Viviani’s Theorem

Yiding Huang
Math199b
Professor Lu
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Contents

Vincenzo Viviani

- Vincenzo Viviani (April 5, 1622 – September 22, 1703) was an Italian mathematician and scientist. He was a pupil of Torricelli and a disciple of Galileo.
- Assitant for Torricelli experiment

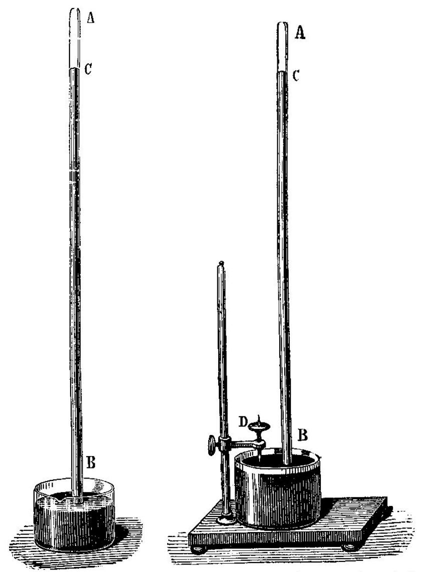
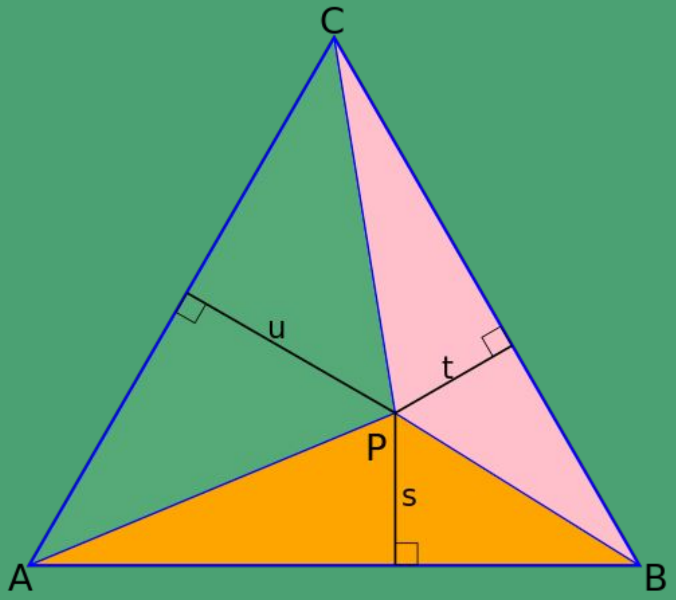


Fig. 275. — Baromètre à cuvette.

Fig. 276. — Baromètre à cuvette et sa pointe.

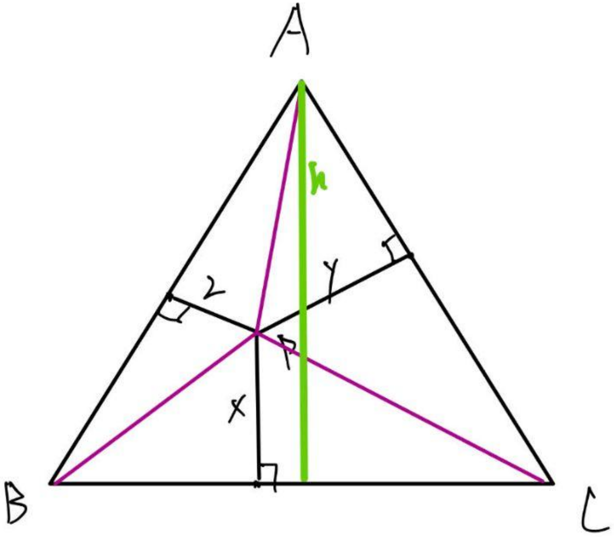
Viviani's Theorem

-The sum of the distances from any interior point to the sides of an equilateral triangle equals the length of the triangle's altitude



u+t+s=h

Proof



$$① \ \triangle ABC = \frac{AB \cdot z}{2} + \frac{AC \cdot y}{2} + \frac{BC \cdot x}{2}$$

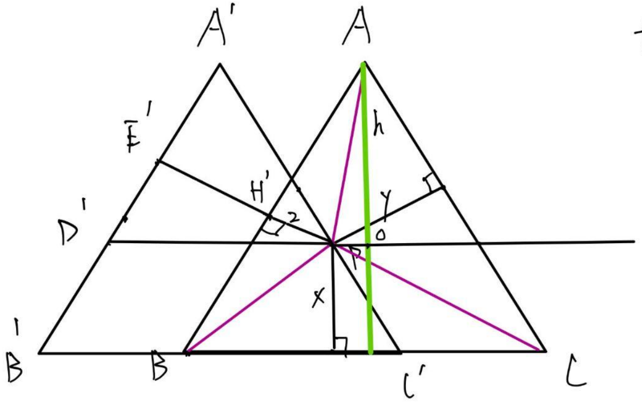
$$= \frac{1}{2} \cdot BC \cdot (x+y+z)$$

$$② \ \triangle ABC = \frac{1}{2} BC \cdot h$$

from ① and ②

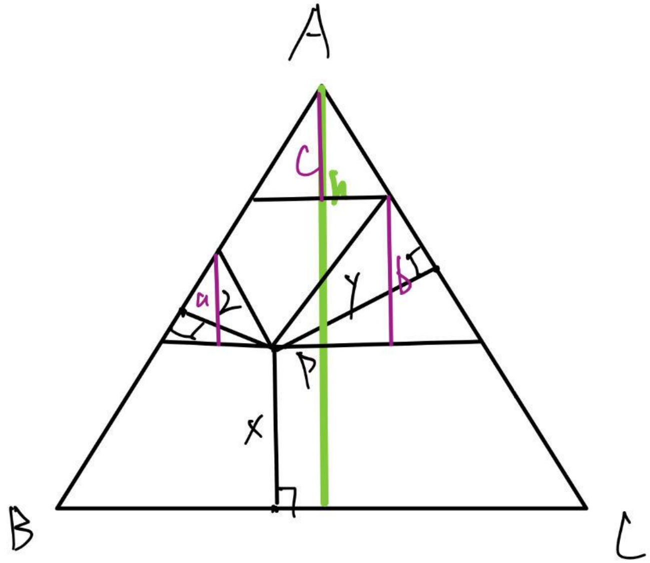
$$x+y+z = h$$

Proof



Let P lie on $\triangle A'B'C'$ which is shifted from $\triangle ABC$. Extend PH' to $A'B'$ at E'. $D'P \parallel B'C'$.

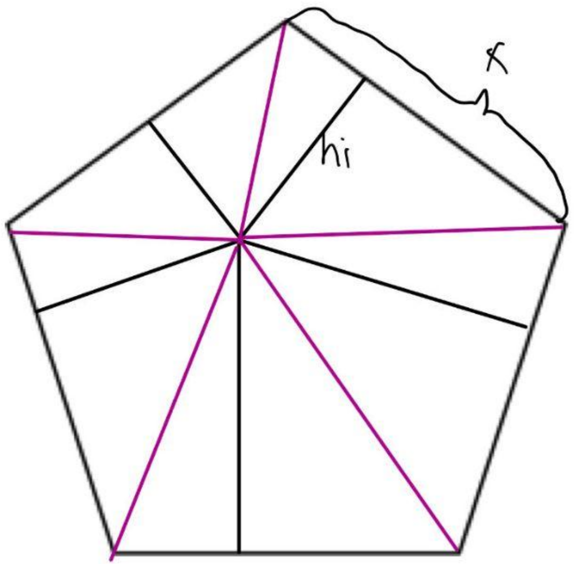
$$\begin{aligned} h &= x + AO \\ &= x + z + E'H' \\ &= x + z + y \end{aligned}$$



Generalization

-The sum of the distances from any point P inside a regular polygon to the sides of the polygon is independent of the location of P.

Proof

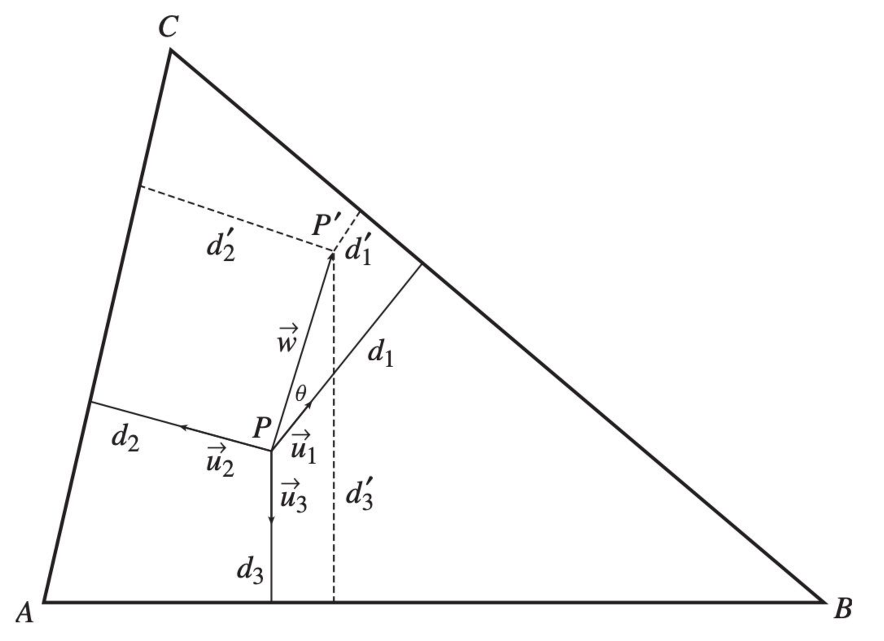


$$S = \sum \frac{1}{2} x \cdot h_i$$

$$= \frac{1}{2} x \sum h_i$$

The converse of Viviani’s Theorem

If, inside ABC, there is a circular region R for which the sum of the distances from a point P in R to the three sides of the triangle is independent of the position of P, then ABC is equilateral.

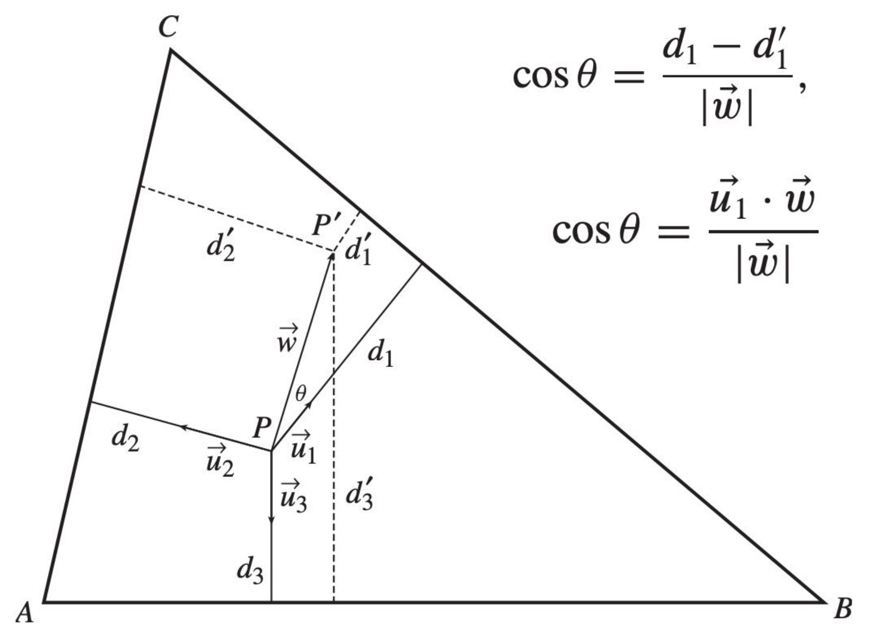


Let P be a point in R, and let u1, u2, and u3 be the unit vectors from P perpendicular to the sides of the triangle (see figure).

Want to show u = u1 + u2 + u3=0.

We prove this by contradiction
Assume the vector u is not 0.
There exist PP' parallel to u denoted by w, and ceta is the angle between w and u1.

d1+d2+d3=d'1+d'2+d'3



Hence, $u_1 \cdot w = d_1 - d'_1$; and by symmetry, $u_2 \cdot w = d_2 - d'_2$ and $u_3 \cdot w = d_3 - d'_3$.

Therefore, $u \cdot w = 0$, and since these two vectors are parallel, it must be that $|u| = 0$, a contradiction.

$u_1 + u_2 + u_3 = 0$.

$(u_1+u_2)^2 = (-u_3)^2$
$|u_1|^2+|u_2|^2+2u_1u_2=1$

It is now straightforward to show that $u_1 \cdot u_2 = u_2 \cdot u_3 = u_3 \cdot u_1 = -1/2$. Consequently, the angle between any pair of these vectors is 2π 3. Therefore the triangl is equilateral.

Thanks