Topic 6
Pascal’s and Brainchon’s Theorem

Math 199
Chen Xu

Content

1 Pascal’s Theorem

2 Brianchon’s Theorem

3 Duality in Geometry

4 Conclusion

Part 1

Pascal's Theorem

Introduction

Physics

- Fluid dynamics and pressure

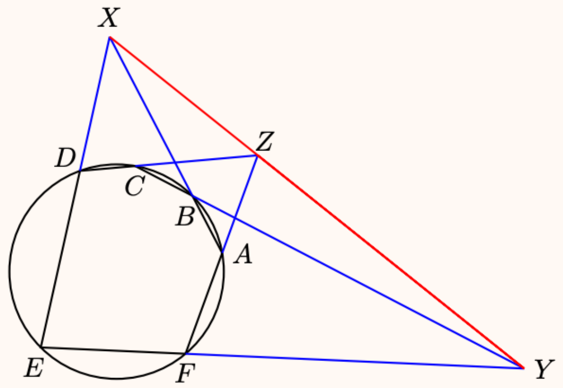
Mathematics

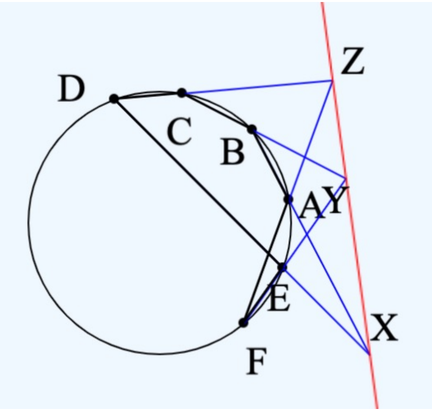
- Pascal’s development of probability theory
- Pascal’s triangle
- **Pascal’s theorem**

Blaise Pascal

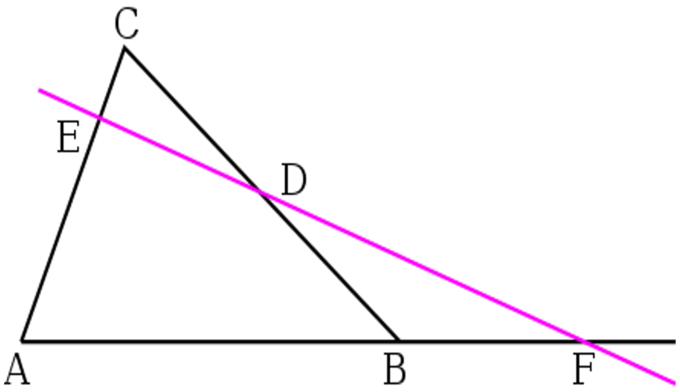
Pascal's Theorem

The hexagon $ABCDEF$ is inscribed to a circle. Assume that AB, DE intersects at X; BC, EF intersects at Y; and CD, FA intersects at Z. Then X, Y, Z are collinear.





Proof



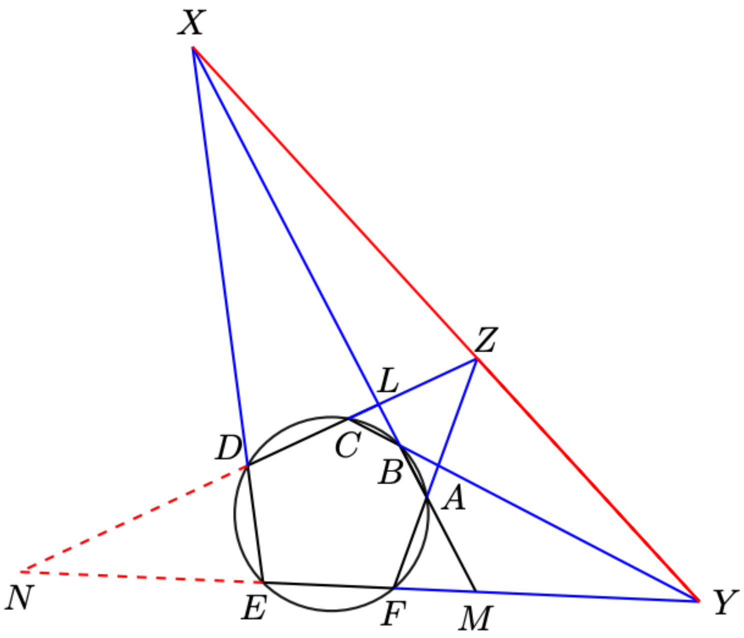
- Menelaus's Theorem

Suppose ΔABC and a transversal line that crosses $BC, AC, and\ AB$ at point $D, E, and\ F$ respectively
$\Longrightarrow \frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1$

- The inverse of Menelaus' Theorem

Suppose points D, E, F are chosen on $BC, AC, and\ AB$ respectively so that $\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1$
$\Longrightarrow D, E, F$ are collinear

Proof



Goal:

Based on the inverse of Menelaus's theorem

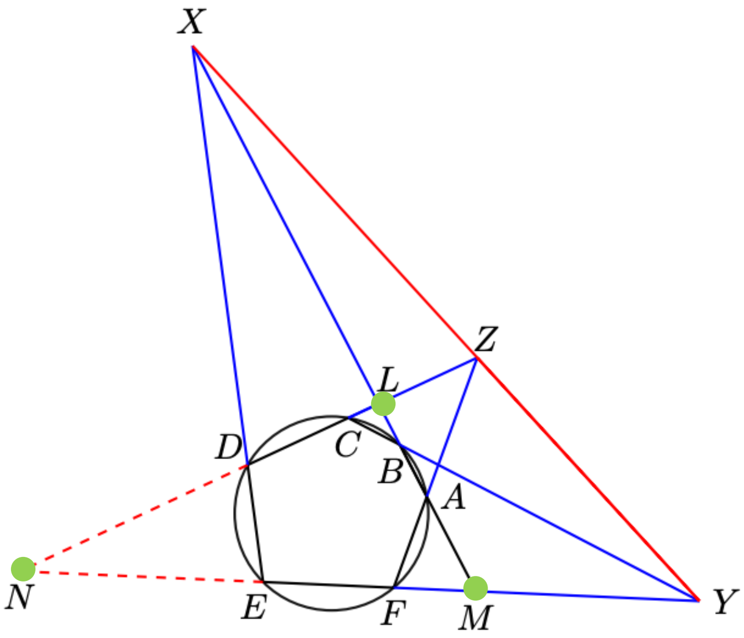
To prove X, Y, Z are collineal

$$\Updownarrow$$

To prove $\frac{LX}{XM} \times \frac{MY}{YN} \times \frac{NZ}{ZL} = 1$

$$\Delta ABC - XYZ$$

Proof



First proof: As in the graph drawn below, let AB and CD intersect at L, BA and EF intersect at M, CD and FE intersect at N.

On $\triangle LMN$, since C, B, Y are collinear, by applying Menelaus' Theorem we obtain

$$\frac{LB}{BM} \cdot \frac{MY}{YN} \cdot \frac{NC}{CL} = 1.$$

Similarly, since F, A, Z are collinear, we obtain

$$\frac{LA}{AM} \cdot \frac{MF}{FN} \cdot \frac{NZ}{ZL} = 1,$$

and since E, D, X are collinear, we also get

$$\frac{ND}{DL} \cdot \frac{LX}{XM} \cdot \frac{ME}{EN} = 1.$$

In the circle $ABCDEF$, by using the Power of Point Theorem, we will get

$$LA \cdot LB = LD \cdot LC,$$

$$NC \cdot ND = NE \cdot NF,$$

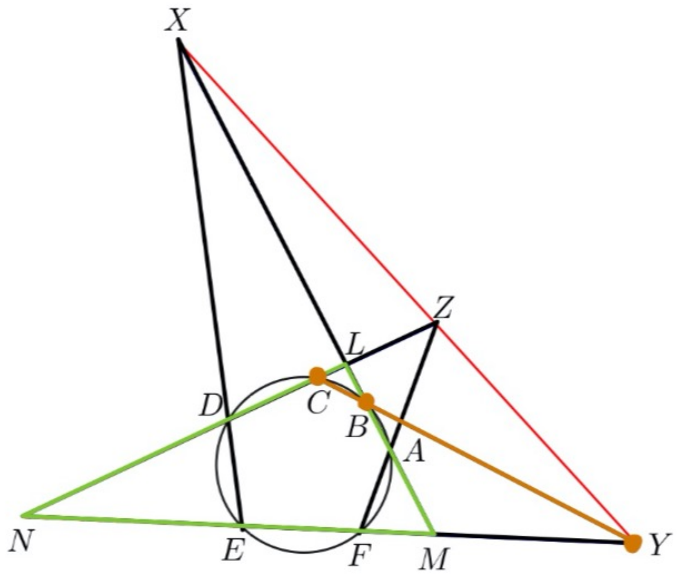
$$MA \cdot MB = MF \cdot ME.$$

Combining the above six equations, we obtain that

$$\frac{LX}{XM} \cdot \frac{MY}{YN} \cdot \frac{NZ}{ZL} = 1$$

Thus, by the inverse of Menelaus' Theorem we conclude that X, Y, Z are collinear.

Proof



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1. $\Delta LMN \ - CBY$

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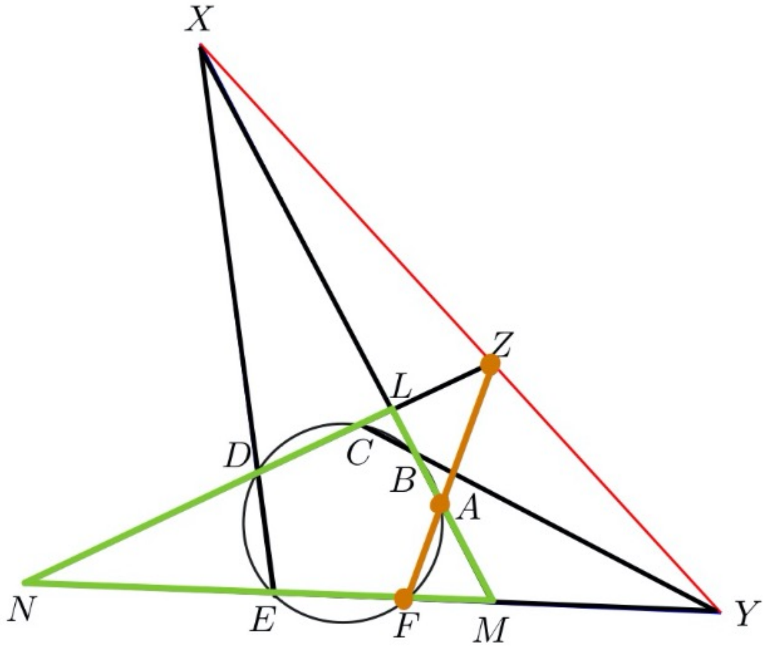

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2. $\Delta LMN - FAZ$

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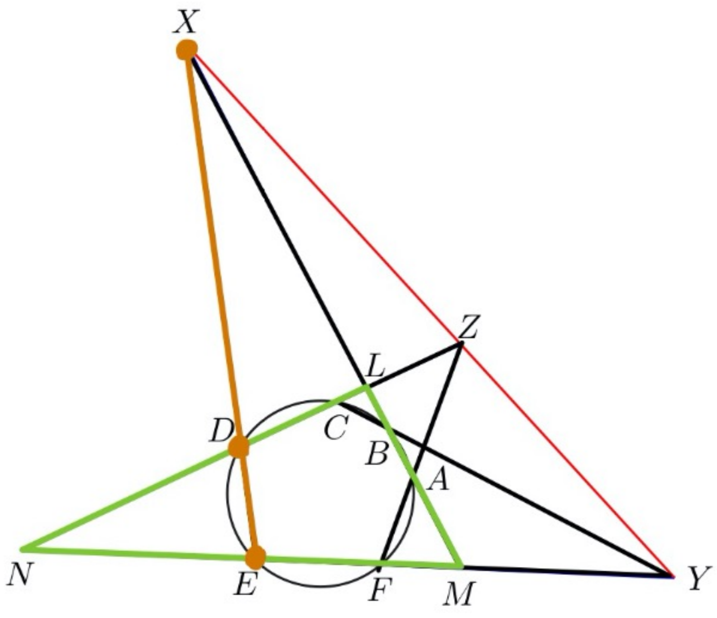

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3. $\Delta LMN - EDX$

In the circle $ABCDEF$, by using the Power of Point Theorem, we will get

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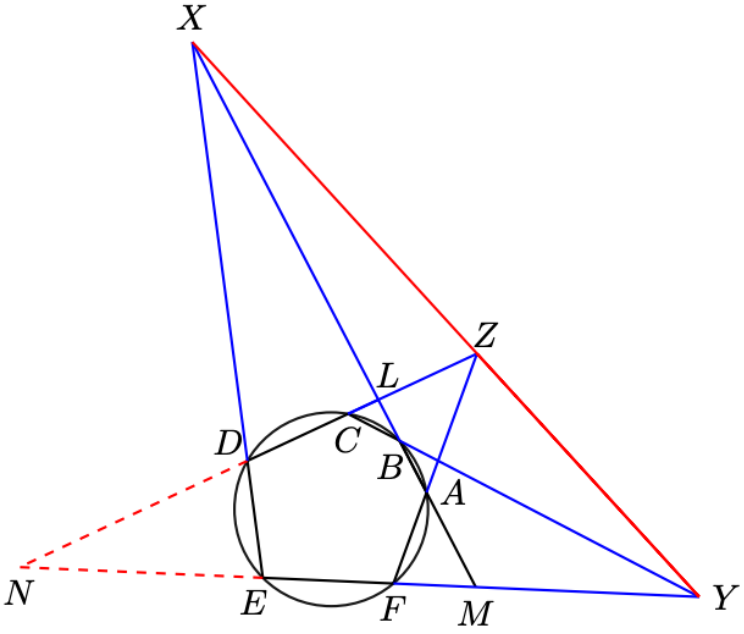
Thus, by the inverse of Menelaus' Theorem we conclude that X, Y, Z are collinear.

Proof



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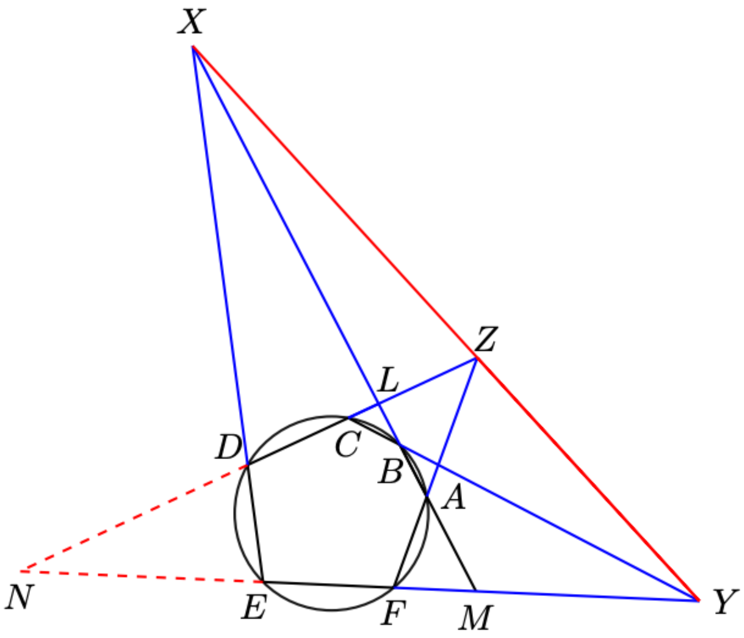

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Combining the above six equations, we obtain that

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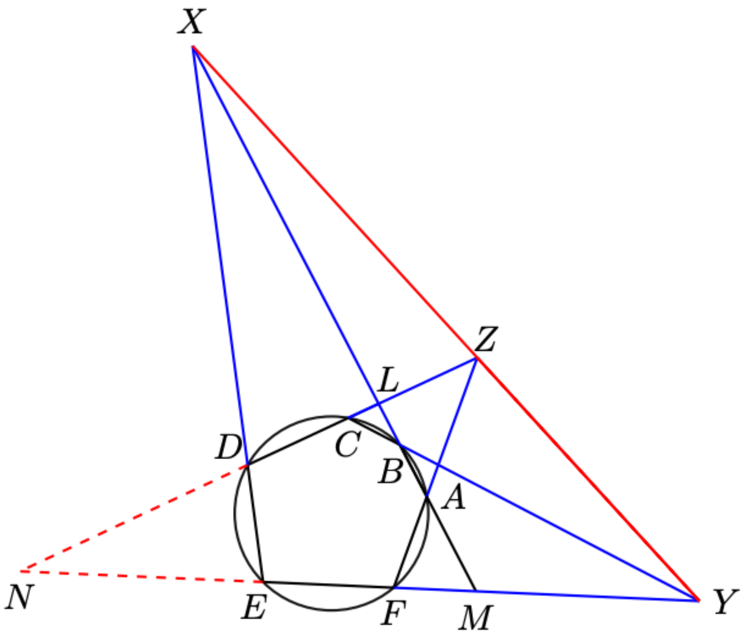
Thus, by the inverse of Menelaus' Theorem we conclude that X, Y, Z are collinear.

Proof



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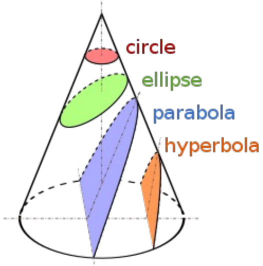
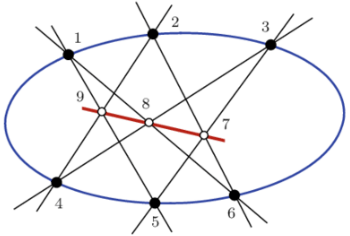
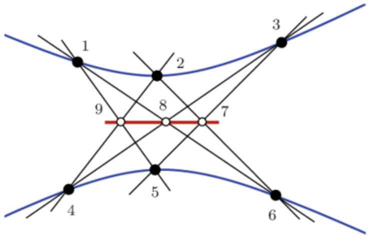
General Pascal’s Theorem

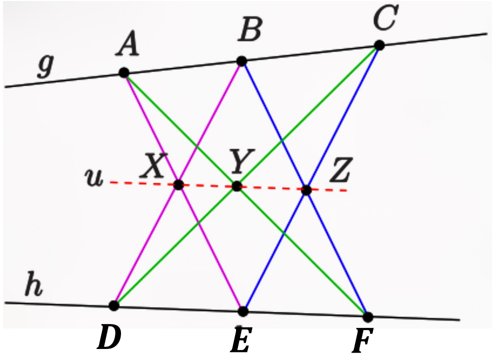
- Pascal’s theorem can be generalized to the case of conic section







- Special case: Pappus’ Hexagon Theorem – when the conic section is degenerated to two lines



The hexagon ECDBFA is inscribed on the two black lines

$$AE \cap BD = X$$
$$AF \cap CD = Y$$
$$BF \cap CE = Z$$

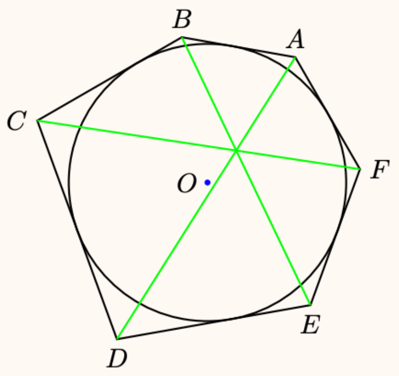
$\Longrightarrow$ the intersection points X, Y, Z are collinear

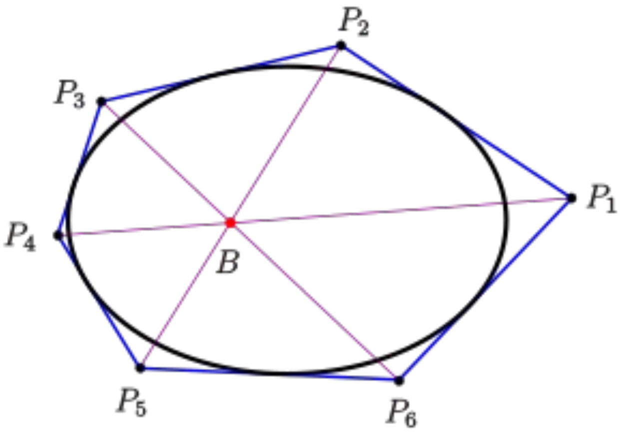
Part 2

Brianchon’s Theorem

Brianchon’s Theorem

The Hexagon $ABCDEF$ is circumscribed on a circle. Then $AD, BE,$ and CF are concurrent.





Proof

- Pole and polar
- Monge’s theorem
- Ceva’s theorem
- Analytic method
- …..

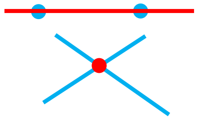
Brianchon’s theorem is the projective dual of Pascal’s theorem

Part 3

Duality in Geometry

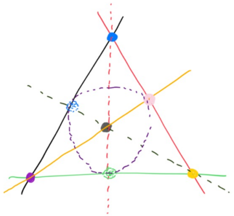
Definition

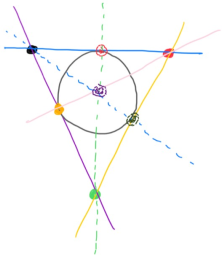
- In the projective geometry of the plane, the words "point" and "line" can be interchanged
  Dual statements: "Two blue points determine a red line"
  "Two blue lines determine a red point"



- In general, abstract projective plane $\Pi = \{P, L, I\}$

  - P is the set of points
  - L is the set of lines
  - $I \subset P \times L$ is the incidence relationship between points and lines



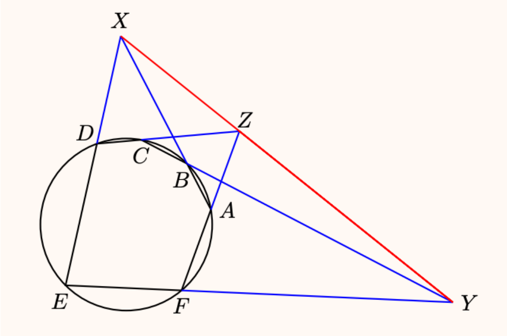


  Duality: we can think instead of L as the set of points and P as the set of lines

- If a statement is true, then the dual statement is true as well.

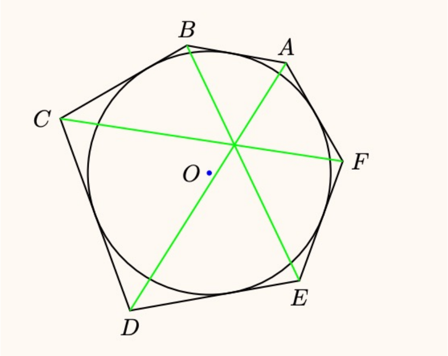
Famous Dual Theorems

Pascal's Theorem



Let A, B, C, D, E and F be any six points on any conic section.
Then the three pairs of lines, AB and DE, BC and EF, CD and FA intersect in three points which are collinear.

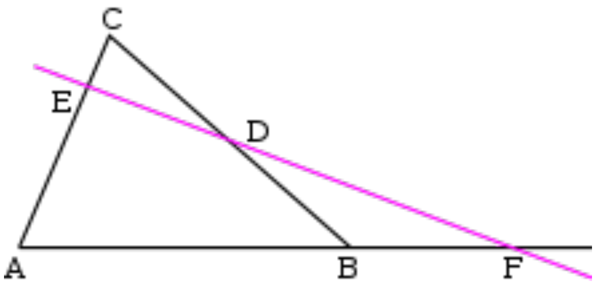
Brianchon's Theorem



Let $ABCDEF$ is a hexagon circumscribed about a conic. Then the lines through three opposite vertices AD, BE, and CF are concurrent.

Famous Dual Theorems

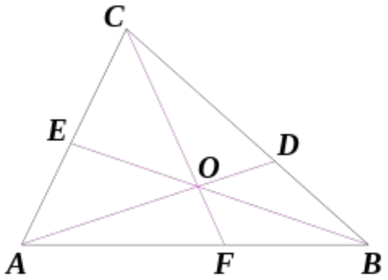
Menelaus’s Theorem



$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1$$

Three points are collinear

Ceva’s Theorem



$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1$$

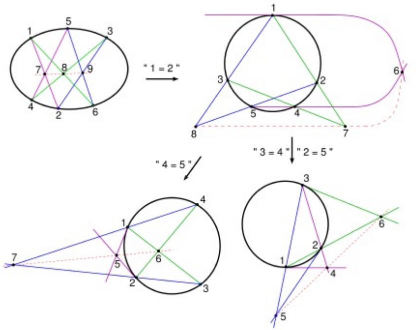
Three lines are concurrent

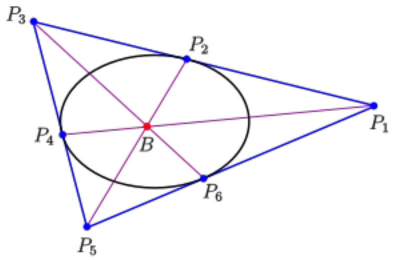
Part 4

Conclusion

Conclusion

- Pascal’s theorem is a generalization of Pappus's hexagon theorem
- There exist 5-point, 4-point and 3-point degenerate cases of Pascal's theorem
- There exist 5-point, 4-point and 3-point degenerate cases of Brianchon’s theorem
- Pascal’s theorem and Brianchon’s theorem are two famous “dual” theorems





Thank You