

# Routh's Theorem

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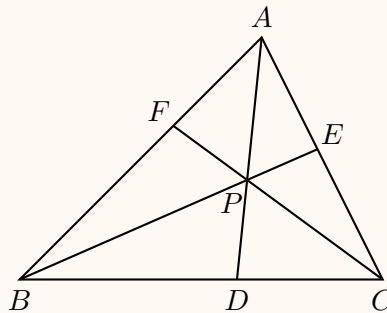
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Routh's Theorem is a generalization of the Ceva's Theorem.

## Theorem 1. (Ceva's Theorem)

In the following triangle  $\triangle ABC$ , let

$$\frac{BD}{DC} = x, \quad \frac{CE}{EA} = y, \quad \frac{AF}{FB} = z.$$



Then,  $AD$ ,  $BE$ , and  $CF$  are concurrent if and only if

$$xyz = 1.$$

For details of the Ceva's Theorem, see [Wikipedia](#), or [Topic 02](#).

We define the following:

## Definition 1. (Cevian)

A Cevian is a line segment which joins a vertex of a triangle with a point on the opposite side (or its extension). For example, in the above picture,  $AD$ ,  $BE$  and  $CF$  are Cevians.

The condition for three general Cevians from the three vertices of a triangle to concur is known as Ceva's theorem.

For a fixed triangle, if the three Cevians are not concurrent, then what is the area of the triangle formed by the pairwise intersections of them? *The Routh's Theorem* provides the answer to this question. If the area is zero, then the triangle is degenerated to a point, and therefore, the theorem is reduced to the Ceva's Theorem. The Routh's Theorem was discovered by *Edward John Routh* in 1896.

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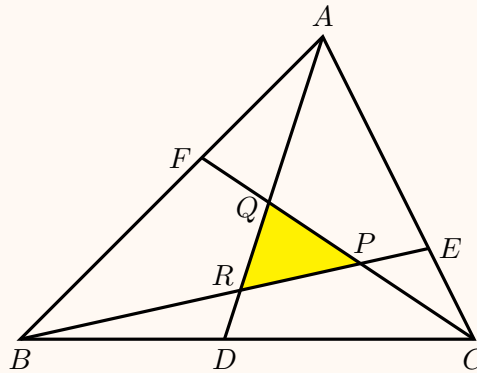
**Theorem 2. (Routh's Theorem)**

In the following  $\triangle ABC$ , let  $D, E, F$  be points on  $BC, CA$ , and  $AB$ , respectively. Assume that

$$\frac{BD}{DC} = x, \quad \frac{CE}{AE} = y, \quad \frac{AF}{FB} = z.$$

Then, the area of  $\triangle PRQ$  formed by the Cevians  $AD, BE$ , and  $CF$  is the area of  $\triangle ABC$  times

$$\frac{(xyz - 1)^2}{(zx + z + 1)(xy + x + 1)(yz + y + 1)}.$$



In particular, if  $xyz = 1$ , then the three Cevians  $AD, BE$ , and  $CF$  are concurrent, and the theorem is reduced to the Ceva's Theorem.

**Proof:** Assume that the area of  $\triangle ABC$  is 1. For  $\triangle ABD$  and line  $FQC$ , using the Menelaus's theorem, we obtain

$$\frac{AF}{FB} \cdot \frac{BC}{CD} \cdot \frac{DQ}{QA} = 1.$$

Then, we can get

$$\frac{QA}{DQ} = \frac{AF}{FB} \cdot \frac{BC}{CD} = z(x + 1).$$

As a result, we have

$$\frac{AQ}{AD} = \frac{z(x + 1)}{1 + z(x + 1)}.$$

Thus, the area of  $\triangle CQA$  is

$$S_{\triangle CQA} = \frac{AQ}{AD} \cdot \frac{DC}{BC} \cdot S_{\triangle ABC} = \frac{z}{zx + z + 1}.$$

Similarly, we get the area of  $\triangle ARB$  and  $\triangle BPC$

$$S_{\triangle ARB} = \frac{x}{xy + x + 1}.$$

$$S_{\triangle BPC} = \frac{y}{yz + y + 1}.$$

Therefore,

$$\begin{aligned} S_{\triangle PQR} &= S_{\triangle ABC} - S_{\triangle CQA} - S_{\triangle ARB} - S_{\triangle BPC} \\ &= 1 - \frac{z}{zx + z + 1} - \frac{x}{xy + x + 1} - \frac{y}{yz + y + 1} \\ &= \frac{(xyz - 1)^2}{(zx + z + 1)(xy + x + 1)(yz + y + 1)}. \end{aligned}$$

The theorem is proved. ■

**Remark** The identity above can be proved by a tedious computation. Here we provide another proof. Let

$$a = xy + x + 1,$$

$$b = yz + y + 1,$$

$$c = zx + z + 1.$$

Let

$$f(x, y, z) = abc - zab - xbc - yca - (xyz - 1)^2.$$

We need to prove that  $f \equiv 0$ . Without loss of generality, we assume that  $x, y, z$  are distinct. Note that for fixed  $y, z$ ,  $f(x, y, z)$  is a quadratic polynomial of  $x$ . Thus we only need to prove the identity for three different values of  $x$ .

First, we assume that  $x = 0$ . Then  $a = 1$ ;  $b = yz + y + 1$ ; and  $c = z + 1$ . Thus we have

$$f(0, y, z) = (yz + y + 1)(z + 1) - z(yz + y + 1) - y(z + 1) - 1 = 0.$$

Next, we assume that  $x = -1/(y + 1)$ . Then  $a = 0$ . From this we conclude that

$$0 = xyz + zx + z$$

and therefore  $c = 1 - xyz$ . Similarly, we have  $bx = xyz - 1$ . Thus

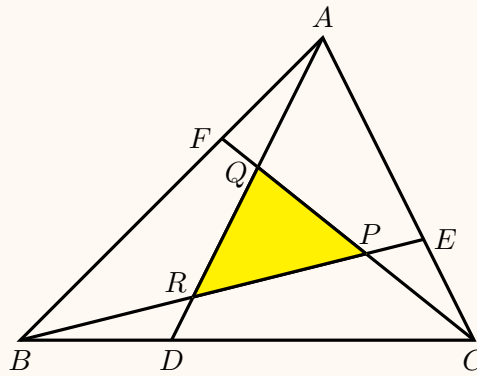
$$f(-1/(y + 1), y, z) = -xbc - (xyz - 1)^2 = 0.$$

By symmetry, we have  $f(-1/(z + 1), y, z) = 0$ . This proves the identity.

A special case of the Routh's Theorem is the following **One-seventh area triangle** problem.

**Theorem 3. (One-seventh Area Triangle)**

*If  $x = y = z = 1/2$  in the above theorem, then the Routh's Theorem is reduced to the popular **one-seventh area triangle** problem.*



*The area of  $\triangle PRQ$  is one-seventh of the area of  $\triangle ABC$ .*

This problem was introduced in the famous book of *Hugo Steinhaus*, *Mathematical Snapshots* (the 1982 version), page 9.