Erdős-Mordell Inequality

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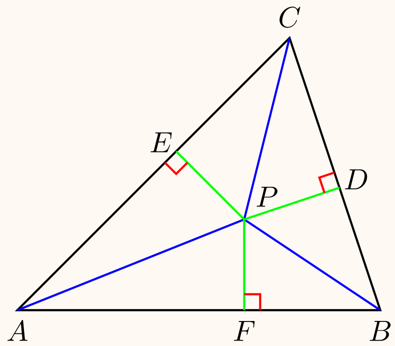
Theorem 1. (Erdős-Mordell Inequality)

Let P be a point inside triangle $\triangle ABC$. Let PD, PE, PF to be orthogonal to AB, BC, CA respectively. Then

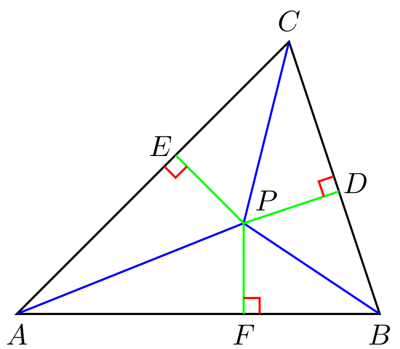
$$PA + PB + PC \geq 2(PD + PE + PF).$$

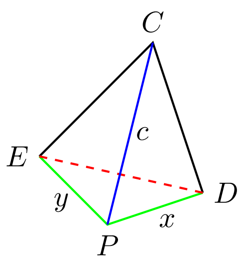


From Wikipedia: "In Euclidean geometry, the Erdős–Mordell inequality states that for any triangle $\triangle ABC$ and point P inside $\triangle ABC$, the sum of the distances from P to the sides is less than or equal to half of the sum of the distances from P to the vertices. It is named after Paul Erdős and Louis Mordell. Erdős (1935) posed the problem of proving the inequality; a proof was provided two years later by L. J. Mordell and D. F. Barrow (1937)." See the Wikipedia for details.

In this artcile, we give three proofs of the inequality. The following one using trigonometry is one of the simplest.

First Proof We shall use law of sines and cosines with inequality of arithmetic and geometric means.





In the above right picture, since $\angle CEP = \angle CDP = 90°$, we can use the law of cosines to obtain

$$ED^2 = x^2 + y^2 - 2xy \cos \angle EPD.$$

Since $\angle EPD = 180^\circ - \angle C = \angle A + \angle B$, we have[a]

$$ED^2 = x^2 + y^2 - 2xy\cos(A+B) = (x \sin B + y \sin A)^2 + (x \cos B - y \cos B)^2.$$

We therefore have

$$ED \geq x \sin B + y \sin A.$$

By law of sines, we have $ED = c \sin C$. Thus we get the inequality

$$c \geq x \frac{\sin B}{\sin C} + y \frac{\sin A}{\sin C}.$$

Similarly, if $PA = a$, $PB = b$, and $PF = z$, then we have

$$a \geq z \frac{\sin B}{\sin A} + y \frac{\sin C}{\sin A}, \qquad b \geq x \frac{\sin C}{\sin B} + z \frac{\sin A}{\sin B}.$$

Therefore, we have

$$a + b + c \geq x\left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B}\right) + y\left(\frac{\sin A}{\sin C} + \frac{\sin C}{\sin A}\right) + z\left(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B}\right).$$

By the Arithmetic-Geometric Inequality, we have

$$\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B} \geq 2, \quad \frac{\sin A}{\sin C} + \frac{\sin C}{\sin A} \geq 2, \quad \frac{\sin B}{\sin A} + \frac{\sin A}{\sin B} \geq 2.$$

Then we have

$$a + b + c \geq 2(x + y + z).$$

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[a] This is called the *Lagrange method of completing square*.

The following area method is more elementary but tricky.

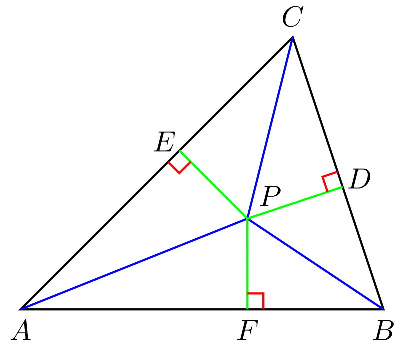
Second Proof We shall use area method to prove the inequality.

Assume that $PA = a, PB = b, PC = c, PD = x, PE = y, PF = z$. Moreover, assume that $BC = p, CA = q$ and $AB = r$. Then since $CP + PF$ is no less than the height of the triangle over AB, we have

$$r(c + z) \geq qy + px + rz,$$

which is simplified to

$$rc \geq qy + px.$$



Note that the above inequality is true for any point P in the cone of $\angle ACB$, even when P is outside of the triangle. So if we fix the triangle $\triangle ABC$ and fix the length $PC = c$, we can allow x, y to vary. Switching x, y, we get

$$rc \geq qx + py,$$

which is

$$c \geq x\frac{q}{r} + y\frac{p}{r}.$$

Similarly, we have

$$b \geq x\frac{r}{q} + z\frac{p}{q}, \qquad a \geq z\frac{q}{p} + y\frac{q}{p}.$$

Therefore

$$a + b + c \geq x\left(\frac{r}{q} + \frac{q}{r}\right) + y\left(\frac{p}{r} + \frac{r}{p}\right) + z\left(\frac{p}{q} + \frac{q}{p}\right) \geq 2(x + y + z).$$

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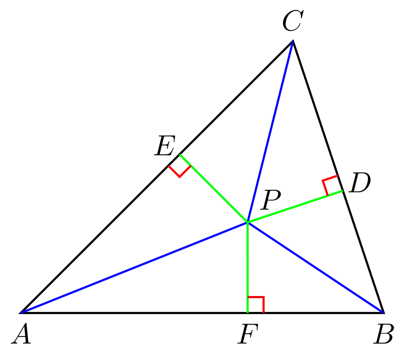
The third proof is essentially algebraic.

Third Proof Here we provide the most important proof using algebra.

As before, we assume that $PA = a, PB = b, PC = c, PD = x, PE = y, PF = z$, $BC = p, C = q$, and $AB = r$. Moreover, we assume that $\angle BPC = 2\alpha, \angle CPA = 2\beta$ and $\angle APB = 2\gamma$. We claim that

$$z \leq \sqrt{ab}\cos\gamma.$$



To prove this, we use the law of cosines to obtain

$$r^2 = a^2+b^2-2ab\cos 2\gamma = (a-b)^2+2ab(1-\cos 2\gamma) \geq 2ab(1-\cos 2\gamma) = 4ab\sin^2\gamma.$$

Thus

$$z = \frac{ab\sin 2\gamma}{r} \leq \sqrt{ab}\cos\gamma.$$

Thus the Erdős-Mordell inequality can be strengthened to the following algebraic inequality

$$a+b+c \geq 2\sqrt{ab}\cos\gamma + 2\sqrt{bc}\cos\alpha + 2\sqrt{ca}\cos\beta.$$

The inequality follows from the following theorem.

Theorem 2. (Generalized AM-GM Inequality)

Let $\alpha+\beta+\gamma = 180°$. Then for any real numbers a, b, c, we have

$$a^2+b^2+c^2 \geq 2ab\cos\gamma + 2bc\cos\alpha + 2ca\cos\beta.$$

If $\alpha = \beta = 90°$ and $\gamma = 0°$, then the inequality becomes

$$a^2+b^2 \geq 2ab.$$

If $\alpha = \beta = \gamma = 60°$, then the inequality becomes

$$a^2+b^2+c^2 \geq ab+bc+ca.$$

If $\alpha = \beta = 0°$ and $\gamma = 180°$, then the inequality becomes

$$a^2+b^2+c^2 \geq -2ab+2bc+2ca.$$

External Link. *See here for more discussions of the above inequality.*

Proof Using the Lagrange Method of Completing Square We have

$$\cos\gamma = -\cos(\alpha+\beta) = -\cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Thus we have

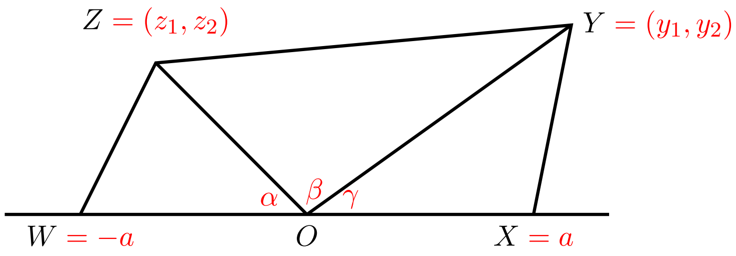
$$\begin{aligned} &a^2+b^2+c^2-2ab\cos\gamma-2bc\cos\alpha-2ca\cos\beta \\ &= a^2+b^2+c^2+2ab\cos\alpha\cos\beta-2ab\sin\alpha\sin\beta-2bc\cos\alpha-2ca\cos\beta \\ &= (c-b\cos\alpha-a\cos\beta)^2+(b\sin\alpha-a\sin\beta)^2 \geq 0. \end{aligned}$$

Proof Using Analytic Geometry In the picture below, we assume that $OX = OW = a$, $OY = b$ and $OZ = c$. The coordinates of Y, Z are marked on the picture. Then by the law of cosines, we have

$$bc\cos\beta = \frac{1}{2}(b^2+c^2-(y_1-z_1)^2-(y_2-z_2)^2) = y_1z_1+y_2z_2.$$

We also have

$$ab\cos\gamma = ay_1, \qquad ca\cos\alpha = -az_1.$$



Therefore, the inequality that needs to be proved is reduced to

$$a^2 + y_1^2 + y_2^2 + z_1^2 + z_2^2 \geq -2az_1 + 2ay_1 + 2y_1z_1 + 2y_2z_2.$$

But this follows from the easy square completing:

$$(a - y_1 + z_1)^2 + (y_2 - z_2)^2 \geq 0.$$

For a regular n-gon, the sum of the distance of the center to the vertexes is equal to $\sec \pi/n$ times of the sum to the sides. This would lead the following generalization of the Erdős-Mordell Inequality to polygons (see here).

Theorem. (Lenhard 1961)

Let $A_1A_2\cdots A_n$ be a convex polygon, and P be an interior point of $A_1A_2\cdots A_n$. Let R_i be the distance from P to the vertex A_i, r_i be the distance from P to the side A_iA_{i+1}. Then

$$\sum_{i=1}^{n} R_i \geq \left(\sec\frac{\pi}{n}\right)\sum_{i=1}^{n} r_i.$$

By using the same method in the above third proof of the Erdős-Mordell Inequality, the following inequality implies the result of Lenhard.

Theorem

Let $a_1, \cdots, a_n$ be nonnegative real numbers. Let $\alpha_1, \cdots, \alpha_n$ be angles such that

$$\sum_{i=1}^{n} \alpha_i = 180^\circ.$$

Then we have

$$\sum_{i=1}^{n} a_i^2 \geq \left(\sec\frac{\pi}{n}\right)\left(\sum_{i=1}^{n-1} a_ia_{i+1}\cos\alpha_i + a_na_1\cos\alpha_n\right).$$

Proof: We can use the same method as above, but using the language of vector algebra in stead of law of cosines. We consider the vectors $\vec{\mathbf{v}}_1, \cdots, \vec{\mathbf{v}}_n \in \mathbb{R}^2$ such that

$\vec{\mathbf{v}}_1 = a_1$, and

$$\vec{\mathbf{v}}_k = (a_k \cos(\alpha_1 + \cdots + a_{k-1}), a_k \sin(\alpha_1 + \cdots + a_{k-1}))$$

for $k = 2, \cdots, n$. Therefore, we have

$$\langle \vec{\mathbf{v}}_i, \vec{\mathbf{v}}_{i+1} \rangle = a_i a_{i+1} \cos \alpha_i$$

for $i = 1, \cdots, n-1$; on the other hand, since $\alpha_1 + \cdots + \alpha_n = \pi$,

$$\langle \vec{\mathbf{v}}_n, \vec{\mathbf{v}}_1 \rangle = a_n a_1 \cos \alpha_n.$$

Thus the inequality we want to prove can be represented by

$$\|\vec{\mathbf{v}}_1\|^2 + \cdots + \|\vec{\mathbf{v}}_n\|^2 \geq \sec \frac{\pi}{n} \left(\sum_{i=1}^{n-1} \langle \vec{\mathbf{v}}_i, \vec{\mathbf{v}}_{i+1} \rangle - \langle \vec{\mathbf{v}}_n, \vec{\mathbf{v}}_1 \rangle \right).$$

If we write $\vec{\mathbf{v}}_i = (p_i, q_i)$ for $i = 1, \cdots, n$, then

$$\langle \vec{\mathbf{v}}_i, \vec{\mathbf{v}}_{i+1} \rangle = p_i p_{i+1} + q_i q_{i+1}$$

for $i = 1, cdots, n-1$, and

$$\langle \vec{\mathbf{v}}_n, \vec{\mathbf{v}}_1 \rangle = p_n p_1 + q_n q_1.$$

Thus the inequality is reduced into the case when $\alpha_1 = \cdots = \alpha_{n-1} = 0$, and $\alpha_n = 180°$:

$$\sum_{i=1}^{n} a_i^2 \geq \left(\sec \frac{\pi}{n} \right) \left(\sum_{i=1}^{n-1} a_i a_{i+1} - a_n a_1 \right).$$

It is really surprising, but you can do the completing the square:

$$\begin{aligned} &\cos \frac{\pi}{n} \sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n-1} a_i a_{i+1} - a_n a_1 \right) \\ =& \sum_{i=1}^{n-2} \frac{1}{2 \sin \frac{i\pi}{n} \sin \frac{(i+1)\pi}{n}} \left(a_i \sin \frac{(i+1)\pi}{n} - a_{i+1} \sin \frac{i\pi}{n} + a_n \sin \frac{\pi}{n} \right)^2. \end{aligned}$$

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External Link. *For details of the proof of the theorem, see for example, here for details.*