LEMOINE CIRCLES

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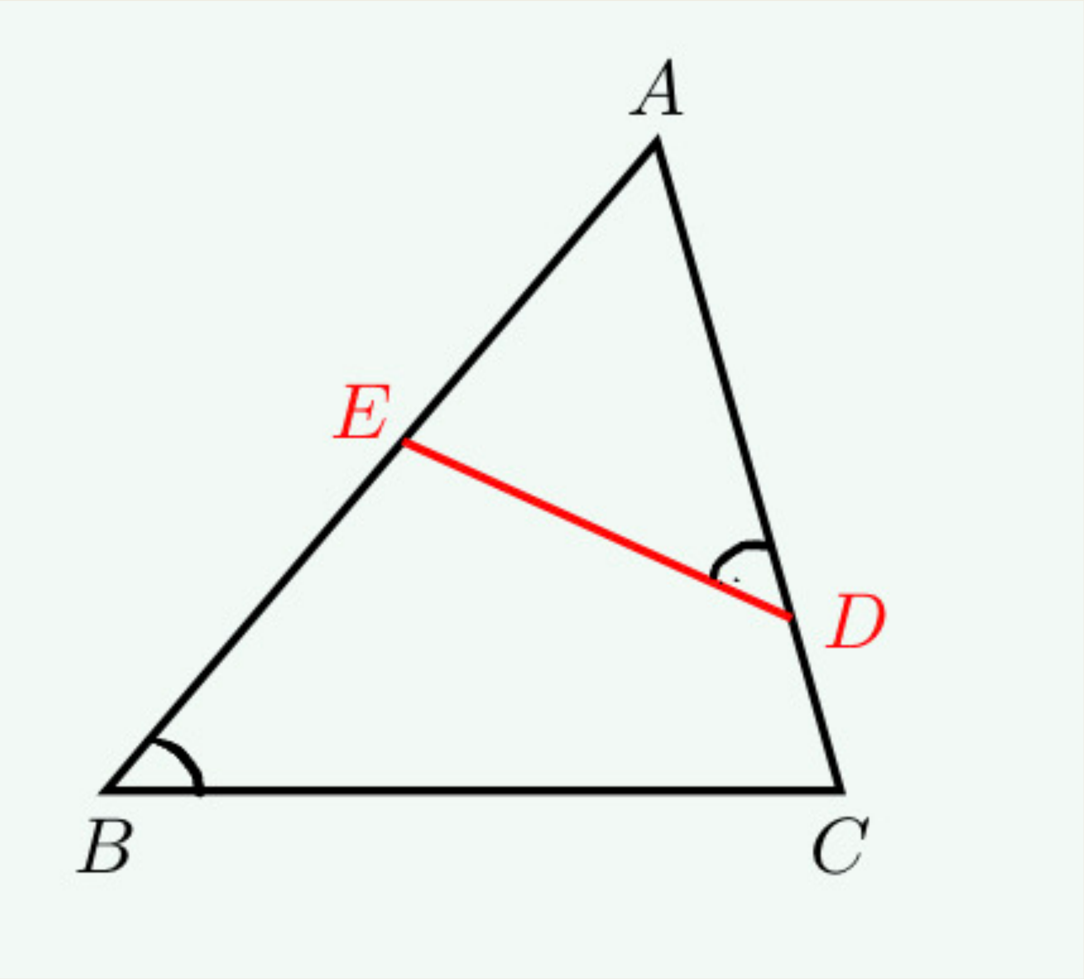
History

The Lemoine circles are named after Émile Lemoine, a French mathematician who discovered them in the 19th century. Émile Lemoine is also known for his work in geometry, particularly on the subject of triangle geometry, and is sometimes referred to as the "father of modern triangle geometry."

Two preliminary concepts

1. Antiparallel Line
2. Symmedian Point

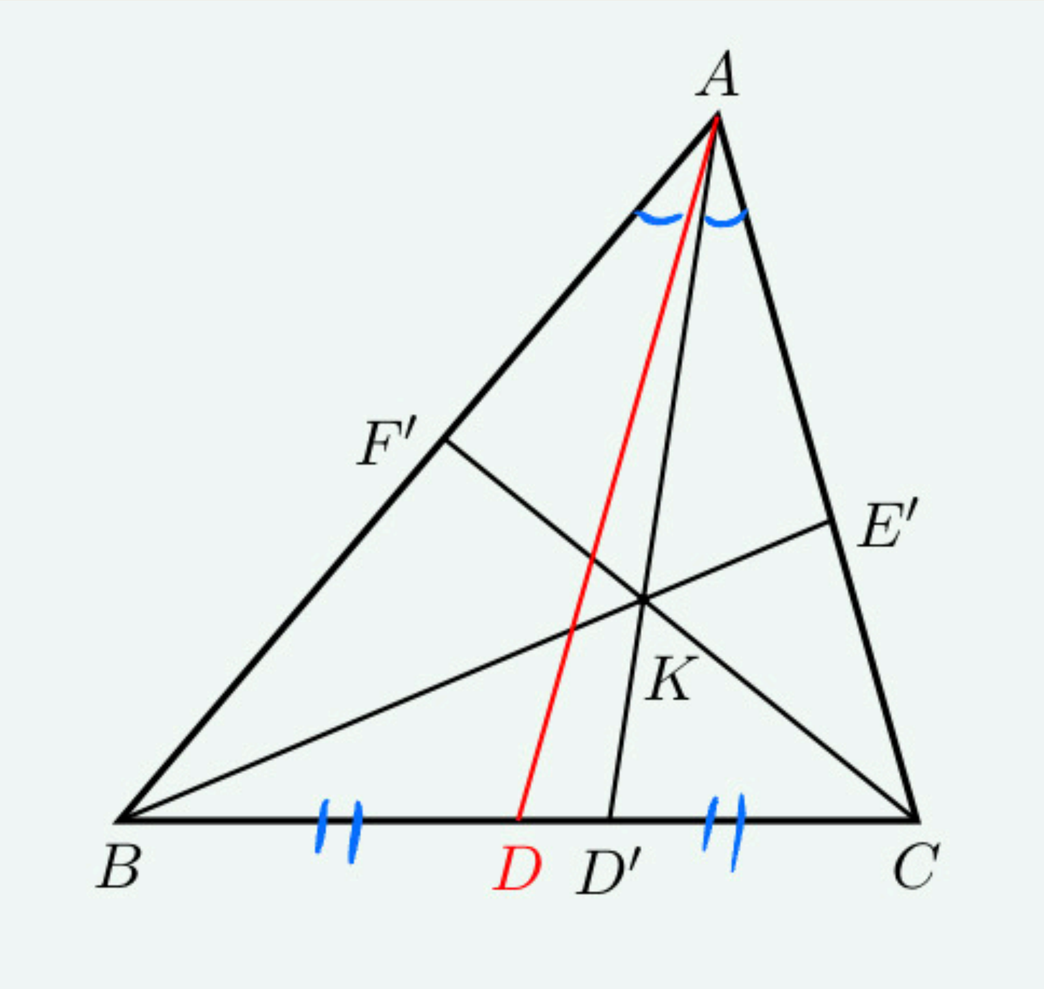
Antiparallel Line



Let D,E be points on AC, AB, if *∠ADE=∠B. The segment D E is called an antiparallel Line.*

When *DE* is antiparallel to *BC*, then *D, E, B, C* is con-cyclic. Also, two line segments which are antiparallel to a third line, then they must be parallel.

Symmedian Point



Symmedian line can also be characterized as follows. Let AD' be the symmedian line on BC. Then

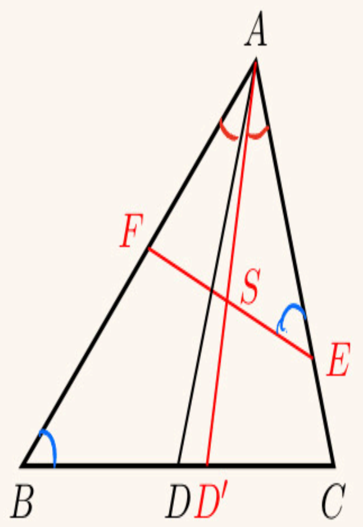
$$\frac{BD'}{D'C} = \frac{AB^2}{AC^2}.$$

Relation between antiparallel line and symmedian line

Theorem 1

In $\triangle ABC$, assume that EF is an antiparallel line and AD' is a symmedian line. EF and AD' intersect at S. Then S is the midpoint of DE, in other words, AS is the median of $\triangle AEF$ on EF.

Conversely, if AS is the median of $\triangle AEF$, then EF is antiparallel to BC.



$\triangle AEF \sim \triangle ABC$

$\because \angle BAD = \angle CAD'$

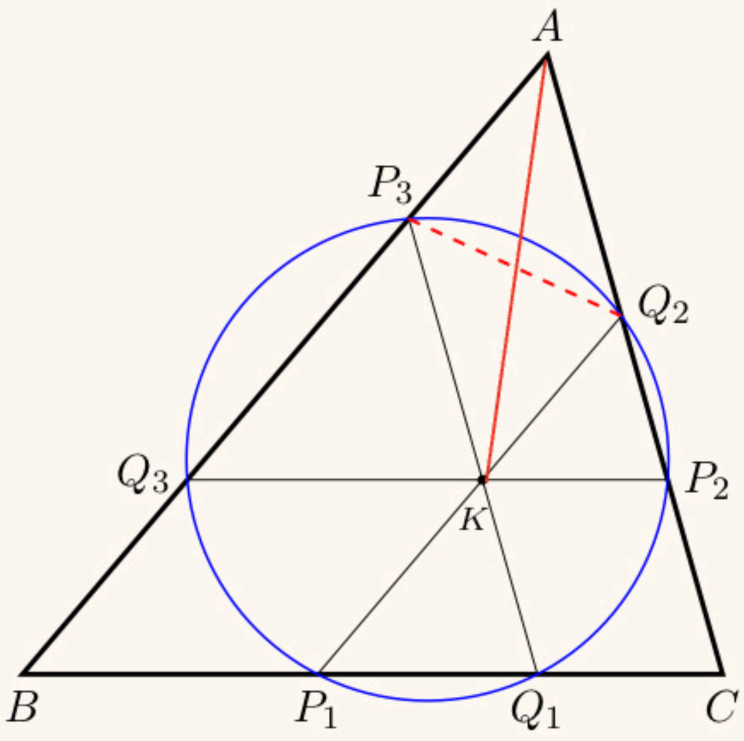
$\therefore \triangle AES \sim \triangle ABD$

$FS = SE$

$E'EF'F$ 为 $\square$

The First Lemoine Circle

Let K be the symmedian point of $\triangle ABC$. Let P_2Q_3, P_3Q_1, P_1Q_2 be parallel lines to BC, CA, AB, respectively. Then $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are con-cyclic.



The circle is known as the First Lemoine Circle.

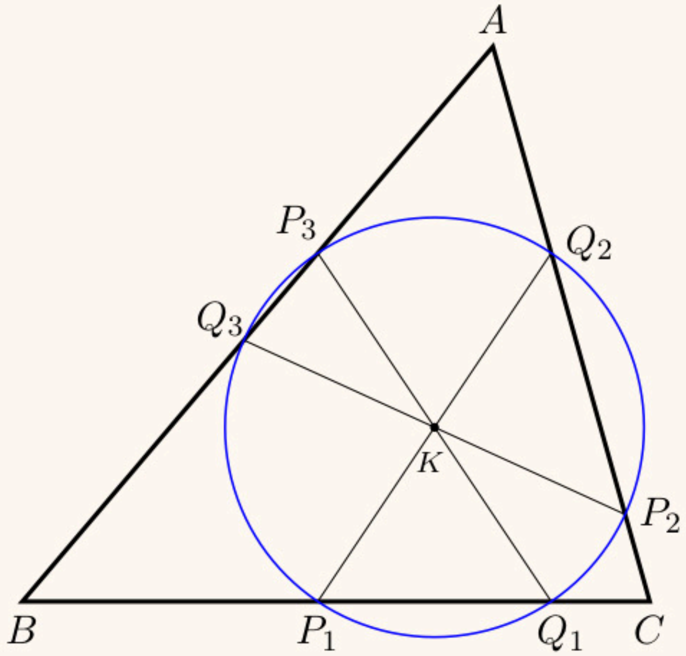
The second Lemoine Circle

Let K be the symmedian point of $\triangle ABC$. Let P_2Q_3, P_3Q_1, P_1Q_2 be antiparallel lines to BC, CA, AB, respectively. Then $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are con-cyclic.

The circle is known as the Second Lemoine Circle. Moreover, we have

$$P_1Q_1 : P_2Q_2 : P_3Q_3 = \cos A : \cos B : \cos C.$$

Therefore the Second Lemoine Circle is also called Cosine Circle.



AK symmedian line

$P_2Q_3 \Rightarrow$ antiparallel

T1 $\Rightarrow KP_2 = KQ_3$ $KP_3 = KQ_1$, $KP_1 = KQ_2$

$\angle P_3Q_3K = \angle C = \angle Q_3P_3K$

$KP_1 = KP_2 = KP_3 = KQ_1 = KQ_2 = KQ$

$P_1Q_1 = 2r\cos A$ $P_2Q_2 = 2r\cos B$

$P_3Q_3 = 2r\cos C$