Miquel Point

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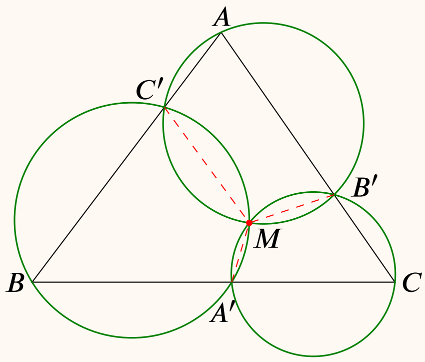
1 Introduction

Miquel Point was named after *Auguste Miquel* (1816-1851), who was a French mathematician. It is one of several important results concerning concurrent circles in Euclidean geometry. The results were published in *Journal de Mathématiques Pures et Appliquées*. See Wikipedia for more information about the Miquel Point.

Theorem 1. (Miquel Point of Triangle)

Let A', B' and C' be points on sides (or extensions) BC, CA, and AB of triangle $\triangle ABC$ respectively. Then the circumcircles of $\triangle AC'B'$, $\triangle BA'C'$, and $\triangle CB'A'$ are concurrent at a point M. The point is called the Miquel Point.



Proof. Assume that the circumcircles $\odot BA'C'$ and $\odot CB'A'$ intersect at a point M (other than A'). In order to prove the theorem, we need to prove that M is on circle $\odot AC'B'$, that is, A, C', M, B' are concyclic.

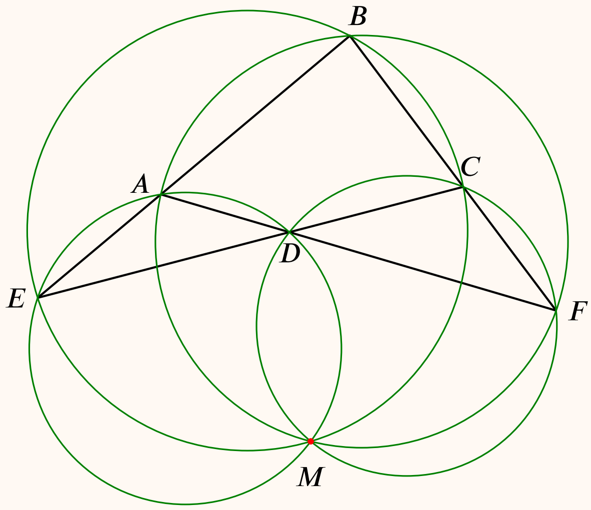
Connect MA', MB' and MC', because A', C, B', M are concyclic, we get $\angle MB'A = \angle MA'C$. Similarly, since C', B, A', M are concyclic, we get $\angle BC'M = \angle MA'C$. Therefore $\angle MB'A = \angle BC'M$, and hence A, C', M, B' are concyclic. ∎

Theorem 2. (Miquel Point of Complete Quadrilateral)

Four lines intersect at six points $ABCDEF$, and the resulting figure is called a complete quadrilateral. A complete quadrilateral contains four triangles. Their circumcircles

[1,2] The authors thank Stephanie Wang for her careful reading and many suggestions.

are concurrent, and the concurrent point is called the Miquel Point of the complete quadrilateral.

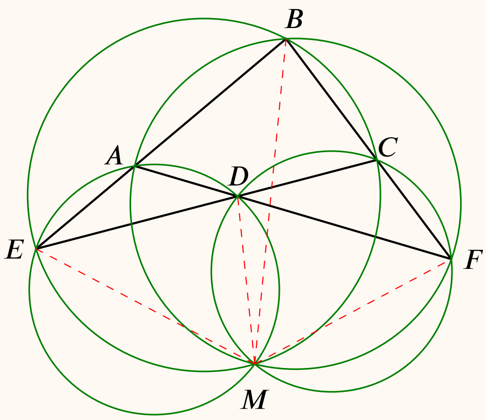


Proof. We consider $\triangle BAF$, points E, D and C are on BA, AF, and BC, respectively. Then by Theorem 1, the circumcircles of $\triangle AED$, $\triangle CDF$ and $\triangle BEC$ are concurrent at Miquel point M. Similarly, if we consider $\triangle BEC$, then the circumcircles of $\triangle AED$, $\triangle CDF$ and $\triangle BAF$ are concurrent at point M'. Since both M and M' are on the circles $\odot AED$, $\odot CDF$, we must have $M' = M$ and thus completes the proof. ■

Theorem 3

In the following picture, $ABCDEF$ is a complete quadrilateral, and point M is the Miquel Point. Then we have

$$MA \cdot MC = MB \cdot MD = ME \cdot MF.$$



Proof. Connect ME, MD, MB and MF. In order to prove the theorem, we need to prove $\triangle MDF \sim \triangle MEB$.

Since $BAMF$ are concyclic, $\angle MFD = \angle MBE$. Similarly, since $AEMD$ are concyclic, $\angle MDF = \angle MEB$. Therefore, $\triangle MDF \sim \triangle MEB$, and hence $MB \cdot MD = ME \cdot MF$. The same argument shows that $ME \cdot MF = MA \cdot MC$. ∎

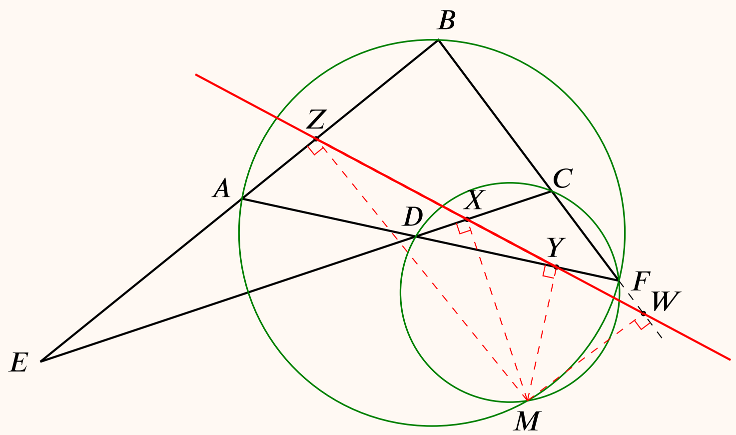
2 Simson Line and Orthocentric Line

Miquel Point is closely related to several other concepts including Simson Line, Orthocentric Line, and Newton Line.

For Simson Line of triangle, see Wikipedia or Topic 5. In the following we define the Simson Line for complete quadrilateral.

Theorem 4

Let M be the Miquel Point of complete quadrilateral $ABCDEF$. Then the pedal points of M to each side of the quadrilateral are collinear. The line is called the Simson Line of the complete quadrilateral.

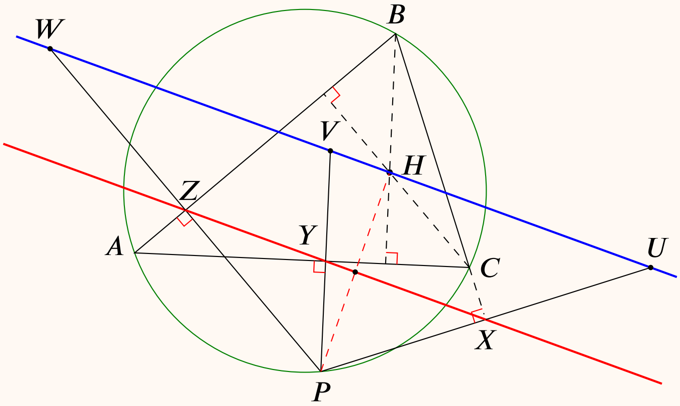


Proof. Let X, Y, Z, W be the projections of M to each side of the quadrilateral. Points X, Y, W are collinear because it is the Simson Line of $\triangle DCF$ with respect to M. Similarly, Points Z, Y, W are collinear because it is the Simson Line of $\triangle ABF$ with respect to M. Therefore X, Y, Z, W are collinear. ∎

Parallel to the Simson Line, we are able to define the *Orthocentric Line* of complete quadrilateral. In order to define such a line, we prove the following result related to the Simson Line of triangle which is of interest by itself.

Theorem 5

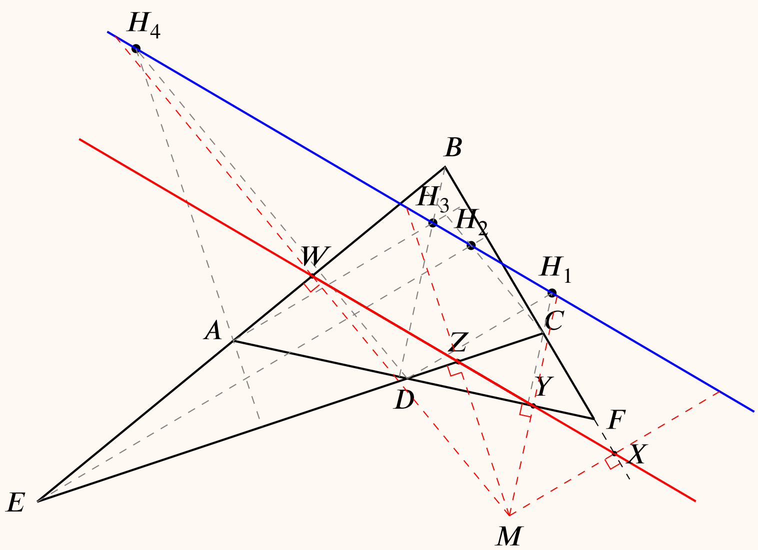
Let P be a point on the circumcircle of $\triangle ABC$, and X, Y, Z be the pedal points from P to BC, CA, and AB, respectively. XYZ is the Simson line with respect to P. Let U, V, W be the symmetric points of P with respect to X, Y, Z, respectively. Then U, V, W, and the orthocenter H of $\triangle ABC$ are collinear.



Proof. Since $UX = XP$, $VY = YP$, and $WZ = ZP$, and since X, Y, Z are collinear, then U, V, W are collinear. To prove that the orthocenter H is on UVW, we use Theorem 3 in Topic 05. By that theorem, the Simson line XYZ bisects PH, and therefore H is on UVW because the Simson Line bisects UP, VP, and WP. ■

Corollary 1

Let $ABCDEF$ be a complete quadrilateral. Let H_1, H_2, H_3, H_4 be the orthocenters of $\triangle DCF$, $\triangle BEC$, $\triangle BAF$, and $\triangle AED$, respectively. Then H_1, H_2, H_3, H_4 are collinear, and the line is called the Orthocentric Line *of the complete quadrilateral.*



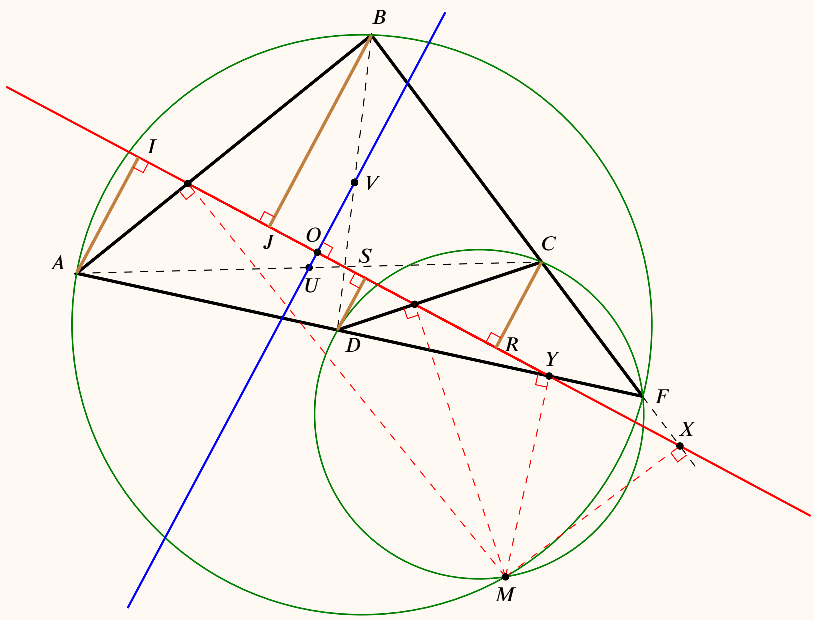
Proof. Let X, Y, Z, W be the projections of the Miquel Point M to each side of $ABCDEF$. Then by the above theorem, the symmetric points of M to X, Y, Z, W are collinear (the blue line in the picture). By using the above theorem again, we know that all the orthocenters H_1, H_2, H_3, H_4 must be on that line. This completes the proof.

For the rest of the article, we shall prove that the Simson Line (and hence the Orthocentric Line) is perpendicular to the Newton Line in a complete quadrilateral. For the definition of Newton Line, see Topic 26.

We provide two proofs here. One proof depends on an interesting result of Simson Line (Theorem 5 of Topic 05), and the other is a direct proof.

Theorem 6

The Newton Line and the Simson Line of a complete quadrilateral are perpendicular.[a]



[a]The red line is the Simson Line, and the blue line UV is Newton Line, where U, V are the midpoints of AC and BD, respectively.

Proof. In the above picture, let I, J, R, S be the projections of A, B, C, D to the Simson Line (the red line), respectively. Let X, Y be the projections of the Miquel Point M to BF and AF, respectively. Then by Theorem 5 of Topic 05, we have

$$IJ = XY = RS.$$

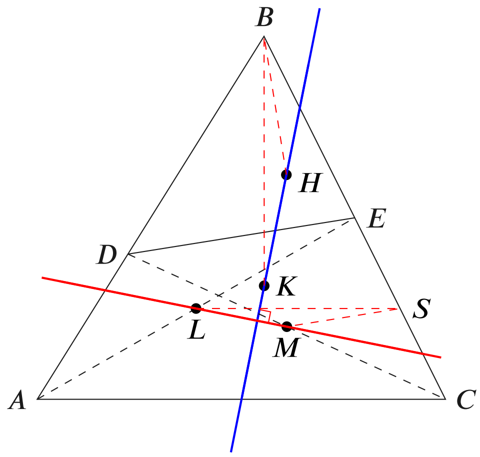
As a result, let O be the midpoint of IR. Then it is also the midpoint of JS.

Since U is the midpoint of AC, OU is the midline of the (crossed) right trapezoid $ACRI$, therefore we conclude that UO is perpendicular to the Simson Line; similarly,

VO is also perpendicular to the same line. Thus U, O, V are collinear and completes the proof of the theorem. ■

We would like to introduce a direct proof of the above important theorem. The technique is related to the article The Usage of Special Techniques of the first author, where the idea is to create a triangle made by midlines (See $\triangle LMS$ in the following).

A Direct Proof. In the following picture, let H, K be the orthocenters of $\triangle BDE$ and $\triangle BAC$, respectively. The line HK is the Orthocentric Line of the complete quadrilateral made from the lines AC, CE, ED and DA. Let L, M be the midpoints of AE and DC, respectively. Then LM is the Newton Line. We need to prove $LM \perp HK$.



We connect BH, BK. Let S be the midpoint of EC. We shall prove that

$$\triangle BHK \sim \triangle SML.$$

Let α be the angle of AC to DE. Let S be the midpoint of EC. Since S is the midpoint of EC, we have $LS \parallel AC$ and $MS \parallel DE$. As a result, $\angle LSM = \alpha$. On the other hand, $BK \perp AC$ and $BH \perp DE$. Thus $\angle KBH = \alpha$ also.

Next, it is obvious that

$$\frac{LS}{MS} = \frac{AC}{DE}.$$

Since K is the orthocenter of $\triangle BAC$, we have $BK = AC \cot \angle B$. By the same argument, $BH = DE \cot \angle B$. Therefore

$$\frac{LS}{MS} = \frac{AC}{DE} = \frac{KB}{HB},$$

and as a result, $\triangle BHK$ and $\triangle SML$ are similar.

We thus conclude that

$$\angle BKH = \angle SLM.$$

Since $LS \parallel AC$ and $BK \perp AC$, we have $LS \perp BK$. Since $BK \perp LS$, we conclude that $HK \perp LM$. ∎