M40002 revision

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You should be familiar with the proofs of the theorems colored in red, because they are likely to be in exam question, or the proofs offer great insight for solving similar questions.

The theorems colored in blue could be useful but you don't really need to know the proof (they are either unseen questions or beyond the syllabus).

For conditions colored in green, try to think of a counter example if that condition is omitted.

1 The real numbers

Condition for equality: $a = b$ iff $\forall \epsilon > 0$, $|a - b| < \epsilon$.

Supremum: A real number s is the supremum for a set $A \subseteq \mathbb{R}$ if:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $b \geq s$.

Infimum: A real number l is the infimum for a set $A \subseteq \mathbb{R}$ if:

(i) l is an lower bound for A;

(ii) if b is any lower bound for A, then $b \leq l$.

Axiom of Completeness: A set $S \subseteq \mathbb{R}$ that is nonempty and bounded above has a supremum.

Greatest lower bound property: The axiom of completeness implies that, a set $S \subseteq \mathbb{R}$ that is nonempty and bounded below has an infimum.

Nested Interval Property: For each $n \in \mathbb{N}$, assume $I_n = [a_n, b_n]$ (what happens when they are open intervals?) and $I_n \supseteq I_{n+1}$, then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Properties of sup and inf:

(i) If s is an upper bound for set A and $s \in A$, then $s = \sup A$

(ii) If $s = \sup A$, the for any $\epsilon > 0$ there exists $a \in A$ s.t. $s - \epsilon < a \leq s$

The analogus results hold for infimum.

Existence of square root: There exists $0 \leq x \in \mathbb{R}$ s.t. $x^2 = a$ for any $a \geq 0$.

Density of $\mathbb{Q}$ in $\mathbb{R}$: For $a, b \in \mathbb{R}$, there exists $r \in \mathbb{Q}$ s.t. $a < r < b$.

Countability: A set A is countable if there exists $f : A \to \mathbb{N}$ that is bijective.

Criterion for Countability: Set A is countable if there exists $f : A \to \mathbb{N}$ that is injective. Set B is uncountable if a subset of B is uncountable.

Cartesian product of countable sets: If set S, T are countable, then $S \times T$ is countable.

Countable union of countable sets: If A_n is countable for $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Diagonal argument (for rationals): $\mathbb{Q}$ is countable.

Diagonal argument: $\mathbb{R}$ is uncountable.

Cardinality of the power set: The set of all subsets of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, is uncountable.

2 Sequences

Convergence of sequence: A sequence (a_n) converges to a real number a if, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |a_n - a| < \epsilon$.

Divergence: A sequence (a_n) diverges iff it does not converge. i.e. $\forall a > 0, \exists \epsilon_0 > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists n \geq N$ s.t. $|a_n - a| \geq \epsilon_0$.

Uniqueness of limit: The limit of a sequence, if exists, must be unique.

Every convergent sequence is bounded.

Supremum as a limit: Suppose $s = \sup A$ exists. Then there exists a sequence (a_n) where $a_n \in A$ s.t. $a_n \to a$.

Rational approximation: Every real number is the limit of some rational sequence.

Algebra of Limits: Let $\lim a_n = a$ and $\lim b_n = b$. Then

(i) $\lim ca_n = ca$ for all $c \in \mathbb{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim a_n b_n = ab$;

(iv) $\lim \frac{a_n}{b_n} = \frac{a}{b}$ when $b \neq 0$.

Order Limit Theoerm: Let $\lim a_n = a$ and $\lim b_n = b$. If $a_n \geq b_n$ for all $n \in \mathbb{N}$, then $a \geq b$. (Note: Even if $a_n > b_n$ we can only infer $a \geq b$).

Squeeze Theorem: If $a_n \leq b_n \leq c_n$ and $\lim a_n = \lim c_n = l$, then $\lim b_n = l$.

Monotone Convergence Theorem: If (a_n) is monotone and bounded, then it converges. The limit is $\sup\{a_n : n \in \mathbb{N}\}$ if a_n is monotone increasing, and is $\inf\{a_n : n \in \mathbb{N}\}$ if a_n is monotone decreasing.

Limit of subsequences: Subsequences of a convergent sequence converge to the same limit as the origin sequence.

Criterion for Divergence of Limit: If two subsequences of a_n converge to different limits, then a_n diverge.

Subsequence formed by odd and even terms: $a_n \to a$ iff $a_{2n} \to a$ and $a_{2n+1} \to a$.

Bolzano-Weierstrass Theorem: Every bounded sequence contains a convergent subseqence.

Cauchy sequence: A sequence (a_n) is called a Cauchy sequence if, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n > m \geq N \Rightarrow |a_n - a_m| < \epsilon$.

Cauchy Criterion: A sequence converge iff it is a Cauchy sequence.

Cesaro Means: If $\lim x_n = l$, then $\lim y_n = l$ where $y_n = \frac{x_1+x_2+\ldots+x_n}{n}$.

3 Series

Series: Let (a_n) be a sequence. Define the partial sum $s_n = \sum_{k=1}^{n} a_n$. The series

$$\sum_{n=1}^{\infty} a_n$$

converges to a if $s_n \to a$.

Algebra of Series: If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

(i) $\sum_{n=1}^{\infty} ca_n = cA$;

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$

Note: We can extend the result to the case where A or/and B are $\pm\infty$ by defining the addition of infinity as: $c + \infty = \infty$, $c - \infty = -\infty$, $\infty + \infty = \infty$, $-\infty - \infty = -\infty$. However, note that $\infty - \infty$ is not defined.

Series as sum of series: We have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{2n} + \sum_{n=1}^{\infty} a_{2n-1}$$

if $\sum_{n=1}^{\infty} a_n$ converges absolutely, or if the three series converge.

We can derive similar results such as

$$a_1 + a_2 + a_3 + a_4 + ... = (a_1 + a_2 + a_4 + a_5 + a_7 + a_8 + ...) + (a_3 + a_6 + a_9 + ...)$$

Do not confuse this result with rearrangement of series.

Cauchy Criterion for Series: The series $\sum_{n=1}^{\infty} a_n$ converges iff given $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n > m \geq N \Rightarrow |a_{m+1} + ... + a_n| < \epsilon$.

Divergence Test: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. (Why doesn't the converse hold?)

Comparison Test: If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges, and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Sandwich Test: If $a_n \leq b_n \leq c_n$ and $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} c_n$ both converge, then $\sum_{n=1}^{\infty} b_n$ converge.

Absolute Convergence Test: If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Alternating Series Test: Let (a_n) be a positive, decreasing sequence and $a_n \to 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

Geometric Series: $\sum_{n=1}^{\infty} r^n$ converges iff $|r| < 1$.

p-Test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Integral Test: Suppose f is a continuous, positive and decreasing function on $[1, \infty)$ and $f(n) = a_n$, then

$$\int_1^{\infty} f(x)dx \quad \text{and} \quad \sum_{n=1}^{\infty} a_n$$

both converge or both diverge.

Ratio Test: For a sequence (a_n), let

$$r = \lim \left| \frac{a_{n+1}}{a_n} \right|$$

If $r < 1$ the series $\sum_{n=1}^{\infty} a_n$ converges, and if $r > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges. (What if $r = 1$?)

Root Test: For a sequence (a_n), let

$$r = \lim |a_n|^{\frac{1}{n}}$$

If $r < 1$ the series $\sum_{n=1}^{\infty} a_n$ converges, and if $r > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Limit Comparison Test: If $\frac{a_n}{b_n} \to l$ and $\sum_{n=1}^{\infty} b_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Limit comparison without absolute convergence: Suppose $\frac{a_n}{b_n} \to l$ but $\sum_{n=1}^{\infty} b_n$ converges conditionally. Then there exists (a_n) s.t. $\sum_{n=1}^{\infty} a_n$ diverges. This is achieved by letting $(a_n) = \frac{1}{n}$ and $b_n = (-1)^n \frac{1}{\sqrt{n}}$.

Rearrangement: Let $\sum_{n=1}^{\infty} a_n$ be a series. A series $\sum_{n=1}^{\infty} b_n$ is said to be a rearrangement of $\sum_{n=1}^{\infty} a_n$ if there exists a bijective $f : \mathbb{N} \to \mathbb{N}$ s.t. $b_k = a_{f(k)}$ for all k.

Rearrangement does not change the limit: If a series converges absolutely, then any rearrangement of this series converges to the same limit.

Rearrangement by a injective function: Let $\sum_{n=1}^{\infty} a_n$ be a series that converges absolutely and $f : \mathbb{N} \to \mathbb{N}$ injective. Then $\sum_{n=1}^{\infty} a_{f(n)}$ converges.

Rearrangement of Conditionally Convergent Series: Suppose $\sum_{n=1}^{\infty} a_n$ converges conditionally, then by suitable rearrangement we can let $\sum_{n=1}^{\infty} a_{f(n)}$ converge to any real number.

Radius of Convergence: Given any power series $\sum_{n=1}^{\infty} a_n z^n$, there exists $R \in [0, \infty]$, called the radius of convergence, s.t.

$|z| < R \Rightarrow \sum_{n=1}^{\infty} a_n z^n$ converges absolutely;

$|z| > R \Rightarrow \sum_{n=1}^{\infty} a_n z^n$ diverges.

Criterion for Radius of Convergence: To show $R \in [0, \infty]$ is the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$, we can show that

$$|z| < R \Rightarrow a_n z^n \to 0 \quad \text{and} \quad |z| > R \Rightarrow a_n z^n \not\to 0$$

4 Basic Topology of $\mathbb{R}$

Open set: A set $O \subseteq \mathbb{R}$ is open if $\forall a \in O$, $\exists \epsilon > 0$ s.t. $(a - \epsilon, a + \epsilon) \subseteq O$.

Limit point: A point x is a limit point of set A if every ϵ-neighborhood of x intersects A at some point other than x.

Limit point in terms of limit: A point x is a limit point of set A if $x = \lim a_n$ for some (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.

Closed set: A set $F \in \mathbb{R}$ is closed if it contains its limit points.

Open and closed sets are complements: A set O is open iff O^c is closed.

Closed set in terms of limit: A set $F \in \mathbb{R}$ is closed iff every Cauchy sequence contained in F has a limit that is in F.

Union/intersection of open sets:

(i) The union of an arbitrary collection of open sets is open;

(ii) The intersection of a finite collection of open sets is open.

Union/intersection of closed sets:

(i) The union of a finite collection of closed sets is closed;

(ii) The intersection of an arbitrary collection of closed sets is closed.

Compactness: A set $K \in \mathbb{R}$ is compact if every sequence in K has a subsequence that converges to a limit in K.

Characterization of Compactness in $\mathbb{R}$: A set is compact iff it is closed and bounded.

5 Functional Limits and Continuity

Functional Limit: We say $\lim_{x\to c} f(x) = L$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Sequence Criterion for Functional Limit: Given a function $f : A \to \mathbb{R}$, the following two statements are equivalent:

(i) $\lim_{x\to c} f(x) = L$;

(ii) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $x_n \to c$, $f(x_n) \to L$.

Algebra of Functional Limits: Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

(i) $\lim_{x\to c} kf(x) = kL$ for all $k \in \mathbb{R}$;

(ii) $\lim_{x\to c}[f(x) + g(x)] = L + M$;

(iii) $\lim_{x\to c} f(x)g(x) = LM$;

(iv) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Divergence Criterion for Functional Limits: If there exists two sequences x_n and y_n with x_n, $y_n \neq c$ and

$$\lim x_n = \lim y_n = c \quad \text{but} \quad \lim f(x_n) \neq \lim f(y_n)$$

then $\lim_{x\to c} f(x)$ does not exist.

Continuity: A function $f : A \to \mathbb{R}$ is continuous at $c \in A$ if

$$\lim_{x\to c} f(x) = f(c)$$

or equivalently: $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$.

Criterion for Discontinuity: Let $f : A \to \mathbb{R}$. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but $f(x_n) \nrightarrow f(c)$, then f is not continuous at c.

Algebraic Continuity Theorem: Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at $c \in A$, then

(i) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$;

(ii) $f(x) + g(x)$ is continuous at c;

(iii) $f(x)g(x)$ is continuous at c;

(iv) $\frac{f(x)}{g(x)}$ is continuous at c, provided that the quotient is defined.

Composition of Continuous functions: If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Exchanging Limit and Continuous Function: If g is continuous then

$$\lim_{x \to a} g(f(x)) = g(\lim_{x \to a} f(x))$$

Preimage of a open set: A function $f : \mathbb{R} \to \mathbb{R}$ is continuous iff for every open set $U \subseteq \mathbb{R}$, the preimage $\{x \in \mathbb{R} : f(x) \in U\}$ is open.

Intermediate Value Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. If $L \in \mathbb{R}$ is between (not equal to) $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ s.t. $f(c) = L$.

Image of a open set: Suppose f is continuous and strictly increasing. Let $U \subseteq \mathbb{R}$ be an open set, then $f(U)$ is open.

Existence of zeros: Every real, odd-degree polynomial has a zero.

Preservation of compact sets: Let $f : A \to \mathbb{R}$ be a continuous function. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.

Extreme Value Theorem: If $f : K \to \mathbb{R}$ is continuous on a compact set K, then there exists $x_0, x_1 \in K$ s.t. $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.

Uniform Continuity: A function $f : A \to \mathbb{R}$ is uniform continuous on A if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall x, y \in A$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

Sequence Criterion for Absence of Uniform Continuity: A function $f : A \to \mathbb{R}$ fails to be uniform continuous on A iff there exists $\epsilon_0 > 0$ and two sequences (x_n), $(y_n) \subseteq A$ s.t.

$$|x_n - y_n| \to 0 \quad \text{but} \quad |f(x_n) - f(y_n)| > \epsilon_0$$

Uniform Continuity on Compact sets: A function that is continuous on a compact set K is uniform continuous on K. Mostly we let K to be an closed interval.

Uniform Continuity on Open Interval: To show f is uniform continuous on (a, b), it suffices to show that f is uniform continuous on $(a, c]$ and $[c, b)$. This also holds when endpoints are at infinity, using this fact, try to show $\sqrt{x}$ uniformly continuous on $[0, \infty)$.

Uniform Continuity of Periodic Functions: A function that is continuous and periodic is uniform continuous.

Continuous Extension Theorem: Let f be a continuous function on (a, b). f is uniform continuous on (a, b) iff we can define $f(a)$ and $f(b)$ s.t. the extended function f is continuous on $[a, b]$.

Lipschitz Functions: A function $f : A \to \mathbb{R}$ is Lipschitz if there exists $M > 0$ s.t.

$$\left| \frac{f(x) - f(y)}{x - y} \right| < M$$

for all $x \neq y \in A$. Intuitively, a function if Lipschitz if the derivative is bounded (however Lipschitz function may not be differentiable, but they are almost everywhere differentiable).

Lipschitz Function is Uniform Continuous.

Function that is Uniform Continuous but not Lipschitz: The function $f(x) = \sqrt{x}$ is uniform continuous on $[0, \infty)$ but it is not Lipschitz, as the derivative near $x = 0$ tends to infinity.

6 The Derivative

Differentiability: Let $f : A \to \mathbb{R}$. Given $c \in A$, the derivative of f at c is defined by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided that the limit exists. In this case we say f is differentiable at c. The derivative can also be written as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiable implies Continuity: If $f : A \to \mathbb{R}$ is differentiable at $c \in A$, then f is continuous at c.

Discontinuous Derivative: The function $f(x) = x^2 \sin(1/x)$ is differentiable at $x = 0$, but

$$\lim_{x \to 0} f'(x)$$

does not exist.

Algebra of Derivative: Suppose f and g are both differentiable at c. Then

(i) $(kf)'(c) = kf'(c)$;

(ii) $(f + g)'(c) = f'(c) + g'(c)$;

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

(iv) $(f/g)'(c) = \dfrac{g(c)f'(c) - f(c)g'(c)}{g(c)^2}$

Chain rule: Suppose $g \circ f$ is defined, f is differentiable at $c \in A$ and g is defined at $f(c)$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c))f'(c)$.

Local Comparison: If $f'(c) > 0$, then there exists $\delta > 0$ s.t.

$$c - \delta < x < c < y < c + \delta \Rightarrow f(x) < f(c) < f(y)$$

an analogus theorem holds when $f'(c) < 0$.

Global Growth: If f is differentiable on (a, b) and $f'(c) > 0$ for all $c \in (a, b)$, then f is strictly increasing on (a, b). If $f'(c) \geq 0$ then f is increasing. However, if f is either increasing or strictly increasing we can only infer that $f'(x) \geq 0$. (Why can't the inequality be strict?)

Local Extremum is a Critical Point: We say f attains a local maximum at c if $\exists \delta > 0$ s.t. $\forall x \in (c-\delta, c+\delta), f(c) \geq f(x)$, and the definition for local minimum is similar. If f attains a local maximum/minimum at c and is differentiable at c, then $f'(c) = 0$.

Interior Extremum Theorem: Let f be differentiable on an open interval (a, b). If f attains a (local) maximum/minimum at some point $c \in (a, b)$, then $f'(c) = 0$.

Rolle's Theorem: Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ s.t. $f'(c) = 0$.

Darboux's Theorem: If f is differentiable on an interval $[a, b]$ and $f'(a) < \alpha < f'(b)$, then there exists $c \in (a, b)$ s.t. $f'(c) = \alpha$.

Nonzero Derivative implies Monotonicity: $f'(x) \neq 0$ for all $x \in (a, b)$ implies f is strictly monotone on (a, b).

Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Generalized Mean Value Theorem: If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ s.t.

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

Nonzero Derivative at Infinity implies the Function Unbounded: Let f be differentiable and $\lim_{x\to\infty} f'(x) = l \neq 0$, then $\lim_{x\to\infty} f(x)$ tends to ∞ or $-\infty$.

L'Hopital's Rule:

Second Derivative Exists implies Local Continuity: $f''(c)$ exists $\Rightarrow f'(c)$ continuous $\Rightarrow f'(c)$ exists near $0 \Rightarrow f$ continuous near 0.

Second Derivative and Critical Point: Let c be a critical point of f.

If $f''(c) > 0$ then f has a local minimum at a;

If $f''(c) < 0$ then f has a local maximum at a.

Convexity: Let $f : I \to \mathbb{R}$ be defined on an interval. We say f is convex on I if, for all $a < x < b \in I$:

$$f(x) \leq \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

If the inequality is strict, we say f is strictly convex.

Convexity implies Continuity: If f is convex on an open interval I, then f is continuous on I.

Increasing Derivative implies Convexity: If f is differentiable on an interval I, and $f'(x)$ is increasing, then f is convex on I. (This is achieved when $f''(x) \geq 0$, as a special case). If $f'(x)$ is strictly increasing then f is strictly convex.

7 The Integral

Lower & Upper Sum: Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Given a partition $P = \{t_0, t_1, ..., t_n\}$ of $[a, b]$, where $a = t_0 < t_1 < ... < t_n = b$, let

$$m_i = \inf f(x) : x \in [t_{i-1}, t_i]$$

$$M_i = \sup f(x) : x \in [t_{i-1}, t_i]$$

The lower sum of f for P is

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

The upper sum of f for P is

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

Refinement increases lower sum and reduces upper sum: We say a partition P' is a refinement of P if $P \subseteq P'$. We have

$$L(f, P) \leq f(f, P') \quad \text{and} \quad U(f, P) \geq U(f, P')$$

Lower sum is always less than upper sum: Given any partition P and Q of $[a, b]$, we have

$$L(f, P) \leq U(f, Q)$$

Integrability: We say a bounded function $f : [a, b] \to \mathbb{R}$ is (Riemann)integrable if

$$\sup\{L(f, P) : P \text{ partition of } [a, b]\} = \inf\{U(f, P) : P \text{ partition of } [a, b]\}$$

and we denote this number by $\displaystyle\int_a^b f(x)dx$.

Criterion for Integrability: Suppose $f : [a, b] \to \mathbb{R}$ is bounded. f is integrable iff $\forall \epsilon > 0$, $\exists P_\epsilon$, a partition of $[a, b]$, s.t.

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

Sequential Criterion for Integrability: Suppose $f : [a, b] \to \mathbb{R}$ is bounded. f is integrable iff there exists a sequence of partitions (P_n) s.t.

$$\lim(U(f, P_n) - L(f, P_n)) = 0$$

Note: $\lim(U(f, P_n) - L(f, P_n)) = 0$ is equivalent to $\lim L(f, P_n) = \lim U(f, P_n)$.

Integrability on subintervals: Let $a < b < c$. Then f is integrable on $[a, c] \Leftrightarrow f$ is integrable on $[a, b]$ and $[b, c]$, and

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Properties of Integrals: Assume f and g are integrable functions on $[a, b]$. Then

(i) $f + g$ is integrable on $[a, b]$, with $\displaystyle\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$;

(ii) kf is integrable on $[a, b]$, with $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)dx$;

(iii) If $m \leq f(x) \leq M$ on $[a, b]$, then $\displaystyle m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$;

(iv) If $f(x) \leq g(x)$ on $[a, b]$, then $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$;

(v) $|f|$ is integrable and $\displaystyle\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$. (If $|f(x)|$ integrable, f may not be integrable).

Continuity implies Integrability: Suppose $f : [a, b] \to \mathbb{R}$ is bounded and continuous on $[a, b]$. Then f is integrable on $[a, b]$.

Fundamental Theorem of Calculus: If f' is integrable on $[a, b]$ then

$$\int_a^b f'(x)dx = f(b) - f(a)$$

If g is integrable on $[a, b]$, then $\displaystyle G(x) = \int_a^x g(t)dt$ is continuous on $[a, b]$. If g is continuous at c, then G is differentiable at c and $G'(c) = g(c)$.

Lebesgue's Theorem: If f is integrable on $[a, b]$, then f is integrable iff the set of points where f is not continuous has measure zero. That is, f is almost-everywhere continuous.