# Euler Line

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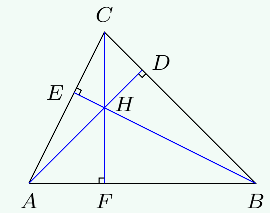
## 1 Introduction

In 1765, Swiss mathematician *Leonhard Euler* discovered that the *orthocenter*, the *circumcenter*, and the *centroid* of a triangle are collinear. The line is thus called the *Euler Line*. Later in 1820s, it was discovered by German mathematician *Karl Feuerbach* that the *nine-point center* is also on the Euler line. So the four important triangle centers are collinear.

We first recall the definition of these special centers of triangle.

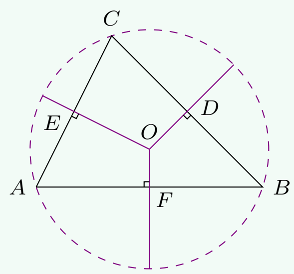
**Definition 1. (Orthocenter)**

*In a triangle, the three altitudes are concurrent, and the concurrent point is called the orthocenter of the triangle. In the following $\triangle ABC$, the intersection $H$ of three altitudes $AD$, $BE$, $CF$ is the orthocenter of $\triangle ABC$.*




**Definition 2. (Circumcenter)**

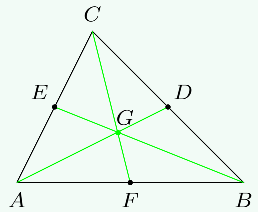
*The circle that passes through three vertices of a triangle is called the circumcircle. The center of the circumcircle is called circumcenter, which is also the intersection of the three perpendicular bisectors of the sides. In the following $\triangle ABC$, the circle that passing through vertices $A,B,C$ is the circumcircle and the circle center $O$ is the circumcenter.*




[1]The author thanks Dr. Zhiqin Lu for his help.

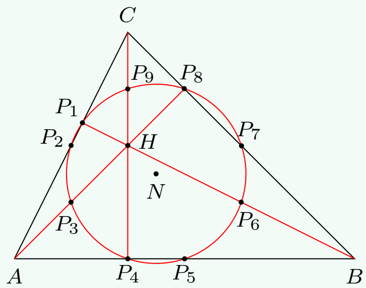
**Definition 3. (Centroid)**

*In a triangle, the intersection of three medians is called* centroid*. In the following* $\triangle ABC$*, the intersection* $G$ *of the three medians* $AD, BE, CF$ *is the centroid of* $\triangle ABC$.




**Definition 4. (Nine-Point Center)**

*In a triangle, the circle that intersects with three midpoints of the sides* ($P_2, P_5, P_7$)*, three feet of the altitudes* ($P_1, P_4, P_8$)*, and three midpoints of the vertices and the orthocenter* ($P_3, P_6, P_9$) *is called a* nine-point circle*, and point* $N$ *is called the* nine-point center.



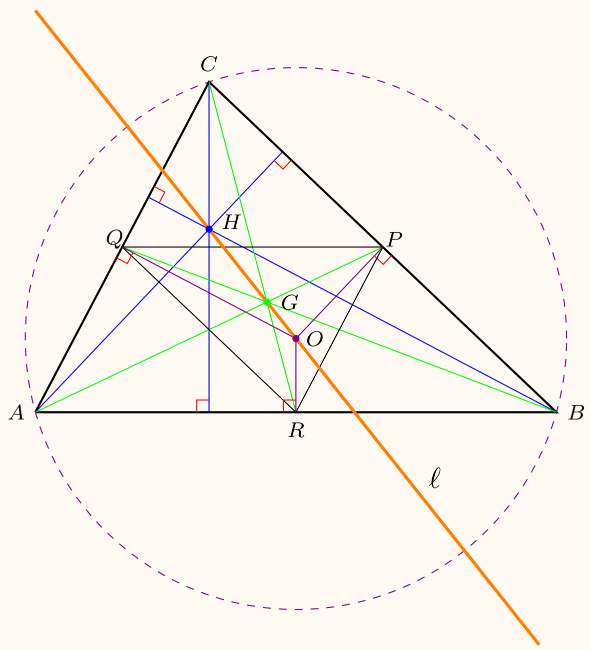

## 2 Euler Line and Its Properties

**Definition 5. (Euler Line)**

*In a triangle, the line that passes through the orthocenter, centroid and circumcenter is called the* Euler Line.

**Theorem 1**

*In the following* $\triangle ABC$*, let* $H$*,* $G$ *and* $O$ *be the orthocenter, the centroid, and the circumcenter, respectively. Then* $H$*,* $G$*,* $O$ *are collinear.*




We give two proofs. The first one is geometric and the second one is more algebraic.

**First proof:** In the $\triangle ABC$ above, let $P, Q, R$ be the midpoints of $BC, AC, AB$. Since $CH \,/\!/\, OR$, it is clear that $\angle HCG = \angle GRO$. By the property of the centroid, $CG : GR = 2 : 1$. Also, since $P$,$Q$,$R$ are midpoints of $\triangle ABC$, we have that $\triangle ABC$ is similar to $\triangle PQR$ with the ratio of $2 : 1$, where the circumcenter $O$ of $\triangle ABC$ is precisely is orthocenter of $\triangle PQR$, which imples that $CH : OR = 2 : 1$. Thus, $\triangle GRO$ is similar to $\triangle GCH$, and therefore $\angle RGO = \angle CGH$. This implies that the orthocenter, centroid and circumcenter are collinear. ■

We use *Trilinear Coordinate System* in the second proof. For the definition of trilinear coordinate system, refer to Wikipedia or Topic 37. From there one can also find the trilinear coordinates of the following centers. Let $A, B, C$ be the three angles of $\triangle ABC$. Then the trilinear coordinates are given by

Orthocenter: $(\sec(A) : \sec(B) : \sec(C))$;

Centroid: $(\csc(A) : \csc(B) : \csc(C))$;

Circumcenter: $(\cos(A) : \cos(B) : \cos(C))$.

**Second proof using trilinear coordinates:** Three points are collinear if and only if

$$\det \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0,$$

where $P_1, P_2, P_3$ are the row vectors representing the trilinear coordinates of these three points. By the above trilinear coordinates of $H, G$ and $O$, we know that they are collinear if and only if

$$\det \begin{bmatrix} \sec(A) & \sec(B) & \sec(C) \\ \csc(A) & \csc(B) & \csc(C) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix} = 0.$$

For the rest of the proof, we verify the above identity. Multiplying the above first row by $(\cos A \cos B \cos C)$ and the second row by $(\sin A \sin B \sin C)$, we get

$$\sigma \overset{def}{=} \det \begin{bmatrix} \sin(B)\sin(C) & \sin(C)\sin(A) & \sin(A)\sin(B) \\ \cos(B)\cos(C) & \cos(C)\cos(A) & \cos(A)\cos(B) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix}. \quad (1)$$

Since $A + B + C = 180°$, we have

$$\cos(A) = -\cos(B+C), \cos(B) = -\cos(C+A), \cos(C) = -\cos(A+B).$$

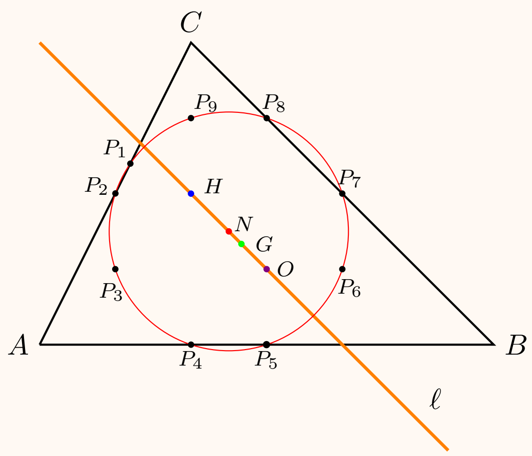
Thus, by the Sum Formula

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B),$$

the first row of the matrix in (1) is equal to the summation of the second and the third rows. As a result, $\sigma = 0$, and this completes the proof.

**Theorem 2**

*The nine-point center is on the Euler Line in $\triangle ABC$. Let $\ell$ be the Euler line, and let $H, G, O$ be the orthocenter, the centroid, and the circumcenter, respectively. Let $N$ be the nine-point center. Then $N$ is on $\ell$.*




**Proof:** The trilinear coordinate for the nine-point center is

$$(\cos(B-C) : \cos(C-A) : \cos(A-B)),$$

according to Wikipedia. We prove that the $O, H, N$ are collinear, which is equivalent to

$$\det \begin{bmatrix} \sec(A) & \sec(B) & \sec(C) \\ \csc(A) & \csc(B) & \csc(C) \\ \cos(B-C) & \cos(C-A) & \cos(A-B) \end{bmatrix} = 0.$$

Similar to the second proof of Theorem 1, we multiply the first row by

$$(\cos A \cos B \cos C)$$

and the second row by

$$(\sin A \sin B \sin C),$$

and get

$$\sigma \overset{def}{=} \det \begin{bmatrix} \sin(B)\sin(C) & \sin(C)\sin(A) & \sin(A)\sin(B) \\ \cos(B)\cos(C) & \cos(C)\cos(A) & \cos(A)\cos(B) \\ \cos(B-C) & \cos(C-A) & \cos(A-B) \end{bmatrix}.$$

Again, by the Sum Formula

$$\cos(A-B) = \sin(A)\sin(B) + \cos(A)\cos(B),$$

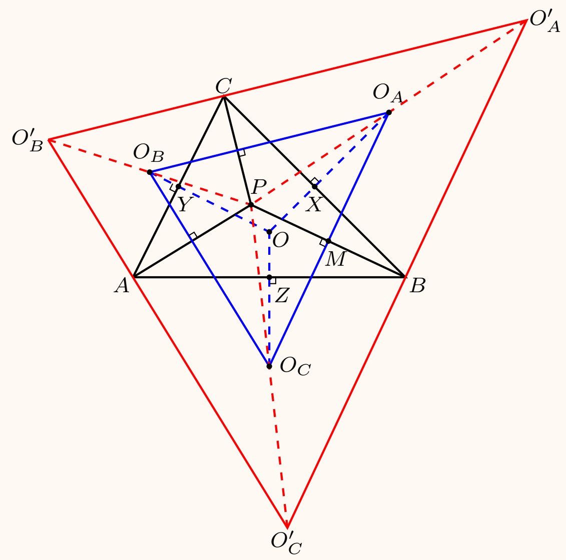
one can show that the three rows are linearly dependent, that is, $\sigma = 0$ ■

## 3 Generalization

In this section, we shall generalize Euler Line from triangle to general polygon. Before we do that, we first establish the following

**Theorem 3**

*Let $P$ be an arbitrary point inside $\triangle ABC$. Let $O_A$, $O_B$, and $O_C$ be the circumcenters of $\triangle PBC$, $\triangle PCA$, and $\triangle PAB$, respectively. Let $O_AX \perp BC$, $O_BY \perp CA$, and $O_C \perp AB$. Then $O_AX, O_BY$, and $O_CZ$ are concurrent at the the circumcenter $O$ of $\triangle ABC$. Moreover, $O$ and $P$ are isogonal conjugate points with respect to $\triangle O_AO_BO_C$.*




**Proof:** Let $\triangle O'_AO'_BO'_C$ be a similar triangle to $\triangle O_AO_BO_C$ with ratio $2:1$ and passes through vertices $A, B, C$, where $O_AO_B \,/\!/\, O'_AO'_B$, $O_BO_C \,/\!/\, O'_BO'_C$, $O_CO_A \,/\!/\, O'_CO'_A$.

Let's first prove that $O$ is the circumcenter of $\triangle ABC$. Since points $O_A$, $O_B$, $O_C$ are circumcenters of $\triangle PBC$, $\triangle PCA$, $\triangle PAB$, respectively, by definition, $O_AO, O_BO, O_CO$ are perpendicular bisectors of $BC, AC, AB$. Thus $O$ is the circumcenter of $\triangle ABC$.

Next we show that point $P$ is the isogonal conjugate point to the circumcenter $O$. Since $O_AO \perp BC$, and $O_AM \perp PB$, then $O_A, X, M, B$ are concyclic. Therefore, $\angle MO_AX = \angle PBC$. Since $O'_AC \perp PC$ and $O'_AB \perp PB$, then $O'_A, C, P, B$ are concyclic. Therefore $\angle PBC = \angle CO'_AP = \angle O_BO_AP$. Thus $O_AP$ and $O_AO$ are isogonal conjugate lines of $\triangle O_AO_BO_C$ and hence $P, O$ are isogonal conjugate points. ■

In order to define the *Generalized Euler Line*, we first define the *Circumcenter of Mass* of a polygon.

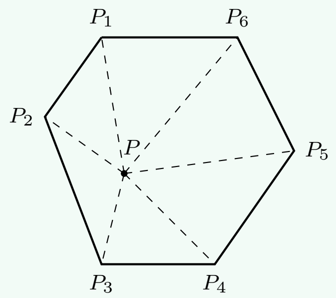
**Definition 6. (Circumcenter of Mass)**

*Let $K$ be an $n$-polygon defined by vertices $P_1, \cdots, P_n$. Let $P$ be an arbitrary point, by connecting $P$ to each $P_i$ , $i = (1, \cdots, n)$, we triangulate $K$ into $n$ triangles $\triangle PP_iP_{i+1}$ where $P_{n+1} = P_1$. Let $O_i$ be the circumcenter of $\triangle PP_iP_{i+1}$. We define the Circumcenter of Mass $(CCM)$ to be*

$$CCM(K) = \sum_{i=1}^{n} \frac{S_i}{S_K} O_i,$$

*where $S_i$ is the area[a] of each $\triangle PP_iP_{i+1}$ and $S_K$ is the area of the polygon $K$ =*

$(P_1, \cdots, P_n)$.

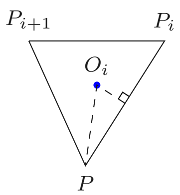



[a] It is in fact the *signed* area defined by $\langle (P_i - P) \times (P_{i+1} - P), \vec{N} \rangle / 2$, where $\vec{N}$ is the normal vector of the Euclidean plane.

**Theorem 4**

*The Circumcenter of Mass $CCM(K)$ of a polygon $K$ is well defined, that is, it is independent to the arbitrary point $P$.*

**Proof:** Let $P_i = (x_i, y_i)$ , $i = (1, \cdots, n)$. $P = (p, q)$. We define $O_i = (u_i, v_i)$, $i = (1, \cdots, n)$ be the circumcenter of $\triangle PP_iP_{i+1}$ for $i = 1, \cdots, n$.




We start by looking at the inner product of $P_i - P$ and $O_i - P$, since $O_i$ is the circumcenter,

$$\begin{aligned} \langle P_i - P, O_i - P \rangle &= |O_i - P||P_i - P| \cos(\angle O_i P P_i) \\ &= \frac{1}{2}|P_i - P|^2 = \frac{1}{2}|P_i|^2 - \langle P_i - P, P \rangle - \frac{1}{2}|P|^2, \end{aligned} \tag{2}$$

and from (2), we have

$$\langle P_i - P, O_i \rangle = \frac{1}{2}|P_i|^2 - \frac{1}{2}|P|^2. \tag{3}$$

Since the above equation is valid for any $i$, we replace $i$ by $i + 1$ to get

$$\langle P_{i+1} - P, O_i \rangle = \frac{1}{2}|P_{i+1}|^2 - \frac{1}{2}|P|^2. \tag{4}$$

Then, multiplying $(y_{i+1} - p)$ to (3) and $(y_i - p)$ to (4) and subtracting, we get

$$\langle (y_{i+1} - p)(P_i - P) - (y_i - p)(P_{i+1} - P), O_i \rangle \tag{5}$$

$$= \frac{1}{2}[(y_{i+1} - p)|P_i|^2 - (y_i - p)|P_{i+1}|^2)] - \frac{1}{2}|P|^2(y_{i+1} - y_i). \tag{6}$$

But (5) is equal to

$$\left\langle \left( (y_{i+1}-p)\begin{bmatrix} x_i - p \\ y_i - q \end{bmatrix} - (y_i - p)\begin{bmatrix} x_{i+1}-p \\ y_{i+1}-q \end{bmatrix} \right), \begin{bmatrix} u_i \\ v_i \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} 2S_i \\ 0 \end{bmatrix}, \begin{bmatrix} u_i \\ v_i \end{bmatrix} \right\rangle = 2S_i u_i.$$

Thus we have

$$S_i u_i = \frac{1}{4}[(y_{i+1}-p)|P_i|^2 - (y_i - p)|P_{i+1}|^2)] - \frac{1}{4}|P|^2(y_{i+1}-y_i). \tag{7}$$

Since $P_{n+1} = P_1$, taking summation, then the terms $\sum(|P|^2(y_{i+1}-y_i))$ and $\sum(|P_i|^2 - |P_{i+1}|^2)$ will cancel out. As a result,

$$\sum_{i=1}^{n} S_i u_i = \frac{1}{4}\sum_{i=1}^{n}[y_{i+1}|P_i|^2 - y_i|P_{i+1}|^2].$$

Similarly, we have

$$\sum_{i=1}^{n} S_i v_i = -\frac{1}{4}\sum_{i=1}^{n}[x_{i+1}|P_i|^2 - x_i|P_{i+1}|^2].$$

Thus,

$$CCM(K) = \sum_{i=1}^{n}\frac{S_i}{S_K}O_i = \frac{1}{4S_K}\begin{bmatrix} \sum_{i=1}^{n}[y_{i+1}|P_i|^2 - y_i|P_{i+1}|^2] \\ -\sum_{i=1}^{n}[x_{i+1}|P_i|^2 - x_i|P_{i+1}|^2] \end{bmatrix}$$

is independent to $P$. ∎

**Definition 7. (Center of Mass)**

*Let $K$ be an $n$-polygon be defined by vertices $P_1, \cdots, P_n$. Let $P$ be an arbitrary point. Connecting $P$ to each $P_i$, we triangulate $K$ into $n$ triangles $\triangle PP_iP_{i+1}$ where $i = 1, \cdots, n$, and $P_{n+1} = P_1$.*

*The Center of Mass $CM(K)$ of $K$ is defined to be*

$$CM(K) = \sum_{i=1}^{n}\frac{S_i}{S_K}G_i$$

*where $S_i$ is the (signed) area of each $\triangle PP_iP_{i+1}$; $G_i = (P_i + P_{i+1} + P)/3$ is the centroid of each $\triangle PP_iP_{i+1}$; and $S_K$ is the area of the polygon $K = (P_1, \cdots, P_n)$.*

We wish to prove the Center of Mass of a polygon is independent of the choice of $P$. We first introduce a helpful lemma.

**Lemma 1**

*Let $P_1, \cdots, P_n$ be vectors in $\mathbb{R}^2$. Then for any $1 \le i, j, k \le n$, we have*

$$\langle P_i \times P_j, \vec{N}\rangle P_k + \langle P_j \times P_k, \vec{N}\rangle P_i + \langle P_k \times P_i, \vec{N}\rangle P_j = 0.$$

**Proof:** Since we are working on $\mathbb{R}^2$, we can write, without loss of generality, that $P_k = \alpha P_i + \beta P_j$. Then we have

$$\begin{aligned}\langle P_i \times P_j, \vec{N}\rangle P_k &= \langle P_i \times P_j, \vec{N}\rangle(\alpha P_i + \beta P_j)\\ &= \alpha\langle P_i \times P_j, \vec{N}\rangle\, P_i + \beta\langle P_i \times P_j, \vec{N}\rangle\, P_j.\end{aligned}$$

On the other hand, we have

$$\langle P_j \times P_k, \vec{N}\rangle P_i = \alpha\langle P_j \times P_i, \vec{N}\rangle\, P_i,$$

and

$$\langle P_k \times P_i, \vec{N}\rangle P_j = \beta\langle P_j \times P_i, \vec{N}\rangle\, P_j.$$

Thus we have

$$\langle P_i \times P_j, \vec{N}\rangle P_k + \langle P_j \times P_k, \vec{N}\rangle P_i + \langle P_k \times P_i, \vec{N}\rangle P_j = 0.$$

■

**Theorem 5**

*The Center of Mass of a polygon is well defined, that is, it is independent to $P$.*

**Proof:** By definition, we have

$$CM(K) = \sum_{i=1}^{n} \frac{\langle (P_i - P) \times (P_{i+1} - P), \vec{N}\rangle(P_i + P_{i+1} + P)}{6S_K}.$$

We expand

$$\begin{aligned}&\sum_{i=1}^{n}\langle (P_i - P) \times (P_{i+1} - P), \vec{N}\rangle(P_i + P_{i+1} + P)\\ &= \langle \sum_{i=1}^{n} P_i \times P_{i+1} - \sum_{i=1}^{n} P_i \times P - \sum_{i=1}^{n} P \times P_{i+1}, \vec{N}\rangle(P_i + P_{i+1} + P)\\ &= \sum_{i=1}^{n}\langle P_i \times P_{i+1}, \vec{N}\rangle(P_i + P_{i+1}) - \sum_{i=1}^{n}\langle P \times (P_{i+1} - P_i), \vec{N}\rangle(P_i + P_{i+1})\\ &+ \sum_{i=1}^{n}\langle P_i \times P_{i+1}, \vec{N}\rangle P + \sum_{i=1}^{n}\langle P \times (P_i - P_{i+1}), \vec{N}\rangle P\\ &= 1 - 2 + 3 + 4.\end{aligned} \tag{8}$$

We shall simplify the above terms. First, we have

$$4 = \sum_{i=1}^{n}\langle P \times (P_i - P_{i+1}), \vec{N}\rangle P = \langle P \times \sum_{i=1}^{n}(P_i - P_{i+1}), \vec{N}\rangle P = 0. \tag{9}$$

Next, using Lemma 1, we get

$$\begin{aligned}
② &= \sum_{i=1}^{n} \langle P \times (P_{i+1} - P_i), \vec{N} \rangle (P_i + P_{i+1}) \\
&= \sum_{i=1}^{n} \langle P \times P_{i+1}, \vec{N} \rangle P_i - \sum_{i=1}^{n} \langle P \times P_i, \vec{N} \rangle P_{i+1} \\
&= \sum_{i=1}^{n} \langle P \times P_{i+1}, \vec{N} \rangle P_i + \sum_{i=1}^{n} \langle P_i \times P, \vec{N} \rangle P_{i+1} \\
&= -\sum_{i=1}^{n} \langle P_{i+1} \times P_i, \vec{N} \rangle P = \sum_{i=1}^{n} \langle P_i \times P_{i+1}, \vec{N} \rangle P = ③.
\end{aligned} \tag{10}$$

Thus we have

$$\begin{aligned}
&\sum_{i=1}^{n} \langle (P_i - P) \times (P_{i+1} - P), \vec{N} \rangle (P_i + P_{i+1} + P) \\
&= ① = \sum_{i=1}^{n} \langle P_i \times P_{i+1}, \vec{N} \rangle (P_i + P_{i+1}),
\end{aligned}$$

and hence

$$CM(K) = \sum_{i=1}^{n} \frac{\langle P_i \times P_{i+1}, \vec{N} \rangle (P_i + P_{i+1})}{6S_K}, \tag{11}$$

which is independant to $P$. ■

**Corollary 1**

*Let $M_i$ be the midpoint of $A_i$ and $A_{i+1}$ for $i = 1, \cdots, n$. Then*

$$CM(K) = \frac{2}{3} \sum_{i=1}^{n} \frac{\langle P_i \times P_{i+1}, \vec{N} \rangle}{S_K} M_i.$$

**Definition 8. (Generalized Euler Line)**

*Let $K$ be a polygon, the Euler Line $\ell$ of the Polygon $K$ is the line that passes through the Center of Mass ($CM$) and the Circumcenter of Mass $CCM(K)$. When the polygon is reduced to triangle, the Generalized Euler Line is reduced to the classical Euler Line.*

**Definition 9. ($t$-Point)**

*Let $\triangle ABC$ be a triangle and let $O, G$ be its circumcenter and centroid, respectively. A point $E_t$ on the Euler line is called a $t$-point, if there is a real number $t$ such that*

$$A_t = t\,G + (1-t)\,O.$$

*For example, the orthocenter $H = A_3$, and the nine-point circle $N = A_{3/2}$.*

*Let $K$ be an $n$-polygon defined by vertices $P_1, \cdots, P_n$. Let $P$ be an arbitrary point, by connecting $P$ to each $P_i$ , $i = (1, \cdots, n)$, we triangulate $K$ into $n$ triangles $\triangle PP_iP_{i+1}$*

*where* $P_{n+1} = P_1$. *Let* $(A_t)_t$ *be the* $t$*-point of* $\triangle PP_iP_{i+1}$. *We define the* *t-point of Mass* *(*$tM$*) to be*

$$tM(K) = \sum_{i=1}^{n} \frac{S_i}{S_K}(A_t)_i,$$

*where* $S_i$ *is the (signed) area of each* $\triangle PP_iP_{i+1}$ *and* $S_K$ *is the area of the polygon* $K = (P_1, \cdots, P_n)$.

Using Theorem 4 and Theorem 5, we then conclude that

**Theorem 6**

*The definition of* $tM(K)$ *is indepedent to the choice of the arbitrary point* $P$. *Moreover,* $tM(K)$ *is on the Generalized Euler Line of* $K$ *for any* $t$.