Monge's Problem

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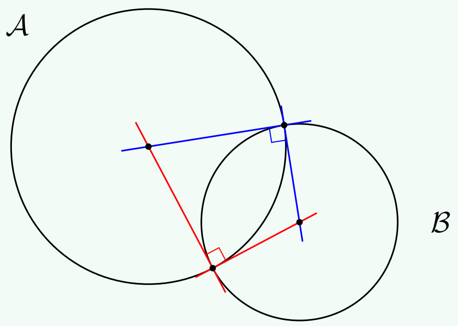
Gaspard Monge (1746-1818) is a French mathematician who invented descriptive geometry. He was introduced before when we discussed and proved the Monge's Theorem. In this topic, we present an interesting problem that the mathematician raised: find the common *orthogonal circle* (also called the *radical circle*) to three given circles. The solution to this problem is not to be confused with the Monge's Theorem on the common external tangents of three circles. Rather, we use the theorem that proves the existence of a *radical center* to three circles.

First, we will introduce some concepts that are crucial to solve Monge's Problem. Then, we will find the solution and show the steps to construct the radical circle.

Definition 1. (Orthogonal Circles)

Two circles are orthogonal if they intersect one another at a right angle.



Notice that the radii of two orthogonal circles form the base and height of a right triangle, and the line connecting the centers form the hypotenuse. By the Pythagorean Theorem we obtain the following property of orthogonal circles:

$$r_A^2 + r_B^2 = d^2,$$

where r_A, r_B represent the radii of circles A and B, respectively, and d represents the length between the centers of the circle.

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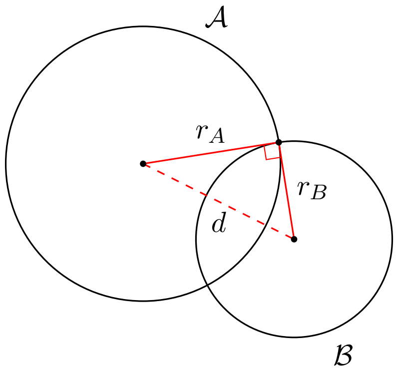


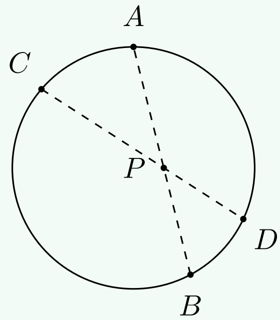
Definition 2. (Power of a Point)

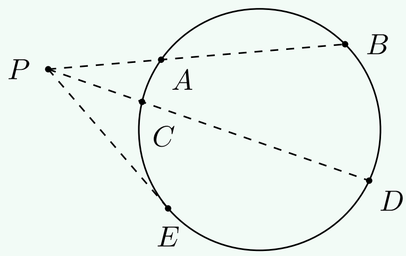
Given a circle centered at O with radius r and some point P on the same plane, the power of point P with respect to the circle is given by

$$PO^2 - r^2.$$

This constant represents the relative distance from point P to the circle. Thus if we draw a line from P that intersects two points on the circle, we get

$$|PO^2 - r^2| = PA \cdot PB = PC \cdot PD = PE^2.$$





When P is inside the circle, the power of P is negative. When P is outside the circle, the power of P is positive and is equal to the squared length of the tangent from P to the circle.

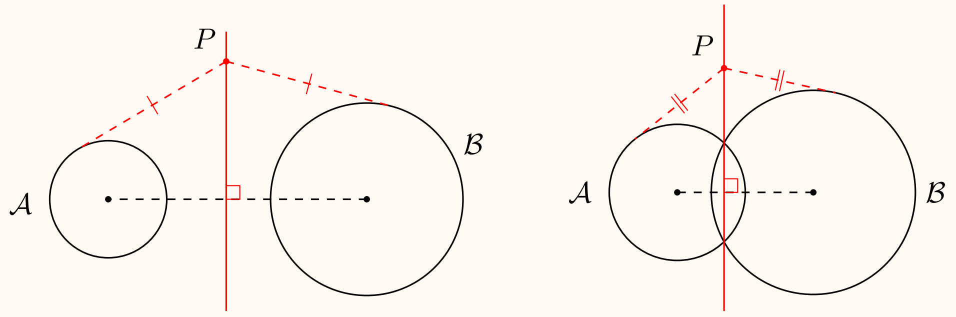
Definition 3. (Radical Axis)

Suppose circle $\mathcal{A}$ has center $O_{\mathcal{A}}$ and radius $r_{\mathcal{A}}$, and circle $\mathcal{B}$ has center $O_{\mathcal{B}}$ and radius $r_{\mathcal{B}}$. Then their radical axis is the set of points that have the same power with respect to the circles:

$$PO_{\mathcal{A}}^2 - r_{\mathcal{A}}^2 = PO_{\mathcal{B}}^2 - r_{\mathcal{B}}^2.$$

Theorem 1. (Radical Axis Theorem)

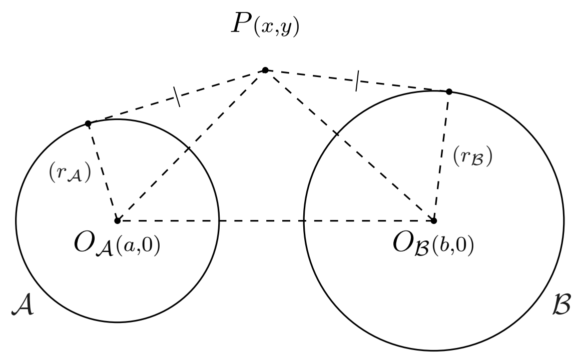
The radical axis of two circles with distinct centers forms a straight line perpendicular to the line joining their centers.



Proof: Using Cartesian coordinates, we draw the line connecting the centers of circles $\mathcal{A}, \mathcal{B}$ on the x-axis. So we can write

$$O_{\mathcal{A}} = (a, 0), \qquad O_{\mathcal{B}} = (b, 0),$$

for centers of $\mathcal{A}$ and $\mathcal{B}$, respectively.



Let $P(x, y)$ be any point on the radical axis, then P has the same power with respect to two circles:

$$\overline{PO_{\mathcal{A}}}^2 - r_{\mathcal{A}}^2 = \overline{PO_{\mathcal{B}}}^2 - r_{\mathcal{B}}^2.$$

Now using the distance formula, we can replace

$$\overline{PO_{\mathcal{A}}} = (x-a)^2 + y^2, \qquad \overline{PO_{\mathcal{B}}} = (x-b)^2 + y^2.$$

Thus, we can rewrite the power equation as

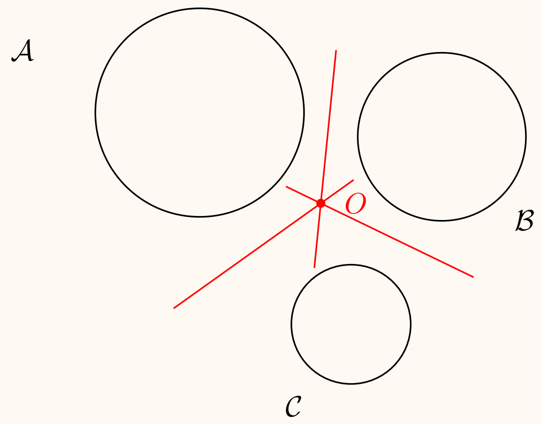
$$\begin{aligned}
(x-a)^2 + y^2 - r_{\mathcal{A}}^2 &= (x-b)^2 + y^2 - r_{\mathcal{B}}^2, \\
x^2 - 2ax + a^2 + y^2 - r_{\mathcal{A}}^2 &= x^2 - 2bx + b^2 + y^2 - r_{\mathcal{B}}^2, \\
-2ax + a^2 - r_{\mathcal{A}}^2 &= -2bx + b^2 - r_{\mathcal{B}}^2, \\
2bx - 2ax &= b^2 - r_{\mathcal{B}}^2 - a^2 + r_{\mathcal{A}}^2, \\
x(2b - 2a) &= b^2 - a^2 + r_{\mathcal{A}}^2 - r_{\mathcal{B}}^2, \\
x &= \frac{b^2 - a^2 + r_{\mathcal{A}}^2 - r_{\mathcal{B}}^2}{2b - 2a}, \\
x &= \frac{b+a}{2} - \frac{r_{\mathcal{B}}^2 - r_{\mathcal{A}}^2}{2(b-a)}.
\end{aligned}$$

This means that any point $P(x, y)$ on the radical axis must satisfy this solution, where $a, b, r_{\mathcal{A}}, r_{\mathcal{B}}$ are constants. Notice that the equation is linear, so every point on the radical axis lies on a straight line. Furthermore, since the points has a fixed x-coordinate, we can conclude that the radical axis is a vertical line perpendicular to the x-axis, which is the line connecting the centers of our circles. ■

Since Monge's Problem asks for three circles, we continue to explore the relationship between the radical axes of three circles.

Theorem 2. (Radical Axis Concurrence Theorem)

Given three circles with distinct and noncollinear centers, the radical axes for each circle pair intersect at one point, known as the power center or the radical center.



Proof: Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are circles with distinct and noncollinear centers $O_{\mathcal{A}}, O_{\mathcal{B}}, O_{\mathcal{C}}$, let P be the point of intersection between the radical axis of $\mathcal{A}, \mathcal{B}$ and the radical axis of $\mathcal{B}, \mathcal{C}$. Then P has the same power with respect to the circles $\mathcal{A}, \mathcal{B}$ and the circles $\mathcal{B}, \mathcal{C}$:

$$\overline{PO_{\mathcal{A}}}^2 - r_{\mathcal{A}}^2 = \overline{PO_{\mathcal{B}}}^2 - r_{\mathcal{B}}^2 \qquad \text{and} \qquad \overline{PO_{\mathcal{B}}}^2 - r_{\mathcal{B}}^2 = \overline{PO_{\mathcal{C}}}^2 - r_{\mathcal{C}}^2,$$

for $r_{\mathcal{A}}, r_{\mathcal{B}}, r_{\mathcal{C}}$ be radii of circles $\mathcal{A}, \mathcal{B}, \mathcal{C}$, respectively. Thus:

$$\overline{PO_{\mathcal{A}}}^2 - r_{\mathcal{A}}^2 = \overline{PO_{\mathcal{C}}}^2 - r_{\mathcal{C}}^2$$

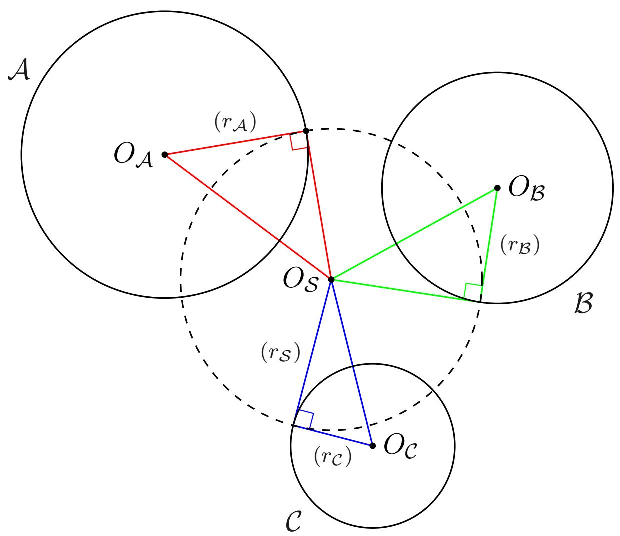
So P has the same power with respect to circles $\mathcal{A}, \mathcal{C}$ and lies on their radical axis. Hence P lies on all three radical axes of the pairs $(\mathcal{A}, \mathcal{B})$, $(\mathcal{B}, \mathcal{C})$, and $(\mathcal{A}, \mathcal{C})$. Then the radical axes must be concurrent at P. ■

Now that we understand the concepts of powers, radical axes, and radical centers, Monge's Problem becomes simple to solve.

Problem. (Monge's Problem)

Given three circles of any radii, find their common orthogonal circle.

Solution: Recall that if we want to draw one circle perpendicular to another, we need the radii of the two circles to be the legs of a right triangle. Suppose we are given three circles $\mathcal{A}, \mathcal{B}, \mathcal{C}$ with radii $r_{\mathcal{A}}, r_{\mathcal{B}}, r_{\mathcal{C}}$ and centers $O_{\mathcal{A}}, O_{\mathcal{B}}, O_{\mathcal{C}}$, respectively. Then, to draw a circle $\mathcal{S}$ perpendicular to all of them, we need three right triangles with one leg being the radius $r_{\mathcal{S}}$ of $\mathcal{S}$ and the center $O_{\mathcal{S}}$ being one vertex connecting the leg and the hypotenuse of the triangles:



$$r_{\mathcal{S}}^2 + r_{\mathcal{A}}^2 = \overline{O_{\mathcal{S}}O_{\mathcal{A}}}^2 \quad \text{and} \quad r_{\mathcal{S}}^2 + r_{\mathcal{B}}^2 = \overline{O_{\mathcal{S}}O_{\mathcal{B}}}^2 \quad \text{and} \quad r_{\mathcal{S}}^2 + r_{\mathcal{C}}^2 = \overline{O_{\mathcal{S}}O_{\mathcal{C}}}^2.$$

Rearranging these equations, we get:

$$r_{\mathcal{S}}^2 = \overline{O_{\mathcal{S}}O_{\mathcal{A}}}^2 - r_{\mathcal{A}}^2 = \overline{O_{\mathcal{S}}O_{\mathcal{B}}}^2 - r_{\mathcal{B}}^2 = \overline{O_{\mathcal{S}}O_{\mathcal{C}}}^2 - r_{\mathcal{C}}^2.$$

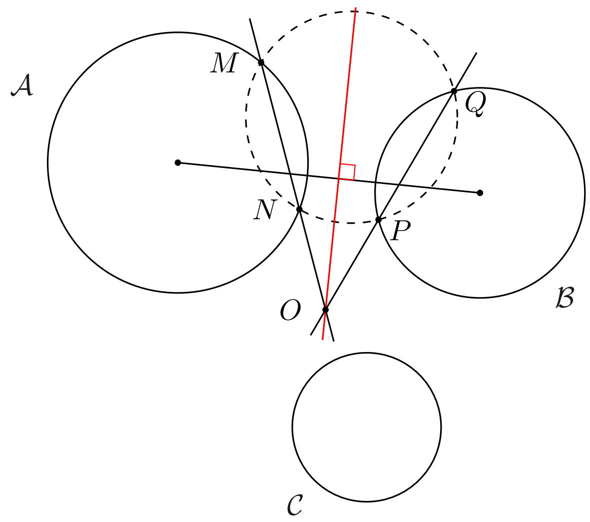
Using Definition 3, we know that the circles $\mathcal{A}, \mathcal{B}, \mathcal{C}$ have the same power at the point $O_{\mathcal{S}}$, which means $O_{\mathcal{S}}$ lies on the radical axes of the three pairs of circles $(\mathcal{A}, \mathcal{B})$, $(\mathcal{B}, \mathcal{C})$, and $(\mathcal{A}, \mathcal{C})$. As a result, the center of the perpendicular circle must be the power center of the three circles. Furthermore, the radius of $\mathcal{S}$ is simply the line connecting $O_{\mathcal{S}}$ to the tangent points on the circles $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

Now that we know how to find the perpendicular circle, let's learn how to draw it by constructing the radical axes and the power center of three circles $\mathcal{A}, \mathcal{B}, \mathcal{C}$:

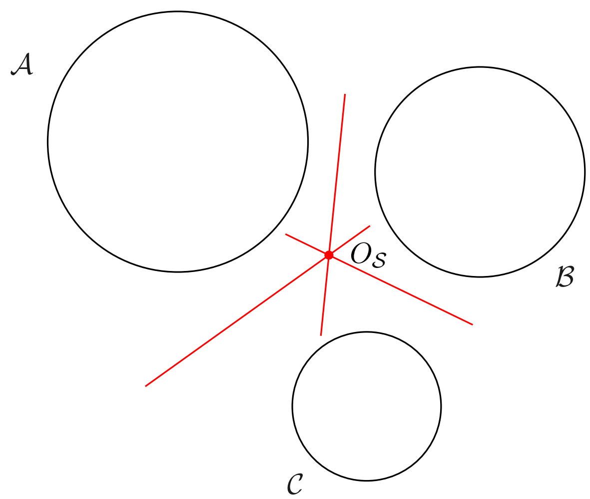
1. Draw a random circle that intersect circles $\mathcal{A}$ and $\mathcal{B}$ at four points, denoted as M, N, P, Q.
2. Draw two secant lines connecting the pair M, N and the pair P, Q. Extend these

lines until they intersect each other at O.

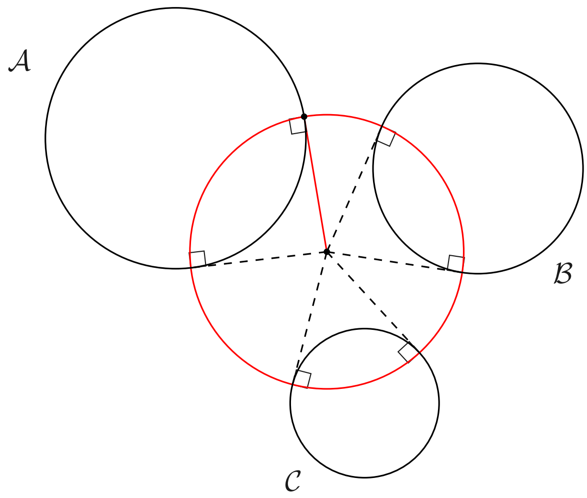
3. Draw a line connecting the centers of circles $\mathcal{A}, \mathcal{B}$.
4. Connect the intersection point O to the line connecting the centers at a perpendicular angle. This line forms the radical axis between the circles $\mathcal{A}, \mathcal{B}$.



5. Repeat steps 1 - 4 to draw the radical axes for the pairs of circles $(\mathcal{B}, \mathcal{C})$ and $(\mathcal{A}, \mathcal{C})$. The point of concurrency is the center of the perpendicular circle, denoted by $O_{\mathcal{S}}$.



6. Draw a tangent line connecting $O_{\mathcal{S}}$ to one of the circles $\mathcal{A}, \mathcal{B}, \mathcal{C}$ for the radius of the perpendicular circle. Using the center $O_{\mathcal{S}}$ and the radius that we found, we can draw our desired circle that is perpendicular to the three given circles.



Hence, we solved Monge's Problem. Since the solution is a circle centered at a point on all three radical axes, it is known as the radical circle.

Keep in mind that the way the three circles are positioned allows for the existence of a radical circle. This is not always true. For instance, if the radical center lies inside one of the three circles $\mathcal{A}, \mathcal{B}, \mathcal{C}$, then we cannot draw a tangent line from the radical center to that circle. In this case, the Monge's Problem is not soluble. ■