Gergonne and Nagel Points

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1 Gergonne Point

Definition 1. (Gergonne Point)

Gergonne Point, a triangle center named after Joseph Diez Gergonne[a], is the perspector[b] of a triangle ΔABC and its contact triangle $\Delta A'B'C'$ (also called Gergonne Triangle). It is also known as Kimberling Center X_7.[c]

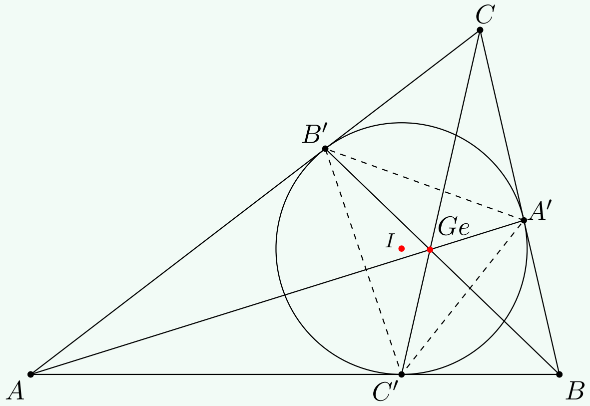

Figure 1: Gergonne Point

[a] *Joseph Diez Gergonne* (19^{th} June 1771 at Nancy, France – 4^{th} May 1859 at Montpellier, France) was a French mathematician and logician.

[b] *Perspector* of two triangles is the point at where the lines joining corresponding vertices of two triangles meet.

[c] *Clark Kimberling* maintained the Encyclopedia of Triangle Centers (ETC), an online list of thousands of points or "centers" associated with the geometry of a triangle.

Theorem 1. (Existence of Gergonne Point)

As shown in Figure 1, line segments $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are concurrent at the Gergonne Point. Thus, the Gergonne Point is well-defined.

Proof. We construct Figure 2 below from Figure 1 by connecting $\overline{IA}$, $\overline{IB}$, $\overline{IC}$, $\overline{IA'}$, $\overline{IB'}$, and $\overline{IC'}$. Then, since A', B', and C' are tangent points of a triangle's incircle and sides, we have

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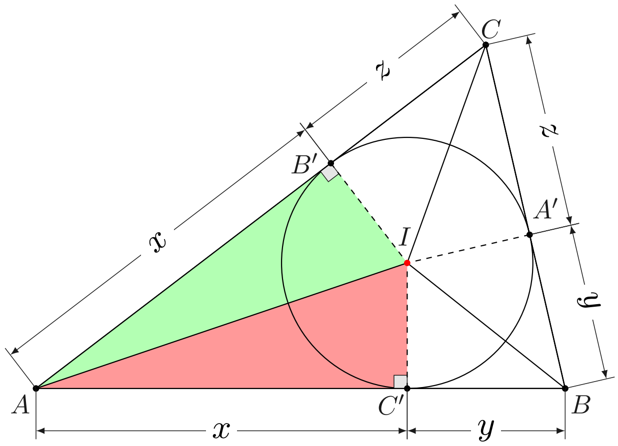

Figure 2: Existence of Gergonne Point

Let's take $\triangle AC'I$ and $\triangle AB'I$ as an example. We notice that

$$\begin{cases} \angle IC'A = \angle IB'A = 90^\circ, & \text{since } \bigcirc I \text{ is tangent to } \overline{AB} \text{ and } \overline{AC}; \\ \overline{IB'} = \overline{IC'}, & \text{since they are both the radius of } \bigcirc I; \\ \overline{AI} = \overline{AI}, & \text{since the side is shared.} \end{cases}$$

Therefore, $\triangle AC'I$ and $\triangle AB'I$ are congruent.

By using the same method, we can also prove $\triangle CB'I \cong \triangle CA'I$ and $\triangle BC'I \cong \triangle BA'I$. Hence, $\overline{AC'} = \overline{AB'} = x$, $\overline{BC'} = \overline{BA'} = y$, and $\overline{CA'} = \overline{CB'} = z$, where $x, y, z \in \mathbb{R}^+$.

By using the Ceva's Theorem, we have

$$\frac{\overline{AC'}}{\overline{C'B}} \cdot \frac{\overline{BA'}}{\overline{A'C}} \cdot \frac{\overline{CB'}}{\overline{B'A}} = \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x} = 1.$$

And therefore we proved that the three cevians $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are concurrent at the same point, which is the Gergonne Point Ge. ■

Definition 2. (Barycentric Coordinate System)

In a reference triangle $\triangle ABC$, the barycentric coordinates (See Topic 37 or here for details) of a point P are an ordered triple of numbers, each of which is proportional to the masses (areas) of $\triangle BPC$, $\triangle APC$, and $\triangle APB$, denoted as $t_1 : t_2 : t_3$ or (t_1, t_2, t_3). The barycentric coordinate system was discovered by Möbius[a] in 1827.

[a] August Ferdinand Möbius (17^{th} November 1790 in Schulpforta, Holy Roman Empire – 26^{th} September 1868 in Leipzig, Germany) was a German mathematician and theoretical astronomer.

Definition 3. (Trilinear Coordinate System)

In a reference triangle $\triangle ABC$, the trilinear coordinates (See Topic 37 or here for details) of a point P is an ordered triple of numbers, each of which is proportional to the directed distance from P to one of the sidelines, denoted as $\alpha : \beta : \gamma$ or (α, β, γ). The trilinear *coordinate system was discovered by Plücker[a] in 1835.*

[a] Julius Plücker (16^{th} June 1801 in Elberfeld, Germany – 22^{th} May 1868 in Bonn, Germany) was a German mathematician and physicist.

Theorem 2. (Barycentric Coordinates of Gergonne Point)

Let a, b, c be the lengths of three sides of $\triangle ABC$, and let s be the semiperimeter. Then the Barycentric Coordinates of Gergonne Point are given by

$$t_1 : t_2 : t_3 = (s-b)(s-c) : (s-c)(s-a) : (s-a)(s-b).$$

Proof. As Figure 3 shows below, we need to determine the mass ratio of $\triangle BGeC$, $\triangle CGeA$, and $\triangle AGeB$. Since we are trying to compare the mass ratio of three parts of the reference triangle, we color them for clarity.

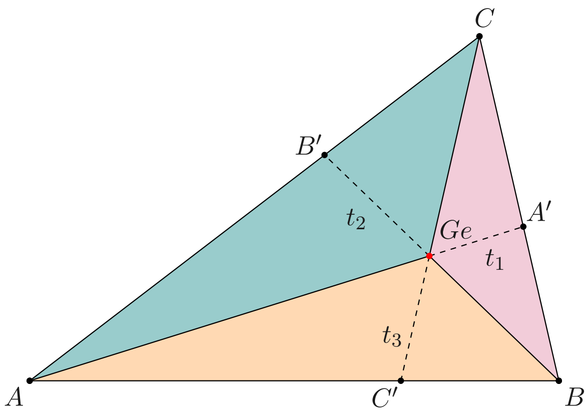

Figure 3: Barycentric Coordinates of Gergonne Point

We have

$$\frac{t_2}{t_1} = \frac{\text{Area of } \triangle CGeA}{\text{Area of } \triangle BGeC} = \frac{\overline{AC'}}{\overline{C'B}}.$$

Similarly, we have that

$$\frac{t_3}{t_2} = \frac{\overline{BA'}}{\overline{A'C}} \quad \text{and} \quad \frac{t_1}{t_3} = \frac{\overline{CB'}}{\overline{B'A}}.$$

Since $\overline{AC'} = \overline{AB'} = x$, $\overline{BC'} = \overline{BA'} = y$, and $\overline{CB'} = \overline{CA'} = z$, we have $a = y+z$, $b = x+z$, $c = x+y$. Solving the equations, we get $x = s-a$, $y = s-b$, $z = s-c$. Thus we have

$$\begin{cases} t_3 : t_1 = x : z = (s-a) : (s-c); \\ t_1 : t_2 = y : x = (s-b) : (s-a); \\ t_2 : t_3 = z : y = (s-c) : (s-b). \end{cases}$$

Combine the above, we conclude

$$t_1 : t_2 : t_3 = (s-b)(s-c) : (s-c)(s-a) : (s-a)(s-b).$$

Theorem 3. (Trilinear Coordinates of Gergonne Point)

Let A, B, C be the angles of $\triangle ABC$. The Trilinear Coordinates of Gergonne Point are

$$\alpha : \beta : \gamma = \sec^2(\frac{A}{2}) : \sec^2(\frac{B}{2}) : \sec^2(\frac{C}{2}).$$

Proof. There is a way to easily obtain the trilinear coordinates of a point by using its barycentric coordinates. We used this method for Nagel Point (See Theorem 7), and the reader can use it similarly for Gergonne Point. But for now, let's provide an independent proof.

As Figure 4 shows below, we need to determine the ratio of $\overline{GeP_1} : \overline{GeP_2} : \overline{GeP_3}$.

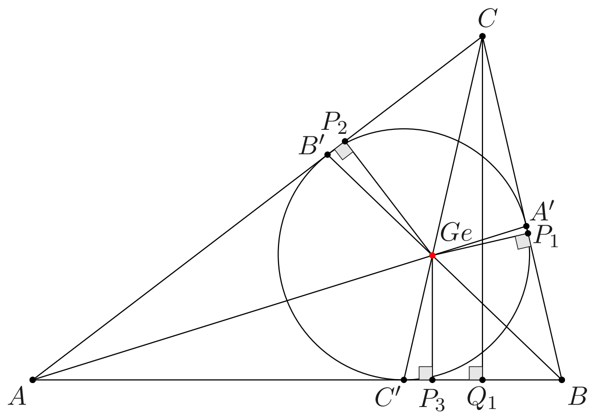

Figure 4: Trilinear Coordinates of Gergonne Point

To simplify our calculation, we define the following values:

$$\begin{cases} a, b, c & \text{respectively the lengths of } \overline{BC}, \overline{CA}, \overline{AB}; \\ H_A, H_B, H_C & \text{respectively the heights of } \triangle ABC \text{ from vertice } A, B, \text{ and } C; \\ \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} & \text{respectively the lengths of } \overline{GeP_1}, \overline{GeP_2}, \text{ and } \overline{GeP_3}. \end{cases}$$

We want to find $\alpha : \beta : \gamma = \tilde{\alpha} : \tilde{\beta} : \tilde{\gamma}$.

By using Menelaus' Theorems for $\triangle CAC'$ and the line segement $\overline{BB'}$, we have

$$\frac{\overline{CB'}}{\overline{B'A}} \cdot \frac{\overline{AB}}{\overline{BC'}} \cdot \frac{\overline{C'Ge}}{\overline{GeC}} = 1.$$

Therefore we have

$$\frac{\overline{GeC}}{\overline{C'Ge}} = \frac{\overline{CB'}}{\overline{B'A}} \cdot \frac{\overline{AB}}{\overline{BC'}}.$$

Let $A_\triangle$ denote the area of $\triangle ABC$, we obtain

$$\frac{\tilde{\gamma}}{H_C} = \frac{\overline{C'Ge}}{\overline{C'C}} = \frac{\overline{C'Ge}}{\overline{C'Ge} + \overline{GeC}}.$$

Thus

$$\tilde{\gamma} = \frac{H_C}{1 + \dfrac{\overline{GeC}}{\overline{C'Ge}}} = \frac{2A_\triangle/\overline{AB}}{1 + \dfrac{\overline{CB'}}{\overline{B'A}} \cdot \dfrac{\overline{AB}}{\overline{BC'}}}$$

$$= 2A_\triangle \cdot \frac{\overline{B'A} \cdot \overline{BC'}}{\overline{AB} \cdot (\overline{B'A} \cdot \overline{BC'} + \overline{CB'} \cdot \overline{AB})}.$$

Since $2s = a + b + c$, where s is the semiperimeter, we have

$$\begin{cases} \overline{AB'} = s - a; \\ \\ \overline{BC'} = s - b; \\ \\ \overline{CB'} = s - c, \end{cases}$$

Thus,

$$\tilde{\gamma} = 2A_\triangle \cdot \frac{(s-a)(s-b)}{c((s-a)(s-b) + (s-c)c)}$$
$$= \frac{8A_\triangle}{-a^2 - b^2 - c^2 + 2ab + 2ac + 2bc} \cdot \frac{(s-a)(s-b)}{c}.$$

Similarly, we can also get

$$\tilde{\beta} = \frac{8A_\triangle}{-a^2 - b^2 - c^2 + 2ab + 2ac + 2bc} \cdot \frac{(s-c)(s-a)}{b};$$
$$\tilde{\alpha} = \frac{8A_\triangle}{-a^2 - b^2 - c^2 + 2ab + 2ac + 2bc} \cdot \frac{(s-c)(s-b)}{a}.$$

Factoring out $\dfrac{8A_\triangle}{-a^2 - b^2 - c^2 + 2ab + 2ac + 2bc}$, we have

$$\alpha : \beta : \gamma = \frac{(s-b)(s-c)}{a} : \frac{(s-c)(s-a)}{b} : \frac{(s-a)(s-b)}{c}$$
$$= \frac{1}{a(s-a)} : \frac{1}{b(s-b)} : \frac{1}{c(s-c)}$$
$$= \frac{1}{a \cdot 1/2(b+c-a)} : \frac{1}{b \cdot 1/2(a+c-b)} : \frac{1}{c \cdot 1/2(a+b-c)}$$
$$= [a(b+c-a)]^{-1} : [b(a+c-b)]^{-1} : [c(a+b-c)]^{-1}.$$

Using the law of cosines and power-reduction formulae, we have

$$a(b+c-a)(a+b+c) = a((b+c)^2 - a^2) = a(2bc + 2bc\cos(A))$$
$$= 2abc(1 + \cos A) = 4abc\cos^2(\frac{A}{2}).$$

Similarly, we obtain

$$b(a+c-b)(a+b+c) = 4abc\cos^2(\frac{B}{2});$$
$$c(a+b-c)(a+b+c) = 4abc\cos^2(\frac{C}{2}).$$

Now we have

$$
\begin{aligned}
&\alpha : \beta : \gamma \\
&= [a(b+c-a)]^{-1} : [b(a+c-b)]^{-1} : [c(a+b-c)]^{-1} \\
&= [a(b+c-a)(a+b+c)]^{-1} : [b(a+c-b)(a+b+c)]^{-1} : [c(a+b-c)(a+b+c)]^{-1} \\
&= [4abc\cos^2(\frac{A}{2})]^{-1} : [4abc\cos^2(\frac{B}{2})]^{-1} : [4abc\cos^2(\frac{C}{2})]^{-1} \\
&= [\cos^2(\frac{A}{2})]^{-1} : [\cos^2(\frac{B}{2})]^{-1} : [\cos^2(\frac{C}{2})]^{-1} \\
&= \sec^2(\frac{A}{2}) : \sec^2(\frac{B}{2}) : \sec^2(\frac{C}{2}).
\end{aligned}
$$

2 Generalized Gergonne Point

The concept of Gergonne Point can be generalized into the following version.

Theorem 4. (Generalized Gergonne Point)

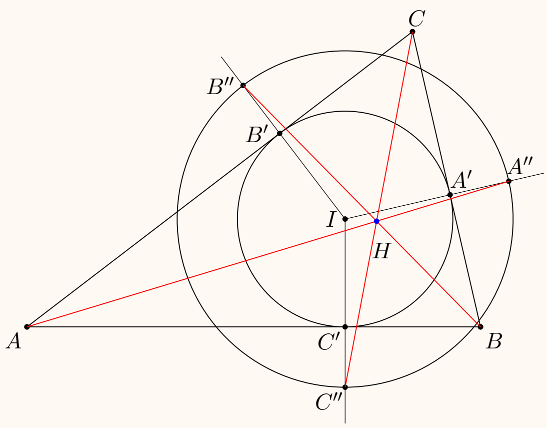

Figure 5: **Generalized Gergonne Point**

As Figure 5 above indicates, let I be the incenter of $\triangle ABC$. *Then we extend the segments* $\overline{IA'}$, $\overline{IB'}$, $\overline{IC'}$, *and enlarge the incircle* I *to make them intersect respectively at points* A'', B'', *and* C''. *Then, lines* $\overline{AA''}$, $\overline{BB''}$, *and* $\overline{CC''}$ *intersects at the same point* H, *which is a Generalized Gergonne Point. In particular, when* $A\&A'$, $B\&B'$, *and* $C\&C'$ *coincide, the point* H *is exactly the Gergonne Point.*

Proof. To prove the theorem, we shall firstly construct a triangle $\triangle V_1V_2V_3$ such that point V_1 is the intersection of the lines perpendicular to segments $\overline{IB''}$ and $\overline{IC''}$, and similarly we can draw points V_2 and V_3. Hence, $\triangle V_1V_2V_3$ and $\triangle ABC$ are similar triangles sharing the same incenter I, shown below in Figure 6.

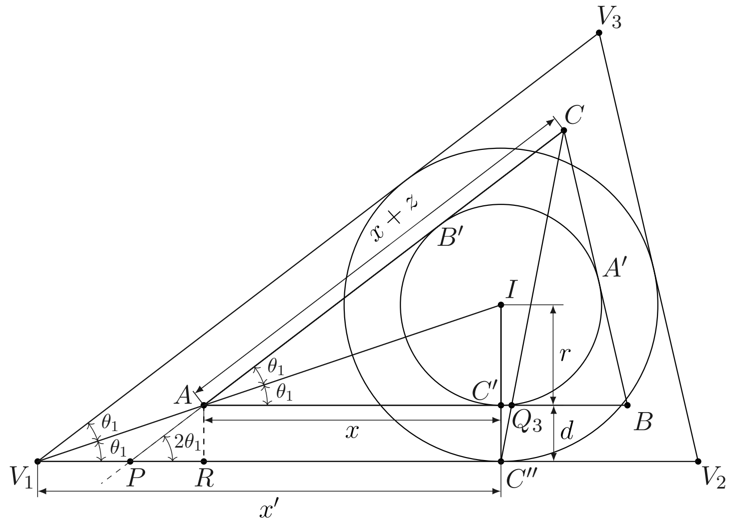

Figure 6: **Existence of Generalized Gergonne Point Part 1**

We then draw two segment $\overline{IV_1}$ and $\overline{IA}$. Now we have

$$\frac{1}{2}\angle CAB = \frac{1}{2}\angle V_3V_1V_2 = \angle CAI = \angle BAI = \angle V_3V_1I = \angle V_2V_1I = \theta_1.$$

$\overline{AB}$ and $\overline{V_1V_2}$ are parallel, because they are both perpendicular to $\overline{IC''}$, and as $\angle IAB = \angle IV_1V_2$, we can easily conclude that point A lies exactly on $\overline{IV_1}$.

Moreover, let the smaller circle has radius r, and the larger circle has radius $r+d$. Then, $\triangle V_1V_2V_3$ and $\triangle ABC$ have a similarity ratio of $r+d : r$. Using the same notations as defined above, we have $\overline{AC'} = x$ and $\overline{AC} = x+z$. Notice that $\triangle IAC'$ and $\triangle IV_1C''$ are similar, then

$$\overline{V_1C''} = x' = \frac{r+d}{r}\cdot\overline{AC'} = \frac{r+d}{r}\cdot x.$$

Extend $\overline{CA}$ to intersect with $\overline{V_1V_2}$ at point P, we have $\angle APR = 2\theta_1$.
Draw a line AR perpendicular to $\overline{V_1V_2}$ at point R, we can calculate that

$$\overline{PC''} = \overline{V_1C''} - \overline{V_1P} = \overline{V_1C''} - \overline{V_1R} + \overline{PR} = x' - d\cdot(\cot 2\theta_1 - \cot\theta_1).$$

To simplify, we let $m_1 = \cot 2\theta_1 - \cot\theta_1$, so

$$\overline{PC''} = x' - dm_1 \qquad \text{and} \qquad \overline{V_1P} = \overline{AP} = dm_1.$$

Now, connect $\overline{CC''}$ to intersect $\overline{AB}$ at Q_3, and we notice that $\triangle CAQ_3$ and $\triangle CPC''$ are also similar, so we have

$$\overline{AQ_3} = \overline{PC''}\cdot\frac{\overline{AC}}{\overline{PC}} = \frac{(x+z)\overline{PC''}}{x+z+dm_1}.$$

Substituting what we obtained for $\overline{PC''}$ back, then we have

$$\overline{AQ_3} = \frac{(x+z)\overline{PC''}}{x+z+dm_1} = \frac{(x+z)(\overline{V_1C''} - \overline{V_1P})}{x+z+dm_1} = \frac{(x+z)(\frac{r+d}{r}\cdot x - dm_1)}{x+z+dm_1}.$$

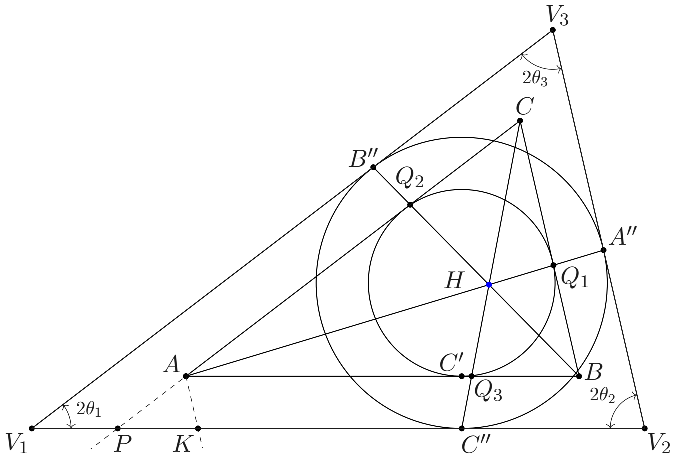

Figure 7: Existence of Generalized Gergonne Point Part 2

Now we do the same thing for the other two sides: draw line segment $\overline{AA''}$ which intersects with $\overline{BC}$ at point Q_1, and draw line segment $\overline{BB''}$ which intersects with $\overline{AC}$ at point Q_2. Let's have $\angle V_1V_2V_3 = 2\theta_2$ and $\angle V_2V_3V_1 = 2\theta_3$, so $m_2 = \cot 2\theta_2 - \cot\theta_2$ and $m_3 = \cot 2\theta_3 - \cot\theta_3$.

Then we can generalize the computation of $\overline{AQ_3}$ to all other five segments:

$$\overline{AQ_3} = \frac{(x+z)(\frac{r+d}{r}\cdot x - dm_1)}{x+z+dm_1};$$

$$\overline{BQ_3} = \frac{(y+z)(\frac{r+d}{r}\cdot y - dm_2)}{y+z+dm_2};$$

$$\overline{BQ_1} = \frac{(x+y)(\frac{r+d}{r}\cdot y - dm_2)}{x+y+dm_2};$$

$$\overline{CQ_1} = \frac{(x+z)(\frac{r+d}{r}\cdot z - dm_3)}{x+z+dm_3};$$

$$\overline{CQ_2} = \frac{(y+z)(\frac{r+d}{r}\cdot z - dm_3)}{y+z+dm_3};$$

$$\overline{AQ_2} = \frac{(x+y)(\frac{r+d}{r}\cdot x - dm_1)}{x+y+dm_1}.$$

Using Ceva's Theorem here, we have

$$\frac{\overline{AQ_3}}{\overline{BQ_3}}\cdot\frac{\overline{BQ_1}}{\overline{CQ_1}}\cdot\frac{\overline{CQ_2}}{\overline{AQ_2}}$$

$$= \frac{\frac{(x+z)(\frac{r+d}{r}\cdot x - dm_1)}{x+z+dm_1}}{\frac{(y+z)(\frac{r+d}{r}\cdot y - dm_2)}{y+z+dm_2}}\cdot\frac{\frac{(x+y)(\frac{r+d}{r}\cdot y - dm_2)}{x+y+dm_2}}{\frac{(x+z)(\frac{r+d}{r}\cdot z - dm_3)}{x+z+dm_3}}\cdot\frac{\frac{(y+z)(\frac{r+d}{r}\cdot z - dm_3)}{y+z+dm_3}}{\frac{(x+y)(\frac{r+d}{r}\cdot x - dm_1)}{x+y+dm_1}}.$$

We can easily recognize that each numerators above and below the main fractional line can be canceled, and hence we only need to calculate the value of

$$\frac{y+z+dm_2}{x+z+dm_1}\cdot\frac{x+z+dm_3}{x+y+dm_2}\cdot\frac{x+y+dm_1}{y+z+dm_3}.$$

Let's go back to the graph above, draw a line crossing point A parallel to $\overline{CB}$, which intersects with $\overline{V_1V_2}$ at point K. We can derive

$$\overline{AP}=dm_1,\overline{AK}=dm_2.$$

Notice that $\triangle PKA$ is similar to $\triangle ABC$, so we have

$$dm_1\cdot(y+z)=dm_2\cdot(x+z).$$

Drawing any similar triangles at other positions, we shall have

$$dm_1\cdot(y+z)=dm_2\cdot(x+z)=dm_3\cdot(x+y)=l.$$

Substituting them back, we simply have

$$\begin{aligned}&\frac{y+z+dm_2}{x+z+dm_1}\cdot\frac{x+z+dm_3}{x+y+dm_2}\cdot\frac{x+y+dm_1}{y+z+dm_3}\\ =&\frac{(x+z)(y+z)+(x+z)dm_2}{(y+z)(x+z)+(y+z)dm_1}\cdot\frac{(x+y)(x+z)+(x+y)dm_3}{(x+z)(x+y)+(x+z)dm_2}\cdot\frac{(y+z)(x+y)+(y+z)dm_1}{(x+y)(y+z)+(x+y)dm_3}\\ =&\frac{(x+z)(y+z)+l}{(y+z)(x+z)+l}\cdot\frac{(x+y)(x+z)+l}{(x+z)(x+y)+l}\cdot\frac{(y+z)(x+y)+l}{(x+y)(y+z)+l}=1.\end{aligned}$$

Hence, we conclude that these three segments $\overline{AA''}$, $\overline{BB''}$, $\overline{CC''}$ obey the Ceva's Theorem and they are concurrent at the same point H, which is the Generalized Gergonne Point. ■

3 Nagel Point

Definition 4. (Nagel Point)

Nagel Point, the Kimberling Center X_8, is named after Christian Heinrich von Nagel[a]. It is the Isotomic Conjugate Point (See Definition 5 of Topic 7 for details) of Gergonne Point.

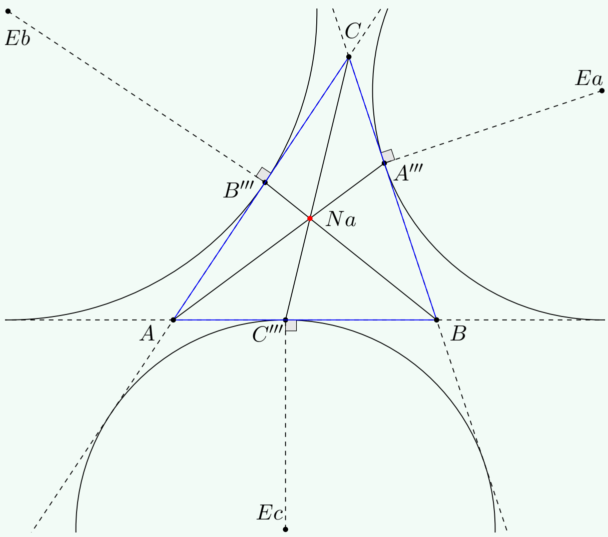

Figure 8: **Nagel Point**

As the diagram above shows, the reference triangle $\triangle ABC$ has three excircles, Ea, Eb, Ec. These three excircles touches $\triangle ABC$ respectively at points A''', B''', C'''. Then $\overline{AA'''}$, $\overline{BB'''}$, and $\overline{CC'''}$ intersect at the Nagel Point.

[a] Christian Heinrich von Nagel (28^{th} February 1803 in Stuttgart, Germany – 27^{th} October 1882 in Ulm, Germany) was a German geometer.

Theorem 5. (Existence of Nagel Point)

As shown in Figure 8, cevians $\overline{AA'''}$, $\overline{BB'''}$, and $\overline{CC'''}$ are concurrent. Thus the Nagel Point is well-defined.

Proof. Consider Figure 9 below. Since the lengths of tangents drawn from an external point to a circle are equal, it is obvious that we have

$$\begin{cases} \overline{CP} = \overline{CQ}; \\ \overline{AP} = \overline{AC'''}; \\ \overline{BQ} = \overline{BC'''}. \end{cases}$$

Therefore, $\overline{CA} + \overline{AC'''} = \overline{CA} + \overline{AP} = \overline{CP} = \overline{CQ} = \overline{CB} + \overline{BQ} = \overline{CB} + \overline{BC'''}$.

Namely, these two points C and C''' exactly divide the circumference of the triangle into two halves. Similar to the notation we used above, let's note half of the circumference of $\triangle ABC$ as s, $\overline{BC} = a$, $\overline{AC} = b$, and $\overline{AB} = c$.

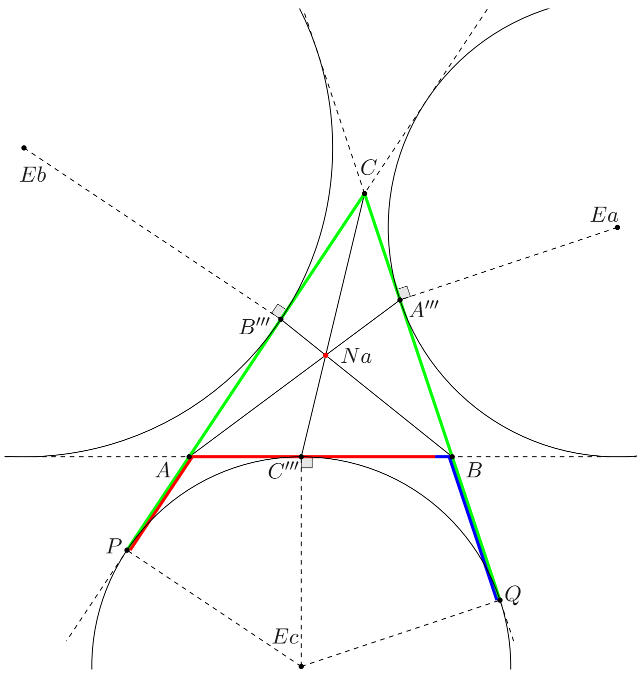

Figure 9: Existence of Nagel Point

Hence, we have $\overline{AC'''} = \overline{CA} + \overline{AC'''} - \overline{CA} = s - b$, and $\overline{BC'''} = \overline{CB} + \overline{BC'''} - \overline{BA} = s - a$.

Likewise, we also have: $\overline{AB'''} = s - c$, $\overline{CB'''} = s - a$, $\overline{CA'''} = s - b$, $\overline{BA'''} = s - c$.

Thus, we shall get

$$\frac{\overline{AC'''}}{\overline{C'''B}} \cdot \frac{\overline{BA'''}}{\overline{A'''C}} \cdot \frac{\overline{CB'''}}{\overline{B'''A}} = \frac{s-b}{s-a} \cdot \frac{s-c}{s-b} \cdot \frac{s-a}{s-c} = 1.$$

Now we observe that Ceva's Theorem holds, and these three cevians $\overline{AA'''}$, $\overline{BB'''}$, and $\overline{CC'''}$ are exactly concurrent at the same point, the Nagel Point. ■

Theorem 6. (Barycentric Coordinates of Nagel Point)

Let s be the semiperimeter of $\triangle ABC$, and let a, b, c be the lengths of three sides of $\triangle ABC$. Then, the Barycentric Coordinates of Nagel Point are

$$t_1 : t_2 : t_3 = (s-a) : (s-b) : (s-c).$$

Proof. The Barycentric Coordinates of Nagel Point was incidentally solved in the proof of Theorem 5, since we successfully derived the lengths of all six segments. Consider Figure 10 below.

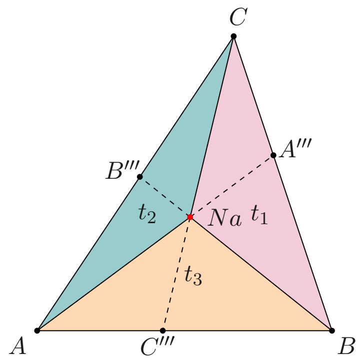

Figure 10: Barycentric Coordinates of Nagel Point

Since $\triangle CNaB$ and $\triangle CNaA$ share a common side, the ratio of their areas is equal to the ratio of $\overline{BC'} : \overline{AC'}$. Similarly, we get

$$\begin{cases} t_1 : t_2 = \overline{BC'''} : \overline{AC'''} = (s-a) : (s-b); \\ t_2 : t_3 = \overline{CA'''} : \overline{BA'''} = (s-b) : (s-c); \\ t_3 : t_1 = \overline{AB'''} : \overline{CB'''} = (s-c) : (s-a). \end{cases}$$

Combine them together, we have

$$t_1 : t_2 : t_3 = (s-a) : (s-b) : (s-c).$$

■

Theorem 7. (Trilinear Coordinates of Nagel Point)

Let A, B, C be the angles of $\triangle ABC$. The Trilinear Coordinates of Nagel Point are

$$\alpha : \beta : \gamma = \csc^2(\frac{A}{2}) : \csc^2(\frac{B}{2}) : \csc^2(\frac{C}{2}).$$

Proof. By the relation between the barycentric and trilinear coordinate systems, from the above theorem, we obtain that the trilinear coordinates of the Nagal Point are given by

$$\alpha : \beta : \gamma = \frac{s-a}{a} : \frac{s-b}{b} : \frac{s-c}{c}.$$

We claim that

$$\sin B + \sin C - \sin A = 4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

To see this, we use various trigonometric formulae to obtain

$$\begin{aligned}
&\sin B + \sin C - \sin A = 2\sin\frac{B+C}{2}\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2} \\
&= 2\cos\frac{A}{2}\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right) = 2\cos\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) \\
&= 4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.
\end{aligned}$$

As a result, using the law of sines, we obtain

$$\frac{s-a}{a} = \frac{2\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin A} = \frac{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin^2\frac{A}{2}}.$$

Similarly, we can obtain

$$\begin{aligned}
\frac{s-b}{b} &= \frac{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin^2\frac{B}{2}}; \\
\frac{s-c}{c} &= \frac{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin^2\frac{C}{2}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\alpha : \beta : \gamma \\
&= \frac{s-a}{a} : \frac{s-b}{b} : \frac{s-c}{c} \\
&= \frac{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin^2\frac{A}{2}} : \frac{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin^2\frac{B}{2}} : \frac{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin^2\frac{C}{2}} \\
&= \frac{1}{\sin^2\frac{A}{2}} : \frac{1}{\sin^2\frac{B}{2}} : \frac{1}{\sin^2\frac{C}{2}} \\
&= \csc^2(\frac{A}{2}) : \csc^2(\frac{B}{2}) : \csc^2(\frac{C}{2}).
\end{aligned}$$