Yff Inequality

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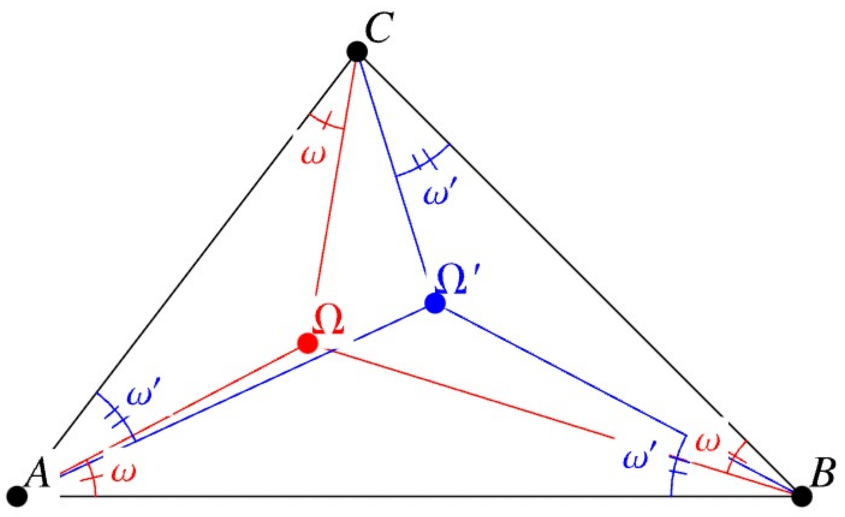
Related Concept - Brocard Point

- There exist exactly one point such that:

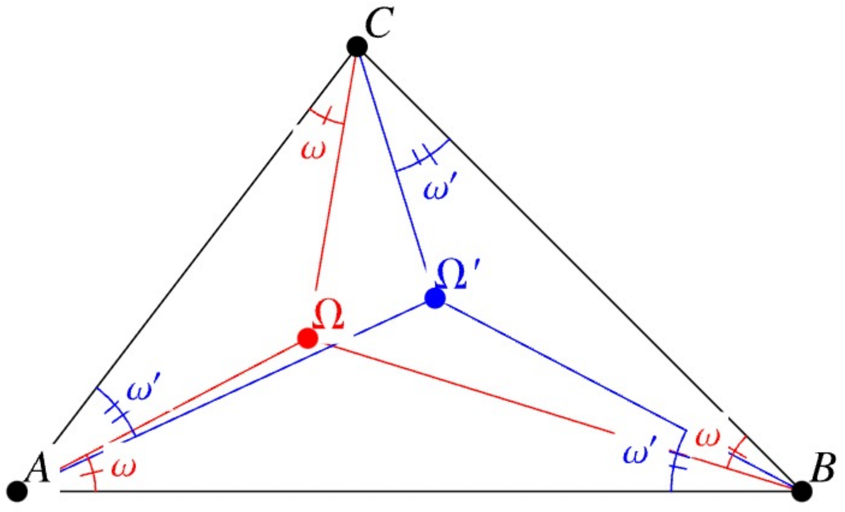
First brocard point Ω
$\angle \Omega AB = \angle \Omega BC = \angle \Omega CA = \omega$
Second brocard point Ω'
$\angle \Omega' BA = \angle \Omega' CB = \angle \Omega' AC = \omega'$



Let's Prove it!

Yff Inequality:

- $2\omega \leq \sqrt[3]{ABC}$



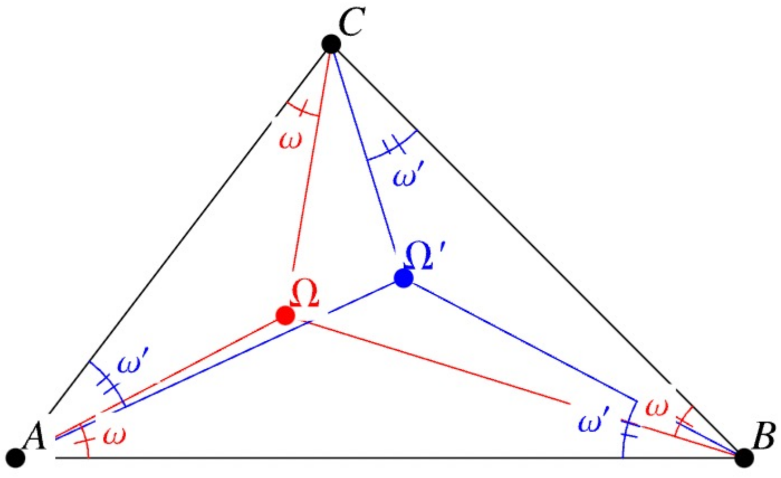
- By Law of Sine:

$$\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$$

$$\implies \frac{a}{b} = \frac{sinA}{sinB}$$

$$\implies \frac{sin\omega}{sinB-\omega} * \frac{sin\omega}{sinC-\omega} * \frac{sin\omega}{sinA-\omega} = \frac{\Omega B}{\Omega A} * \frac{\Omega C}{\Omega B} * \frac{\Omega A}{\Omega C} = 1$$

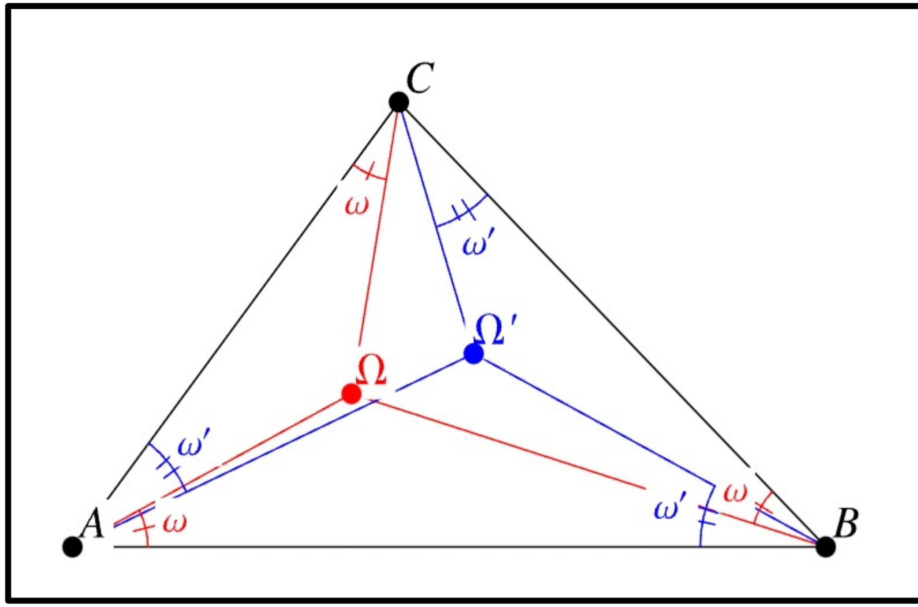
$$\implies \sin^3\omega = sin(B-\omega)sin(C-\omega)sin(A-\omega)$$



- Let f(x)=ln ($sine^x$- ω)

$\Longrightarrow$ f"(x)= $\frac{e^x \cdot [cos(e^x - \omega) sin(e^x - \omega) - e^x]}{sin^2(e^x - w)} \leq -\omega \leq 0$

$\Longrightarrow$ f(x) is concave



By **Jensen Inequality**,

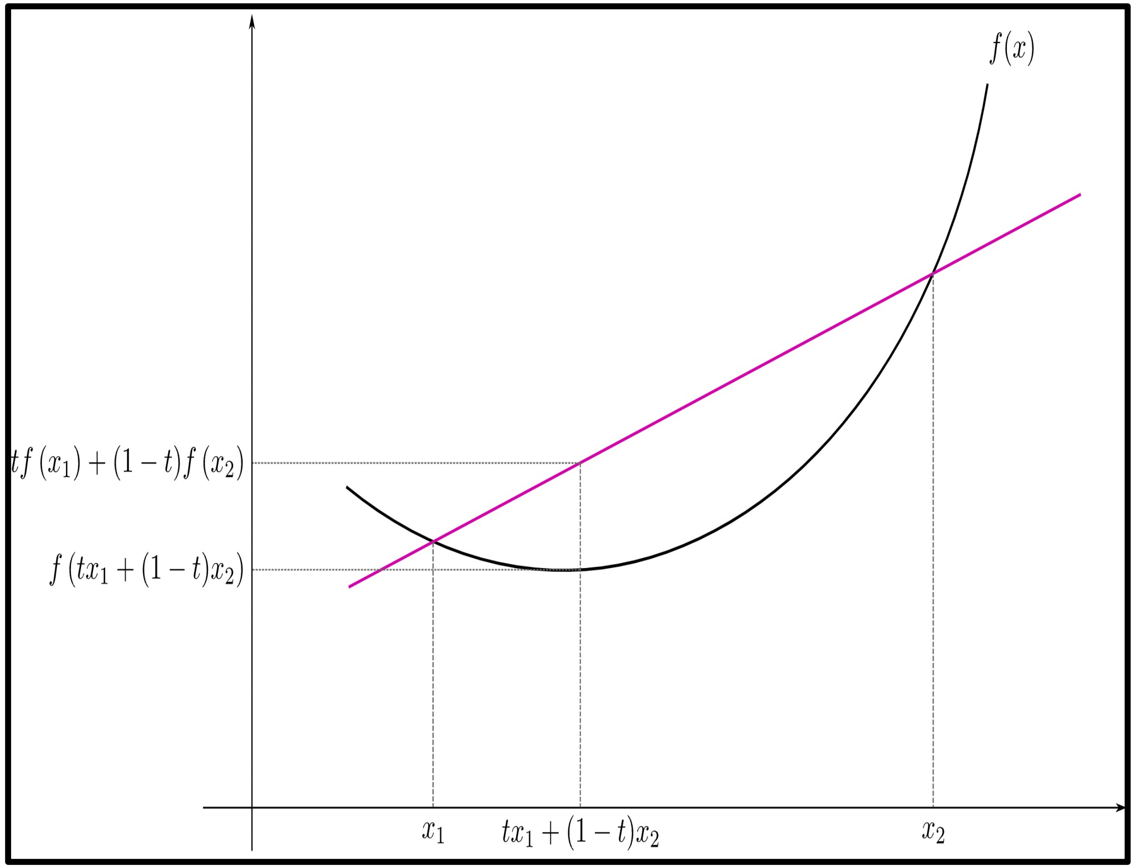
$\Longrightarrow sin(B-\omega)sin(C-\omega)sin(A-\omega) \leq sin^3(\sqrt[3]{ABC}-\omega)$

$\Longrightarrow sin^3\omega \leq sin^3(\sqrt[3]{ABC}-\omega)$

Related concept- Jensen Inequality

- If function is concave, have:

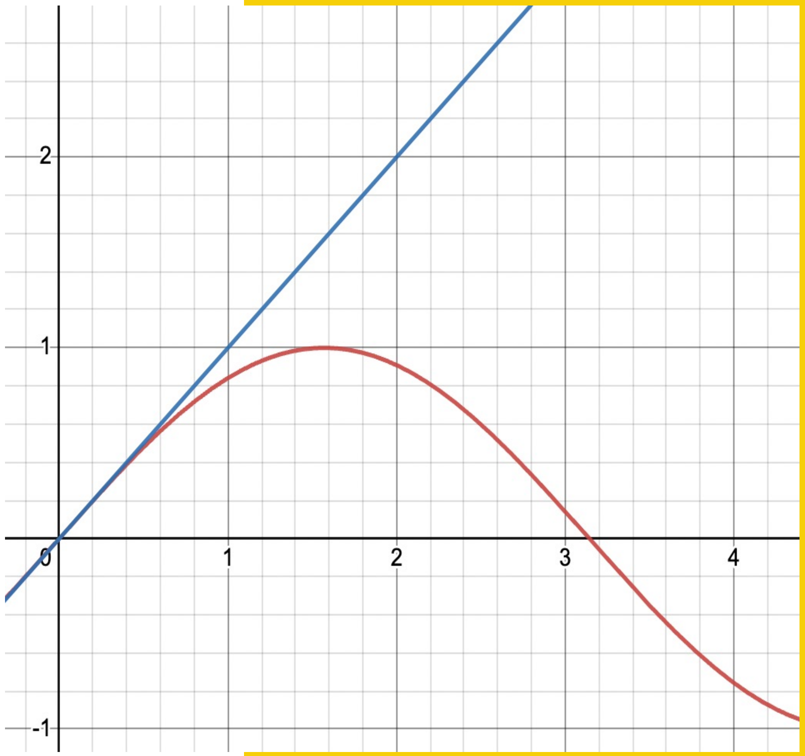
$$\varphi\left(\frac{\sum a_i x_i}{\sum a_i}\right) \geq \frac{\sum a_i \varphi(x_i)}{\sum a_i}.$$



For $sin^3\omega \leq sin^3(\sqrt[3]{ABC} - \omega)$,
we have $\omega \leq \frac{6}{\pi}$ and also $2\omega \leq \frac{(A+B+C)}{3}$(*arithmetic*)

$\Rightarrow \sqrt[3]{ABC} - \omega \in [0, \frac{A+B+C}{3}]$
By $sinx \leq x$ for $x \in [0, \infty]$,

Therefore, we get that $2\omega \leq \sqrt[3]{ABC}$(*geometric*)



Some thought

- How about for hamonic mean?

Theorem 3 (Lu)

Let $\alpha, \beta, \gamma > 0$, and let $\alpha + \beta + \gamma = \pi$. Then

$$\cot\alpha + \cot\beta + \cot\gamma - \cot\left(\frac{1}{2} \cdot \frac{3}{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}}\right) \geq 0.$$

- if we let f(x)=ln $(sine^{x}\text{-}\ \omega)+\frac{1}{2}\lambda\omega x^2 \Rightarrow$ f"(x) $\leq -\omega + \lambda\omega \leq 0$

  By Jensen inequality,

  $$\Longrightarrow \frac{sinx}{sin(\sqrt[3]{ABC}-\omega)} \leq e^{-\frac{1}{2}\lambda\omega(\frac{1}{3}(ln^2 A+ln^2 B+ln^2 C)-\sqrt[3]{ln(ABC)^2})} \leq 1(when\lambda = 1)$$

Citation Page

[1]Brocard Angle. Wolfram Mathword, Apr 23, 2010

https://mathworld.wolfram.com/BrocardAngle.html

[2] Zhiqin Lu, Generalized Yff Inequality. /lu.math.uci.edu/msi/

[3]Marian Dincă, A Direct Proof of the Yff's Conjecture. /https://vixra.org/pdf/1008.0037v1.pdf/