The Morley's Miracle

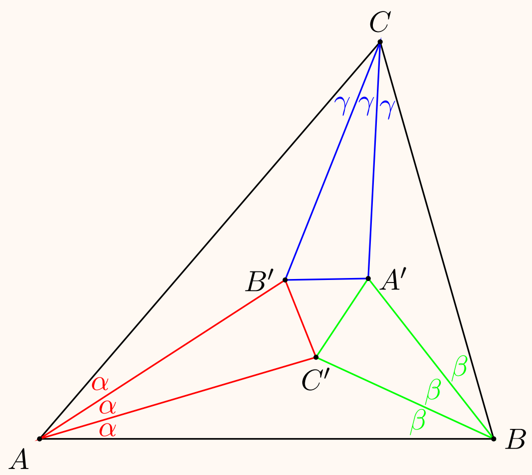
Zhiqin Lu, zlu@uci.edu

(last updated: January 11, 2022)

The Morley's Theorem states that in any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle. The theorem was discovered by Frank Morley in 1899.

Theorem 1. (Morley's Miracle)

In the following picture, the red, green, and blue lines are the angle trisectors of the corresponding angles. Then $\Delta A'B'C'$ is an equilateral triangle.



There are a lot different proofs of the Morley's Theorem. The following trigonometry method is one of the simplest.

Solution: Let R be the circumradius of the ΔABC. Then

$$AB = 2R\sin C = 2R\sin 3\gamma.$$

Using the law of sines, we have

$$AC' = \frac{AB}{\sin(\alpha+\beta)} \cdot \sin\beta = 2R\frac{\sin 3\gamma}{\sin(\alpha+\beta)}\sin\beta.$$

Using the formula $\sin 3x = 3\sin x - 4\sin^3 x$, we get

$$AC' = 2R\frac{\sin 3(\alpha+\beta)}{\sin(\alpha+\beta)}\sin\beta = 2R(3-4\sin^2(\alpha+\beta))\sin\beta.$$

The key is to simplify the above expression. We have

$$\begin{aligned}
&3-4\sin^2(\alpha+\beta) = 3-2(1+\cos 2(\alpha+\beta))\\
&= 2(\cos\frac{\pi}{3} + \cos 2(\alpha+\beta)) = 4\cos(\frac{\pi}{6}+\alpha+\beta)\cos(\alpha+\beta-\frac{\pi}{6}).
\end{aligned}$$

Note that $\alpha+\beta+\gamma=\pi/3$, we have

$$3-4\sin^2(\alpha+\beta)=2\sin\gamma\cos(\frac{\pi}{6}-\gamma).$$

Thus

$$AC'=8R\sin\beta\sin\gamma\cos(\frac{\pi}{6}-\gamma).$$

Using the same method, we have

$$AB'=8R\sin\beta\sin\gamma\cos(\frac{\pi}{6}-\beta).$$

Let $\sigma=8R\sin\beta\sin\gamma$. Then by law of cosines, we have

$$(B'C')^2=\sigma^2(\cos^2(\frac{\pi}{6}-\gamma)+\cos^2(\frac{\pi}{6}-\beta)-2\cos(\frac{\pi}{6}-\gamma)\cos(\frac{\pi}{6}-\beta)\cos\alpha).$$

We have

$$\begin{aligned}
&\sigma^2(\cos^2(\frac{\pi}{6}-\gamma)+\cos^2(\frac{\pi}{6}-\beta)-2\cos(\frac{\pi}{6}-\gamma)\cos(\frac{\pi}{6}-\beta)\cos\alpha)\\
&=\frac{1+\cos(\frac{\pi}{3}-2\gamma)}{2}+\frac{1+\cos(\frac{\pi}{3}-2\beta)}{2}-(\cos(\frac{\pi}{3}-\gamma-\beta)+\cos(\beta-\gamma))\cos\alpha\\
&=1+\frac{1}{2}(\cos(\frac{\pi}{3}-2\gamma)+\cos(\frac{\pi}{3}-2\beta))-\cos^2\alpha-\cos(\beta-\gamma)\cos\alpha\\
&=\sin^2\alpha.
\end{aligned}$$

Thus

$$B'C'=8R\sin\alpha\sin\beta\sin\gamma.$$

By symmetry, $C'A'=A'B'=B'C'=8R\sin\alpha\sin\beta\sin\gamma$.

Remark We are able to avoid using law of cosines, which is complicated. We observe that

$$\frac{AB'}{AC'}=\frac{\cos(\frac{\pi}{6}-\beta)}{\cos(\frac{\pi}{6}-\gamma)}=\frac{\sin(\frac{\pi}{3}+\beta)}{\sin(\frac{\pi}{3}+\gamma)}.$$

Thus we have $\angle B'C'A=\frac{\pi}{3}+\beta$. Similarly, $\angle BC'A'=\frac{\pi}{3}+\alpha$. Thus

$$\angle A'B'C'=2\pi-(\frac{\pi}{3}+\beta)-(\frac{\pi}{3}+\gamma)-(\pi-\alpha-\beta)=\frac{\pi}{3}.$$

In the following, we use the uniqueness method to give a purely plane geometric proof.

Solution: The usual way is to draw the triangle $\triangle ABC$, and then draw the angle trisectors to construct points A', B', C'. Note that the points A', B', C' are *uniquely* determined by A, B, C.

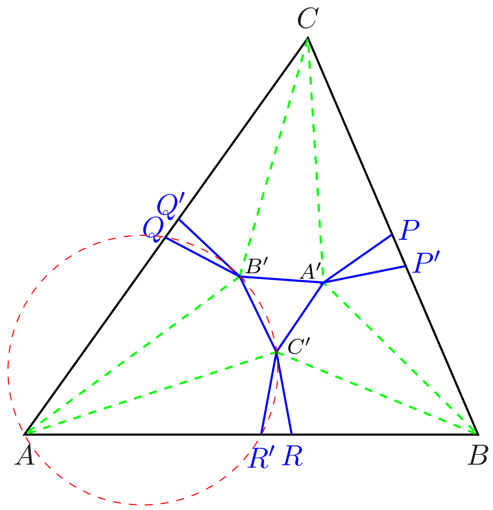
So we shall be able to prove the theorem using the following method. Let $\angle A+\angle B+\angle C=180°$ be given. If we are able to start from an equilateral $\triangle A'B'C'$ to construct a triangle $\triangle ABC$ with the given angles, we complete the proof.

Let $\angle A=3\alpha$, $\angle B=3\beta$, and $\angle C=3\gamma$. Then $\alpha+\beta+\gamma=\pi/3$. In the picture below, starting with an equilateral triangle $\triangle A'B'C'$ with side length 1, we construct points P, Q, R, P', Q', R' such that

$$\angle B'C'R' = \angle QB'C' = \pi - 2\alpha,$$
$$\angle C'A'P' = \angle RC'A' = \pi - 2\beta,$$
$$\angle A'B'Q' = \angle PA'B' = \pi - 2\gamma,$$

and

$$A'P = A'P' = B'Q = B'Q' = C'R = C'R' = 1.$$



Connecting PP', QQ' and RR' we shall construct $\triangle ABC$.

We need to prove

$$\angle A = 3\alpha, \qquad \angle B = 3\beta, \qquad \angle C = 3\gamma,$$

and $AB', AC', BC', BA', CA', CB'$ are the corresponding angle trisectors.

We compute

$$\angle RC'R' = 2\pi - (\pi - 2\alpha) - (\pi - 2\beta) - \pi/3 = \pi/3 - 2\gamma$$

Since $\triangle C'R'R$ is isosceles, we conclude that $\angle C'R'R = \pi/3 + \gamma$. Similarly, $\angle CQB' = \pi/3 + \beta$. Thus, from the fact that the summation of the angles of the pentagon $AQB'C'R'$ is 3π, we conclude that $\angle A = 3\alpha$.

By the construction, the quadrilateral $QB'C'R'$ is an isosceles trapezoid. Thus it must be con-cyclic. The point A must be on the circle because if A is outside the circle, we must have $\angle B'AB < 2\alpha$, and $\angle CAB' < \alpha$; and if A is inside the circle, we have the reverse inequalities, which is a contradiction to the fact that $\angle A = 3\alpha$. Thus the green lines are angle trisectors and this completes the proof.

■

External Link. *Further readings on Morley's Theorem.*

External Link. *Bookmark this website for your college junior year: a group theoretic proof of Morley's Theorem by the Fields Medalist A. Connes: A new proof of Morley's theorem.*