The Ptolemy's Theorem and Kelvin Transform

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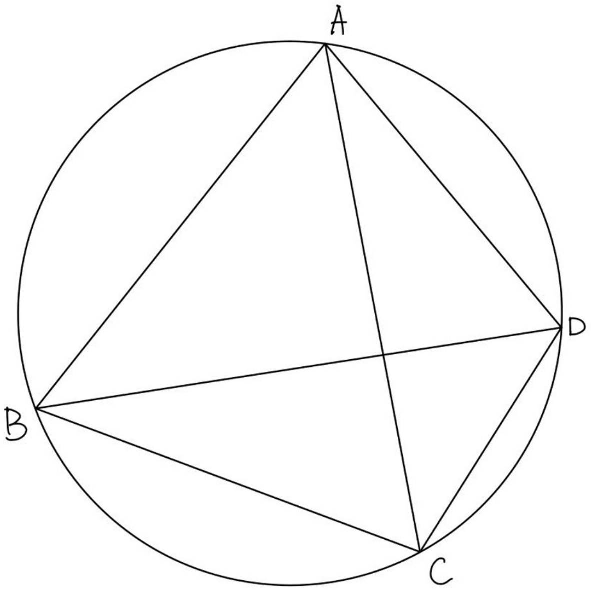
Ptolemy's Theorem

Let ABCD be a cyclic quadrilateral, then:

$AC \cdot BD = AB \cdot CD + AD \cdot BC$



Proof 1

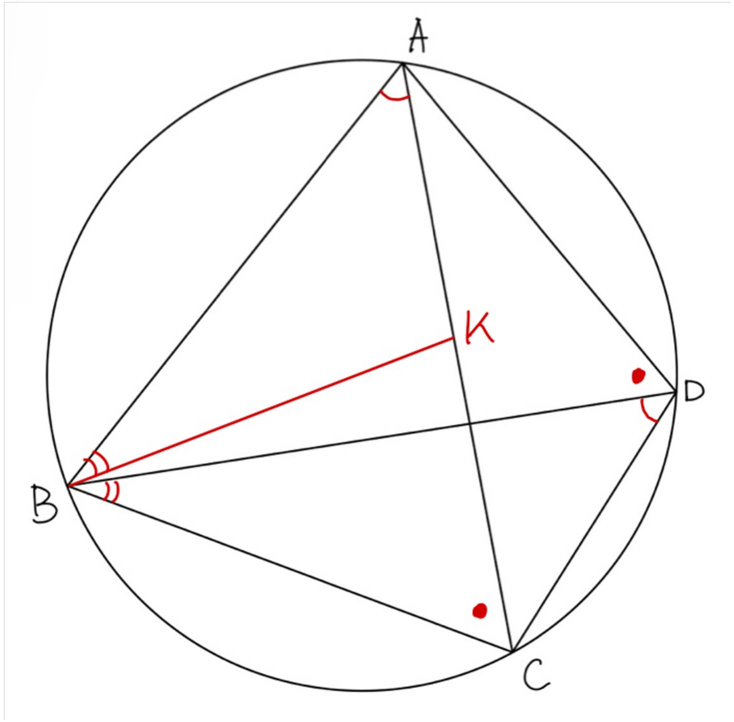
Define point K on AC, ∠ABK = ∠DBC

A, B, C, D are concyclic -> ∠BAK = ∠BDC

Thus ΔABK is similar to ΔDBC

$$\frac{AB}{BD} = \frac{AK}{CD}$$

$$AB \cdot CD = BD \cdot AK$$

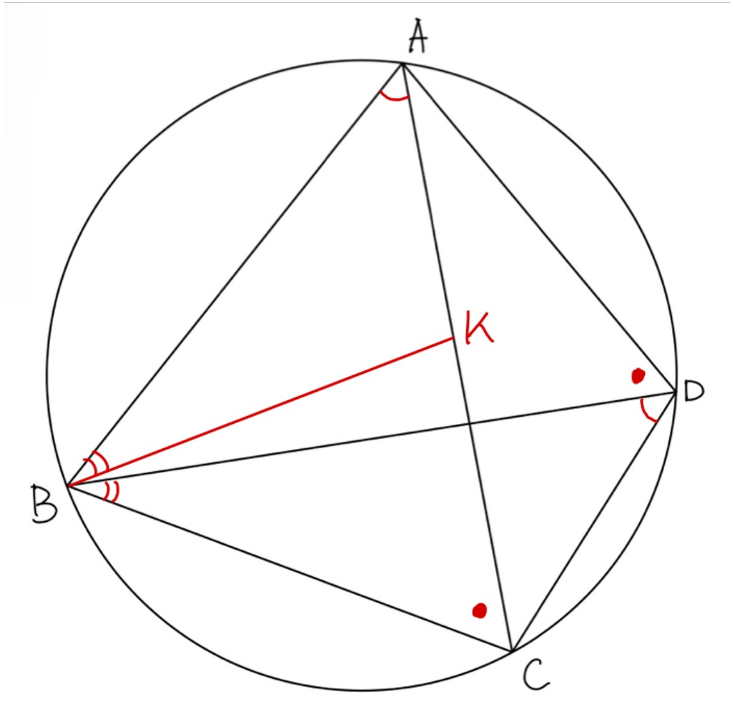


Proof 1

ΔKBC is similar to ΔABD

$$\frac{BC}{BD} = \frac{KC}{AD}$$

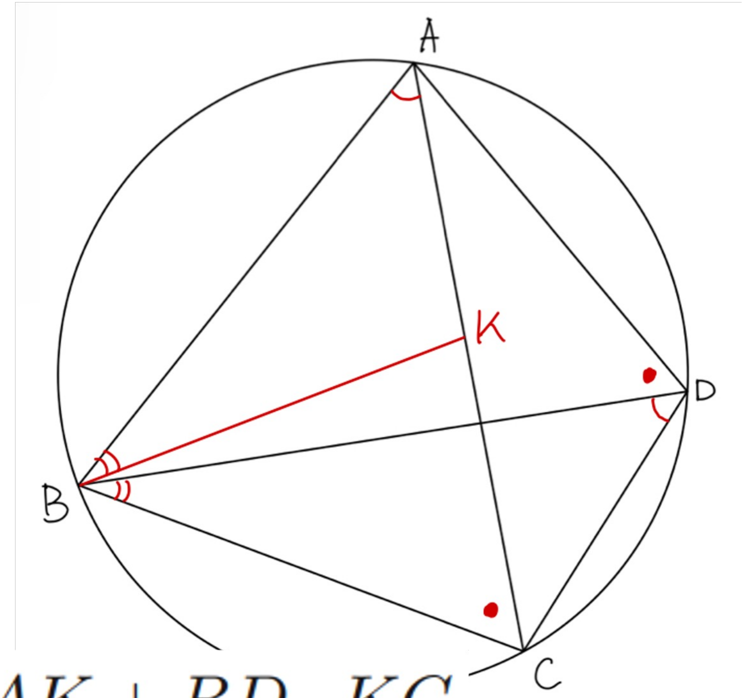
$$AD \cdot BC = BD \cdot KC$$



Proof 1

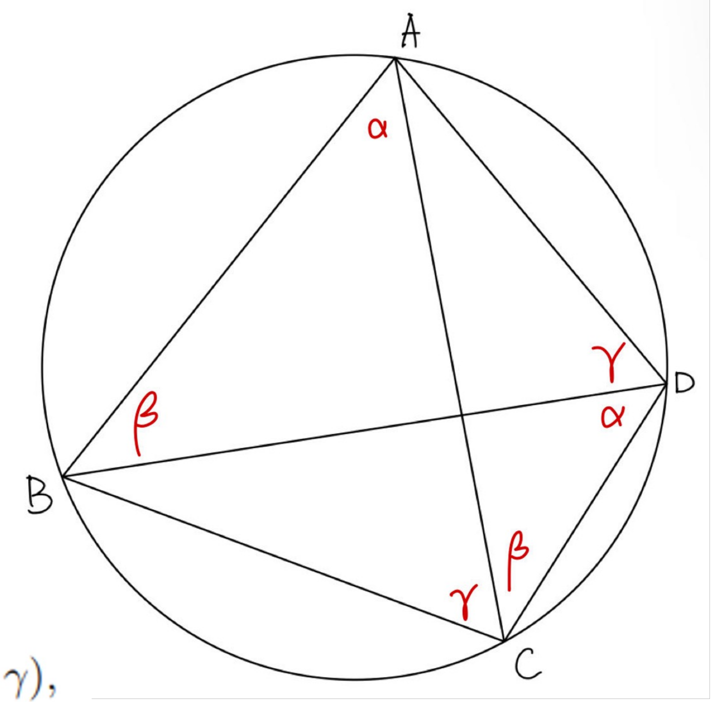
$$AB \cdot CD = BD \cdot AK$$

$$AD \cdot BC = BD \cdot KC$$



$$AB \cdot CD + AD \cdot BC = BD \cdot AK + BD \cdot KC$$
$$= BD \cdot AC$$

Proof 2



For concyclic triangles:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

So

$$AB = 2R\sin(\gamma), \quad BC = 2R\sin(\alpha),$$

$$CD = 2R\sin(180 - \alpha - \beta - \gamma) = 2R\sin(\alpha + \beta + \gamma),$$

$$DA = 2R\sin(\beta).$$

$$AC = 2R\sin(\alpha + \gamma), \quad BD = \sin(\beta + \gamma).$$

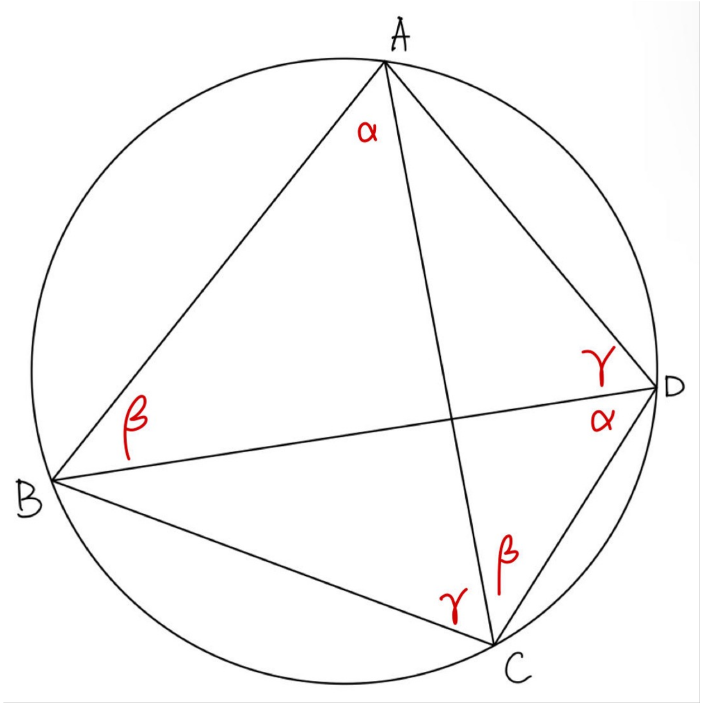
Proof 2

For concyclic triangles:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

So

$$\sin(\alpha+\gamma)\cdot\sin(\beta+\gamma)$$
$$= \sin(\alpha)\cdot\sin(\beta) + \sin(\gamma)\cdot\sin(\alpha+\beta+\gamma)$$



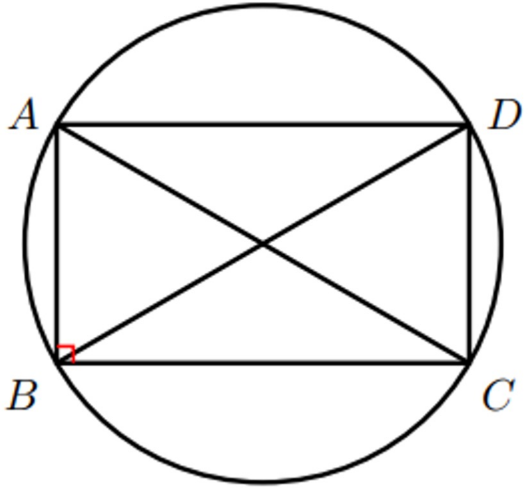
Pythagorean Theorem

Let ΔABC be a right triangle. ∠ABC = 90◦ , then:

$$AB^2 + BC^2 = AC^2$$

By the Ptolemy Theorem:

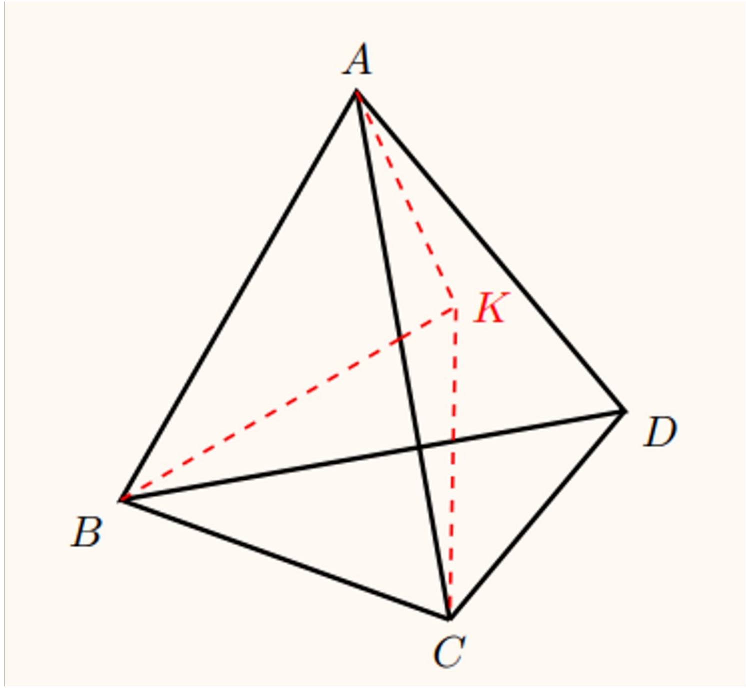
$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$



Ptolemy Inequality

Let ABCD be a quadrilateral, then:

$AC \cdot BD \leq AB \cdot CD + AD \cdot BC$

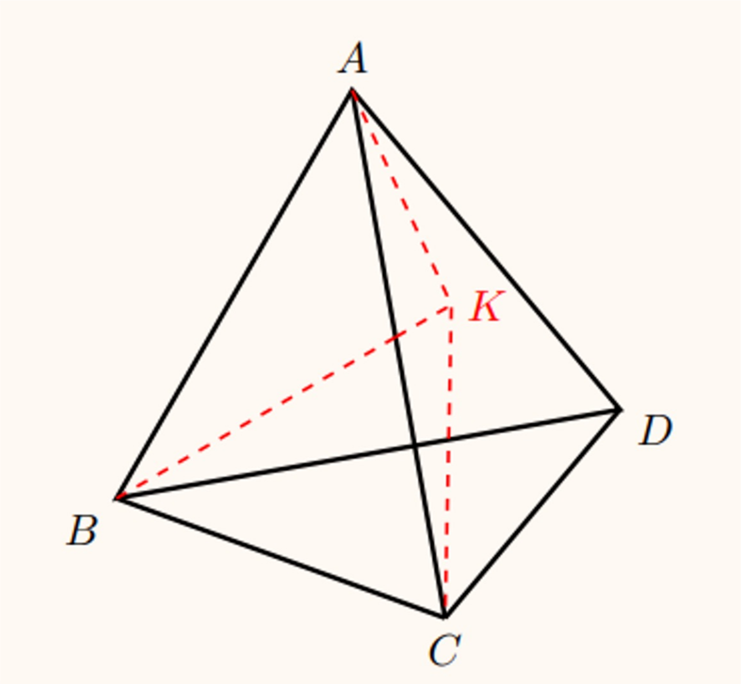


Ptolemy Inequality

$$\angle ABK = \angle DBC$$

$$\frac{AB}{DB} = \frac{BK}{BC}.$$

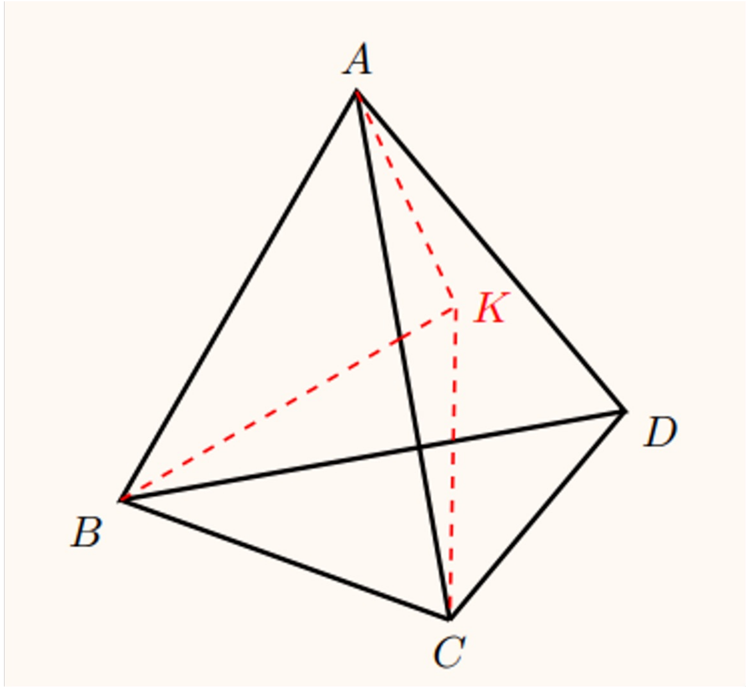
$$\frac{AB}{BK} = \frac{DB}{BC}. \qquad \frac{AK}{CD} = \frac{AB}{BD} \qquad \frac{AD}{KC} = \frac{DB}{BC}.$$



Ptolemy Inequality

$$AB \cdot CD = BD \cdot AK,$$

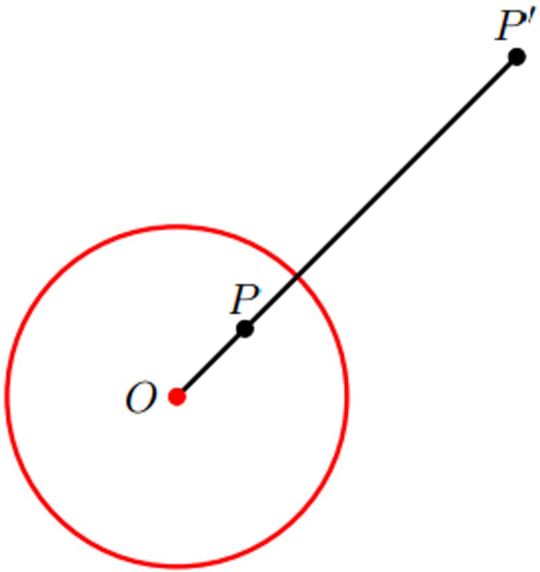
$$AD \cdot BC = BD \cdot KC.$$



$$AB \cdot CD + AD \cdot BC = BD \cdot (AK + KC) \geq BD \cdot AC.$$

Kelvin Transform

$$OP \cdot OP' = r^2.$$

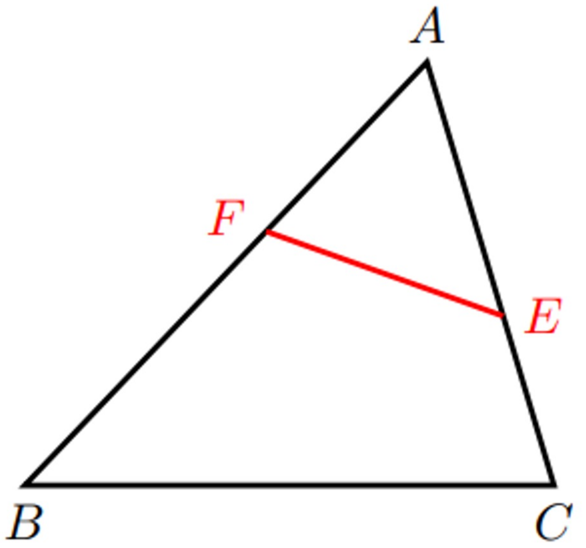


Preparation

$$AE \cdot AC = 1, \quad AF \cdot AB = 1.$$

$$AE \cdot AC = AF \cdot AB,$$

$$\frac{AE}{AB} = \frac{AF}{AC}.$$



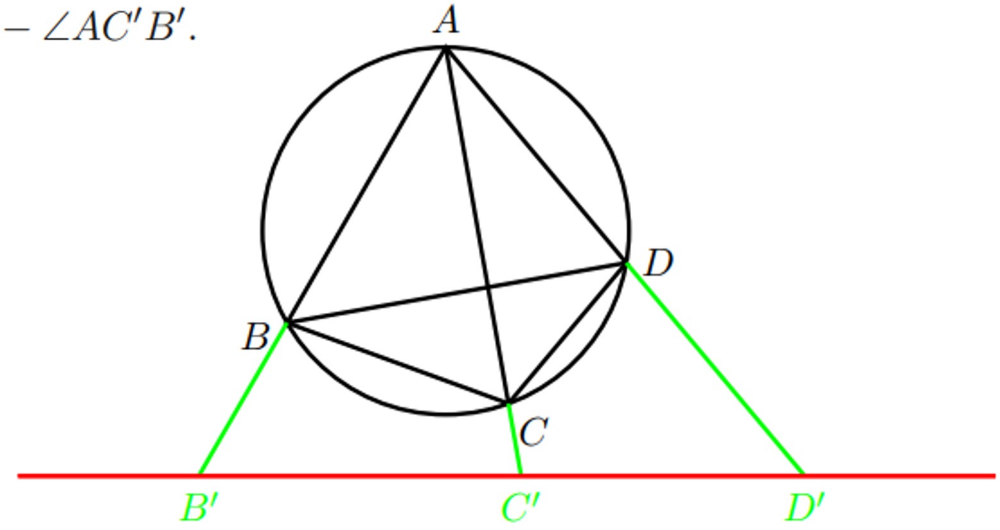
Line EF is called an *anti-parallel* line.

$$BC = \frac{AB}{AE} \cdot EF. \qquad BC = \frac{EF}{AE \cdot AF}.$$

Proof 3

$$AB \cdot AB' = AC \cdot AC' = AD \cdot AD' = 1.$$

$$\angle AC'B' = \angle ABC = 180^\circ - \angle ADC = 180^\circ - \angle AC'B'.$$



$$B'C' = \frac{BC}{AB \cdot AC}$$

$$B'D' = \frac{BD}{AB \cdot AD}$$

$$C'D' = \frac{CD}{AC \cdot AD}$$

$$\frac{BD}{AB \cdot AD} = \frac{BC}{AB \cdot AC} + \frac{CD}{AC \cdot AD}$$

The End