MORELY'S TRIANGLE IS PERSPECTIVE TO THE TRIANGLE

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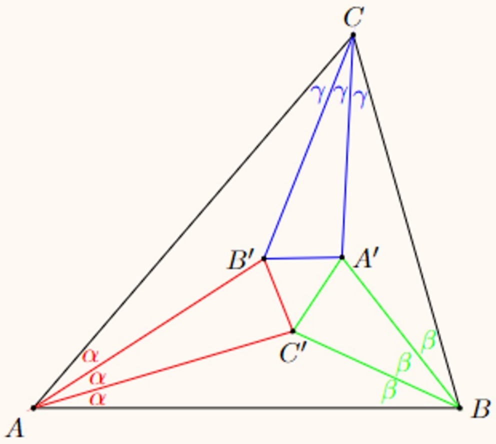
Introduction

What is Morley’s Theorem?

- In any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle

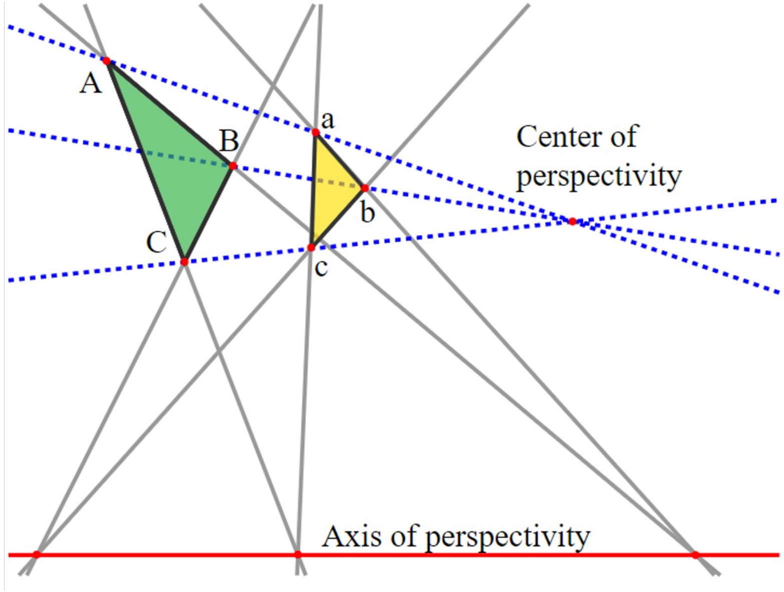
Theorem 1. (Morley’s Miracle)

In the following picture, the red, green, and blue lines are the angle trisectors of the corresponding angles. Then $\Delta A'B'C'$ is an equilateral triangle.



What Does Perspective Means?

Two Triangles are perspective from a line if the extensions of their three pairs of corresponding sides meet in Collinear points. The line joining these points is called the Perspective Axis.



Proof

From the proof of Morley's theorem. We know that the first Morley triangle, has vertices given in trilinear coordinates relative to a triangle ABC as follows:

A'-vertex = 1 : 2 cos(C/3) : 2 cos(B/3)

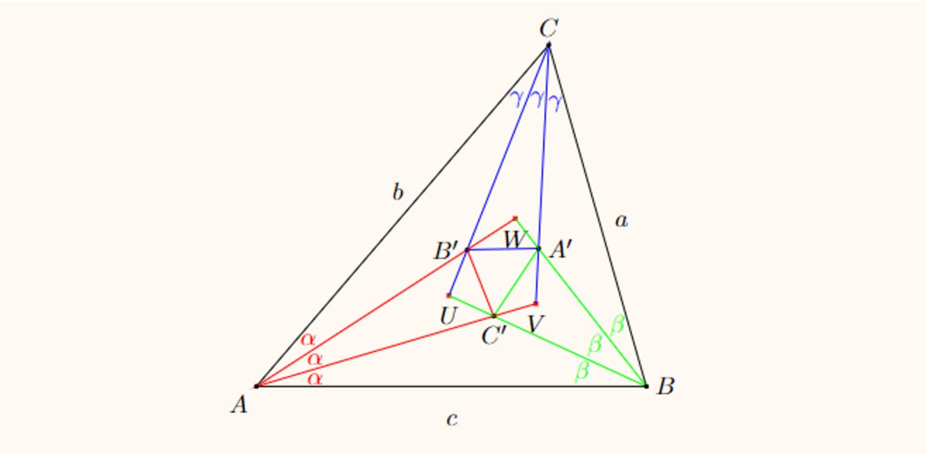
B'-vertex = 2 cos(C/3) : 1 : 2 cos(A/3)

C'-vertex = 2 cos(B/3) : 2 cos(A/3) : 1

A = 1 : 0 : 0

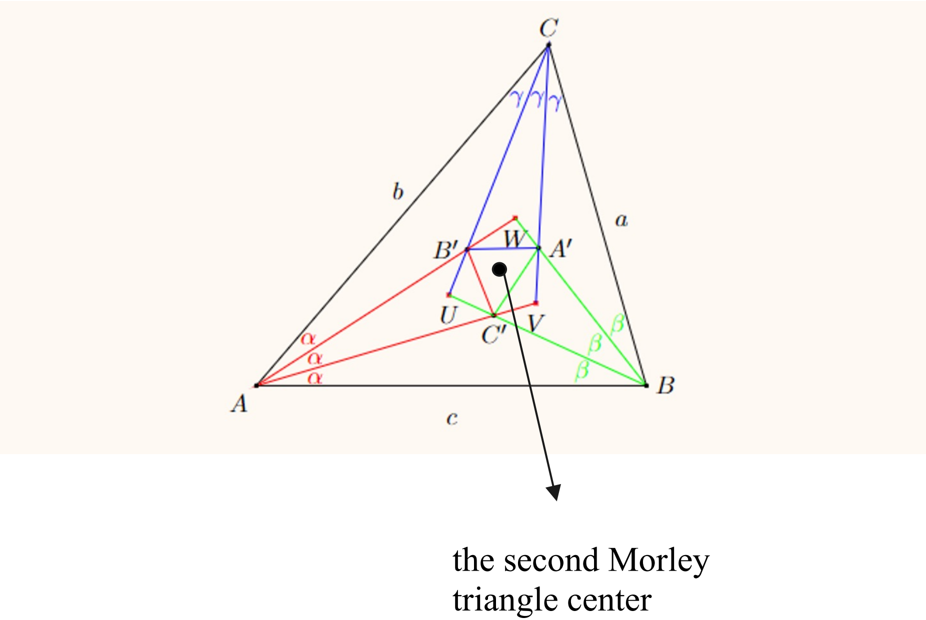
B = 0 : 1 : 0

C = 0 : 0 : 1



Proof (Cont.)

By the theorem of P. Yff, 1967, we know that the Morley equilateral triangle $\Delta A'B'C'$ is perspective to the original triangle ΔABC and the center of the perspective is called the second Morley triangle center (Denote as O). It has trilinear coordinates $\sec(\alpha) : \sec(\beta) : \sec(\gamma)$

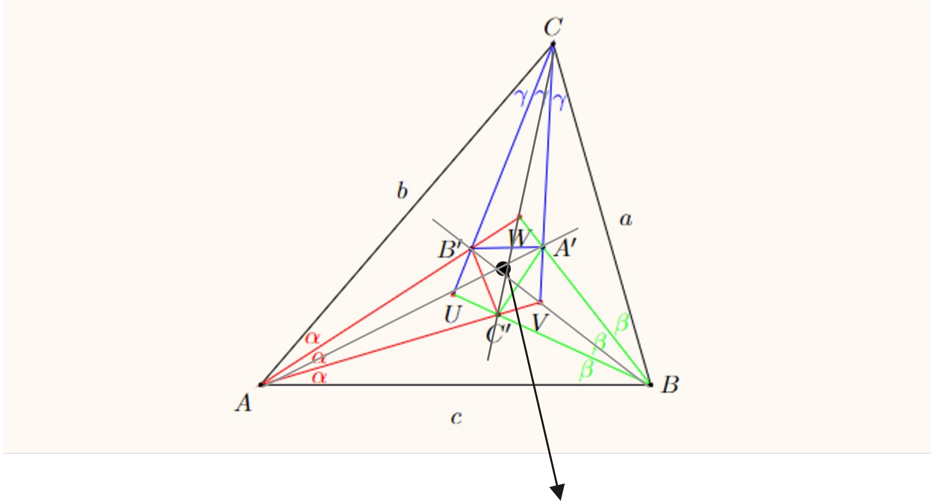


Proof (Cont.)

We can prove this by using the trilinear coordinates. By trilinear coordinates, three points:

P = (p : q : r), U = (u : v : w), X = (x : y : z) are collinear if and only if the determinant

$$\begin{vmatrix} p & q & r \\ u & v & w \\ x & y & z \end{vmatrix} = 0$$



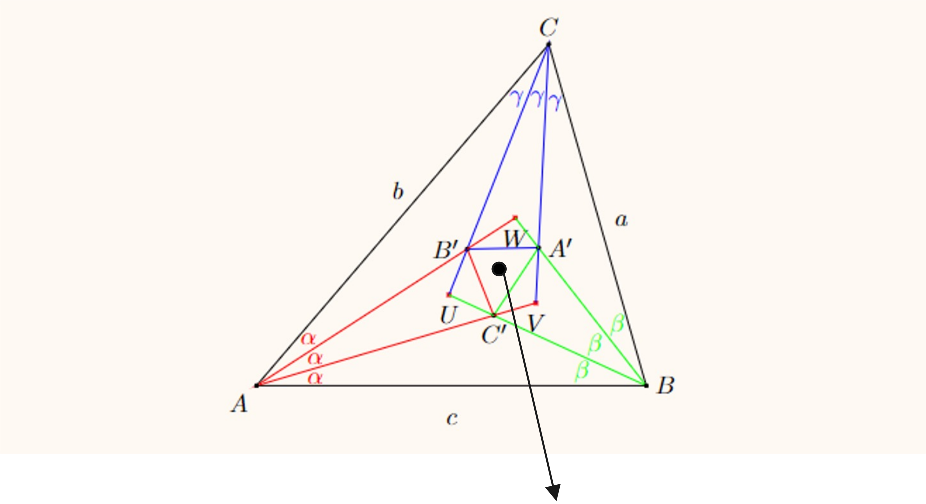
the second Morley triangle center

Proof (Cont.)

$$\begin{vmatrix} 1 & 2\cos(\gamma) & 2\cos(\beta) \\ \sec(\alpha) & \sec(\beta) & \sec(\gamma) \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 1(0) - 2\cos(\gamma)(-\sec(\gamma)) + 2\cos(\beta)(-\sec(\alpha))$$

$$= 2 - 2 = 0$$



the second Morley triangle center

This shows that point A, A’, and O are collinear. Using the same method, we can show that B, B’, O; C, C’, O are also collinear.

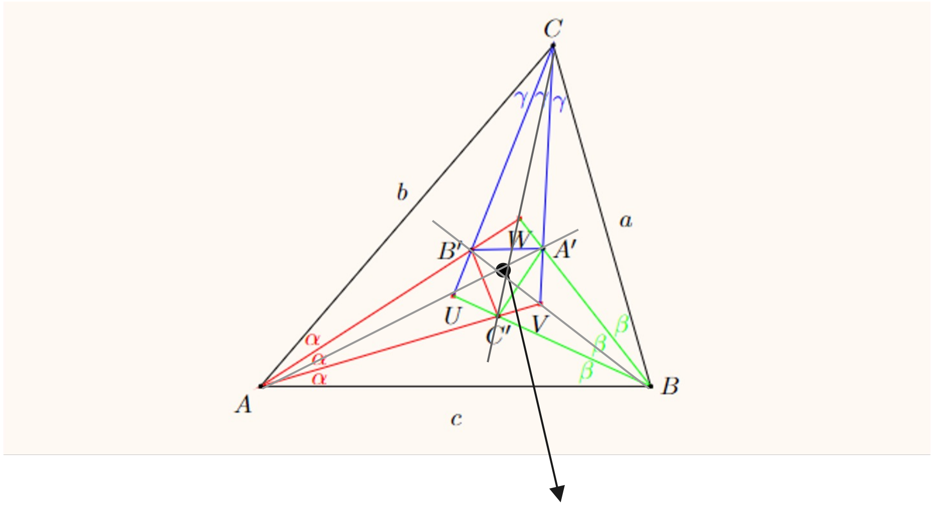
Proof (Use of Desargue’s Theorem)

Let A1B1C1 and A2B2C2 be two triangles. Consider two conditions:

1. Lines A1A2, B2B2, C1C2 joining the corresponding vertices are concurrent.
2. Points ab, bc, ca of intersection of the (extended) sides A1B1 and A2B2, B1C1 and B2C2, C1A1 and C2A2, respectively, are collinear.

Desargues' Theorem claims that 1. implies 2. It's dual asserts that 1. follows from 2. In particular, the dual to Desragues' theorem coincides with its converse.

Hence, since AA’, BB’, CC’ all intersect tersect at a point O, in other words, they are concurrent. This implies that triangle ABC and triangle A’B’C’ are perspective.



CITATION

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