On Ceva's and Melenaus' Theorems

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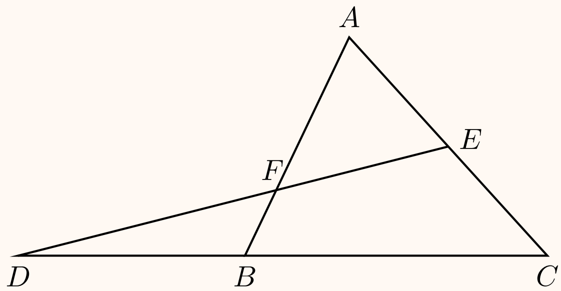
1 The Ceva's and Menelaus' Theorems

The Menelaus Theorem, named for *Menelaus of Alexandria*, is a very powerful theorem in proving collinearity of points on the Euclidean plane.

Theorem 1. (Menelaus' Theorem)

In the following $\triangle ABC$*,* D, E, F *are points on* BC*,* CA*, and* AB*, respectively. Assume that* D, E, F *are collinear. Then*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$



Conversely, if the above equation is valid, then the points D, E, F *are collinear.*

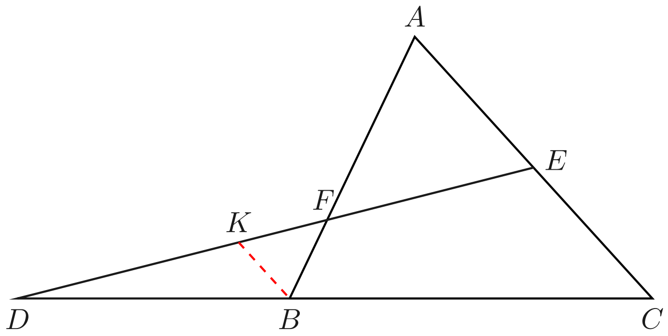
External Link. *Here is the Menelaus Theorem in Wikipedia.*

Proof: In the picture below, we draw BK parallel to CA. The idea is to represent all three ratios as ratios on line DE. First, we have

$$\frac{AF}{FB} = \frac{FE}{KF}, \qquad \frac{BD}{DC} = \frac{DK}{DE}$$

We also have

$$\frac{CE}{EA} = \frac{CE}{BK} \cdot \frac{BK}{EA} = \frac{DE}{DK} \cdot \frac{KF}{FE}.$$



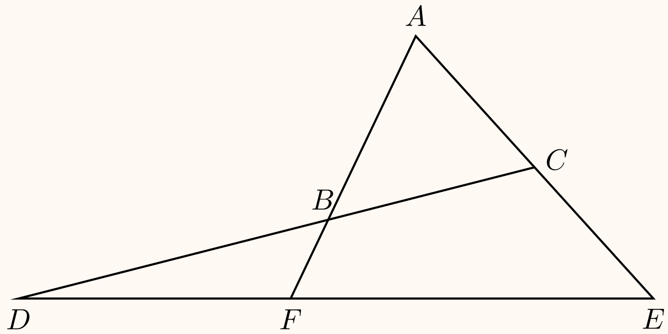
Putting the above all together, we prove the theorem. ∎

There is a variant of the Menelaus Theorem when the line DEF intersect the extended lines of the three sides of the triangle. The theorem is called the *External Menelaus Theorem*.

Theorem 2. (External Menelaus' Theorem)

In the following $\triangle ABC$, D, E, F are points on the extended lines of BC, CA, and AB, respectively. Assume that D, E, F are collinear. Then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$



Conversely, if the above equation is valid, then the points D, E, F are collinear.

Proof: We draw BK parallel to AE. By triangle similarity we have

$$\frac{BD}{DC} = \frac{BK}{CE},$$

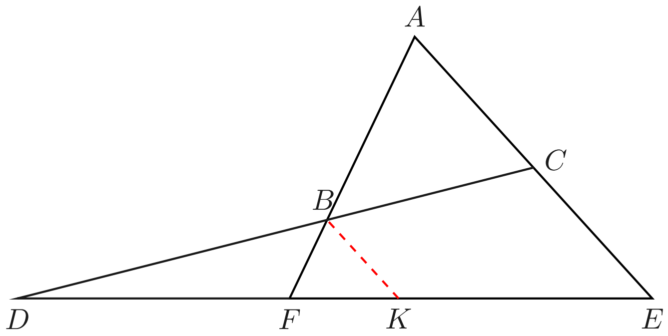
and

$$\frac{AF}{FB} = \frac{AE}{BK}.$$

Thus after multiple the above two equations together we will obtain

$$\frac{BD}{DC} \cdot \frac{AF}{FB} = \frac{AE}{CE},$$

which implies the external Menelaus Theorem.

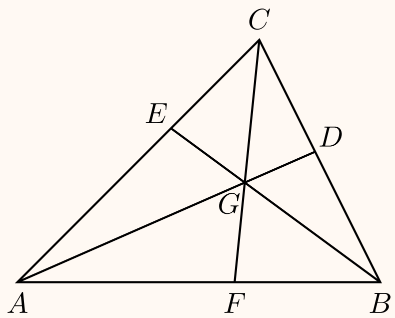


Dual to the Menelaus Theorem, the following Ceva Theorem, attributed to Giovanni Ceva, an Italian hydraulic engineer and mathematician, is very powerful in proving the concurrency of lines.

Theorem 3. (Ceva's Theorem)

In the following picture, the lines AD, BE, CF are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$



Conversely, if the above equation is valid, then AD, BE, CF are concurrent[a].

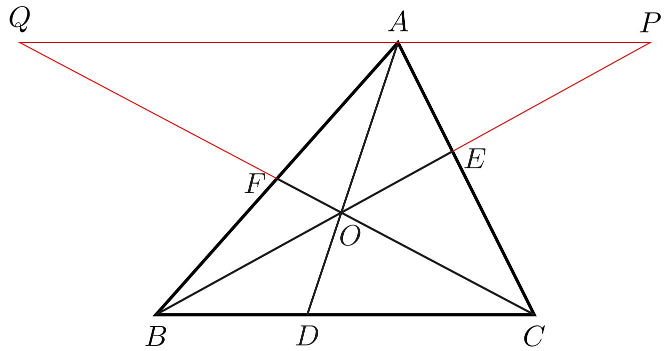
[a] The lines AD, BE, CF are concurrent or parallel. For the sake of simplicity, we often omit to state the latter case.

Proof: The Ceva Theorem can be proved using similar triangle properties. In the following picture, we have

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{QA}{BC} \cdot \frac{BD}{DC} \cdot \frac{BC}{AP} = \frac{QA}{DC} \cdot \frac{BD}{AP}.$$

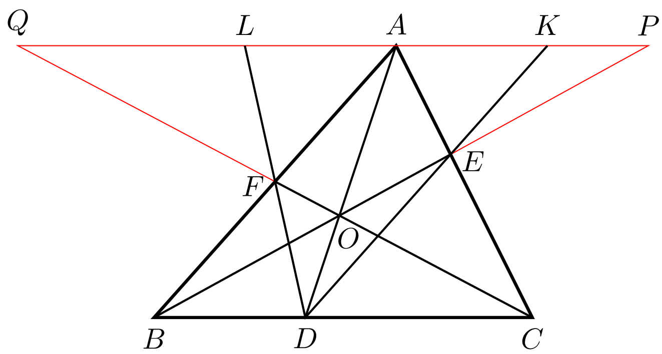
We thus have

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{QA}{DC} \cdot \frac{BD}{AP} = \frac{AO}{OD} \cdot \frac{OD}{AO} = 1.$$



External Link. *Here is the Ceva Theorem mentioned in Wikipedia.*

Example 1 In the following picture, E, F are points on AC and AB, respectively. DE, DF intersect on the line LK at K and L, respectively. Assume that $LK \parallel BC$. Prove that $AK = AL$.



Proof: Completing the above picture by the red lines. Since

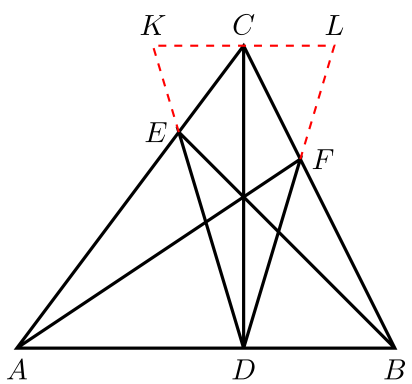
$$AL = \frac{AF}{FB} \cdot BD, \qquad AK = \frac{AE}{EC} \cdot DC,$$

by the Ceva Theorem, we have

$$\frac{AL}{AK} = 1,$$

that is, $AL = AK$.

Example 2 In the following picture, assume that $CD \perp AB$, then $\angle EDC = \angle FDC$.



Proof: Using the result of the above example, we conclude that $\triangle DKL$ is isosceles, which implies our result.

■

2 Proofs Using Area Method

Proof of Ceva's Theorem Using Area Method: We use the pictture in Theorem 3 to give an alternate proof to show that if AD, BE,CF are concurrent, then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1.$$

Actually it is easy to see that

$$\frac{BD}{DC}=\frac{[\triangle BAG]}{[\triangle CAG]},$$

$$\frac{CE}{EA}=\frac{[\triangle CBG]}{[\triangle ABG]},$$

and

$$\frac{AF}{FB}=\frac{[\triangle ACG]}{[\triangle BCG]}.$$

Multiplying the above equations together, we get

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1,$$

which finishes the proof.

■

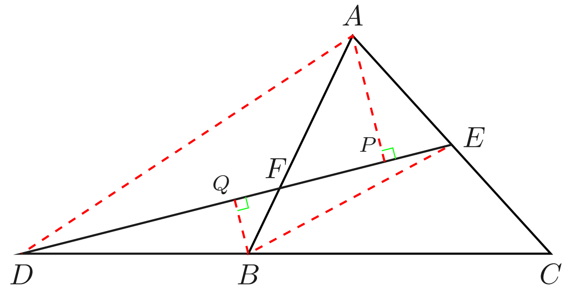
It is also possible to prove the Menelaus' Theorem using the similar area method as well.

Proof of Menelaus Theorem Using Area Method: In the following picture, we have

$$\frac{AF}{FB}=\frac{AP}{BQ}=\frac{[\triangle DAE]}{[\triangle DBE]}.$$

Thus we have

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{[\triangle DAE]}{[\triangle DBE]}\cdot\frac{[\triangle DBE]}{[\triangle DCE]}\cdot\frac{[\triangle DCE]}{[\triangle DAE]}=1.$$



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3 The Algebra Behind the Ceva's and Menelaus' Theorems

In this section, we seek the algebra behind both Ceva and Menelaus Theorems using vector algebra. We start with the Menelaus Theorem. Let A, B, C be represented by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, respectively. Assume that

$$\frac{BD}{DC} = \lambda, \qquad \frac{CE}{EA} = \mu, \qquad \frac{AF}{FB} = \nu.$$

Let D, E, F be represented by vectors $\mathbf{d}, \mathbf{e}, \mathbf{f}$. Then we have

$$\mathbf{e} = \frac{1}{1+\mu}\mathbf{c} + \frac{\mu}{1+\mu}\mathbf{a}, \qquad \mathbf{f} = \frac{1}{1+\nu}\mathbf{a} + \frac{\nu}{1+\nu}\mathbf{b}.$$

The vector $\mathbf{d}$ is a little special. Since D is on the extended line segment BC, we have

$$\mathbf{d} = \frac{1}{1-\lambda}\mathbf{b} + \frac{-\lambda}{1-\lambda}\mathbf{c}$$

Assume that D, E, F are collinear. Then we can write one vector, say $\mathbf{d}$, as a linear combination of $\mathbf{e}, \mathbf{f}$, that is, we can have

$$\mathbf{d} = c_1\mathbf{e} + c_2\mathbf{f},$$

which is

$$\frac{1}{1-\lambda}\mathbf{b} + \frac{-\lambda}{1-\lambda}\mathbf{c} = \frac{c_1}{1+\mu}\mathbf{c} + \frac{c_1\mu}{1+\mu}\mathbf{a} + \frac{c_2}{1+\nu}\mathbf{a} + \frac{c_2\nu}{1+\nu}\mathbf{b}.$$

From the above equation, we get

$$\frac{1}{1-\lambda} = \frac{c_2\nu}{1+\nu}, \qquad \frac{-\lambda}{1-\lambda} = \frac{c_1}{1+\mu}, \qquad \frac{c_1\mu}{1+\mu} + \frac{c_2}{1+\nu} = 0.$$

From the above first two equations, we have

$$c_1 = \frac{-\lambda(1+\mu)}{1-\lambda}, \qquad c_2 = \frac{1+\nu}{(1-\lambda)\nu}.$$

Substituting these into the third equation and simplifying, we get $\lambda\mu\nu = 1$. This completes the proof of the Menelaus Theorem.

Now we use the above algebraic result to prove the Ceva Theorem.

Algebraic Proof of Ceva's Theorem: Let ℓ_1, ℓ_2, ℓ_3 be the equations of the lines BC, CA, AB[a]. Let the points D, E, F be represented by the vectors $\mathbf{d}, \mathbf{e}, \mathbf{f}$. Then

$$\mathbf{d} = \frac{1}{1+\lambda}\mathbf{b} + \frac{\lambda}{1+\lambda}\mathbf{c}, \qquad \mathbf{e} = \frac{1}{1+\mu}\mathbf{c} + \frac{\mu}{1+\mu}\mathbf{a}, \qquad \mathbf{f} = \frac{1}{1+\nu}\mathbf{a} + \frac{\nu}{1+\nu}\mathbf{b},$$

where we assume that

$$\frac{BD}{DC} = \lambda, \qquad \frac{CE}{EA} = \mu, \qquad \frac{AF}{FB} = \nu,$$

and by our assumption, $\lambda\mu\nu = 1$.

It is a straightforward computation to show that

$$\ell_2 - \frac{\ell_2(\mathbf{b})}{\lambda\ell_3(\mathbf{c})}\ell_3, \qquad \ell_3 - \frac{\ell_3(\mathbf{c})}{\mu\ell_1(\mathbf{a})}\ell_1, \qquad \ell_1 - \frac{\ell_1(\mathbf{a})}{\nu\ell_2(\mathbf{b})}\ell_2.$$

represent the lines AD, BE, CF, respectively. Since
$$\frac{\ell_2(\mathbf{b})}{\lambda\ell_3(\mathbf{c})}\cdot\frac{\ell_3(\mathbf{c})}{\mu\ell_1(\mathbf{a})}\cdot\frac{\ell_1(\mathbf{a})}{\nu\ell_2(\mathbf{b})}=1,$$
we know
$$\ell_2-\frac{\ell_2(\mathbf{b})}{\lambda\ell_3(\mathbf{c})}\ell_3$$
can be written as a linear combination of
$$\ell_3-\frac{\ell_3(\mathbf{c})}{\mu\ell_1(\mathbf{a})}\ell_1,\qquad \ell_1-\frac{\ell_1(\mathbf{a})}{\nu\ell_2(\mathbf{b})}\ell_2.$$
Thus AD, BE, CF are concurrent. ■

[a] ℓ_j are liner functions, for example, if ℓ is an equation of a line, then $\ell(\mathbf{x}) = ax + by - c$, where $\mathbf{x} = (x, y)$.

External Link. *By the above discussion, we know that even though the Menelaus and Ceva Theorems appear to be different, the algebra foundation behind both are the same. In Projective Geometry, we call such pair of theorems dual theorems.*

See here for a unified proof using cross ratios.

The algebra above has the following generalization.

Theorem 4

Let $n \geq 3$. Let $\mathbf{a}_1, \cdots, \mathbf{a}_n$ be vectors and $\lambda_1, \cdots, \lambda_n$ be real numbers not equal to 1. Define
$$\mathbf{d}_i=\frac{1}{1-\lambda_i}\mathbf{a}_i-\frac{\lambda_i}{1-\lambda_i}\mathbf{a}_{i+1}$$
for $1 \leq i \leq n-1$, and
$$\mathbf{d}_n=\frac{1}{1-\lambda_n}\mathbf{a}_n-\frac{\lambda_n}{1-\lambda_n}\mathbf{a}_1.$$
Then if any $(n-1)$ subset of $\mathbf{d}_1, \cdots, \mathbf{d}_n$ is collinear, then $\mathbf{d}_1, \cdots, \mathbf{d}_n$ is collinear if and only if
$$\lambda_1\cdots\lambda_n=1.$$

Proof: We define
$$c_1=-\frac{\lambda_n(1-\lambda_1)}{1-\lambda_n},$$
and
$$c_{i+1}=-\frac{\lambda_n(1-\lambda_{i+1})}{1-\lambda_n}\lambda_1\cdots\lambda_i$$
for $1 \leq i \leq n-1$.

Then we have
$$c_1+\cdots+c_{n-1}=1$$

if $\lambda_1 \cdots \lambda_n = 1$. Moreover, we have

$$\mathbf{d}_n = \sum_{i=1}^{n-1} c_d \mathbf{d}_i.$$

The theorem is proved.

∎

Using the above result, we have the following generalization of Menelaus Theorem.

Theorem 5. (Generalized Menelaus' Theorem)

Let $n \geq 3$. Let the vertexes of an n-gon be $A_1, \cdots, A_n$. Let $D_1, \cdots, D_n$ be points on the lines $A_1A_2, \cdots, A_{n-1}A_n, A_nA_1$, respectively. Assume that any $(n-1)$ subset of $D_1, \cdots, D_n$ is collinear. Then $D_1, \cdots, D_n$ is collinear if and only if

$$\frac{A_1D_1}{D_1A_2} \cdot \frac{A_2D_2}{D_2A_3} \cdots \frac{A_{n-1}D_{n-1}}{D_{n-1}A_n} \cdot \frac{A_nD_n}{D_nA_1} = 1.$$