The Ptolemy's Theorem and Kelvin Transform

Zhiqin Lu, zlu@uci.edu, Felix Yu, felixyu2627@gmail.com
Xianfu Liu, xianful@uci.edu
(last updated: June 18, 2022)

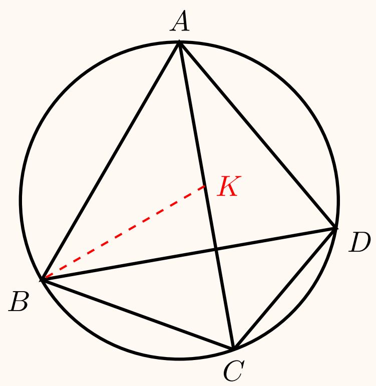
Ptolemy's Theorem is a relation between the four sides and two diagonals of a cyclic quadrilateral. It was discovered by *Claudius Ptolemy*, a Greek mathematician, astronomer, astrologer, geographer, and music theorist about 2000 years ago.

Here is the Wikipedia link of the Ptolemy's Theorem.

Theorem 1. (Ptolemy's Theorem)

In the following picture, let $ABCD$ be a cyclic quadrilateral. Then

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$



Proof. We define a point K on AC such that $\angle ABK = \angle DBC$. Since A, B, C, D are concyclic, we must have $\angle BAK = \angle BDC$. As a result, ΔABK is similar to ΔDBC. Thus we have

$$\frac{AB}{BD} = \frac{AK}{CD},$$

which is equivalent to

$$AB \cdot CD = BD \cdot AK. \tag{1}$$

Using the same method, we obtained that ΔKBC is similar to ΔABD and hence

$$\frac{BC}{BD} = \frac{KC}{AD},$$

or

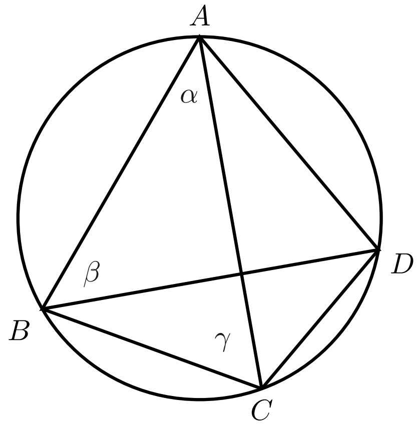
$$AD \cdot BC = BD \cdot KC. \tag{2}$$

Therefore by (1) and (2), we have

$$AB \cdot CD + AD \cdot BC = BD \cdot AK + BD \cdot KC = BD \cdot AC.$$

The Ptolemy's Theorem can also be proved using trigonometry.

Proof Using Trigonometry. We define $\angle BAC = \alpha$, $\angle DBA = \beta$, $\angle ACB = \gamma$. Therefore $\angle BDC = \alpha$, $\angle DCA = \beta$, $\angle ADB = \gamma$.



Using the law of sines we have

$$AB = 2R\sin(\gamma), \quad BC = 2R\sin(\alpha),$$

$$CD = 2R\sin(180 - \alpha - \beta - \gamma) = 2R\sin(\alpha + \beta + \gamma),$$

$$DA = 2R\sin(\beta).$$

Similarly, we have

$$AC = 2R\sin(\alpha + \gamma), \quad BD = \sin(\beta + \gamma).$$

Putting it all together, we conclude that the Ptolemy's Theorem is equivalent to the following identity.

$$2R\sin(\alpha+\gamma)\cdot 2R\sin(\beta+\gamma) = 2R\sin(\alpha)\cdot 2R\sin(\beta) + 2R\sin(\gamma)\cdot 2R\sin(\alpha+\beta+\gamma).$$

Cancelling $4R^2$, we obtain

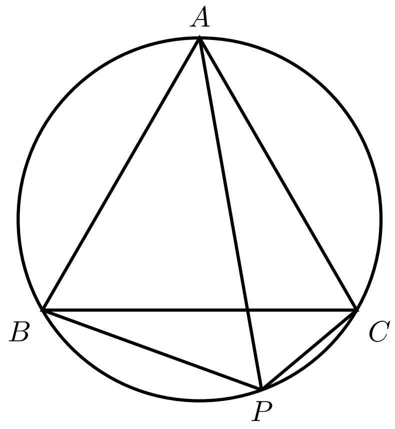
$$\sin(\alpha + \gamma)\cdot\sin(\beta + \gamma) = \sin(\alpha)\cdot\sin(\beta) + \sin(\gamma)\cdot\sin(\alpha + \beta + \gamma).$$

The above formula can easily be proved using the Product-to-sum formulas.

In the following, we provide some interesting corollaries of the Ptolemy's Theorem.

Example 1 In the following picture, ΔABC is an equilateral triangle. Let P be a point on the arc BC. Prove that

$$PA = PB + PC.$$



Proof. Using the Ptolemy Theorem, we have

$$BC \cdot AP = AB \cdot CP + BP \cdot AC.$$

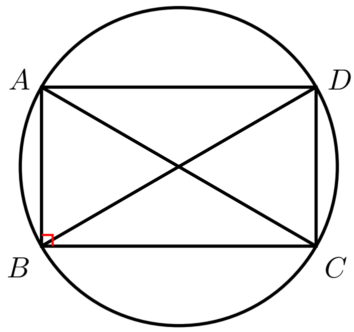
Since ΔABC is equilateral, we have $AB = BC = CA$. Thus we have

$$PA = PB + PC.$$

■

Example 2 (Pythagorean Theorem) Let ΔABC be a right triangle. $\angle ABC = 90°$. Then

$$AB^2 + BC^2 = AC^2.$$



Proof. We are able to construct a rectangle $ABCD$ which is inscribed in the circle. By the Ptolemy Theorem, we have

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

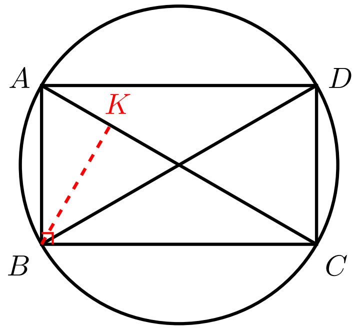
Since $BD = AC$, $AB = CD$ and $AD = BC$, we get the conclusion. ■

Example 3 (Einstein's Proof of the Pythagorean Theorem) In the following picture, draw $BK \perp AC$. Then we have

$$AB^2 = AK \cdot AC, \qquad BC^2 = CK \cdot AC.$$

Therefore

$$AB^2 + BC^2 = AK \cdot AC + CK \cdot AC = AC^2.$$

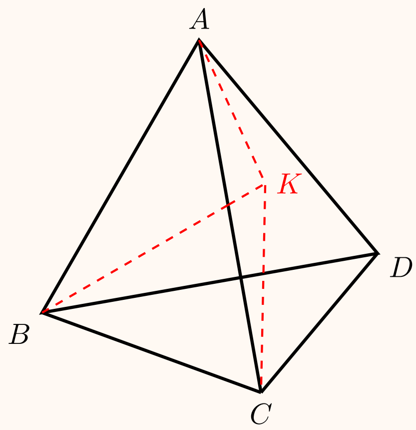


Theorem 2. (Ptolemy Inequality)

Let $ABCD$ be a quadrilateral (not necessarily concyclic). Then

$$AC \cdot BD \leq AB \cdot CD + AD \cdot BC.$$

The equality is valid if and only if A, B, C, D are concyclic.



Proof. We use the similar method as before. But this time K does not have to be on the line AC.

Define the point K by letting $\angle ABK = \angle DBC$, and

$$\frac{AB}{DB} = \frac{BK}{BC}.$$

Then ΔABK is similar to ΔDBC. If we rewrite this equation as

$$\frac{AB}{BK} = \frac{DB}{BC}.$$

Then from $\angle ABD = \angle KBC$, we conclude that ΔABD is similar to ΔKBC. Thus we have

$$\frac{AK}{CD} = \frac{AB}{BD}, \quad \frac{AD}{KC} = \frac{DB}{BC}.$$

By taking the cross products, we have

$$AB \cdot CD = BD \cdot AK, \qquad AD \cdot BC = BD \cdot KC.$$

Thus

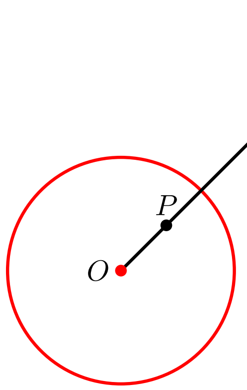
$$AB \cdot CD + AD \cdot BC = BD \cdot (AK + KC) \geq BD \cdot AC.$$

If the equality is valid, then we must have $AK + KC = AC$. Therefore K must be on the line AC. This implies $\angle BDC = \angle BAK = \angle BAC$ and hence A, B, C, D are concyclic.

In the above proof of the Ptolemy Inequality, we used the triangle inequality only. Is it possible that the Ptolemy Inequality in our Universe is merely the triangle inequality in another Universe?

In the following, we define the *Kelvin transform*. Kelvin transform is a geometric transform. It contains a fixed point O,and a constant $r > 0$. Point P is mapped to Point P' such that

$$OP \cdot OP' = r^2.$$



Here the point O is called the *center* of the transform, and r is called the *radius* of the transform.

Example 4 In the following ΔABC, assume that E, F are points on AC and AB respectively. Assume that

$$AE \cdot AC = 1, \quad AF \cdot AB = 1.$$

Then E, F are the Kelvin transforms of C, B, respectively, with center A and radius 1.

Since

$$AE \cdot AC = AF \cdot AB,$$

we have

$$\frac{AE}{AB} = \frac{AF}{AC}.$$

As a result, ΔAEF is similar to ΔABC. Line EF is called an *anti-parallel* line.

From the similarity of ΔAEF and ΔABC, we have

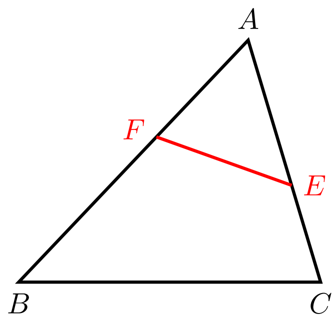
$$\frac{BC}{EF} = \frac{AB}{AE}.$$

Thus we have

$$BC = \frac{AB}{AE} \cdot EF.$$

Noting that $AB \cdot AF = 1$, we obtain

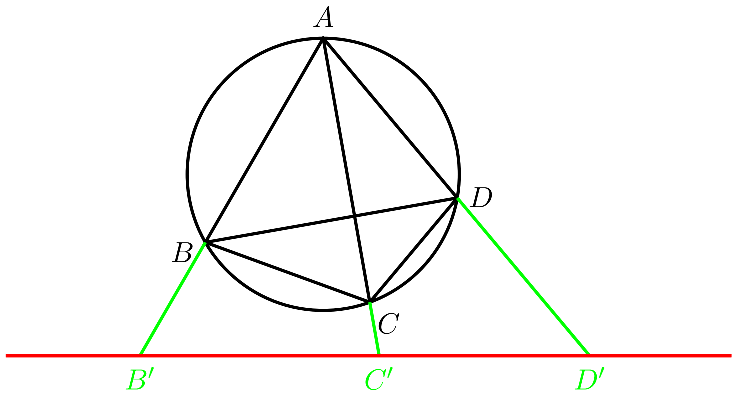
$$BC = \frac{EF}{AE \cdot AF}. \tag{3}$$



With the above preparation, we give the third proof of the Ptolemy's Theorem using the Kelvin transform.

Proof. In the following picture, we use A as the center and $r = 1$ as the radius of the Kelvin transform. Assume that B', C', D' are the Kelvin transform of B, C, D, respectively. Then

$$AB \cdot AB' = AC \cdot AC' = AD \cdot AD' = 1. \tag{4}$$



Since BC is an anti-parallel line of $B'C'$, and CD is an anti-parallel line of $C'D'$, we have

$$\angle AC'B' = \angle ABC = 180^\circ - \angle ADC = 180^\circ - \angle AC'B'.$$

Thus B', C', D' are collinear. Using (3), we have

$$B'C' = \frac{BC}{AB \cdot AC}, \quad C'D' = \frac{CD}{AC \cdot AD}, \quad B'D' = \frac{BD}{AB \cdot AD}.$$

Since $B'D' = B'C' + C'D'$, we get

$$\frac{BD}{AB \cdot AD} = \frac{BC}{AB \cdot AC} + \frac{CD}{AC \cdot AD}.$$

This proves the Ptolemy's Theorem. ■

The Ptolemy Theorem in our Universe, after the Kelvin Transform, becomes a trivial fact that $B'D' = B'C' + C'D'$.

Remark Using the same method as above, we can prove the Ptolemy Inequality as well. In Topic 5, both the Ptolemy's Theorem and the Ptolemy Inequality can be proved using the Simson Line.