The Lemoine Axis

Math 199
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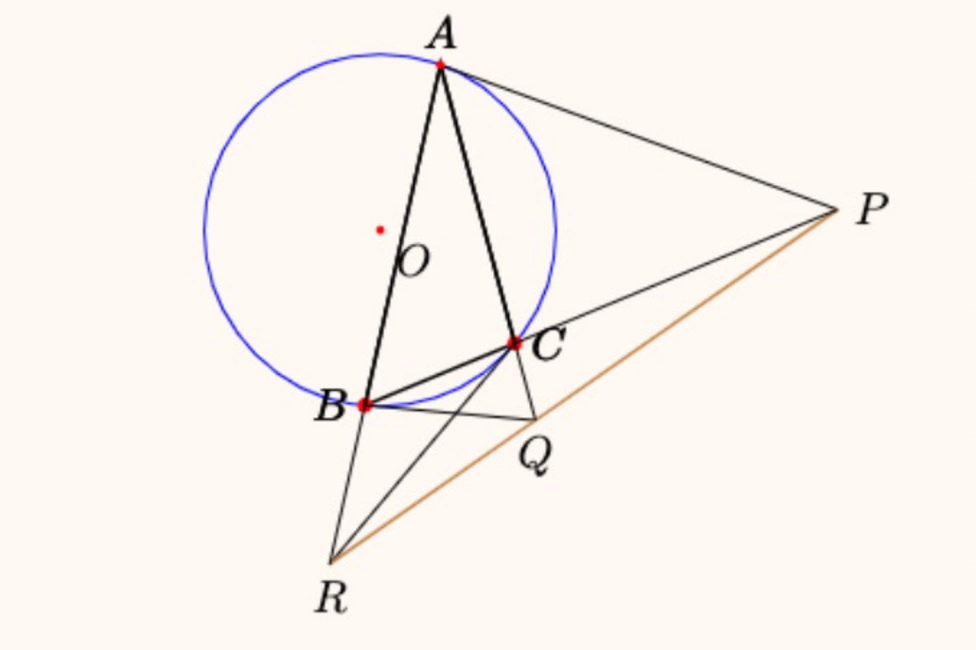
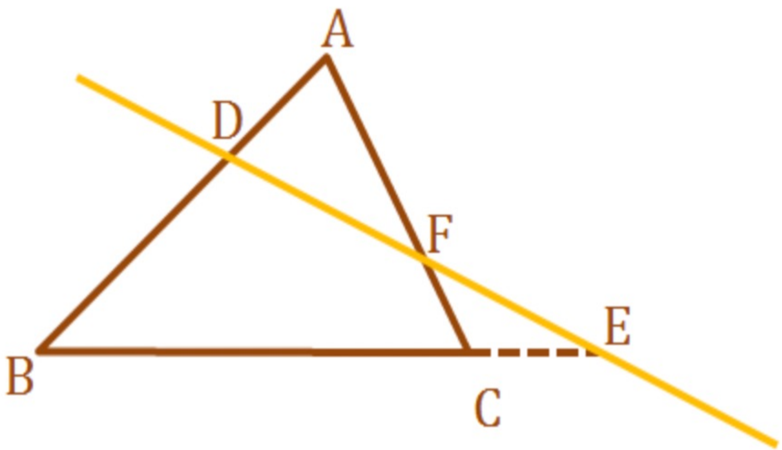
Definition

The **Lemoine line** is the line passing through the intersection points of the tangents to the circumcircle of a triangle at the vertices of the triangle. These intersection points are collinear, and this line is called the Lemoine line .



Useful Theorem



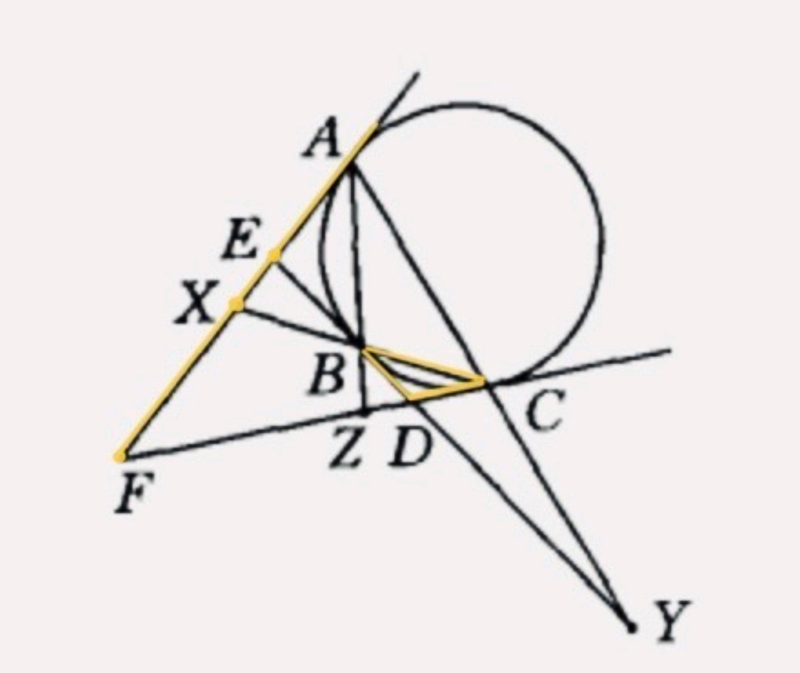
Menelaus' theorem states that if a line intersects △ABC or extended sides at points D, E, and F, the following statement holds:

$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1$$

Converse of Menelaus' Theorem: Suppose three points D,E,F are on sides (or extension) AB,BC,AC respectively, such that 1or 3 of them are in the extensions of the sides. Then points D,E,F are collinear if and only if:

$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1$$

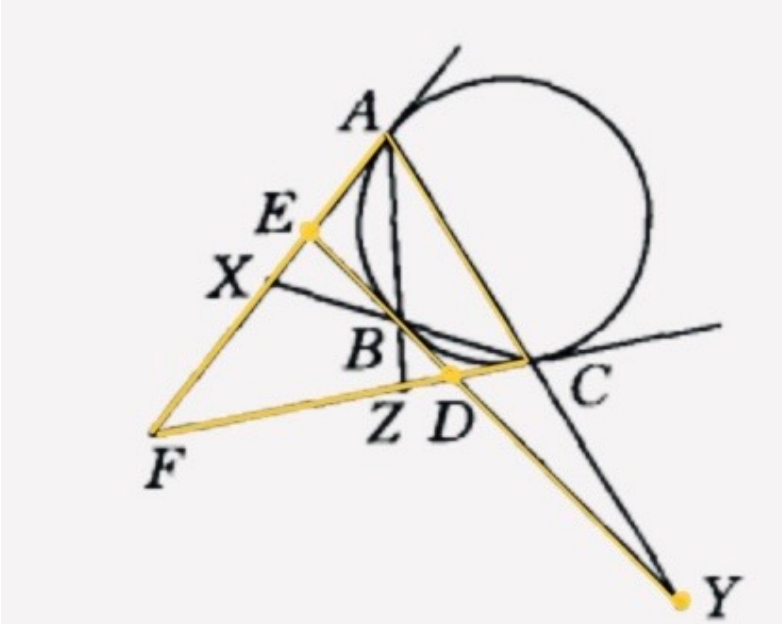
Proof:



XEF intersects $\triangle$BCD

$$\frac{XB}{XC} \cdot \frac{FC}{FD} \cdot \frac{ED}{EB} = 1$$

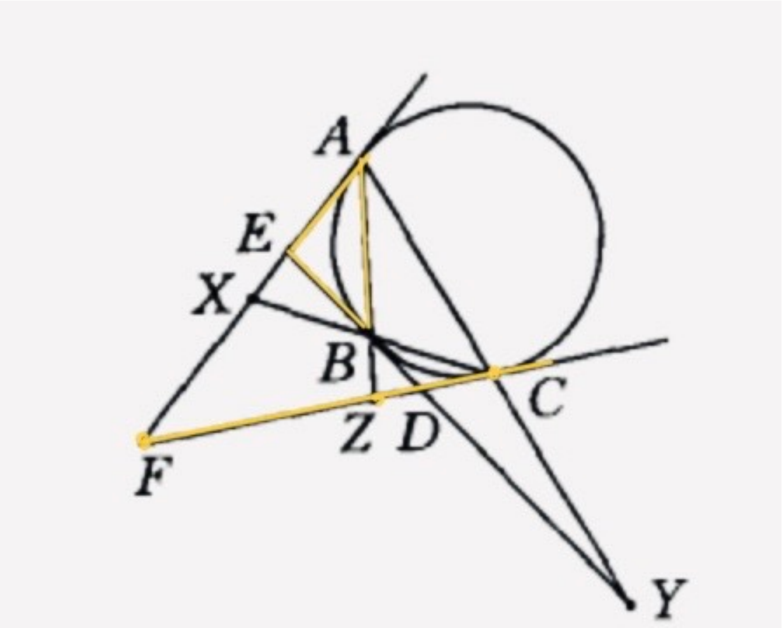
Proof:



YDE intersects △ACF

$$\frac{YC}{YA} \cdot \frac{EA}{EF} \cdot \frac{DF}{DC} = 1$$

Proof:

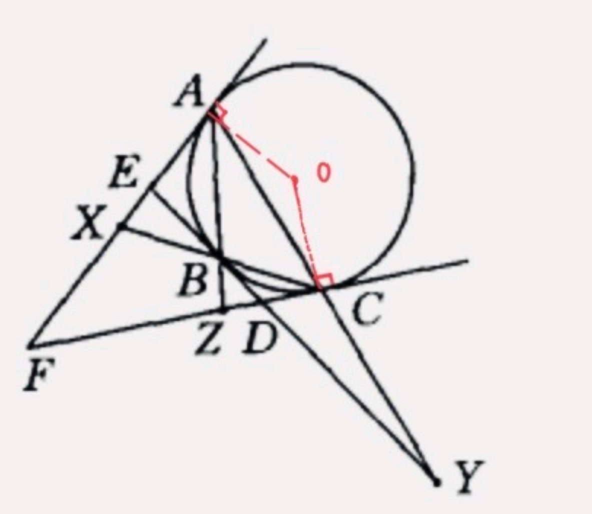


ZDF intersects $\triangle$ABE

$$\frac{ZA}{ZB} \cdot \frac{DB}{DE} \cdot \frac{FE}{FA} = 1$$

Proof:

$$\frac{XB}{XC}\cdot\frac{FC}{FD}\cdot\frac{ED}{EB}\cdot\frac{YC}{YA}\cdot\frac{EA}{EF}\cdot\frac{DF}{DC}\cdot\frac{ZA}{ZB}\cdot\frac{DB}{DE}\cdot\frac{FE}{FA}=1$$



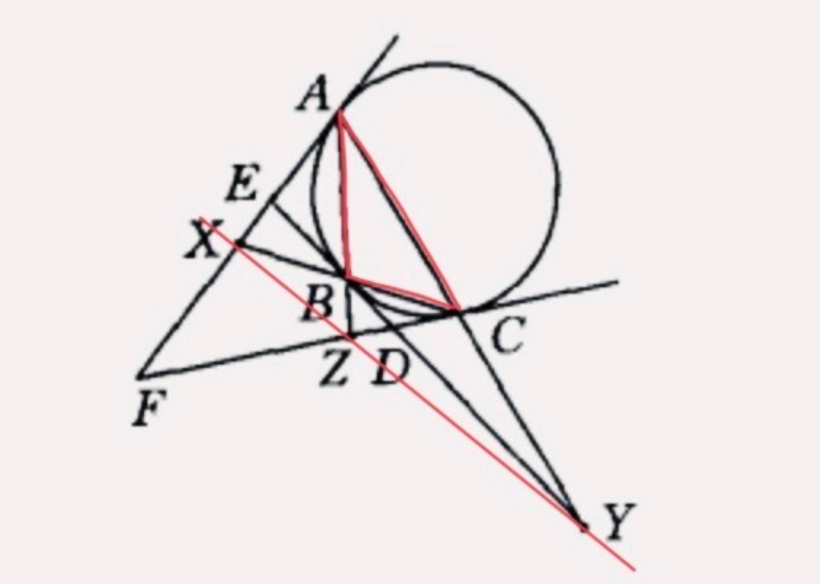
$$\frac{XB}{XC}\cdot\frac{FC}{FA}\cdot\frac{YC}{YA}\cdot\frac{EA}{EB}\cdot\frac{ZA}{ZB}\cdot\frac{DB}{DC}=1$$

since △ACF, △DBC, △EAB are Isosceles triangles

DB=BC AE=BE AF=CF

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=1$$

Proof:



$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=1$$

Since X, Y, and Z are located on the extensions of the three sides of △ABC respectively, according to Menelaus' theorem, it follows that the three points X, Y, and Z are collinear.

Thank you for watching