Topic 22-Spieker's Theorem

UC Irvine-Math199C

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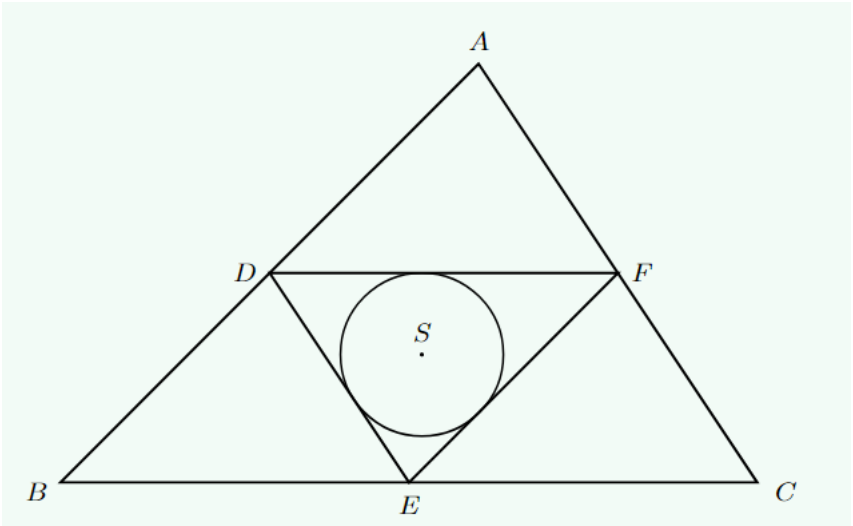
Spieker Circle

- It is named after the 19th-century German geometer Theodor Spieker. He is a secondary school teacher and his geometry textbook inspired Einstein's interest in higher mathematics.

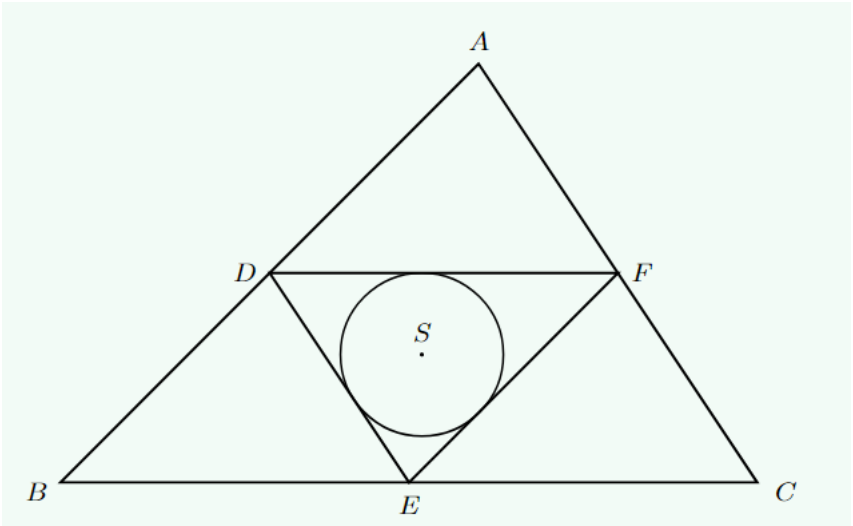
Definition



The Spieker circle is the incircle of the midpoint triangle of a triangle. The Spieker circle's center is called the Spieker center.

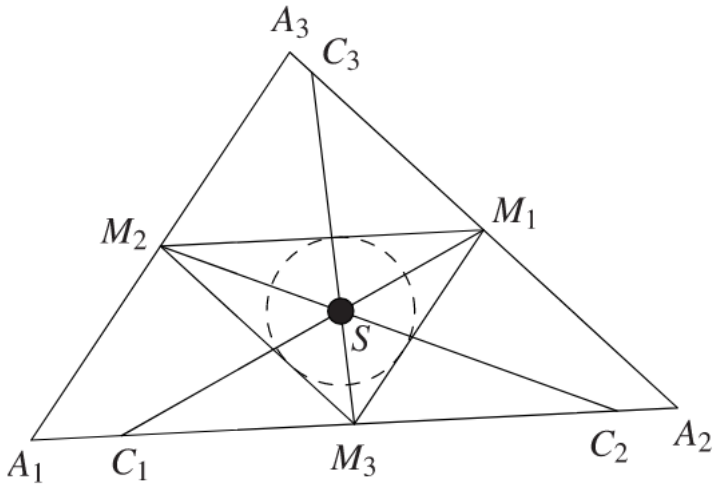
Triangle DEF is the midpoint triangle of triangle ABC. Then the Spiker incircle of triangle DEF is Spieker circle. And the center of the circle is S which is called Spieker center.

Center of gravity



- Spieker center S is also the center of gravity of triangle ABC's perimeter in physics.
- We can assume triangle ABC has uniform-density boundary. Then points D, E, F is center of gravity of AB,BC,AC.
- We also assume triangle DEF was connected by weightless rods. Then the gravitational force on points D, E, F will be concentrated to the incenter of triangle DEF.

Cleavance Center

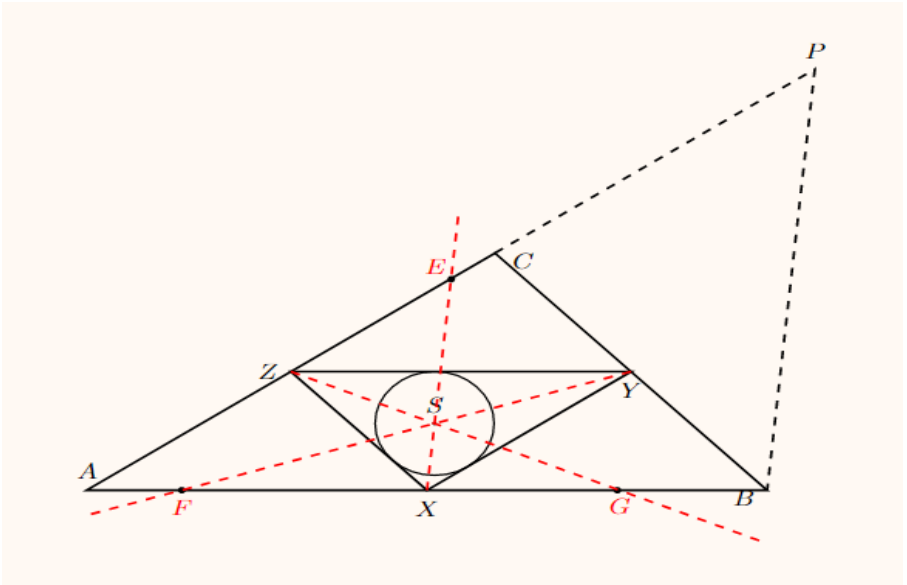


Cleavance Center is the point where all three cleavers of the triangle intersect each other which is also the Spieker center.

Cleaver is a perimeter bisectors with an endpoint at a side's midpoint. M_2C_2, M_1C_1, M_3C_3 are cleavers. We can see M_2C_2 divides triangle $A_1A_2A_3$'s perimeter. Hence,

$M_2A_3+A_3A_2+A_2C_2=A_1C_2+A_1M_2$

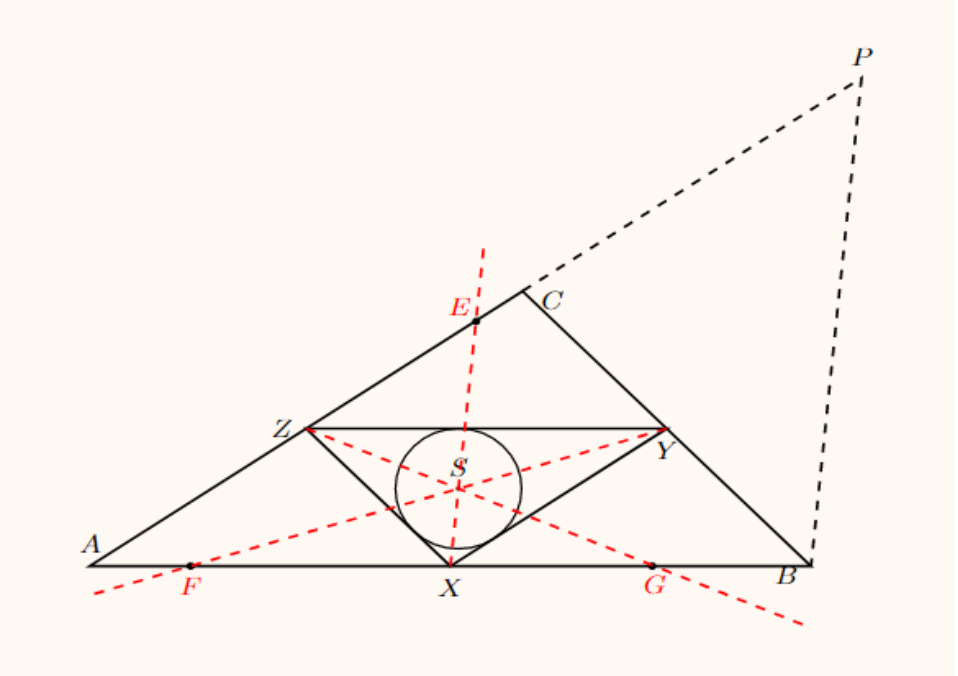
Proof



Made a extension line PC=BC.

First, we prove ZCXY is a parallelogram.
Then we prove angle AEX= angle CPB.
Hence we get triangle AEX is similar to triangle APB, and similar ratio is 1:2.
Then we get AE=EP=EC+BC.

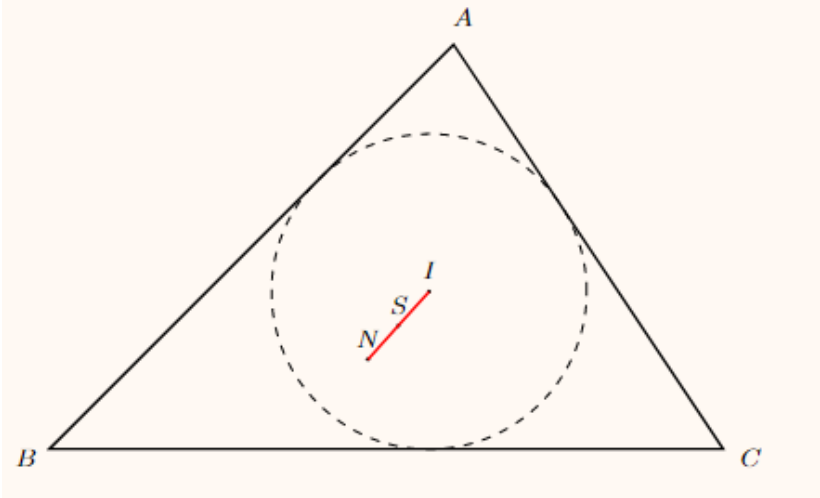
Proof-continue



With this method, we can prove YF and ZG are also the cleaver of the triangle ABC.

Then we can know the Cleavance center is the Spieker center.

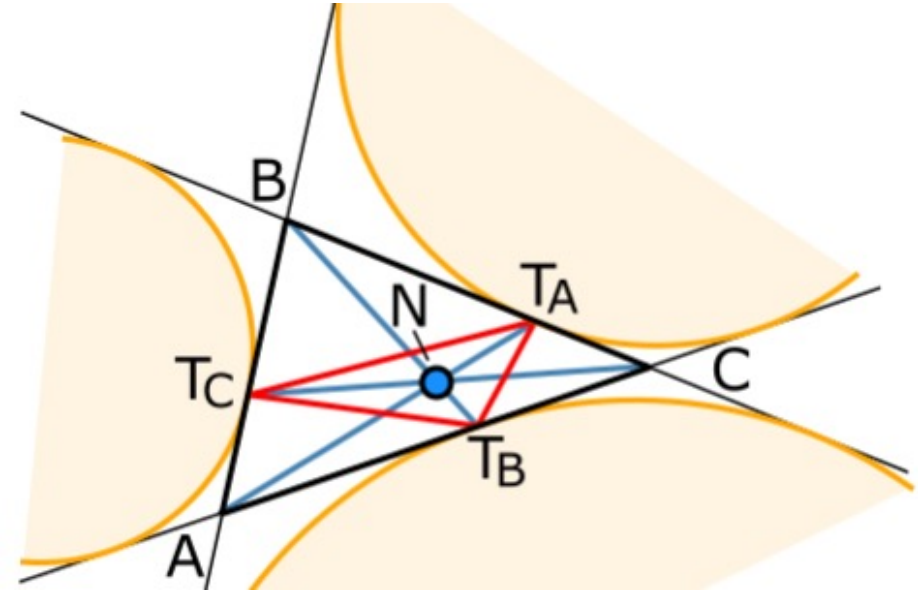
Nagel point and Spieker center



The incenter I of the ΔABC, Nagel point N and Spieker center S can form a line which is called Nagel line in the ΔABC.

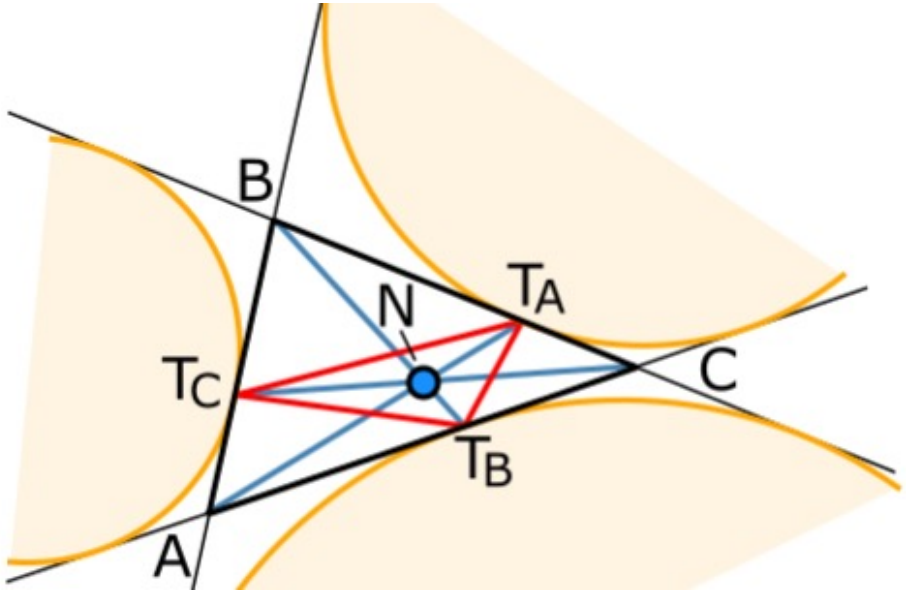
The Spieker center will always be on this line

What is Nagel point



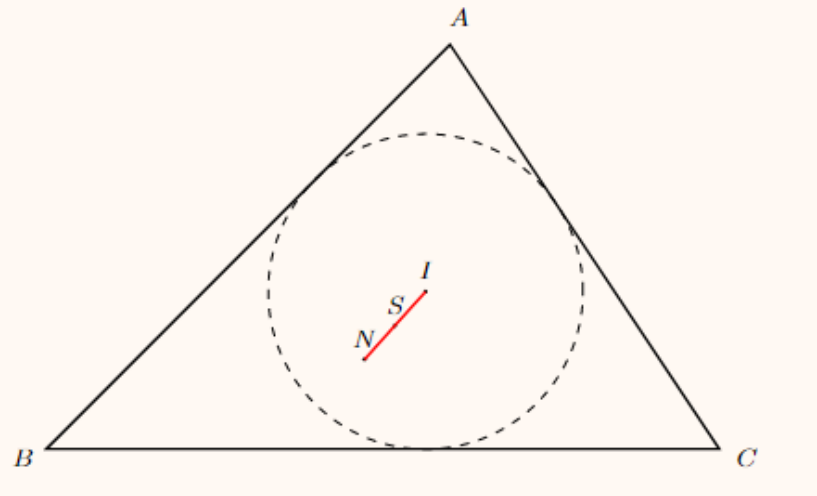
- The Nagel point (named after Christian Heinrich von Nagel) is a triangle center. It is the concurrent point of all three triangle splitters.
- In picture above, T_AA, T_CC, T_BB are splitters. They intersect at the same point which is the Nagel point.

What is Splliters



- Splitter is a line segment through one of the vertices of a triangle that bisects the perimeter of the triangle.
- We can see TAA from the vertice A and bisect the perimeter of triangle ABC
- Similar with cleavers but different. Because cleavers also bisect the perimeter but instead emanate from the midpoint of one of the triangle's sides.

Proof



- To prove that point I, S and N are in the same line. We need get their trilinear coordinates.

Trilinear coordinates describes the ratio distance of the point from the three sides of the triangle.

Three points are collinear if and only if the determinant of there trilinear coordinates equals to zero.

Proof continue

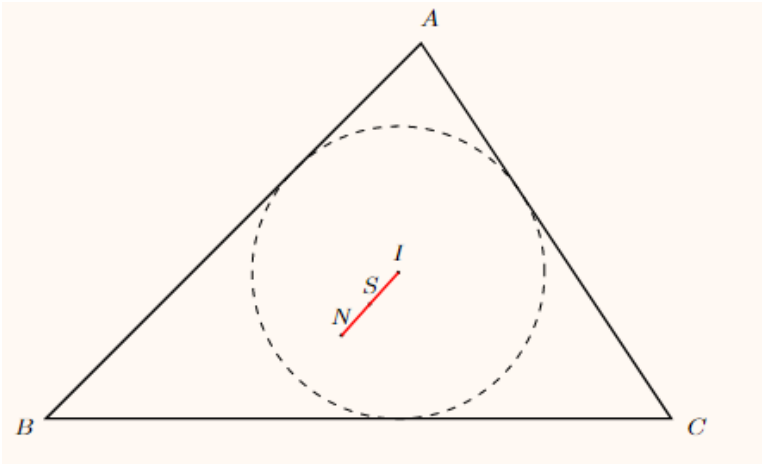
$$P = p : q : r$$
$$U = u : v : w$$
$$X = x : y : z$$

For example, we assume the trilinear coordinates of point P, U, X are above.
Then if

$$D = \begin{vmatrix} p & q & r \\ u & v & w \\ x & y & z \end{vmatrix} = 0$$

Point P, U, X are on the same line.

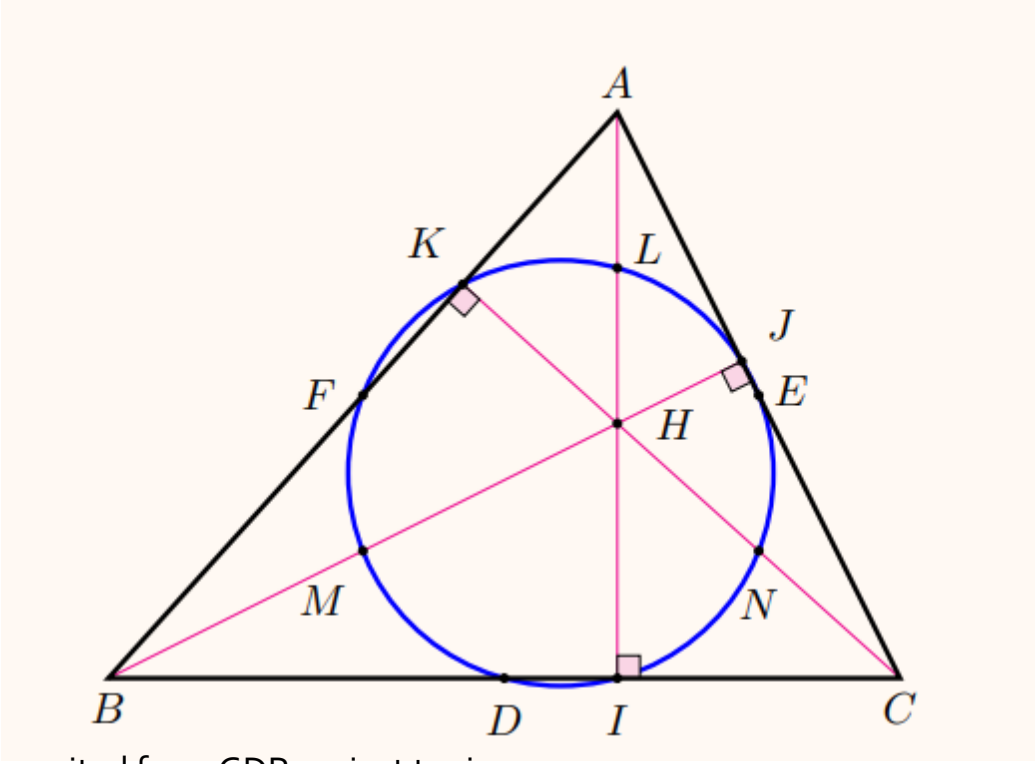
Proof continue



Then with this method, we can get point I, S, N's trilinear coordinates from Wikipedia then calculate the determinant of their trilinear coordinates equals to zero.

$$det \begin{bmatrix} bc(b+c) & ca(c+a) & ab(a+b) \\ (b+c-a)/a & (c+a-b)/b & (a+b-c)/c \\ 1 & 1 & 1 \end{bmatrix} = 0$$

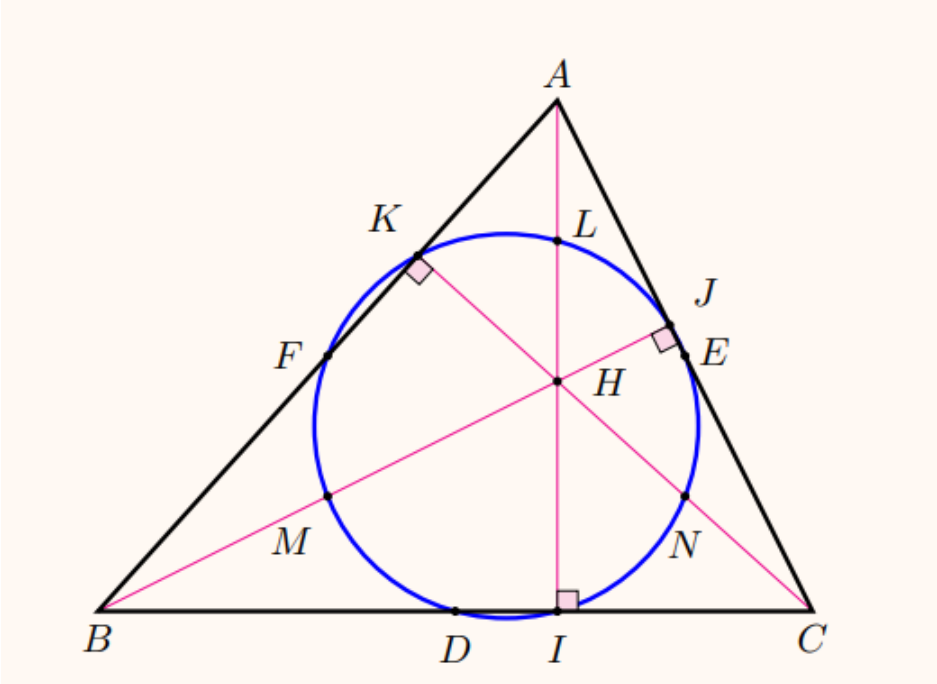
Analogue with Nine-point circle



cited from GDP project topic 13.

- The nine-point circle with the Euler line and the Spieker circle with the Nagel line are analogous to each other that having dual-like similarities.

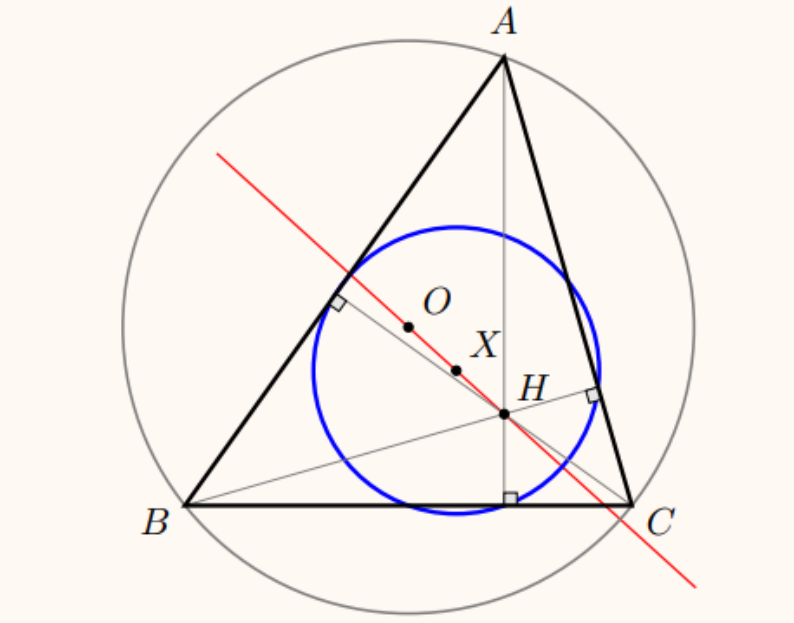
What is Nine-point circle



- Consider any triangle ΔABC and these nine points: the 3 midpoints of each of its sides E, D, F, the 3 feet of each altitude K, I, J, and the 3 midpoints L, M, N, of the line segments between the orthocenter H and each vertex. These nine points are concyclic and they form the nine-point circle of ΔABC

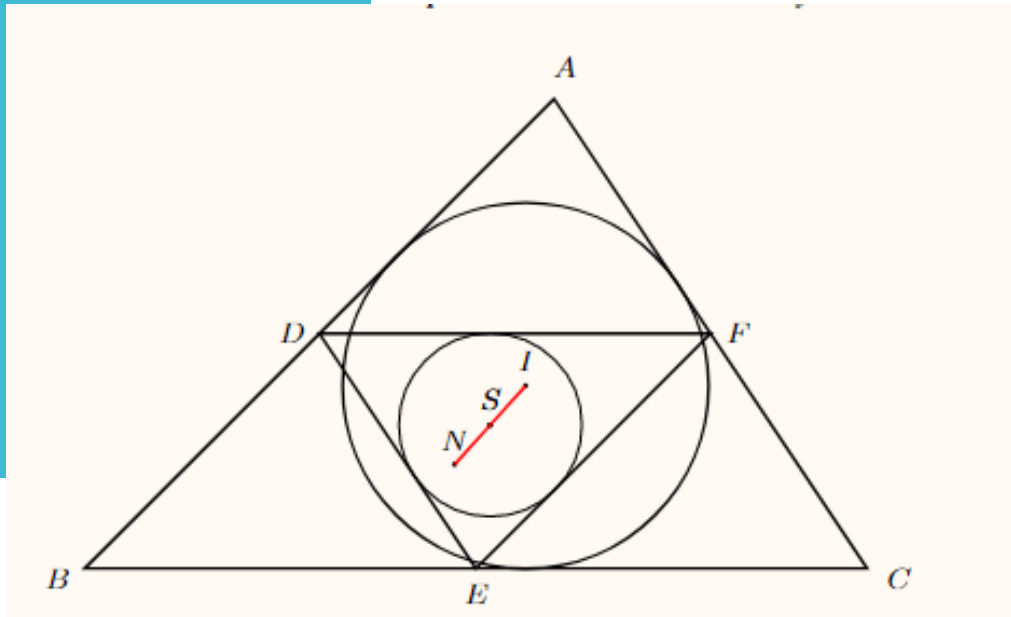
(Cited from GDP project topic 13.)

Analogue 1



- Given △ABC, the center of its nine-point circle X lies on the Euler line, midway between the orthocenter H and the circumcenter O.

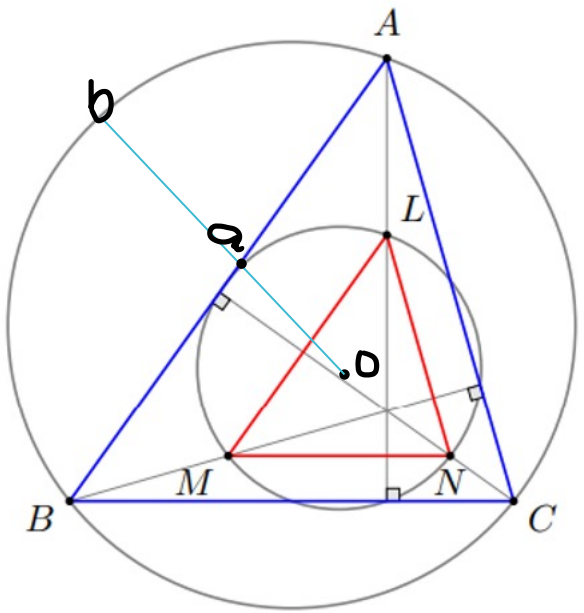
cited from GDP project topic 13.

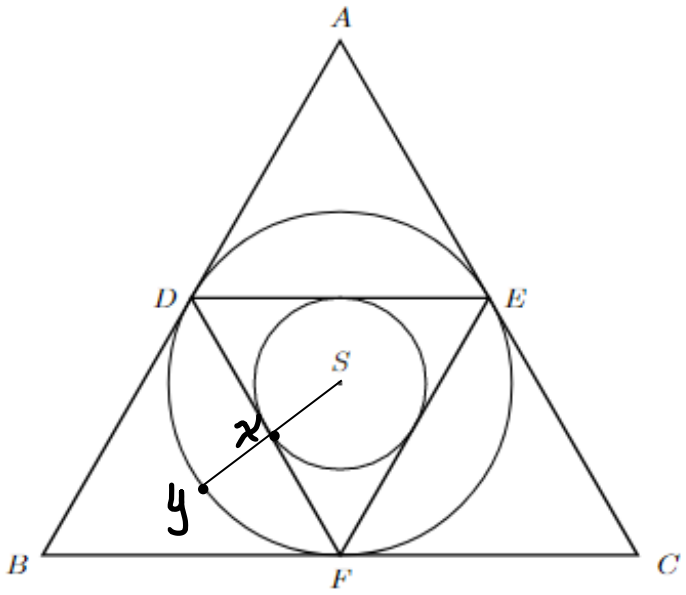


The center of the Spieker circle S is the midpoint of the segment joining the incenter I and Nagel point of a triangle N.

Analogue 2

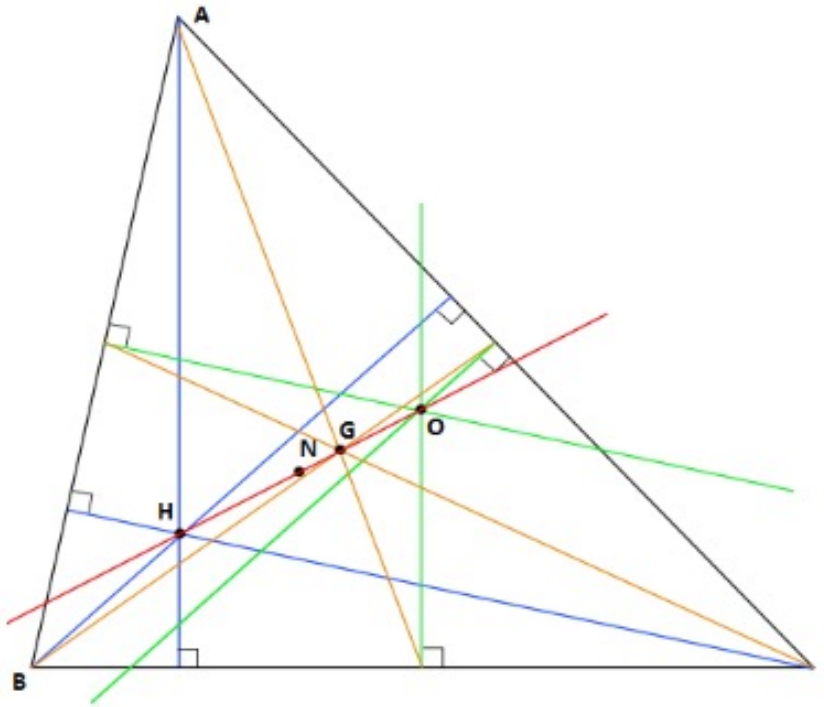
- The radius of the nine-point circle is one-half the radius of the circumcircle of a triangle.(aO=1/2 bO)
- The radius of the Spieker circle is one-half the radius of the incircle of a triangle.(Sx=1/2 Sy)





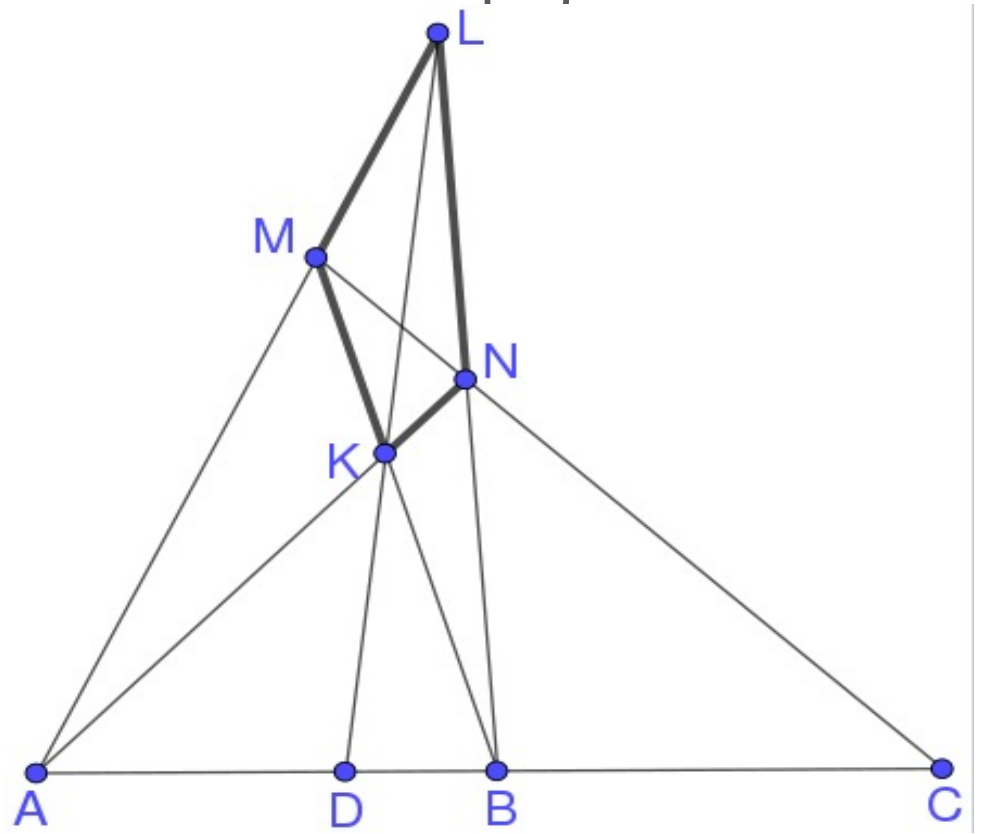
Analogue 3

- The center(N) of the nine-point circle is the projective harmonic conjugate of the circumcenter (O) with respect to the orthocenter(H) and the centroid(G) of a triangle.(which means HO: GO= NH:GN)



What is projective harmonic conjugate

- Given three collinear points A, B, C, let L be a point not lying on their join and let any line through C meet LA, LB at M, N respectively. If AN and BM meet at K, and LK meets AB at D, then D is called the harmonic conjugate of C with respect to A, B.
- (Cited from Wikipedia)

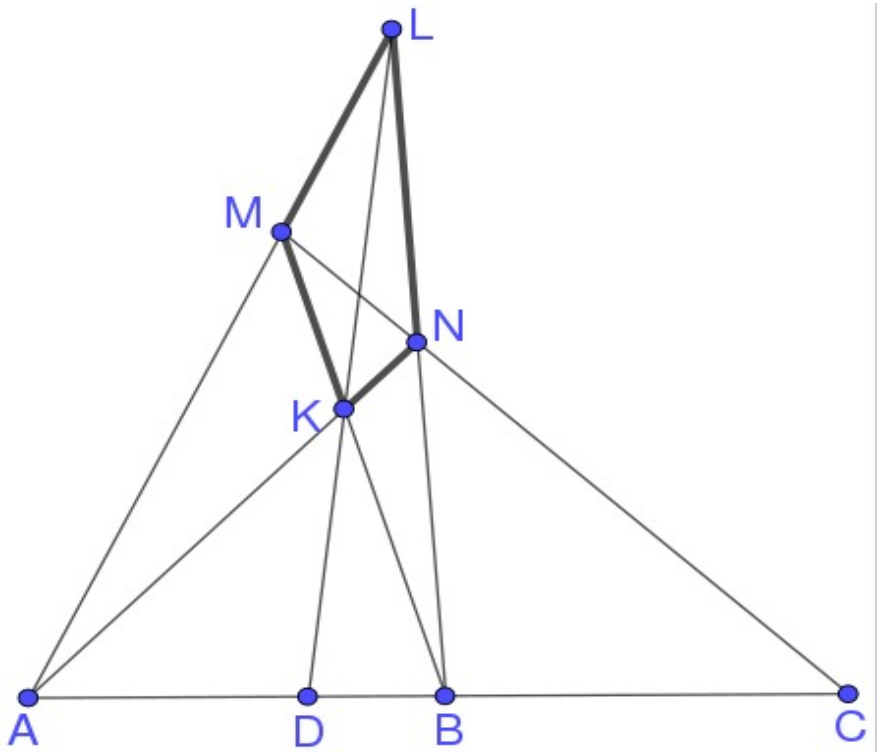


projective harmonic conjugate

- D always divides the segment AB internally in the same proportion as C divides AB externally. That is:

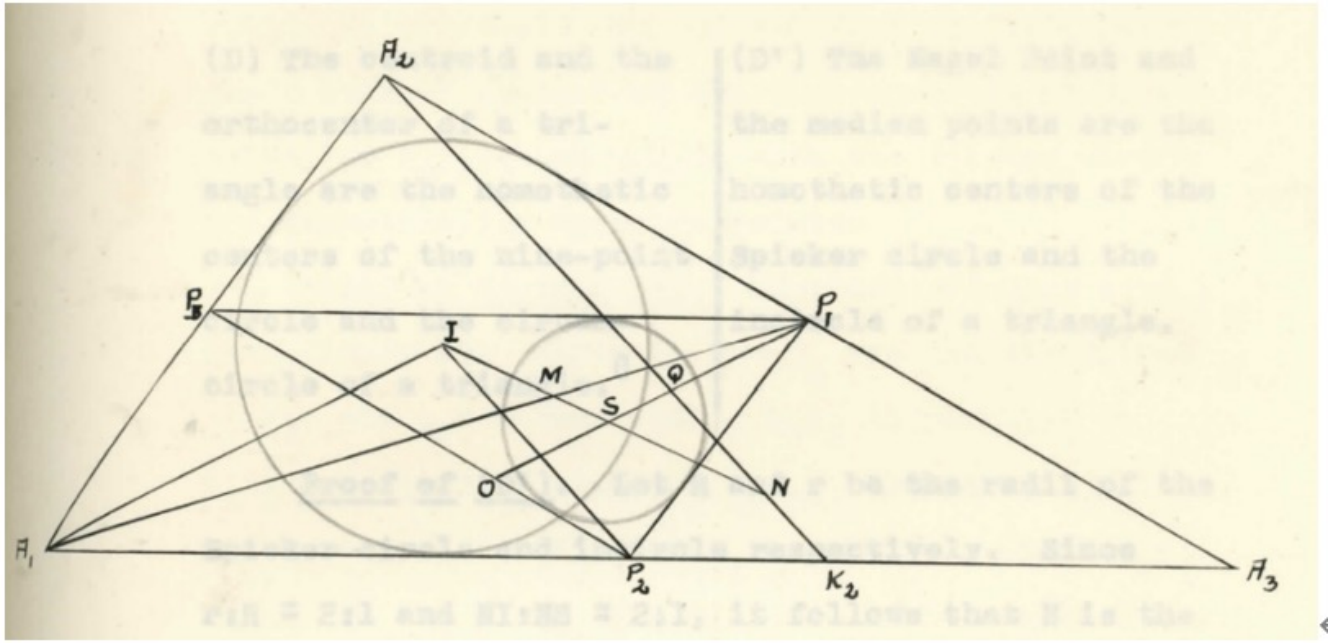
$$|AC| : |BC| = |AD| : |DB|$$

(Cited from Wikipedia)



Analogue 3

- The center(S) of the Spieker circle is the projective harmonic conjugate of the incenter(I) with respect to the median point(M) and the Nagel point of a triangle(N).(Which means IN:SN=IM:MS)



Reference

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ThankYou!