# Lemoine Circles

Zhiqin Lu, zlu@uci.edu

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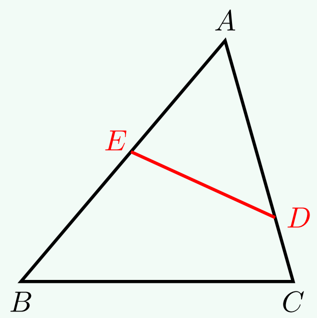
## 1 Preliminaries

There are two preliminary concepts that are closely related to the Lemoine Circles. The first of which is the *antiparallel line*.

**Definition 1**

*In the following $\triangle ABC$, let $D, E$ be points on $AC, AB$, respectively. The segment $DE$ is called an antiparallel line, if $\angle ADE = \angle B$.*




**Remark** The terminology "antiparallel" comes from the fact that $\angle AED$ and $\angle B$ are the corresponding angles, and if $DE$ were parallel to $BC$, then they could have been equal. We call $\angle ADE$ and $\angle B$ a pair of *anti-corresponding angles*. When anti-corresponding angles are equal, then two lines are antiparallel.

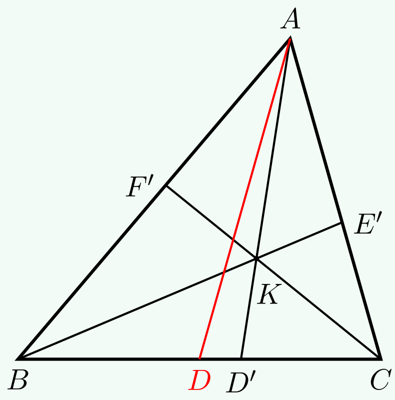
When $DE$ is antiparallel to $BC$, then $D, E, B, C$ is con-cyclic. Also, two line segments which are antiparallel to a third line, then they must be parallel.

Note that unlike the concept of parallel line, antiparallel line is only defined with respect to a reference triangle.

The second related concept is the *Symmedian Point*, which is also known as the *Lemoine Point* or the *Grebe Point*.

**Definition 2**

*In $\triangle ABC$, let $AD$ be the median on $BC$ (that is, $BD = DC$). Line $AD'$ is called a symmedian line on $BC$, if $AD'$ is the isogonal line of $AD$, that is $\angle BAD = \angle CAD'$. Three symmedian lines $AD', BE', CF'$ are concurrent at the point $K$, which is called the symmedian point of the triangle.*




**Remark** Symmedian line can also be characterized as follows. Let $AD'$ be the symmedian line on $BC$. Then

$$\frac{BD'}{D'C} = \frac{AB^2}{AC^2}.$$

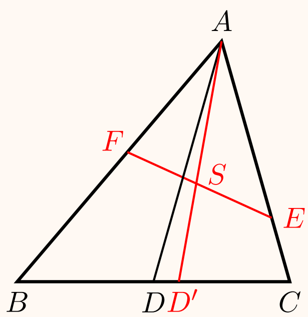
One can prove this property by using the Law of Sines. It then follows from the Ceva Theorem that the three symmedian lines are concurrent.

Symmedian point is one of the most important triangle centers. More about symmedian lines and symmedian point can be found at Wikipedia, and also, Topic 16.

The following is a simple but useful relation between antiparallel line and symmedian line.

**Theorem 1**

*In $\triangle ABC$, assume that $EF$ is an antiparallel line and $AD'$ is a symmedian line. $EF$ and $AD'$ intersect at $S$. Then $S$ is the midpoint of $DE$, in other words, $AS$ is the median of $\triangle AEF$ on $EF$.*
*Conversely, if $AS$ is the median of $\triangle AEF$, then $EF$ is antiparallel to $BC$.*




**Proof:** By assumption, $\triangle AEF \sim \triangle ABC$. Since $\angle BAD = \angle CAD'$, we conclude that $\triangle AES \sim \triangle ABD$. Thus $FS = SE$.

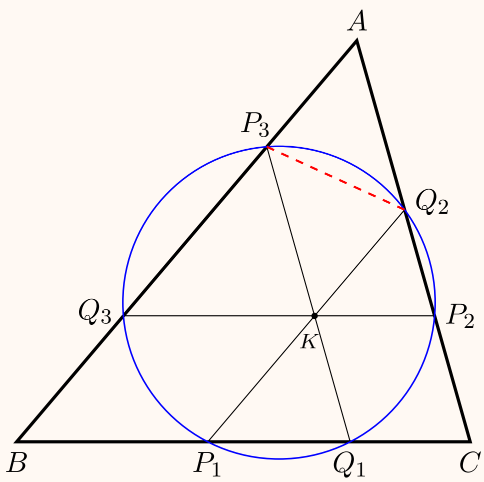
We can use the uniqueness to prove the inverse theorem. If $EF$ is not antiparallel, we can draw an antiparallel line $E'F'$ passing $S$. Then $EF$, $E'F'$ bisect mutually and hence $E'EF'F$ is a parallelogram, which is not possible since $AB$ is not parallel to $AC$. ■

## 2 The Lemoine Circles and Their Properties

Émile Michel Hyacinthe Lemoine (1840-1912) is a French geometer and an civil engineer. In 1873, at the meeting of the *Association Francaise pour l'Avancement des Sciences* held in Lyons, Lemoine presented a paper entitled *Sur quelues proprié tés d'un point remarquable du triangle*. In that paper he called attention to the point of intersection of the symmedians of a triangle and described some of its more important properties. He also introduced the special circles named for him.

**Theorem 2. (The First Lemoine Circle)**

*Let $K$ be the symmedian point of $\triangle ABC$. Let $P_2Q_3, P_3Q_1, P_1Q_2$ be parallel lines to $BC, CA, AB$, respectively. Then $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are con-cyclic.*




*The circle is known as the First Lemoine Circle.*

**Proof:** We draw $AK$ and $P_3Q_2$. By assumption, $AP_3KQ_2$ is a parallelogram. Thus $AK$ and $P_3Q_2$ bisect mutually. By Theorem 1, $P_3Q_2$ is an antiparallel line. Since $P_2Q_3 \parallel BC$, $P_3Q_2$ is antiparallel to $P_2Q_3$. Thus $P_2, Q_2, P_3, Q_3$ are con-cyclic. Similarly, $P_1, Q_1, P_3, Q_3$ and $P_1, Q_1, P_2, Q_2$ are con-cyclic respectively. By the Davies Theorem below, we conclude that these six points are con-cyclic. ■

**Theorem 3. (Davies Theorem)**

*On $\triangle ABC$, let $P_1, Q_1$ be two points on $BC$; $P_2, Q_2$ be two points on $CA$; and $P_3, Q_3$ be two points on $AB$. Assume that for any $1 \leq i, j \leq 3$, $i \neq j$, $P_i, Q_i, P_j, Q_j$ are con-cyclic.*

*Then these six points $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are con-cyclic.*

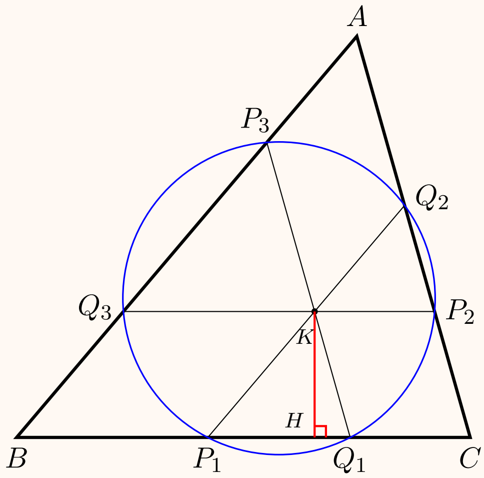
For a proof of the theorem, see Topic 28.

**Theorem 4**

*Let $P_iQ_i$ (i=1,2,3) be the chords of the First Lemoine Circle made from the three sides of the triangle $\triangle ABC$. Then*

$$P_1Q_1 : P_2Q_2 : P_3Q_3 = BC^3 : CA^3 : AB^3.$$

*In other words, the lengths of the chords are proportional to the cubic of the three sides of the triangle. Because of this, the First Lemoine Circle is also called Triplicate-ratio Circle.*




**Proof:** Let $BC = a, CA = b, AB = c$. By Topic 16, the distance of the symmedian point to each side is proportional to the length of side. Thus $KH = ka$. Since $\triangle KP_1Q_1 \sim \triangle ABC$, we must have

$$\frac{P_1Q_1}{BC} = \frac{KH}{h_1},$$

where $h_1$ is the height of $\triangle ABC$ on $BC$. Let $S$ be the area of $\triangle ABC$. Then $h_1 = 2S/a$.Therefore

$$P_1Q_1 = 2kSa^3.$$

Thus

$$P_1Q_1 : P_2Q_2 : P_3Q_3 = a^3 : b^3 : c^3.$$

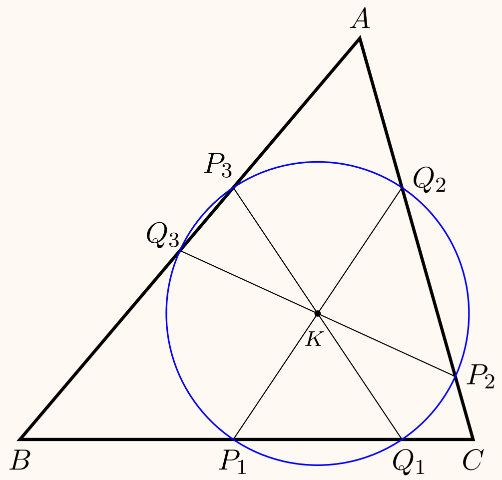
■

**Theorem 5. (The Second Lemoine Circle)**

*Let $K$ be the symmedian point of $\triangle ABC$. Let $P_2Q_3, P_3Q_1, P_1Q_2$ be antiparallel lines to $BC, CA, AB$, respectively. Then $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are con-cyclic.*
*The circle is known as the Second Lemoine Circle. Moreover, we have*

$$P_1Q_1 : P_2Q_2 : P_3Q_3 = \cos A : \cos B : \cos C.$$

*Therefore the Second Lemoine Circle is also called Cosine Circle.*




**Proof:** Since $AK$ is a symmedian line and $P_2Q_3$ is antiparallel, by Theorem 1, $KP_2 = KQ_3$. Similarly, $KP_3 = KQ_1$, and $KP_1 = KQ_2$.

On the other hand, we have $\angle P_3Q_3K = \angle C = \angle Q_3P_3K$. Therefore $\triangle KP_3Q_3$ is an isosceles triangle. Similarly, $\triangle KP_1Q_1$ and $KP_2Q_2$ are isosceles triangles. Summarizing, we have

$$KP_1 = KP_2 = KP_3 = KQ_1 = KQ_2 = KQ_3.$$

In particular, the six points $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are con-cyclic with $K$ as the center of the circle.

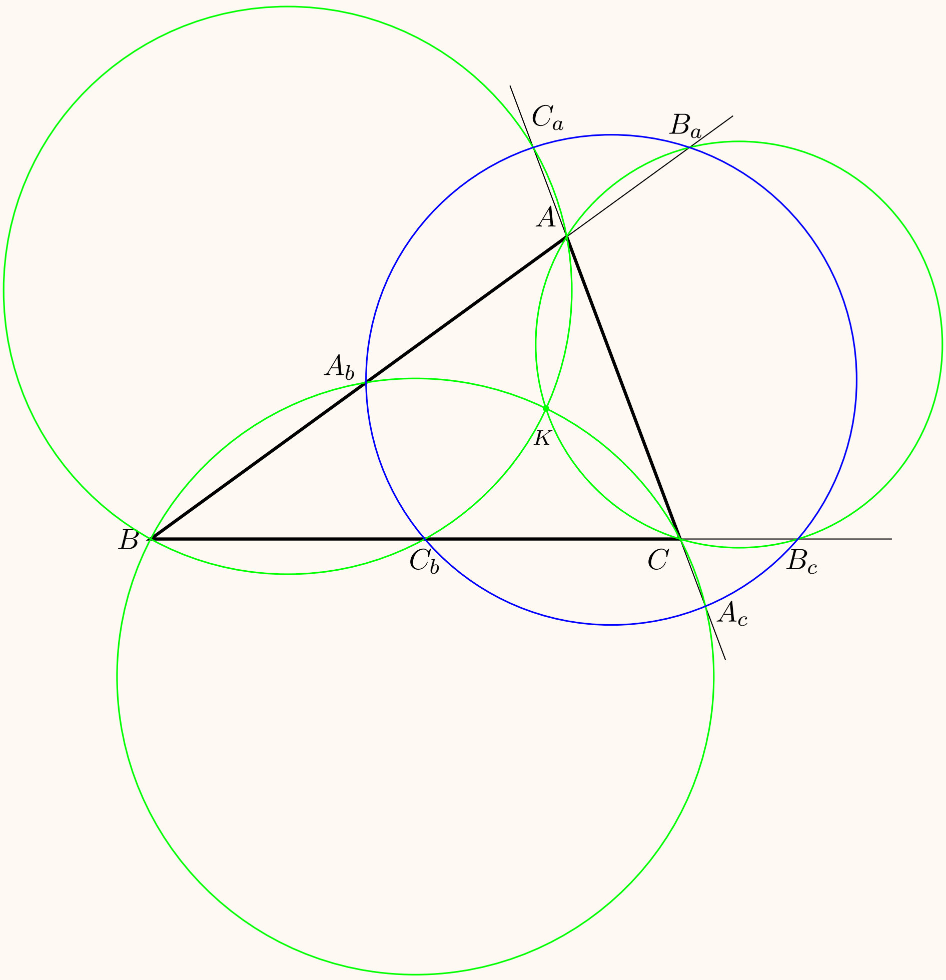
Let $r$ be the radius of the Second Lemoine Circle. Then $P_1Q_1 = 2r\cos A$, $P_2Q_2 = 2r\cos B$, and $P_3Q_3 = 2r\cos C$, which justifies the name of *Cosine Circle*.

In 2002, Jean-Pierre Ehrmann defined the so-call the *Third Lemoine Circle*. See the paper of Darij Grinberg for details.

**Theorem 6. (The Third Lemoine Circle)**

*Let $K$ be the symmedian point of $\triangle ABC$. Let $A_b, A_c$ be the second intersection points of the circumscribed circle of $\triangle KBC$ with $AB, AC$, respectively; let $B_a, B_c$ be the second intersection points of the circumscribed circle of $\triangle KCA$ with $BA, BC$, respectively; and let $C_a, C_b$ be the second intersection points of the circumscribed circle of $\triangle KAB$ with*

*$CA, CB$, respectively. Then the six points $A_b, A_c, B_a, B_c, C_a, C_b$ are con-cyclic.*

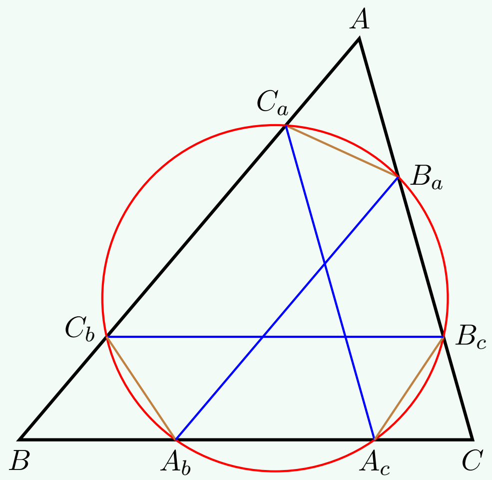



**Remark** In the mathoverflow, the *Lemoine-Lozada Circles* are discussed, see here. People proposed to call it the *Fourth Lemoine Circle*. However, the circle doesn't belong to the family of Tucker Circles (see below).

## 3 Tucker Circles

**Definition 3. (Tucker Hexagon and Tucker Circles)**

*In $\triangle ABC$, let $A_c$ be a point on $BC$. Let $A_cB_c$ be antiparallel to $AB$; let $B_cC_b$ be parallel to $BC$; let $C_bA_b$ be antiparallel to $AC$; let $A_bB_a$ to be parallel to $AB$; let $B_aC_a$ be antiparallel to $BC$. Then $C_aA_c$ must be parallel to $CA$. The hexagon $A_cB_cC_bA_bB_aC_a$ is called the Tucker Hexagon. Moreover, the hexagon is inscribed in a circle, which is called the Tucker Circle.*




Since $A_c$ is an arbitrary point on $BC$, we in fact get a family of circles, called *Tucker Circles*. The First, Second and Third Lemoine Circles belong to the Tucker Circles. For more details on Tucker Circles, see Topic 29.