M40007 revision notes

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The exam style of applied math is very predictable. It will always have one question about random walk and one question about point-source current. The other two questions may cover topics such as electric circuit, spring mass system, orthogonal eigenvectors, and Laplacian matrix. Topics that are not examined in the past papers include method of relaxation, minimization and one-dimensional continuum limit.

1 Graphs and incident matrix

For a graph G with m edges and n nodes, the incident matrix A is $m \times n$.

The dimension of the right null space, $\dim(\text{null}(A))$, is the number of connected graphs in G. Particularly, if G itself is connected then $\dim(\text{null}(A)) = 1$. We see that the dimension of the right null space of A is at least one. In fact, A always have the right null vector

$$\vec{x_0} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad A\vec{x_0} = 0$$

The rank of A can be calculated by

$$\text{rk}(A) = n - \dim(\text{null}(A))$$

The dimension of the left null space, $\dim(\text{null}(A^T))$, is the number of independent loops in graph G. Instead of counting, this can be calculated by

$$\dim(\text{null}(A^T)) = m - \text{rk}(A)$$

Define the node variable x_i to be the potential of node i and let

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Define the edge variable w_a to be the internal current flowing across edge a and let

$$\vec{w} = \begin{bmatrix} w_a \\ w_b \\ \vdots \\ w_p \end{bmatrix}$$

Ohm's law states that, if current in edge a flows from node i to node j, then

$$w_a = c_a(x_i - x_j)$$

where c_a is the conductance of edge a. Define the diagonal matrix

$$C = \begin{bmatrix} c_a & & & \\ & c_b & & \\ & & \ddots & \\ & & & c_p \end{bmatrix}$$

as the conductance matrix. We have

$$\vec{w} = -CA\vec{x}$$

Let $\vec{f_I}$ denote the total current diverging out of each node. Then

$$\vec{f_I} = -A^T\vec{w}$$

Putting them all together

$$\vec{f_I} = -A^T\vec{w} = A^TCA\vec{x}$$

The **Laplacian** matrix is defined by

$$K = A^TCA$$

By **conservation of current**, $\vec{f} = \vec{f_I}$ where $\vec{f}$ is the external current into each node. Hence we have

$$K\vec{x} = \vec{f}$$

The **Divergence theorem** states that

$$\vec{x_0}^T\vec{f_I} = 0$$

This can be shown by

$$\vec{x_0}^T\vec{f_I} = \vec{x_0}^T(-A^T\vec{w}) = -(A\vec{x_0})^T\vec{w} = 0$$

From the theorem we can deduce that, in order for $K\vec{x} = \vec{f}$ to have a solution, the entries of $\vec{f}$ must add up to 0.

2 Properties of the Laplacian matrix

The Laplacian is a $n \times n$ symmetric matrix and always has a right null vector $\vec{x_0}$. K_{ii} is the sum of all conductance of the edges connected to node i, and K_{ij} is the sum of all conductance of the edges connecting node i and node j, times -1. This can be summarized as

$$K = D - W$$

The right null space of K is same as the right null space of A. To show this result, first note that

$$\text{null}(A) \subseteq \text{null}(K)$$

Suppose there is some vector $\vec{v} \in \text{null}(K)$ s.t. $v \notin \text{null}(A)$, then $A\vec{x} \neq \vec{0}$. Let $\vec{y} = CA\vec{x} \neq \vec{0}$ (since C is invertible), we have $A^T\vec{y} = \vec{0}$. However, note that $\vec{y} \in \text{Im}(A)$ and $\vec{y} \in \text{null}(A^T)$.

By **Fundamental Theorem of Linear Algebra**, $\text{null}(A^T)$ and $\text{Im}(A)$ are orthogonal complements, so their intersection is $\{\vec{0}\}$, hence $\vec{y} = \vec{0}$, contradiction. Thus

$$\text{null}(K) \subseteq \text{null}(A)$$

3 Dirichlet problem and effective conductance

The **boundary nodes** refer to the nodes with non-zero external current, and at other nodes we say KCL holds. The Dirichlet problem solves the case when two boundary nodes, node 1 and node 2, are set to unit voltage and grounded, respectively ($x_1 = 1$ and $x_2 = 0$) so external current flows in via node 1 and leaves the circuit via node 2, i.e.

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} f \\ -f \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Using Schur's complements we can write

$$\begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix} \begin{bmatrix} \hat{e} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} \hat{f} \\ 0 \end{bmatrix}$$

and solve for $\hat{x}$ and f.

The effective conductance of a circuit, given two boundary nodes x_+ and x_-, satisfy

$$f = C_{eff}(x_+ - x_-)$$

When $x_+ = 1$ and $x_- = 0$ we have $f = C_{eff}$, thus solving the Dirichlet problem yields the effective conductance.

However, Schur's complement is too complicated. In exam, one should first look for symmetry and eliminate conductors with no current flowing through, then collapse conductors in parallel and in series. Recall that for parallel conductors, effective conductance is $c_a + ... + c_p$. As for conductor in series, the effective conductance is $\frac{1}{1/c_a + ... + 1/c_p}$

If collapsing didn't work, use harmonic potentials as discussed in section 4.

4 Random walks and circuit

We define the **hitting probability** p_i to be the probability that, starting at node i and do random walk, reaching node x_+ before reaching node x_-.

For a given circuit we can link the node potential and hitting probability as follows:

Define the **hopping probability** p_{ij} as the probability that, while doing random walk, move from node i to node j. If we let

$$p_{ij} = \frac{c_{ij}}{\sum\limits_k c_{ik}}$$

where c_{ij} is the conductance of the edge connecting node i and j, then the hitting probability p_i equals to x_i when x_+, x_- are set to 1 and 0, respectively. Note that this is exactly the Dirichlet problem.

To solve for hitting probabilities, note that at boundary nodes: $p_+ = 1$, $p_- = 0$; At non-boundary nodes:

$$p_i = \sum_k p_{ik} p_k$$

That is, the hitting probability at non-boundary nodes is a weighted average of its neighbors. Solving the system of linear equations yields p_i, and thus yields x_i.

The escape probability, P_{esc}, is the probability that starting at node + and do random walk, reaching node - before returning to node +.

If we define the hopping probabilities the same as above, then

$$P_{esc} = \frac{C_{eff}}{c_+} \quad \text{where} \quad c_+ = \sum_k c_{+k}$$

It is very likely for the exam to involve a random walk on a straight line. Consider the graph

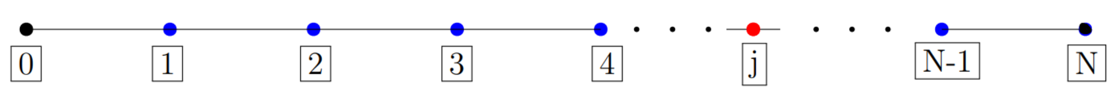

Figure 1: A straight graph with N+1 nodes and uniform conductance

Suppose the question asks the probability that, starting at node j, reaching node N before reaching node 0.

Let x_+ denote node N and x_- denote node 0. To find the hitting probability p_j, we solve the system of linear equations

$$\begin{cases} p_0 = 0 \\ p_1 = \frac{1}{2}p_0 + \frac{1}{2}p_2 \\ p_2 = \frac{1}{2}p_1 + \frac{1}{2}p_3 \\ \quad\vdots \\ p_{N-1} = \frac{1}{2}p_{N-2} + \frac{1}{2}p_N \\ p_N = 1 \end{cases}$$

Thus $p_i - p_{i-1} = p_{i+1} - p_i$ for $i = 1, ..., N-1$, by setting the difference to be D we have

$$p_i = (p_i - p_{i-1}) + ... + (p_1 - p_0) + p_0 = iD$$

Since $p_n = 1$, we obtain $D = \frac{1}{n}$ and $p_j = \frac{j}{D}$.

Now suppose the question asks the probability that, starting at node j, reaching node 0 or N before returning to node j. This the escape probability problem where x_+ is node j and x_- is node 0 and N. We can ground node 0 and N and merge them together, so that x_- correspond to a single node. Now the circuit consists two conductors in parallel, with conductance $\frac{1}{j}$ and $\frac{1}{N-j}$. Thus

$$P_{esc} = \frac{C_{eff}}{c_+} = \frac{\frac{1}{j} + \frac{1}{N-j}}{2} = \frac{N}{2j(N-j)}$$

5 Spring mass system

The spring mass system is similar to electric circuit. Each mass (and wall) represents a node and the springs represent edges. In spring mass system we no longer cares about conductance, but spring constants. Let

$$\vec{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

be the vector of displacements and vector of external forces on each mass. Suppose also K is the Laplacian matrix. Then

$$K\vec{\phi} = \vec{f}$$

holds if the system is in equilibrium.

The following formula holds when the system is not in equilibrium:

$$\vec{f} - K\vec{\phi} = M\frac{d^2\vec{\phi}}{dt^2}$$

where the diagonal matrix

$$M = \begin{bmatrix} m_a & & & \\ & m_b & & \\ & & \ddots & \\ & & & m_p \end{bmatrix}$$

consists the masses of each mass.

Ideally the displacement of all walls are 0 so we don't have to calculate them. Thus in the equation we can discard all entries related to the walls and obtain

$$\vec{f} - K'\vec{\phi} = M\frac{d^2\vec{\phi}}{dt^2}$$

(Note that for simplicity I only changed K to K') we call K' the reduced Laplacian matrix.

For instance, the equation of the following system

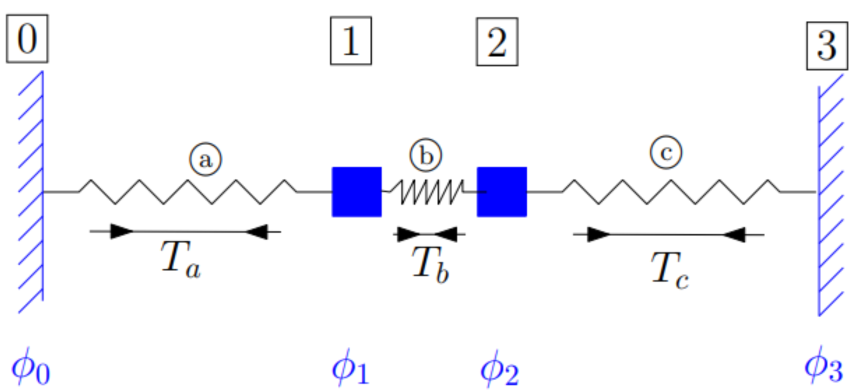

Figure 2: A spring mass system with two fixed walls

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} m_0 \frac{d^2\phi_0}{dt^2} \\ m_1 \frac{d^2\phi_1}{dt^2} \\ m_2 \frac{d^2\phi_2}{dt^2} \\ m_3 \frac{d^2\phi_3}{dt^2} \end{bmatrix}$$

can be reduced to

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} m_1 \frac{d^2\phi_1}{dt^2} \\ m_2 \frac{d^2\phi_2}{dt^2} \end{bmatrix}$$

since $\phi_0 = \phi_3 = 0$.

In the case of free oscillations (no external force) and suppose all masses are 1, we have

$$-K'\vec{\phi} = \frac{d^2\vec{\phi}}{dt^2}$$

Let $\vec{\phi} = \vec{\phi_0} e^{iwt}$. Then we have

$$-K' \vec{\phi_0} e^{iwt} = (iw)^2 \vec{\phi_0} e^{iwt}$$

$$K' \vec{\phi_0} = w^2 \vec{\phi_0}$$

This is an eigenvalue problem. $w = \pm\sqrt{\lambda}$ are the **natural frequencies** of the system.

6 Circulant and tridiagonal matrices

The $n \times n$ **shift matrix** is defined by

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & & & & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

The effect of this matrix is shifting every entry down and moving the bottom entry to the top, so clearly $S^n = I$. Any matrix that can be written as

$$a_0 I + a_1 S + a_2 S^2 + ... + a_{n-1} S^{n-1}$$

is called a **circulant matrix**. We are interested in the $n \times n$ circulant matrix

$$K = \begin{bmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & & 0 \\ 0 & -1 & 2 & & 0 \\ \vdots & & & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 2 \end{bmatrix}$$

K is symmetric, so it has n eigenvectors and eigenvalues of K have multiplicity 1. K and S commute because

$$KS = (2I - S - S^{n-1})S = S(2I - S - S^{n-1}) = SK$$

If $K\vec{v} = \lambda'\vec{v}$, then

$$K(S\vec{v}) = SK\vec{v} = S(\lambda'\vec{v}) = \lambda'(S\vec{v})$$

Hence $\vec{v}$ and $S\vec{v}$ are eigenvectors of K with eigenvalue λ'. Since the multiplicity of λ' is 1, we know $S\vec{v} = \lambda\vec{v}$. Thus we proved K and S share eigenvectors.

Now

$$S^n \vec{v} = \lambda^n \vec{v} = \vec{v}$$

hence $\lambda = w^j$ where $j = 0, 1, .., n-1$ and $w = e^{2\pi i/n}$. Thus we deduce the eigenvectors of S, also the eigenvectors of K, are

$$\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}, \quad \begin{bmatrix}1\\w\\w^2\\\vdots\\w^{n-1}\end{bmatrix}, \quad \begin{bmatrix}1\\w^2\\w^4\\\vdots\\w^{2(n-1)}\end{bmatrix}, ..., \begin{bmatrix}1\\w^{n-1}\\w^{2(n-1)}\\\vdots\\w^{(n-1)(n-1)}\end{bmatrix},$$

To calculate the eigenvalues with respect to K, we can simply compute the first entry of $K\vec{v_j}$:

$$\lambda_j = 2 - w^j - w^{j(n-1)} = 2 - (w^j + w^{-j}) = 2 - 2\cos\left(\frac{\pi j}{N+1}\right)$$

Denote the eigenvectors as $\vec{e_0}, \vec{e_1}, \vec{e_2}, ..., \vec{e_{n-1}}$. By the **Spectral theorem**, they form an orthogonal basis (with respect to the complex dot product). Since

$$\begin{aligned} &\vec{e_j}^\dagger \vec{e_j} \\ &= \begin{bmatrix}1\\w^j\\w^{2j}\\\vdots\\w^{(n-1)j}\end{bmatrix}^\dagger \begin{bmatrix}1\\w^j\\w^{2j}\\\vdots\\w^{(n-1)j}\end{bmatrix} \\ &= \begin{bmatrix}1\\1/w^j\\1/w^{2j}\\\vdots\\1/w^{(n-1)j}\end{bmatrix}^T \begin{bmatrix}1\\w^j\\w^{2j}\\\vdots\\w^{(n-1)j}\end{bmatrix} \\ &= n \end{aligned}$$

the norm of each vector is $\sqrt{n}$.

The following question is fairly likely to occur in exam:

Suppose the Laplacian matrix of a electric circuit/spring mass system is the circulant matrix K. Let

$$\vec{x} = \sum_{j=0}^{n-1} a_j \vec{e_j}$$

calculate the coefficients a_j.

Apply K on both sides:

$$\vec{f} = K\vec{x} = \sum_{j=0}^{n-1} a_j K \vec{e_j} = \sum_{j=0}^{n-1} a_j \lambda_j \vec{e_j}$$

Using the orthogonality of $\vec{e_j}$:

$$\vec{e_m}^{\dagger}\vec{f} = \vec{e_m}^{\dagger}\sum_{j=0}^{n-1} a_j\lambda_j\vec{e_j} = a_m\lambda_m\sqrt{n}$$

$$a_m = \frac{\vec{e_m}^{\dagger}\vec{f}}{\lambda_m\sqrt{n}}$$

The $n \times n$ tridiagonal matrix has form

$$K' = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & & 0 \\ 0 & -1 & 2 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{bmatrix}$$

To illustrate the process of finding the eigenvectors we consider the case when $n = 3$:

$$K_3' = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

The circulant matrix

$$K_8 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

has eigenvectors

$$\begin{bmatrix} 1 & w^j & w^{2j} & w^{3j} & w^{4j} & w^{5j} & w^{6j} & w^{7j} \end{bmatrix}^T$$

for $j = 0, 1, ..., 7$ and $w = e^{2\pi i/8}$. Notice that $w^4 = -1$ so the eigenvector is

$$\begin{bmatrix} 1 & w^j & w^{2j} & w^{3j} & (-1)^j & w^{5j} & w^{6j} & w^{7j} \end{bmatrix}^T$$

If $K_8\vec{v_j} = \lambda\vec{v_j}$, then taking complex conjugate on each side yields

$$K_8^*\vec{v_j}^* = \lambda_j^*\vec{v_j}^*$$

But since K_8 and λ_j are real, this is just

$$K_8 \vec{v_j}^* = \lambda_j \vec{v_j}^*$$

So $\vec{v_j}$ and $\vec{v_j}^*$ are eigenvectors with same eigenvalue λ. Their linear combination

$$\vec{\psi_j} = \frac{1}{2i}(\vec{v_j} - \vec{v_j}^*) = \mathrm{Im}(\vec{v_j})$$

$$= \begin{bmatrix} 0 \\ \mathrm{Im}(w) \\ \mathrm{Im}(w^2) \\ \mathrm{Im}(w^3) \\ 0 \\ \mathrm{Im}(w^5) \\ \mathrm{Im}(w^6) \\ \mathrm{Im}(w^7) \end{bmatrix}$$

is a real eigenvector with eigenvalue λ_j. Plugging in, we have

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ \mathrm{Im}(w) \\ \mathrm{Im}(w^2) \\ \mathrm{Im}(w^3) \\ 0 \\ \mathrm{Im}(w^5) \\ \mathrm{Im}(w^6) \\ \mathrm{Im}(w^7) \end{bmatrix} = \lambda_j \begin{bmatrix} 0 \\ \mathrm{Im}(w) \\ \mathrm{Im}(w^2) \\ \mathrm{Im}(w^3) \\ 0 \\ \mathrm{Im}(w^5) \\ \mathrm{Im}(w^6) \\ \mathrm{Im}(w^7) \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \mathrm{Im}(w) \\ \mathrm{Im}(w^2) \\ \mathrm{Im}(w^3) \end{bmatrix} = \lambda_j \begin{bmatrix} \mathrm{Im}(w) \\ \mathrm{Im}(w^2) \\ \mathrm{Im}(w^3) \end{bmatrix}$$

Thus the eigenvectors of K_3' are

$$\begin{bmatrix} \mathrm{Im}(w) \\ \mathrm{Im}(w^2) \\ \mathrm{Im}(w^3) \end{bmatrix} = \begin{bmatrix} \sin(2\pi j/8) \\ \sin(4\pi j/8) \\ \sin(6\pi j/8) \end{bmatrix}$$

for $j = 1, 2, 3$. Note that

$$\begin{bmatrix} \mathrm{Im}(w^5) \\ \mathrm{Im}(w^6) \\ \mathrm{Im}(w^7) \end{bmatrix} = \begin{bmatrix} \sin(10\pi j/8) \\ \sin(12\pi j/8) \\ \sin(14\pi j/8) \end{bmatrix} = \begin{bmatrix} \sin(2\pi j/8 + \pi j) \\ \sin(4\pi j/8 + \pi j) \\ \sin(6\pi j/8 + \pi j) \end{bmatrix} = \pm \begin{bmatrix} \sin(2\pi j/8) \\ \sin(4\pi j/8) \\ \sin(6\pi j/8) \end{bmatrix}$$

so we don't have to consider them.

To summarize, the $n \times n$ tridiagonal matrix has orthogonal eigenvectors

$$\vec{e_j} = \begin{bmatrix} \sin(\frac{\pi j}{N+1}) \\ \sin(\frac{2\pi j}{N+1}) \\ \vdots \\ \sin(\frac{N\pi j}{N+1}) \end{bmatrix} \quad \text{for} \quad j = 1, ..., N$$

with eigenvalues

$$\lambda_j = 2 - 2\cos\left(\frac{\pi j}{N+1}\right)$$

and each eigenvector has norm

$$\sqrt{\frac{N+1}{2}}$$

7 Two-dimensional conductors and complex analysis

Consider a two-dimensional conductor with uniform conductivity $\hat{c}$.

Suppose its voltage distribution is given by the scalar field $\varphi(x, y)$ and the current density is given by the vector field $\vec{J}(x, y) = \begin{bmatrix} J^{(x)}(x, y) \\ J^{(y)}(x, y) \end{bmatrix}$. Finally, let $\vec{F}(x, y)$ be the external current density at (x, y). We have

$$\vec{J} = -\hat{c} \begin{bmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{bmatrix} = -\hat{c}\nabla\varphi \quad \text{and} \quad \vec{F} = \nabla\vec{J}$$

Combining we have

$$-\hat{c}\nabla^2\varphi = \vec{F}$$

Thus

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

where KCL holds. Such φ is said to be a harmonic function.

$\varphi(x, y)$ is a harmonic function iff it can be written as $\mathrm{Re}(h(z))$ where $z = x + yi$. To check $\mathrm{Re}(h(z))$ is harmonic, we compute the partial derivatives:

$$\frac{\partial h}{\partial x} = \frac{dh}{dz}\frac{\partial z}{\partial x} = h'(z)$$

$$\frac{\partial h}{\partial y} = \frac{dh}{dz}\frac{\partial z}{\partial y} = h'(z)i$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial h'(z)}{\partial z}\frac{\partial z}{\partial x} = h''(z)$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{\partial h'(z)i}{\partial y}\frac{\partial z}{\partial y} = h''(z)i^2 = -h''(z)$$

Hence $\nabla^2 h = 0$, and since $\varphi = \frac{1}{2}(h(z) + \overline{h(z)})$, $\nabla^2 \varphi = 0$.

The relationship between $\vec{J}$ and $h(z)$ is given by

$$J^{(x)} - iJ^{(y)} = -\hat{c}\frac{dh}{dz}$$

A common exam question is to check if the voltage is zero at certain areas of the conductor. Suppose $\varphi = \text{Re}(\ln(R(z))$, the technique is to show that $\overline{R(z)} = \frac{1}{R(z)}$ so $|R(z)| = 1$. Write $R(z) = e^{i\theta}$, it is clear that

$$\varphi = \text{Re}(\ln(R(z)) = \text{Re}(\ln(e^{i\theta})) = \text{Re}(i\theta) = 0$$

The total current passing through a boundary can be calculated as

$$I = \int_{\text{boundary}} \begin{bmatrix} J^{(x)} \\ J^{(y)} \end{bmatrix} \cdot \hat{n} ds$$

where $\hat{n}$ is the unit normal vector at each point on the boundary. By expressing the normal vector in terms of complex numbers, i.e. write $\begin{bmatrix} x \\ y \end{bmatrix}$ as $x + yi$, we obtain

$$I = \int_{\text{boundary}} \text{Re}((J^{(x)} - iJ^{(y)}) \cdot n) ds$$

$$I = \int_{\text{boundary}} \text{Re}\left(-\hat{c}\frac{dh}{dz} \cdot n\right) ds$$

Finally, if

$$h(z) = -\frac{m}{2\pi}\ln(z - z_0) + \hat{h}(z)$$

where $\hat{h}(z)$ is non-singular at z_0, then there is a point source of current strength m at z_0.