# Nine-Point Circle

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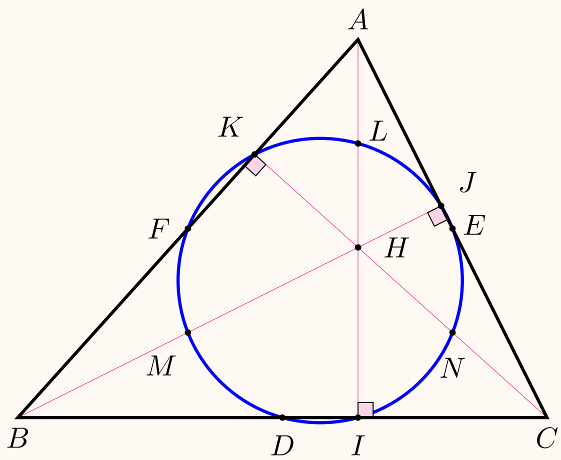
The *Nine-Point Circle* of any triangle is defined by these nine concyclic points:

- the 3 midpoints of each side of the triangle;
- the 3 feet of each altitude;
- the 3 midpoints of the line segments between the orthocenter and each vertex of the triangle.

The discovery of the nine-point circle is credited to *Karl Wilhelm Feuerbach*, though he only recognized that the 3 midpoints of the side lengths and the 3 feet of the altitudes were concyclic. The nine-point circle is also known as *Feuerbach's Circle*, *Euler's Circle*, and *Terquem's Circle*.

**Theorem 1. (Nine-Point Circle Theorem)**

*Consider any triangle $\triangle ABC$ and these nine points: the 3 midpoints of each of its sides, the 3 feet of each altitude, and the 3 midpoints of the line segments between the orthocenter and each vertex. These nine points are concyclic and they form the nine-point circle of $\triangle ABC$.*




Before proving this theorem, we need some preparation.

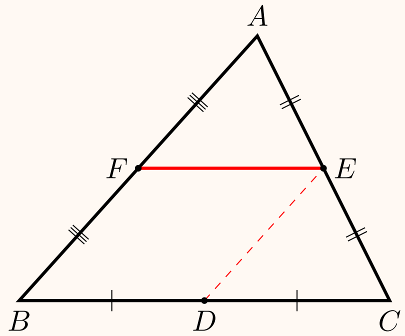
**Definition 1. (Midline)**

*A midline of a triangle is a line segment whose endpoints are the midpoints of two sides of the triangle.*

[1,2]The authors thank Dr. Zhiqin Lu for his help.

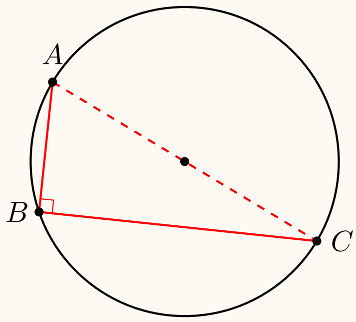
**Theorem 2. (The Midline Theorem)**

*The segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length of the third side.*




**Theorem 3. (Thale's Theorem)**

*If A, B, and C are distinct points on a circle where the line $\overline{AC}$ is a diameter, then the angle $\angle ABC$ is a right angle.*

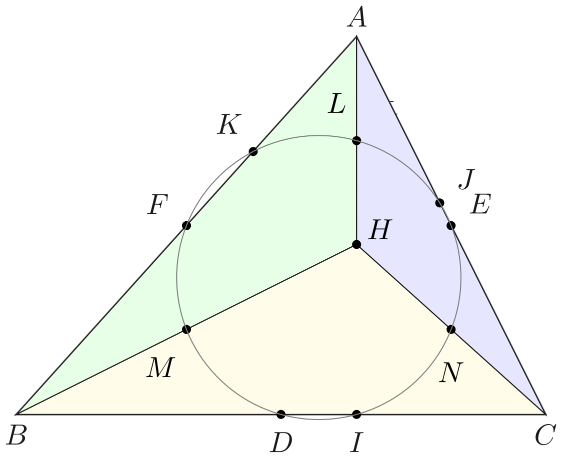



**External Link.** *Further readings on the Midline Theorem and Thale's Theorem.*

**Proof of Feuerbach's Theorem (Theorem 1).** First, we let:

- $D, E, F$ be midpoints of $\overline{BC}, \overline{CA}, \overline{AB}$, respectively;
- $I, J, K$ be feet of altitudes from $A, B, C$, respectively;
- $L, M, N$ be the midpoints between the orthocenter and the vertices $A, B, C$, respectively ;
- $H$ be the intersection point between three altitudes, or the orthocenter of $\triangle ABC$.

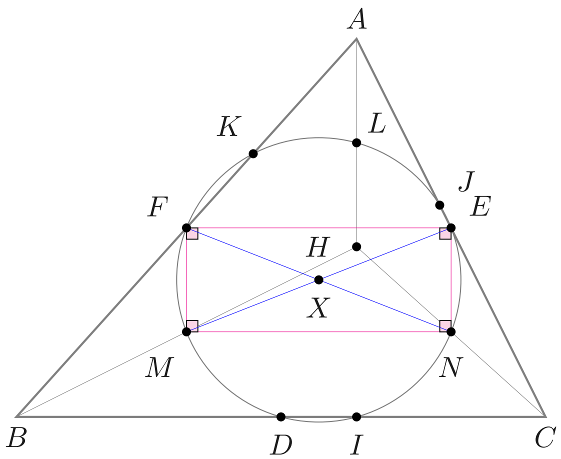
To prove the Nine-Point Circle Theorem, we show that the nine points have equal distance from the circle's center. This can be done by dividing $\triangle ABC$ into smaller triangles $\triangle AHB, \triangle CHA$, and $\triangle BHC$ and analyze them individually, as drawn below:



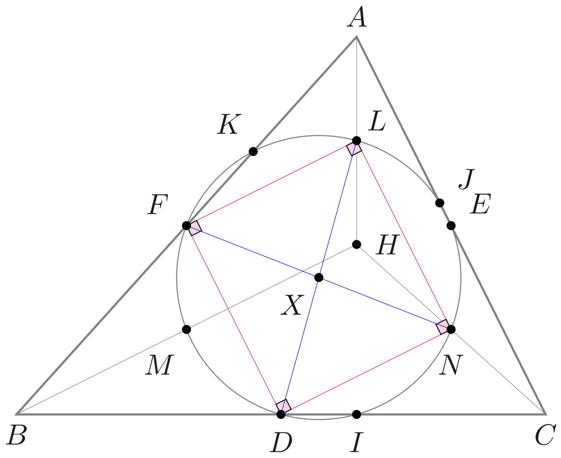

By definition, the line segment $\overline{FM}$ is a midline of $\triangle AHB$, $\overline{EN}$ is a midline of $\triangle AHC$, $\overline{MN}$ is a midline of $\triangle BHC$, and $\overline{FE}$ is a midline of $\triangle ABC$. Thus, we can make the following conclusions using Midline Theorem:

- $\overline{FM} \parallel \overline{AH} \parallel \overline{EN}$ and $\overline{FM} = \frac{1}{2}\overline{AH} = \overline{EN}$;
- $\overline{MN} \parallel \overline{BC} \parallel \overline{FE}$ and $\overline{MN} = \frac{1}{2}\overline{BC} = \overline{FE}$.

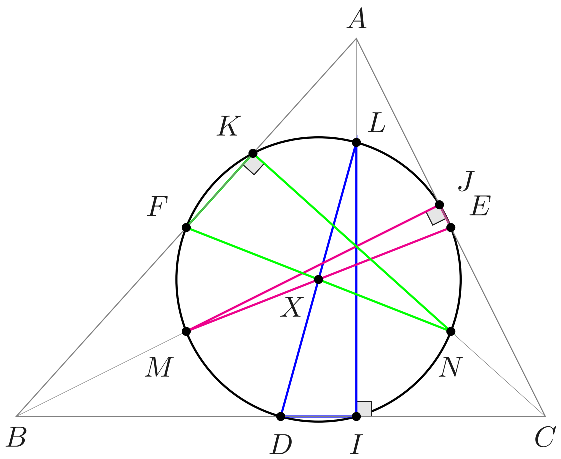
Finally, since $\overline{AI}$ is an altitude of $\triangle ABC$, we have $\overline{AH} \perp \overline{BC}$, which implies $(\overline{FM} \parallel \overline{EN}) \perp (\overline{FE} \parallel \overline{MN})$. Hence, $EFMN$ generates a rectangle with two diagonals intersecting at $X$. This proves that $E$, $F$, $M$, and $N$ are concyclic with respect to center $X$.




With the same logic, we prove that $DNLF$ creates another rectangle by considering the midlines $\overline{FL}$, $\overline{LN}$, $\overline{DN}$, and $\overline{FD}$ of the same triangles as above. We can conclude that the points $D$, $F$, $L$, and $N$ are concyclic. Furthermore, since rectangles $DNLF$ and $EFMN$ share the common diagonal $\overline{FN}$, they are both centered at $X$. Thus, $D$, $E$, $F$, $L$, $M$, and $N$ are all concyclic.



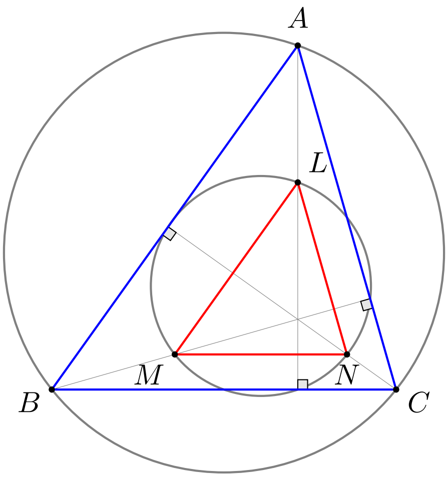

Lastly, to prove $J, I, K$ are concyclic with the other six points, consider the triangle $\triangle DIL$. We have $\overline{DI} \perp \overline{LI}$ because $\overline{BC} \perp \overline{AI}$. Now $\overline{DL}$ constitutes the diameter of the circle centered at $X$, thus $I$ must lie on the same circle by Thale's Theorem. Similar argument can be applied to triangles $\triangle EJM$ and $\triangle FKN$ to show that $J$ is concyclic with $E, M$ and $K$ is concyclic with $F, N$.




Thus, the nine points: $D$, $E$, $F$, $I$, $J$, $K$, $L$, $M$, and $N$ are all concyclic.

■

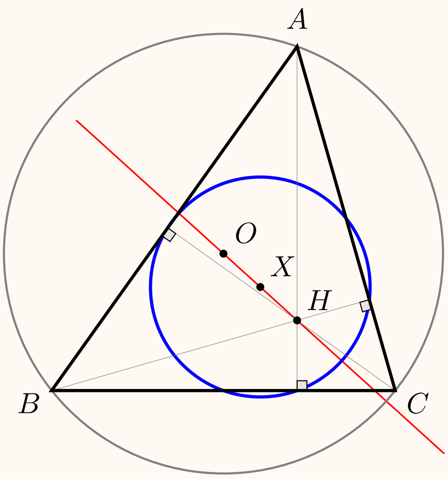
One important property of the nine-point circle is that its radius equals to half of the circumcircle radius. This is easy to observe since the circumcircle is generated by $\triangle ABC$ while the nine-point circle is generated by $\triangle LMN$, whose sides are half of the sides of $\triangle ABC$ by the Midline Theorem (See below).



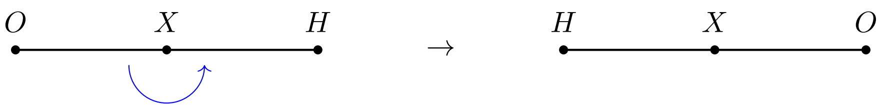

Below we will discuss other important theorems relating to the nine-point circle.

**Theorem 4. (Property of the Nine Point Circle and Euler's Line)**

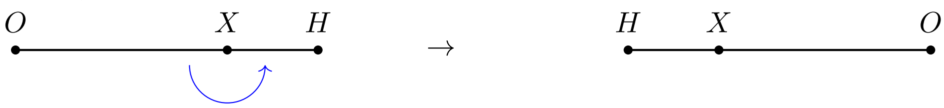
*Given $\triangle ABC$, the center of its nine-point circle $X$ lies on the Euler line, midway between the orthocenter $H$ and the circumcenter $O$.*



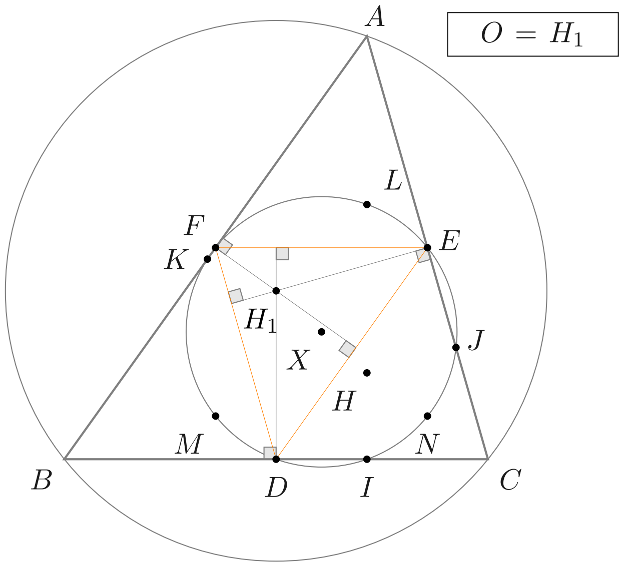

To prove that the nine-point center $X$ is the midpoint of $\overline{OH}$, we will show that $O$ maps onto $H$ when we rotate the line segment by 180° about point $X$, as described below:



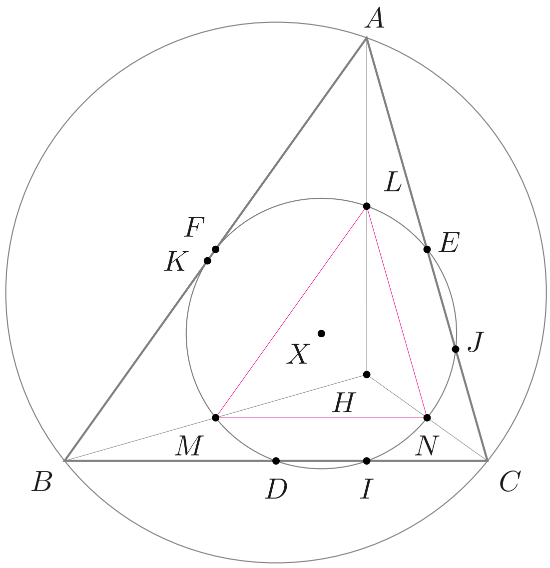

In contrast, if $X$ is not the midpoint of $O$ and $H$, the rotation by 180° about $X$ will yield the following result:



**Proof of Theorem 4.** First, connect the three midlines of $\triangle ABC$. Then $\triangle DEF$ is known as the *medial triangle*. If we draw the orthocenter $H_1$ of $\triangle DEF$ then $H_1$ is the intersection of the perpendicular bisectors of $\triangle ABC$. This is because midlines are parallel to their corresponding third side, thus the heights of the medial triangle are perpendicular to the original triangle sides. By definition, $H_1$ is the circumcenter $O$ of $\triangle ABC$.




Next, draw $\triangle LMN$, which is similar to and has the same orthocenter $H$ as $\triangle ABC$. Since $\triangle LMN$ is also generated by three midlines, its sides are half of the sides of $\triangle ABC$. In other words, $\triangle LMN = \triangle DEF$, where vertex $L$ corresponds with $D$, $M$ corresponds with $E$, and $N$ corresponds with $F$.

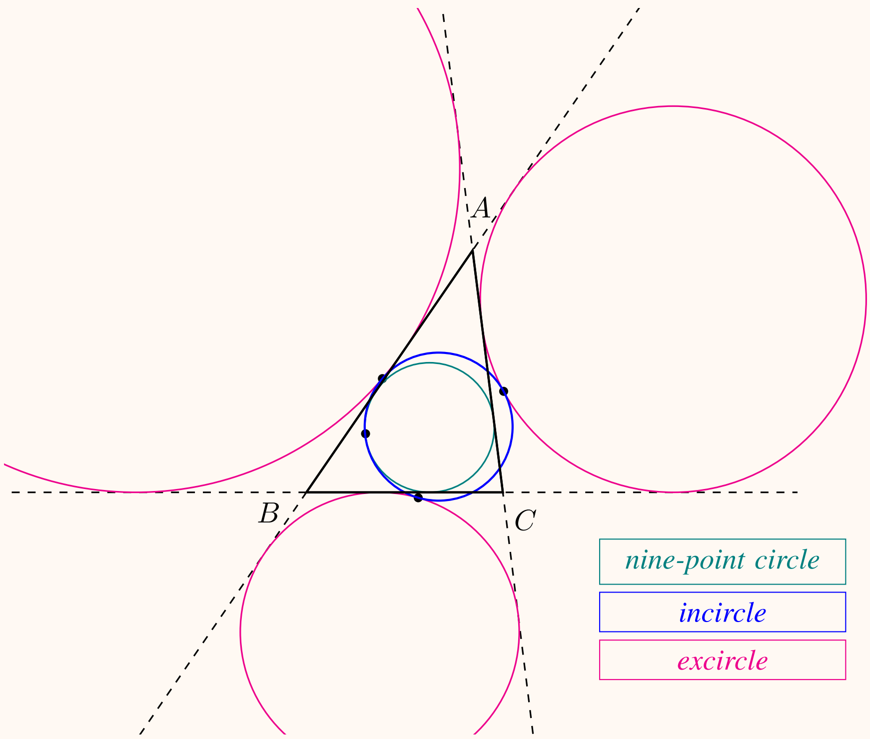



Furthermore, we can conclude that $\triangle DEF$ is essentially $\triangle LMN$ after being rotated $180°$ about the center $X$. This is because the pairs $(M, E)$, $(F, N)$, and$(D, L)$ are *diametrically opposite points*, which means they make up the diameter of the nine-point circle. Thus, rotate $D$ by $180°$ about $X$ will get us to $L$ and vice versa. This logic applies to the entire triangles $\triangle DEF$ and $\triangle LMN$, including their orthocenters. Since the orthocenter of $\triangle DEF$ is the circumcenter of $\triangle ABC$ and the orthocenter of $\triangle LMN$ is the orthocenter of $\triangle ABC$, we have proven that $X$ is the midpoint of $O$ and $H$. ■

Another interesting theorem on the nine-point circle was also discovered by Feuerbach in the same year. In this theorem, he recognized the importance of a triangle's incircle and excircles in relation to its nine-point circle.

**Theorem 5. (Feuerbach's Theorem)**

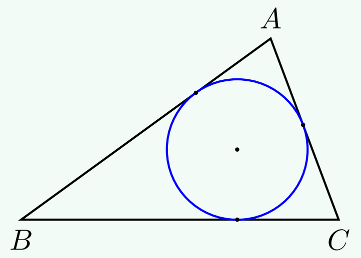
*The nine-point circle of a triangle is internally tangent to the incircle and externally tangent to the three excircles.*




Before proving Feuerbach's Theorem, we will go over the definitions and properties of incircles and excircles of a given triangle.

**Definition 2. (Incircles)**

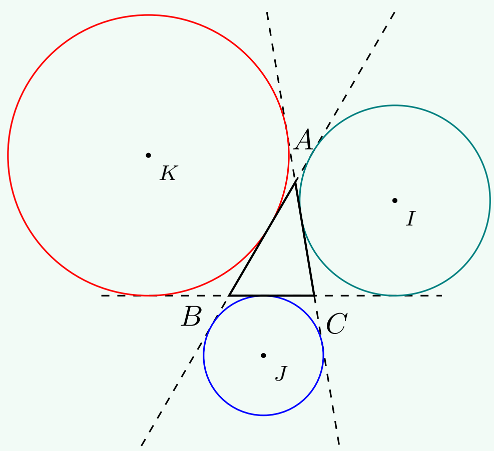
*Incircle is the circle that is tangent to all three sides of a triangle.*




*The center of the incircle is called incenter.*

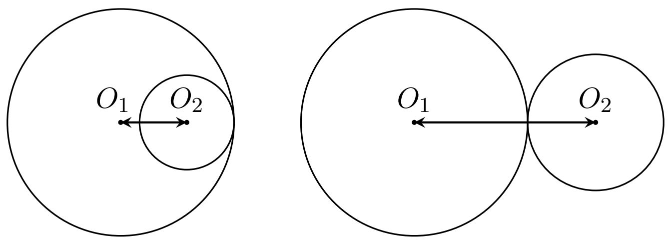
**Definition 3. (Excircles)**

*Excircle is circle that is tangent to one side of the triangle and to the extension of the remaining two sides.*




*There are three excircles, and their centers are called the excenters.*

If two circles are internally tangent, then the distance between the two circle centers is equal to the subtraction of two radii; If two circles are externally tangent, then the distance between the two circle centers will equal the sum of the radii.



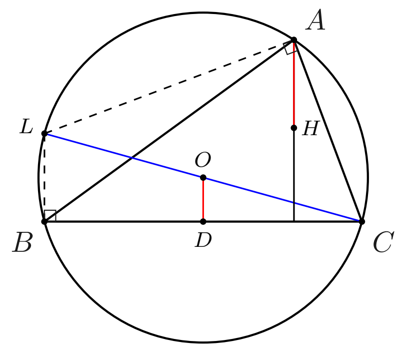

The following theorems will help us complete the proof.

**Theorem 6. (Relationship Between Circumcenter and Orthocenter)**

*Given a triangle $\triangle ABC$, the distance from its circumcenter to the midpoint of one side is equal to half the distance from the orthocenter to the opposite vertex.*

**Proof.** Let $O$ be the the circumcenter and $H$ be the orthocenter of $\triangle ABC$. If $D$ is the midpoint of $\overline{BC}$, then we want to prove that $\overline{OD} = \frac{1}{2}\overline{AH}$.

- Draw $\overline{LC}$ as the diameter of the circumcircle. By Thale's Theorem, since $A$ and $B$ are points on the circumcircle, $\angle LBC = 90°$ and $\angle LAC = 90°$.
- $\overline{OD}$ is a midline of $\triangle LBC$, thus $\overline{OD} = \frac{1}{2}\overline{LB}$ by the Midline Theorem.
- In addition, we have $(\overline{LA} \parallel \overline{BH}) \perp \overline{AC}$ and $(\overline{AH} \parallel \overline{LB}) \perp \overline{BC}$ by the proof of Theorem 1. This means that $ALBH$ forms a parallelogram and so $AH = LB = 2 \cdot OD$.




There are several ways to prove Feuerbach's Theorem. In the following, we shall use basic geomertric method to prove it.

**Proof of Feuerbach's Theorem (Theorem 5).** Let $O$ be the circumcenter; $H$ be the orthocenter; $I$ be the incenter; and $N$ be the nine-point center of $\triangle ABC$. To prove that the nine-point circle is internally tangent to the incircle, we will show that the segment $\overline{IN}$ is equal to $\frac{1}{2}R - r$, where $R$ represents the circumcircle radius and $r$ represents the incircle radius.

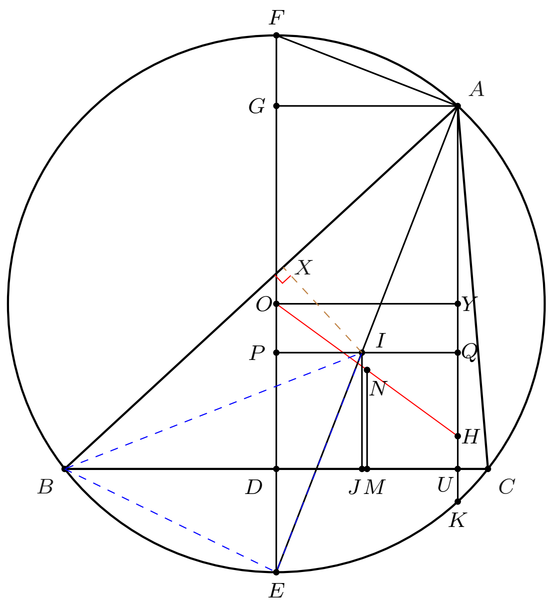
First, connect the midpoint $D$ of $\overline{BC}$ with $O$ and extend the line to meet the circumcircle at $F$ and $E$. Then $\overline{EDOF}$ is the diameter of $\triangle ABC$'s circumcircle. Let $G$ be a point on this diameter where $\overline{AG} \perp \overline{EF}$. Draw height $\overline{AU}$ perpendicular to $\overline{BC}$ and let $\overline{AU}$ intersects the circumcircle at $K$. Notice that $\overline{EF} \parallel \overline{AK}$. Let $Y$ be the midpoint of $\overline{AK}$, then $\overline{OY} \perp \overline{AK}$.

Next, draw segment $\overline{EA}$. Using the *Inscribed Angle Theorem* on the circumcircle, we know:

$$\angle BAE = \frac{1}{2}\angle BOE = \frac{1}{2}\angle EOC = \angle FOC$$

Therefore $\overline{EA}$ bisects $\angle BAC$ and passes through the incenter $I$.

Now, let $J, M$ be the projections of $I, N$ to $\overline{BC}$. Draw a line through the incenter $I$ that is perpendicular to both $\overline{EF}$ and $\overline{AK}$. Let $P$ be its intersection with $\overline{EF}$ and $Q$ its intersection with $\overline{AK}$.




By Pythagorean Theorem, we have

$$IN^2 = (IJ - NM)^2 + MJ^2.$$

We shall compute the quantities of the above right side one-by-one. First $IJ = r$, the radius of the incircle. Next, since $NM$ is the mid-line of the trapezoid $DUHO$, we know $NM = (OD + HU)/2$. By Theorem 6, we know that $NM = AK/4$.

It is slightly complicated to to compute $JM$. We first observe that $JM = DM - DJ = JU - MU$. Since $DM = MU$, we have

$$JM^2 = DM^2 - DJ \cdot JU = \frac{DU^2}{4} - PI \cdot IQ.$$

We claim that

$$PI \cdot IQ = r \cdot FG. \tag{1}$$

Assuming the claim is true, then

$$IN^2 = (IJ - NM)^2 + MJ^2 = (r - \frac{R - FG}{2})^2 + \frac{DU^2}{4} - r \cdot FG.$$

Expanding the right side above, we have

$$IN^2 = (\frac{R}{2} - r)^2 - (\frac{R}{2} - r) \cdot FG + \frac{FG^2}{4} + \frac{DU^2}{4} - r \cdot FG.$$

Since

$$FG^2 + DU^2 = FA^2 = FE \cdot FG = 2R \cdot FG,$$

we get

$$IN^2 = (\frac{R}{2} - r)^2,$$

and hence $IN = R/2 - r$. Therefore the nine-point circle is internally tangent to the incircle.

It remains to verify Claim (1). Let $IX \perp AB$. Then $\triangle AXI \sim \triangle EPI$. Thus we have

$$\frac{IX}{PI} = \frac{AI}{EI}.$$

We connect $IB$, $BE$. It is not hard to compute $\angle IBE = \angle IBC + \angle CBE = (\angle A + \angle B)/2$. Since $\angle AEB = \angle C$, we have $\angle BIE = (\angle A + \angle B)/2 = \angle IBE$. Thus $\triangle EIB$ is isosceles. As a result, $IE = BE = FA$.

Since $\triangle FGA \sim \triangle IQA$, we have

$$\frac{AI}{AF} = \frac{IQ}{FG}.$$

Combining the above equations, we get

$$\frac{r}{PI} = \frac{IX}{PI} = \frac{IQ}{FG},$$

completing the proof of the claim.

Using similar lines as above, it follows that the nine-point circle is externally tangent to the three excircles. ∎

**Definition 4. (Feuerbach Point)**

*The tangent point of the nine-point circle and the incircle is called the Feuerbach Point.*