SYMMEDIAN POINT

MATH199C
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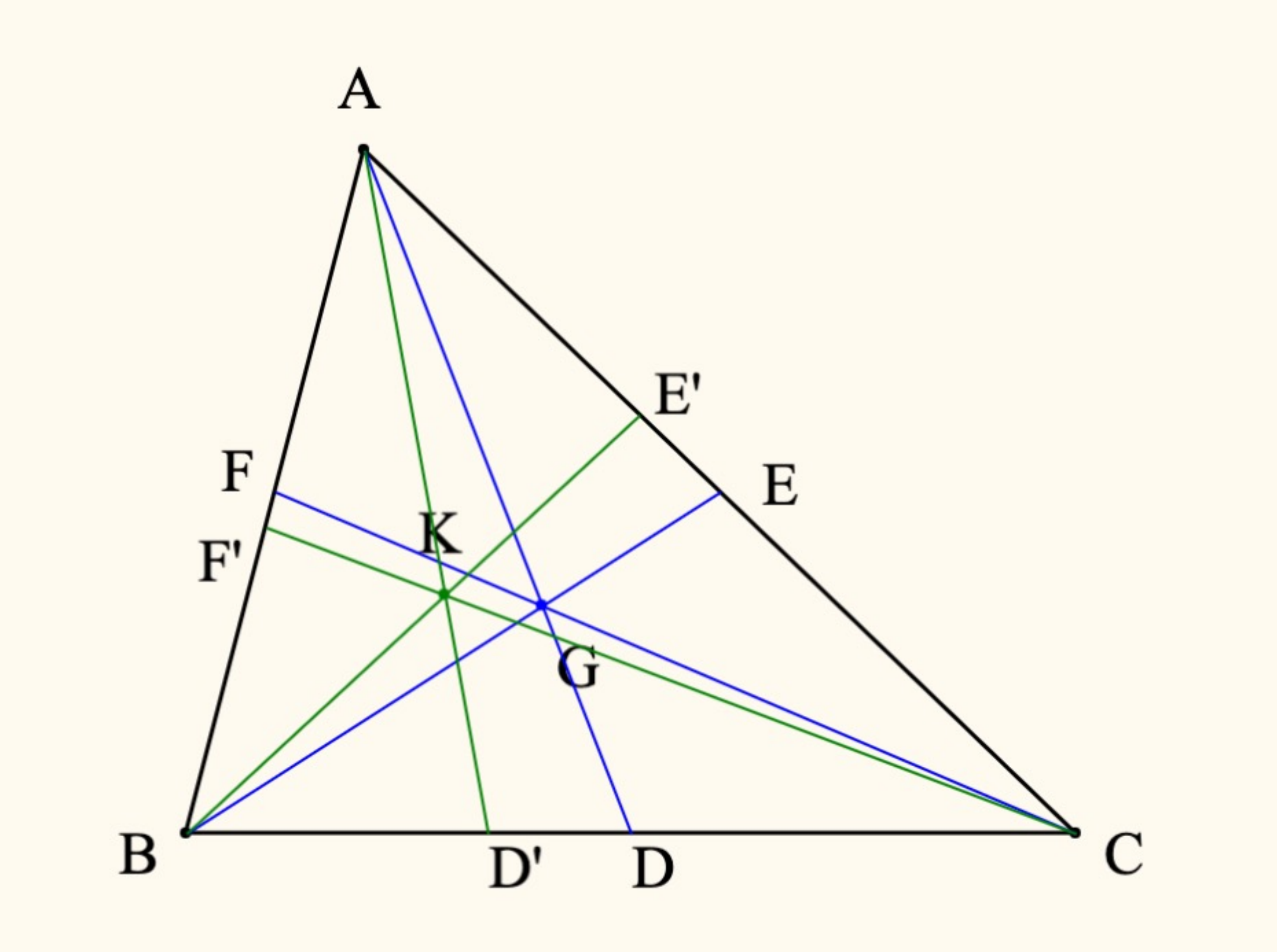
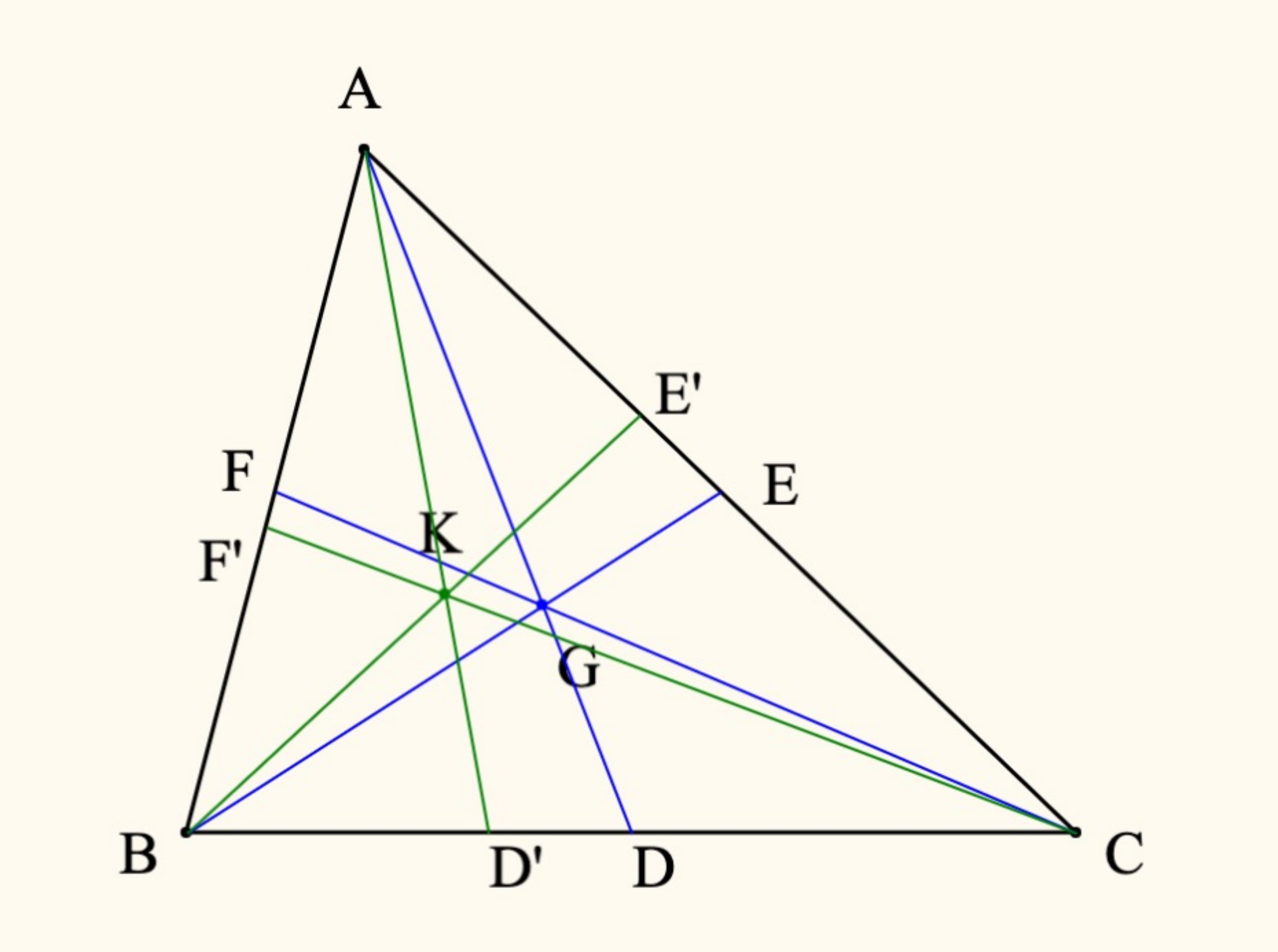
Symmedian

- symmedians are three particular lines associated with every triangle. They are constructed by taking a **median** of the triangle, and reflecting the line over the corresponding angle bisector (the line through the same vertex that divides the angle there in half)
- $\angle D'AC = \angle BAD$, $\angle E'BA = \angle CBE$, $\angle F'CA = \angle FCB$. These lines are called $\triangle ABC$'s Symmedians.



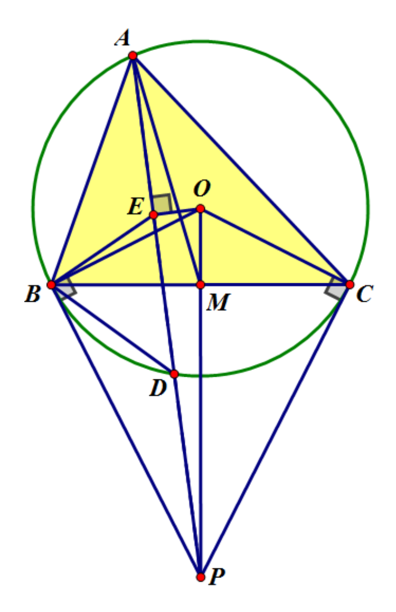
Symmedian Point

- Symmedian point is the intersection of the three Symmedians
- AD, BE, CF are the median point, which is the point G
- AD′ , BE′ , CF′ are the symmedian point which is the point K



AM: median of triangle ABC.
D: intersection of line AP and circle O.
E: midpoint of AD.

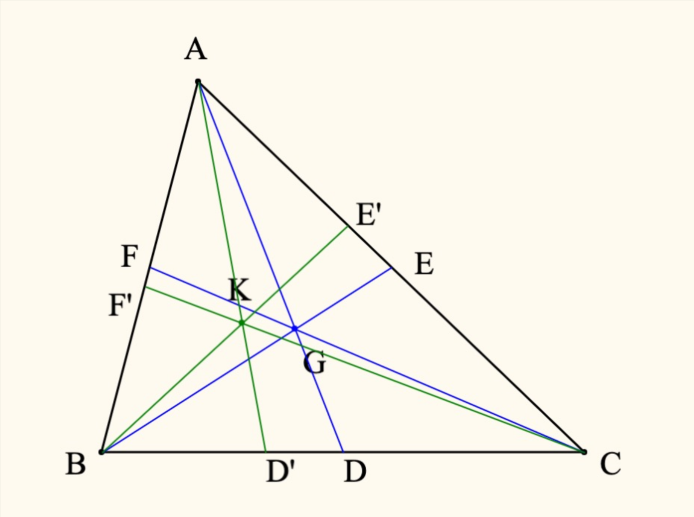
∠OEP= ∠OBP = 90 degree, so OEBP is a concylic quadrilateral.
∠BEP= ∠BOP=1/2 ∠BOC= ∠BAC,
So triangle BED and triangle BAC are similar. Since E and M are midpoints of AD and BC. Triangle BDA and MCA are similar or ∠BAD = ∠MAC
AD is the symmedian of triangle ABD.



Construct Symmedian Point

Let ABC be a triangle inscribed in a circle with center O. Draw tangents of the circle (O) at *B* and *C*, and they meet at point P. Then AP is the symmedian of triangle ABC.

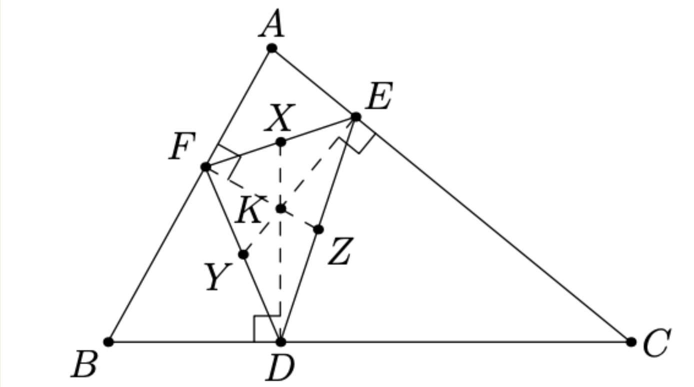
- **Theorem:** Three symmedians of a triangle are concurrent.
- By Ceva's theorem, we know that the lines AD' , BE' , CF' are concurrent if and only if $BD'/ D'C \cdot CE'/ E'A \cdot AF' /F'B = 1$.
- By definition, symmedians are lines that are isogonal to the corresponding medians of a triangle. $BD/DC \cdot BD' /D'C = (AB /CA)^2$ which implies $BD'/D'C = (AB/CA)^2$
- Similarly, we have $CE' /E'A = (BC /AB)^2$, $(AF'/ F'B) = (CA /BC)^2$.
- we have $(BD' /D'C) \cdot (CE'/ E'A) \cdot (AF'/ F'B) = (AB /CA)^2 \cdot (BC /AB)^2 \cdot (CA/ BC)^2 = 1$.



- Lemoine’s Pedal Triangle Theorem
- The symmedian point K of triangle ABC is the only point in the plane of ABC which is the centroid of its own pedal triangle.
- Ratio Lemma: $$\frac{XE}{XF} = \frac{KE}{KF} \cdot \frac{\sin XKE}{\sin XKF}.$$
- ∠XKE = ∠C and ∠XKF = ∠B since the quadrilaterals KDCE and KF BD are cyclic; thus, we conclude that

$$\frac{XE}{XF} = \frac{AC}{AB} \cdot \frac{\sin C}{\sin B} = \frac{AC}{AB} \cdot \frac{AB}{AC} = 1.$$

- This proves that X is the midpoint of EF

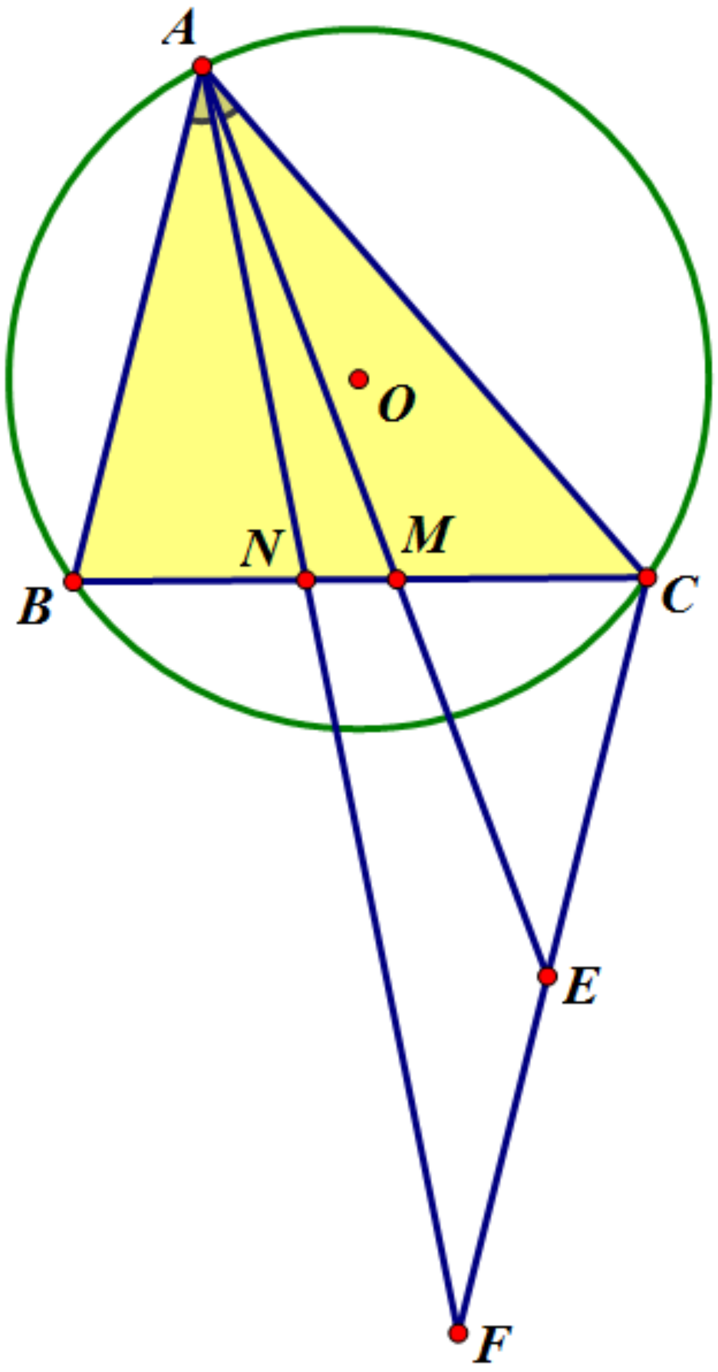


From C, draw a line parallel to ABAB, and it intersects AM, AN at E, FE,F, respectively. Then ∠CAM = ∠BAN = ∠CFA, which gives us triangle CEA and triangle $CAF \Longrightarrow CE\times CF=(CA)$^2.

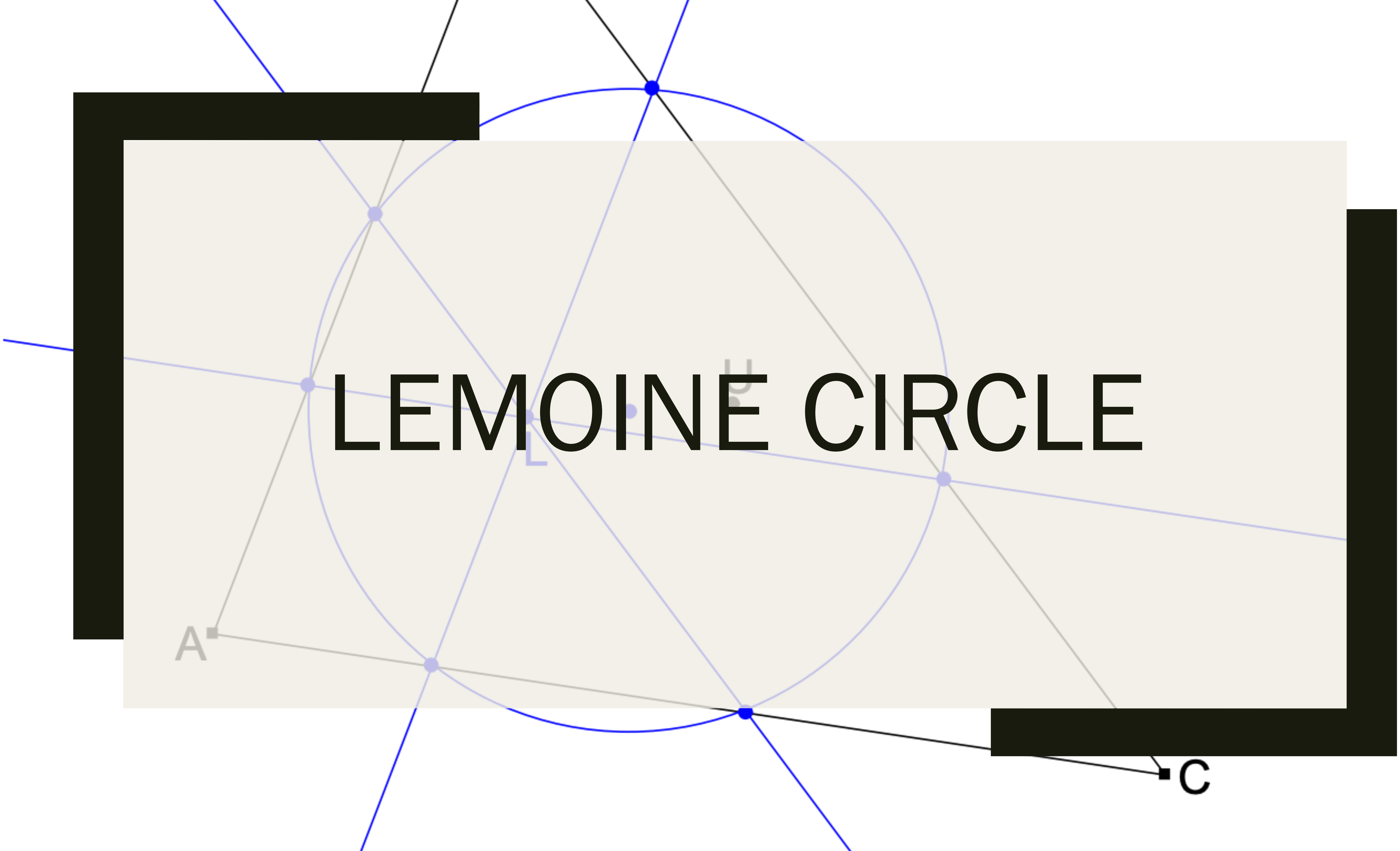
By Intercept Theorem,

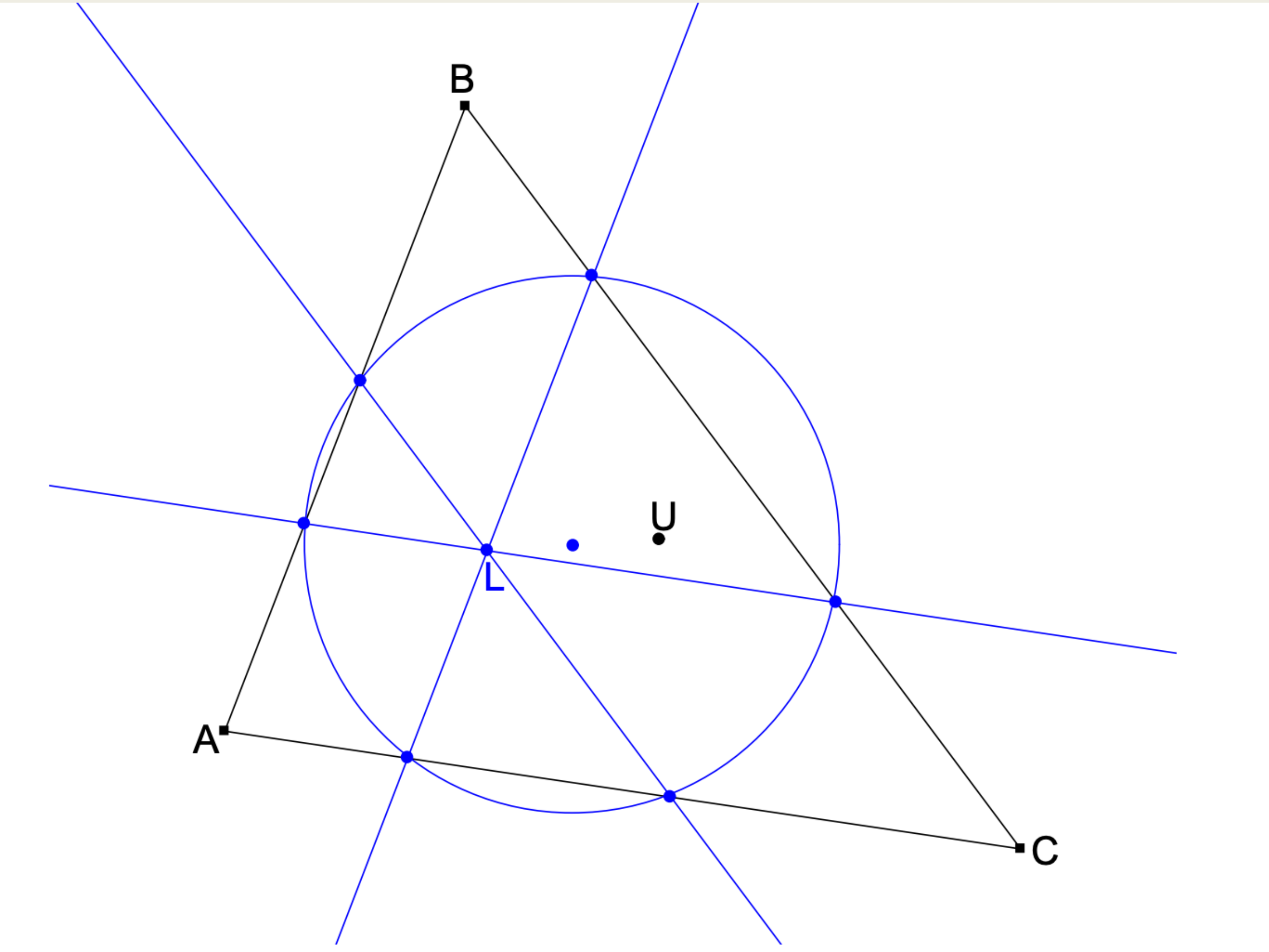
NB/NC =AB/CF and MB/MC = AB/CE

$$\frac{NB}{NC}\times\frac{MB}{MC}=\frac{AB^2}{CE\times CF}=\frac{AB^2}{AC^2}.$$



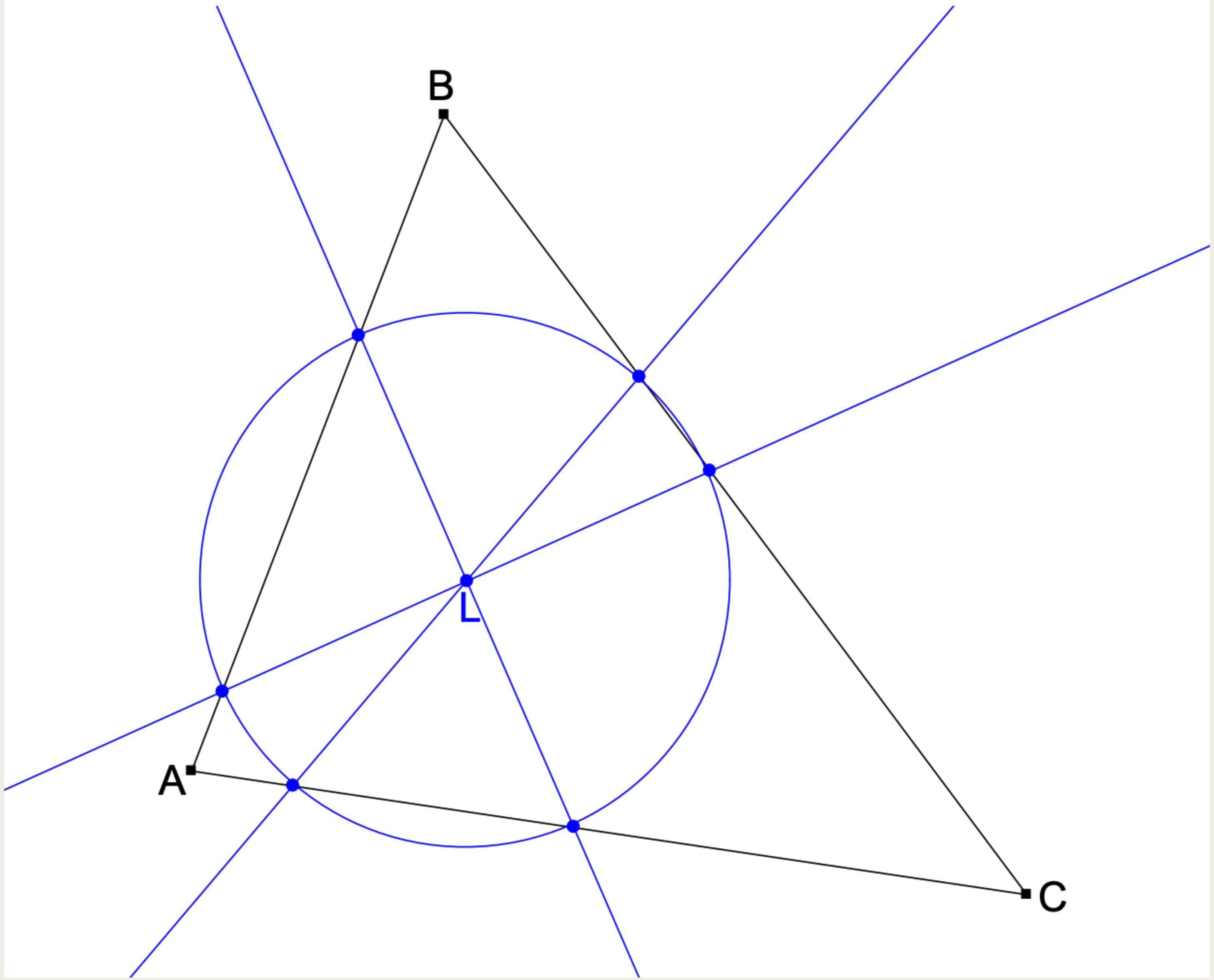
LEMOINE CIRCLE





First Lemoine circle of triangle ABC

- Let the parallels to the lines BC; BC; CA; CA; AB; AB through L meet the lines CA; AB; AB; BC; BC; CA at six points. These six points lie on one circle,
- This circle is a Tucker circle, and its center is the midpoint of the segment UL, where U is the circumcenter of triangle ABC



second Lemoine circle

- Let the Antiparallels to the lines BC; BC; CA; CA; AB; AB through L meet the lines CA; AB; AB; BC; BC; CA at six points. These six points lie on one circle, the so-called **second Lemoine circle**