# Quadrilateral Area Formulas

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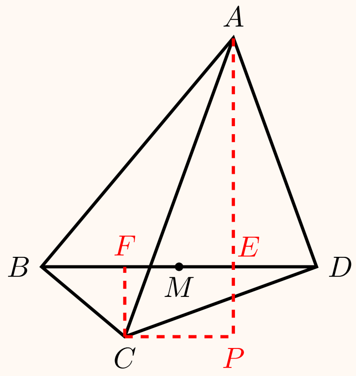
There is no simple area formula for a general quadrilateral. This is partially due to the *instability of quadrilateral*, that is, unlike the case of triangle, four sides do not completely determine a unique quadrilateral.

In the following *Bretschneider's Formula*, the area is represented by the four sides and the two diagonals of a quadrilateral.

**Theorem 1. (Bretschneider Formula)**

*In the following convex quadrilateral $ABCD$. Assume that $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, and $BD = f$. Then the area $S$ of $ABCD$ is given by*

$$S = \frac{1}{4}\sqrt{4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2}$$




**Proof.** Draw $AP \perp BD$ and $CP \perp AP$. The key computation should be $AP$, because

$$S = \frac{1}{2}BD \cdot AP. \tag{1}$$

Let $M$ be the midpoint of $BD$. By the Pythagorean Theorem, we have

$$AB^2 - AD^2 = BE^2 - ED^2 = (BE + ED)(BE - ED).$$

Thus

$$a^2 - d^2 = f(BE - ED) = f(BM + ME - (MD - ME)) = 2f \cdot ME.$$

Similarly, we have

$$c^2 - b^2 = 2f \cdot FM.$$

[1]The author thanks Stephanie Wang for her careful reading and many suggestions.

Therefore, we have

$$a^2 - b^2 + c^2 - d^2 = 2f \cdot EF.$$

Thus

$$AP = \sqrt{AC^2 - EF^2} = \sqrt{e^2 - \frac{1}{4f^2}(a^2 - b^2 + c^2 - d^2)^2}.$$

The Bretschneider's Formula follows from the above equation and (1). ■

An algebraic variant of Bretschneider's Formula is the following:

**Corollary 1. (Coolidge Formula)**

*Let $p = (a+b+c+d)/2$ be the semiperimeter of the quadrilateral. Then the area $S$ is given by*

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d) - \frac{1}{4}(ac+bd-ef)(ac+bd-ef)}.$$

**Proof.** We have the following algebraic identity

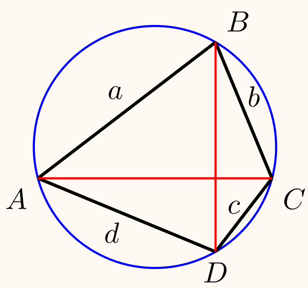
$$\begin{aligned}
&4(ac+bd)^2 - (a^2 - b^2 + c^2 - d^2)^2 \\
&= (2ac + 2bd + a^2 - b^2 + c^2 - d^2)(2ac + 2bd - a^2 + b^2 - c^2 + d^2) \\
&= ((a+c)^2 - (b-d)^2)((b+d)^2 - (a-c)^2) \\
&= (a+c+b-d)(a+c-b+d)(b+d+a-c)(B+d-a+c) \\
&= 16(p-a)(p-b)(p-c)(p-d).
\end{aligned}$$

The corollary then follows from Bretschneider's Formula. ■

A quadrilateral is called *cyclic*, if the four vertices all lie on a single circle.

**Corollary 2. (Brahmagupta's formula)**

*Assume that $ABCD$ is a convex cyclic quadrilateral with sides $a, b, c, d$, respectively.*




*Then the area of $ABCD$ is given by*

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)},$$

*where $p = (a+b+c+d)/2$ is the semiperimeter.*

**Proof.** Using the Ptolemy's Theorem (see Wikipeida or Topic 10), we have

$$ef = ad + bd.$$

Therefore the formula follows from the Coolidge formula (Corollary 1). ∎

**Corollary 3. (Heron's Formula)**

*Assume that the three sides of a triangle are $a, b, c$, respectively. Then the area of the triangle is given by*

$$S = \sqrt{p(p-a)(p-b)(p-c)},$$

*where $p$ is the semiperimeter.*

**Proof.** We take $d = 0$ in Brahmagupta's formula. When one side of a quadrilateral is degenerate to zero, the quadrilateral is degenerate to a triangle, and the Heron's formula follows. ∎

A polygon is called *tangential*, or *circumscribed*, if it is a convex polygon that contains an inscribed circle.

For the rest of this article, we introduce area formulas for tangential quadrilaterals and general tangential polygons.

**Theorem 2**

*Let $P = A_1 \cdots A_n$ be a tangential $n$-polygon with $r$ being the radius of the inscribed circle, where $A_i$ are the vertexes for $1 \leq i \leq n$. Let $a_i = A_iA_{i+1}$ for $1 \leq i \leq n-1$ and $a_n = A_nA_1$ be the side lengths of the polygon. Then the area of the polygon is given by*

$$S = \frac{1}{2}\Big(\sum_{i=1}^{n} a_i\Big)r.$$

The proof is obvious and left to the reader.

Before applying the above theorem into quadrilateral, we first study the following example.

**Problem**

*A circle is inscribed in quadrilateral $ABCD$, tangent to $\overline{AB}$ at $P$ and to $\overline{CD}$ at $Q$. Given that $AP = 19$, $PB = 26$, $CQ = 37$, and $QD = 23$, find the square of the radius of the circle.*

This is the Problem No. 10 in the 2000 AIME (II).

**Solution.** Call the center of the circle $O$. By drawing the lines from $O$ tangent to the sides and from $O$ to the vertices of the quadrilateral, four pairs of congruent right triangles are formed. Thus

$$\angle AOP + \angle POB + \angle COQ + \angle QOD = 180,$$

or

$$(\arctan(\frac{19}{r}) + \arctan(\frac{26}{r})) + (\arctan(\frac{37}{r}) + \arctan(\frac{23}{r})) = 180°.$$

Take the tangent of both sides and use the identity for $\tan(A+B)$ to get

$$\tan(\arctan(\frac{19}{r}) + \arctan(\frac{26}{r})) + \tan(\arctan(\frac{37}{r}) + \arctan(\frac{23}{r})) = 0.$$

Use the identity for $\tan(A+B)$ again to get

$$\frac{\frac{45}{r}}{1 - 19 \cdot \frac{26}{r^2}} + \frac{\frac{60}{r}}{1 - 37 \cdot \frac{23}{r^2}} = 0.$$

Solving the above equation, we get $r^2 = 647$.

In the above solution, we used the formula

$$\alpha + \beta = \arctan\left(\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\right).$$

It turns out that we have the following generalization of the above formula. Let $x_1, \cdots, x_n$ be real numbers. As is well-known,

$$\tan(\arctan x_1 + \arctan x_2) = \frac{x_1 + x_2}{1 - x_1 x_2}.$$

Thus we have

$$\tan(\arctan x_1 + \arctan x_2 + \arctan x_3) = \frac{\tan(\arctan x_1 + \arctan x_2) + x_3}{1 - \tan(\arctan x_1 + \arctan x_2)x_3}.$$

Using the formula again, we get

$$\tan(\arctan x_1 + \arctan x_2 + \arctan x_3) = \frac{\frac{x_1+x_2}{1-x_1x_2} + x_3}{1 - \frac{x_1+x_2}{1-x_1x_2}x_3}$$
$$= \frac{x_1 + x_2 + x_3 - x_1x_2x_3}{1 - (x_1x_2 + x_2x_3 + x_3x_1)}.$$

In general, we define the *elementary symmetric polynomials* by

$$\begin{aligned}
\sigma_0 &= 1 \\
\sigma_1 &= x_1 + \cdots + x_n = \sum_i x_i \\
\sigma_2 &= \sum_{i<j} x_i x_j \\
&\cdots\cdots \\
\sigma_k &= \sum_{j_1<j_2<\cdots<j_k} x_{j_1} \cdots x_{j_k} \\
&\cdots\cdots \\
\sigma_n &= x_1 \cdots x_n
\end{aligned}$$

and $\sigma_r = 0$ if $r > n$. Then we have the following result.

**Theorem 3**

*Using the above notations, we have*

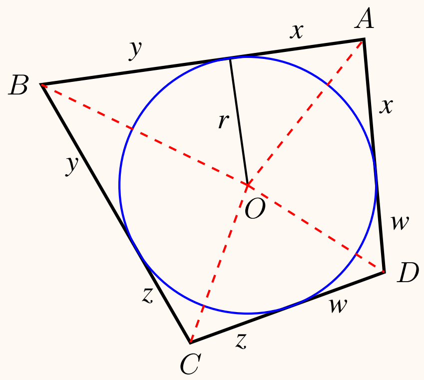
$$\tan\left(\sum_i \arctan x_i\right) = \frac{\sigma_1 - \sigma_3 + \sigma_5 - \cdots}{1 - \sigma_2 + \sigma_4 - \cdots}.$$

The theorem can be proved using the mathematical induction. We omit the proof here.

**Theorem 4**

*In the following quadrilateral $ABCD$ inscribed by the circle of radius $r$, we assume that the lengths of the tangents are $x, y, z, w$, respectively. Then the area $S$ is given by*

$$S = \sqrt{(x+y+z+w)(xyz+yzw+zwx+wxy)}.$$




**Proof.** We connect $OA, OB, OC, OD$. Then

$$S = S_{\triangle OAB} + S_{\triangle OBC} + S_{\triangle OCD} + S_{\triangle ODA} = (x+y+z+w)r.$$

In order to express $r$ in terms of $a, b, c, d$, we observe that

$$\angle OAB + \angle OBC + \angle OCD + \angle ODA = 360^\circ.$$

Thus we have

$$\arctan\frac{x}{r} + \arctan\frac{y}{r} + \arctan\frac{z}{r} + \arctan\frac{w}{r} = 180^\circ.$$

By Theorem 3, we have

$$0 = \sigma_1 - \sigma_3,$$

where $\sigma_1 = (x+y+z+w)/r$, and $\sigma_3 = (xyz+yzw+zwx+wxy)/r^3$. Thus we solve

$$r = \sqrt{\frac{xyz+yzw+zwx+wxy}{x+y+z+w}}.$$

The formula then follows from Theorem 2.

A quadrilateral is called *bicentric*, if it is both tangential and cyclic.

**Corollary 4**

*Let $a, b, c, d$ be the side lengths of a bicentric quadrilateral. Then its area is given by*

$$S = \sqrt{abcd}.$$

**Proof.** We use the picture in Theorem 4, where the lengths of tangent lines from the vertexes are $x, y, z, w$. Then the semiperimeter $p$ is given by

$$p = x + y + z + w.$$

By definition, $a = x + y$, $b = y + z$, $c = z + w$, and $d = w + x$. Then we have $p - a = z + w = c$ and similarly, $p - b = d$, $p - c = a$, and $p - d = b$. Thus the corollary follows from Brahmagupta's formula (Corollary 2). ■