

Euler Line

Chuxiangbo Wang¹, chuxianw@uci.edu

(last updated: June 15, 2022)

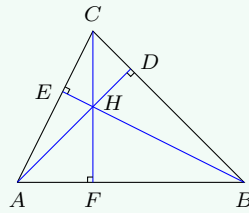
1 Introduction

In 1765, Swiss mathematician *Leonhard Euler* discovered that the *orthocenter*, the *circumcenter*, and the *centroid* of a triangle are collinear. The line is thus called the *Euler Line*. Later in 1820s, it was discovered by German mathematician *Karl Feuerbach* that the *nine-point center* is also on the Euler line. So the four important triangle centers are collinear.

We first recall the definition of these special centers of triangle.

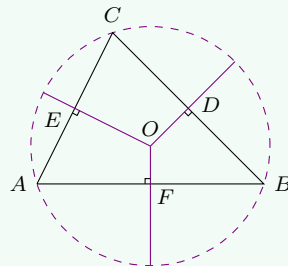
Definition 1. (Orthocenter)

In a triangle, the three altitudes are concurrent, and the concurrent point is called the *orthocenter* of the triangle. In the following $\triangle ABC$, the intersection H of three altitudes AD , BE , CF is the orthocenter of $\triangle ABC$.



Definition 2. (Circumcenter)

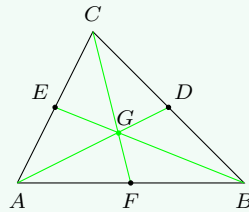
The circle that passes through three vertices of a triangle is called the *circumcircle*. The center of the circumcircle is called *circumcenter*, which is also the intersection of the three perpendicular bisectors of the sides. In the following $\triangle ABC$, the circle that passing through vertices A, B, C is the circumcircle and the circle center O is the circumcenter.



¹The author thanks Dr. Zhiqin Lu for his help.

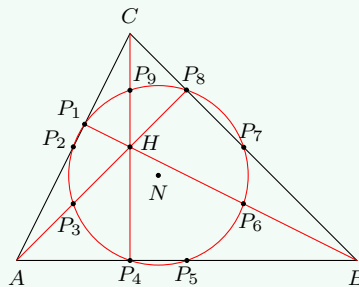
Definition 3. (Centroid)

In a triangle, the intersection of three medians is called **centroid**. In the following $\triangle ABC$, the intersection G of the three medians AD, BE, CF is the centroid of $\triangle ABC$.



Definition 4. (Nine-Point Center)

In a triangle, the circle that intersects with three midpoints of the sides (P_2, P_5, P_7), three feet of the altitudes (P_1, P_4, P_8), and three midpoints of the vertices and the orthocenter (P_3, P_6, P_9) is called a **nine-point circle**, and point N is called the **nine-point center**.



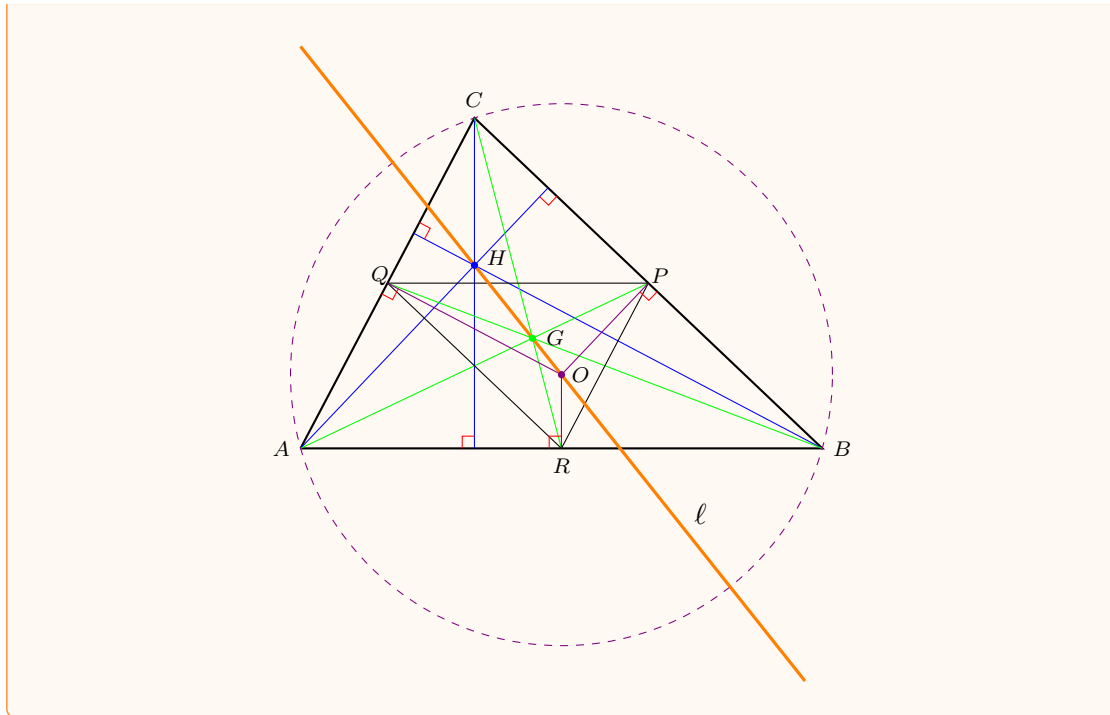
2 Euler Line and Its Properties

Definition 5. (Euler Line)

In a triangle, the line that passes through the orthocenter, centroid and circumcenter is called the **Euler Line**.

Theorem 1

In the following $\triangle ABC$, let H , G and O be the orthocenter, the centroid, and the circumcenter, respectively. Then H , G , O are collinear.



We give two proofs. The first one is geometric and the second one is more algebraic.

First proof: In the $\triangle ABC$ above, let P, Q, R be the midpoints of BC, AC, AB . Since $CH \parallel OR$, it is clear that $\angle HCG = \angle GRO$. By the property of the centroid, $CG : GR = 2 : 1$. Also, since P, Q, R are midpoints of $\triangle ABC$, we have that $\triangle ABC$ is similar to $\triangle PQR$ with the ratio of $2 : 1$, where the circumcenter O of $\triangle ABC$ is precisely the orthocenter of $\triangle PQR$, which implies that $CH : OR = 2 : 1$. Thus, $\triangle GRO$ is similar to $\triangle GCH$, and therefore $\angle RGO = \angle CGH$. This implies that the orthocenter, centroid and circumcenter are collinear. ■

We use *Trilinear Coordinate System* in the second proof. For the definition of trilinear coordinate system, refer to [Wikipedia](#) or [Topic 37](#). From there one can also find the trilinear coordinates of the following centers. Let A, B, C be the three angles of $\triangle ABC$. Then the trilinear coordinates are given by

Orthocenter: $(\sec(A) : \sec(B) : \sec(C))$;

Centroid: $(\csc(A) : \csc(B) : \csc(C))$;

Circumcenter: $(\cos(A) : \cos(B) : \cos(C))$.

Second proof using trilinear coordinates: Three points are collinear if and only if

$$\det \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0,$$

where P_1, P_2, P_3 are the row vectors representing the trilinear coordinates of these

three points. By the above trilinear coordinates of H, G and O , we know that they are collinear if and only if

$$\det \begin{bmatrix} \sec(A) & \sec(B) & \sec(C) \\ \csc(A) & \csc(B) & \csc(C) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix} = 0.$$

For the rest of the proof, we verify the above identity. Multiplying the above first row by $(\cos A \cos B \cos C)$ and the second row by $(\sin A \sin B \sin C)$, we get

$$\sigma \stackrel{def}{=} \det \begin{bmatrix} \sin(B) \sin(C) & \sin(C) \sin(A) & \sin(A) \sin(B) \\ \cos(B) \cos(C) & \cos(C) \cos(A) & \cos(A) \cos(B) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix}. \quad (1)$$

Since $A + B + C = 180^\circ$, we have

$$\cos(A) = -\cos(B + C), \cos(B) = -\cos(C + A), \cos(C) = -\cos(A + B).$$

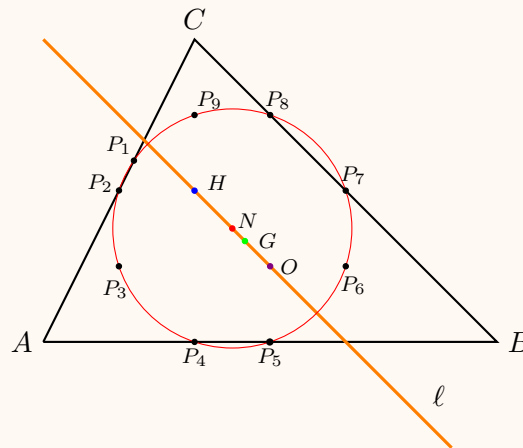
Thus, by the Sum Formula

$$\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B),$$

the first row of the matrix in (1) is equal to the summation of the second and the third rows. As a result, $\sigma = 0$, and this completes the proof. ■

Theorem 2

The nine-point center is on the Euler Line in $\triangle ABC$. Let ℓ be the Euler line, and let H, G, O be the orthocenter, the centroid, and the circumcenter, respectively. Let N be the nine-point center. Then N is on ℓ .



Proof: The trilinear coordinate for the nine-point center is

$$(\cos(B - C) : \cos(C - A) : \cos(A - B)),$$

according to [Wikipedia](#). We prove that the O, H, N are collinear, which is equivalent

to

$$\det \begin{bmatrix} \sec(A) & \sec(B) & \sec(C) \\ \csc(A) & \csc(B) & \csc(C) \\ \cos(B-C) & \cos(C-A) & \cos(A-B) \end{bmatrix} = 0.$$

Similar to the second proof of Theorem 1, we multiply the first row by

$$(\cos A \cos B \cos C)$$

and the second row by

$$(\sin A \sin B \sin C),$$

and get

$$\sigma \stackrel{\text{def}}{=} \det \begin{bmatrix} \sin(B) \sin(C) & \sin(C) \sin(A) & \sin(A) \sin(B) \\ \cos(B) \cos(C) & \cos(C) \cos(A) & \cos(A) \cos(B) \\ \cos(B-C) & \cos(C-A) & \cos(A-B) \end{bmatrix}.$$

Again, by the Sum Formula

$$\cos(A-B) = \sin(A) \sin(B) + \cos(A) \cos(B),$$

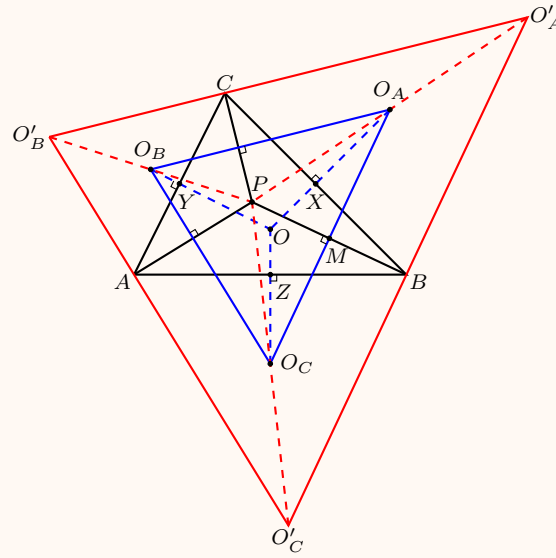
one can show that the three rows are linearly dependent, that is, $\sigma = 0$ ■

3 Generalization

In this section, we shall generalize Euler Line from triangle to general polygon. Before we do that, we first establish the following

Theorem 3

Let P be an arbitrary point inside $\triangle ABC$. Let O_A , O_B , and O_C be the circumcenters of $\triangle PBC$, $\triangle PCA$, and $\triangle PAB$, respectively. Let $O_AX \perp BC$, $O_BY \perp CA$, and $O_CZ \perp AB$. Then O_AX , O_BY , and O_CZ are concurrent at the circumcenter O of $\triangle ABC$. Moreover, O and P are isogonal conjugate points with respect to $\triangle O_AO_BO_C$.



Proof: Let $\triangle O_A O_B O_C$ be a similar triangle to $\triangle O_A O_B O_C$ with ratio 2 : 1 and passes through vertices A, B, C , where $O_A O_B \parallel O_A' O_B'$, $O_B O_C \parallel O_B' O_C'$, $O_C O_A \parallel O_C' O_A'$.

Let's first prove that O is the circumcenter of $\triangle ABC$. Since points O_A, O_B, O_C are circumcenters of $\triangle PBC, \triangle PCA, \triangle PAB$, respectively, by definition, $O_A O, O_B O, O_C O$ are perpendicular bisectors of BC, AC, AB . Thus O is the circumcenter of $\triangle ABC$.

Next we show that point P is the isogonal conjugate point to the circumcenter O . Since $O_A O \perp BC$, and $O_A M \perp PB$, then O_A, X, M, B are concyclic. Therefore, $\angle MO_A X = \angle PBC$. Since $O_A' C \perp PC$ and $O_A' B \perp PB$, then O_A', C, P, B are concyclic. Therefore $\angle PBC = \angle CO_A' P = \angle O_B O_A P$. Thus $O_A P$ and $O_A O$ are isogonal conjugate lines of $\triangle O_A O_B O_C$ and hence P, O are isogonal conjugate points.

■

In order to define the *Generalized Euler Line*, we first define the *Circumcenter of Mass* of a polygon.

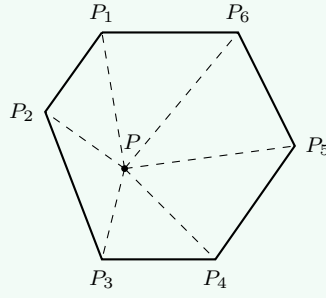
Definition 6. (Circumcenter of Mass)

Let K be an n -polygon defined by vertices P_1, \dots, P_n . Let P be an arbitrary point, by connecting P to each P_i , $i = (1, \dots, n)$, we triangulate K into n triangles $\triangle P P_i P_{i+1}$ where $P_{n+1} = P_1$. Let O_i be the circumcenter of $\triangle P P_i P_{i+1}$. We define the *Circumcenter of Mass (CCM)* to be

$$CCM(K) = \sum_{i=1}^n \frac{S_i}{S_K} O_i,$$

where S_i is the area^a of each $\triangle P P_i P_{i+1}$ and S_K is the area of the polygon $K =$

(P_1, \dots, P_n) .

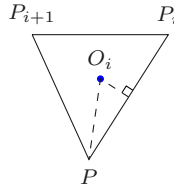


^aIt is in fact the *signed* area defined by $\langle (P_i - P) \times (P_{i+1} - P), \vec{N} \rangle / 2$, where \vec{N} is the normal vector of the Euclidean plane.

Theorem 4

The Circumcenter of Mass $CCM(K)$ of a polygon K is well defined, that is, it is independent to the arbitrary point P .

Proof: Let $P_i = (x_i, y_i)$, $i = (1, \dots, n)$. $P = (p, q)$. We define $O_i = (u_i, v_i)$, $i = (1, \dots, n)$ be the circumcenter of $\triangle PP_iP_{i+1}$ for $i = 1, \dots, n$.



We start by looking at the inner product of $P_i - P$ and $O_i - P$, since O_i is the circumcenter,

$$\begin{aligned} \langle P_i - P, O_i - P \rangle &= |O_i - P| |P_i - P| \cos(\angle O_i P P_i) \\ &= \frac{1}{2} |P_i - P|^2 = \frac{1}{2} |P_i|^2 - \langle P_i - P, P \rangle - \frac{1}{2} |P|^2, \end{aligned} \quad (2)$$

and from (2), we have

$$\langle P_i - P, O_i \rangle = \frac{1}{2} |P_i|^2 - \frac{1}{2} |P|^2. \quad (3)$$

Since the above equation is valid for any i , we replace i by $i + 1$ to get

$$\langle P_{i+1} - P, O_i \rangle = \frac{1}{2} |P_{i+1}|^2 - \frac{1}{2} |P|^2. \quad (4)$$

Then, multiplying $(y_{i+1} - p)$ to (3) and $(y_i - p)$ to (4) and subtracting, we get

$$\langle (y_{i+1} - p)(P_i - P) - (y_i - p)(P_{i+1} - P), O_i \rangle \quad (5)$$

$$= \frac{1}{2} [(y_{i+1} - p)|P_i|^2 - (y_i - p)|P_{i+1}|^2] - \frac{1}{2} |P|^2 (y_{i+1} - y_i). \quad (6)$$

But (5) is equal to

$$\begin{aligned} & \left\langle \left((y_{i+1} - p) \begin{bmatrix} x_i - p \\ y_i - q \end{bmatrix} - (y_i - p) \begin{bmatrix} x_{i+1} - p \\ y_{i+1} - q \end{bmatrix} \right), \begin{bmatrix} u_i \\ v_i \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} 2S_i \\ 0 \end{bmatrix}, \begin{bmatrix} u_i \\ v_i \end{bmatrix} \right\rangle = 2S_i u_i. \end{aligned}$$

Thus we have

$$S_i u_i = \frac{1}{4} [(y_{i+1} - p)|P_i|^2 - (y_i - p)|P_{i+1}|^2] - \frac{1}{4} |P|^2 (y_{i+1} - y_i). \quad (7)$$

Since $P_{n+1} = P_1$, taking summation, then the terms $\sum (|P|^2 (y_{i+1} - y_i))$ and $\sum (|P_i|^2 - |P_{i+1}|^2)$ will cancel out. As a result,

$$\sum_{i=1}^n S_i u_i = \frac{1}{4} \sum_{i=1}^n [y_{i+1}|P_i|^2 - y_i|P_{i+1}|^2].$$

Similarly, we have

$$\sum_{i=1}^n S_i v_i = -\frac{1}{4} \sum_{i=1}^n [x_{i+1}|P_i|^2 - x_i|P_{i+1}|^2].$$

Thus,

$$CCM(K) = \sum_{i=1}^n \frac{S_i}{S_K} O_i = \frac{1}{4S_K} \begin{bmatrix} \sum_{i=1}^n [y_{i+1}|P_i|^2 - y_i|P_{i+1}|^2] \\ - \sum_{i=1}^n [x_{i+1}|P_i|^2 - x_i|P_{i+1}|^2] \end{bmatrix}$$

is independent to P . ■

Definition 7. (Center of Mass)

Let K be an n -polygon be defined by vertices P_1, \dots, P_n . Let P be an arbitrary point. Connecting P to each P_i , we triangulate K into n triangles $\triangle PP_i P_{i+1}$ where $i = 1, \dots, n$, and $P_{n+1} = P_1$.

The **Center of Mass** $CM(K)$ of K is defined to be

$$CM(K) = \sum_{i=1}^n \frac{S_i}{S_K} G_i$$

where S_i is the (signed) area of each $\triangle PP_i P_{i+1}$; $G_i = (P_i + P_{i+1} + P)/3$ is the centroid of each $\triangle PP_i P_{i+1}$; and S_K is the area of the polygon $K = (P_1, \dots, P_n)$.

We wish to prove the Center of Mass of a polygon is independent of the choice of P . We first introduce a helpful lemma.

Lemma 1

Let P_1, \dots, P_n be vectors in \mathbb{R}^2 . Then for any $1 \leq i, j, k \leq n$, we have

$$\langle P_i \times P_j, \vec{N} \rangle P_k + \langle P_j \times P_k, \vec{N} \rangle P_i + \langle P_k \times P_i, \vec{N} \rangle P_j = 0.$$

Proof: Since we are working on \mathbb{R}^2 , we can write, without loss of generality, that $P_k = \alpha P_i + \beta P_j$. Then we have

$$\begin{aligned}\langle P_i \times P_j, \vec{N} \rangle P_k &= \langle P_i \times P_j, \vec{N} \rangle (\alpha P_i + \beta P_j) \\ &= \alpha \langle P_i \times P_j, \vec{N} \rangle P_i + \beta \langle P_i \times P_j, \vec{N} \rangle P_j.\end{aligned}$$

On the other hand, we have

$$\langle P_j \times P_k, \vec{N} \rangle P_i = \alpha \langle P_j \times P_i, \vec{N} \rangle P_i,$$

and

$$\langle P_k \times P_i, \vec{N} \rangle P_j = \beta \langle P_j \times P_i, \vec{N} \rangle P_j.$$

Thus we have

$$\langle P_i \times P_j, \vec{N} \rangle P_k + \langle P_j \times P_k, \vec{N} \rangle P_i + \langle P_k \times P_i, \vec{N} \rangle P_j = 0.$$

■

Theorem 5

The Center of Mass of a polygon is well defined, that is, it is independent to P .

Proof: By definition, we have

$$CM(K) = \sum_{i=1}^n \frac{\langle (P_i - P) \times (P_{i+1} - P), \vec{N} \rangle (P_i + P_{i+1} + P)}{6S_K}.$$

We expand

$$\begin{aligned}& \sum_{i=1}^n \langle (P_i - P) \times (P_{i+1} - P), \vec{N} \rangle (P_i + P_{i+1} + P) \\ &= \langle \sum_{i=1}^n P_i \times P_{i+1} - \sum_{i=1}^n P_i \times P - \sum_{i=1}^n P \times P_{i+1}, \vec{N} \rangle (P_i + P_{i+1} + P) \\ &= \sum_{i=1}^n \langle P_i \times P_{i+1}, \vec{N} \rangle (P_i + P_{i+1}) - \sum_{i=1}^n \langle P \times (P_{i+1} - P_i), \vec{N} \rangle (P_i + P_{i+1}) \quad (8) \\ &+ \sum_{i=1}^n \langle P_i \times P_{i+1}, \vec{N} \rangle P + \sum_{i=1}^n \langle P \times (P_i - P_{i+1}), \vec{N} \rangle P \\ &= \textcircled{1} - \textcircled{2} + \textcircled{3} + \textcircled{4}.\end{aligned}$$

We shall simplify the above terms. First, we have

$$\textcircled{4} = \sum_{i=1}^n \langle P \times (P_i - P_{i+1}), \vec{N} \rangle P = \langle P \times \sum_{i=1}^n (P_i - P_{i+1}), \vec{N} \rangle P = 0. \quad (9)$$

Next, using Lemma 1, we get

$$\begin{aligned}
 \textcircled{2} &= \sum_{i=1}^n \langle P \times (P_{i+1} - P_i), \vec{N} \rangle (P_i + P_{i+1}) \\
 &= \sum_{i=1}^n \langle P \times P_{i+1}, \vec{N} \rangle P_i - \sum_{i=1}^n \langle P \times P_i, \vec{N} \rangle P_{i+1} \\
 &= \sum_{i=1}^n \langle P \times P_{i+1}, \vec{N} \rangle P_i + \sum_{i=1}^n \langle P_i \times P, \vec{N} \rangle P_{i+1} \\
 &= - \sum_{i=1}^n \langle P_{i+1} \times P_i, \vec{N} \rangle P = \sum_{i=1}^n \langle P_i \times P_{i+1}, \vec{N} \rangle P = \textcircled{3}.
 \end{aligned} \tag{10}$$

Thus we have

$$\begin{aligned}
 &\sum_{i=1}^n \langle (P_i - P) \times (P_{i+1} - P), \vec{N} \rangle (P_i + P_{i+1} + P) \\
 &= \textcircled{1} = \sum_{i=1}^n \langle P_i \times P_{i+1}, \vec{N} \rangle (P_i + P_{i+1}),
 \end{aligned}$$

and hence

$$CM(K) = \sum_{i=1}^n \frac{\langle P_i \times P_{i+1}, \vec{N} \rangle (P_i + P_{i+1})}{6S_K}, \tag{11}$$

which is independent to P . ■

Corollary 1

Let M_i be the midpoint of A_i and A_{i+1} for $i = 1, \dots, n$. Then

$$CM(K) = \frac{2}{3} \sum_{i=1}^n \frac{\langle P_i \times P_{i+1}, \vec{N} \rangle}{S_K} M_i.$$

Definition 8. (Generalized Euler Line)

Let K be a polygon, the Euler Line ℓ of the Polygon K is the line that passes through the Center of Mass (CM) and the Circumcenter of Mass $CCM(K)$. When the polygon is reduced to triangle, the Generalized Euler Line is reduced to the classical Euler Line.

Definition 9. (t -Point)

Let $\triangle ABC$ be a triangle and let O, G be its circumcenter and centroid, respectively. A point E_t on the Euler line is called a t -point, if there is a real number t such that

$$A_t = tG + (1 - t)O.$$

For example, the orthocenter $H = A_3$, and the nine-point circle $N = A_{3/2}$.

Let K be an n -polygon defined by vertices P_1, \dots, P_n . Let P be an arbitrary point, by connecting P to each P_i , $i = (1, \dots, n)$, we triangulate K into n triangles $\triangle PP_i P_{i+1}$

where $P_{n+1} = P_1$. Let $(A_t)_t$ be the t -point of $\triangle PP_iP_{i+1}$. We define the t -point of Mass (tM) to be

$$tM(K) = \sum_{i=1}^n \frac{S_i}{S_K} (A_t)_i,$$

where S_i is the (signed) area of each $\triangle PP_iP_{i+1}$ and S_K is the area of the polygon $K = (P_1, \dots, P_n)$.

Using Theorem 4 and Theorem 5, we then conclude that

Theorem 6

The definition of $tM(K)$ is independent to the choice of the arbitrary point P . Moreover, $tM(K)$ is on the Generalized Euler Line of K for any t .