M50003 revision

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Upper triangular block matrix: Define the upper triangular block matrix

$$
A = \begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{pmatrix}
$$

then

(1) $\det(A) = \det(A_1)...\det(A_n)$

(2) For any polynomial p:

$$
p(A) = \begin{pmatrix} p(A_1) & & & * \\ & p(A_2) & & \\ & & \ddots & \\ & & & p(A_n) \end{pmatrix}
$$

Similar matrices: If A and B are similar, then they share the same characteristic polynomial, minimal polynomial, eigenvalues, nullity, geometric multiplicities, rank, and trace.

Direct sum of two subspaces: Suppose V_1, V_2 are subspaces of f.d. vector space V. Then $V = V_1 \oplus V_2$ is equivalent to the following conditions:

(1) $\forall \underline{v} \in V$, there is exactly one way to write $\underline{v} = \underline{v_1} + \underline{v_2}$, where $\underline{v_1} \in V_1$, $\underline{v_2} \in V_2$;

(2) $V = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$;

(3) $V = V_1 + V_2$ and $\dim(V_1) + \dim(V_2) = \dim(V)$.

(4) $V_1 \cap V_2 = \{0\}$ and $\dim(V_1) + \dim(V_2) = \dim(V)$.

Direct sum of n subspaces: Suppose $V_1, ..., V_n$ are subspaces of f.d. vector space V. Then $V = V_1 \oplus ... \oplus V_n$ is equivalent to the following conditions:

(1) $\forall \underline{v} \in V$, there is exactly one way to write $\underline{v} = \underline{v_1} + ... + \underline{v_n}$, where $\underline{v_i} \in V_i$ for every i;

(2) $\dim(V) = \sum_{i=1}^{n} \dim(V_i)$, and if B_i is a basis of V_i, then $B = B_1 \cup ... \cup B_n$ is a basis of V.

Direct sum and block diagonal matrix: Suppose $T : V \to V$ linear and $V = V_1 \oplus ... \oplus V_k$, where each V_i is T-invariant. Let B_i be the basis of V_i, and thus $B = B_1 \cup ... \cup B_k$ is a basis of V. Then $[T]_B$ is the block diagonal matrix

$$[T]_B = \begin{pmatrix} [T_{V_1}]_{B_1} & & & \\ & [T_{V_2}]_{B_2} & & \\ & & \ddots & \\ & & & [T_{V_n}]_{B_n} \end{pmatrix}$$

Algebraic and geometric multiplicities: If λ is an eigenvalue of linear map $T : V \to V$, then $g(\lambda) \leq a(\lambda)$.

Equivalent conditions for diagonalisability: Let $n = \dim(V)$, $T : V \to V$ linear, $\lambda_1, ..., \lambda_r$ are distinct eigenvalues, and

$$c_T(x) = \prod_{i=1}^{r}(x - \lambda_i)^{a(\lambda_i)}$$

then T diagonalisable is equivalent to the following conditions:

(1) $\sum_{i=1}^{r} g(\lambda_i) = n$;

(2) $g(\lambda_i) = a(\lambda_i)$ for all i;

(3) $V = E_{\lambda_1}(T) \oplus ... \oplus E_{\lambda_r}(T)$, where $E_\lambda(T) = \text{Ker}(T - \lambda I)$.

Dimension of quotient space: $\dim(V/U) = \dim(V) - \dim(U)$

Basis of quotient space: If V has basis $\underline{v_1}, ..., \underline{v_n}$ and the subspace W has basis $\underline{v_1}, ..., \underline{v_m}$, then $W + \underline{v_{m+1}}, ..., W + \underline{v_n}$ is a basis for V/W. Likewise, given the basis of W and V/W we can find a basis for V.

Matrix and invariant subspace: Let $T : V \to V$ linear, W is a T-invariant subspace of V. If W has basis $B_W = \{\underline{w_1}, ..., \underline{w_r}\}$, V/W has basis $\overline{B} = \{W + \underline{v_1}, ..., W + \underline{v_s}\}$, so V has basis $B = \{\underline{w_1}, ..., \underline{w_r}, \underline{v_1}, ..., \underline{v_s}\}$. Then

$$[T]_B = \begin{pmatrix} [T_W]_{B_W} & * \\ 0 & [\overline{T}]_{\overline{B}} \end{pmatrix}$$

Characteristic polynomial of restriction and quotient operator: Suppose W is a T-invariant space. Let T_W be the restriction operator on W and $\overline{T}$ be the quotient operator on V/W. Then

$$c_T(x) = c_{T_W}(x)c_{\overline{T}}(x)$$

Triangularisation theorem: Let $n = \dim(V)$ over a field $\mathbb{F}$ and $T : V \to V$ linear. Suppose the characteristic polynomial factorizes in $\mathbb{F}[x]$ as a product of linear factors:

$$c_T(x) = \prod_{i=1}^{n}(x - \lambda_i), \quad \lambda_i \in \mathbb{F}$$

then there exists a basis B s.t. $[T]_B$ upper triangular.

Find the basis that triangularises a linear map: Let $T : V \to V$ linear. The following steps find the basis B s.t. $[T]_B$ is upper triangular.

(1) Find an eigenvector $\underline{v_1}$ of T. Extend to a basis $\underline{v_1}, \underline{w_1}, ..., \underline{w_m}$ of V;

(2) Let $V_1 = \text{span}(\underline{v_1})$, thus $B_1 = \{V_1 + \underline{w_1}, ..., V_1 + \underline{w_m}\}$ is a basis of V/V_1;

(3) Consider $\overline{T}_1 : V/V_1 \to V/V_1$. By calculating $[\overline{T}_1]_{B_1}$, find an eigenvector $V_1 + \underline{v_2}$ of $\overline{T}$;

(4) Let $V_2 = \text{span}(\underline{v_1}, \underline{v_2})$. Follow (2), (3) to find an eigenvector $V_2 + \underline{v_3}$ of $\overline{T}_2 : V/V_2 \to V/V_2$;

(5) Repeat to find $\underline{v_i}$. $B = \{v_1, ..., v_n\}$ is the desired basis.

Note: Suppose $\dim(V) = n$, then after finding the first $n - 1$ basis vectors, we can pick any vector $\underline{v_n}$ that makes the list of vectors LI, because the last vector only affects the last column of $[T]_B$, but $[T]_B$ will be upper triangular regardless of the last column.

Cayley-Hamilton theorem: $c_T(T) = 0$

Euclid algorithm: Let $f, g \in \mathbb{F}[x]$ with $\deg(g) \geq 1$. Then $\exists q, r \in \mathbb{F}[x]$ s.t.

$$f = gq + r$$

where $r = 0$ or $\deg(r) < \deg(g)$.

Bezout lemma: If $d = \gcd(f, g)$, then $\exists s, r \in \mathbb{F}[x]$ s.t.

$$d = rf + sg$$

Properties of polynomials: Let $T : V \to V$ linear, p and q are polynomials. Then

(1) $\text{Ker}(p(T))$ and $\text{Im}(p(T))$ are T-invariant;

(2) $(pq)(T)(\underline{v}) = [p(T)q(T)](\underline{v}) = p(q(T)(\underline{v}))$;

(3) $\text{lcm}(f, g) \cdot \gcd(f, g) = fg$.

Irreducible polynomials:

(1) If p is irreducible, $p|fg$, then $p|f$ or $p|g$;

(2) All irreducible monic polynomials in $\mathbb{C}[x]$ takes form $x - a$; All irreducible monic polynomials in $\mathbb{R}[x]$ takes form $x - a$ or $x^2 + bx + c$, where $b^2 - 4c < 0$;

(3) The irreducible polynomials in $\mathbb{F}_2[x]$ with degree ≤ 4 are

$$x, x + 1, x^2 + 1, x^3 + x + 1, x^3 + x^2 + 1, x^4 + x + 1, x^4 + x^3 + 1, x^4 + x^3 + x^2 + x + 1$$

Properties of minimal polynomial: Let $T : V \to V$ be linear. Then

(1) $m_T(x)|c_T(x)$;

(2) If λ is an eigenvalue of T, then $m_T(\lambda) = 0$;

(3) If $p(x)$ is an irreducible factor of $c_T(x)$, then $p(x)|m_T(x)$.

Companion matrix: Let $p(x) = x^n + ... + a_1 x + a_0$. The companion matrix

$$C(p(x)) = \begin{pmatrix} 0 & 0 & ... & 0 & -a_0 \\ 1 & 0 & ... & 0 & -a_1 \\ 0 & 1 & ... & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & 1 & -a_{n-1} \end{pmatrix}$$

has characteristic polynomial and minimal polynomial $= p(x)$.

Diagonal block matrix: Define the diagonal block matrix

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{pmatrix}$$

then

(1) If λ is an eigenvalue of A, then its geometric multiplicity is the sum of geometric multiplicities in the sub blocks:

$$\dim(\mathrm{Ker}(A - \lambda I)) = \dim(\mathrm{Ker}(A_1 - \lambda I)) + ... + \dim(\mathrm{Ker}(A_n - \lambda I))$$

(2) The minimal polynomial of A is

$$m_A(x) = \mathrm{lcm}(m_{A_1}(x), ..., m_{A_n}(x))$$

Primary decomposition: Let V be f.d. over $\mathbb{F}$, the minimal polynomial of $T : V \to V$ be m_T. Suppose the factorization of the minimal polynomial into irreducible factors be

$$m_T(x) = \prod_{i=1}^{k} f_i(x)^{n_i}$$

Define $V_i = \mathrm{Ker}(f_i(T)^{n_i})$. Then

(1) $V = V_1 \oplus ... \oplus V_k$;

(2) Each V_i are T-invariant, and $m_{T_i}(x) = f_i(x)^{n_i}$.

Diagonalisability and minimal polynomial: The linear map $T : V \to V$ over $\mathbb{F}$ is diagonalisable iff $m_T(x)$ can be factored into distinct linear factors:

$$m_T(x) = \prod_{i=1}^{k} (x - \lambda_i)$$

where λ_i are distinct.

Existence of nontrivial invariant subspace: The linear map $T : V \to V$ over $\mathbb{F}$ has a nontrivial invariant subspace (that is, $\exists W \neq 0$ or V that is T-invariant) iff $m_T(x)$ is reducible in $\mathbb{F}[x]$.