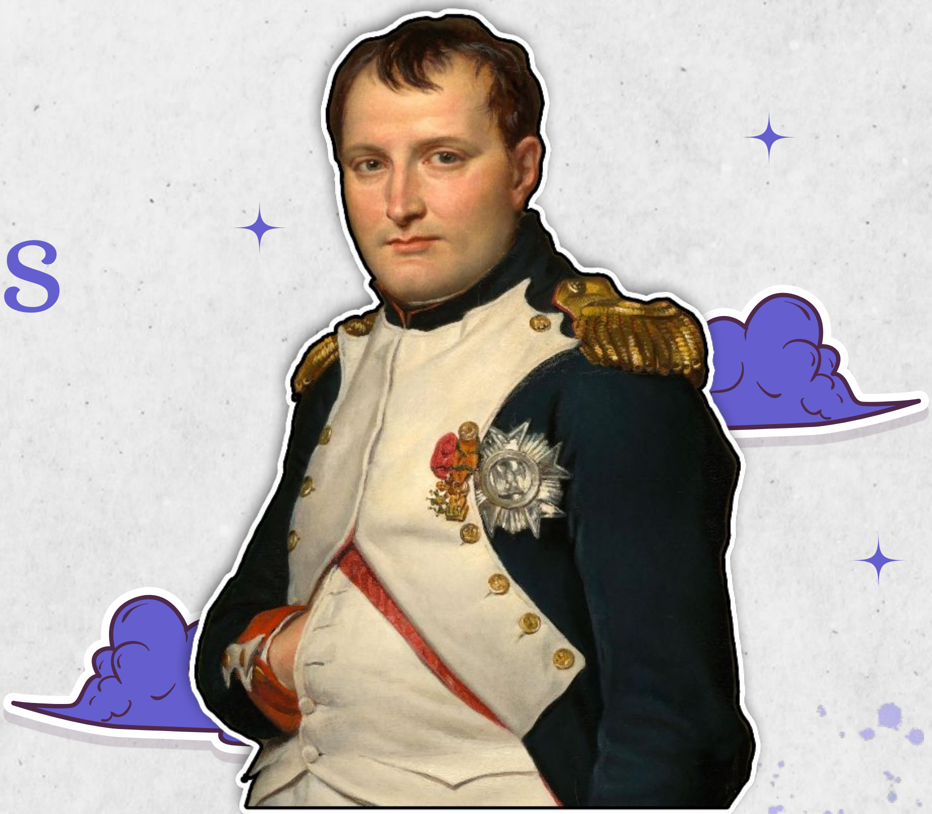
Topic 9:
Napoleon's Theorem



MATH 199B

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Starting Question

The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3}$, 5, *and* $\sqrt{37}$, *as shown, is* $m\sqrt{p}/n$, *where* m, n, *and* p *are positive integers,* m *and* n *are relatively prime, and* p *is not divisible by the square of any prime. Find* $m+n+p$.

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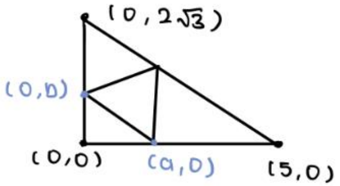
If x, y satisfy $px + qy = 1$, then minimal value of $\sqrt{x^2+y^2}$ is $\frac{1}{\sqrt{p^2+q^2}}$.

Distance between point (x_0, y_0) and line $px+qy+r=0$ is $\frac{|px_0+qy_0+r|}{\sqrt{p^2+q^2}}$

=> distance between origin & any point (x,y) on $px+qy=1$ is at least $\frac{1}{\sqrt{p^2+q^2}}$

Now let the right triangle vertices be $(0,0)$, $(5,0)$, $(0,2\sqrt{3})$



let $(a,0)$, $(0,b)$ be 2 vertices of equilateral Δ on the legs of right Δ

=> 3rd vertice of equilateral Δ = $\left(\frac{a+b\sqrt{3}}{2}, \frac{a\sqrt{3}+b}{2}\right)$

& lies on hypotenuse $\frac{x}{5}+\frac{y}{2\sqrt{3}}=1$

=> a, b must satisfy

$$\frac{a+b\sqrt{3}}{10}+\frac{2\sqrt{3}+b}{4\sqrt{3}}=1$$

$$\Rightarrow \frac{7}{20}a+\frac{11\sqrt{3}}{60}b=1$$

By lemma

min of $\sqrt{a^2+b^2}$ =

$$\frac{1}{\sqrt{\left(\frac{7}{20}\right)^2+\left(\frac{11\sqrt{3}}{60}\right)^2}}=\frac{10\sqrt{3}}{\sqrt{67}}$$

min area of equilateral Δ =

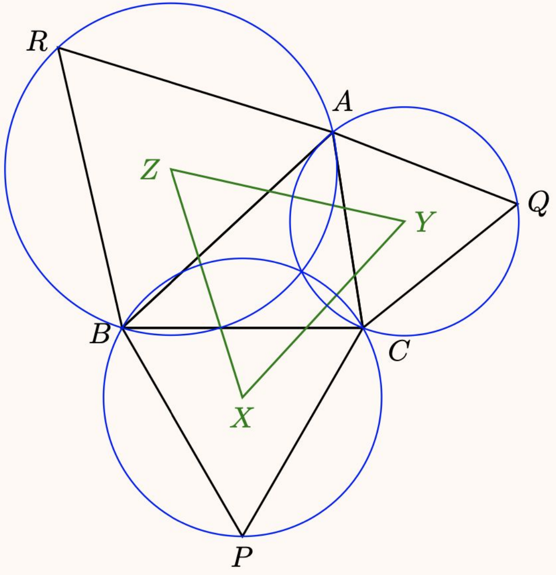
$$\frac{\sqrt{3}}{4}\cdot\left(\frac{10\sqrt{3}}{\sqrt{67}}\right)^2=\frac{75\sqrt{3}}{67}$$

$75+3+67=\boxed{145}$

How to determine the minimal inscribed equilateral triangle of a given triangle?

Theorem 1. (Napoleon's Theorem)

In the following, $\triangle BCP$, $\triangle CAQ$, and $\triangle ABR$ are equilateral triangle. Let X, Y, Z be the centers of $\triangle BCP$, $\triangle CAQ$, and $\triangle ABR$ respectively. Then $\triangle XYZ$ is equilateral.



Theorem 1. (Napoleon's Theorem)

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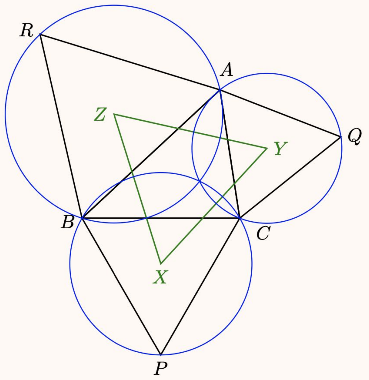


Proof:

Assume A, B, C correspond to complex numbers a, b, c.

Let $\sigma = e^{\pi i/3}$.

-Then complex number of point $P=(b - c)\sigma + c$

-Complex number of the center $X=1/3(b+c+(b-c)\sigma+c)$

$=1/3((1+\sigma)b+(2-\sigma)c)$.

-By this we can conclude:

$Y=1/3((1 + \sigma)c + (2 - \sigma)a)$

$Z=1/3 ((1 + \sigma)a + (2 - \sigma)b)$

We would need to prove : $|Z-X|=|Y-X|$

$Z-X =1/3((1+\sigma)a+(1-2\sigma)b-(2-\sigma)c)$

$Y-X = 1/3((2-\sigma)a-(1+\sigma)b-(1-2\sigma)c)$

Let a=0

$Z-X=\frac{1}{3}(0+(1-2\sigma)b-(2-\sigma)c)$

$=\frac{1}{3}((1-2\sigma)b-(2-\sigma)c)$

$Y-X=\frac{1}{3}(0-(1+\sigma)b-(1-2\sigma)c)$

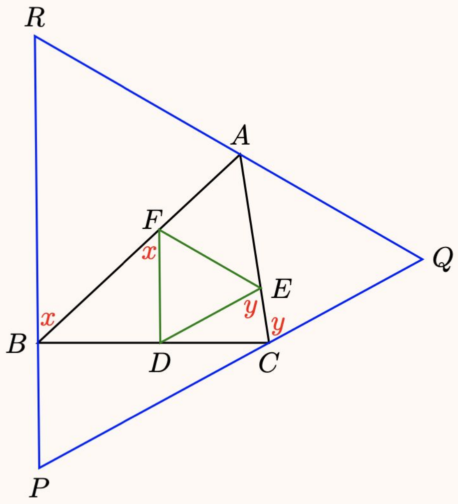
$=\frac{1}{3}((-1-\sigma)b+(-1+2\sigma)c)$

$|Z-X|=|Y-X|= \sqrt{3}/3(|b-\sigma c|)$

Theorem

Assume that $\triangle DEF$ is an inscribed equilateral triangle. Assume also that $DE \parallel PQ$, $EF \parallel QR$, and $FD \parallel RP$. Then

$$DE \cdot PQ = \frac{4}{\sqrt{3}} \operatorname{Area}(\triangle ABC).$$



Now assume △PQR is an equilateral triangle. Let ∠A = α, ∠B = β, and ∠C = γ. Let R be the radius of the circumscribed circle of △ABC.
Then Let DE=a and QR=b
Using the law of sines

b=RA+AQ=AB/sin 60(sinx) +AC/sin 60(sin y)
=4R/√3 (sinx sinγ + siny sinβ)

BC=BD+DC=a/sinβ(sinx) +a/sinγ(sin y)

$$a = \frac{BC}{\frac{\sin x}{\sin\beta} + \frac{\sin y}{\sin\gamma}} = \frac{2R\sin\alpha}{\frac{\sin x}{\sin\beta} + \frac{\sin y}{\sin\gamma}}.$$

Therefore, we have

$$ab = \frac{8R^2}{\sqrt{3}} \sin\alpha \sin\beta \sin\gamma.$$

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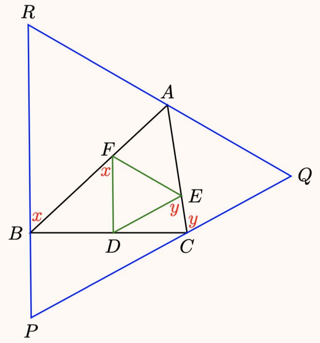
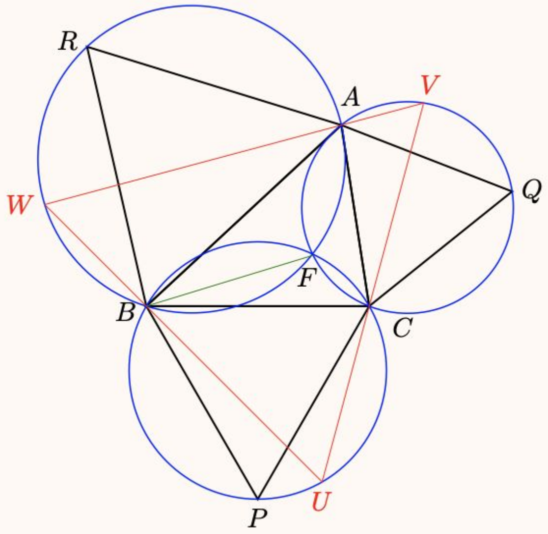


Theorem

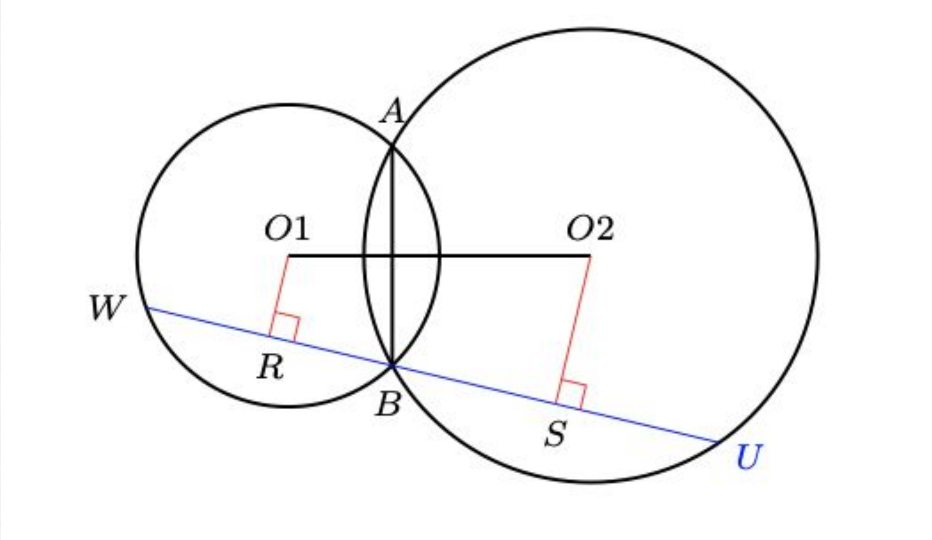
In the following picture, starting from a point U on the circumscribed circle of $\triangle BCP$. Connecting UB intersecting at W, and UC intersecting on V. Then U, A, W are collinear. Moreover, $\triangle UVW$ is equilateral.



$\angle U = \angle V = \angle W = 60$

$\angle VAW = 180$

Thus U, A, W are collinear.



$W\,U = 2\,RS \le 2O_1O_2$.

$W\,U$ will be maximized when $W\,U \perp AB$. When $W\,U \perp AB$, we get the maximum circumscribed equilateral triangle

Hence the minimal inscribed triangle can be located.