The Ptolemy's Theorem and Kelvin Transform

Zhiqin Lu, zlu@uci.edu

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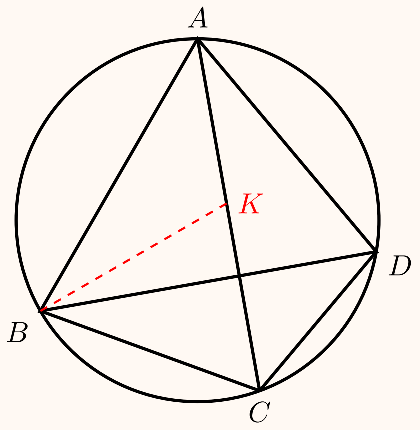
Theorem 1. (Ptolemy Theorem)

Let $ABCD$ be a cyclic quadrilateral. Then

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$



External Link. *The above result is known as the the Ptolemy Theorem. See the history note about the theorem in the Wikipedia.*

Solution: We define a point K on AC such that $\angle ABK = \angle DBC$. Since A, B, C, D are cyclic, we must have $\angle BAK = \angle BDC$. As a result, ΔABK is similar to ΔDBC. Thus we have

$$\frac{AB}{BD} = \frac{AK}{CD},$$

which is equivalent to

$$AB \cdot CD = BD \cdot AK. \tag{1}$$

Using the same method, we obtained that ΔKBC is similar to ΔABD and hence

$$\frac{BC}{BD} = \frac{KC}{AD},$$

or

$$AD \cdot BC = BD \cdot KC. \tag{2}$$

Therefore by (1) and (2), we have

$$AB \cdot CD + AD \cdot BC = BD \cdot AK + BD \cdot KC = BD \cdot AC.$$

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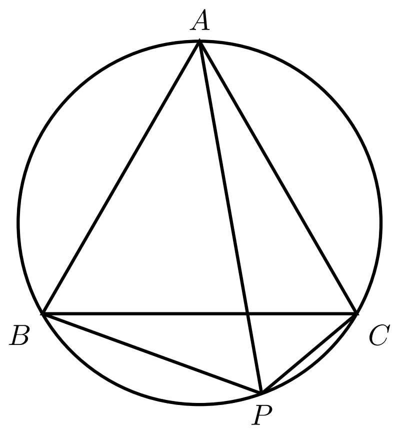
We have the following examples about the theorem.

Example 1 In the following picture, ΔABC is an equilateral triangle. Let P be a point on the arc BC. Prove that

$$PA = PB + PC.$$



Solution: Using the Ptolemy Theorem, we have

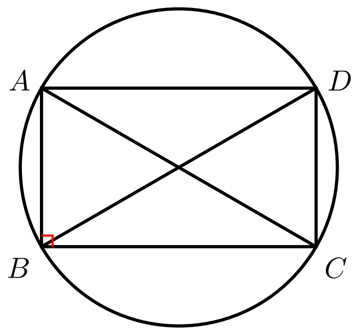
$$BC \cdot AP = AB \cdot CP + BP \cdot AC.$$

Since ΔABC is equilateral, we have $AB = BC = CA$. Thus we have

$$PA = PB + PC.$$

Example 2 (Pythagorean Theorem) Let ΔABC be a right triangle. Then

$$AB^2 + BC^2 = AC^2.$$



Solution: By the Ptolemy Theorem, we have

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

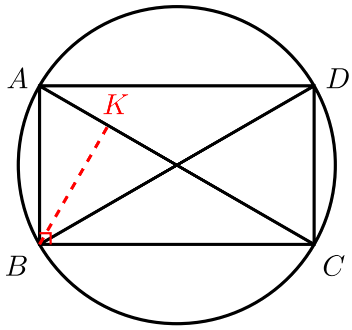
Since $BD = AC$, $AB = CD$ and $AD = BC$, we get the conclusion.

Example 3 (Einstein's Proof of the Pythagorean Theorem) Draw $BK \perp AC$. Then we have

$$AB^2 = AK \cdot AC, \qquad BC^2 = CK \cdot AC.$$

Therefore

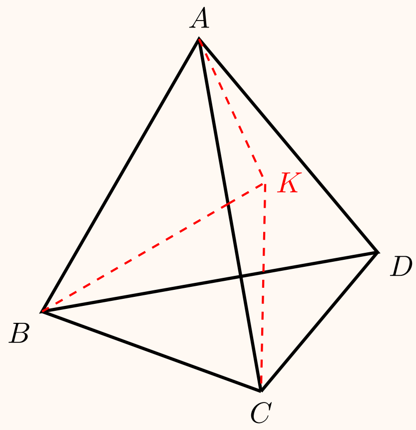
$$AB^2 + BC^2 = AK \cdot AC + CK \cdot AC = AC^2.$$



Theorem 2. The Ptolemy Inequality

Let $ABCD$ be a quadrilateral (not necessarily con-cyclic). Then

$$AC \cdot BD \leq AB \cdot CD + AD \cdot BC.$$



Solution: We use the similar method as before. But this time K does not have to be on the line AC.

Define the point K as follows. We let $\angle ABK = \angle DBC$, and assume that

$$\frac{AB}{DB} = \frac{BK}{BC}.$$

Then ΔABK is similar to ΔDBC. If we rewrite this equation as

$$\frac{AB}{BK} = \frac{DB}{BC}.$$

Then from $\angle ABD = \angle KBC$, we conclude that ΔABD is similar to ΔKBC. Thus we have

$$\frac{AK}{CD} = \frac{AB}{BD}, \quad \frac{AD}{KC} = \frac{DB}{BC}.$$

By taking the cross products, we have

$$AB \cdot CD = BD \cdot AK, \qquad AD \cdot BC = BD \cdot KC.$$

Thus

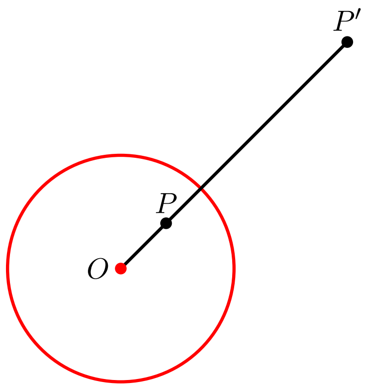
$$AB \cdot CD + AD \cdot BC = BD \cdot (AK + KC) \geq BD \cdot AC.$$

■

In the above proof of the Ptolemy Inequality, we used the triangular inequality only. Is it possible that the Ptolemy Inequality in our Universe is merely the triangular inequality in another Universe?

In the following, we define the Kelvin transform. Kelvin transform is a geometric transform. It contains a fixed point O,and a constant $k > 0$. Point P is mapped to Point P' such that

$$OP \cdot OP' = k^2.$$



Here the point O is called the *center* of the transform, and k is called the *radius* of the transform.

The transform is also known as the *Inverse Transform*.

Example 4 In the following ΔABC. Assume that E, F are points on AC and AB respectively. Assume that

$$AE \cdot AC = 1, \quad AF \cdot AB = 1.$$

Then E is the Kelvin transform of C with center A and radius 1, and F is the Kelvin transform of B with center A and radius 1.

Since

$$AE \cdot AC = AF \cdot AB,$$

we have

$$\frac{AE}{AB} = \frac{AF}{AC}.$$

As a result, ΔAEF is similar to ΔABC. Line EF is called an *anti-parallel* line.

From this, we have

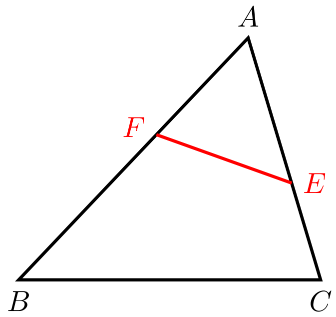
$$\frac{BC}{EF} = \frac{AB}{AE}.$$

Thus we have

$$BC = \frac{AB}{AE} \cdot EF.$$

Noting that $AB \cdot AF = 1$, we obtain

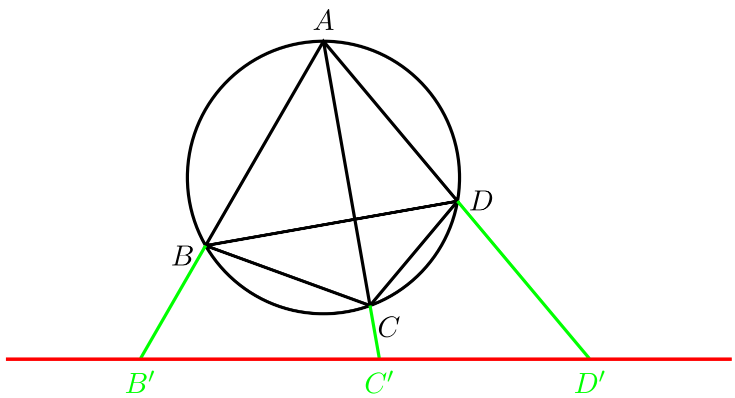
$$BC = \frac{EF}{AE \cdot AF}. \tag{3}$$



With the above preparation, let's give the Ptolemy Theorem a proof using the Kelvin transform.

Solution: In the following picture, we use A as the center and any positive number k (we take $k = 1$) as the radius to do the Kelvin transform. Assume that B', C', D' are the transform of B, C, D, respectively. Then

$$AB \cdot AB' = AC \cdot AC' = AD \cdot AD' = 1. \tag{4}$$



Using the above example, we have

$$\angle AC'B' = \angle ABC = 180^\circ - \angle ADC = 180^\circ - \angle AC'B'.$$

Thus B', C', D' are collinear. Using (3), we have

$$B'C' = \frac{BC}{AB \cdot AC}, \qquad C'D' = \frac{CD}{AC \cdot AD}, \qquad B'D' = \frac{BD}{AB \cdot AD}.$$

From $B'D' = B'C' + C'D'$, we get

$$\frac{BD}{AB \cdot AD} = \frac{BC}{AB \cdot AC} + \frac{CD}{AC \cdot AD}.$$

The Ptolemy Theorem follows from the above equation. ■

So the Ptolemy Theorem in our Universe, after the Kelvin Transform, becomes a trivial fact that $B'D' = B'C' + C'D'$.

Remark Using the Kelvin Transformation, we can prove the Ptolemy Inequality as well. In fact, the Ptolemy Inequality is also true in 3-dimensional space. Using the Kelvin Transformation, one can prove the three dimensional version.