Trilinear Coordinate System

Natasha Xiao[1], jxiao12@uci.edu

(last updated: April 8, 2022)

1 Introduction

Given a fixed triangle, we can define a coordinate system called *trilinear coordinate system* to the Euclidean plane – with respect to the triangle. Such a coordinate system is useful in triangle geometry because it adapts the shape of the given triangle.

Trilinear coordinate system was introduced by *Plücker* in 1835. The point on the Euclidean plane is represented by homogeneous coordinates $(x : y : z)$ or (x, y, z), where the latter has different meaning than the homogeneous coordinates in projective geometry.

Before introducing the trilinear coordinate system, we first need to introduce the concept of *signed distance*, which is a generalization of the distance from a point to a line.

Definition 1. (Signed Distance)

Let

$$px + qy + r = 0$$

represent a line L in the Euclidean plane. Let $P = (x_0, y_0)$ be a point. Then the signed distance of P to the line L is defined to be

$$\mathrm{dist}(P, L) = \frac{px_0 + qy_0 + r}{\sqrt{p^2 + q^2}}.$$

We make two remarks on the signed distance. First, $|\mathrm{dist}(P, L)|$ is the distance of the point to the line by analytic geometry. Second, the signed distance depends not only on the point and the line, but also on the *orientation* of the line. Both $ax + by + c = 0$ and $-ax - by - c = 0$ represent the same line, but the corresponding signed distances differ by a negative sign.

Definition 2. (Trilinear Coordinate System)

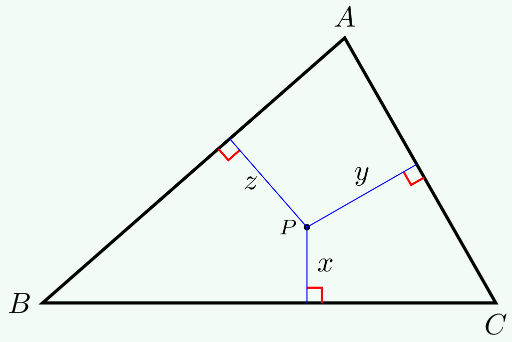
Let $\triangle ABC$ be a fixed triangle. We shall choose the orientations of the lines BC, CA, AB so that any point in the interior of the triangle would have positive signed distances to all three sides.

Let P be any point in the plane. The trilinear coordinates of P are the ratios of its signed distances to the three sides BC, CA, AB.

In the following picture, P is an interior point of $\triangle ABC$. Let (x, y, z) be the distances

[1]The author thanks Dr. Zhiqin Lu for his help. She also thanks Stephanie Wang for her careful reading and many suggestions.

(=signed distances) of P to BC, CA, AB, respectively. Then, the trilinear coordinates of P is given by (x, y, z), or $(x : y : z)$.



2 Basic Properties of the Trilinear Coordinate System

The definition in the last section leaves a theoretical question whether trilinear coordinates are well-defined or not.

Theorem 1

Trilinear coordinate system is indeed a coordinate system. For homogeneous coordinates (x, y, z) and (x_1, y_1, z_1), if there is a non-zero number k such that $x = kx_1, y = ky_1, z = kz_1$, then (x, y, z) and (x_1, y_1, z_1) represent the same point.

Proof. We use the picture in Definition 2. Let a, b, c be the lengths of BC, CA, AB, respectively. Let $\tilde{x}, \tilde{y}, \tilde{z}$ be the signed distances to BC, CA, AB, respectively. We claim that

$$a\tilde{x} + b\tilde{y} + c\tilde{z} = 2\Delta, \tag{1}$$

where Δ is the area of $\triangle ABC$.

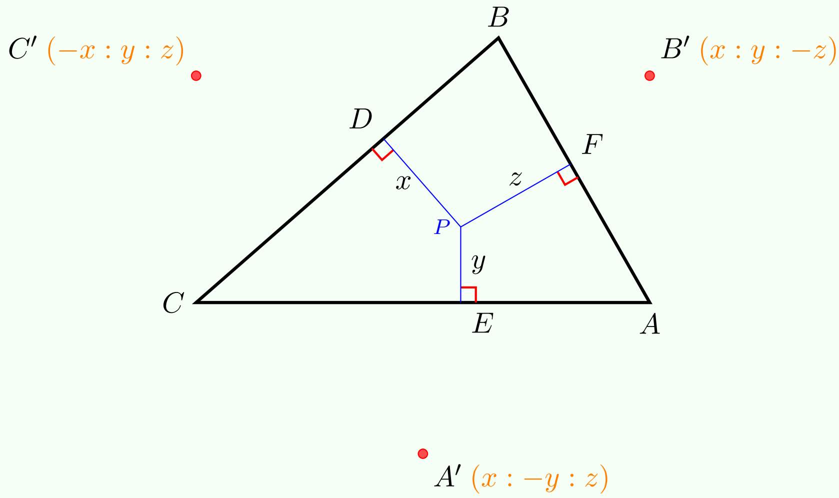
To prove the above identity, we observe that the signed distances $\tilde{x}, \tilde{y}, \tilde{z}$ are linear functions of P, hence so is $a\tilde{x} + b\tilde{y} + c\tilde{z}$. When P is an interior point of $\triangle ABC$, we then have

$$a\tilde{x} + b\tilde{y} + c\tilde{z} = 2S_{\triangle PBC} + 2S_{\triangle PCA} + 2S_{\triangle PAB} = 2\Delta.$$

Therefore, such a linear function must be equal to the constant 2Δ. This proves the claim.

By (1), we conclude that if $(x, y, z) = k(\tilde{x}, \tilde{y}, \tilde{z})$, then k must be positive. Thus (x, y, z) and (x_1, y_1, z_1) represent the same point. ∎

Remark It is impossible for all three trilinear coordinates to be non-positive.



- The coordinates of vertices A, B, and C of the triangle are $(1 : 0 : 0)$, $(0 : 1 : 0)$, and $(0 : 0 : 1)$, respectively.
- The **positive** and **negative** signs of the *trilinear coordinate* components can be determined according to the following rules:
  - The component x of a point P is positive if it is on the same side of the edge BC as vertex A, and negative when P and vertex A are on opposite sides of BC.
  - The component y of a point P is positive if it is on the same side of the edge CA as vertex B, and negative when P and vertex B are on opposite sides of CA.
  - The component z of a point P is positive if it is on the same side of the edge AB as vertex C, and negative when P and vertex C are on opposite sides of AB.

External Link. *Here is the introduction of the Trilinear Coordinate System in Wikipedia.*

3 The Trilinear Coordinates of the Five Triangle Centers

Theorem 2

Given a triangle $\triangle ABC$ with three vertices A, B, and C, each vertex corresponds to its opposite side length a, b and c, respectively.

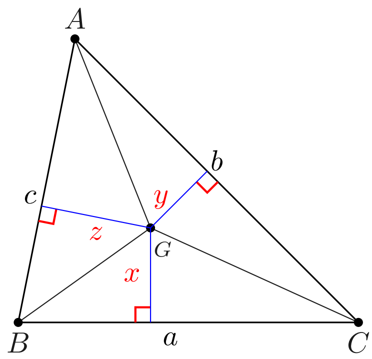
The trilinear coordinates of the five triangle centers:

- *The trilinear coordinates for the incenter is $(1 : 1 : 1)$; and three excenters are $(-1 : 1 : 1)$, $(1 : -1 : 1)$, and $(1 : 1 : -1)$, respectively.*
- *The trilinear coordinates for the centroid is $(\csc(A) : \csc(B) : \csc(C))$ or $(a^{-1} : b^{-1} : c^{-1})$.*
- *The trilinear coordinates for the circumcenter is $(\cos(A) : \cos(B) : \cos(C))$.*
- *The trilinear coordinates for the orthocenter is $(\sec(A) : \sec(B) : \sec(C))$.*

Proof. It is well known that the (signed) distances of the incenter to the three sides are equal. Therefore the trilinear coordinates of the incenter is $(1, 1, 1)$.

Similarly, the distances of any excenters to the three sides are equal as well. However, one of the signed distances is negative. Thus the coordinates of the excenters are $(-1 : 1 : 1)$, $(1 : -1 : 1)$, and $(1 : 1 : -1)$, respectively.

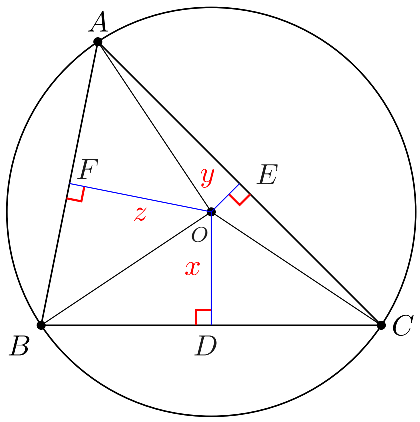
Let G be the centroid of $\triangle ABC$ in the following picture. Let x, y, z be the distances to the three sides BC, CA and AB, respectively. By the property of the centroid, the area of $\triangle BGC$ = the area of $\triangle AGC$ = the area of $\triangle AGB$, which implies $ax = by = cz$.



Therefore, the trilinear coordinate for the centroid of $\triangle ABC$ is

$$(x : y : z) = (a^{-1} : b^{-1} : c^{-1}) = (\csc(A) : \csc(B) : \csc(C)).$$

Now we turn to the circumcenter. In the following picture, let O be the circumcenter.



By definition, the circumcenter is the center of the circumcircle, which is the circle passing through the three vertices $A, B,$ and C. Since $\angle BAC$ is an inscribed angle, we have

$$\angle A = \frac{1}{2}\angle BOC = \angle BOD.$$

Therefore,

$$x = R\cos A,$$

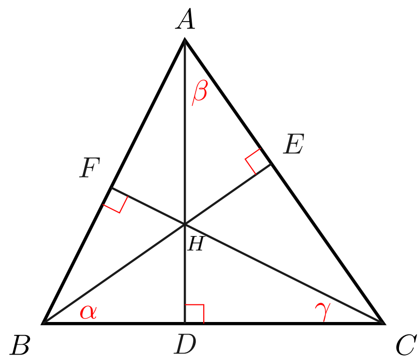
where R is the radius of the circumcircle. Similarly, we have

$$y = R\cos B, \quad z = R\cos C.$$

Thus, the *trilinear coordinates* of the circumcenter of $\triangle ABC$ are

$$(\cos A : \cos B : \cos C).$$

Finally, we turn to the orthocenter. In the following picture, let H be the orthocenter of $\triangle ABC$.



Let $\angle EBC = \alpha$, $\angle CAD = \beta$ and $\angle FCB = \gamma$. Then

$$\begin{aligned} HD &= BD\tan\alpha = BD\cot C \\ &= AB\cos B\cot C = \frac{2R\cos B\cos C\cos A}{\cos A} \\ &= 2R\cos B\cos C. \end{aligned}$$

Similarly, we have

$$HE = 2R\cos A\cos C, \quad FH = 2R\cos A\cos B$$

Therefore, the *trilinear coordinates* of H are

$$(R\cos B\cos C : R\cos A\cos C : R\cos A\cos B) = (\sec A : \sec B : \sec C).$$

4 Converting Between Cartesian and Trilinear Coordinates

In this section, we prove a transformation formula between *Trilinear coordinates* and *Cartesian coordinates*.

Theorem 3

Let a, b, c be the side lengths of $\triangle ABC$, respectively, and let $(x : y : z)$ be the trilinear coordinates of a point P. Let $\vec{A}, \vec{B}, \vec{C}$ be the vector representations of vertexes A, B, C, *respectively. Then we have*

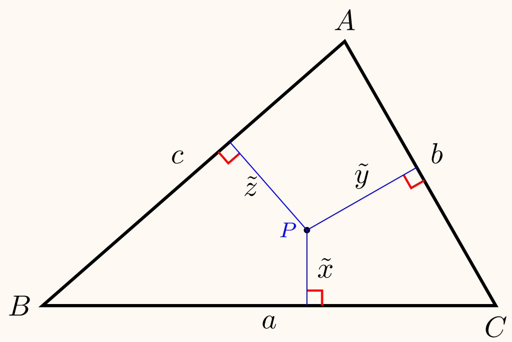
$$\vec{P} = \frac{ax}{ax+by+cz}\vec{A} + \frac{by}{ax+by+cz}\vec{B} + \frac{cz}{ax+by+cz}\vec{C},$$

where $\vec{P}$ is the vector representation of P. Conversely, let

$$\vec{P} = k_1\vec{A} + k_2\vec{B} + k_3\vec{C},$$

where k_1, k_2, k_3 are real numbers such that $k_1 + k_2 + k_3 = 1$. Then the trilinear coordinates of P are given by

$$(a^{-1}k_1 : b^{-1}k_2, c^{-1}k_3).$$



Proof. We make the following observation regarding to the signed distance. Let ℓ be the equation of a line L and let P be a point. Then by Definition 1, the signed distance of P to the line is given by $c(\ell)\ell(\vec{P})$, where $c(\ell)$ is a constant depending only on ℓ.

Let $\tilde{x}, \tilde{y}, \tilde{z}$ be the signed distances of P to BC, CA and AB, respectively, and let $\ell_{BC} = 0$ be the equations of the line BC. Then

$$\tilde{x} = c(\ell_{BC})\ell_{BC}(\vec{P}).$$

Writing $\vec{P} = k_1\vec{A} + k_2\vec{B} + k_3\vec{C}$, with $k_1 + k_2 + k_3 = 1$, we have

$$\tilde{x} = k_1 c(\ell_{BC})\ell_{BC}(\vec{A})$$

because $\ell_{BC}(\vec{B}) = \ell_{BC}(\vec{C}) = 0$. Let h_1 be the height of $\triangle ABC$ over the side BC. Then $h_1 = c(\ell_{BC})\ell_{BC}(\vec{A})$ and hence $\tilde{x} = k_1h_1$. Similarly, we have $\tilde{y} = k_2h_2$ and $\tilde{z} = k_3h_3$, where h_2, h_3 are the heights over CA, AB, respectively.

Let $(x : y : z)$ be the trilinear coordinates of P. Then

$$\frac{ax}{ax+by+cz} = \frac{a\tilde{x}}{a\tilde{x}+b\tilde{y}+c\tilde{z}}.$$

Since $a\tilde{x} + b\tilde{y} + c\tilde{z} = 2\Delta$ by (1), we have

$$\frac{ax}{ax+by+cz} = k_1.$$

Similarly, we have

$$\frac{by}{ax+by+cz} = k_2, \quad \frac{cz}{ax+by+cz} = k_3.$$

This proves the first part of the theorem. For the second part, we just need to observe that

$$h_1 : h_2 : h_3 = a^{-1} : b^{-1} : c^{-1}.$$

Thus

$$(\tilde{x} : \tilde{y} : \tilde{z}) = (k_1h_1 : k_2h_2 : k_3h_3) = (a^{-1}k_1 : b^{-1}k_2, c^{-1}k_3).$$

5 Example

Theorem 4. (Euler Line)

In any non-equilateral triangle, the orthocenter, the centroid, and the circumcenter are collinear. The line is called the Euler Line.

Proof. We first remark that three points P_i $(i = 1, 2, 3)$ with trilinear coordinates $(x_i : y_i : z_i)$ $(i = 1, 2, 3)$, respectively, are collinear if and only if

$$\det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = 0.$$

By Theorem 2, in order to prove the theorem, we only need to prove

$$\det \begin{bmatrix} \sec(A) & \sec(B) & \sec(C) \\ \csc(A) & \csc(B) & \csc(C) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix} = 0.$$

Multiplying the 1st row by $(\cos A \cos B \cos C)$ and the 2nd row by $(\sin A \sin B \sin C)$, we get

$$\sigma \overset{def}{=} \det \begin{bmatrix} \sin(B)\sin(C) & \sin(C)\sin(A) & \sin(A)\sin(B) \\ \cos(B)\cos(C) & \cos(C)\cos(A) & \cos(A)\cos(B) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix}.$$

Subtracting the first row by the second row, we get

$$\sigma = \begin{bmatrix} -\cos(B+C) & -\cos(C+A) & -\cos(A+B) \\ \cos(B)\cos(C) & \cos(C)\cos(A) & \cos(A)\cos(B) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix}.$$

Since $\angle A + \angle B + \angle C = 180°$, we have

$$\cos(A) = -\cos(B+C), \cos(B) = -\cos(C+A), \cos(C) = -\cos(A+B).$$

Thus

$$\sigma = \begin{bmatrix} \cos(A) & \cos(B) & \cos(C) \\ \cos(B)\cos(C) & \cos(C)\cos(A) & \cos(A)\cos(B) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix} = 0,$$

completing the proof of the theorem.