Erdős-Mordell Inequality

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Math 199

CATALOGUE

01

Background

01 Introduction

Background PREFACE

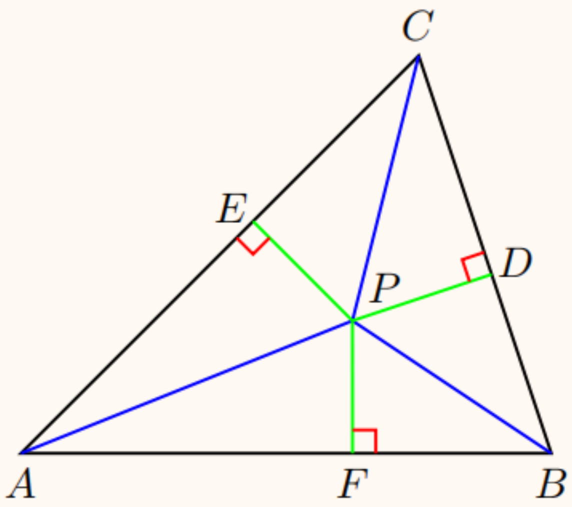
From Wikipedia: “In Euclidean geometry, the Erdős–Mordell inequality states that for any triangle $\triangle ABC$ and point P inside $\triangle ABC$, the sum of the distances from P to the sides is less than or equal to half of the sum of the distances from P to the vertices. It is named after Paul Erdős and Louis Mordell. Erdős (1935) posed the problem of proving the inequality; a proof was provided two years later by L. J. Mordell and D. F. Barrow (1937).” See the Wikipedia for details.

In this artical, we give three proofs of the inequality. The following one using trigonometry is one of the simplest.

01 conclusion

Let P be a point inside triangle $\triangle ABC$. Let PD, PE, PF to be orthogonal to AB, BC, CA respectively. Then

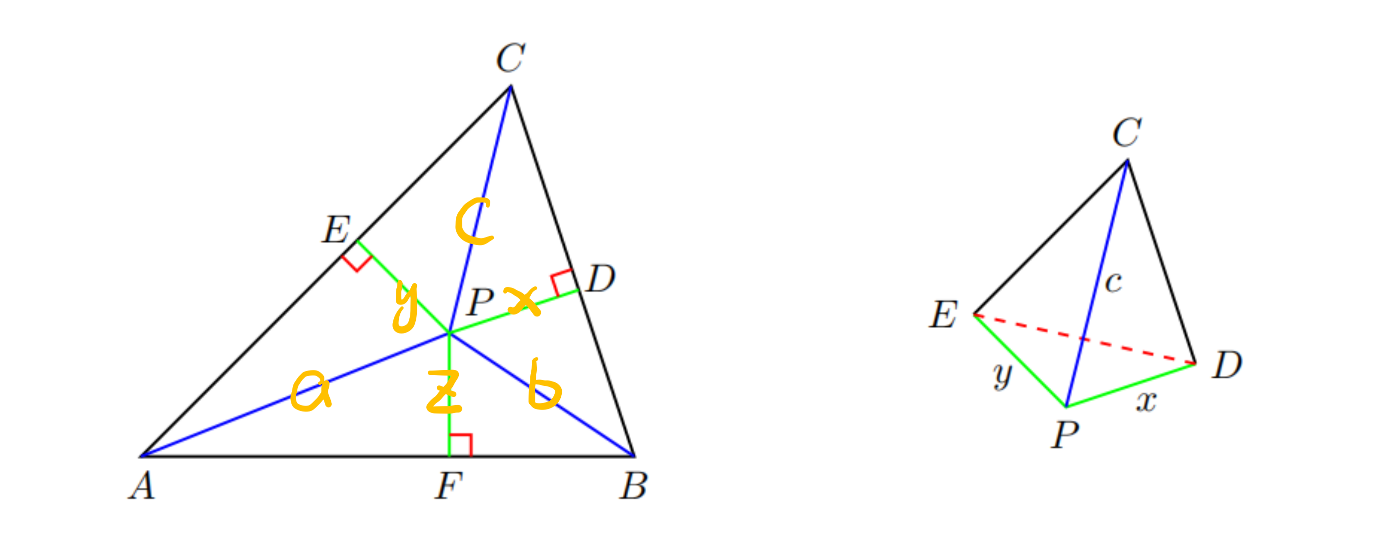
$$PA + PB + PC \geq 2(PD + PE + PF).$$



02

First Proof

02 Process of the equation



Step1

By the law of Cosine, $ED^2 = x^2 + y^2 - 2xycos\angle EPD$.

Since $\angle EPD = 180° - \angle ACB = \angle A + \angle B$, then we have
$x^2 + y^2 - 2xycos(A + B) = (xsinB + ysinA)^2 + (xcosB - ycosA)^2$

Proof. $ED^2 = x^2 + y^2 - 2xy(cos(A)cos(B) - sin(A)sin(B)) =$
$x^2 + y^2 - 2xycosAcosB - 2xysinAsinB =$
$x^2sin^2B + x^2cos^2B + y^2sin^2A + y^2cos^2A - 2xycosAcosB - 2xysinAsinB =$
$(x^2sin^2B - 2xysinAsinB + y^2sin^2A) + (x^2cos^2B + 2xycosAcosB + y^2cos^2A) =$
$(xcosB + ycosA)^2 + (xsinB - ysinA)^2$ □

We therefore have $ED^2 \geq (xsinB + ysinA)^2$, then $ED \geq xsinB + ysinA$

Step2

By law of sines, we have $ED = c\sin C$. Thus we get the inequality

$$c \geq x\frac{\sin B}{\sin C} + y\frac{\sin A}{\sin C}.$$

Similarly, if $PA = a$, $PB = b$, and $PF = z$, then we have

$$a \geq z\frac{\sin B}{\sin A} + y\frac{\sin C}{\sin A}, \qquad b \geq x\frac{\sin C}{\sin B} + z\frac{\sin A}{\sin B}.$$

02 Process of the equation (cont.)

Step3

Therefore, we have

$$a+b+c \geq x\left(\frac{\sin B}{\sin C}+\frac{\sin C}{\sin B}\right)+y\left(\frac{\sin A}{\sin C}+\frac{\sin C}{\sin A}\right)+z\left(\frac{\sin B}{\sin A}+\frac{\sin A}{\sin B}\right)$$

By the Arithmetic-Geometric Inequality, we have

$$\frac{\sin B}{\sin C}+\frac{\sin C}{\sin B} \geq 2, \quad \frac{\sin A}{\sin C}+\frac{\sin C}{\sin A} \geq 2, \quad \frac{\sin B}{\sin A}+\frac{\sin A}{\sin B} \geq 2.$$

Then we have

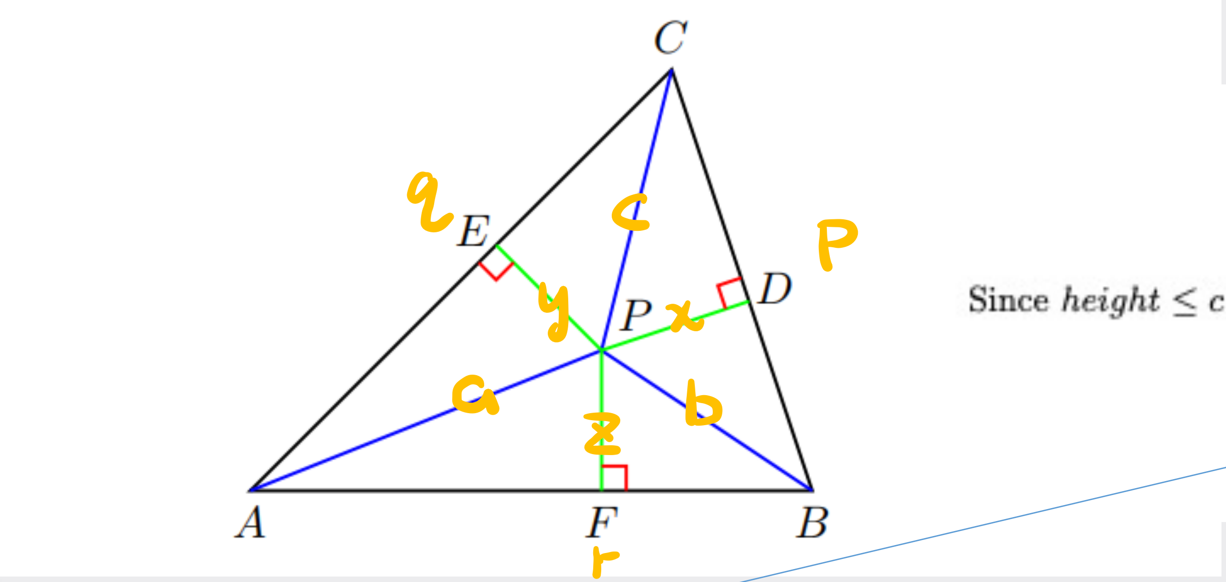
$$a+b+c \geq 2(x+y+z).$$

where the Arithmetic-Geometric Inequality is $a+b \geq 2\sqrt{ab}$, thus, $\frac{sinB}{sinC}+\frac{sinC}{sinB} \geq 2$.

03

Second Proof

03 We shall use area method to prove the inequality



$$\frac{1}{2}\cdot height\cdot r=\frac{1}{2}\cdot qy+\frac{1}{2}\cdot px+\frac{1}{2}\cdot rz$$
$$r\cdot height=qy+px+rz$$

Since $height\le c+z$

$$r(c+z)\ge qy+px+rz$$
$$rc+rz\ge qy+px+rz$$
$$rc\ge qy+px$$

Setting

Assume that $PA=a$,
$PB=b$, $PC=c$, $PD=x$,
$PE=y$, $PF=z$.
$BC=p$, $CA=q$, $AB=r$.

equation

$r(c+z)\ge qy+px+rz$,
$rc\ge qy+px$,
$rc\ge qx+py$,

Deduce

$c\ge xq/r+yp/r$,
$b\ge xr/q+zp/q$,
$a\ge zq/p+yq/p$.

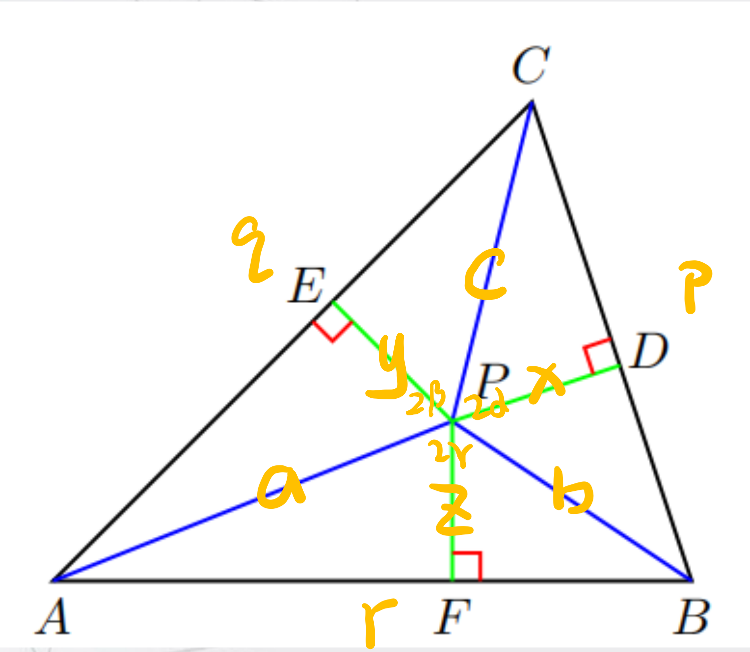
QED

$$a+b+c\ge x\left(\frac{r}{q}+\frac{q}{r}\right)+y\left(\frac{p}{r}+\frac{r}{p}\right)+z\left(\frac{p}{q}+\frac{q}{p}\right)\ge 2(x+y+z).$$

04

Third Proof

04 Reasoning and calculation



1. Setting

$PA = a,\ PB = b,\ PC = c,\ PD = x,\ PE = y,\ PF = z,\ BC = p,\ AC = q$, and $AB = r$. $\angle BPC = 2\alpha,\ \angle CPA = 2\beta$ and $\angle APB = 2\gamma$.

2. Calculating

By the law of cosine, $r^2 = a^2 + b^2 - 2abcos(2\gamma) = (a-b)^2 - 2ab - 2abcos(2\gamma) = (a-b)^2 + 2ab(1-cos(2\gamma)) \geq 2ab(1-cos(2\gamma)) = 2ab(1-(1-2sin^2\gamma)) = 4absin^2\gamma$

thus we have $r^2 \geq 4absin^2\gamma$, which is to say $r \geq 2\sqrt{(ab)}sin\gamma$

$area = \frac{1}{2}absin(2\gamma) = \frac{1}{2}ab \cdot 2sin\gamma cos\gamma = absin\gamma cos\gamma$

$z = \frac{area}{r} = \frac{absin\gamma cos\gamma}{r} \leq \sqrt{ab}cos\gamma$

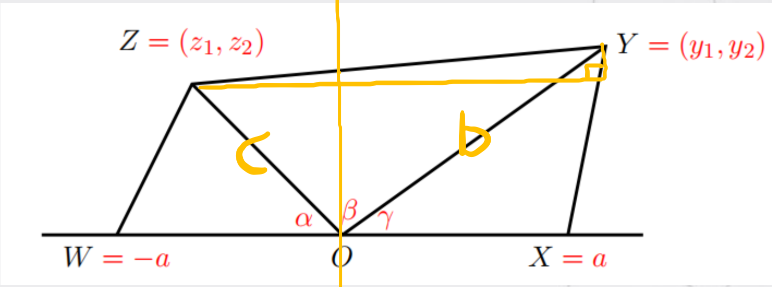
3.Conclusion

$a + b + c \geq 2\sqrt{ab}\cos\gamma + 2\sqrt{bc}\cos\alpha + 2\sqrt{ca}\cos\beta$

04 Proof of the above conclusion

- **1**. *Let* $\alpha + \beta + \gamma = 180◦$
  $a\text{^}2 + b\text{^}2 + c\text{^}2 \geq 2ab \cos \gamma + 2bc \cos \alpha + 2ca \cos \beta.$
- **2.** If $\alpha = \beta = 90◦$ and $\gamma = 0◦$
  $a\ 2 + b\ 2 \geq 2ab.$
- **3.** If $\alpha = \beta = \gamma = 60◦$
  $a\ 2 + b\ 2 + c\ 2 \geq ab + bc + ca.$
- **4.** If $\alpha = \beta = 0◦$ and $\gamma = 180◦$
  $a\ 2 + b\ 2 + c\ 2 \geq -2ab + 2bc + 2ca.$

04 Proof Using Analytic Geometry



Setting

Step1

$OX = OW = a,$
$OY = b,\ OZ = c$

Step2

distance $O \rightarrow ZY$

Equation

$bc\cos\beta = \frac{1}{2}(b^2+c^2-(y_1-z_1)^2-(y_2-z_2)^2) =$
$\frac{1}{2}(b^2+c^2-((y_1-z_1)^2+(y_2-z_2)^2)) = y_1z_1+y_2z_2$

similarly, $ab\cos\gamma = ay_1,$

$y_1^2-2y_1z_1+z_1^2+y_2^2-2y_2z_2+z_2^2$

$ca\cos\alpha = -az_1.$

b^2 z^2

Step3

Application

$a^2+b^2+c^2 \geq 2ab\cos\gamma + 2bc\cos\beta + 2ac\cos\alpha \geq 2ay_1 + 2(y_1z_1+y_2z_2) - 2az_1$

Step4

Conclusion

$a^2+y_1^2+y_2^2+z_1^2+z_2^2 = a^2+(y_1^2+y_2^2)+(z_1^2+z_2^2) = a^2+b^2+c^2 \geq$
$-2az_1+2y_1z_1+2y_2z_2+2ay_1$

05

Theorem

Let $A_1A_2\cdots A_n$ be a convex polygon, and P be an interior point of $A_1A_2\cdots A_n$. Let R_i be the distance from P to the vertex A_i, r_i be the distance from P to the side A_iA_{i+1}. Then

$$\sum_{i=1}^{n} R_i \geq \left(\sec\frac{\pi}{n}\right)\sum_{i=1}^{n} r_i.$$

By using the same method in the above third proof of the Erdős-Mordell Inequality

Let $a_1,\cdots,a_n$ be nonnegative real numbers. Let $\alpha_1,\cdots,\alpha_n$ be angles such that

$$\sum_{i=1}^{n}\alpha_i = 180^\circ.$$

Then we have

$$\sum_{i=1}^{n} a_i^2 \geq \left(\sec\frac{\pi}{n}\right)\left(\sum_{i=1}^{n-1} a_ia_{i+1}\cos\alpha_i + a_na_1\cos\alpha_n\right).$$

Proof :

Let : $\vec{\mathbf{v}}_1 = a_1$, and

$$\vec{\mathbf{v}}_k = (a_k \cos(\alpha_1 + \cdots + a_{k-1}), a_k \sin(\alpha_1 + \cdots + a_{k-1}))$$

for $k = 2, \cdots, n$. Therefore, we have

$$\langle \vec{\mathbf{v}}_i, \vec{\mathbf{v}}_{i+1} \rangle = a_i a_{i+1} \cos \alpha_i$$

Get

for $i = 1, \cdots, n-1$; on the other hand, since $\alpha_1 + \cdots + \alpha_n = \pi$,

$$\langle \vec{\mathbf{v}}_n, \vec{\mathbf{v}}_1 \rangle = a_n a_1 \cos \alpha_n.$$

Thus the inequality we want to prove can be represented by

$$\|\vec{\mathbf{v}}_1\|^2 + \cdots + \|\vec{\mathbf{v}}_n\|^2 \geq \sec \frac{\pi}{n} \left(\sum_{i=1}^{n-1} \langle \vec{\mathbf{v}}_i, \vec{\mathbf{v}}_{i+1} \rangle - \langle \vec{\mathbf{v}}_n, \vec{\mathbf{v}}_1 \rangle \right).$$

Next

If we write $\vec{\mathbf{v}}_i = (p_i, q_i)$ for $i = 1, \cdots, n$, then

$$\langle \vec{\mathbf{v}}_i, \vec{\mathbf{v}}_{i+1} \rangle = p_i p_{i+1} + q_i q_{i+1}$$

for $i = 1, cdots, n-1$, and

$$\langle \vec{\mathbf{v}}_n, \vec{\mathbf{v}}_1 \rangle = p_n p_1 + q_n q_1.$$

Thus the inequality is reduced into the case when $\alpha_1 = \cdots = \alpha_{n-1} = 0$, and $\alpha_n = 180^\circ$:

$$\sum_{i=1}^{n} a_i^2 \geq \left(\sec\frac{\pi}{n}\right)\left(\sum_{i=1}^{n-1} a_i a_{i+1} - a_n a_1\right)$$

It is really surprising, but you can do the completing the square:

$$\begin{aligned}&\cos\frac{\pi}{n}\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n-1} a_i a_{i+1} - a_n a_1\right)\\ =&\sum_{i=1}^{n-2}\frac{1}{2\sin\frac{i\pi}{n}\sin\frac{(i+1)\pi}{n}}\left(a_i\sin\frac{(i+1)\pi}{n} - a_{i+1}\sin\frac{i\pi}{n} + a_n\sin\frac{\pi}{n}\right)^2.\end{aligned}$$

QED！！！

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THANK YOU