Newton Line

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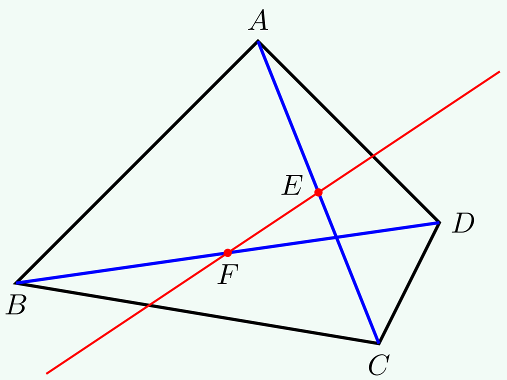
1 Introduction

On a convex quadrilateral which is not a parallelogram, the line formed by the midpoints of the two diagonals is called *Newton Line*, or *Newton-Gauss Line*.

Definition 1. (Newton Line)

Let $ABCD$ be a quadrilateral. Let E and F be the midpoints of the diagonals AC and BD, respectively. If $E \neq F$, then the line defined by EF is called the Newton Line.



If $E = F$, that is, if $ABCD$ is a parallelogram, then the Newton Line is not defined.

2 The First Property of Newton Line

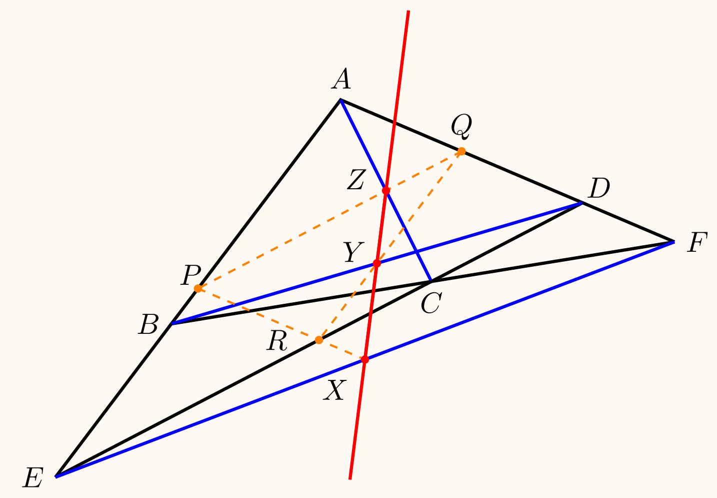
Definition 2. (Complete Quadrilateral)

A complete quadrilateral is a system of four lines, no three of which pass through the same point, and the six points of intersection of these lines.

Theorem 1

The midpoints of the three diagonals of a complete quadrilateral lie on the Newton Line.

[1]The author thanks Dr. Zhiqin Lu for his help.



Proof. Let X, Y, and Z be midpoints of line segments EF , BD and AC, respectively. We want to show X , Y and Z are collinear. Let P, Q and R be the the midpoints of line segments AE, AD and ED, respectively. Connect PZ, ZQ, QY, YR, PR and RX.

PR is the mid-line in $\triangle EAD$. Thus

$$PR \parallel AD, \; PR = \frac{1}{2}AD.$$

Similarly, we can prove that in $\triangle ABD$, $\triangle EBD$, $\triangle ACD$, $\triangle ACE$, and $\triangle DEF$,

$$QY \parallel AB, \; QY = \frac{1}{2}AB, \quad RY \parallel BE, \; RY = \frac{1}{2}BE,$$
$$QZ \parallel CD, \; QZ = \frac{1}{2}CD, \quad PZ \parallel CE, \; PZ = \frac{1}{2}CE,$$
$$RX \parallel DF, \; RX = \frac{1}{2}DF.$$

By the fact that both PR and RX are parallel to AF, we conclude that P, R, and X are collinear. Similarly, R, Y, Q and Q, Z, P are collinear, respectively. In particular, $PX = \frac{1}{2}AF$. Hence in $\triangle AEF$,

$$PX \parallel AF, \; PX = \frac{1}{2}AF.$$

Since BCF is a line crossing $\triangle AED$, by Menelaus' Theorem, we have

$$\frac{AF}{FD} \cdot \frac{CD}{CE} \cdot \frac{EB}{AB} = 1.$$

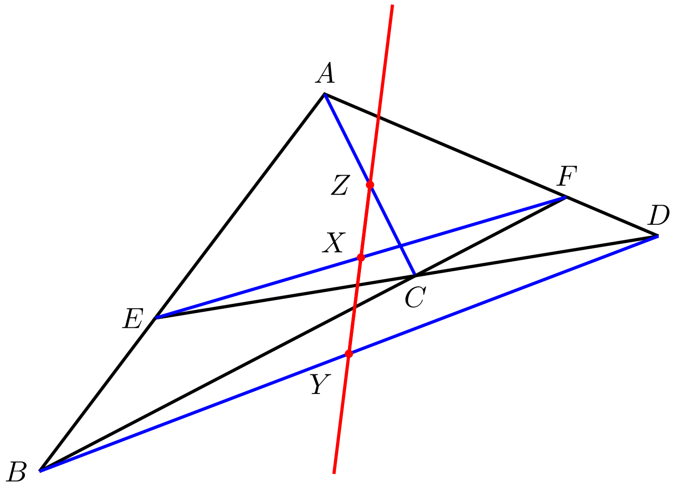
Substituting the proportion relations of the above, we get

$$\frac{PX}{RX} \cdot \frac{RY}{QY} \cdot \frac{QZ}{PZ} = 1.$$

By applying Menelaus' Theorem on $\triangle PQR$, we conclude that X, Y, and Z are collinear. ■

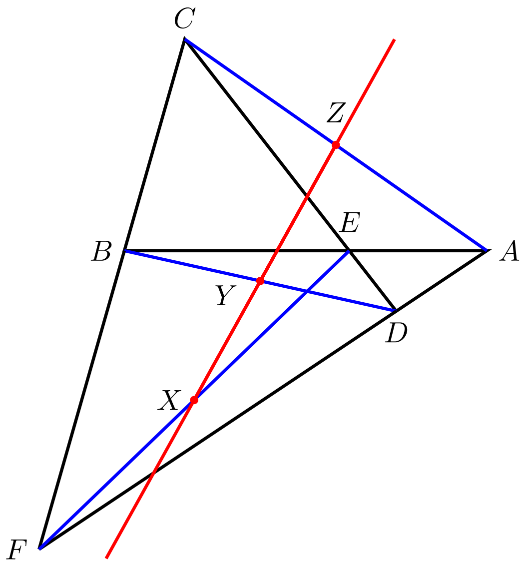
Remark The above theorem is also true when $ABCD$ is not convex. There are two cases:

If $ABCD$ is a concave quadrilateral, then we obtain the following picture.



In this case, $AECF$ is convex. Therefore, we apply Theorem 1 on $AECF$, the result holds.

If $ABCD$ is a crossed quadrilateral, then we have the following picture.

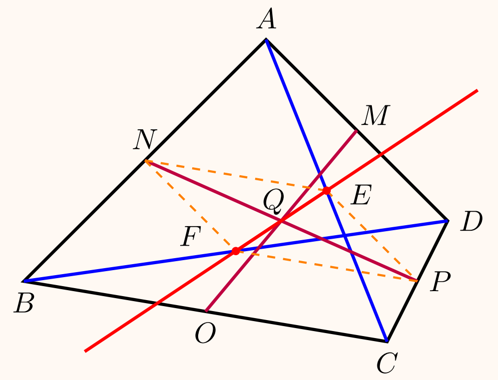


In this case, $BEDF$ is convex. Therefore, we apply Theorem 1 on $BEDF$, the result holds.

3 Other properties of Newton Line

Theorem 2

Let N, O, P, and M be the midpoints of segments AB, BC, CD, and AD. Let point Q be the intersection of NP and MO. Then Q is on the Newton line EF, and $EQ = FQ$.



Proof. Connect FN, FP, EN, and EP. Since N and E are the midpoints of line segments AB and AC. Therefore, NE is the mid-line in $\triangle ABC$. Hence

$$NE \parallel BC,$$

and

$$NE = \frac{1}{2}BC.$$

Similarly,

$$FP \parallel BC,\ EP \parallel AD,\ NF \parallel AD.$$

$$FP = \frac{1}{2}BC,\ EP = \frac{1}{2}AD,\ NF = \frac{1}{2}AD.$$

Then we get

$$NE \parallel PF,\ NF \parallel EP,$$

and hence the quadrilateral $NFPE$ is a parallelogram. The diagonals EF and NP intersect at a point Q'.

Similarly, we can prove that the quadrilateral $FOEM$ is also a parallelogram. The diagonals EF and MO intersect at a point Q''.

Since both Q' and Q'' lie on the Newton Line EF and

$$EQ' = FQ',\ EQ'' = FQ'',$$

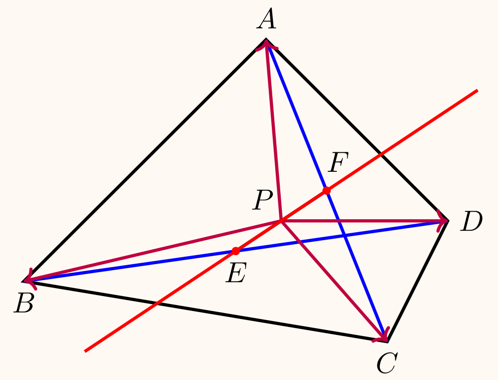
then $Q' = Q'' = Q$, completing the proof. ∎

Theorem 3

Let A, B, C, D be four fixed points on the Euclidean plane, and let P be a moving point. Then the locus of P satisfying the following equation is the Newton Line of the quadrilateral $ABCD$;

$$\frac{1}{2}\cdot\overrightarrow{PA}\times\overrightarrow{PB}+\frac{1}{2}\cdot\overrightarrow{PC}\times\overrightarrow{PD}=\frac{1}{2}\cdot\overrightarrow{PD}\times\overrightarrow{PA}+\frac{1}{2}\cdot\overrightarrow{PB}\times\overrightarrow{PC}, \tag{1}$$

where $\times$ is the cross product of vectors.



Proof. By abuse of notation, we use vertex notation as vector notation also. Therefore, we have, for example, $\overrightarrow{PA} = A - P$. Using this, the equation in the theorem can be written as

$$(A-P)\times(B-P)+(C-P)\times(D-P) = (D-P)\times(A-P)+(B-P)\times(C-P).$$

Noting that $P \times P = 0$, expanding the above equation, we obtain

$$P \times (A + C - B - D) = -\frac{1}{2}(A + C) \times (B + D), \tag{2}$$

which is a linear equation. Therefore the locus of P is a straight line.

In order to prove that the line is the Newton Line, we need to prove that points E, F are on the line. To see this, we let $P = E = \frac{1}{2}(B + D)$. Then we have

$$P \times (A + C - B - D) = -\frac{1}{2}(A + C) \times (B + D), \tag{3}$$

and hence E is on Line 2. Similarly, we let $P = F = \frac{1}{2}(A + C)$. Then F satisfies (2) as well. Therefore Line 2 passes both E, F, and it is the Newton Line. ∎

Remark Using the Newton Line Equation (3), we can give a vector-geometry proof of Theorem 1. We use the picture in Theorem 1. Since the midpoint of EF is $\frac{1}{2}(E + F)$, the identity we need to prove is

$$(E + F) \times (A + C - B - D) = -(A + C) \times (B + D).$$

Since E, B, A are collinear, there is a real number λ_1 such that

$$E = B + \lambda_1(A - B).$$

Similarly, there are real numbers $\lambda_2, \lambda_3, \lambda_4$ such that

$$E = C + \lambda_2(C - D);$$

$$F = C + \lambda_3(C - B);$$

$$F = D + \lambda_4(A - D).$$

Using the above equations, we have

$$E \times (A + C - B - D) = E \times (A - B) + E \times (C - D)$$
$$= B \times (A - B) + C \times (C - D) = -A \times B - C \times D,$$

and

$$F \times (A + C - B - D) = F \times (C - B) + F \times (A - D)$$
$$= C \times (C - B) + D \times (A - D) = -C \times B - A \times D.$$

Therefore, we have

$$(E + F) \times (A + C - B - D) = -A \times B - C \times D - C \times B - A \times D = -(A + C) \times (B + D),$$

completing the proof.

Remark The theorem characterizes Newton Line using vector notation. Note that for three vectors A, B, C on the Euclidean plane, the area of $\triangle ABC$ can be given by

$$S_{\triangle ABC} = \frac{1}{2}\|(B - A) \times (C - A)\|.$$

Let $\vec{N}$ be the unit normal vector of the Euclidean plane. Then we can use the expression

$$\frac{1}{2}\langle (B - A) \times (C - A), \vec{N} \rangle$$

to denote the *signed area* of $\triangle ABC$. If the vector $C - A$ can be obtained by rotating $B - A$ counter-clockwise by an angle of no more than $180°$, then the signed area is non-negative, and is equal to the (unsigned) area of $\triangle ABC$.

If P is a point inside a convex quadrilateral $ABCD$, then the signed areas in (1) are all positive, and hence we can rewrite (1) as

$$S_{\triangle PAB} + S_{\triangle PCD} = S_{\triangle PDA} + S_{\triangle PBC}. \tag{4}$$

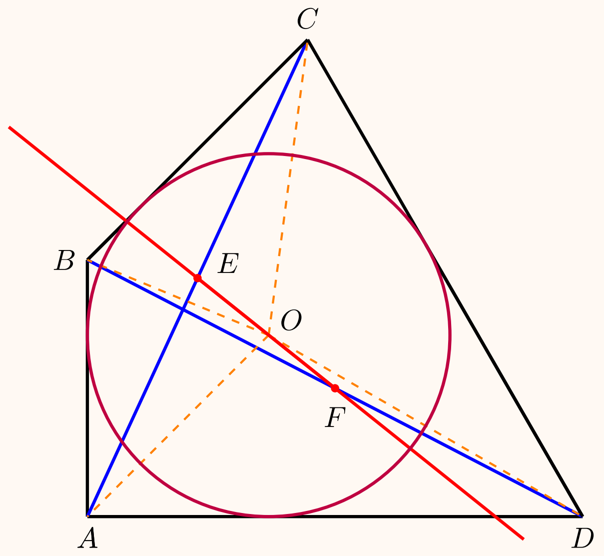
In that case, Theorem is reduced to the following Anne's Theorem (by Pierre-Leon Anne, see Anne' Theorem for details).

Theorem 4. (Anne's Theorem)

Let P be a point inside a convex quadrilateral and let P be on the Newton Line. Then (4) holds.

Corollary 1

Let $ABCD$ be a tangential quadrilateral, circle O is inscribed in $ABCD$. Then point O is on the Newton Line of $ABCD$.



Proof. Connect OA, OB, OC, and OD. According to the property of tangential quadrilateral, we obtain

$$AB + CD = AD + BC.$$

We also get that the altitudes of $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle AOD$ are the radius r of circle O. Therefore,

$$\frac{1}{2} \cdot r \cdot (AD + BC) = \frac{1}{2} \cdot r \cdot (AB + CD),$$

which means

$$S_{\triangle AOD} + S_{\triangle BOC} = S_{\triangle AOB} + S_{\triangle COD}.$$

By Theorem 3, the center of the circle O on the Newton Line. ■