol transforming

$$|\psi_{AB}\rangle \to |\phi_{AB}\rangle$$

has the following form:

- Alice performs a generalized measurement on her system and then sends the result to Bob.
- Bob performs an operation on his system, conditional on the measurement result.

Transformations between 2 qubit states.

As a simple application of the result, suppose Alice and Bob each possess a three-dimensional quantum system, with respective orthonormal bases denoted $|1\rangle, |2\rangle, |3\rangle$. Define states $|\psi\rangle$ and $|\phi\rangle$ of their joint system by

$$|\psi\rangle \equiv \sqrt{\frac{1}{2}}|11\rangle + \sqrt{\frac{2}{5}}|22\rangle + \sqrt{\frac{1}{10}}|33\rangle, \qquad \vec{\lambda} = \left(\frac{1}{2}, \frac{2}{5}, \frac{1}{10}\right)$$

$$|\phi\rangle \equiv \sqrt{\frac{3}{5}}|11\rangle + \sqrt{\frac{1}{5}}|22\rangle + \sqrt{\frac{1}{5}}|33\rangle. \qquad \vec{\mu} = \left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right)$$

$$\boxed{\begin{gathered} 1/2 < 3/5 \\ 1/2 + 2/5 > 3/5 + 1/5 \\ 1/2 + 2/5 + 1/10 = 3/5 + 1/5 + 1/5 \end{gathered}}$$

$$\boxed{\begin{gathered} \vec{\lambda} \not\prec \vec{\mu} \\ \vec{\mu} \not\prec \vec{\lambda} \end{gathered}}$$

It follows from Theorem 1 that neither $|\psi\rangle \to |\phi\rangle$ nor $|\phi\rangle \to |\psi\rangle$, providing an example of essentially different types of entanglement, from the point of view of local operations and classical communication. We will say that $|\psi\rangle$ and $|\phi\rangle$ are *incomparable*.

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Entanglement-Assisted Local Manipulation of Pure Quantum States

Daniel Jonathan and Martin B. Plenio
Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom
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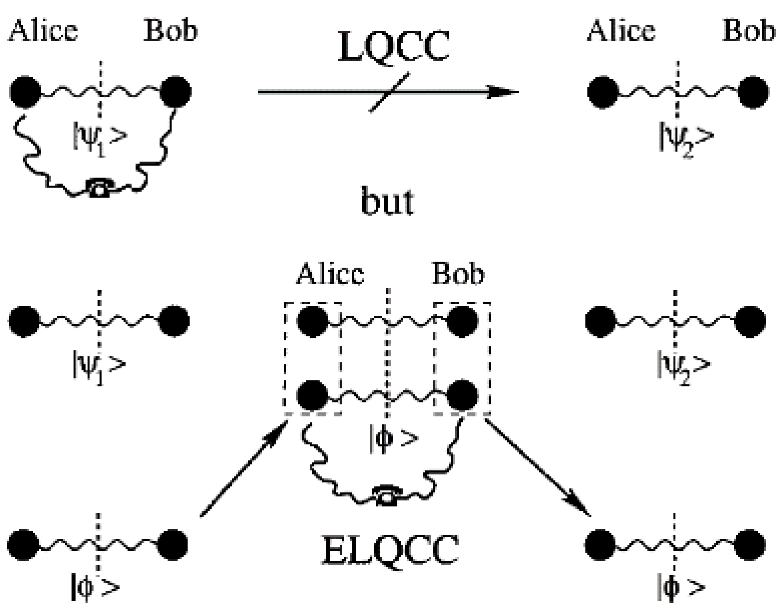
We demonstrate that local transformations on a composite quantum system can be enhanced in the presence of certain entangled states. These extra states act much like *catalysts* in a chemical reaction: they allow otherwise impossible local transformations to be realized, without being consumed in any way. In particular, we show that this effect can considerably improve the efficiency of entanglement concentration procedures for finite states.

Martin Plenio



- Alice and Bob share a finite bipartite system in state $|\psi_1\rangle$.
- Using only LQCC they are not able to convert this state into $|\psi_2\rangle$ with certainty.
- However, for a suitably chosen entangled state $|\phi\rangle$, they can always make the transformation $|\psi_1\rangle|\phi\rangle \to |\psi_2\rangle|\phi\rangle$.
- The state $|\phi\rangle$ is separable before and after the transformation and so need only be borrowed.

Quantum catalysis: Example

- It is not possible to transform between the two states

$$\boxed{\begin{aligned}|\psi_1\rangle &= \sqrt{0.4}\,|0_A\rangle|0_B\rangle+\sqrt{0.4}\,|1_A\rangle|1_B\rangle+\sqrt{0.1}\,|2_A\rangle|2_B\rangle+\sqrt{0.1}\,|3_A\rangle|3_B\rangle \\ |\psi_2\rangle &= \sqrt{0.5}\,|0_A\rangle|0_B\rangle+\sqrt{0.25}\,|1_A\rangle|1_B\rangle+\sqrt{0.25}\,|2_A\rangle|2_B\rangle\end{aligned}}$$

  using local operations and classical communication with 100% certainty because neither majorizes the other : $0.4 < 0.5$ and $0.8 > 0.75$.

- For $\boxed{|\phi\rangle = \sqrt{0.6}\,|4_C\rangle|4_D\rangle+\sqrt{0.4}\,|5_C\rangle|5_D\rangle,}$ the Schmidt coefficients of the (four particle) product states $|\psi_1\rangle|\phi\rangle$ and $|\psi_2\rangle|\phi\rangle$ are :

$\|\psi_1\rangle\|\phi\rangle$	$\|0_A4_C\rangle\|0_B4_D\rangle$	$\|1_A4_C\rangle\|1_B4_D\rangle$	$\|0_A5_C\rangle\|0_B5_C\rangle$	$\|1_A5_C\rangle\|1_B5_D\rangle$	$\|2_A4_C\rangle\|2_B4_D\rangle$	$\|3_A4_C\rangle\|3_B4_D\rangle$	$\|2_A5_C\rangle\|2_B5_D\rangle$	$\|3_A5_C\rangle\|3_B5_D\rangle$
λ_i	0.24	0.24	0.16	0.16	0.06	0.06	0.04	0.04
$\|\psi_2\rangle\|\phi\rangle$	$\|0_A4_C\rangle\|0_B4_D\rangle$	$\|0_A5_C\rangle\|0_B5_D\rangle$	$\|1_A4_C\rangle\|1_B4_D\rangle$	$\|2_A4_C\rangle\|2_B4_D\rangle$	$\|1_A5_C\rangle\|1_B5_D\rangle$	$\|2_A5_C\rangle\|2_B5_D\rangle$	$\|3_A4_C\rangle\|3_B4_D\rangle$	$\|3_A5_C\rangle\|3_B5_D\rangle$
μ_i	0.3	0.2	0.15	0.15	0.1	0.1	0.0	0.0

- Hence, we have $\vec{\lambda} \prec \vec{\mu}$.

- The transformation $\boxed{\begin{aligned}|\psi_1\rangle|\phi\rangle \to |\psi_3\rangle = \sqrt{0.3}\,|0_A4_C\rangle|0_B4_D\rangle+\sqrt{0.2}\,|1_A4_C\rangle|1_B4_D\rangle+\sqrt{0.15}\,|0_A5_C\rangle|0_B5_D\rangle \\ +\sqrt{0.15}\,|1_A5_C\rangle|1_B5_D\rangle+\sqrt{0.1}\,|2_A4_C\rangle|2_B4_D\rangle+\sqrt{0.1}\,|3_A4_C\rangle|3_B4_D\rangle\end{aligned}}$

  can therefore be achieved by Nielsen's theorem.

- Futher local unitary operations allow $|\psi_3\rangle \to |\psi_2\rangle|\phi\rangle$.
- The state $|\phi\rangle$ therefore acts as a ***catalyst*** enabling the transformation $|\psi_1\rangle|\phi\rangle \to |\psi_2\rangle|\phi\rangle$ and therefore the transformation $|\psi_1\rangle \to |\psi_2\rangle$.

Entanglement concentration

Charles Bennett *et al* Phys. Rev. A **53** 2046 (1996).

- Is it possible to increase the entanglement of a bipartite state by just measuring one of the two particles?
- Consider the following measurement operator for a bipartite state with particles *A* and *B*:

$$\boxed{\hat{M} = \lambda_1 \hat{M}_1 + \lambda_2 \hat{M}_2,}$$

where

$$\boxed{\begin{aligned}\hat{M}_1 &= \tan\theta \left|0_A\right\rangle\left\langle 0_A\right| + \left|1_A\right\rangle\left\langle 1_A\right|,\\ \hat{M}_2 &= \sqrt{1-\tan^2\theta}\left|0_A\right\rangle\left\langle 0_A\right|.\end{aligned}}$$

This is an example of a type of generalised measurement called a **POVM**. We cover POVMs in the next lecture.

- M_1 and M_2 are mutually exclusive measurement events.
- If $\hat{M}_1$ occurs, wave function collapse causes: $\left|\psi\right\rangle \rightarrow \dfrac{\hat{M}_1\left|\psi\right\rangle}{\left\langle\psi\right|\hat{M}_1^\dagger\hat{M}_1\left|\psi\right\rangle^{1/2}}.$
- If $\hat{M}_2$ occurs, wave function collapse causes: $\left|\psi\right\rangle \rightarrow \dfrac{\hat{M}_2\left|\psi\right\rangle}{\left\langle\psi\right|\hat{M}_2^\dagger\hat{M}_2\left|\psi\right\rangle^{1/2}}.$

Entanglement concentration

Charles Bennett *et al*
Phys. Rev. A **53** 2046 (1996).

- We take the initial state to be in Schmidt form:

$$\boxed{|\psi_{AB}\rangle = \cos\theta|0_A\rangle|0_B\rangle + \sin\theta|1_A\rangle|1_B\rangle.}$$

$\boxed{\theta - \text{small}}$

- The result of measurement M_1 occurring is:

$$\begin{aligned} M_1|\psi\rangle &= \left(\tan\theta|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|\right)\left(\cos\theta|0_A\rangle|0_B\rangle + \sin\theta|1_A\rangle|1_B\rangle\right), \\ &= \sin\theta|0_A\rangle|0_B\rangle + \sin\theta|1_A\rangle|1_B\rangle, \\ &= \sqrt{2}\sin\theta \cdot \frac{1}{\sqrt{2}}\left(|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle\right). \end{aligned}$$

Maximally entangled state

- The result of measurement M_2 occurring is:

$$\begin{aligned} M_2|\psi\rangle &= \sqrt{1-\tan^2\theta}\,|0_A\rangle\langle 0_A|\left(\cos\theta|0_A\rangle|0_B\rangle + \sin\theta|1_A\rangle|1_B\rangle\right), \\ &= \sqrt{\cos 2\theta}\,|0_A\rangle|0_B\rangle. \end{aligned}$$

Separable state

Hence, we can increase the entanglement of a bipartite state by measuring one of the two particles, some of the time.

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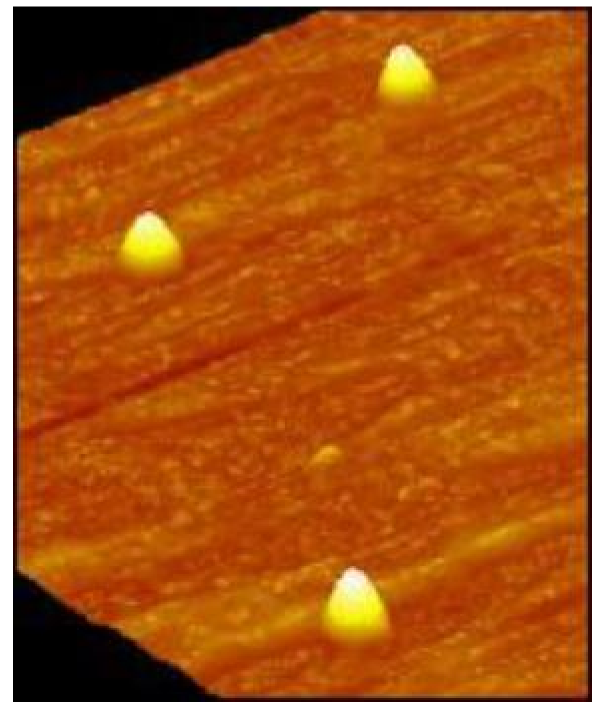
LETTERS

A semiconductor source of triggered entangled photon pairs

R. M. Stevenson[1], R. J. Young[1,2], P. Atkinson[2], K. Cooper[2], D. A. Ritchie[2] & A. J. Shields[1]

Entangled photon pairs are an important resource in quantum optics[1], and are essential for quantum information[2] applications such as quantum key distribution[3,4] and controlled quantum logic operations[5]. The radiative decay of biexcitons—that is, states consisting of two bound electron–hole pairs—in a quantum dot has been proposed as a source of triggered polarization-entangled photon pairs[6]. To date, however, experiments have indicated that a splitting of the intermediate exciton energy yields only classically correlated emission[7–9]. Here we demonstrate triggered photon pair emission from single quantum dots suggestive of polarization entanglement. We achieve this by tuning the splitting to zero, through either application of an in-plane magnetic field or careful control of growth conditions. Entangled photon pairs generated 'on demand' have significant fundamental advantages over other schemes[10–13], which can suffer from multiple pair emission, or require post-selection techniques or the use of photon-number discriminating detectors. Furthermore, control over the pair generation time is essential for scaling many quantum information schemes beyond a few gates. Our results suggest that a triggered entangled photon pair source could be implemented by a simple semiconductor light-emitting diode[14].



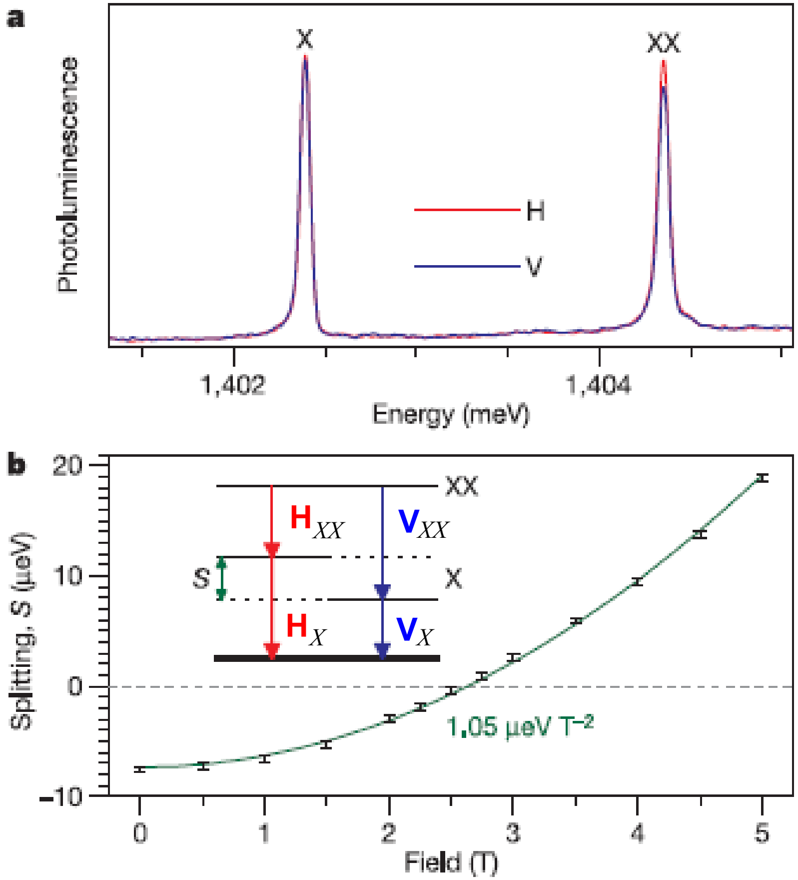
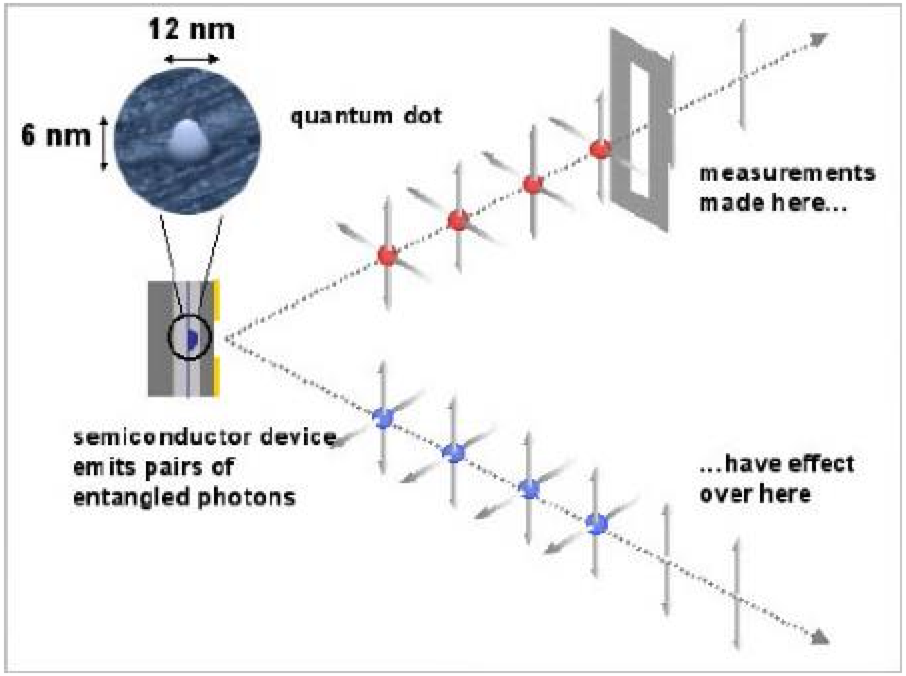


Figure 1 | Polarized photoluminescence spectra from single quantum dots. **a**, Vertically (blue) and horizontally (red) polarized photoluminescence for a single quantum dot with small polarization splitting. The features correspond to emission by the exciton (X) and biexciton (XX) state. **b**, Polarization splitting, *S*, as a function of in-plane magnetic field for a single dot with 'inverted' *S* at 0 T. The green line shows a quadratic fit to the data with a coefficient of 1.05 μeV T^{-2}. Inset shows the level diagram of the radiative decay of the biexciton state. The competing two photon decay paths are distinguished only by the polarization of the photons, indicated by the arrow colour, and the splitting, *S*, of the intermediate exciton level. Error bars span two standard deviations from the fitted line.

The radiative decay of the biexciton state (XX) in a quantum dot emits a pair of photons, with polarization determined by the spin of the intermediate exciton state (X). In an ideal quantum dot with degenerate X states, the polarization of the XX photon is predicted to be entangled with that of the X photon, forming the state $(|H_{XX}H_X> + |V_{XX}V_X>)/\sqrt{2}$, where H and V denote the polarization of the XX and X photons[6].



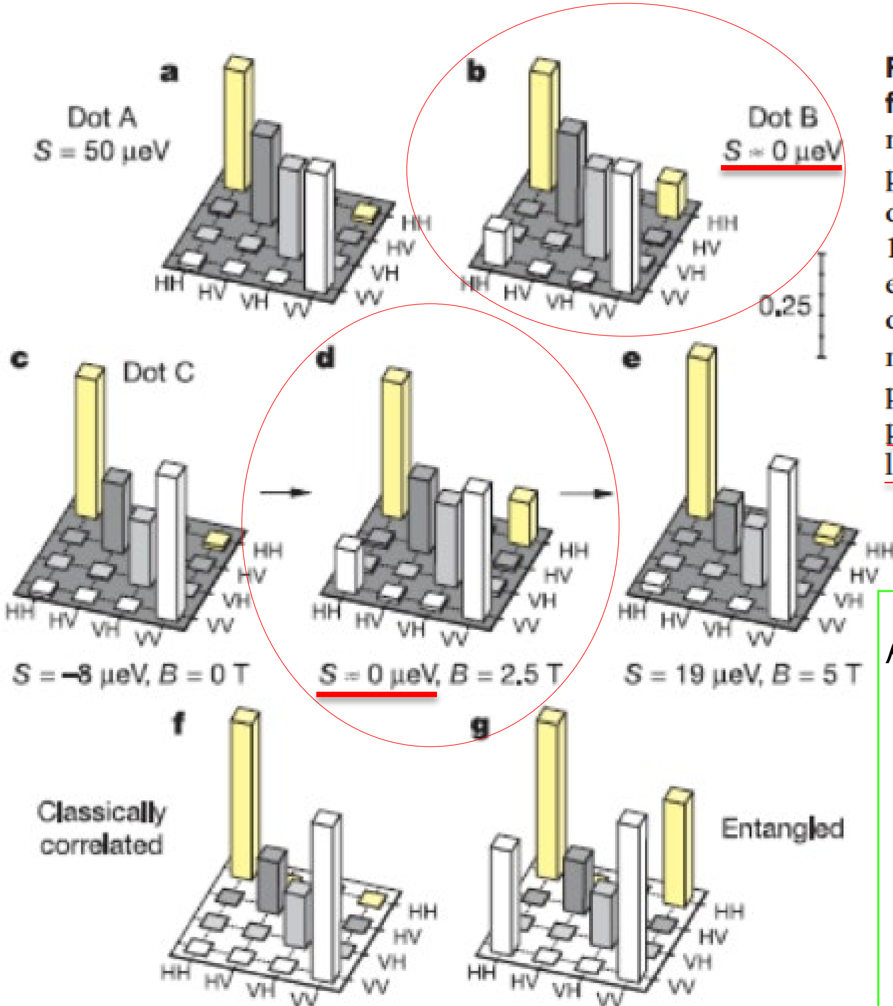


Figure 3 | Density matrices for the biexciton–exciton two-photon cascade from conventional and degenerate quantum dots. a–e, Real parts of measured density matrices corresponding to reference dot A with polarization splitting, $S = 50\,\mu\text{eV}$ (**a**), dot B with $S \approx 0\,\mu\text{eV}$ at 0 T (**b**), and dot C, with S tuned by the magnetic field to be $-8\,\mu\text{eV}$ (**c**), $0\,\mu\text{eV}$ (**d**) and $19\,\mu\text{eV}$ (**e**). The imaginary components are not shown, and were zero within experimental error. Density matrices **b** and **d** feature strong outer off diagonal elements associated with entangled photon pair states, which are not present in the reference case (a). **f**, **g**, Density matrices representing the predicted state for ideal classically correlated (**f**) and entangled (**g**) photon pairs, including 50% contribution from uncorrelated background light.

$$\rho = \frac{1}{\sqrt{2}}\left(|H_{XX}\rangle|H_X\rangle + |V_{XX}\rangle|V_X\rangle\right)\frac{1}{\sqrt{2}}\left(\langle V_X|\langle V_{XX}| + \langle H_X|\langle H_{XX}|\right),$$

$$= \frac{1}{2}\left(|H_{XX}\rangle|H_X\rangle\langle H_X|\langle H_{XX}| + |H_{XX}\rangle|H_X\rangle\langle V_X|\langle V_{XX}|\right.$$

$$\left. + |V_{XX}\rangle|V_X\rangle\langle H_X|\langle H_{XX}| + |V_{XX}\rangle|V_X\rangle\langle V_X|\langle V_{XX}|\right),$$

$$= \begin{pmatrix} 1 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 1 \end{pmatrix}.$$