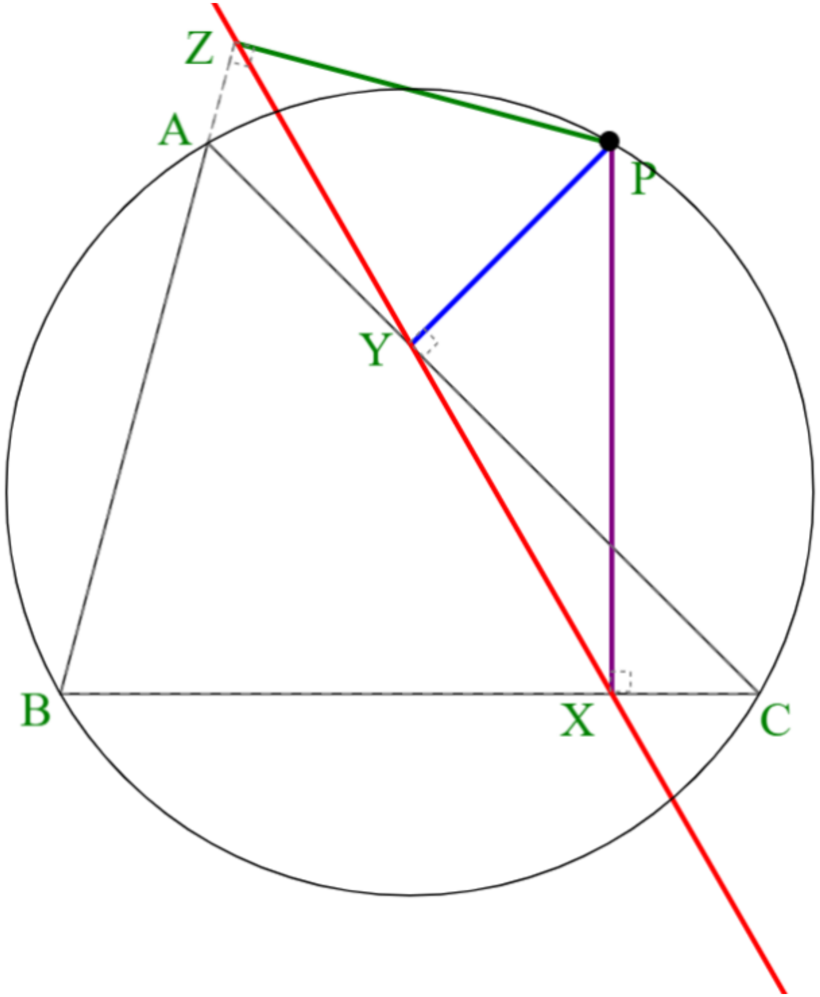
Topic 5:

Simson Line

Mingyan Xu



Contents

1. History

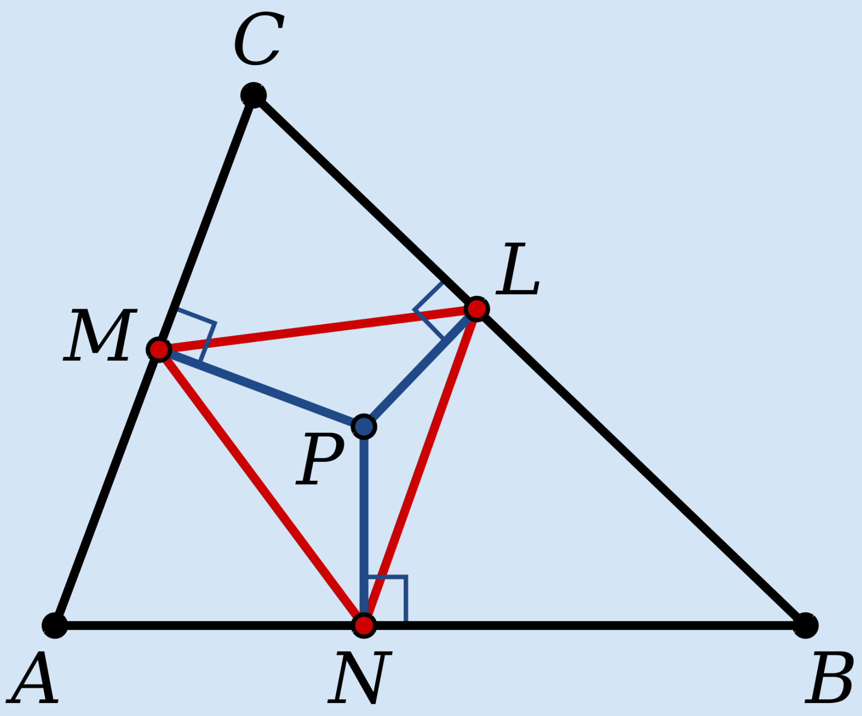
In the field of geometry, the Simson Line is a term used to describe a mathematical concept that has its origins in the work of William Wallace, a mathematician who lived in the late 18th century.

However, the line is often **mistakenly attributed** to Robert Simson, who was a prominent mathematician from Scotland lived over a century earlier. That’s why this concept is called Simson Line.

Robert Simson

William Wallace

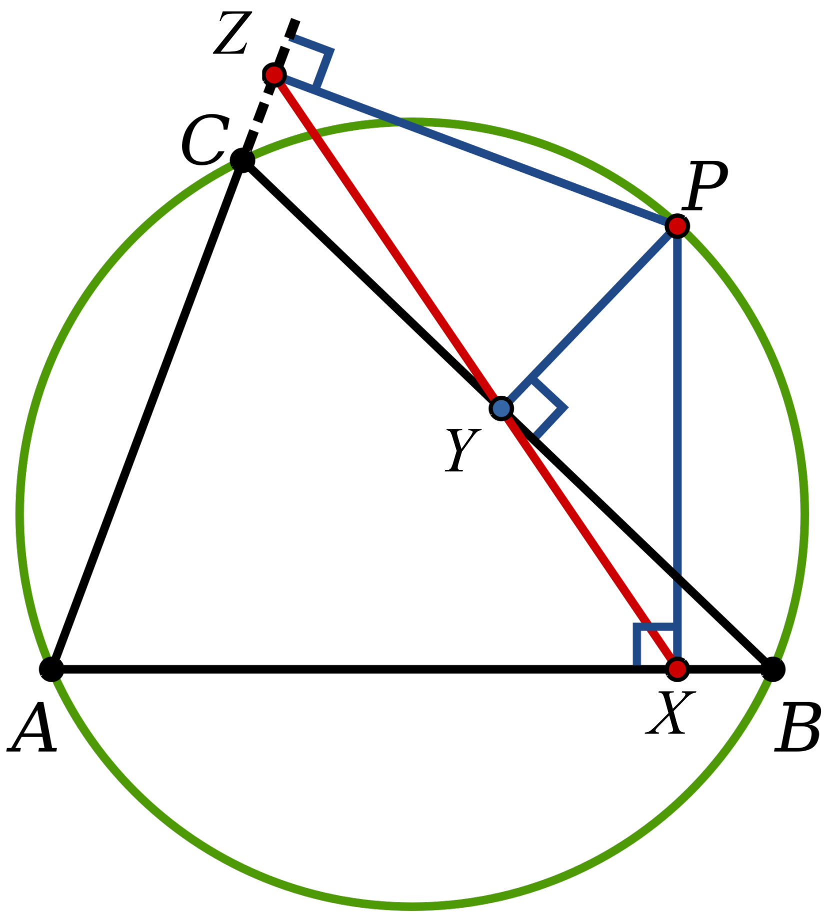
2. Introduction : Pedal Triangle



- Consider a triangle ABC, and a point P that is not one of the vertices A, B, C.
- Drop perpendiculars from P to the three sides of the triangle.
- Label L, M, N the intersections of the lines from P with the sides BC, AC, AB. The pedal triangle is then $\triangle LMN$

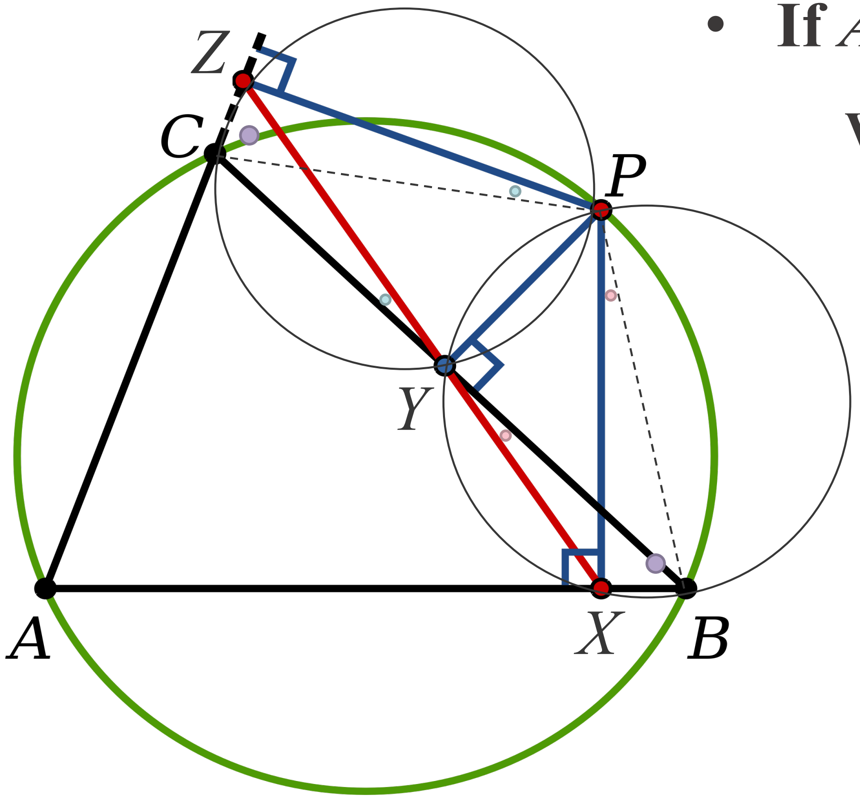
2. Introduction : Simson Line

- Let P be an arbitrary point, and let X, Y , Z be the projections of P to the lines BC, CA and AB, respectively. Then X, Y , Z are collinear **if and only if** P lies on the circumcircle of $\triangle ABC$.



- It is a special case of pedal triangle.

3. Proof (forward)



- **If A, B, C, P are concyclic, prove X, Y, Z are collinear.**

Want to prove: $\angle CYZ = \angle BYX$

$\angle CYZ = \angle CPZ = 90^\circ - \angle ZCP$

$\angle BYX = \angle BPX = 90^\circ - \angle PBX$

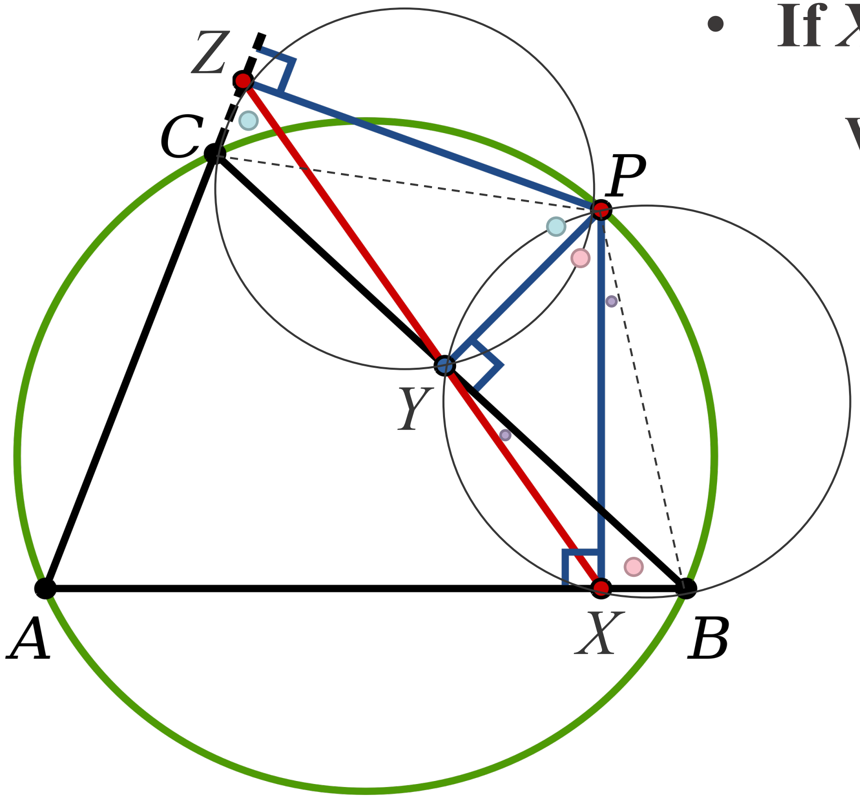
$\angle PBX = \angle ZCP$

$\angle PBX + \angle BPX = \angle ZCP + \angle CPZ = 90^\circ$

$\angle BPX = \angle CPZ$

Therefore, $\angle CYZ = \angle BYX$

3. Proof (backward)



- **If X, Y, Z are collinear, prove A, B, C, P are concyclic.**

Want to prove: $\angle CPB + \angle A = 180$

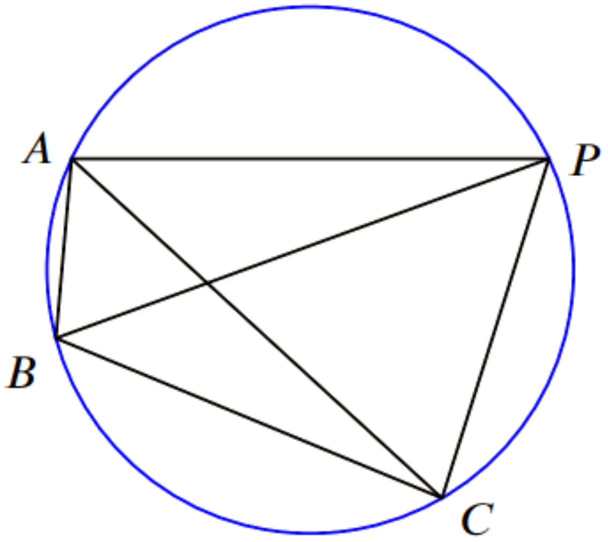
$\angle AZX = \angle CPY$

$\angle YPX = \angle YBX$ $\quad\quad$ $\angle BYX = \angle BPX$

$$\begin{aligned}\angle CPB &= \angle CPY + \angle YPX + \angle BPX \\ &= \angle AZX + (180^\circ - \angle YXB) \\ &= 180^\circ - \angle A\end{aligned}$$

4. Property: Topic 10

- For a cyclic quadrilateral, the product of the diagonals equals the sum of the products of the opposite sides.
- $AC \cdot BP = AB \cdot CP + AP \cdot BC.$



By the law of sines, we have

$$ZY = AP \cdot \sin\angle ZPY = AP \cdot \sin\angle BAC = \frac{AP \cdot BC}{2R},$$

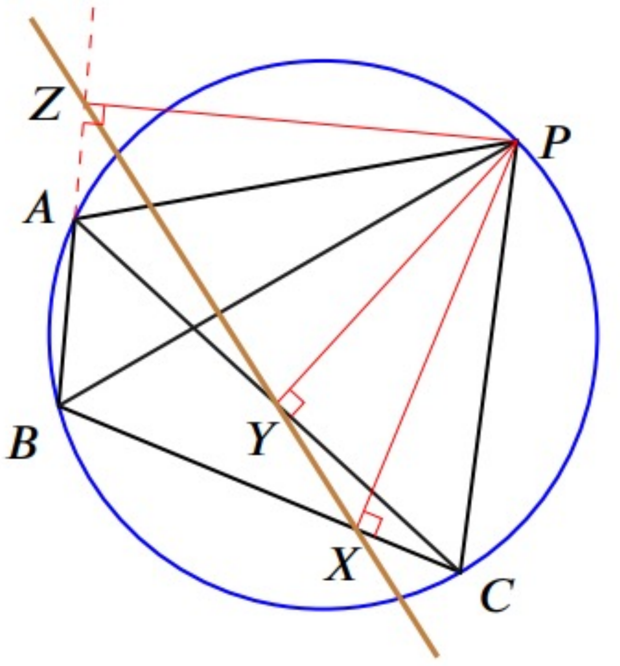
where R is the radius of circumcircle. Similarly, we have

$$YX = \frac{CP \cdot AB}{2R}, \quad ZX = \frac{AC \cdot BP}{2R}.$$

Since X, Y, Z are collinear, $ZY + YX = ZX$. Therefore

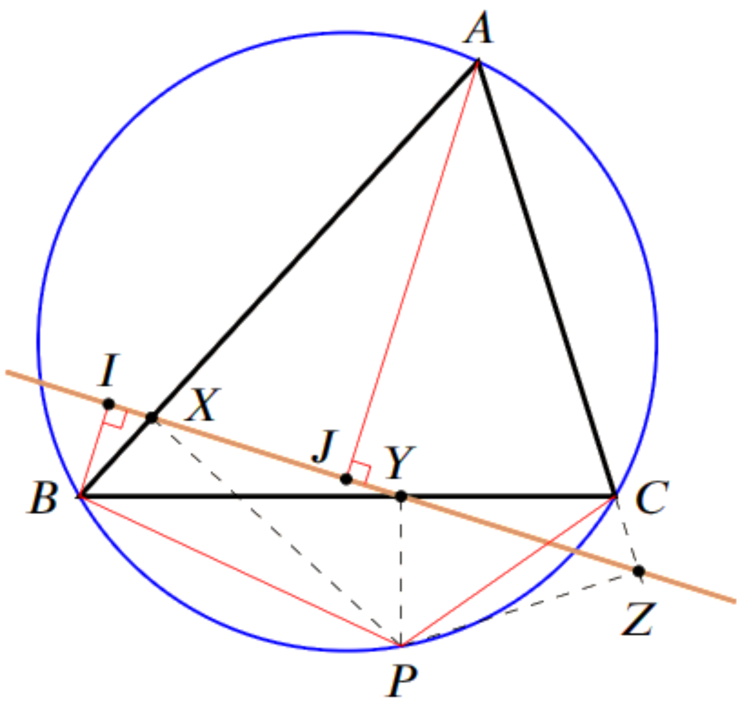
$$\frac{AC \cdot BP}{2R} = \frac{AP \cdot BC}{2R} + \frac{CP \cdot AB}{2R},$$

which implies the Ptolemy Theorem.



4. Property

- In the right picture, let point I, J be the pedal points of B, A to the Simson line, respectively.
- Then the line segment $IJ = YZ$.



Proof. In the above picture, $BXYP$ is concyclic. Therefore $\angle PBY = \angle PXY$, and because $PX \perp AB$ and $AJ \perp XY$, we then get $\angle PXY = \angle XAJ$.

By the law of sines, we have

$$AB \cdot \sin \angle PBY = AX \cdot \sin \angle XAJ + XB \cdot \sin \angle IBX = XJ + IX = IJ.$$

Since $PYCZ$ concyclic, we then have

$$YZ = PC \cdot \sin \angle ACB = 2R \cdot \sin \angle PBY \cdot \sin \angle ACB = AB \cdot \angle PBY = IJ,$$

where R is the radius of circumcircle of $\triangle ABC$. ■

THANKS for listening