M50003 revision

Teddy Wu

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Upper triangular block matrix: Define the upper triangular block matrix

$$A = \begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{pmatrix}$$

then

(1) $\det(A) = \det(A_1 \cdots \det(A_n)$

(2) For any polynomial p:

$$p(A) = \begin{pmatrix} p(A_1) & & & * \\ & p(A_2) & & \\ & & \ddots & \\ & & & p(A_n) \end{pmatrix}$$

Similar matrices: If A and B are similar, then they share the same characteristic polynomial, minimal polynomial, eigenvalues, nullity, geometric multiplicities, rank, and trace.

Direct sum of two subspaces: Suppose V_1, V_2 are subspaces of f.d. vector space V. Then $V = V_1 \oplus V_2$ is equivalent to the following conditions:

(1) $\forall \underline{v} \in V$, there is exactly one way to write $\underline{v} = \underline{v_1} + \underline{v_2}$, where $\underline{v_1} \in V_1$, $\underline{v_2} \in V_2$;

(2) $V = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$;

(3) $V = V_1 + V_2$ and $\dim(V_1) + \dim(V_2) = \dim(V)$.

(4) $V_1 \cap V_2 = \{0\}$ and $\dim(V_1) + \dim(V_2) = \dim(V)$.

Direct sum of n subspaces: Suppose $V_1, \cdots, V_n$ are subspaces of f.d. vector space V. Then $V = V_1 \oplus \cdots \oplus V_n$ is equivalent to the following conditions:

(1) $\forall \underline{v} \in V$, there is exactly one way to write $\underline{v} = \underline{v_1} + \cdots + \underline{v_n}$, where $\underline{v_i} \in V_i$ for every i;

(2) $\dim(V) = \sum_{i=1}^{n} \dim(V_i)$, and if B_i is a basis of V_i, then $B = B_1 \cup ... \cup B_n$ is a basis of V.

Direct sum and block diagonal matrix: Suppose $T : V \to V$ linear and $V = V_1 \oplus \cdots \oplus V_k$, where each V_i is T-invariant. Let B_i be the basis of V_i, and thus $B = B_1 \cup ... \cup B_k$ is a basis of V. Then $[T]_B$ is the block diagonal matrix

$$[T]_B = \begin{pmatrix} [T_{V_1}]_{B_1} & & & \\ & [T_{V_2}]_{B_2} & & \\ & & \ddots & \\ & & & [T_{V_n}]_{B_n} \end{pmatrix}$$

Algebraic and geometric multiplicities: If λ is an eigenvalue of linear map $T : V \to V$, then $g(\lambda) \leq a(\lambda)$.

Equivalent conditions for diagonalisability: Let $n = \dim(V)$, $T : V \to V$ linear, $\lambda_1, \cdots, \lambda_r$ are distinct eigenvalues, and

$$c_T(x) = \prod_{i=1}^{r} (x - \lambda_i)^{a(\lambda_i)}$$

then T diagonalisable is equivalent to the following conditions:

(1) $\sum_{i=1}^{r} g(\lambda_i) = n$;

(2) $g(\lambda_i) = a(\lambda_i)$ for all i;

(3) $V = E_{\lambda_1}(T) \oplus \cdots \oplus E_{\lambda_r}(T)$, where $E_\lambda(T) = \mathrm{Ker}(T - \lambda I)$.

Dimension of quotient space: $\dim(V/U) = \dim(V) - \dim(U)$

Basis of quotient space: If V has basis $\underline{v_1}, \cdots, \underline{v_n}$ and the subspace W has basis $\underline{v_1}, \cdots, \underline{v_m}$, then $W + \underline{v_{m+1}}, \cdots, W + \underline{v_n}$ is a basis for V/W.

Conversely, given the basis of W being $\{\underline{v_1}, \cdots, \underline{v_m}\}$ and basis of V/W being $\{W + \underline{w_1}, \cdots, W + \underline{w_n}\}$, a basis for V is

$$\{\underline{v_1}, \cdots, \underline{v_m}, \underline{w_1}, \cdots, \underline{w_n}\}$$

Matrix and invariant subspace: Let $T : V \to V$ linear, W is a T-invariant subspace of V. If W has basis $B_W = \{\underline{w_1}, ..., \underline{w_r}\}$, V/W has basis $\overline{B} = \{W + \underline{v_1}, ..., W + \underline{v_s}\}$, so V has basis $B = \{\underline{w_1}, ..., \underline{w_r}, \underline{v_1}, ..., \underline{v_s}\}$. Then

$$[T]_B = \begin{pmatrix} [T_W]_{B_W} & * \\ 0 & [\overline{T}]_{\overline{B}} \end{pmatrix}$$

Characteristic polynomial of restriction and quotient operator: Suppose W is a T-invariant space. Let T_W be the restriction operator on W and $\overline{T}$ be the quotient operator on V/W. Then

$$c_T(x) = c_{T_W}(x) c_{\overline{T}}(x)$$

Triangularisation theorem: Let $n = \dim(V)$ over a field $\mathbb{F}$ and $T : V \to V$ linear. Suppose the characteristic polynomial factorizes in $\mathbb{F}[x]$ as a product of linear factors:

$$c_T(x) = \prod_{i=1}^{n} (x - \lambda_i), \quad \lambda_i \in \mathbb{F}$$

then there exists a basis B s.t. $[T]_B$ upper triangular.

Note: The equivalent condition of T being upper triangularisable is $m_T(x)$ can be factorized into linear factors.

Find basis that triangularises a linear map: Let $T : V \to V$ linear. The following steps find the basis B s.t. $[T]_B$ is upper triangular.

(1) Find an eigenvector $\underline{v_1}$ of T. Extend to a basis $\underline{v_1}, \underline{w_1}, \cdots, \underline{w_m}$ of V;

(2) Let $V_1 = \text{span}(\underline{v_1})$, thus $B_1 = \{V_1 + \underline{w_1}, \cdots, V_1 + \underline{w_m}\}$ is a basis of V/V_1;

(3) Consider $\overline{T}_1 : V/V_1 \to V/V_1$. By calculating $[\overline{T}_1]_{B_1}$, find an eigenvector $V_1 + \underline{v_2}$ of $\overline{T}$;

(4) Let $V_2 = \text{span}(\underline{v_1}, \underline{v_2})$. Follow (2), (3) to find an eigenvector $V_2 + \underline{v_3}$ of $\overline{T}_2 : V/V_2 \to V/V_2$;

(5) Repeat to find $\underline{v_i}$. $B = \{v_1, \cdots, v_n\}$ is the desired basis.

Note: Suppose $\dim(V) = n$, then after finding the first $n-1$ basis vectors, we can pick any vector $\underline{v_n}$ that makes the list of vectors LI, because the last vector only affects the last column of $[T]_B$, but $[T]_B$ will be upper triangular regardless of the last column.

Cayley-Hamilton theorem: $c_T(T) = 0$

Euclid algorithm: Let $f, g \in \mathbb{F}[x]$ with $\deg(g) \geq 1$. Then $\exists q, r \in \mathbb{F}[x]$ s.t.

$$f = gq + r$$

where $r = 0$ or $\deg(r) < \deg(g)$.

Bezout lemma: If $d = \gcd(f, g)$, then $\exists s, r \in \mathbb{F}[x]$ s.t.

$$d = rf + sg$$

Properties of polynomials: Let $T : V \to V$ linear, p and q are polynomials. Then

(1) $\text{Ker}(p(T))$ and $\text{Im}(p(T))$ are T-invariant;

(2) $(pq)(T)(\underline{v}) = [p(T)q(T)](\underline{v}) = p(q(T)(\underline{v}))$;

(3) $\text{lcm}(f, g) \cdot \gcd(f, g) = fg$.

Irreducible polynomials:

(1) If p is irreducible, $p|fg$, then $p|f$ or $p|g$;

(2) All irreducible monic polynomials in $\mathbb{C}[x]$ takes form $x - a$; All irreducible monic polynomials in $\mathbb{R}[x]$ takes form $x - a$ or $x^2 + bx + c$, where $b^2 - 4c < 0$;

(3) The irreducible polynomials in $\mathbb{F}_2[x]$ with degree ≤ 4 are

$$x, x+1, x^2+x+1, x^3+x+1, x^3+x^2+1, x^4+x+1, x^4+x^3+1, x^4+x^3+x^2+x+1$$

Properties of minimal polynomial: Let $T : V \to V$ be linear. Then

(1) $m_T(x) | c_T(x)$;

(2) If $p(T) = 0$, then $m_T | p$;

(3) If λ is an eigenvalue of T, then $m_T(\lambda) = 0$;

(4) If $p(x)$ is an irreducible factor of $c_T(x)$, then $p(x) | m_T(x)$;

(5) Suppose W is T-invariant. Then m_{T_W} and $m_{\overline{T}}$ both divide m_T.

Companion matrix: Let $p(x) = x^n + \cdots + a_1 x + a_0$. The companion matrix

$$C(p(x)) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

has characteristic polynomial and minimal polynomial $= p(x)$.

Diagonal block matrix: Define the diagonal block matrix

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{pmatrix}$$

Then

(1) If λ is an eigenvalue of A, then its geometric multiplicity is the sum of geometric multiplicities in the sub blocks:

$$\dim(\mathrm{Ker}(A - \lambda I)) = \dim(\mathrm{Ker}(A_1 - \lambda I)) + \cdots + \dim(\mathrm{Ker}(A_n - \lambda I))$$

(2) The minimal polynomial of A is

$$m_A(x) = \mathrm{lcm}(m_{A_1}(x), \cdots, m_{A_n}(x))$$

Primary decomposition: Let V be f.d. over $\mathbb{F}$, the characteristic and minimal polynomial of $T : V \to V$ be c_T and m_T. Suppose the factorization of the characteristic and minimal polynomial into irreducible factors be

$$c_T(x) = \prod_{i=1}^{k} f_i(x)^{n_i}, \quad m_T(x) = \prod_{i=1}^{k} f_i(x)^{m_i}$$

Define $V_i = \mathrm{Ker}(f_i(T)^{m_i})$. Then

(1) $V = V_1 \oplus \cdots \oplus V_k$;

(2) Each V_i are T-invariant, and $c_{T_i}(x) = f_i(x)^{n_i}$, $m_{T_i}(x) = f_i(x)^{m_i}$.

This theorem says the whole space can be decomposed into several mutually independent subspaces, each of which corresponds to a single irreducible factor. We can choose a basis for each subspace and combine them, obtaining a basis for V. If we do a change of basis under these vectors, then the matrix of the transformation becomes block diagonal (justified by previous thms about direct sum).

Diagonalisability and minimal polynomial: The linear map $T : V \to V$ over $\mathbb{F}$ is diagonalisable iff $m_T(x)$ can be factored into distinct linear factors:

$$m_T(x) = \prod_{i=1}^{k}(x - \lambda_i)$$

where λ_i are distinct.

Existence of nontrivial invariant subspace:

(1). If $m_T(x)$ is reducible in $\mathbb{F}[x]$, then $T : V \to V$ over $\mathbb{F}$ has a nontrivial invariant subspace (that is, $\exists W \neq 0$ or V that is T-invariant)

(2). If $c_T(x)$ is irreducible in $\mathbb{F}[x]$, then $T : V \to V$ over $\mathbb{F}$ has no nontrivial invariant subspace

Jordan block: The matrix

$$J = J_n(x) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$$

satisfy

(1) $c_J(x) = m_J(x) = (x - \lambda)^n$;

(2) $a(\lambda) = n, g(\lambda) = 1$;

(3) $(J - \lambda I)^i$ has rank $n - i$. Particularly, $(J - \lambda I)^n = 0$.

Jordan Canonical Form: Let A be $n \times n$ matrix and suppose $c_A(x)$ factors as a product of linear factors over $\mathbb{F}$, then

(1) A is similar to a matrix of form

$$J = J_{n_1}(\lambda_1) \oplus J_{n_2}(\lambda_2) \oplus \cdots \oplus J_{n_k}(\lambda_k)$$

where $\sum n_i = n$;

(2) J is unique apart from changing the order of Jordan blocks.

Determine the JCF: Suppose the JCF of A is J. Write J as

$$J = (J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_a}(\lambda)) \oplus (J_{m_1}(\mu) \oplus \cdots \oplus J_{m_b}(\mu)) \oplus \cdots$$

To determine the λ-blocks, use the following results:

(1) $n_1 + \cdots + n_a = a(\lambda)$;

(2) Number of λ-blocks $= g(\lambda)$;

(3) $\max(n_1, \cdots, n_a) = r$, where r is the highest power s.t. $(x - \lambda)^r$ dividing $m_A(x)$;

(4) When the above results still unable to identify a JCF, rewrite all λ-blocks as

$$J(\lambda) = J_{n_1}(\lambda)^{a_1} \oplus J_{n_2}(\lambda)^{a_2} \oplus \cdots \oplus J_{n_r}(\lambda)^{a_r}$$

find the dimension of generalized eigenvector spaces

$$d_i = \dim(\mathrm{Ker}(A - \lambda I)^i)$$

now

$$d_1 \text{ is the number of } \lambda\text{−blocks;}$$

$$d_2 - d_1 \text{ gives the number of } \lambda\text{-blocks with size} \geq 2;$$

$$d_3 - d_2 \text{ gives the number of } \lambda\text{-blocks with size} \geq 3;$$

$$\cdots$$

so we can identify $J(\lambda)$. Repeat the process for all eigenvalues, we can identify the JCF.

Determine the Jordan basis: Given any $n \times n$ matrix A with minimal polynomial

$$m_T(x) = \prod_{i=1}^{k} (x - \lambda_i)^{m_i}$$

use primary decomposition to find a basis $B = \{v_i, \cdots, v_n\}$ s.t. after the change of basis (view A as a linear map) we get a block diagonal matrix $A_1 \oplus \cdots \oplus A_k$, where each A_i has minimal polynomial $(x - \lambda_i)^{m_i}$. Thus we can assume A has minimal polynomial $(x - \lambda)^m$.

Let $S = A - \lambda I$. The Jordan basis for S also works for A.

For each $i \geq 1$, find the generalized eigenspace

$$N_i = \mathrm{Ker}(S^i)$$

This gives a nested vector space

$$0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq N_{r-1} \subsetneq N_r = V$$

Find a basis $\{\underline{v_1}, \cdots, \underline{v_{a_r}}\}$ for $V \text{mod} N_{r-1}$. They are cyclic vectors for the blocks $J_r(0)^{a_r}$. So we get some Jordan basis vectors

$$\cdots, S^2(\underline{v_1}), S(\underline{v_1}), \underline{v_1}, \cdots, S^2(\underline{v_{a_r}}), S(\underline{v_{a_r}}), \underline{v_{a_r}}$$

If we get enough basis vectors we are done. Otherwise, consider the set of vectors

$$S(\underline{v_1}), \cdots, S(\underline{V_{a_r}})$$

which are LI in $N_{r-1} \text{mod} N_{r-2}$. Extend to a basis

$$S(\underline{v_1}), \cdots, S(\underline{V_{a_r}}), \underline{w_1}, \cdots, \underline{w_{a_{r-1}}}$$

for $N_{r-1} \text{mod} N_{r-2}$. Now $\underline{w_1}, \cdots, \underline{w_{a_{r-1}}}$ are cyclic vectors for blocks $J_{r-1}(0)^{a_{r-1}}$.

Repeat the algorithm until we get enough basis vectors corresponding to all λ-blocks. Do this for all blocks, we obtain a Jordan basis for $A_1 \oplus \cdots \oplus A_k$.

Use these Jordan basis as coefficients of linear combination of vectors in B to find the Jordan basis of the original matrix A.

Property of annihilator: Suppose $f(x)$ is the annihilator of the cyclic subspace $Z(\underline{v}, T)$. Then $f(T)(\underline{w}) = 0$ for all $\underline{w} \in Z(\underline{v}, T)$.

Properties of cyclic subspace: Let $Z = Z(\underline{v}, T)$ be a cyclic subspace of V, and suppose Z has annihilator $f(x)$ with degree k. Then $B = \{\underline{v}, T(\underline{v}), \cdots, T^{k-1}(\underline{v})\}$ is a basis for Z. Further more, the matrix of $T|_Z$ is the companion matrix $C(f)$ with respect to the basis B, so the minimal polynomial of $T|_Z$ is $f(x)$.

Rational Canonical Form: Let A be any $n \times n$ matrix. Let $m_A(x)$ be factorized as

$$m_A(x) = \prod_{i=1}^{t} f_i(x)^{k_i}$$

where $f_1(x), ..., f_t(x) \in \mathbb{F}[x]$ are distinct irreducible polynomials. Then A is similar to a block diagonal matrix of form

$$\begin{aligned} R = C\big(f_1(x)^{k_{11}}\big) \oplus \cdots \oplus C\big(f_1(x)^{k_{1r_1}}\big) \oplus \cdots \\ \oplus C\big(f_t(x)^{k_{t1}}\big) \oplus \cdots \oplus C\big(f_t(x)^{k_{tr_t}}\big) \end{aligned}$$

where $k_i = k_{11} \geq \cdots \geq k_{ir_i}$ for each i.

Determine the RCF: Let $T : V \to V$ with V f.d.. Suppose in the factorization of the minimal polynomial we have a term $f(x)^k$. We first find the RCF corresponding to that term. The RCF of all other terms can be found similarly. Thus, now we assume $T : V \to V$ has minimal polynomial $f(x)^k$.

The RCF corresponding to $f(x)^k$ has form $C\big(f(x)^{k_{11}}\big) \oplus \cdots \oplus C\big(f(x)^{k_{1r_1}}\big)$. Rewrite as

$$C(f)^{a_1} \oplus C(f^2)^{a_2} \oplus \cdots \oplus C(f^k)^{a_k}$$

Suppose the degree of $f(x)$ is d. Then from the equation above we can infer an equation

$$da_1 + 2da_2 + \cdots + kda_k = \dim(V)$$

If we write out the vector spaces spanned by the basis vectors corresponding to each diagonal block, we get

$$V = (Z_{11} \oplus \cdots \oplus Z_{1a_1}) \oplus \cdots \oplus (Z_{k1} \oplus \cdots \oplus Z_{ka_k})$$

Apply $f(T)^{k-1}$ to both sides. It sends Z_{ij} to 0 if $i < k$, and it sends Z_{kj} to a vector space of dimension d. None of the resulting vector spaces overlap, so

$$\dim(f(T)^{k-1}(V)) = \mathrm{rank}(f(T)^{k-1}|_V) = a_k d$$

And similarly

$$\dim(f(T)^{k-2}(V)) = \mathrm{rank}(f(T)^{k-2}|_V) = 2a_{k-1}d + a_k d$$

using these equations to find the values of $a_1, \cdots, a_k$.

In a more general case, say $V = V_1 \oplus V_2$, minimal polynomial factors as $f(x)^{k_1} g(x)^{k_2}$. We need to find $\mathrm{rank}(f(T)^{k-1}|_{V_1})$ and $\mathrm{rank}(g(T)^{k-1}|_{V_2})$. The trick is to use

$$\mathrm{rank}(f(T)^{k-1}|_V) = \mathrm{rank}(f(T)^{k-1}|_{V_1}) + \mathrm{rank}(f(T)^{k-1}|_{V_2})$$

Which simplifies to

$$\mathrm{rank}(f(T)^{k-1}) = \mathrm{rank}(f(T)^{k-1}|_{V_1}) + \dim(V_2)$$

Since $f(x)^{k-1}$ and $g(x)$ are co-prime.