VIVIANI’S THEOREM

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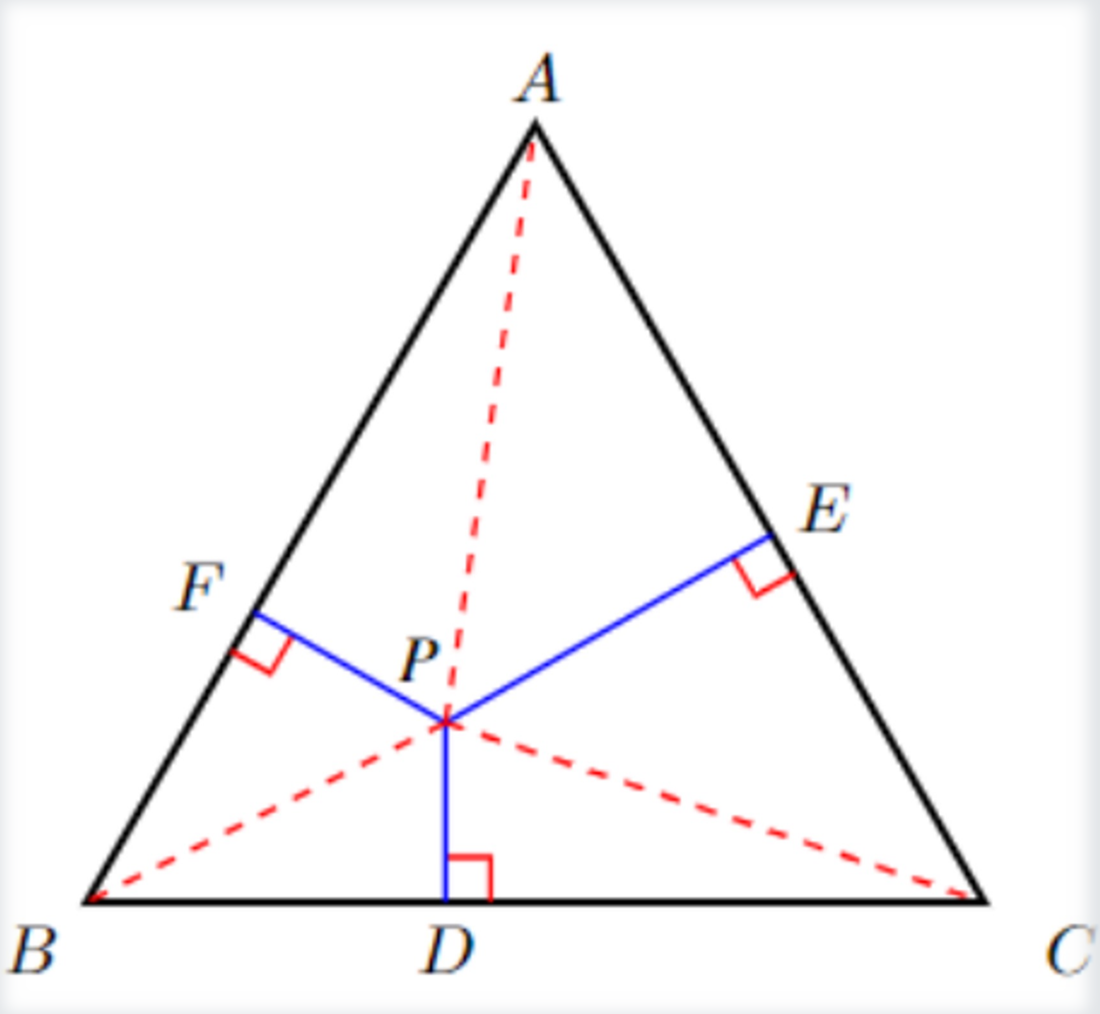
Established by Vincenzo Viviani (1622–1703), an Italian mathematician and scientist who was a student of Galileo.

Theorem:

In an equilateral triangle, the sum of distance from an arbitrary point to the three sides is always constant, and equal to the height of the triangle.



PROOF

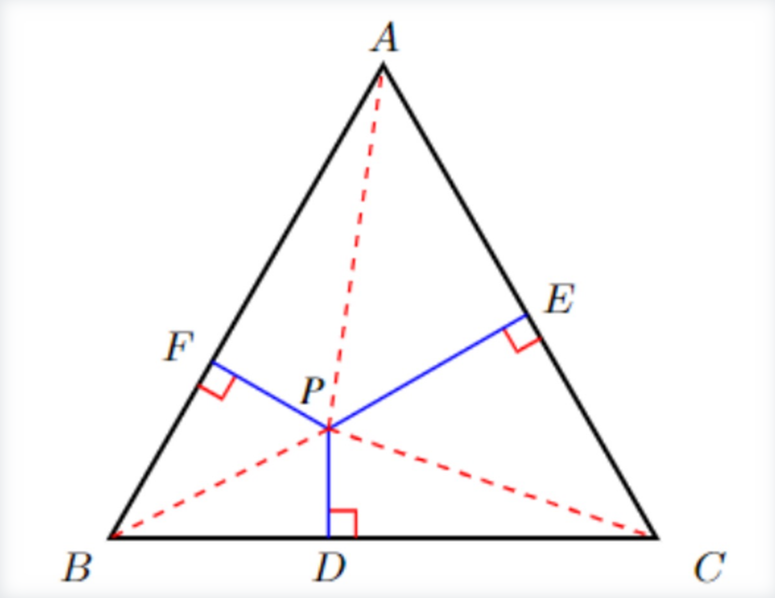
$$S_{\triangle ABC} = S_{\triangle BPC} + S_{\triangle CPA} + S_{\triangle APB},$$

we have

$$\frac{\sqrt{3}}{4}a^2 = S_{\triangle ABC} = \frac{a \cdot PD}{2} + \frac{a \cdot PE}{2} + \frac{a \cdot PF}{2}.$$

We thus conclude that

$$PD + PE + PF = \frac{\sqrt{3}}{2}a,$$



What if the point P is not in the triangle?

SIGNED DISTANCE

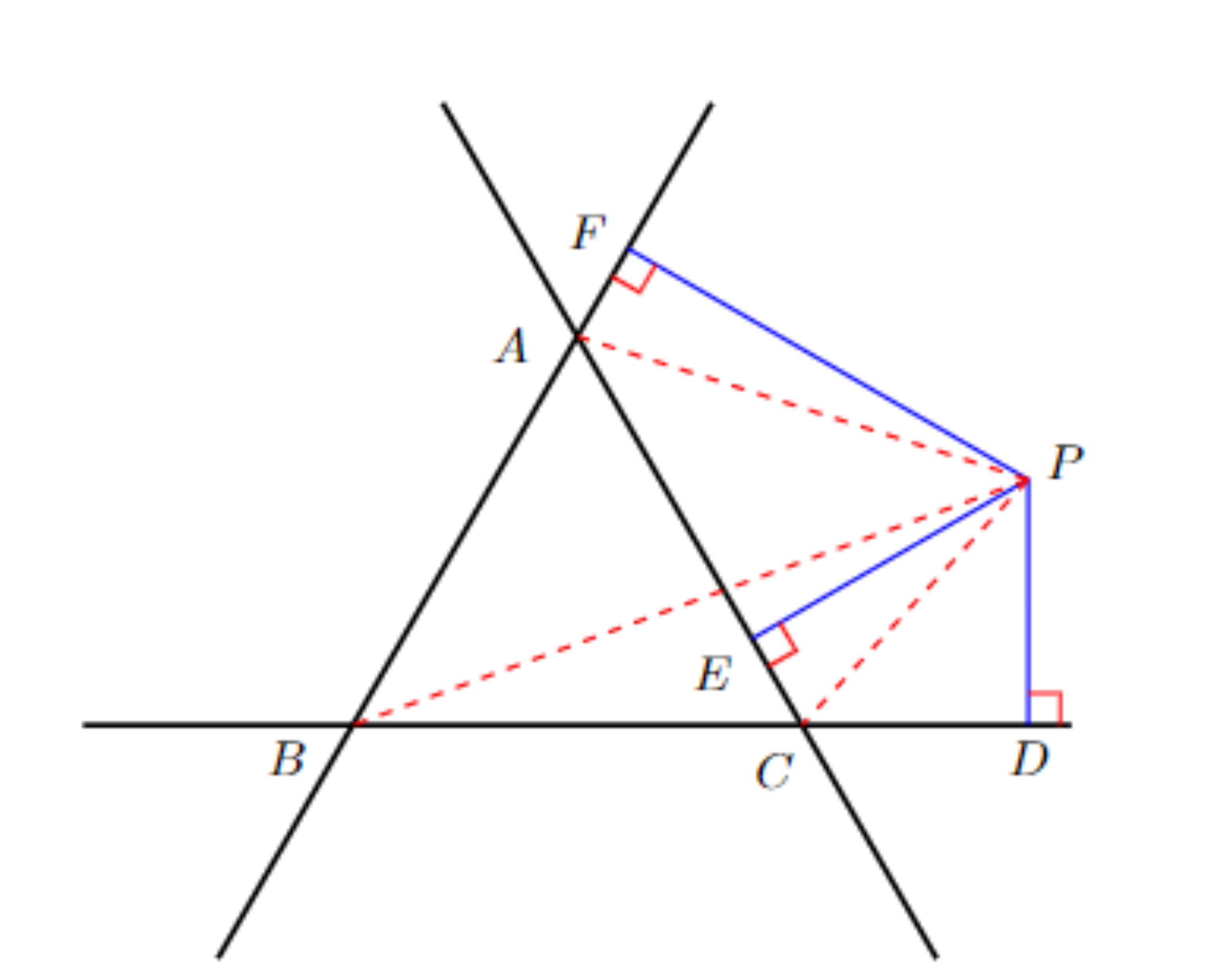
Definition. (Signed Distance)

On a Cartesian coordinate system, let the equation of a line L to be *ax + by + c = 0*, and let *P = (x0, y0)* be a point. Then the signed distance of the point P to L is given by

$$\frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}}$$

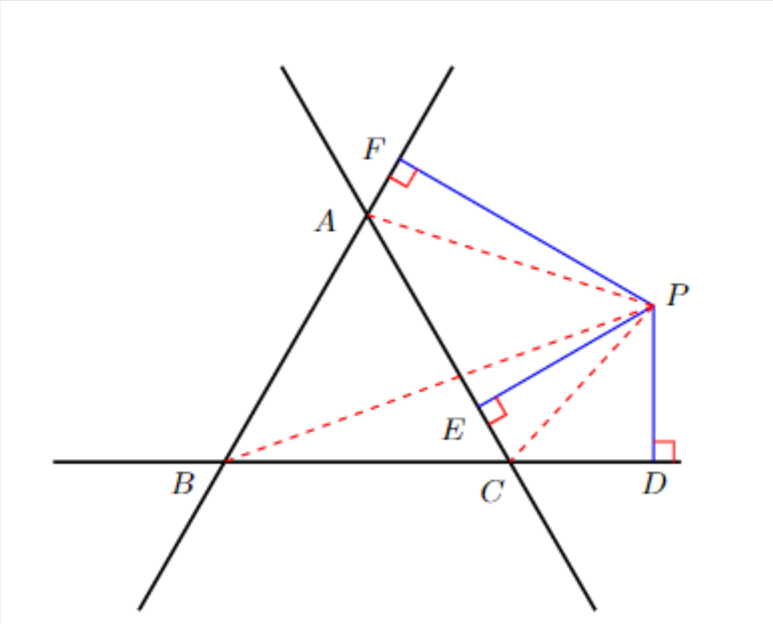
Notice: that the signed distance depends not only on the point and the line, but also depends on the orientation of the line: both ax + by + c = 0 and −ax − by − c = 0 represent the same line, but the corresponding signed distances differ by a negative sign.

Let ΔABC be a fixed triangle. We define the orientations of the lines *BC*, *CA* and *AB* in such a way that the signed distances of *A* to *BC*; *B* to *CA*; and *C* to *B*, respectively, are all positive.



VIVIANI’S THEOREM ON SIGNED DISTANCE

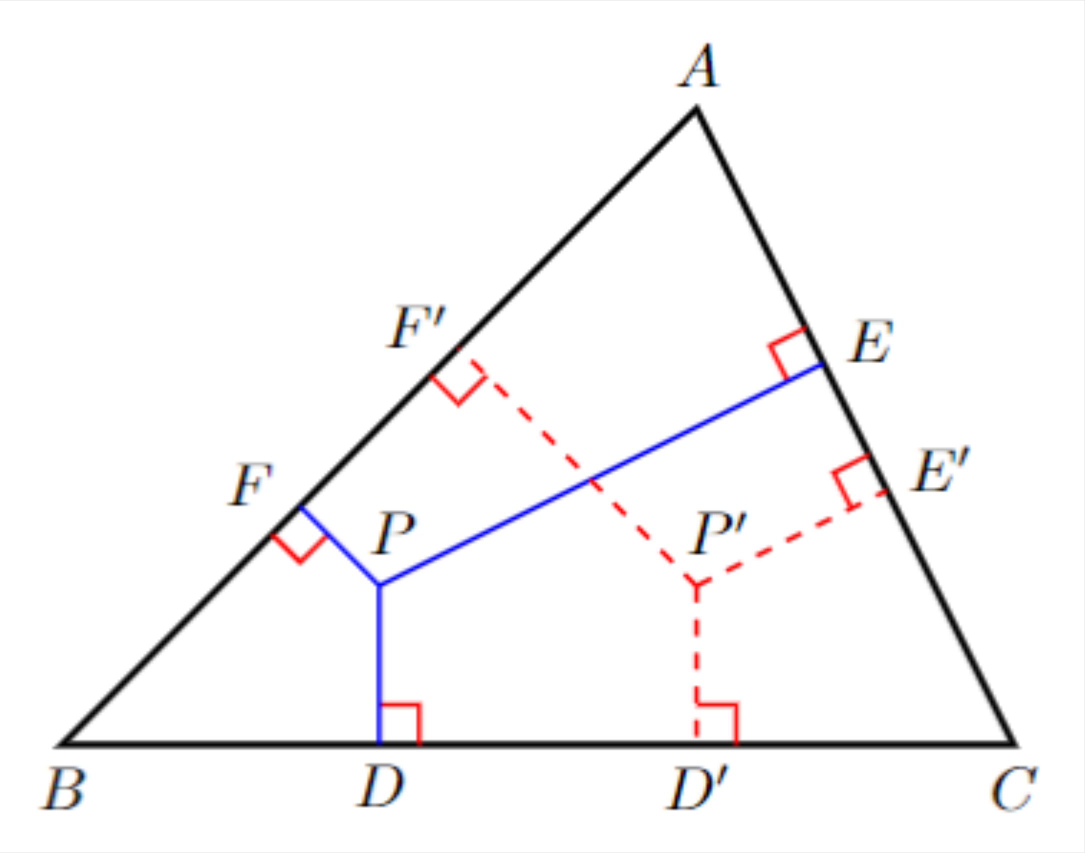
Using the above orientations, then the sum of the signed distances from any point to the sides of an equilateral triangle equals the length of the triangle’s altitude, regardless the position of P.



Does the converse of the Viviani's Theorem hold?

THE CONVERSE VIVIANI'S THEOREM

Let P be a point inside a fixed $\triangle ABC$, and let PD, PE, PF be the distances from P to BC, CA and AB, respectively. If PD + PE + PF is a constant, then $\triangle ABC$ is equilateral.

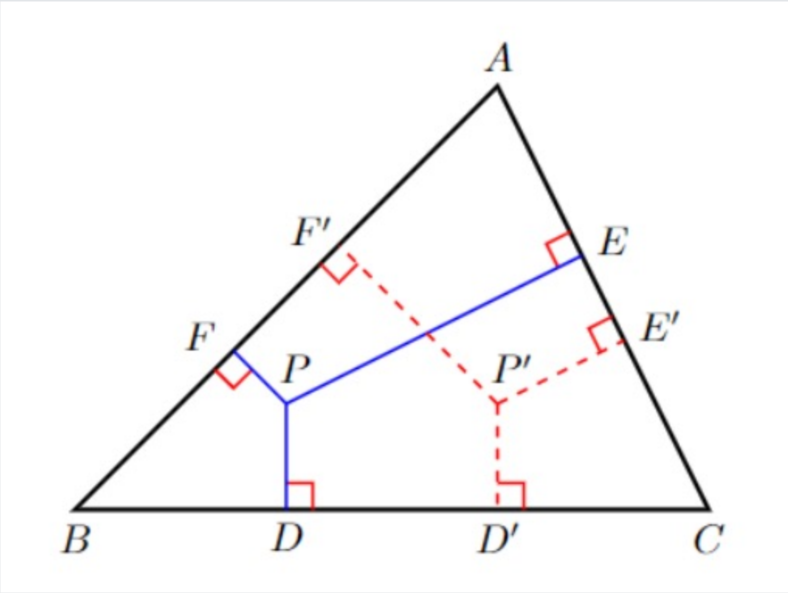


PROOF:

$a = BC,\ b = CA$ and $c = AB$
$\beta = S\Delta ABC$
$x = PD,\ y = PE,\ z = PF$
$\alpha = PD + PE + PF = x + y + z.$

Since
$S\Delta ABC = S\Delta BPC + S\Delta APC + S\Delta APB$,
$a \cdot x + b \cdot y + c \cdot z = 2\beta.$

System of linear equations of three variables
$x + y + z = \alpha,$
$a \cdot x + b \cdot y + c \cdot z = 2\beta.$



PROOF: (CONTINUING)

$$\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \end{bmatrix}$$

The coefficient matrix

If △ABC is not equilateral, then without loss of generality, we may assume a != b.

As a result, M is a matrix of Rank 2, and its solution set is one dimensional and represents a line.

However, the solution set is supposed to be two dimensional, and represents every point in the triangle.

Thus, there is a contradiction.

Acknowledgement

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CITATION

- 1. Wikipedia. (2022). Viviani’s theorem. en.wikipedia.org/wiki/Viviani%27s_theorem.