Brocard Circle

Ningyue Xu

01

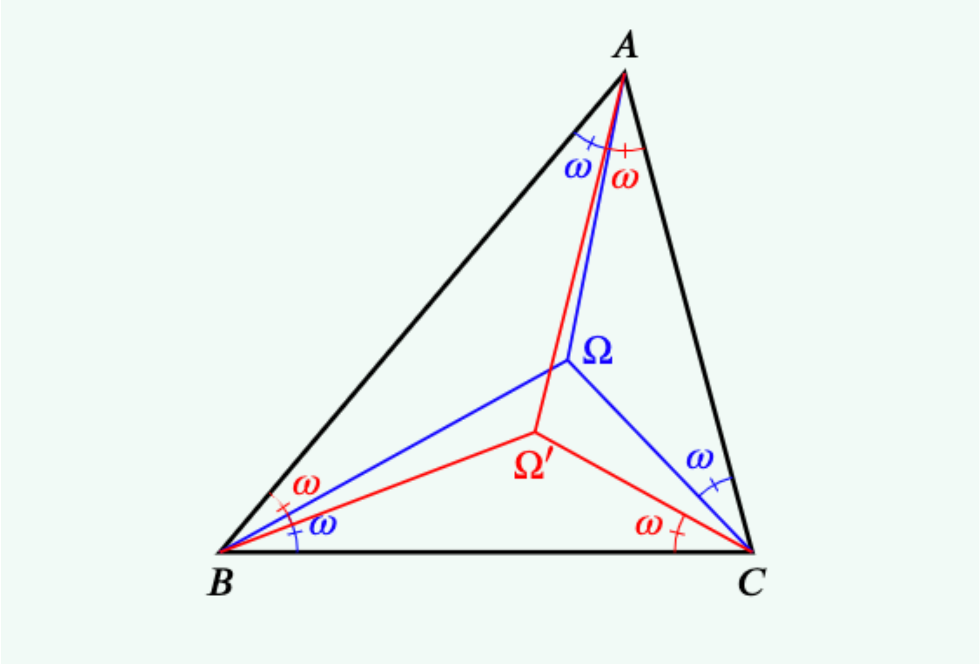
Introduction

Brocard point

Let Ω be a point inside the $\triangle ABC$. If
$\angle \Omega AB = \angle \Omega BC = \angle \Omega CA = \omega$,
then Ω is called the Brocard Point (or the First Brocard Point).

ω is called the Brocard Angle.

Similarly, we can define the Second Brocard Point as the point Ω ′ such that
$\angle \Omega' CB = \angle \Omega' AC = \angle \Omega' BA = \omega$



Brocard Triangle

In geometry, the Brocard triangle of a triangle is a triangle formed by the intersection of lines from a vertex to its corresponding Brocard point and a line from another vertex to its corresponding Brocard point. Given the coordinate triangles $\triangle A1A2A3$ and the first and second Brocard points Ω , Ω'

let

B1=A2 Ω ∩ A3 Ω'

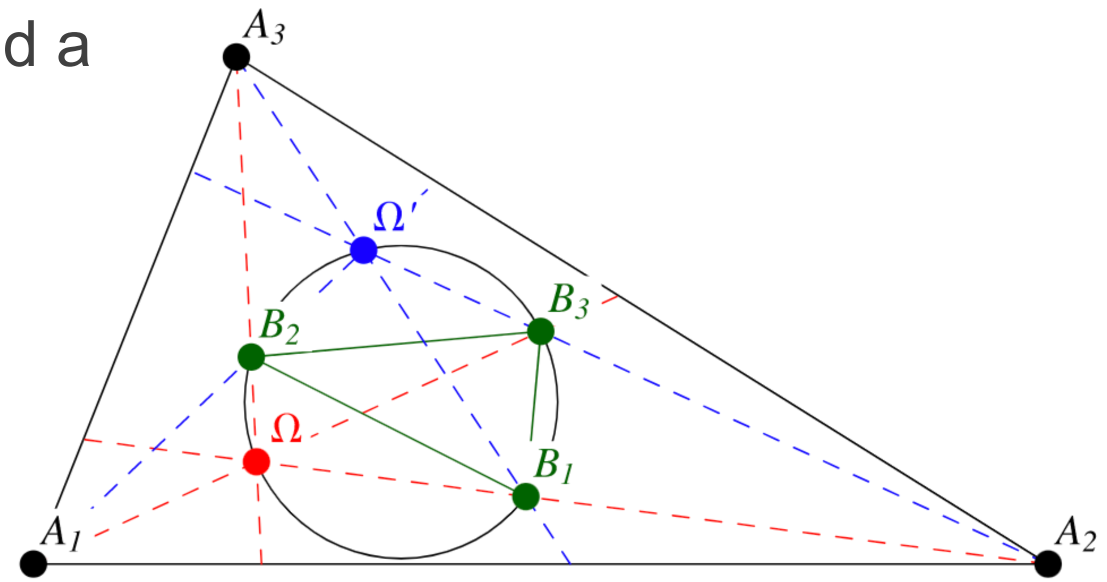
B2=A3 Ω ∩ A1Ω'

B3= A1 Ω ∩ A2Ω'

then the $\triangle B1B2B3$, is called the first Brocard triangle

which is similar to $\triangle A1A2A3$.

It is inscribed in the Brocard circle.



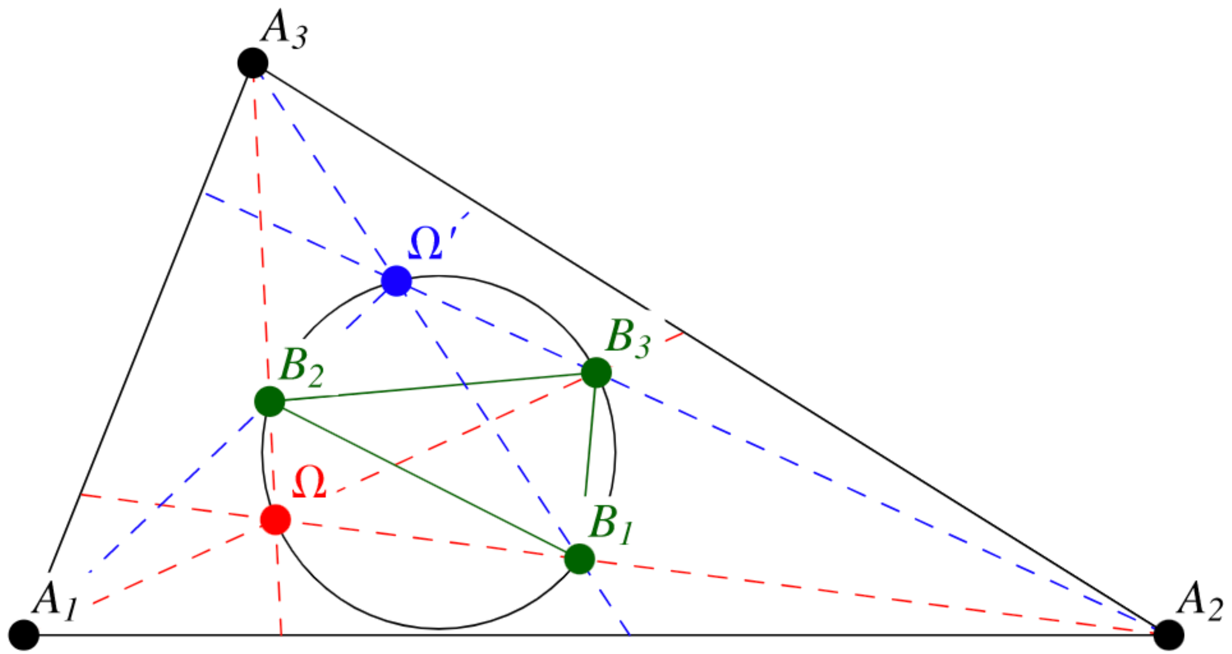
Are they on the same circle?

Yes!!

Brocard Circle

The Brocard circle, also known as the seven-point circle, is the circle having the line segment connecting the circumcenter O and symmedian point K of a triangle $\triangle ABC$ as its diameter (known as the Brocard diameter).

This circle also passes through the first and second Brocard points respectively. It also passes through Kimberling centers X_i for i=3, 6, 1083, and 1316. Brocard Triangle is inscribed in the Brocard circle.



$$a\,b\,c\left(\alpha^2+\beta^2+\gamma^2\right)=a^3\,\beta\gamma+b^3\,\gamma\alpha+c^3\,\alpha\beta$$

Why are they on the same circle?

We'll solve this problem algebraically next!!!!

02

c o o r d i n a t e s

Trilinear coordinates

Trilinear coordinates are a coordinate system used in plane geometry to describe the location of points, especially for points within a triangle. This coordinate system is defined by the three sides of the triangle.
In triangle ABC, the trilinear coordinates of a point P are defined as α:β:γ, where α, β, and γ are the respective distances from P to the sides BC, CA, and AB. It's important to note that these distances are typically normalized such that α/a + β/b + γ/c = 1, where a, b, and c represent the lengths of the sides BC, CA, and AB respectively.
For example, the trilinear coordinates of the vertices A, B, and C of triangle ABC are 1:0:0, 0:1:0, and 0:0:1 respectively. Special points of the triangle, like the incenter, circumcenter, centroid, etc., have specific trilinear coordinates.
The trilinear coordinates of the circumcenter are a(b^2 + c^2 – a^2) : b(c^2 + a^2 – b^2) : c(a^2 + b^2 – c^2)
The trilinear coordinates of the symmedian point are a : b : c

At this point, the easiest way to prove this is for us to find the trilinear coordinates of the circle and bring in the trilinear coordinates of each point and find that they all match

$$a\,b\,c\left(\alpha^2+\beta^2+\gamma^2\right)=a^3\,\beta\gamma+b^3\,\gamma\alpha+c^3\,\alpha\beta$$

Barycentric coordinates

Closely related to the trilinear coordinates are the Barycentric coordinates. In the context of a triangle, barycentric coordinates are also known as area coordinates or areal coordinates, Given a triangle ABC and a point P in its plane, we can define three new triangles: PBC, PCA and PAB. Then, the Barycentric coordinates of the point P can be defined as

u = [PBC] / [ABC]
v = [PCA] / [ABC]
w= [PAB] / [ABC]
is called the area coordinate or Barycentric coordinates of the point P, and is denoted as (u:v:w)

Conversion between barycentric and trilinear coordinates
A point with trilinear coordinates x : y : z has barycentric coordinates ax : by : cz where a, b, c are the side lengths of the triangle. Conversely, a point with barycentrics (u:v:w) has trilinears u/a:v/b:w/c

03

Seven-point circle

Brocard point's coordinates

Theorem 13. (Circle) *The general equation of a circle is*

$$-a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0$$

for constants u, v, w.

Proof. Let the circle have center (i, j, k) and radius r. Then we use the Distance formula and note that this is

$$-a^2(y - j)(z - k) - b^2(z - k)(x - i) - c^2(x - i)(y - j) = r^2.$$

Expanding yields

$$-a^2yz - b^2zx - c^2xy + Lx + My + Nz = C$$

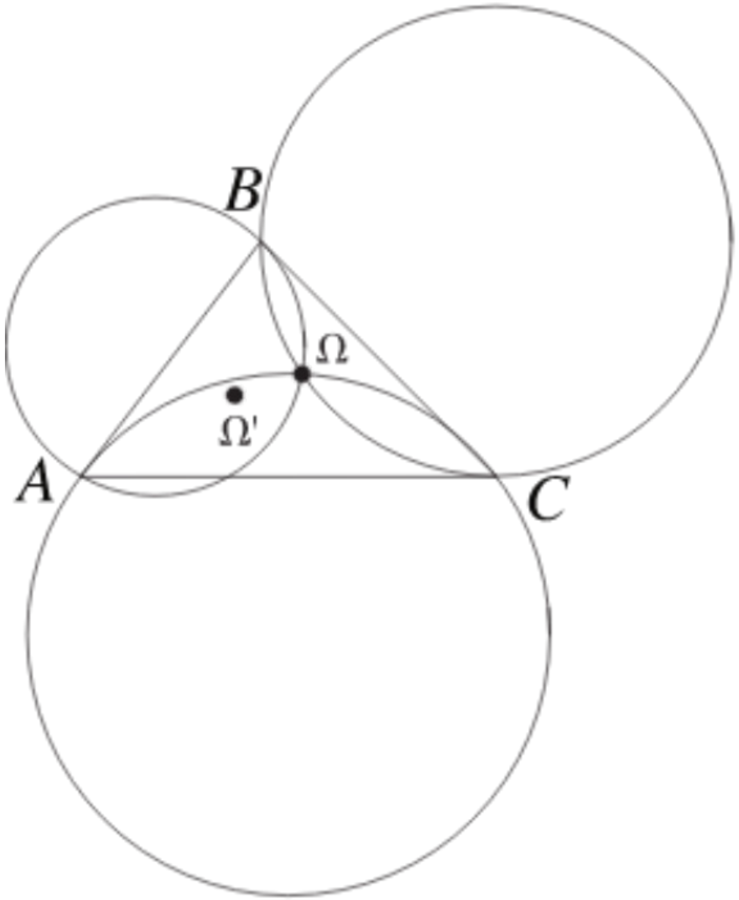
for constants L, M, N, C. Since $x + y + z = 1$, we rewrite the righthand side as $C(x + y + z)$, and subtracting yields

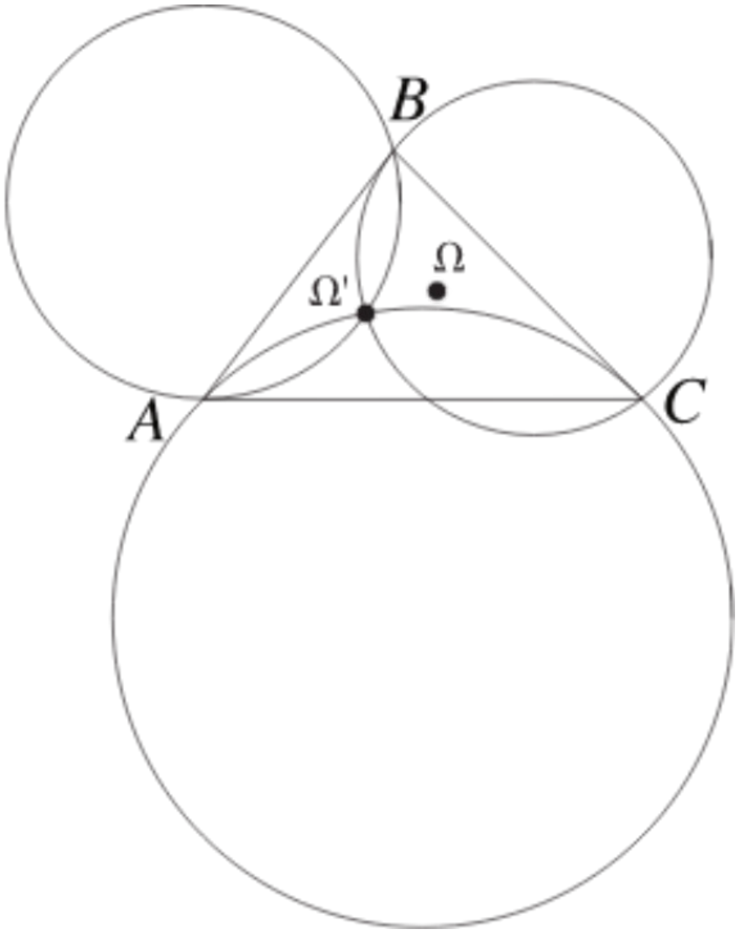
$$-a^yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0,$$

where $u = L - C, v - M - C, w = N - C$. $\square$

Brocard point's coordinates

As in the diagram right, form a circle through points A and B, tangent to edge BC of the triangle (the center of this circle is at the point where the perpendicular bisector of AB meets the line through point B that is perpendicular to BC). Symmetrically, form a circle through points B and C, tangent to edge AC, and a circle through points A and C, tangent to edge AB. These three circles have a common point, the first Brocard point of triangle ABC.





Brocard point's coordinates

let's first use the equation of a circle to find the coordinates of the Brocard points. Consider a circle that passes through the vertices A and B, and is tangent to the side AC at vertex A. This circle is denoted as CAAB.

Since this circle passes through points A and B, the equation of the circle is a^2yz + b^2zx + c^2*xy* - *wz*(x + y + z) = 0. Since this circle is tangent to the side AC at A, we have z^2= 0, which simplifies the equation to 0. This means we have w = b^2

So, CAAB becomes $a^2yz + b^2zx + c^2xy - b^2z(x+y+z) = 0.$

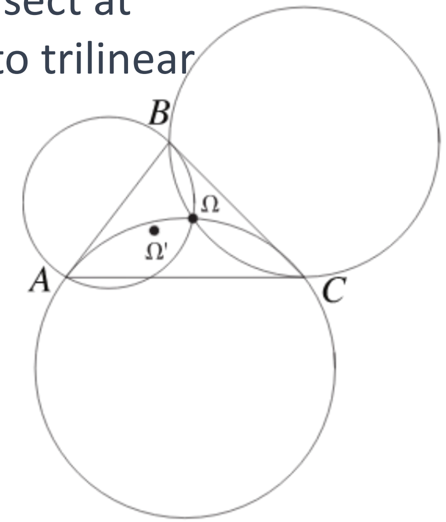
Similarly, we can derive the equations for others:

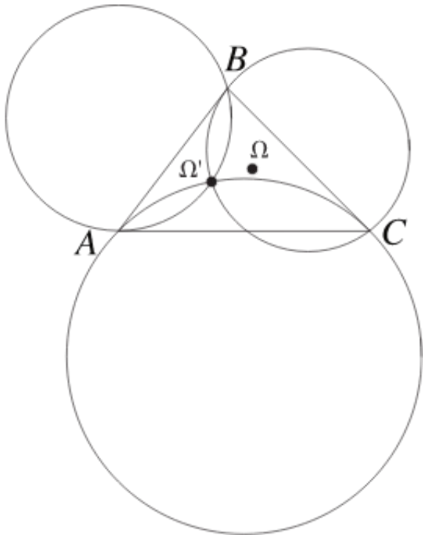
$$C_{BBC}: a^2yz + b^2zx + c^2xy - c^2x(x+y+z) = 0$$

$$C_{CCA}: a^2yz + b^2zx + c^2xy - a^2y(x+y+z) = 0$$

By solving these three equations together, we can find that these circles intersect at one point. Note that these are the barycentric coordinates. After converting to trilinear coordinates, we have $\Omega = \left(\frac{b}{c} : \frac{c}{a} : \frac{a}{b}\right)$

Similarly, we can get Ω' = $\left(\frac{c}{b} : \frac{a}{c} : \frac{b}{a}\right)$





Circle function

$$a\,b\,c\left(\alpha^2+\beta^2+\gamma^2\right)=a^3\ \beta\gamma+b^3\ \gamma\alpha+c^3\ \alpha\beta$$

The trilinear coordinates of the symmedian point are a : b : c=sinA ： sinB ： sinC

The trilinear coordinates of the circumcenter are a(b^2 + c^2 – a^2) : b(c^2 + a^2 – b^2) : c(a^2 + b^2 – c^2)

```
[2]
from sympy import symbols, simplify

a, b, c = symbols('a b c')

alpha1, beta1, gamma1 = a, b, c
alpha2, beta2, gamma2 =  a*(b**2 + c**2 - a**2), b*(c**2 + a**2 - b**2), c*

equation = lambda alpha, beta, gamma: a*b*c*(alpha**2 + beta**2 + gamma**2)

result1 = simplify(equation(alpha1, beta1, gamma1))
result2 = simplify(equation(alpha2, beta2, gamma2))

print(result1)
print(result2)
```

```
0
0
```

Brocard point's coordinates

$$a\,b\,c\left(\alpha^2+\beta^2+\gamma^2\right)=a^3\,\beta\gamma+b^3\,\gamma\alpha+c^3\,\alpha\beta$$

$$\Omega=\left(\frac{b}{c}:\frac{c}{a}:\frac{a}{b}\right)$$

$$\Omega'=\left(\frac{c}{b}:\frac{a}{c}:\frac{b}{a}\right)$$

[2]

```
from sympy import symbols, simplify

a, b, c = symbols('a b c')

alpha1, beta1, gamma1 = c/b, a/c, b/a
alpha2, beta2, gamma2 = b/c, c/a, a/b

equation = lambda alpha, beta, gamma: a*b*c*(alpha**2 + beta**2 + gamma**2)

result1 = simplify(equation(alpha1, beta1, gamma1))
result2 = simplify(equation(alpha2, beta2, gamma2))

print(result1)
print(result2)
```

```
0
0
```

Brocard triangle 's coordinates

The trilinear vertex matrix is

$$\begin{bmatrix} a\,b\,c & c^3 & b^3 \\ c^3 & a\,b\,c & a^3 \\ b^3 & a^3 & a\,b\,c \end{bmatrix}.$$

$$a\,b\,c\left(\alpha^2+\beta^2+\gamma^2\right)=a^3\ \beta\gamma+b^3\ \gamma\alpha+c^3\ \alpha\beta$$

[7]

```
from sympy import symbols, simplify

a, b, c = symbols('a b c')

points_trilinear_coords = [
    {'alpha': a*b*c, 'beta': c**3, 'gamma': b**3},
    {'alpha': c**3, 'beta': a*b*c, 'gamma': a**3},
    {'alpha': b**3, 'beta': a**3, 'gamma': a*b*c},
]
equation = lambda alpha, beta, gamma: a*b*c*(alpha**2 + beta**2 + gamma**2)

for i, coords in enumerate(points_trilinear_coords, start=1):
    result = simplify(equation(coords['alpha'], coords['beta'], coords['gam
    print(f'Result for point {i}: {result}')
```

```
Result for point 1: 0
Result for point 2: 0
Result for point 3: 0
```

Circle function

$$a^2yz + b^2zx + c^2xy = \frac{a^2b^2c^2(x+y+z)}{a^2+b^2+c^2}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right).$$

The area coordinates of the symmedian point are a^2 : b^2 : c^2

The area coordinates of the circumcenter are a^2(b^2 + c^2 – a^2) : b^2(c^2 + a^2 – b^2) : c^2(a^2 + b^2 – c^2)

```python
from sympy import symbols, simplify

a, b, c = symbols('a b c')

x = a**2 * (b**2 + c**2 - a**2)
y = b**2 * (c**2 + a**2 - b**2)
z = c**2 * (a**2 + b**2 - c**2)

expr = a**2*y*z + b**2*z*x + c**2*x*y - (a**2 * b

expr = expr.subs({x: x, y: y, z: z})

simplified_expr = simplify(expr)

print(simplified_expr)
```

```
0
```

[1]

```python
from sympy import symbols, simplify

a, b, c = symbols('a b c')

x = a**2
y = b**2
z = c**2

expr = a**2*y*z + b**2*z*x + c**2*x*y - (a**2 * b**2 * c**2 * (x + y + z)) * ((x/a**2 + y/b**2 + z/c**2)

expr = expr.subs({x: x, y: y, z: z})

simplified_expr = simplify(expr)

print(simplified_expr)
```

```
0
```

Brocard point's coordinates

$$a^2yz + b^2zx + c^2xy = \frac{a^2b^2c^2(x+y+z)}{a^2+b^2+c^2}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right).$$

Point 1: (a*(b/c), b*(c/a), c*(a/b)) = (ab/c, bc/a, ca/b)
Point 2: (a*(c/b), b*(a/c), c*(b/a)) = (ac/b, ab/c, bc/a)

```
[5]
from sympy import symbols, simplify, Rational

a, b, c = symbols('a b c')

expr = a**2 * symbols('y') * symbols('z') + b**2 * symbols('z') * symbols('x') + c**2 * symbols('x') * s

trilinear1 = (b/c, c/a, a/b)
trilinear2 = (c/b, a/c, b/a)

barycentric1 = (a * trilinear1[0], b * trilinear1[1], c * trilinear1[2])
barycentric2 = (a * trilinear2[0], b * trilinear2[1], c * trilinear2[2])

simplified_expr1 = simplify(expr.subs({symbols('x'): barycentric1[0], symbols('y'): barycentric1[1], syn
simplified_expr2 = simplify(expr.subs({symbols('x'): barycentric2[0], symbols('y'): barycentric2[1], syn

print("Point 1:", simplified_expr1)
print("Point 2:", simplified_expr2)
```

```
Point 1: 0
Point 2: 0
```

Brocard triangle 's coordinates

$$a^2yz + b^2zx + c^2xy = \frac{a^2b^2c^2(x+y+z)}{a^2+b^2+c^2}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right).$$

The trilinear vertex matrix is

$$\begin{bmatrix} a\,b\,c & c^3 & b^3 \\ c^3 & a\,b\,c & a^3 \\ b^3 & a^3 & a\,b\,c \end{bmatrix}.$$

[7]

```python
from sympy import symbols, simplify

a, b, c = symbols('a b c')
x, y, z = symbols('x y z')

original_expr = a**2*y*z + b**2*z*x + c**2*x*y - (a**2 * b**2 * c**2 * (x + y + z)) * ((x/a**2 + y/b**2

points_trilinear = [(a*b*c, c**3, b**3), (c**3, a*b*c, a**3), (b**3, a**3, a*b*c)]

for point in points_trilinear:
    X, Y, Z = point

    x_bary = X*a
    y_bary = Y*b
    z_bary = Z*c

    expr = original_expr.subs({x: x_bary, y: y_bary, z: z_bary})

    simplified_expr = simplify(expr)
    print(simplified_expr)
```

```
0
0
0
```

They are on the same circle

Thank you