The Morley’s Miracle

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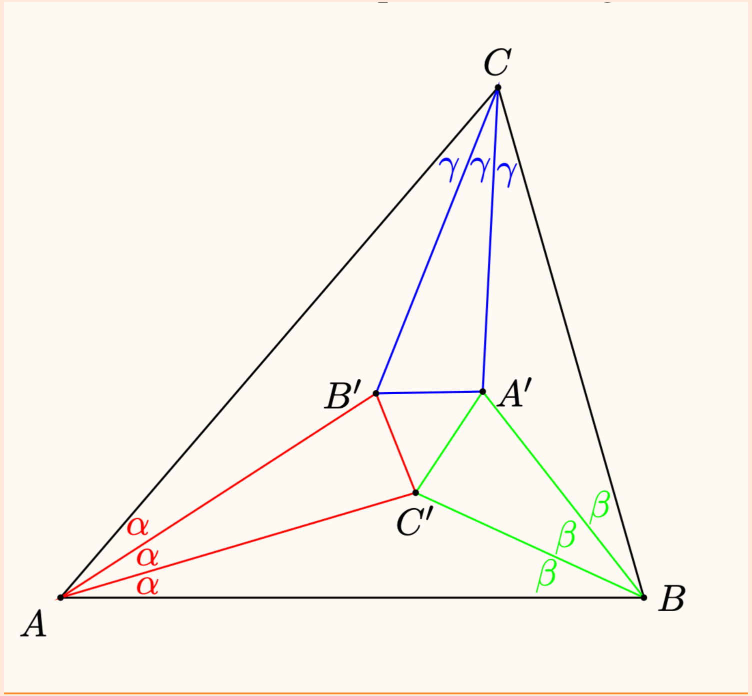
Abstract

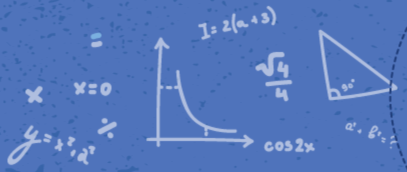
The Morley's Theorem states that in any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle. The theorem was discovered by Frank Morley in 1899.

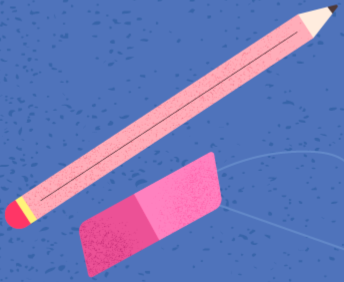
A simple example：

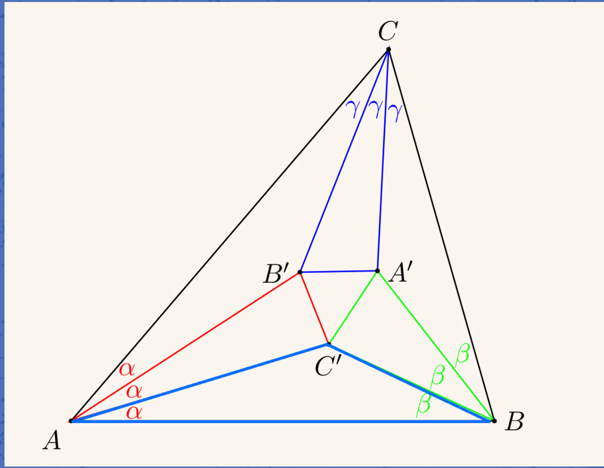


The trigonometry method

one of the simplest method







Solution: Let R be the circumradius of the ΔABC. Then

$$AB = 2R\sin C = 2R\sin 3\gamma.$$

Using the law of sines, we have

$$AC' = \frac{AB}{\sin(\alpha+\beta)} \cdot \sin\beta = 2R\frac{\sin 3\gamma}{\sin(\alpha+\beta)} \sin\beta.$$

Using the formula $\sin 3x = 3\sin x - 4\sin^3 x$, we get

$$AC' = 2R\frac{\sin 3(\alpha+\beta)}{\sin(\alpha+\beta)} \sin\beta = 2R(3 - 4\sin^2(\alpha+\beta))\sin\beta.$$

Formula notes

Product-to sum- identities

$\sin(\alpha+\beta) = \sin\alpha \sin\beta + \cos\alpha \sin\beta$ ①

$\sin(\alpha-\beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$ ②

$\cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$ ③

$\cos(\alpha-\beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$ ④

① + ② $\Rightarrow \sin\alpha \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

① - ② $\Rightarrow \cos\alpha \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$

③ + ④ $\Rightarrow \cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$

④ - ③ $\Rightarrow \sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

Let $A = \alpha+\beta$, $B = \alpha-\beta$ $\Rightarrow \begin{cases} \alpha = \frac{A+B}{2} \\ \beta = \frac{A-B}{2} \end{cases}$

Sum-to-product identities

① + ②:

$\sin A + \sin B = 2\sin\frac{A+B}{2} \cos\frac{A-B}{2}$

① - ②:

$\sin A - \sin B = 2\cos\frac{A+B}{2} \sin\frac{A-B}{2}$

③ + ④:

$\cos A + \cos B = 2\cos\frac{A+B}{2} \cos\frac{A-B}{2}$

④ - ③:

$\cos A - \cos B = -(\cos B - \cos A) = -2\sin\frac{A+B}{2} \sin\frac{A-B}{2}$

Note that $\alpha + \beta + \gamma = \pi/3$,

$$3 - 4\sin^2(\alpha+\beta) = 3 - 2(1 + \cos 2(\alpha+\beta))$$
$$= 2(\cos\frac{\pi}{3} - \cos 2(\alpha+\beta)) = 4\cos(\frac{\pi}{6} + \alpha + \beta)\cos(\alpha + \beta - \frac{\pi}{6}).$$

$$3 - 4\sin^2(\alpha+\beta) = 2\sin\gamma\cos(\frac{\pi}{6} - \gamma).$$

$$AC' = 8R\sin\beta\sin\gamma\cos(\frac{\pi}{6} - \gamma).$$

Similarly

$$AB' = 8R\sin\beta\sin\gamma\cos(\frac{\pi}{6} - \beta).$$

Let $\sigma = 8R\sin\beta\sin\gamma$. Then by law of cosines, we have

$$(B'C')^2 = \sigma^2(\cos^2(\frac{\pi}{6} - \gamma) + \cos^2(\frac{\pi}{6} - \beta) - 2\cos(\frac{\pi}{6} - \gamma)\cos(\frac{\pi}{6} - \beta)\cos\alpha).$$

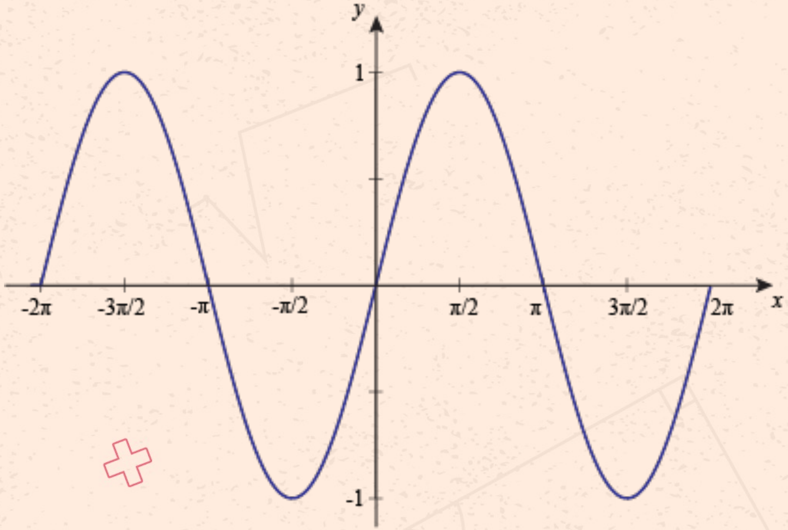
We have

$$\begin{aligned}
&\sigma^2(\cos^2(\frac{\pi}{6} - \gamma) + \cos^2(\frac{\pi}{6} - \beta) - 2\cos(\frac{\pi}{6} - \gamma)\cos(\frac{\pi}{6} - \beta)\cos\alpha) \\
&= \frac{1 + \cos(\frac{\pi}{3} - 2\gamma)}{2} + \frac{1 + \cos(\frac{\pi}{3} - 2\beta)}{2} - (\cos(\frac{\pi}{3} - \gamma - \beta) + \cos(\beta - \gamma))\cos\alpha \\
&= 1 + \frac{1}{2}(\cos(\frac{\pi}{3} - 2\gamma) + \cos(\frac{\pi}{3} - 2\beta)) - \cos^2\alpha - \cos(\beta - \gamma)\cos\alpha \\
&= \sin^2\alpha.
\end{aligned}$$

Thus

$$B'C' = 8R\sin\alpha\sin\beta\sin\gamma.$$

By symmetry, $C'A' = A'B' = B'C' = 8R\sin\alpha\sin\beta\sin\gamma$.



The angles in a triangle add up to less than 180 degrees

AB 'and AC' are strictly increasing at angles less than 90 degrees

Remark We are able to avoid using law of cosines, which is complicated. We observe that

$$\frac{AB'}{AC'} = \frac{\cos(\frac{\pi}{6} - \beta)}{\cos(\frac{\pi}{6} - \gamma)} = \frac{\sin(\frac{\pi}{3} + \beta)}{\sin(\frac{\pi}{3} + \gamma)}.$$

Thus we have $\angle B'C'A = \frac{\pi}{3} + \beta$. Similarly, $\angle BC'A' = \frac{\pi}{3} + \alpha$. Thus

$$\angle A'B'C' = 2\pi - (\frac{\pi}{3} + \beta) - (\frac{\pi}{3} + \gamma) - (\pi - \alpha - \beta) = \frac{\pi}{3}.$$

The uniqueness method

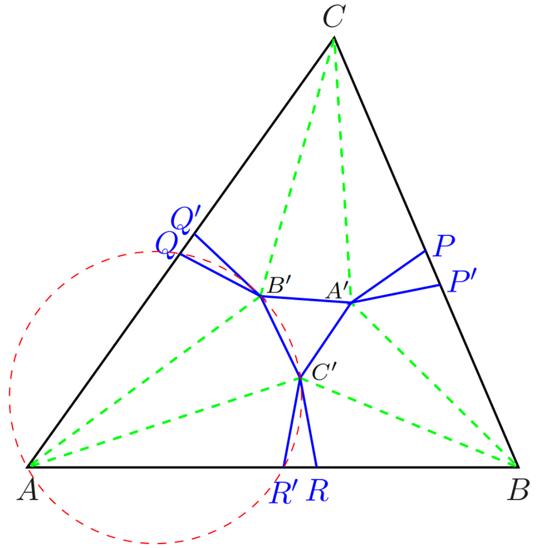
give a purely plane geometric proof

starting with an equilateral triangle $\triangle A'B'C'$ with side length 1,

P, Q, R, P', Q', R' such that

$$\angle B'C'R' = \angle QB'C' = \pi - 2\alpha,$$
$$\angle C'A'P' = \angle RC'A' = \pi - 2\beta,$$
$$\angle A'B'Q' = \angle PA'B' = \pi - 2\gamma,$$

$$A'P = A'P' = B'Q = B'Q' = C'R = C'R' = 1.$$



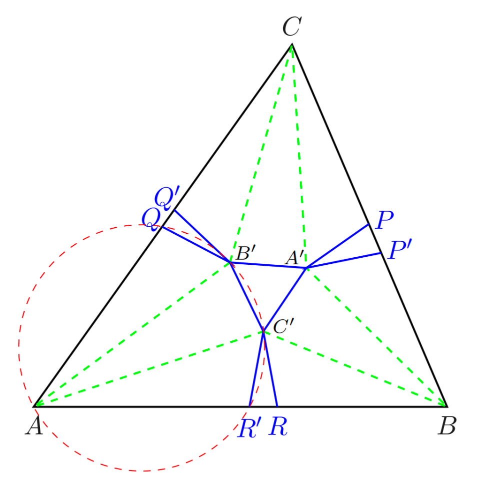
Connecting PP', QQ' and RR' we shall construct $\triangle ABC$.

We need to prove :
∠A = 3α, ∠B = 3β, ∠C = 3γ,
and AB′, AC′, BC′, BA′, CA′, CB′are the corresponding angle trisectors

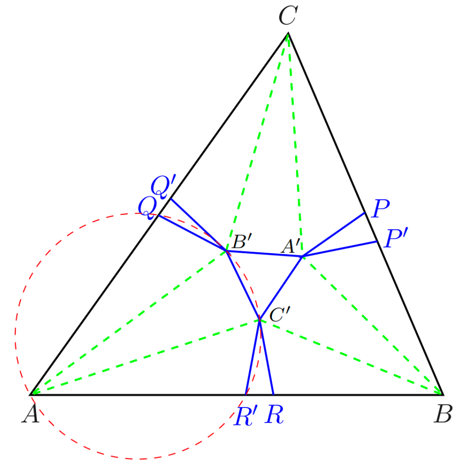
We compute

$$\angle RC'R' = 2\pi - (\pi - 2\alpha) - (\pi - 2\beta) - \pi/3 = \pi/3 - 2\gamma$$

Since $\triangle C'R'R$ is isosceles, we conclude that $\angle C'R'R = \pi/3 + \gamma$. Similarly, $\angle CQB' = \pi/3 + \beta$. Thus, from the fact that the summation of the angles of the pentagon $AQB'C'R'$ is 3π, we conclude that $\angle A = 3\alpha$.



By the construction, the quadrilateral $QB'C'R'$ is an isosceles trapezoid. Thus it must be con-cyclic. The point A must be on the circle because if A is outside the circle, we must have $\angle B'AB < 2\alpha$, and $\angle CAB' < \alpha$; and if A is inside the circle, we have the reverse inequalities, which is a contradiction to the fact that $\angle A = 3\alpha$. Thus the green lines are angle trisectors and this completes the proof.



THANKS!