Topic 13
Nine-Point Circle

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Background

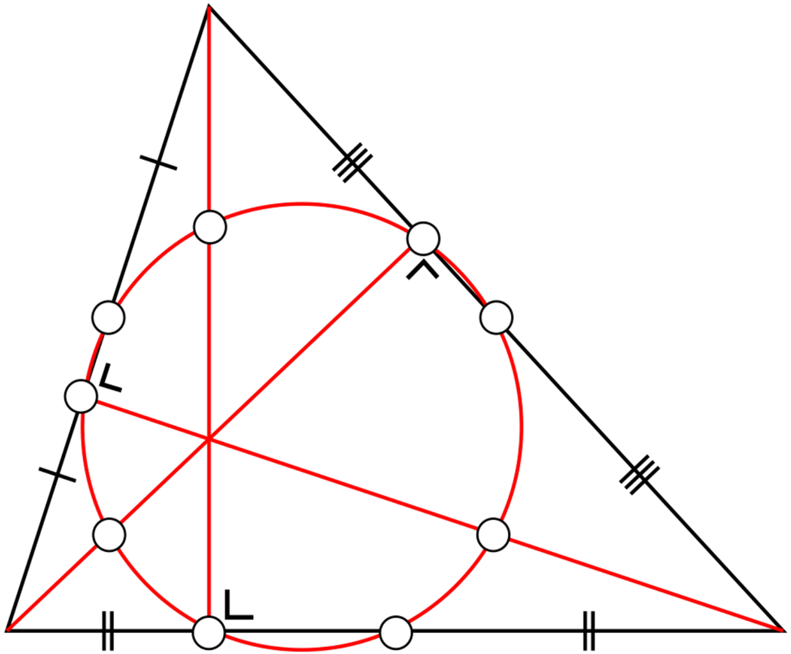
Definition: In geometry, the **nine-point circle** is a circle that can be constructed for any given triangle. It is so named because it passes through nine significant concyclic points defined from the triangle.

These Nine points are:

The midpoint of each side of the triangle

The foot of each altitude

The midpoint of the line segment from each vertex of the triangle to the orthocenter (where the three altitudes meet; these line segments lie on their respective altitudes)



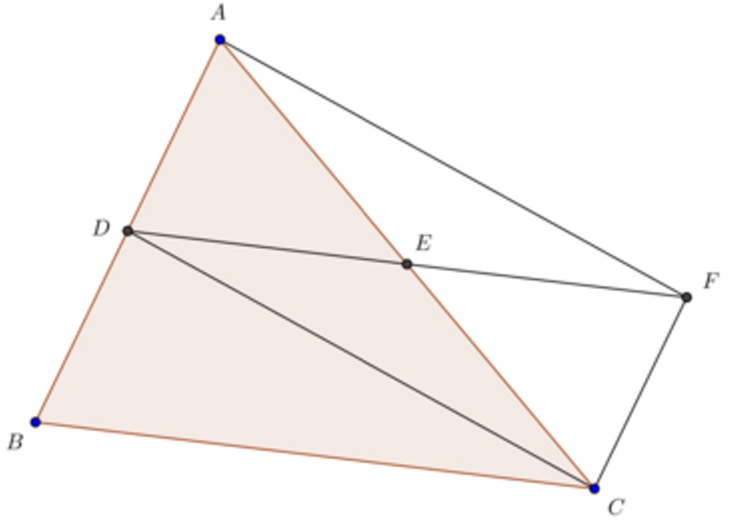
Background

- The nine-point circle is also known as **Feuerbach's circle** (after Karl Wilhelm Feuerbach), **Euler's circle** (after Leonhard Euler), **Terquem's circle** (after Olry Terquem),
- In 1765, **Leonhard Euler** proved that "the hypocentered and hypocentered triangles have a common external circle (six-point circle)." Many people mistakenly believe that the nine-point circle was discovered by Euler, so this circle is also known as Euler's circle.
- The first person to prove the nine-point circle was **Poncelet** (1821).
- In 1822, **Karl Wilhelm Feuerbach** also discovered the nine-point circle and concluded that "the nine-point circle is tangent to the inner and side tangent circles of the triangle", so the Germans called this circle the Feuerbach circle and the four tangent points the Feuerbach points

Basic Concept

Midline Theorem:

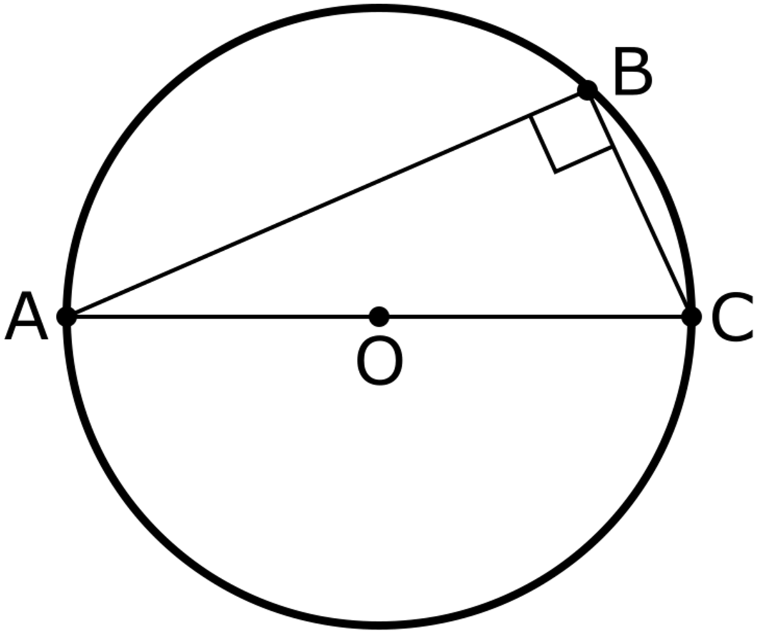
The midline of a triangle is parallel to the third side of that triangle and half its length.



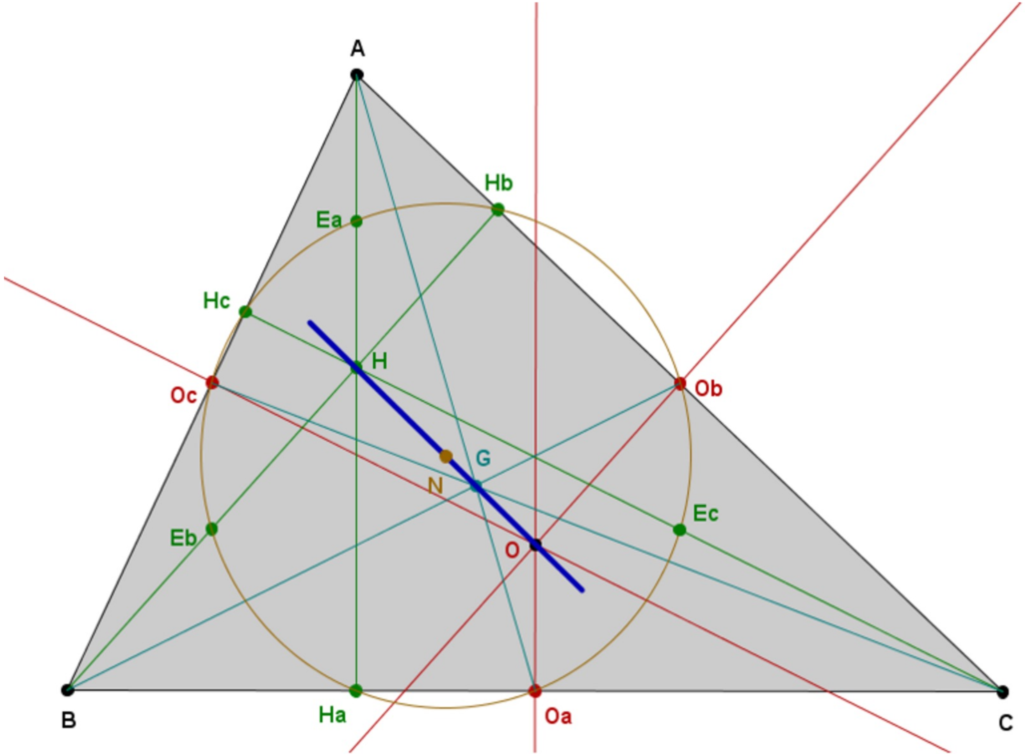
Basic Concept

Thales's theorem:

In geometry, **Thales's theorem** states that if A, B, and C are distinct points on a circle where the line AC is a diameter, the angle ABC is a right angle.

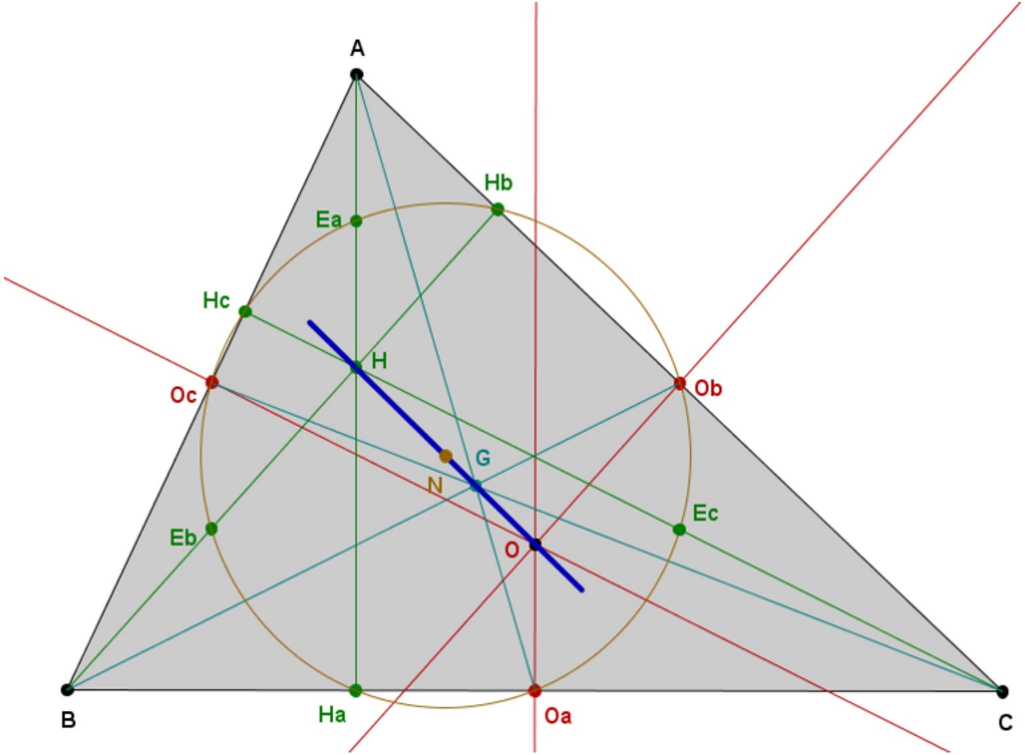


Proof 1



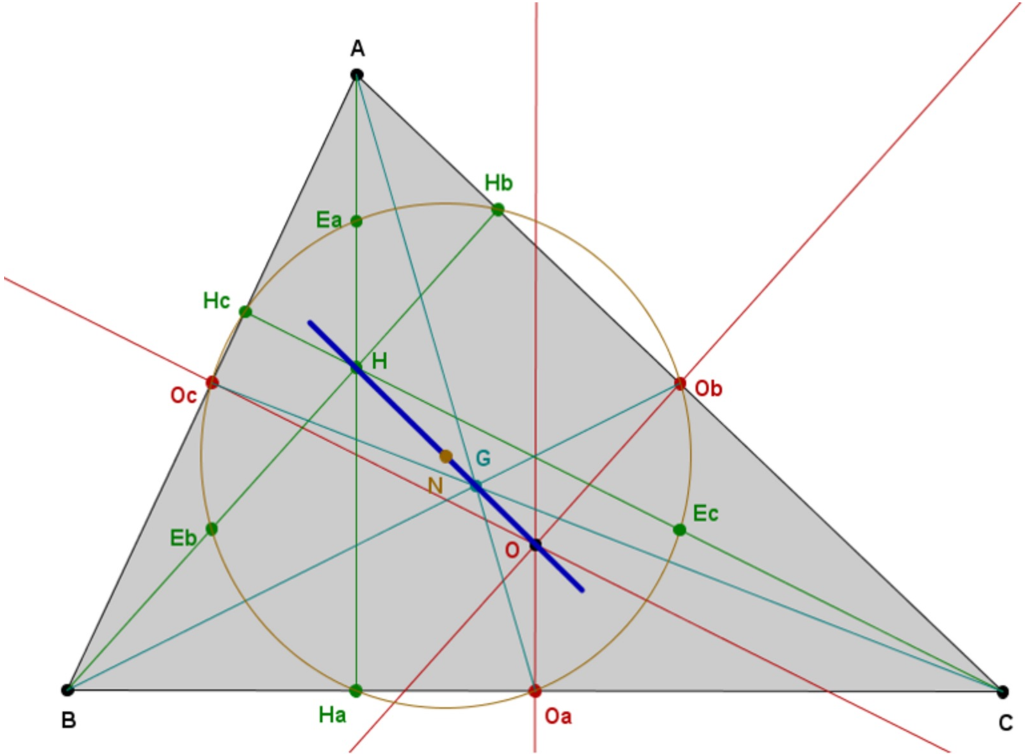
Since O_c is the midpoint of AB and E_b is the midpoint of BH, O_cE_b is parallel to AH. Using similar logic, we see that O_bE_c is also parallel to AH. Since E_b is the midpoint of HB and E_c is the midpoint of HC, E_bE_c is parallel to BC, which is perpendicular to AH. Similar logic gives us that O_bO_c is perpendicular to AH as well. Therefore $O_bO_cE_bE_c$ is a rectangle, which is a cyclic figure. The diagonals O_bE_b and O_cE_c are diagonals of the circumcircle. Similar logic to the above gives us that $O_aO_cE_aE_c$ is a rectangle with a common diagonal to $O_bO_cE_bE_c$. Therefore the circumcircles of the two rectangles are identical. We can also gain that rectangle $O_aO_bE_aE_b$ is also on the circle.

Proof 1



We now have a circle with the points O_a, O_b, O_c, E_a, E_b, and E_c on it, with diameters O_aE_A, O_bE_b, and O_cE_c. We now note that $\angle E_aH_aO_a = \angle E_bH_bO_b = \angle E_cH_cO_c = 90°$. Therefore H_a, H_b, and H_c are also on the circle. We now have a circle with the midpoints of the sides on it, the three midpoints of the segments joining the vertices of the triangle to its orthocenter on it, and the three feet of the altitudes of the triangle on it. Therefore, the nine points are on the circle, and the nine-point circle exists.

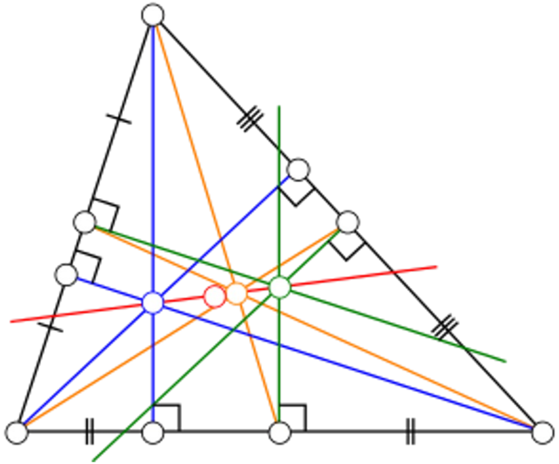
Proof 2



We know that the reflection of the orthocenter about the sides and about the midpoints of the triangle's sides lie on the circumcircle. Thus, consider the homothety centered at H with ratio $1/2$. It maps the circumcircle of $\triangle ABC$ to the nine-point circle, and the vertices of the triangle to its Euler points. Hence proved.

Euler Line

It is a central line of the triangle, and it passes through several important points determined from the triangle, including the orthocenter, the circumcenter, the centroid, the Exeter point and the center of the nine-point circle of the triangle.



Euler's line, with the center of the nine-point circle
Medians (intersect at the centroid)
Altitudes (intersect at the orthocenter)
Perpendicular lines from the side midpoints (intersect at the circumcenter)

Proof

Let ABC be a triangle. A proof of the fact that the circumcenter O, the centroid G and the orthocenter H are **collinear** relies on free vectors. We start by stating the prerequisites. First, G satisfies the relation

$$\vec{GA} + \vec{GB} + \vec{GC} = 0.$$

This follows from the fact that the absolute barycentric coordinates of G are $\frac{1}{3} : \frac{1}{3} : \frac{1}{3}$. Further, the problem of Sylvester[7] reads as

$$\vec{OH} = \vec{OA} + \vec{OB} + \vec{OC}.$$

Now, using the vector addition, we deduce that

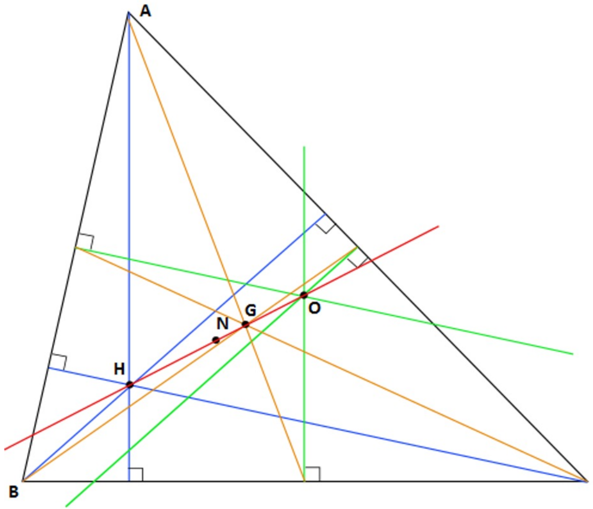
$$\vec{GO} = \vec{GA} + \vec{AO} \,(\text{in triangle } AGO),\ \vec{GO} = \vec{GB} + \vec{BO} \,(\text{in triangle } BGO),\ \vec{GO} = \vec{GC} + \vec{CO} \,(\text{in triangle } CGO).$$

By adding these three relations, term by term, we obtain that

$$3 \cdot \vec{GO} = \left(\sum_{\text{cyclic}} \vec{GA}\right) + \left(\sum_{\text{cyclic}} \vec{AO}\right) = 0 - \left(\sum_{\text{cyclic}} \vec{OA}\right) = -\vec{OH}.$$

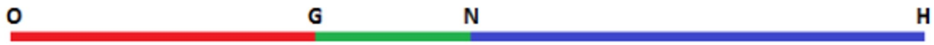
In conclusion, $3 \cdot \vec{OG} = \vec{OH}$, and so the three points O, G and H (in this order) are collinear.

Properties



The proof above shows more than collinearity: since H is sent to O through a scale factor of $-\frac{1}{2}$, we have $HG = 2GO$. In fact, more is true: if N is the nine-point center of the triangle, we have

$$HG = 2GO, \quad ON = NH, OG = 2GN, \quad NH = 3GN.$$



Relative distances of triangle centers. The red line is twice as long as the green line, and the blue line is three times as long as the green line.

Slope of Euler Line

Additionally, if A_1 is the foot of the altitude from A to BC and M_a is the midpoint of BC (with B_1 and M_b defined analogously), then

> The intersection of A_1M_b and B_1M_a lies on the Euler line.

The slope of the Euler line relates to the slope of the sides in a nice way: If m_1, m_2, m_3 are the slopes of the three sides of a triangle ABC, and m_e is the slope of the Euler line, then

$$m_1m_2 + m_2m_3 + m_3m_1 + m_1m_e + m_2m_e + m_3m_e + 3m_1m_2m_3m_e + 3 = 0,$$

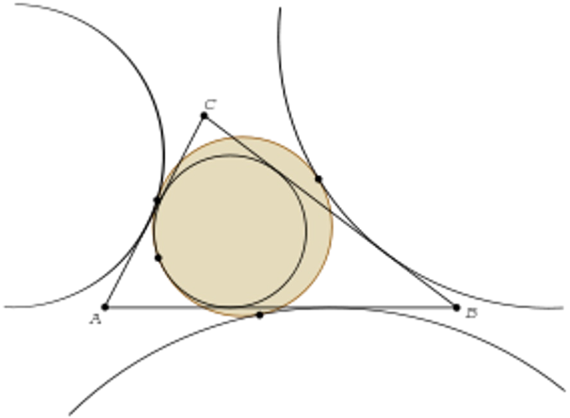
or equivalently,

$$m_e = -\frac{m_1m_2 + m_2m_3 + m_3m_1 + 3}{m_1 + m_2 + m_3 + 3m_1m_2m_3}.$$

Feuerbach’s Theorem

The **nine-point circle** of a triangle is internally tangent to the **incircle** and externally tangent to the three **excircles**

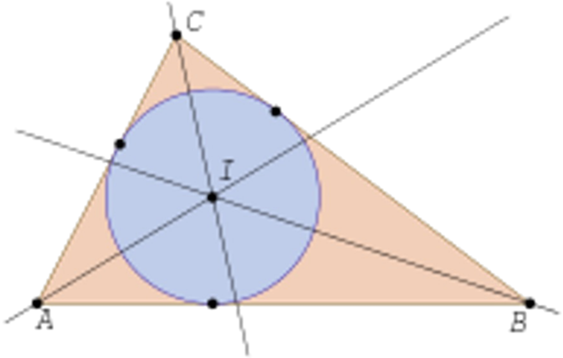
The tangent point of the nine-point circle and the incircle is called the **Feuerbach Point.**



Basic Concept

Incircle is the circle that is tangent to all three sides of a triangle.

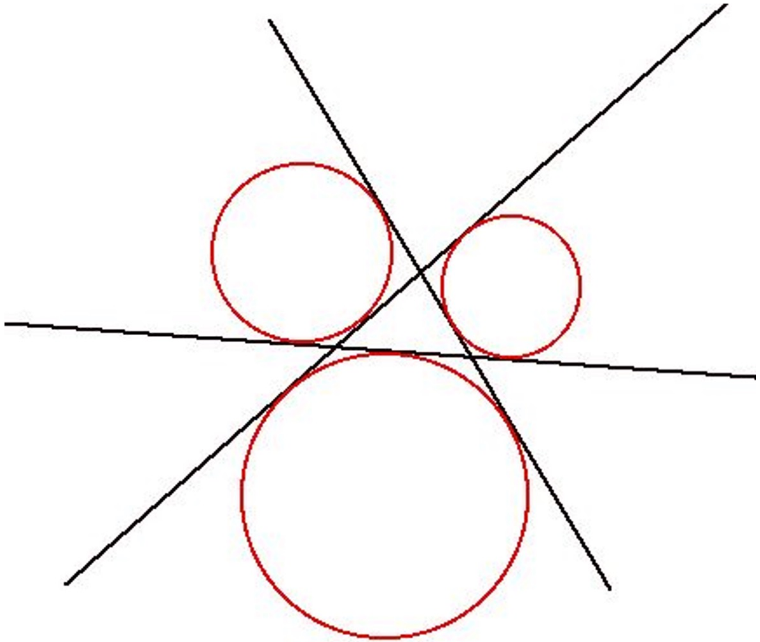
The center of the incircle is called incenter.



Basic Concept

Excircle is circle that is tangent to one side of the triangle and to the extension of the remaining two sides.

There are three excircles, and their centers are called the excenters



Proof

An Elementary Proof of Feuerbach's Theorem.

Let O be the centre of the circumscribing circle of $\triangle ABC$, A_1 the middle point of BC, and EA_1OF the diameter at right angles to BC. Draw AX perpendicular to BC and produce it to meet the circle in K. Let H be the orthocentre of $\triangle ABC$; join OH and bisect it in N, the centre of the nine-point circle.

Draw OY perpendicular to and bisecting AK.

Join EA, which bisects $\angle BAC$ and contains the incentre I; draw ID, NM perpendicular to BC. Join AF and draw AG perpendicular to EF; also draw PIQ parallel to BC and meeting EF in P and AX in Q.

Then we have $AH = 2OA_1$, $HK = 2HX$, $AI \,.\, IE = 2Rr$.

Also from similar triangles $\dfrac{PI}{IE} = \dfrac{FG}{AF}$ and $\dfrac{IQ}{AI} = \dfrac{AF}{FE}$.

Thus $\dfrac{PI \,.\, IQ}{AI \,.\, IE} = \dfrac{FG}{FE}$, so that $\dfrac{PI \,.\, IQ}{2R \,.\, r} = \dfrac{FG}{2R}$, and $PI \,.\, IQ = r \,.\, FG$.

Now the projection of IN on $FE = ID - NM = r - \frac{1}{2}(OA_1 + HX)$
$= r - \frac{1}{4}(AH + HK) = r - \frac{1}{2}AY.$

Proof

Hence the square of this projection $= r^2 - r \,.\, AY + \frac{1}{4}AY^2$

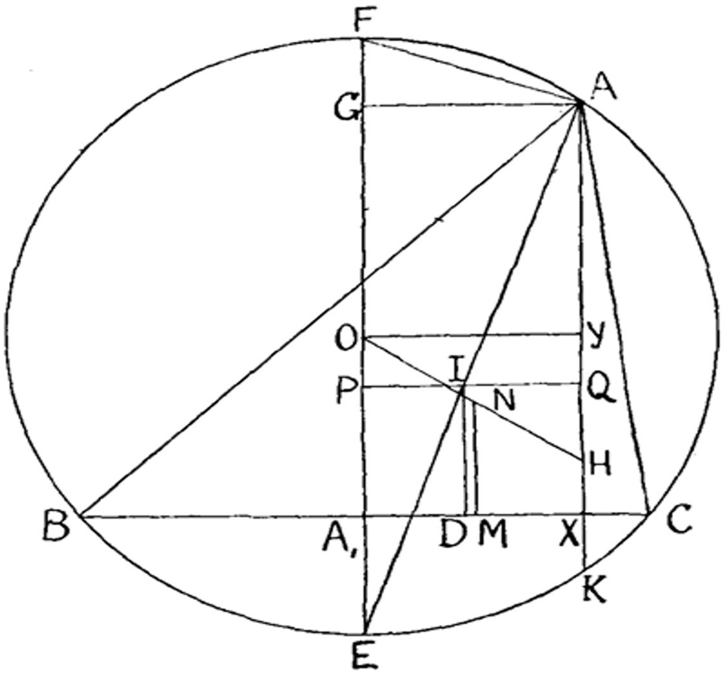
$$= r^2 - r \,.\, GO + \tfrac{1}{4}AY^2 \quad \ldots\ldots\ldots(1)$$

Again, the square of the projection of

$$IN \text{ on } BC = DM^2 = A_1M^2 - A_1D \,.\, DX$$

$$= \tfrac{1}{4}A_1X^2 - PI \,.\, IQ$$

$$= \tfrac{1}{4}OY^2 - r \,.\, FG \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$



Proof

Adding the results (1) and (2) we get

$$I_1N^2 = \tfrac{1}{4}(AY^2 + OY^2) - r(FG + GO) + r^2$$
$$= \tfrac{1}{4}R^2 - r \,.\, R + r^2.$$

Thus $IN = \frac{1}{2}R - r$, and the theorem is proved for the incircle. The proof for an excircle proceeds on exactly similar lines.

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Thank You for watching

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