Pascal and Brainchon Theorems

Math 199

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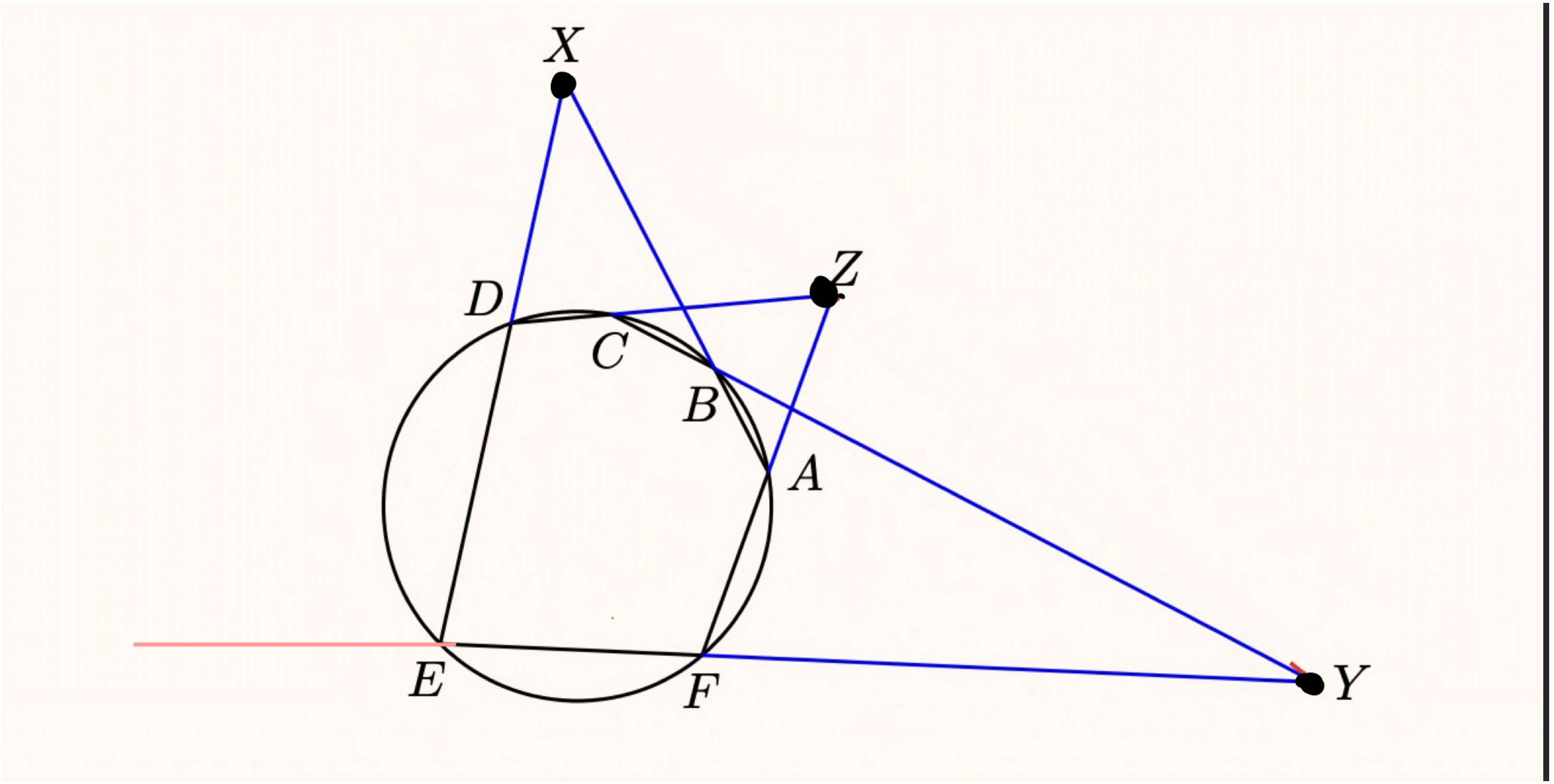
Contents

- Pascal's Theorem
- Related Theorem (Desargues' Theorem)
- Brianchon's Theorem
- Generation & Special case & Degeneration

Basic Information

- In hexagon ABCDEF
- Line AB Intersects Line DE at point X
- Line BC Intersects Line EF at Point Y
- Line DC Intersects Line AF at Point Z

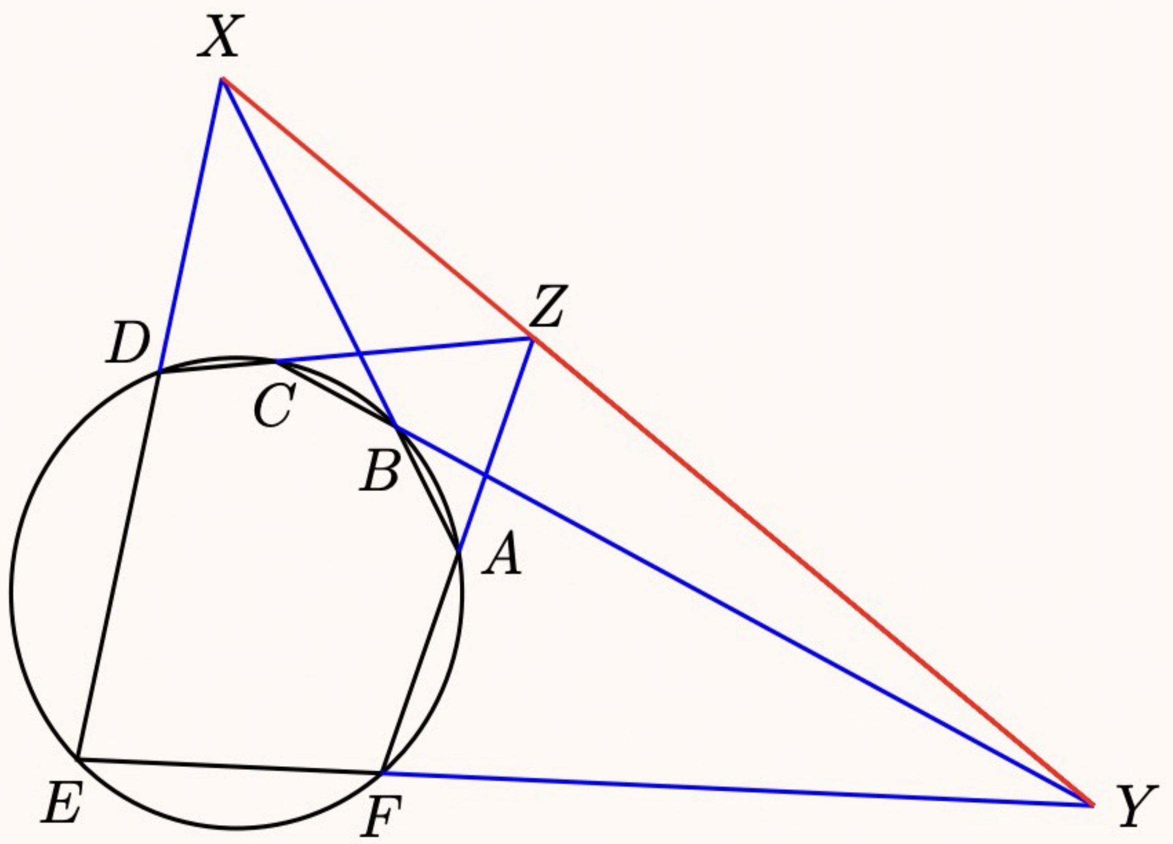


Q: Are X, Z, and Y collinear?

Yes !!!!!! → **Pascal Theorems**

Pascal Theorems

- *The hexagon* ABCDEF *is inscribed to a circle. Assume that* AB, DE *intersects at* X*;* BC, EF *intersects at* Y*; and* CD, F A *intersects at* Z*. Then* X, Y, and Z are collinear.

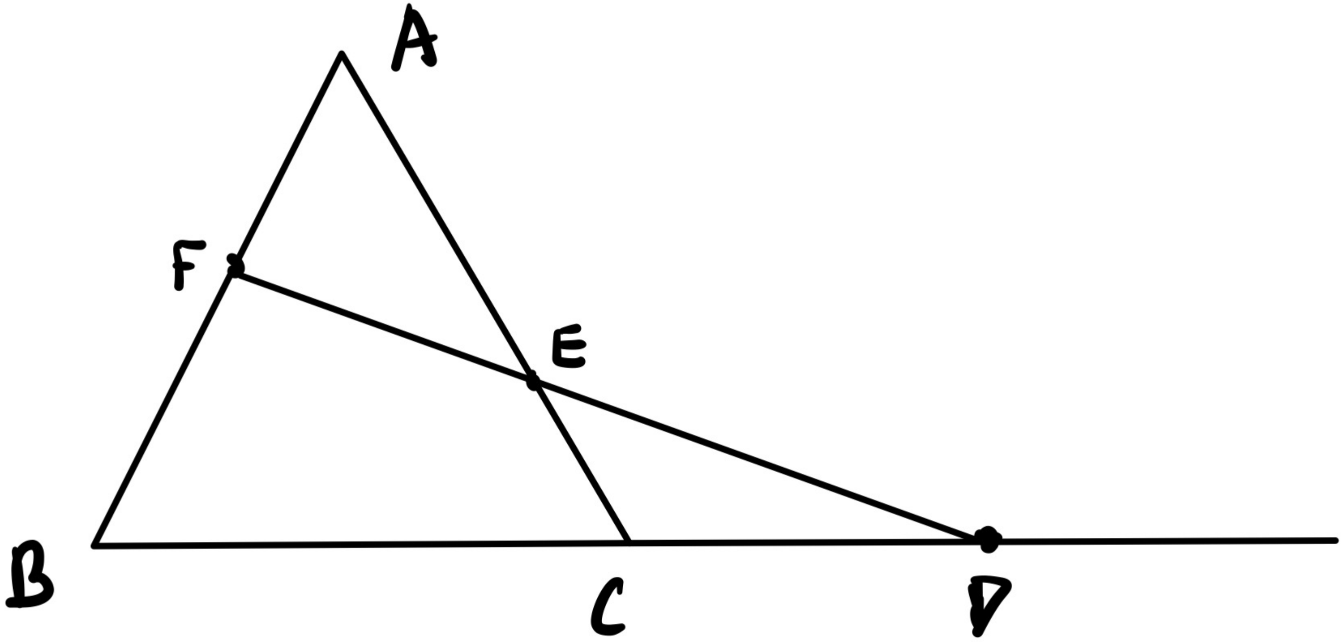


Review

Menelaus' s Theorem

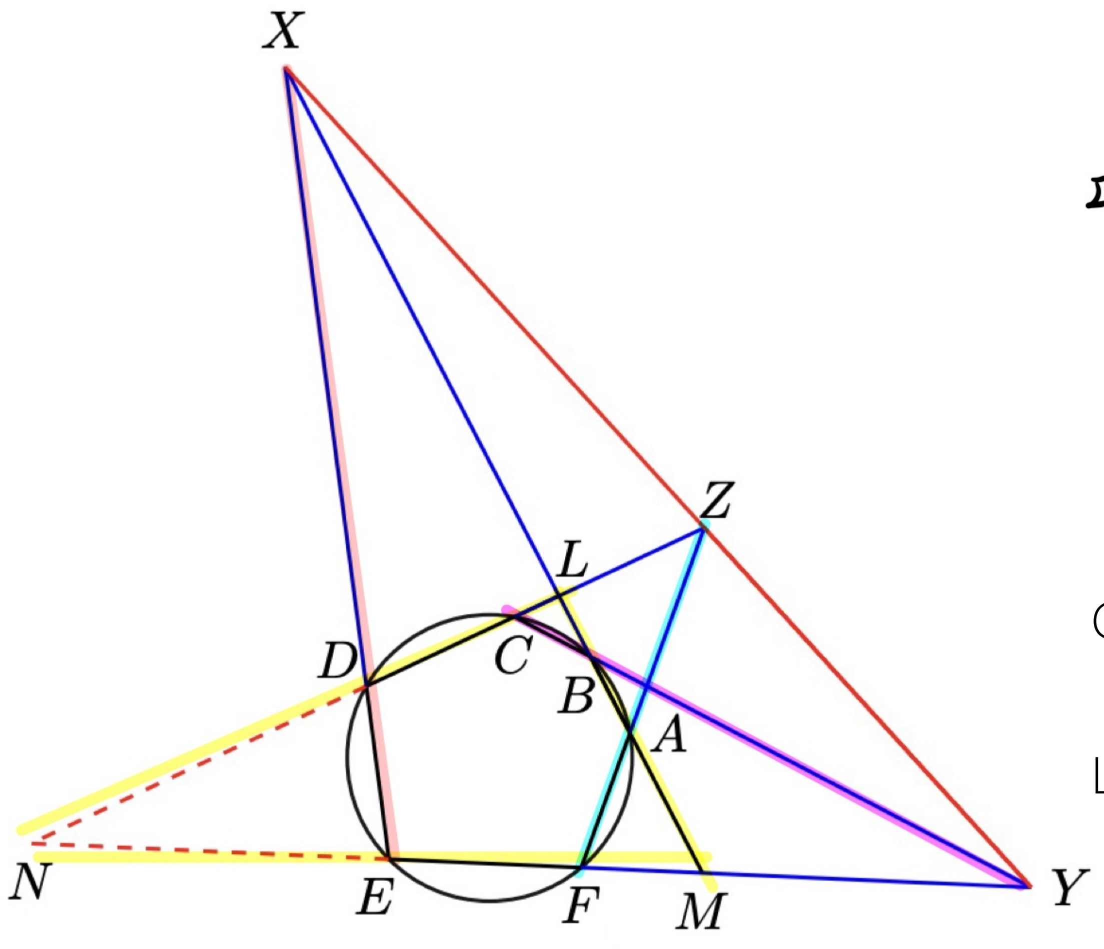
- ΔABC
- a transversal line EFD that crosses (points E, F, and D are collinear)

AF/FB=BD/DC=CE/EA=1



Proof

- As in the graph drawn below, let AB and CD intersect at L, BA, and EF intersect at M, and CD and FE intersect at N.



in ΔLMN (Menelau's Theorem)

C, B, Y collinear $\Rightarrow \frac{LB}{BM} \cdot \frac{MY}{YN} \cdot \frac{NC}{CL} = 1$

F, A, Z collinear $\Rightarrow \frac{LA}{AM} \cdot \frac{MF}{FN} \cdot \frac{NZ}{ZL} = 1$

E, D, X collinear $\Rightarrow \frac{ND}{DL} \cdot \frac{LX}{XM} \cdot \frac{ME}{EN} = 1$

In circle ABCDEF,

power of point Theorem

$LA \cdot LB = LD \cdot LC$

$NC \cdot ND = NE \cdot NF$

$MA \cdot MB = MF \cdot ME$

Combine six equations

⬇

LX/XM=MY/YN=NZ/ZL=1 (in Δ ZNY)

⬇

Points X, Z, and Y are collinear. By Inverse of Menelaus's Theorem

Pappus’ Theorem (Special Case)

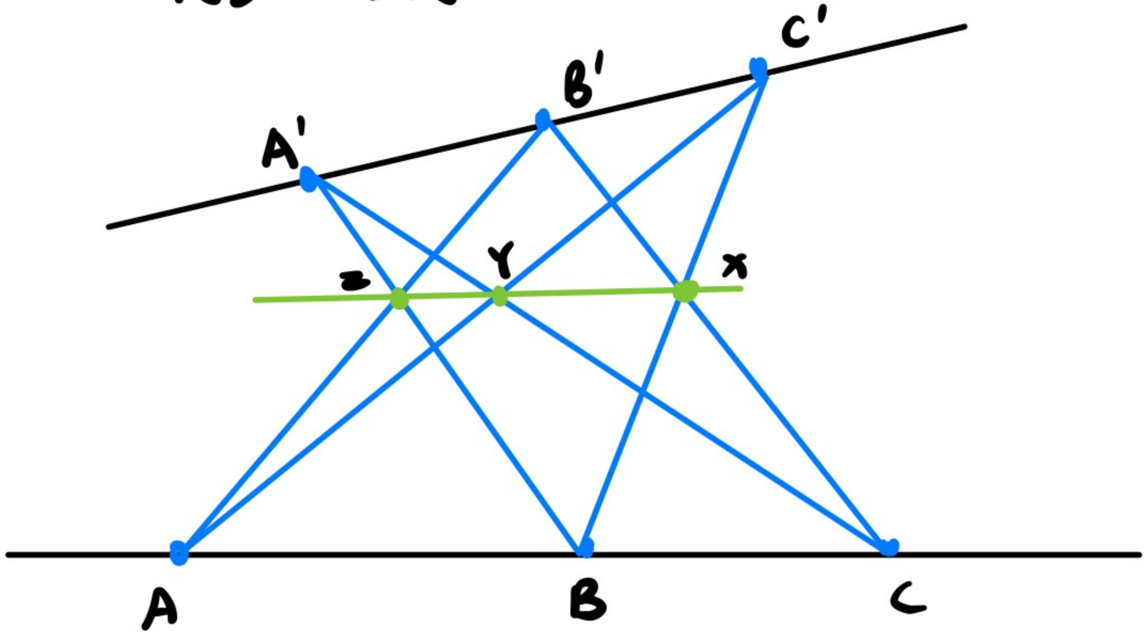
The Hexagon BC’AB’CA’ is inscribed on the two black lines or a circle.

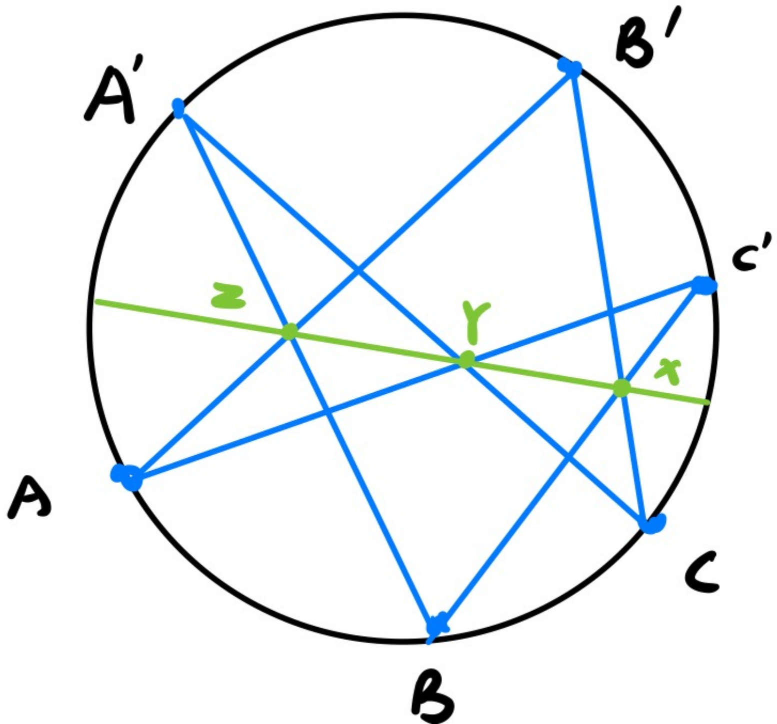
Assume $BC' \cap B'C \Rightarrow X$

$CA' \cap A'C \Rightarrow Y$

$AB' \cap B'A \Rightarrow Z$

$\Rightarrow$ Then X Y, Z are collinear



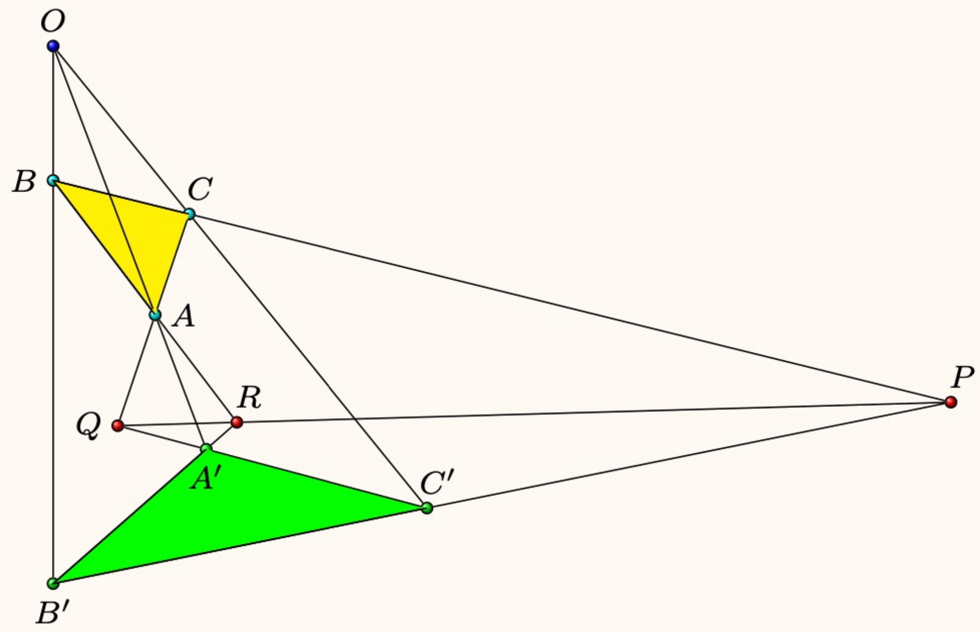


Desargues'　Theorem (Related Theorem)

- Self-"dual" Theorem

Theorem 1. (Desargues' Theorem)

We consider triangles $\triangle ABC$ and $\triangle A'B'C'$. Assume that lines BC, $B'C'$ intersect at P, CA and $C'A'$ intersect at Q, and AB, $A'B'$ intersect at R. Then P, Q, R are collinear if and only if AA', BB' and CC' are concurrent.



Desargues' Theorem & Pascal's Theorem

- Extend Line EF to EA′
- Extend Line DE to DA
- Extend Line BC to BA
- Extend Line BA to BA′

ABC & A′B′C′

X Y Z are collinear
(Pascal's Theorem)

AA′, BB′, and CC″
are concurrent.

DE & BA at X

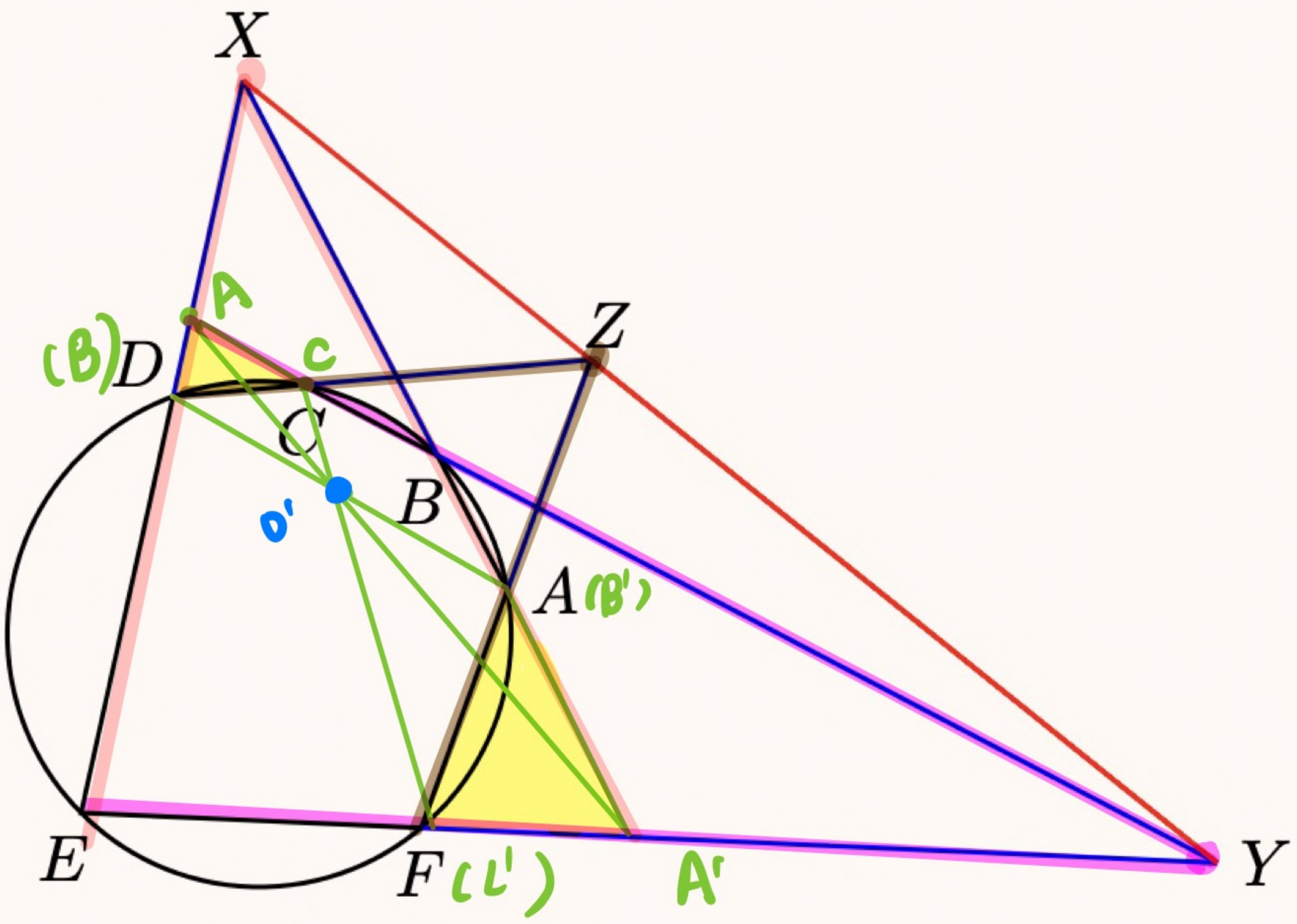
AB & A′B′ at X

DC & AF at Z

BC & B′C′ at Z

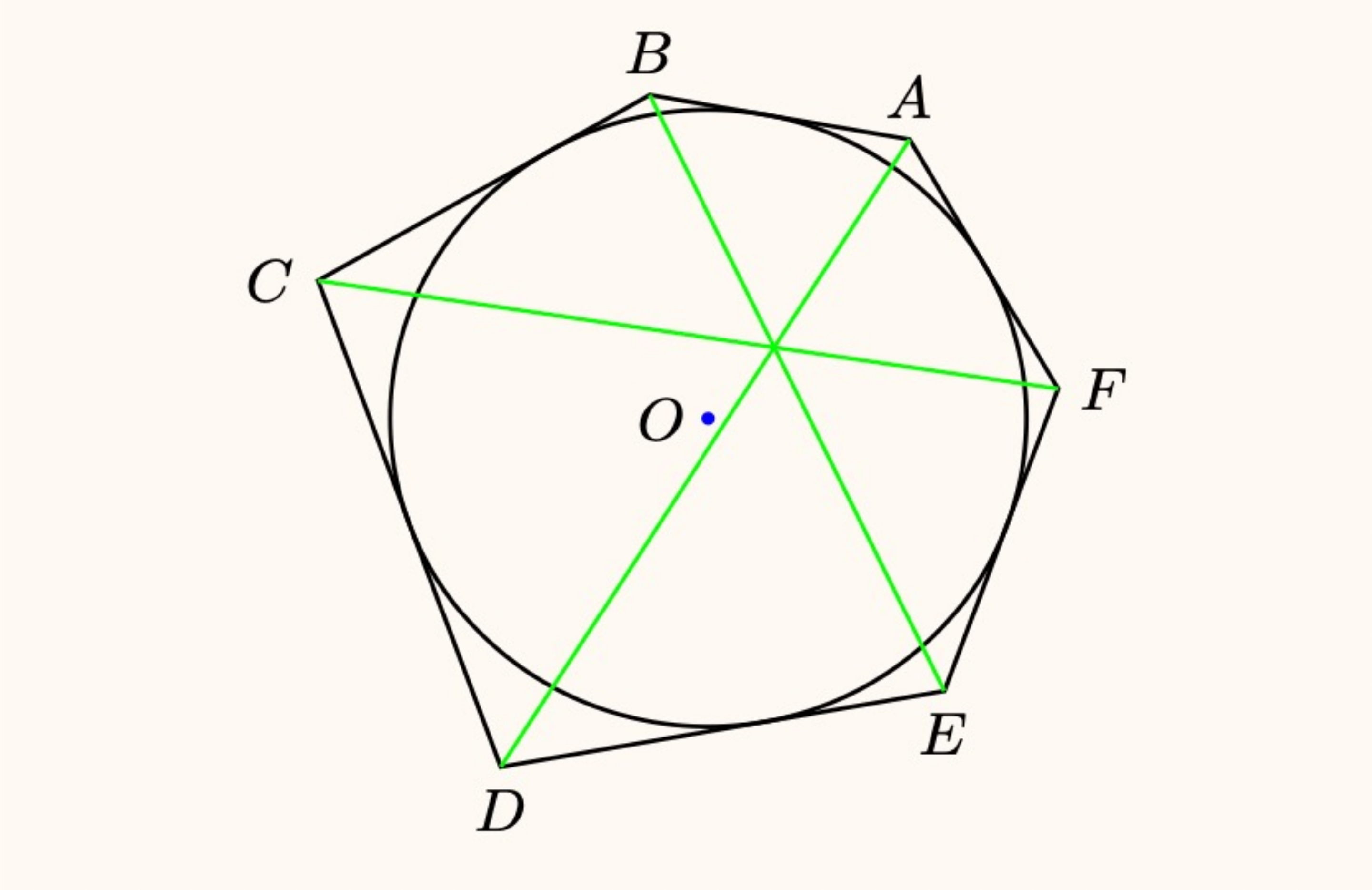
BC & EF at Y

AC & A′C′ at Y



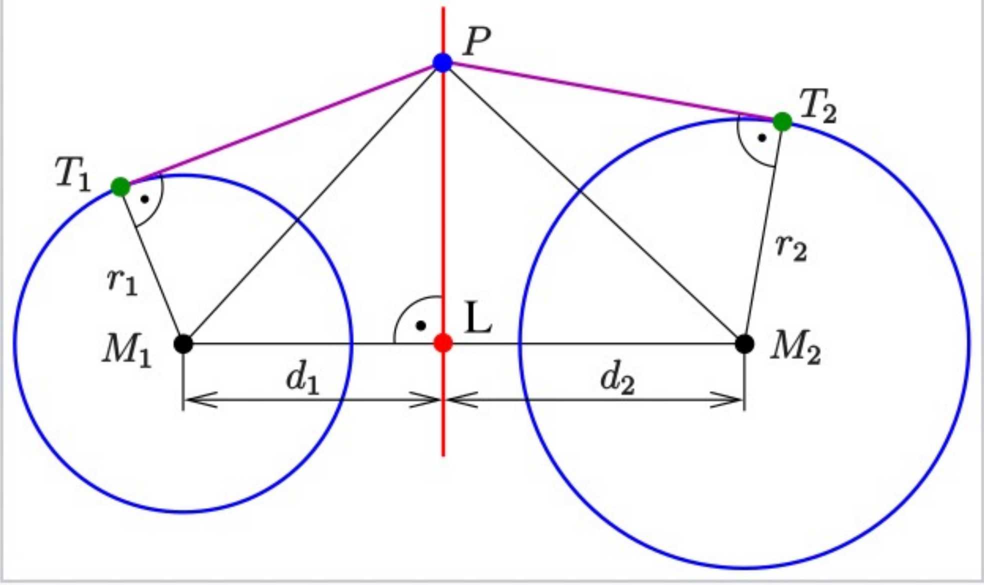
Brainchon Theorems

- The Hexagon ABCDEF is circumscribed on a circle. The AD, BE, and CF is concurrent.



Related Theorems

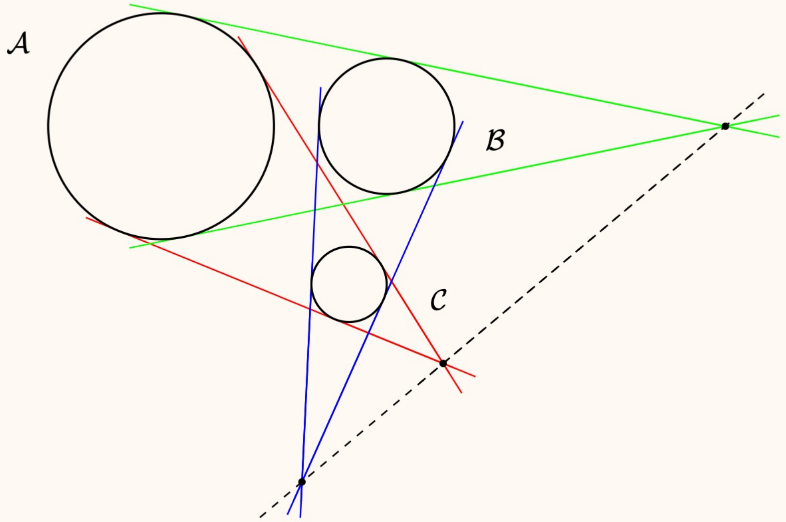
Radical Axis: the line such that tangents drawn from any point of the line to two given circles are equal in length.

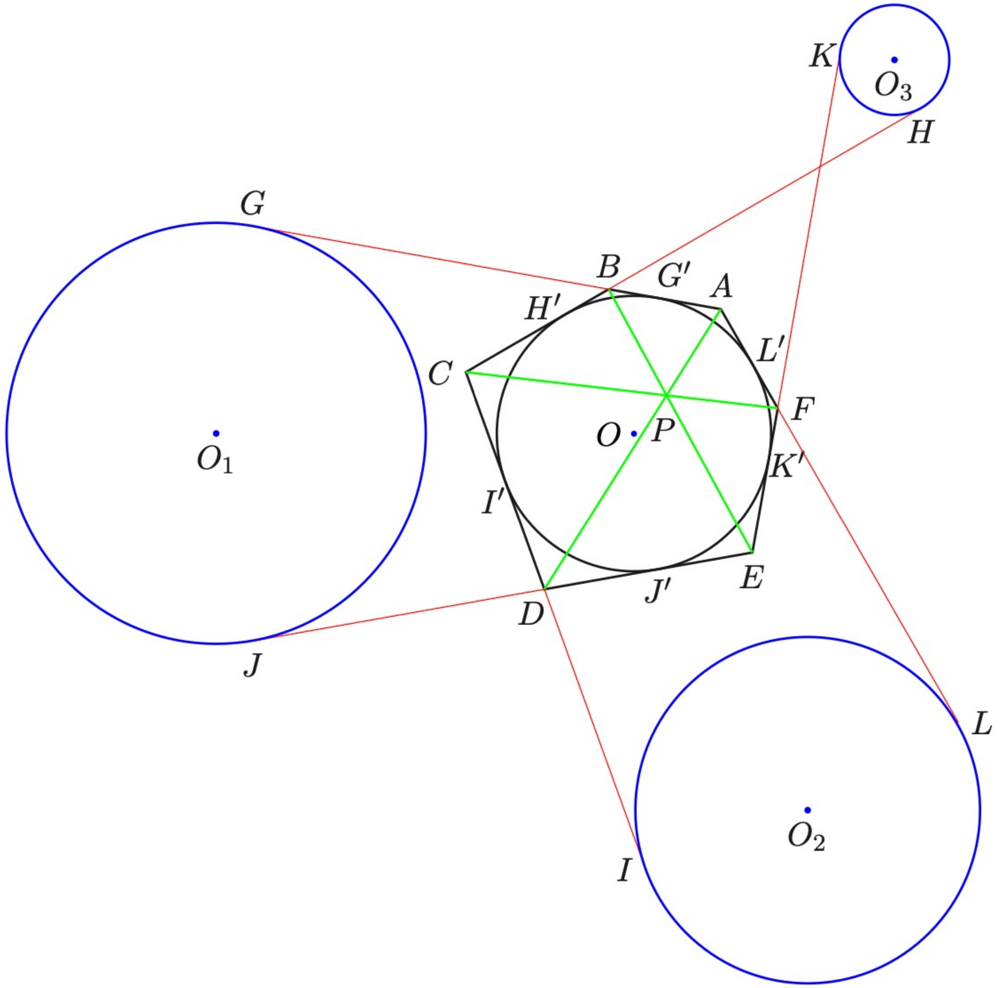


Monge's Theorem

Theorem 1. (Monge's Theorem)

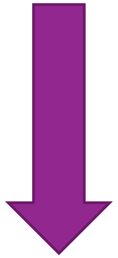
Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be non-overlapping circles with different radii. For each pair of circles, draw their common external tangents. Then, the points of intersection of those tangent lines are collinear.





Euclidean Geometry proof:

- Line AD is the radical axis of O1, O2
- Line BE is the radical axis of O1, O3
- Line CF is the radical axis of O2, O3



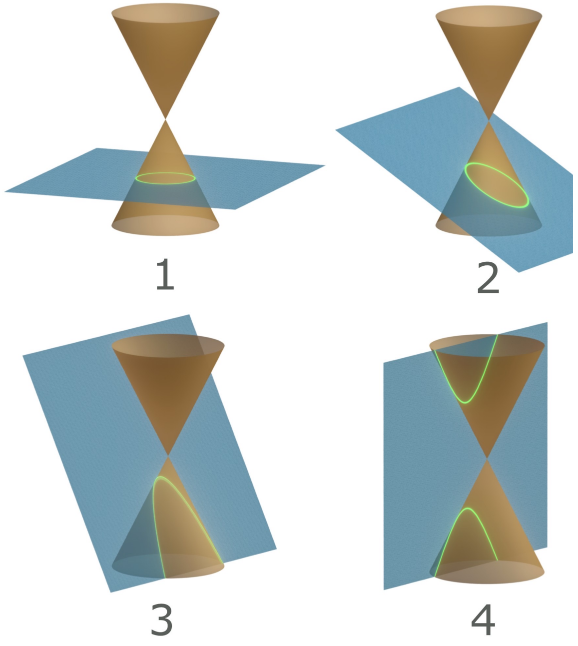
Monge’ s Theorem

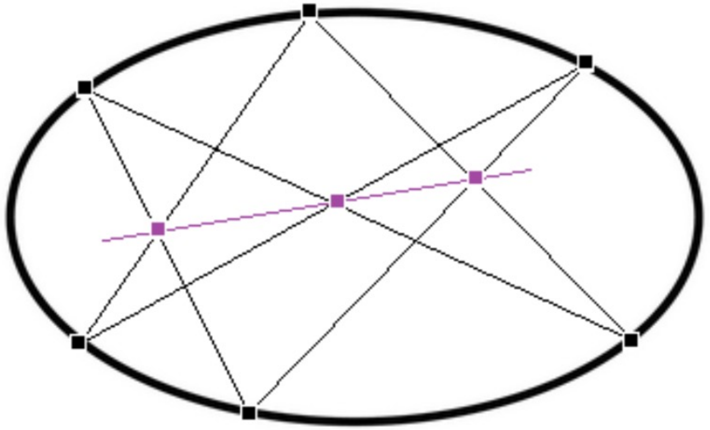
Lines AD, BE, and CF are concurrent

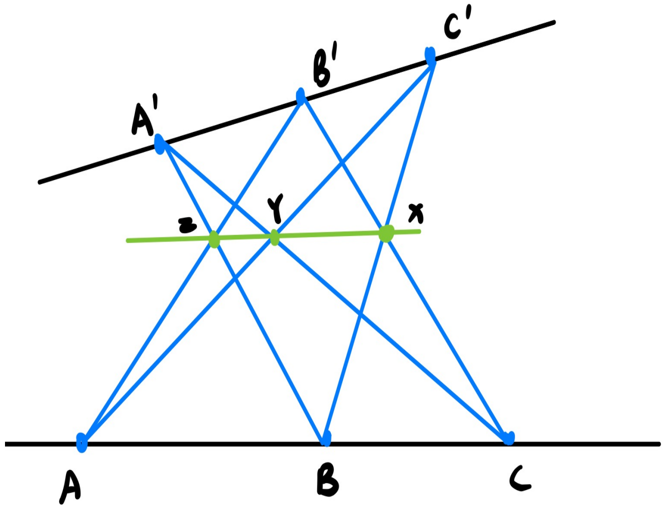
Q: Are those two theorems true in the conic section?

Hint : The conic section includes a circle, ellipse, parabola, and hyperbola.

Yes!!!

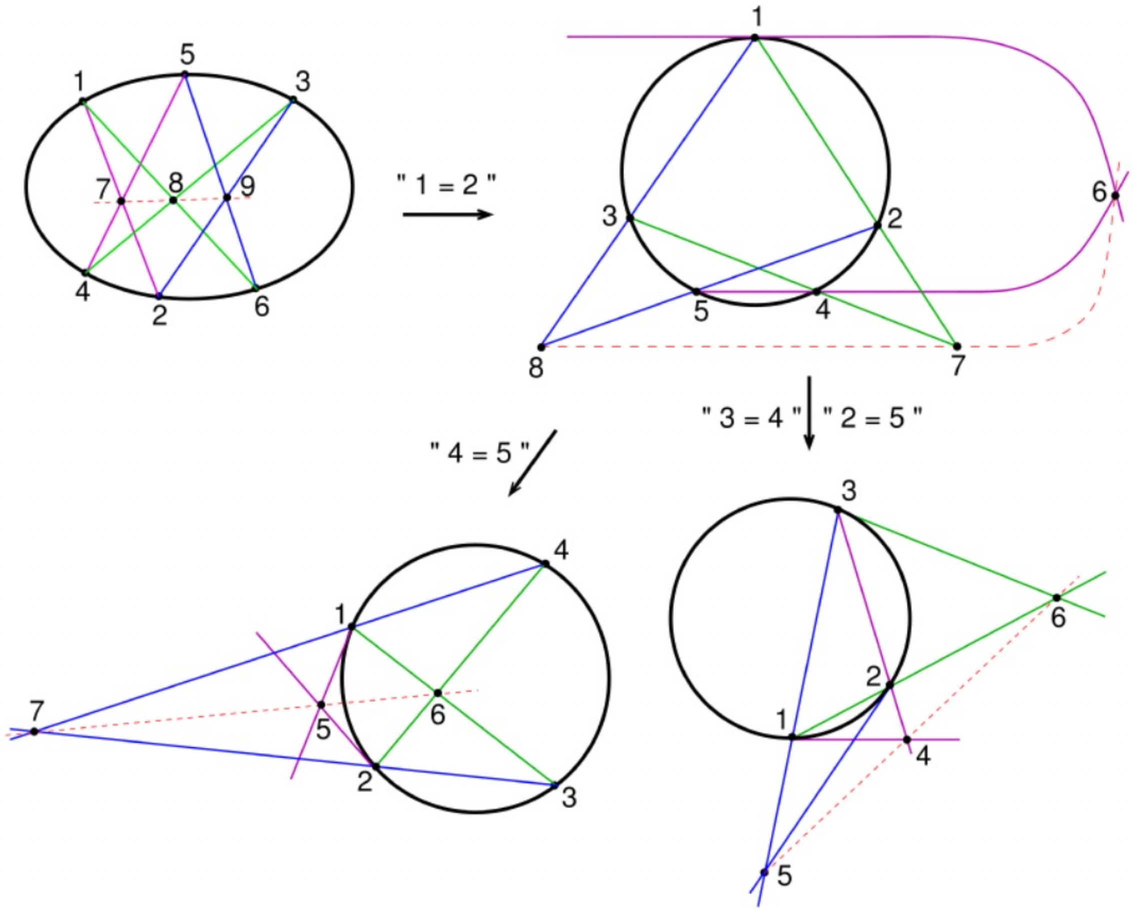






Q: Are exist 5-point, 4-point, and 3-point degenerate cases of Pascal's theorem and Brianchon’ s theorem?

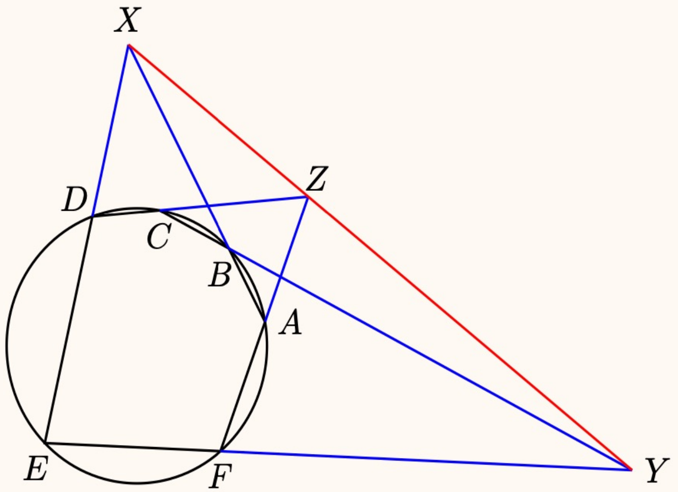
Yes!!!



Conclusion

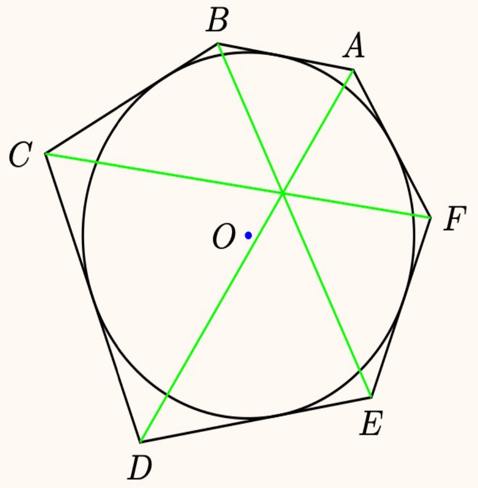
Theorem 2. (Pascal's Theorem)

The hexagon $ABCDEF$ is inscribed to a circle. Assume that AB, DE intersects at X; BC, EF intersects at Y; and CD, FA intersects at Z. Then X, Y, Z are collinear.



Theorem 4. (Brianchon's Theorem)

The Hexagon $ABCDEF$ is circumscribed on a circle. Then AD, BE, and CF are concurrent.



Two "Dual" Theorems

Thank You For Listening