Topic 7
Isogonal Conjugate and Isotomic Conjugate Points

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Introduction

Notes:

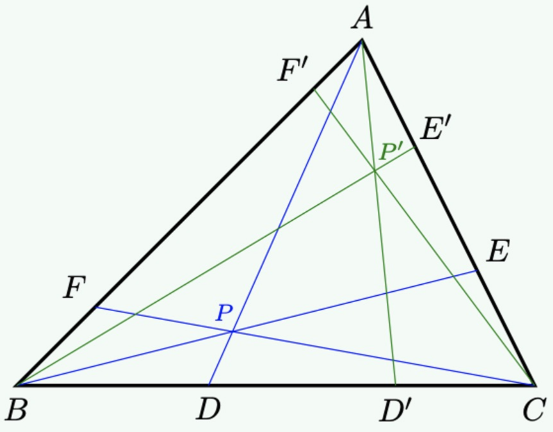
*Isogonal: having similar angles

*Conjugate: here imply a special relationship between two points

Definition 1. (Isogonal Conjugate Points)

Let P be any point. Assume that AP intersects BC at D; BP intersects CA at E; and CP intersects AB at F. The line AD' is called the isogonal conjugate line of AD, if $\angle CAD' = \angle BAD$. Let BE' and CF' be the corresponding isogonal conjugate lines similarly defined. Then AD', BE', CF' are concurrent at a point P', which is called the isogonal conjugate point of P.

Isogonal points are reflexive, that is, if P' is the isogonal conjugate point of P, then P is the isogonal conjugate point of P'.



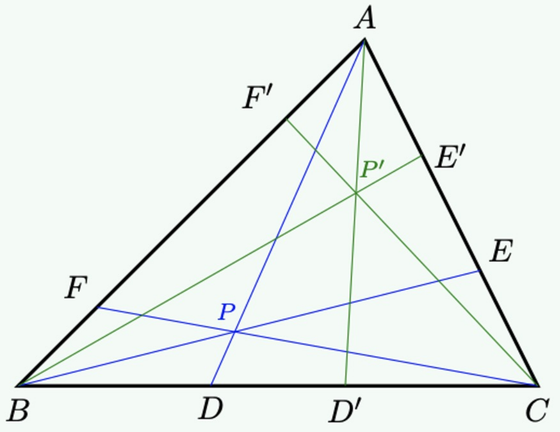
Notes:

*Isotomic: having equal but reflected cevians

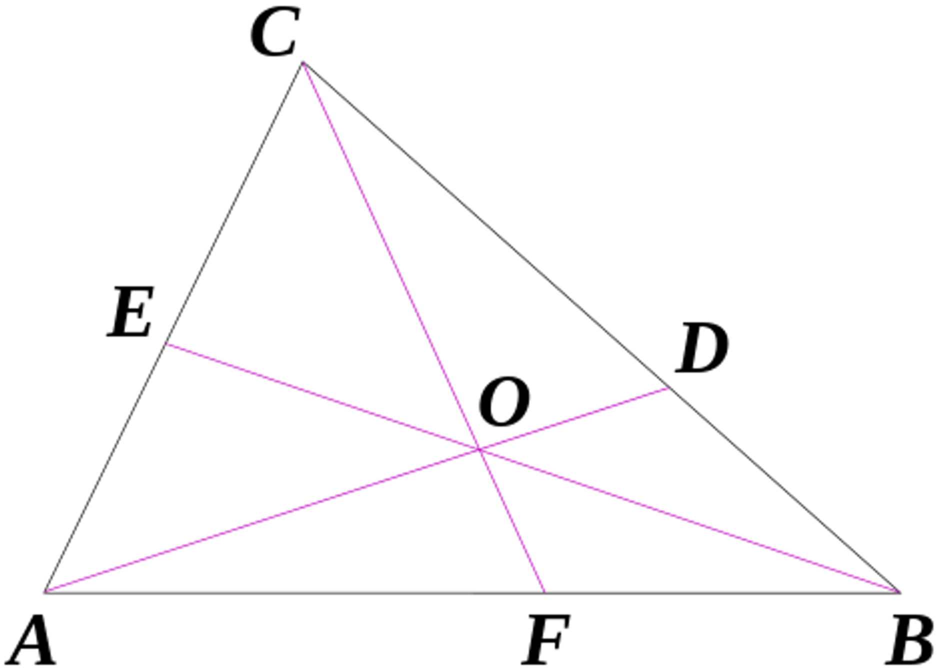
Definition 2. (Isotomic Conjugate Points)

Let P be any point. Assume that AP intersects BC at D; BP intersects CA at E; and CP intersects AB at F. The line AD' is called the isotomic conjugate line of AD, if $BD = D'C$. Let BE' and CF' be the corresponding isotomic conjugate lines similarly defined. Then AD', BE', CF' are concurrent at P'. P' is called the isotomic conjugate point of P.

Isotomic points are reflexive, that is, if P' is the isotomic conjugate point of P, then P is the isotomic conjugate point of P'.



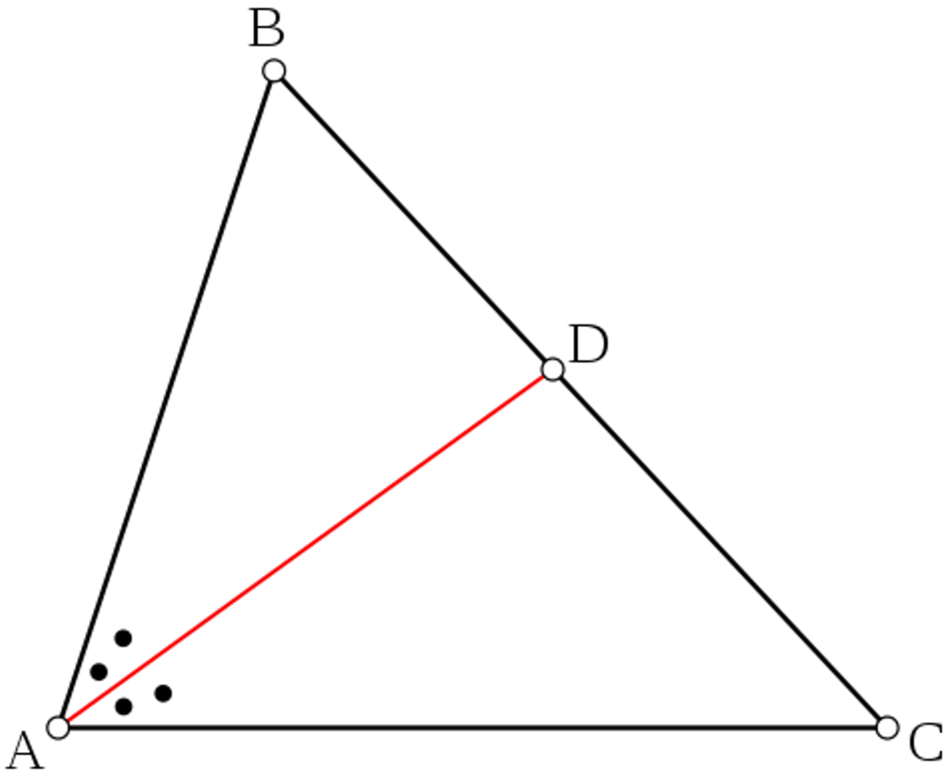
Theorem



Ceva’s Theorem

In Euclidean geometry, Ceva's theorem is a theorem about triangles. Given a triangle △ABC, let the lines AO, BO, CO be drawn from the vertices to a common point O (not on one of the sides of △ABC), to meet opposite sides at D, E, F respectively. (The segments AD, BE, CF are known as cevians.) Then, using signed lengths of segments,

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$



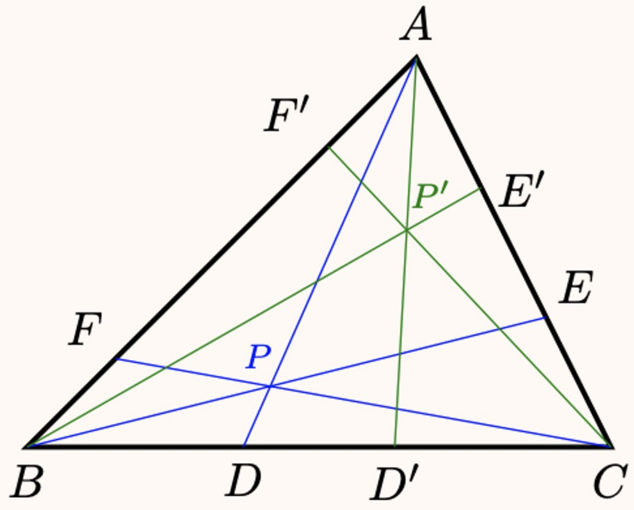
Angle bisector theorem

Consider a triangle △ABC. Let the angle bisector of angle ∠ A intersect side BC at a point D between B and C. The angle bisector theorem states that the ratio of the length of the line segment BD to the length of segment CD is equal to the ratio of the length of side AB to the length of side AC:

$$\frac{|BD|}{|CD|} = \frac{|AB|}{|AC|},$$

Theorem 1

Assume that AD, BE, CF are concurrent at P. Then their isotomic conjugate lines AD', BE', CF' are concurrent at a point P'.



Proof By definition of the isotomic conjugate lines, we have

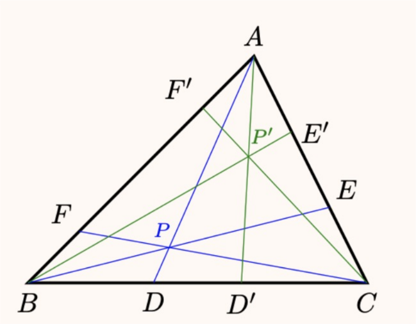
$$\frac{BD'}{D'C}=\frac{DC}{BD}=\left(\frac{BD}{DC}\right)^{-1}.$$

BD = D'C

BD' = BD + DD' = DD' + D'C = DC

Similarly, we have

$$\frac{CE'}{E'A}=\left(\frac{CE}{EA}\right)^{-1}, \qquad \frac{AF'}{F'B}=\left(\frac{AF}{FB}\right)^{-1}.$$



Thus we have

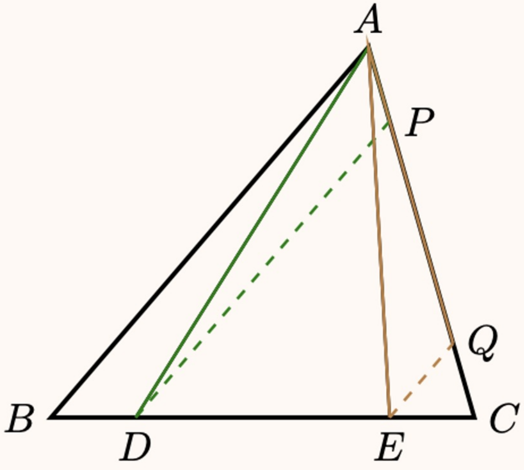
$$\frac{BD'}{D'C}\cdot\frac{CE'}{E'A}\cdot\frac{AF'}{F'B}=\left(\frac{BD}{DC}\right)^{-1}\cdot\left(\frac{CE}{EA}\right)^{-1}\cdot\left(\frac{AF}{FB}\right)^{-1}=1,$$

and hence AD', BE', CF' are concurrent. Here we used the Ceva's Theorem and its converse. ∎

Theorem 2

In the following picture, assume that $\angle BAD = \angle EAC$. Prove that

$$\frac{BD}{DC} \cdot \frac{BE}{EC} = \left(\frac{AB}{AC}\right)^2.$$



In particular, if AD is the angle bisector of $\angle A$, then the theorem is reduced to the Angel Bisector Theorem.

Proof The easiest way to prove the result is to use the Law of Sines. But in what follows, we provide a pure geometric proof.

Draw $DP \parallel EQ \parallel BA$ interesting on AC on P and Q, respectively. We have $\angle ADP = \angle BAD = \angle EAQ$ and $\angle DAP = \angle BAE = \angle AEQ$. Thus $\triangle ADP \sim \triangle EAQ$. As a result,

$$\frac{AP}{EQ} = \frac{DP}{AQ}.$$

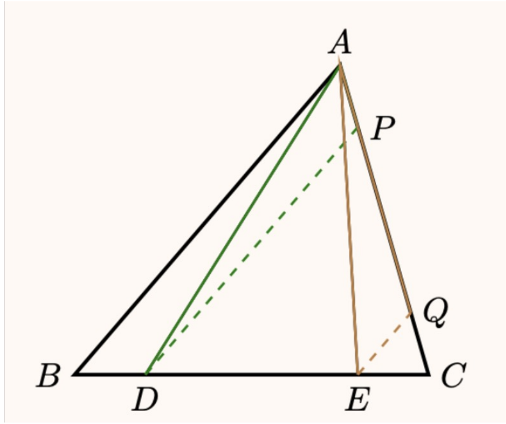
We therefore have

$$\frac{BD}{DC} \cdot \frac{BE}{EC} = \frac{AP \cdot AQ}{DC \cdot EC} = \frac{EQ \cdot DP}{DC \cdot EC}.$$

But

$$\frac{EQ}{EC} = \frac{AB}{AC}, \qquad \frac{DP}{DC} = \frac{AB}{AC}.$$

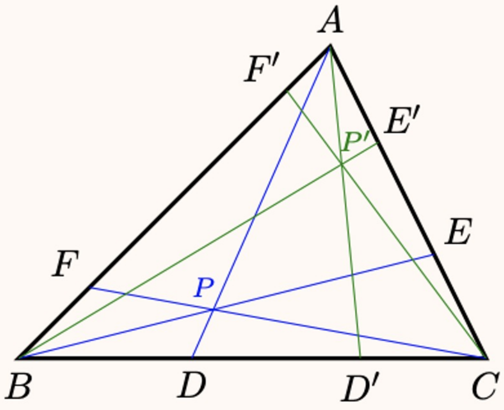
This completes the proof. ■



Since parallel,
△QEC~△ABC~△DPC

Theorem 3

Assume that AD, BE, CF are concurrent at P. Then their isogonal conjugate lines AD', BE', CF' are concurrent at a point P'.



Proof By Theorem 2,

$$\frac{BD}{DC} \cdot \frac{BD'}{D'C} = \left(\frac{AB}{AC}\right)^2, \quad \frac{CE}{EA} \cdot \frac{CE'}{E'A} = \left(\frac{BC}{CA}\right)^2, \quad \frac{AF}{FB} \cdot \frac{AF'}{F'B} = \left(\frac{CA}{BC}\right)^2.$$

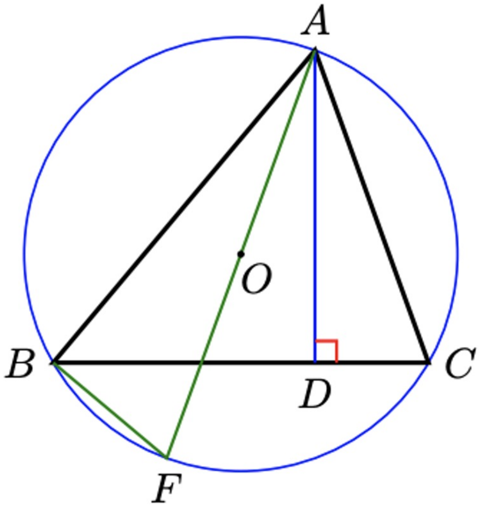
The result then follows from the Ceva's Theorem, similar to the proof of the previous theorem. ∎

Example 1 (Typical Isogonal Lines) In the following picture, AF is a diameter of the circle (where O is the circumcenter). $AD \perp BC$. Then since $\angle BFA = \angle BCA$, we have

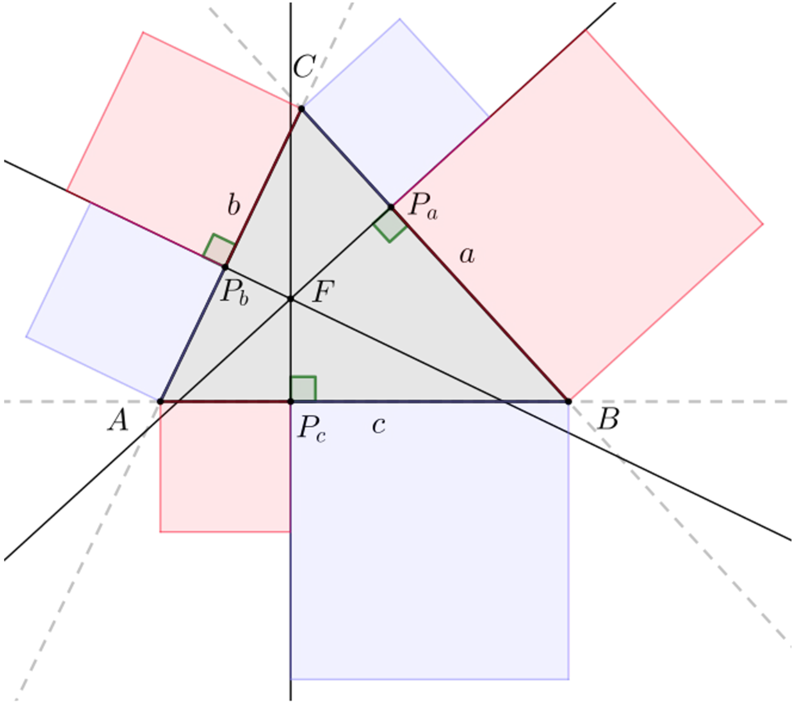
$$\angle BAF = \angle DAC.$$

Since AF is a diameter
AB⊥BF, and BFA = BCA

Thus AD and AF are isogonal lines.

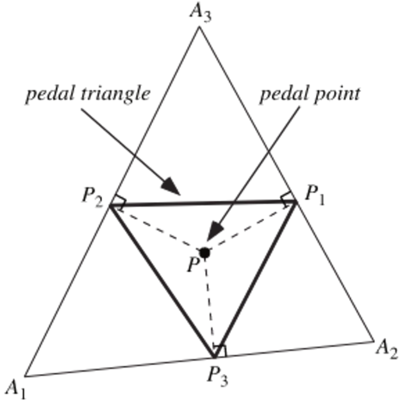


Based on the above Example 1, we can give the second proof of the above theorem using the Carnot's Theorem.



Carnot's theorem

*pedal points



For a triangle $\triangle ABC$ with sides a, b, c consider three lines that are perpendicular to the triangle sides and intersect in a common point F. If P_a, P_b, P_c are the pedal points of those three perpendiculars on the sides a, b, c, then the following equation holds:

$$|AP_c|^2 + |BP_a|^2 + |CP_b|^2 = |BP_c|^2 + |CP_a|^2 + |AP_b|^2$$

Second Proof In the following picture, let X, Y, Z be the projections of P to BC, CA, AB, respectively. The $\triangle XYZ$ is called the *pedal triangle* (see here for more details of pedal triangles).

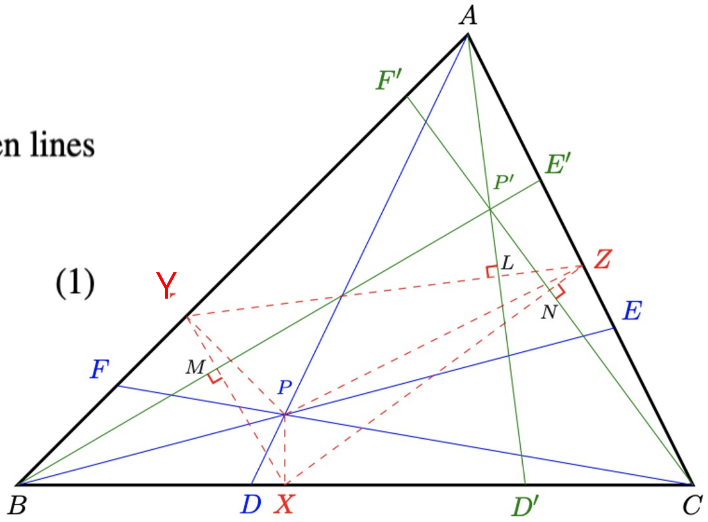
The key observation here is that the isogonal conjugate lines AD', BE', CF' are perpendicular to the corresponding sides of the pedal triangle $\triangle XYZ$. This can be proved using the following argument: since $PY \perp AB, PZ \perp AC$, $AYPZ$ is concyclic. Thus

$$\angle LAZ + \angle AZY = \angle YAP + \angle APY = 90^\circ.$$

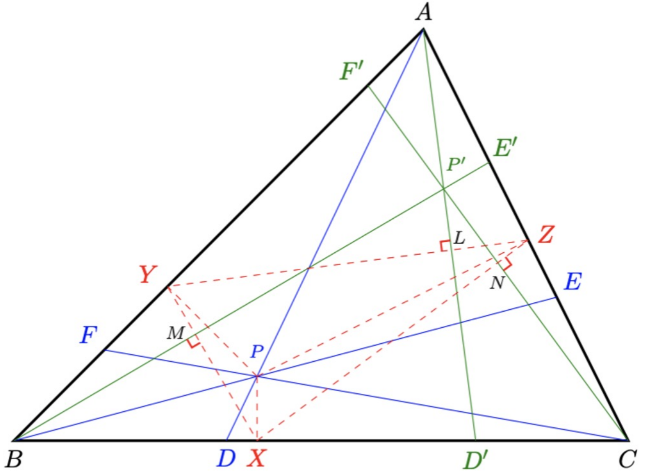
Thus $AD' \perp ZY$. Similarly, $BE' \perp XY$, and $CF' \perp AB$.

By the Carnot's Theorem (see also Topic 35), we know that the three green lines AD', BE', CF' are concurrent if

$$XL^2 - LZ^2 + ZN^2 - NX^2 + XM^2 - MY^2 = 0. \qquad (1)$$



AL⊥YZ, thus
AY^2-YL^2 = AZ^2-LZ^2=AL^2



However, we have

$$YL^2 - LZ^2 = AY^2 - AZ^2,$$

$$ZN^2 - NX^2 = CZ^2 - CX^2,$$

$$XM^2 - MY^2 = BX^2 - BY^2.$$

Therefore, Equation (1) is valid if and only if

$$AY^2 - BY^2 + BD^2 - CD^2 + CE^2 - AE^2 = 0,$$

but this follows from the Carnot's Theorem again and the fact that PX, PY, PZ are concurrent.

THANKS FOR WATCHING!

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