Brocard Point

Yff’s Inequality

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Introduction

Brocard Point

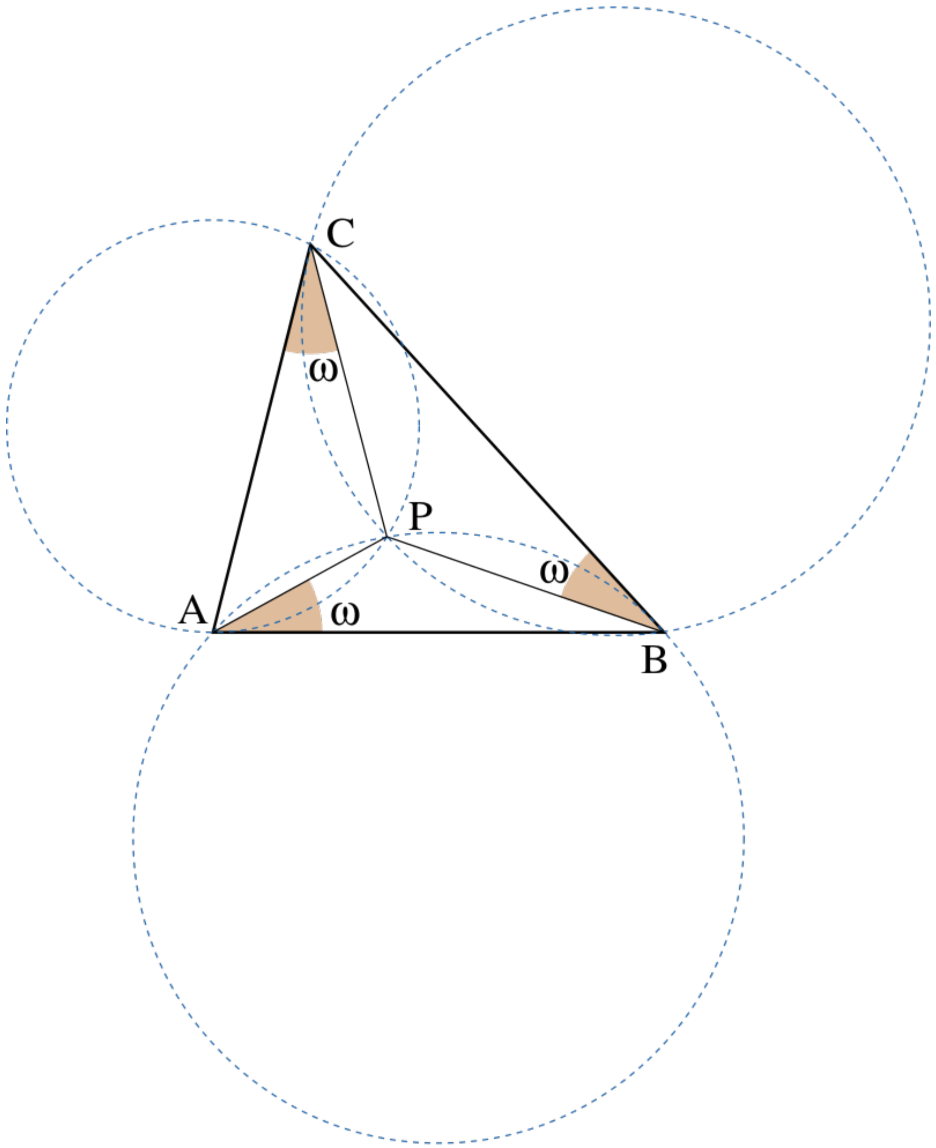
In a triangle ABC with sides a, b, and c, there is exactly one point P such that the line segments AP, BP, and CP form the same angle, ω, with the respective sides c, a, and b, namely that

$$\angle PAB = \angle PBC = \angle PCA = \omega$$

The point P is the first **Brocard point** of the triangle ABC and the ω is called the **Brocard angle** of the triangle. This angle has the property

$$\cot\omega = \cot A + \cot B + \cot C$$

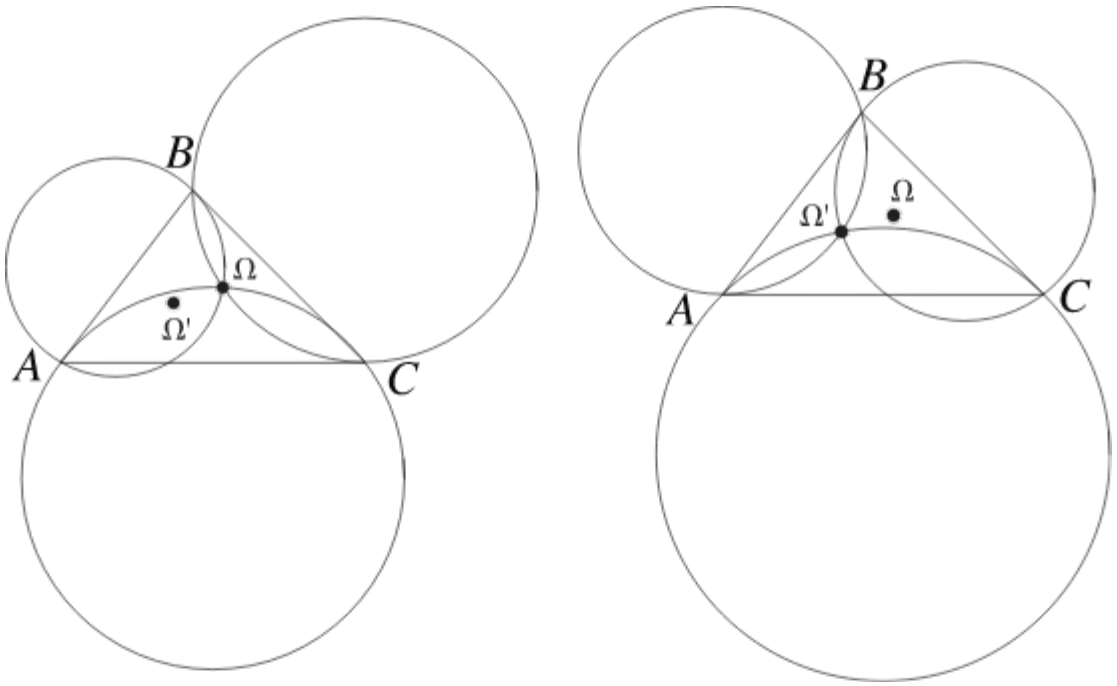
where A,B,C are three vertex angles in the triangles



Introduction

How to find Brocard Point

As in the diagram right, form a circle through points A and B, tangent to edge BC of the triangle (the center of this circle is at the point where the perpendicular bisector of AB meets the line through point B that is perpendicular to BC). Symmetrically, form a circle through points B and C, tangent to edge AC, and a circle through points A and C, tangent to edge AB. These three circles have a common point, the first Brocard point of triangle ABC.



Introduction

Yff’s Inequality

Yff's inequality was introduced by the American mathematician Peter Yff in 1963.
let ΔABC *be a triangle, let* ω *be the brocard angle of* ΔABC*. Then, Yff’s inequality is*

$$8\omega^3 \leq ABC$$

where A,B,C are three angles of the triangle measure in radians.

Method 1 : Abi-Khuzam Inequality

Given : $\cot\omega = \cot A + \cot B + \cot C$

$\csc^2\omega = \csc^2 A + \csc^2 B + \csc^2 C,$

$\omega \leq \frac{\pi}{6}$

By considering a decreasing steadily function $\frac{\sin x}{x}$ *in the interval*

$0 < x < \frac{\pi}{2}$

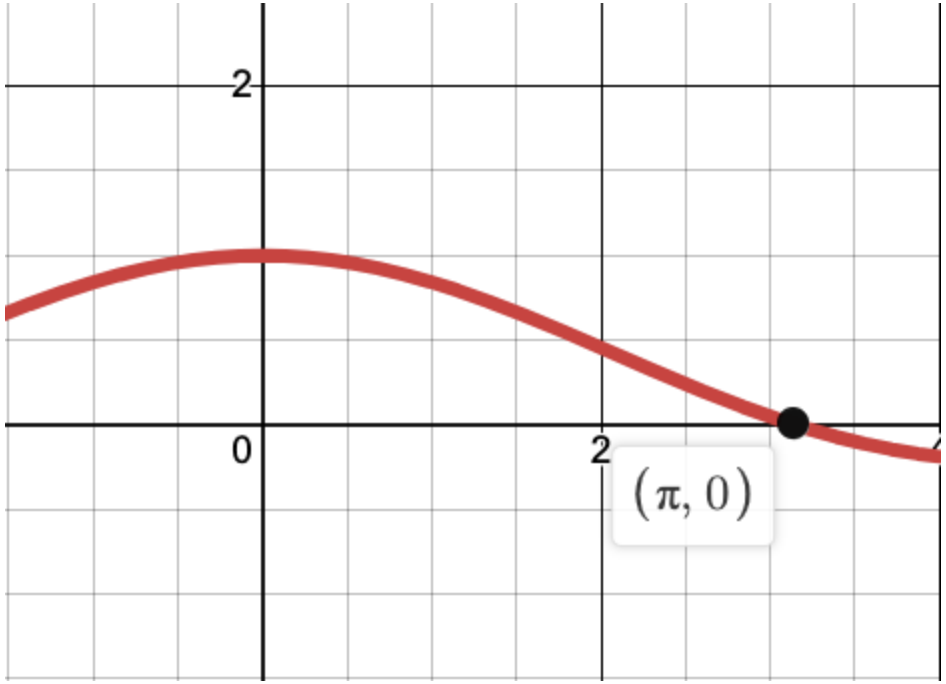
then, since the $\omega \leq \frac{\pi}{6}$, *it implies* $\frac{\sin\omega}{\omega} \geq \frac{3}{\pi}$, *and* $\frac{\pi}{3\omega} \geq \csc\omega.$

Applying the inequality between the arithmetic and geometric mean, we can obtain:

$$\frac{\csc^2 A+\csc^2 B+\csc^2 C}{3} \geq \sqrt[3]{\csc A \csc B \csc C}^2$$

then, we can further get:

$$\left(\frac{\pi}{3\omega}\right)^2 \geq \csc^2\omega \geq 3\left(\sqrt[3]{\csc A \csc B \csc C}\right)^2$$



Method 1: Abi-Khuzam Inequality

Let $\sin x = x \prod_{n=1}^{\infty}[1-(\frac{x}{\pi n})^2] = x\ P(x)$*,*

Then we have

$$P(x_1)P(x_2)P(x_3) = \prod_{i=1}^{3}\prod_{n=1}^{\infty}[1-(\frac{x_i}{\pi n})^2]$$

after using inequality between arithmetic-geometiric mean, we can get :

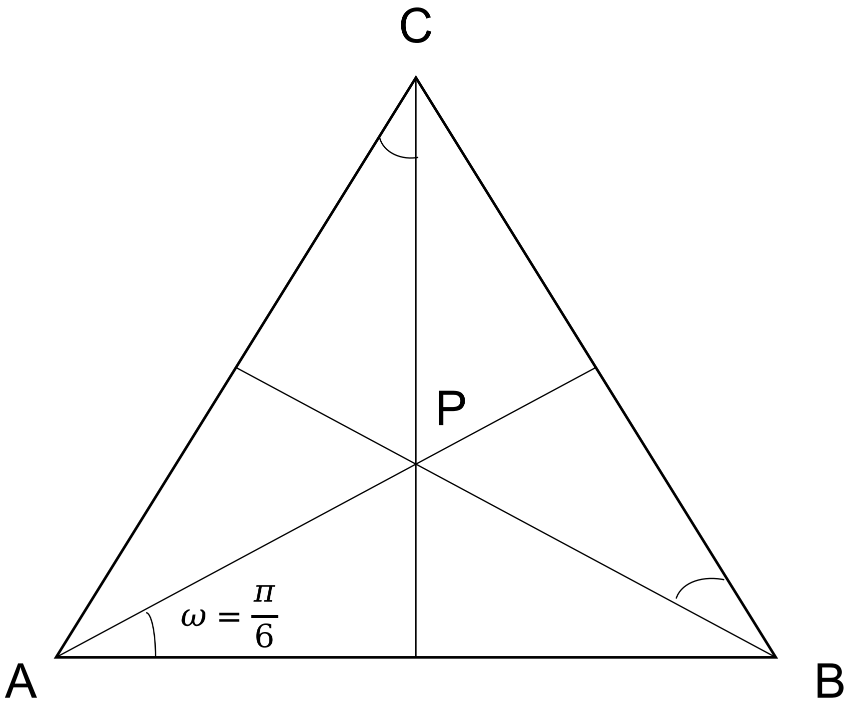
$$P(x_1)P(x_2)P(x_3) \le \prod_{n=1}^{\infty}[1-(\frac{1}{3n})^2]^3 = [P(\frac{\pi}{3})]^3 = (\frac{3\sqrt{3}}{2\pi})^3$$

Abi-Khuzam inequality *states :*

$$\sin A \sin B \sin C \le ABC(\frac{3\sqrt{3}}{2\pi})^3$$

Then, We can use $\sin x = \frac{1}{\csc x}$ *compute this formula, finally, we get:*

$$2\omega \le \sqrt[3]{ABC}$$



Method 2 : Marian Dincă Function

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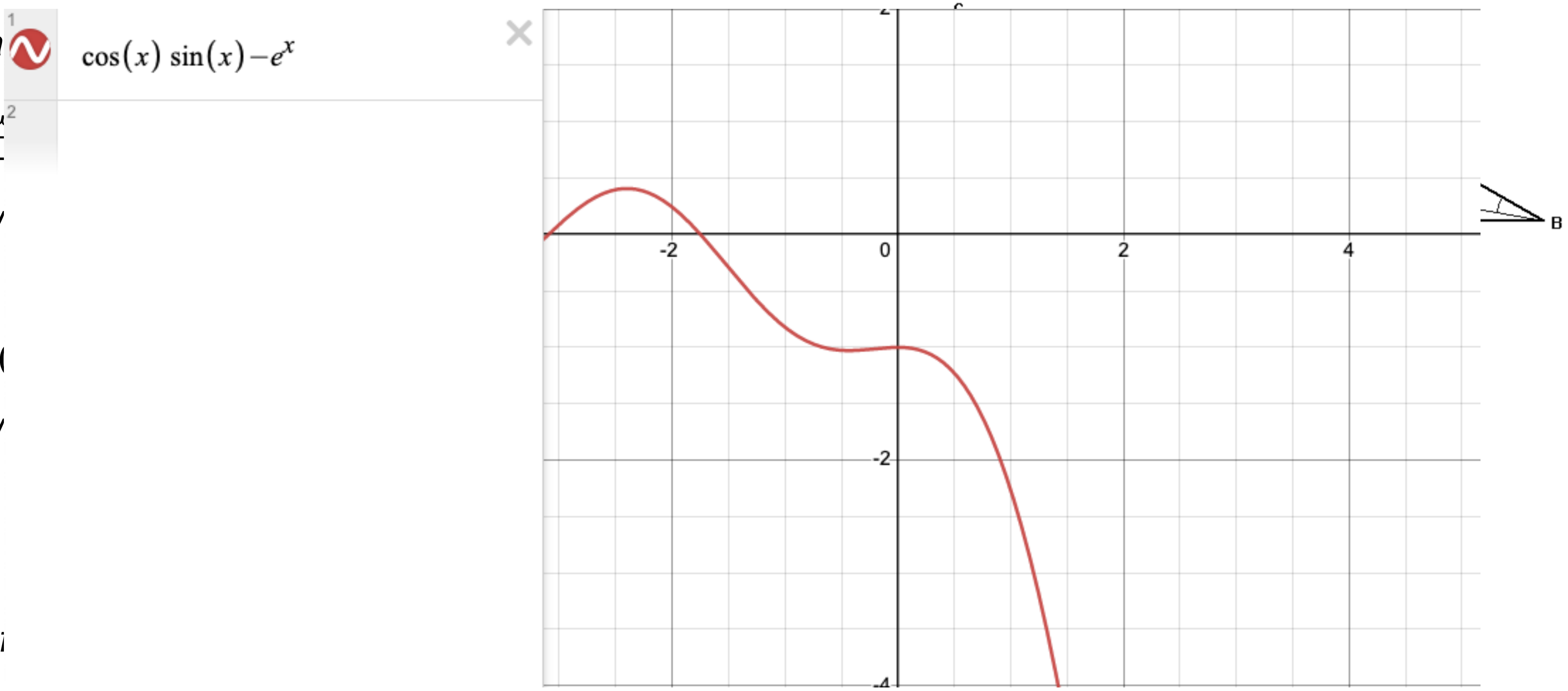
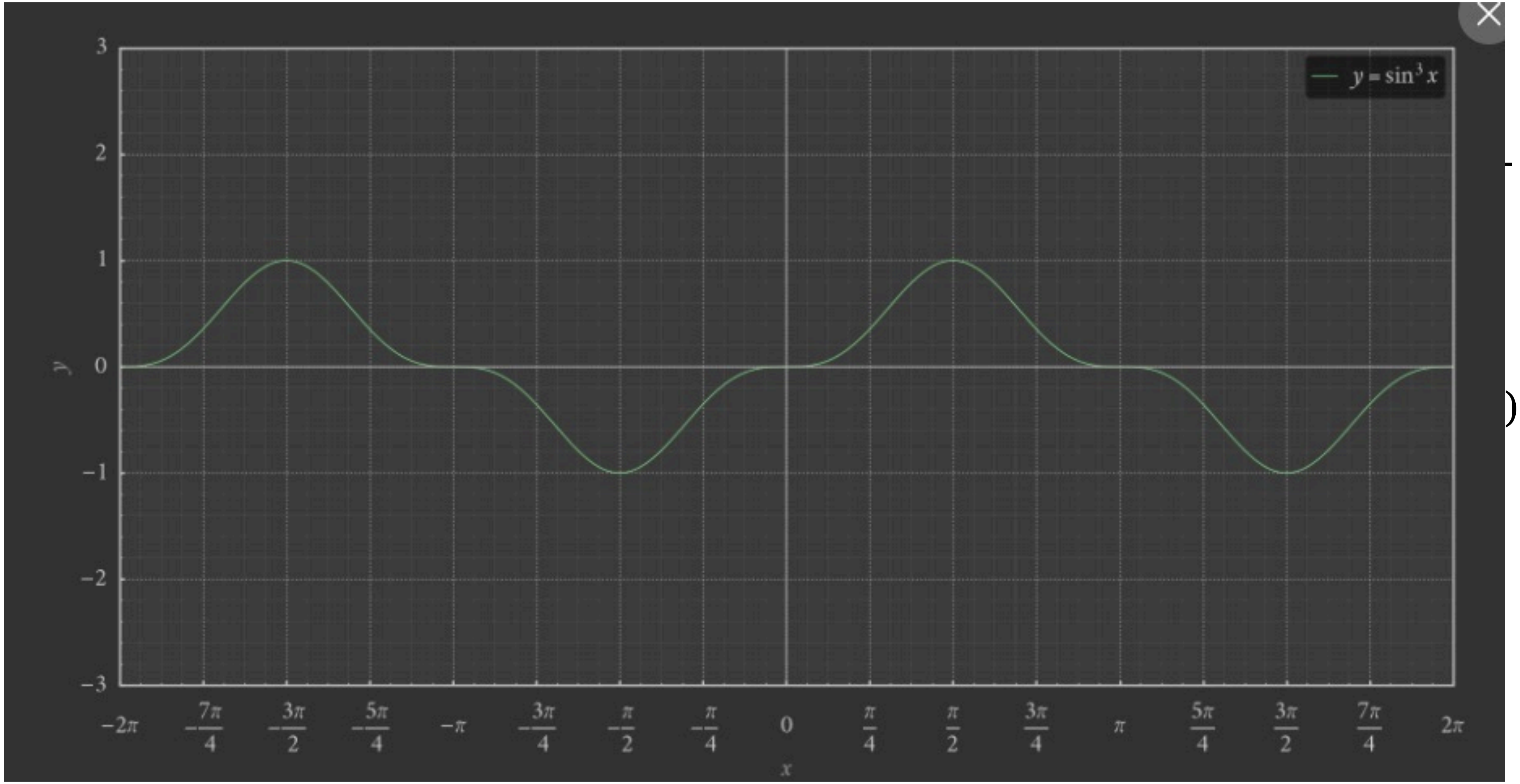


Method 2: Marian Dincă Function



$\omega)]$

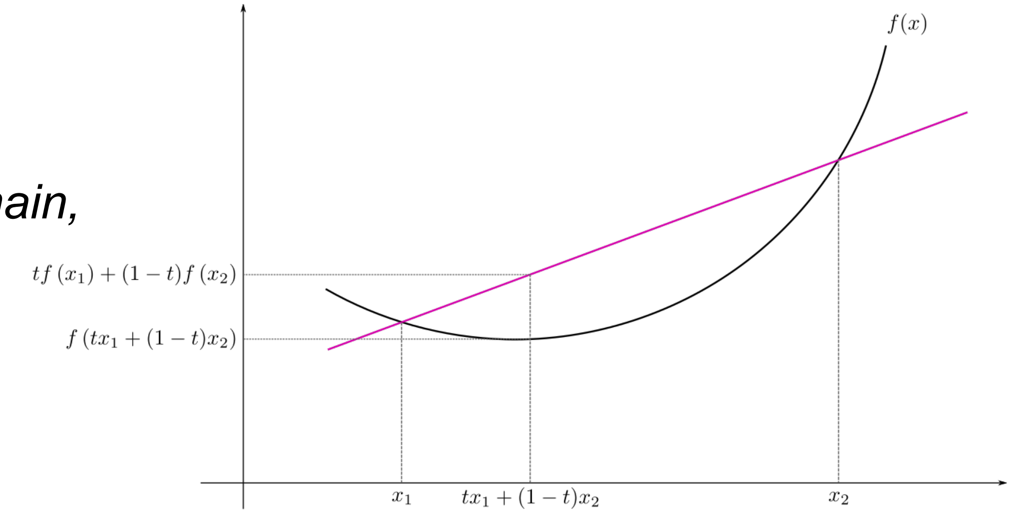
$),$

Addition inequality

Jensen's inequality

For a real convex function f(x), numbers $x_1, x_2 \cdots\cdots x_n$ in its domain, and positive scalar α_i, Jensen's inequality states as:

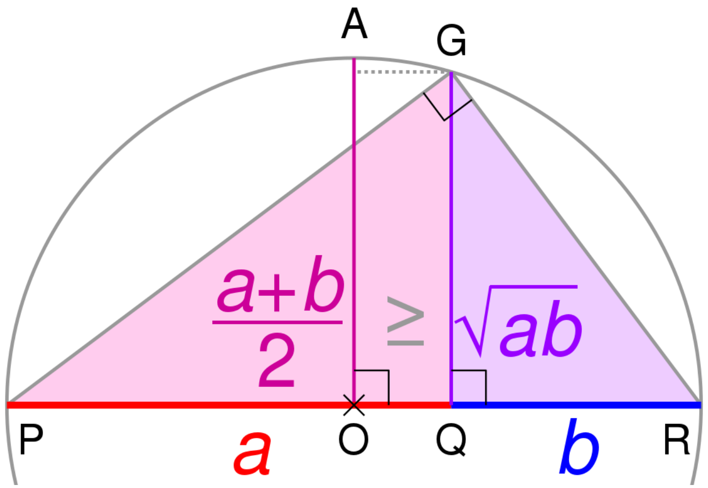
$$f\left(\frac{\sum \alpha_i x_i}{\sum \alpha_i}\right) \leq \frac{\sum \alpha_i f(x_i)}{\sum \alpha_i}$$



Inequality of arithmetic and geometric means

we have that for any list of n nonnegative real numbers $x_1, x_2 \cdots\cdots x_n$,

$$\frac{x_1 + x_2 + \cdots\cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots\cdots x_n}$$



THANKS