# Pascal's and Brianchon's Theorems

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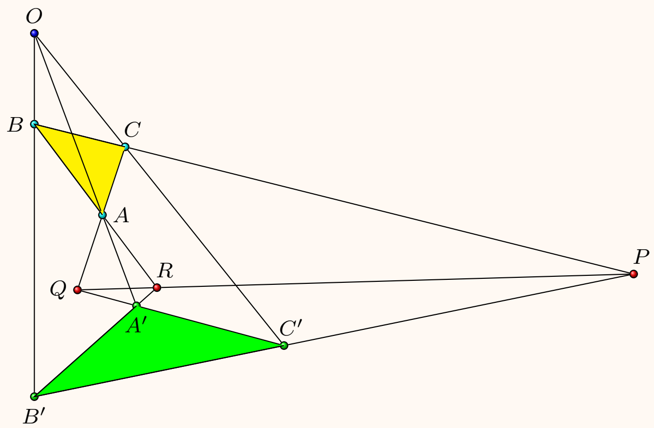
## 1 Desargues' Theorem

The Pascal's and Brainchon's Theorems are two famous "dual" theorems in Projective Geometry. But let's first introduce a related classical theorem.

**Theorem 1. (Desargues' Theorem)**

*We consider the triangles $\triangle ABC$ and $\triangle A'B'C'$. Assume that the lines $BC$, $B'C'$ intersect at $P$, $CA$ and $C'A'$ intersect at $Q$, and $AB$, $A'B'$ intersect at $R$. Then $P, Q, R$ are collinear if and only if $AA'$, $BB'$ and $CC'$ are concurrent.*




The line $PQR$ is called the *axis of perspectivity*, and the point $O$ is called the *center of perspectivity*. The theorem is known as *Perspective principle* in painting. See Wikipedia for details.

**Proof:** Here we provide a proof using the Menelaus' Theorem only. On $\triangle OB'C'$, since $B, C, P$ are collinear, we must have

$$\frac{OB}{BB'} \cdot \frac{B'P}{PC'} \cdot \frac{C'C}{CO} = 1.$$

On $\triangle OC'A'$, since $C, A, Q$ are collinear, we have

$$\frac{C'C}{CO} \cdot \frac{OA}{AA'} \cdot \frac{A'Q}{QC'} = 1.$$

On $\triangle OA'B'$, since $A, B, R$ are collinear, we have

$$\frac{OB}{BB'} \cdot \frac{B'R}{RA'} \cdot \frac{A'A}{AO} = 1.$$

From the above 3 equations, we get

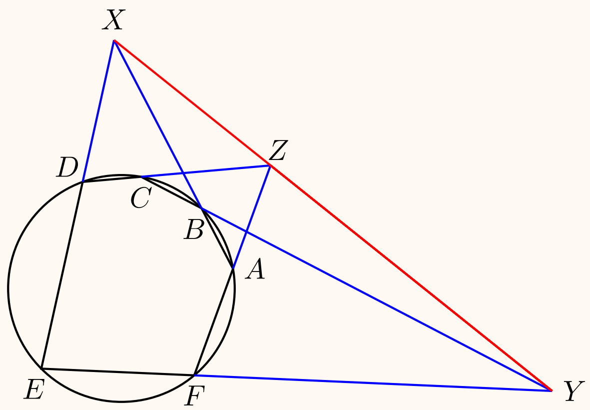
$$\frac{A'Q}{Q'C} \cdot \frac{C'P}{PB'} \cdot \frac{B'R}{RA'} = 1.$$

Therefore, using the Menelaus Theorem on $\triangle A'B'C'$, we conclude that $P, Q, R$ are collinear. ■

## 2 Pascal's Theorem

**Theorem 2. (Pascal's Theorem)**

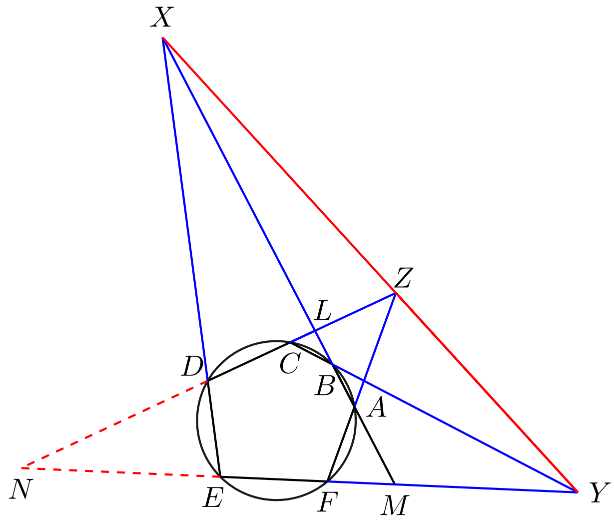
*The hexagon $ABCDEF$ is inscribed to a circle. Assume that $AB$, $DE$ intersects at $X$; $BC$, $EF$ intersects at $Y$; and $CD$, $FA$ intersects at $Z$. Then $X, Y, Z$ are collinear.*




**Remark** The above is called *Pascal's Theorem*, which was discovered by the French mathematician *Blaise Pascal*, when he was 16 years old. The theorem can be generalized to the case of conic section (see Wikipedia). When the conic section is degenerated to two lines, it is also called the *Pappus' Hexagon Theorem* below – some people believed that Euclid knew this theorem before Pappus.

We first give a purely Euclidean Geometry proof.

**Proof:** As in the graph drawn below, let $AB$ and $CD$ intersects at $L$, $BA$ and $EF$ intersects at $M$, $CD$ and $FE$ intersects at $N$.




On $\triangle LMN$, since $C, B, Y$ are collinear, applying Menelaus Theorem we obtain

$$\frac{LB}{BM} \cdot \frac{MY}{YN} \cdot \frac{NC}{CL} = 1.$$

Similarly, since $F, A, Z$ are collinear, we obtain

$$\frac{LA}{AM} \cdot \frac{MF}{FN} \cdot \frac{NZ}{ZL} = 1,$$

and since $E, D, X$ are collinear, we also get

$$\frac{ND}{DL} \cdot \frac{LX}{XM} \cdot \frac{ME}{EN} = 1.$$

In the circle $ABCDEF$, using the Power of Point Theorem, we will get

$$LA \cdot LB = LD \cdot LC,$$

$$NC \cdot ND = NE \cdot NF,$$

$$MA \cdot MB = MF \cdot ME.$$

Combining the above 6 equations, we obtain that

$$\frac{LX}{XM} \cdot \frac{MY}{YN} \cdot \frac{NZ}{ZL} = 1$$

Thus, by the inverse of Menelaus theorem we conclude that $X, Y, Z$ are collinear.

We are able to use algebraic method to prove Pascal Theorem as well. However, it is surprising that the algebra behind Pascal Theorem is about the factorization of cubic polynomial.

**Second proof:** We assume the circle is the unit circle. Let the equations for

$$AB, BC, CD, DE, EF, FA$$

be $\ell_1, \ell_2, \cdots, \ell_6$. These functions $\ell_j$ are linear functions. As a result, we consider two cubic polynomials $\ell_1\ell_3\ell_5$ and $\ell_2\ell_4\ell_6$. Obviously, these two polynomials pass the nine points $A, B, C, D, E, F, X, Y, Z$.

We choose a general point $P$ in the circle. Choose a number $\lambda$ such that

$$(\ell_1\ell_3\ell_5 + \lambda\ell_2\ell_4\ell_6)(P) = 0.$$

Here is a fundamental question: in general, if a cubic curve doesn't vanishing identically on the unit circle, then what is the maximum number of intersections? The answer is 6, and we shall prove it.

We can use complex numbers to write any cubic polynomials as

$$f(z) = Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3 + Ez^2 + Fz\bar{z} + G\bar{z}^2 + Hz + I\bar{z} + J = 0.$$

Let $z = e^{i\theta}$ be a point on the unit disk, then $\bar{z} = 1/z$. If we multiply the above equation by $z^3$ on both sides, we get a degree 6 polynomial of single variables. In general, such a 6-degree polynomial has at most 6 roots. Since $f(z)$ vanishes on severn points $A, B, C, D, E, F, P$ on the unit circle, it must be vanishing identically on the circle. As a result, we can factorize it as

$$f(z) = (|z|^2 - 1)\ell(z),$$

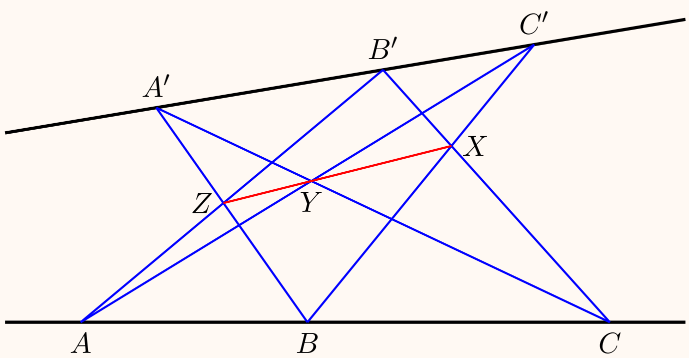
where, by the degree consideration, $\ell(z)$ must be linear. Since $\ell$ passes $X, Y, Z$, we conclude that $X, Y, Z$ are collinear.

■

As mentioned above, Pascal Theorem can be generalized to the case of conic curves. The above algebraic proof can be used to prove the general conic Pascal Theorem as well. A special case, where the conic curve is degenerated to two lines, is also called *Pappus' Theorem*.

**Theorem 3. (Pappus' Theorem)**

*In the following picture, the Hexagon $BC'AB'CA'$ is inscribed on the two black lines. Assume that $BC', B'C$ intersect at $X$; $CA', A'C$ intersect at $Y$, and $AB', B'A$ intersect at $Z$. Then $X, Y, Z$ are collinear.*

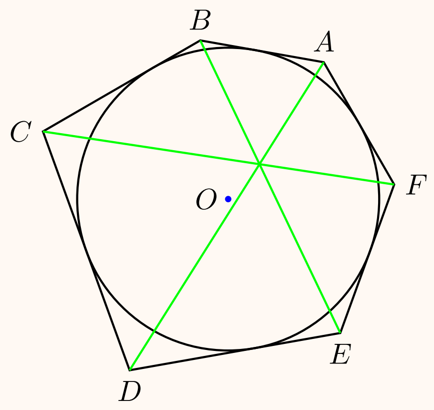



## 3 Brianchon's Theorem

In this section, we introduce the *Brianchon's Theorem* on circumscribed hexagon.

**Theorem 4. (Brianchon's Theorem)**

*The Hexagon $ABCDEF$ is circumscribed on a circle. Then $AD, BE$, and $CF$ are concurrent.*

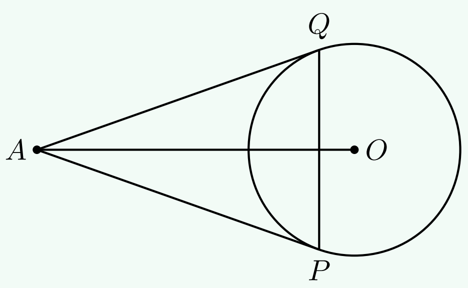



**Remark** Pascal's Theorem and Brainchon's Theorem are the two famous "dual" theorems. Using the following concepts of *Pole and Polar*, we can prove that Pascal and Brianchon Theorems are equivalent.

**Definition. (Pole and Polar)**

*Let $O$ be the unit circle. The the pair $(A, PQ)$ is called the pair of pole and polar, where $A$ is the pole, and $PQ$ is the polar.*
*Let $(x_0, y_0)$ be the coordinates of $A$. Then the equation of $PQ$ is*

$$x_0x + y_0y - 1 = 0.$$




**Proof of the Brianchon Theorem (using pole and polar):** See the graph below: let the coordinates of $A_i$ be $(x_i, y_i)$ for $1 \le i \le 6$. Then the equations for $B_6B_1$ is

$$\ell_1(x, y) = x_1x + y_1y - 1.$$

Similarly, the equations for $B_iB_{i+1}$ for $1 \le i \le 5$ are

$$\ell_i(x, y) = x_ix + y_iy - 1.$$

Using the Pascal Theorem, there is a number $\lambda$ such that

$$\ell_1\ell_3\ell_5 + \lambda\ell_2\ell_4\ell_6 = C(x^2 + y^2 - 1)(px + qy - 1),$$

where $C$ is a constant. We claim $(p, q)$ is on the lines $A_1A_4$, $A_2A_5$, and $A_3A_6$.

In order to prove this, let $P = (p_1, q_1)$ be the intersection of $B_1B_6$ and $B_3B_4$. Then we have

$$x_1p_1 + y_1q_1 - 1 = 0, \qquad x_4p_1 + y_4q_1 - 1 = 0.$$

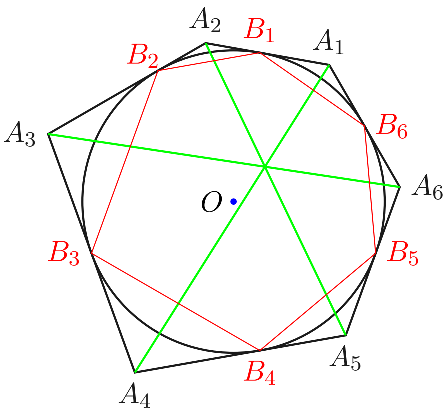
Moreover, we have

$$pp_1 + qq_1 - 1 = 0.$$

Thus the three points $A_1, A_4$ and $(p, q)$ are on the line

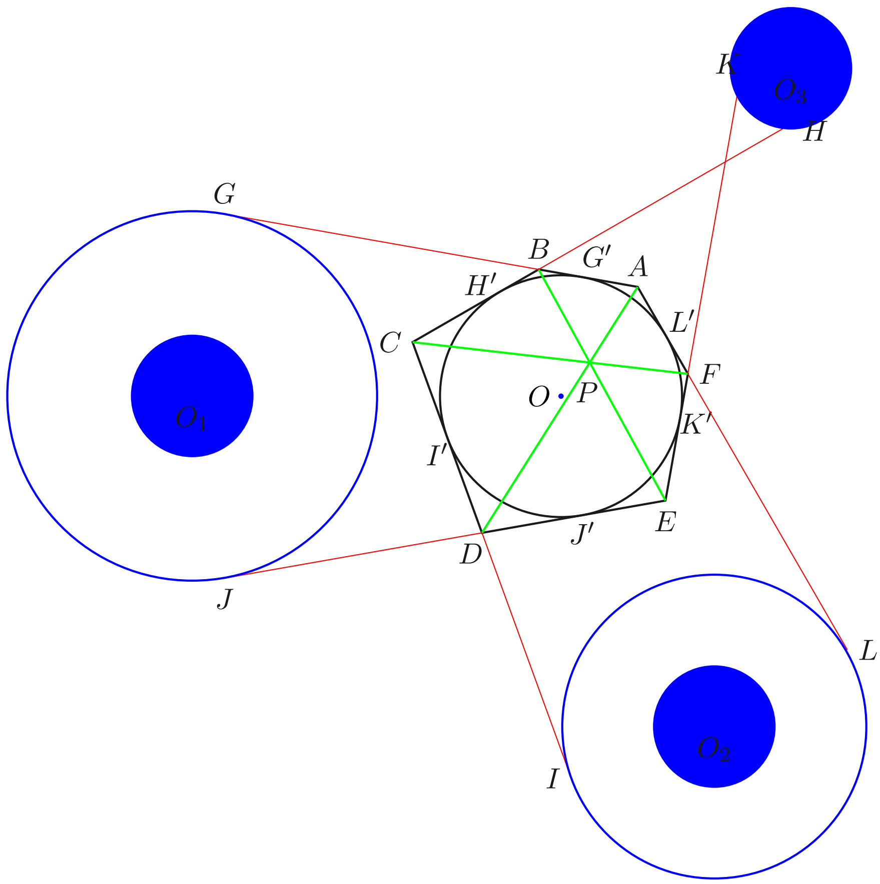
$$p_1x + q_1y - 1 = 0.$$

This completes the proof.




**Remark** By the above proof, we also know that Pascal and Brianchon Theorems are equivalent.

**A Euclidean Geometry proof:** We shall use the Monge's Theorem of radical axes to prove the result.




In the above theorem, let $G', H', I', J', K', L'$ be the tangent points of the lines $AB, BC, CD, DE, FE$ and $FA$ respectively. Define $G, H, I, J, K, L$ such that

$$GG' = HH' = II' = JJ' = KK' = LL'.$$

Define circles $O_1, O_2$ and $O_3$ such that $GG', JJ'$ are tangent lines to $O_1$, $II', LL'$ are tangent lines to $O_2$, and $KK', HH'$ are tangent to $O_3$. It is well known that $AD$ is the radical axis of $O_1, O_2$; $BE$ is the radical axis of $O_3, O_1$; and $CF$ is the radical axis of $O_2, O_3$. By the Monge Theorem, $AD$, $BE$ and $CF$ are concurrent.

**External Link.** *Here is the Wikipedia link to the Brianchon's Theorem.*