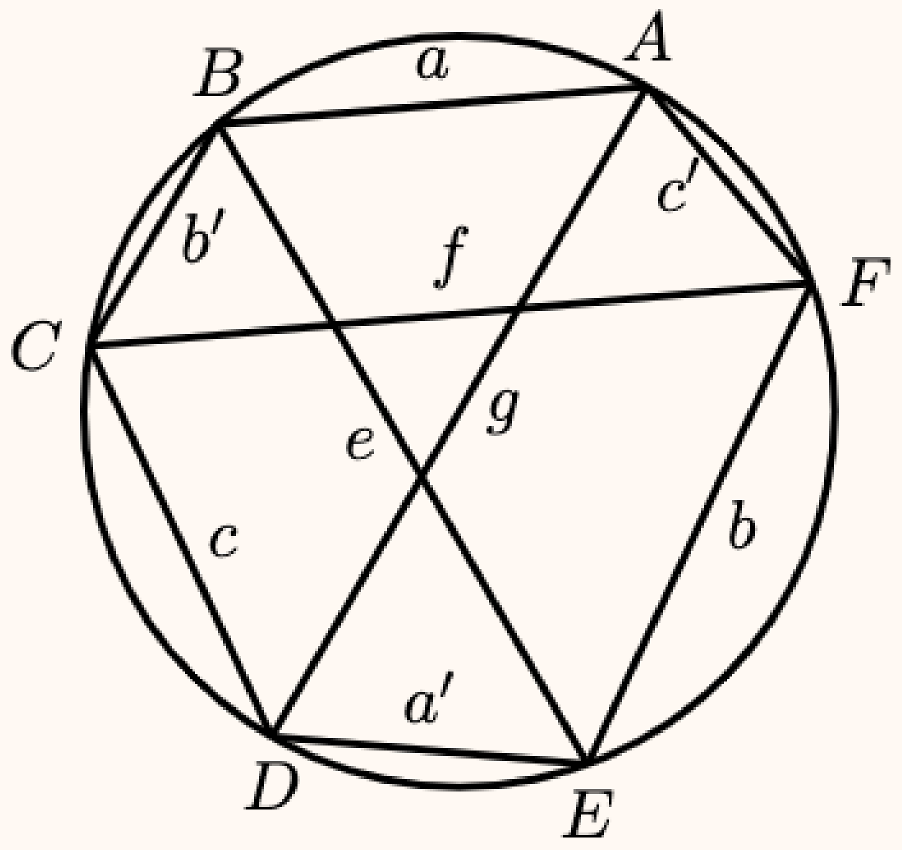
Fuhrmann's Theorem

UC Irvine - Math 199B

Songhan(Hanson) Hu

Theorem 1. (Fuhrmann's Theorem)

Let $ABCDEF$ be a convex concyclic hexagon. Let a, b', c, a', b, c' be the side lengths of AB, BC, CD, DE, EF, FA, respectively. Let e, f, g be the lengths of the main diagonals AD, BE, CF, respectively (See picture below).



Then

$$efg = aa'e + bb'f + cc'g + abc + a'b'c' \tag{1}$$

Review: Ptolemy's Theorem

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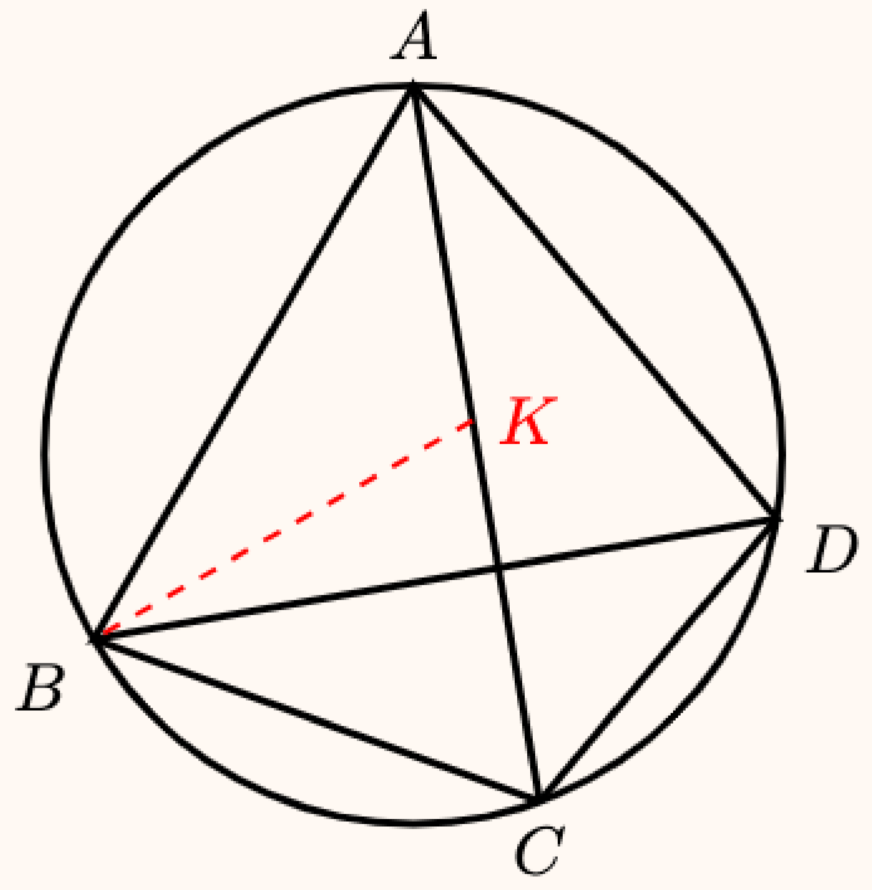
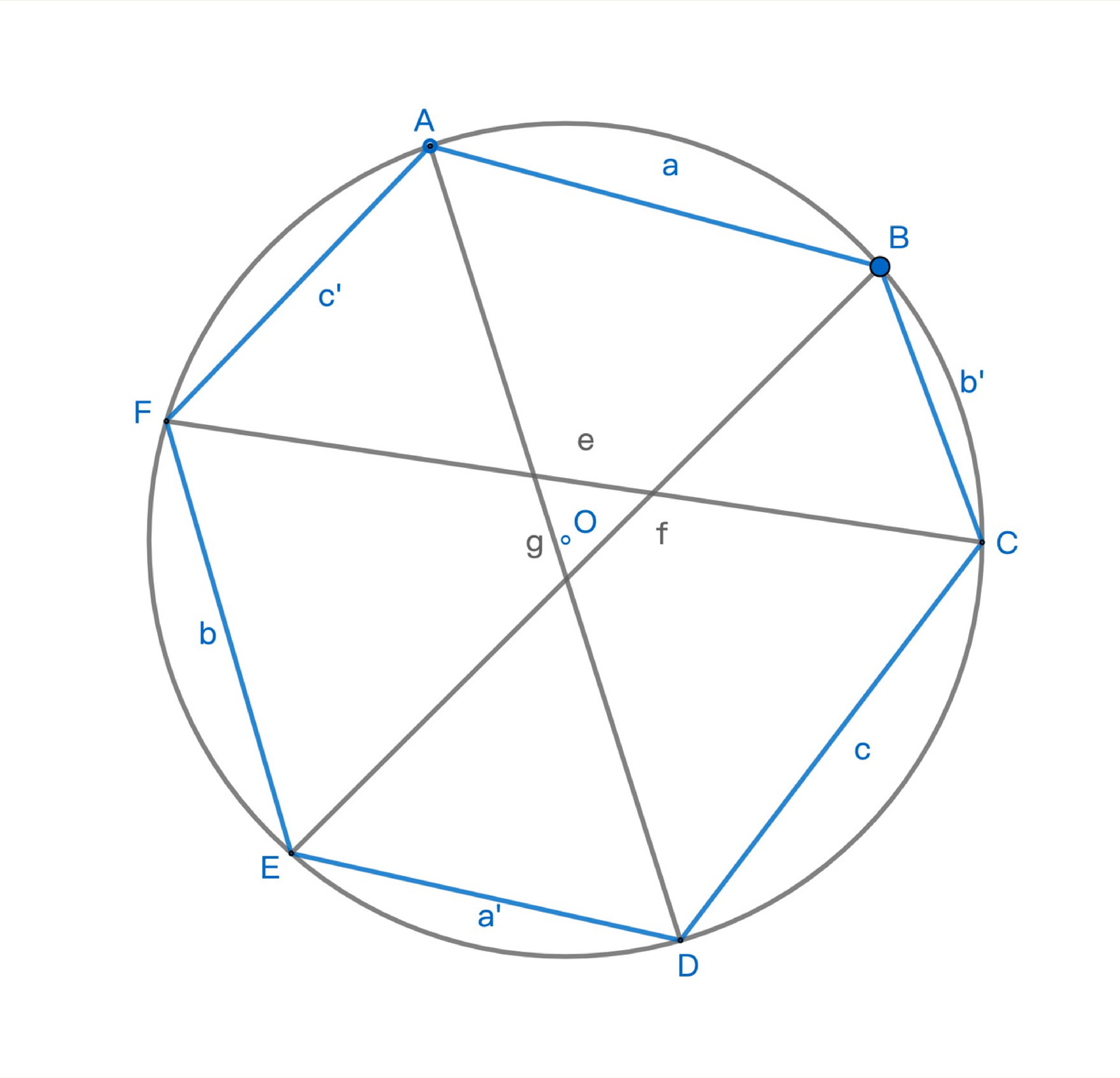
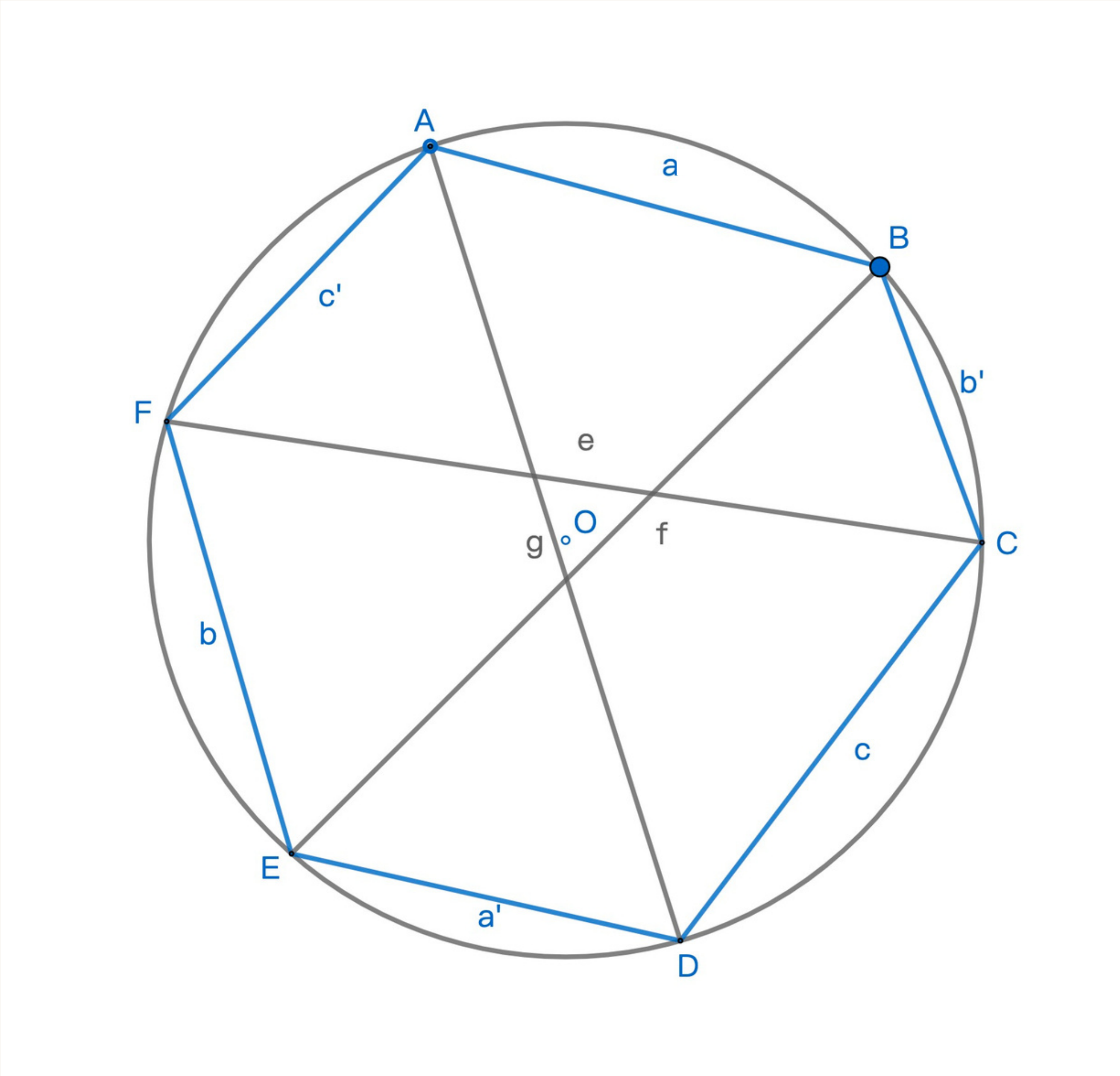
Concept and Definition

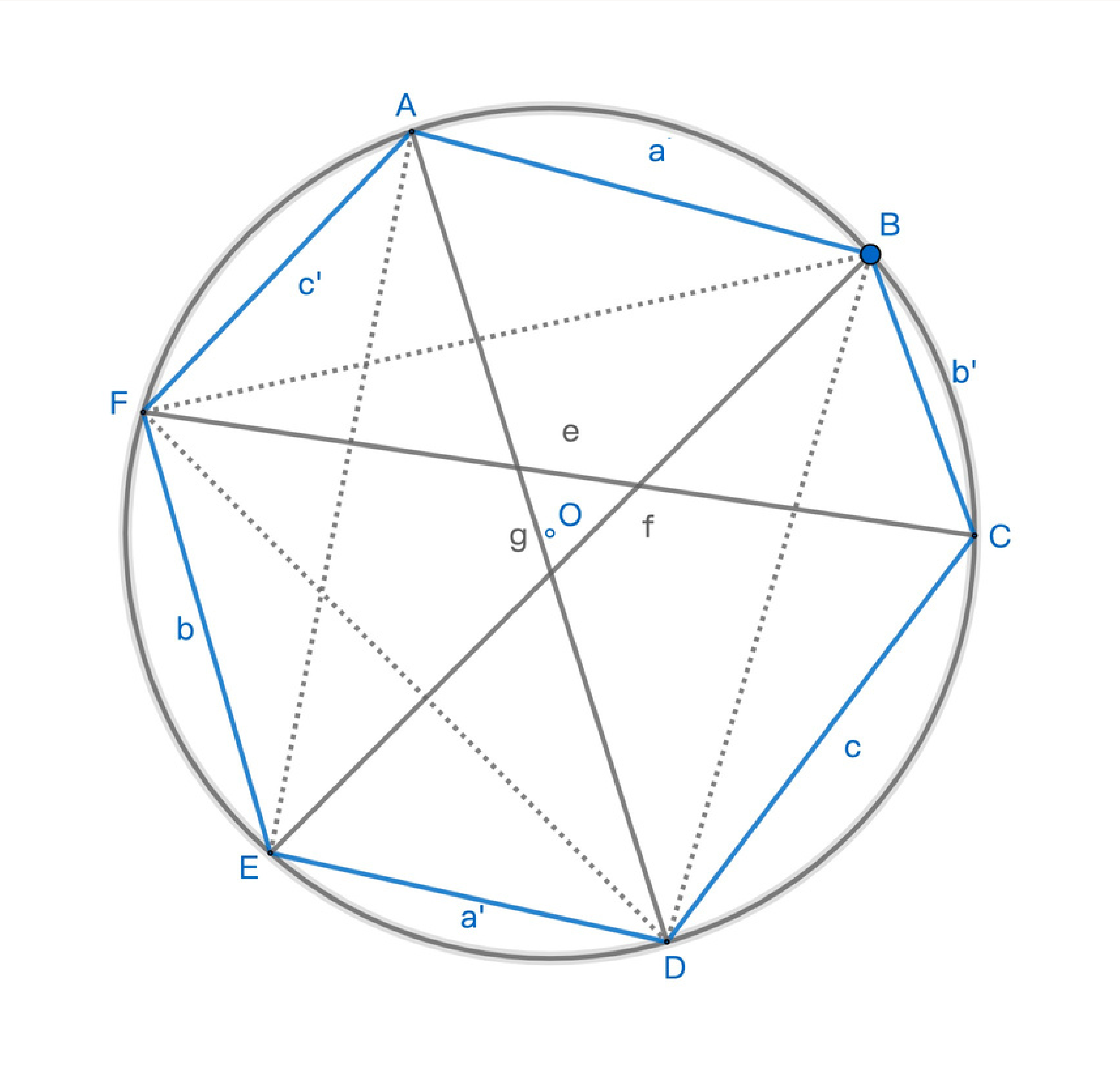
In the following picture, let $ABCD$ be a cyclic quadrilateral. Then

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$



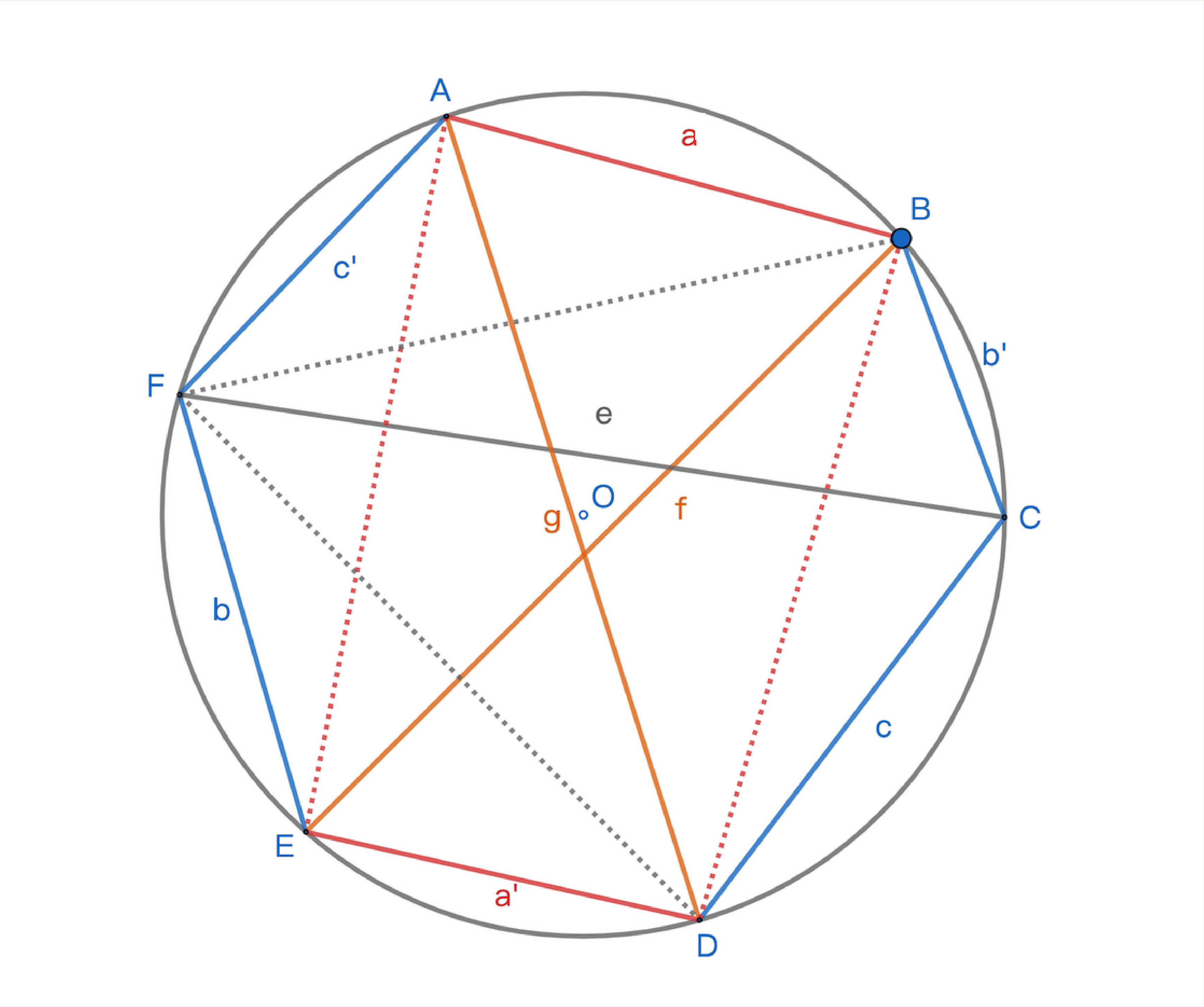

A
a
B
c'
b'
F
e
O
g
f
C
b
c
E
a'
D





Goal: Find a quadrilateral that diagonals composed by the main diagonals in the hexogon

$$efg = aa'e + bb'f + cc'g + abc + a'b'c'$$



Quadrilateral ABDE ---> Ptolemy's Theorem

$$gf = AE * BD + a' * a$$

$$egf = AE * BD * e + a' * a * e$$

$$AE = \frac{ab+c'f}{BF}$$

$$BD = \frac{b'g+ac}{AC} \quad \Rightarrow \quad BD = \frac{b'g+ac}{ae+b'c'} \cdot BF$$

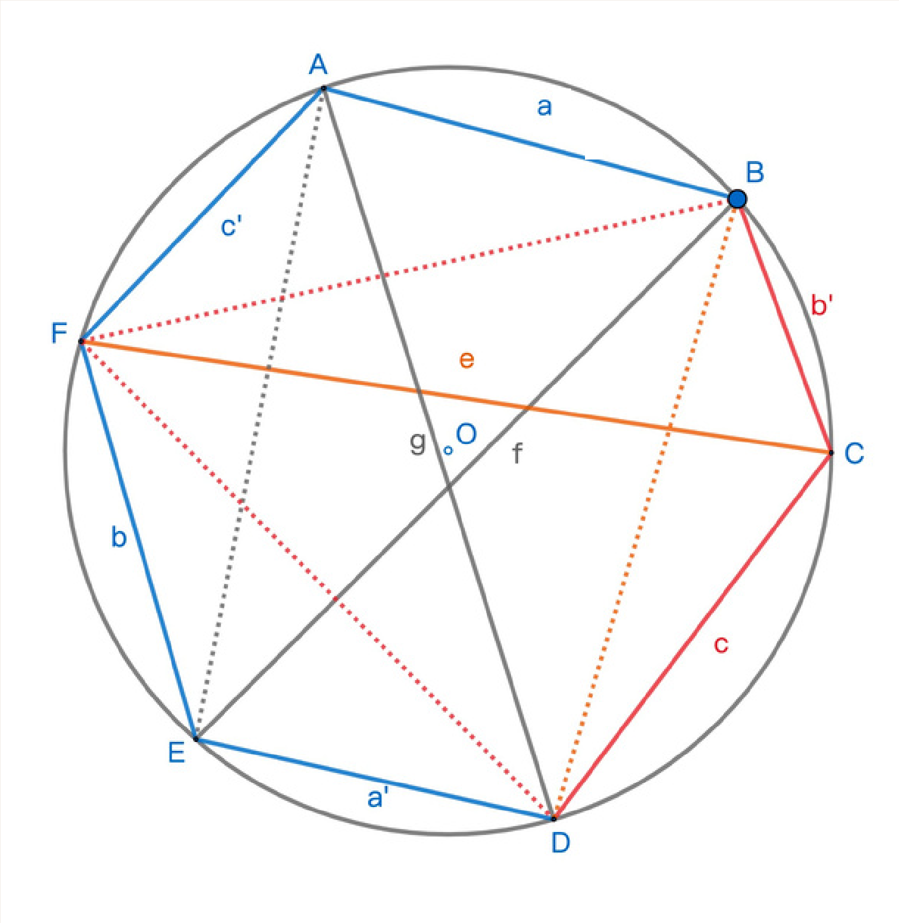
$$\frac{1}{AC} = \frac{BF}{ae+b'c'}$$

$$fg = (ab+c'f)\left(\frac{b'g+ac}{ae+b'c'}\right)e + aa'$$

$$fg(ae+b'c') = (ab+c'f')(b'g+ac) + aa' \quad (ae+b'c')$$

$$aefg + b'c'\ fg = abb'\ g + b'c'f'g + a^2bc + acc'\ f + a^2a'e + aa'bc' .$$

$\times$



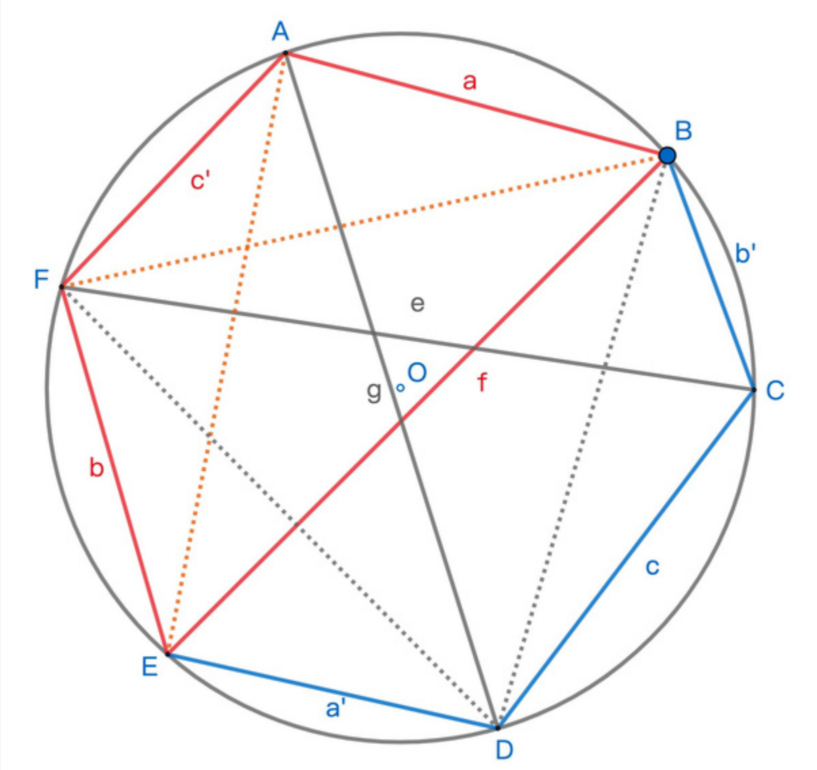
$$egf = AE * BD * e + a' * a * e$$

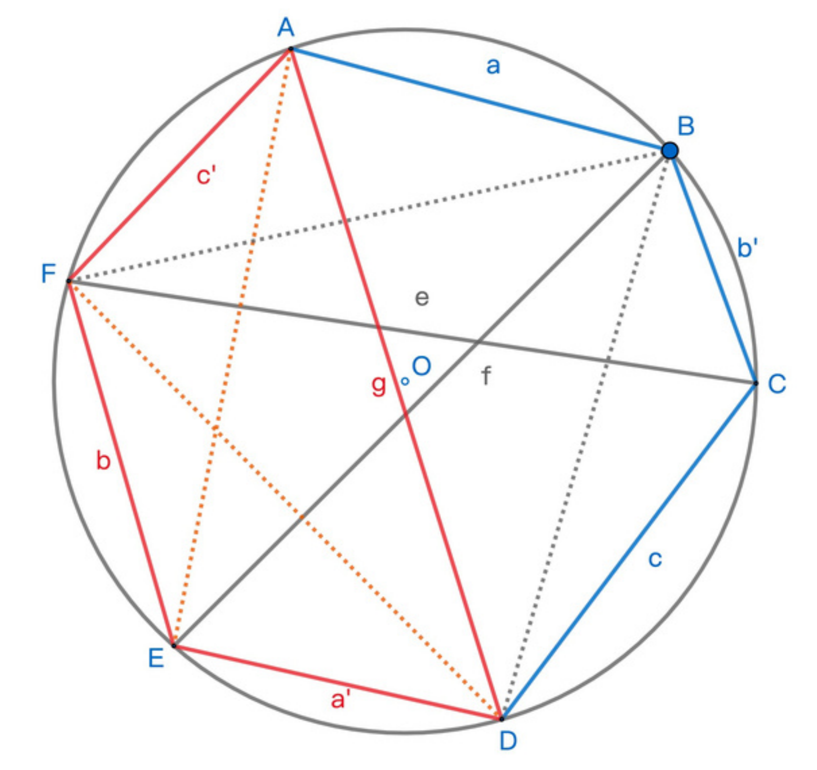
by quadrilateral BCDF:

$$BD * e = b' * DF + c * BF$$

$$\longrightarrow \quad efg = AE * (b' * DF + c * BF) + a * a' * e$$

$$\longrightarrow \quad efg = AE * DF * b' + BF * AE * c + a * a' * e$$





$$efg = aa'e + bb'f + cc'g + abc + a'b'c'$$

Summary of this proof

Intuition 1:

Find the correct quadrilateral to connect the main diagonals

Intuition 2:

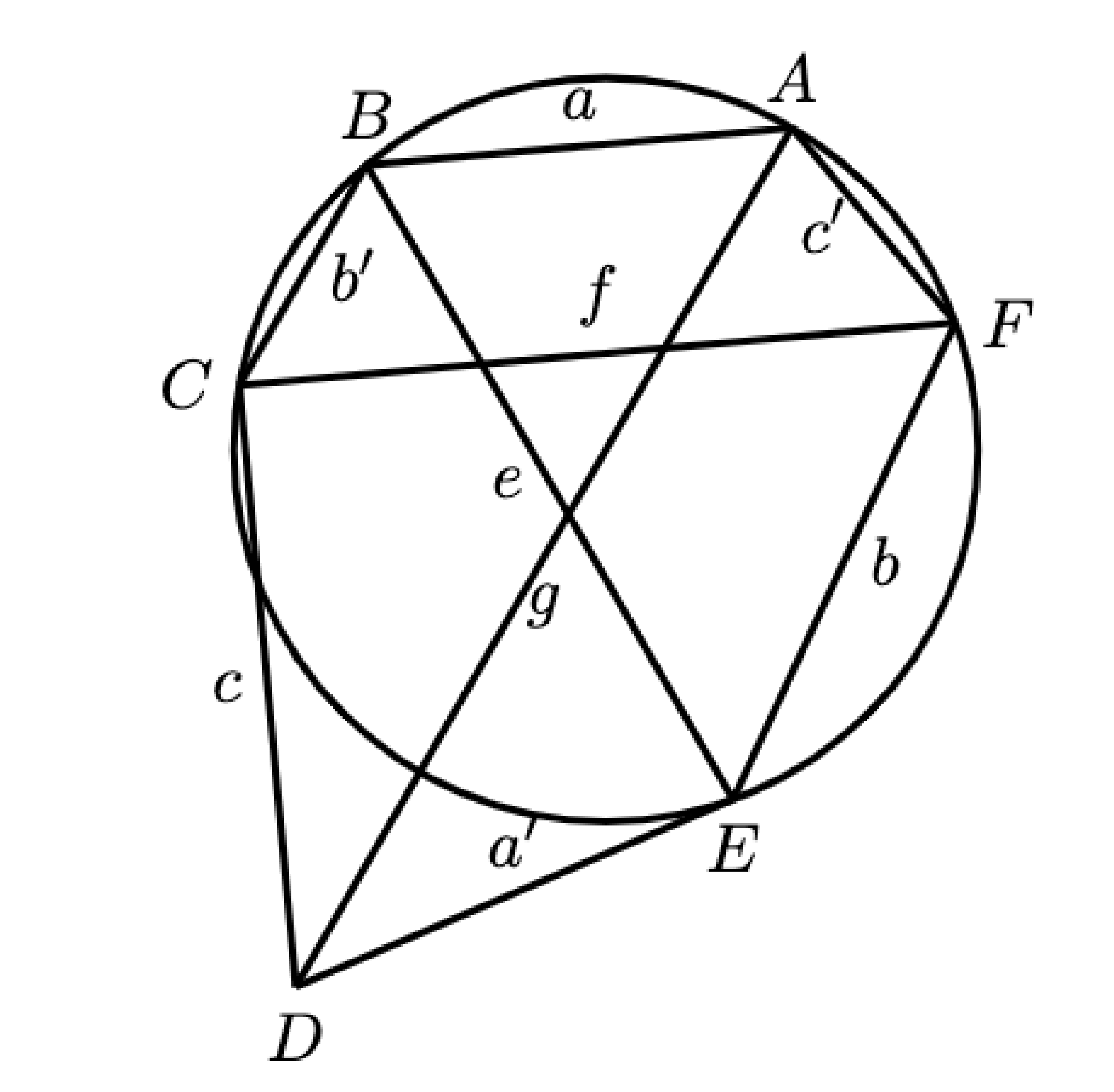
Eliminate Terms

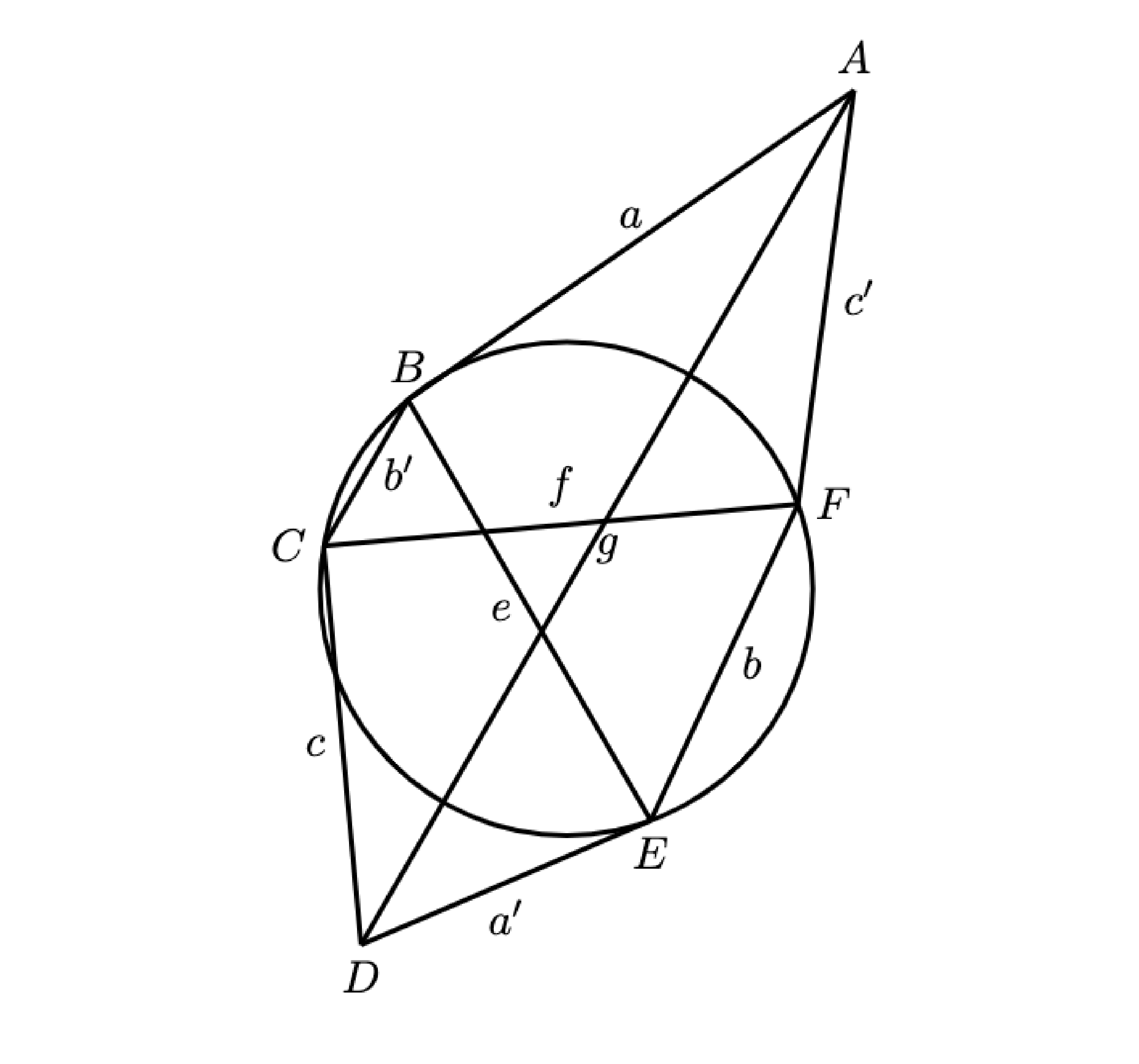
Difficulty:

Eliminate terms based on BD * e

Extension 1: how if it is
not cyclic

Not cyclic: one or more vertices is not on a circle.





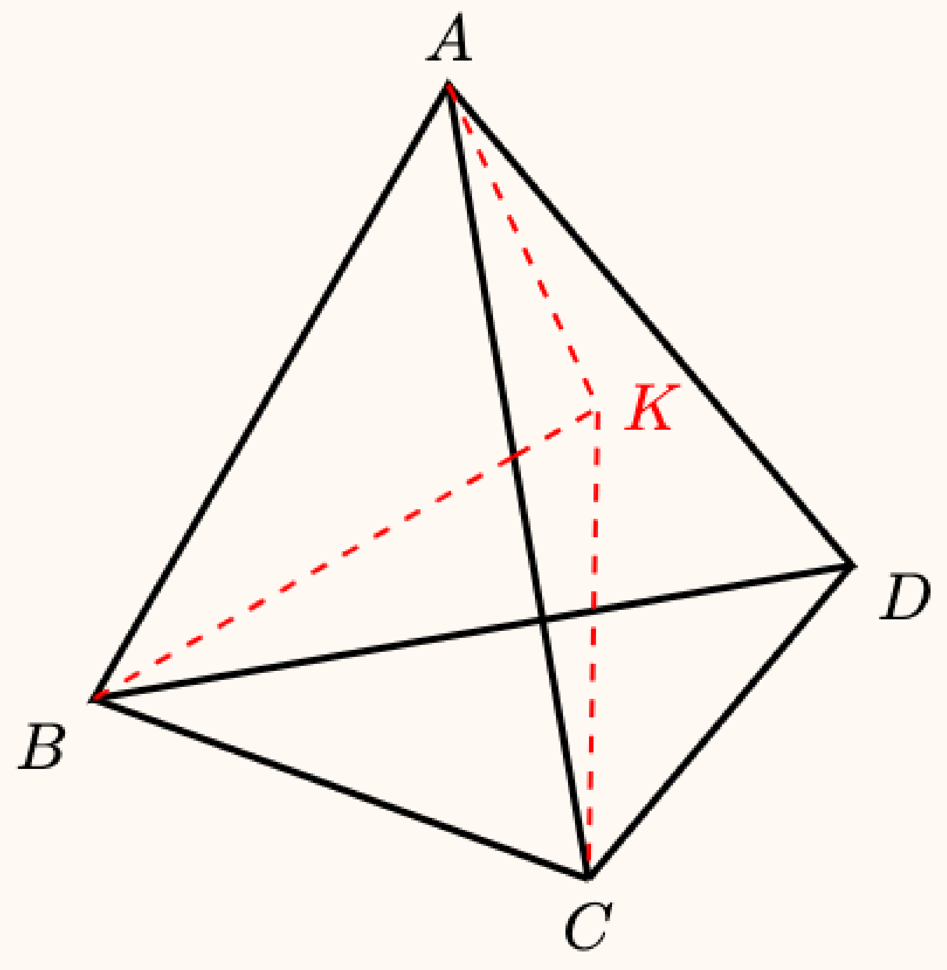
what is changed?

Review: Ptolemy's Inequality

Let $ABCD$ be a quadrilateral (not necessarily concyclic). Then

$$AC \cdot BD \leq AB \cdot CD + AD \cdot BC.$$

The equality is valid if and only if A, B, C, D are concyclic.



PROOF:

PTOLEMY INEQUALITY IN R^N

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Proposition: Kelvin Equaility

The Kelvin Equality relates the norms of two vectors x and y in a specific way:

$$\left\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\| = \frac{\|x-y\|}{\|x\|\,\|y\|}$$

Solution: According to Kelvin Transformation, for vectors x, y

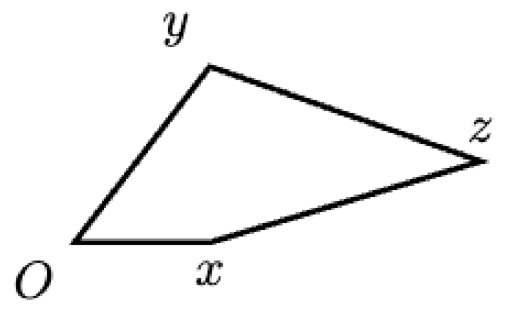
$$\left\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\| = \frac{\|x-y\|}{\|x\|\,\|y\|} \tag{2}$$

By triangle Inequality, we have:

$$\left\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\| \le \left\| \frac{x}{\|x\|^2} - \frac{z}{\|z\|^2} \right\| + \left\| \frac{z}{\|z\|^2} - \frac{y}{\|y\|^2} \right\|$$

Apply the equation 2

$$\|x-y\|\,\|z\| \le \|x-z\|\,\|y\| + \|y-z\|\,\|x\| \tag{3}$$



which is Ptolemy's Inequality in R^n. ∎

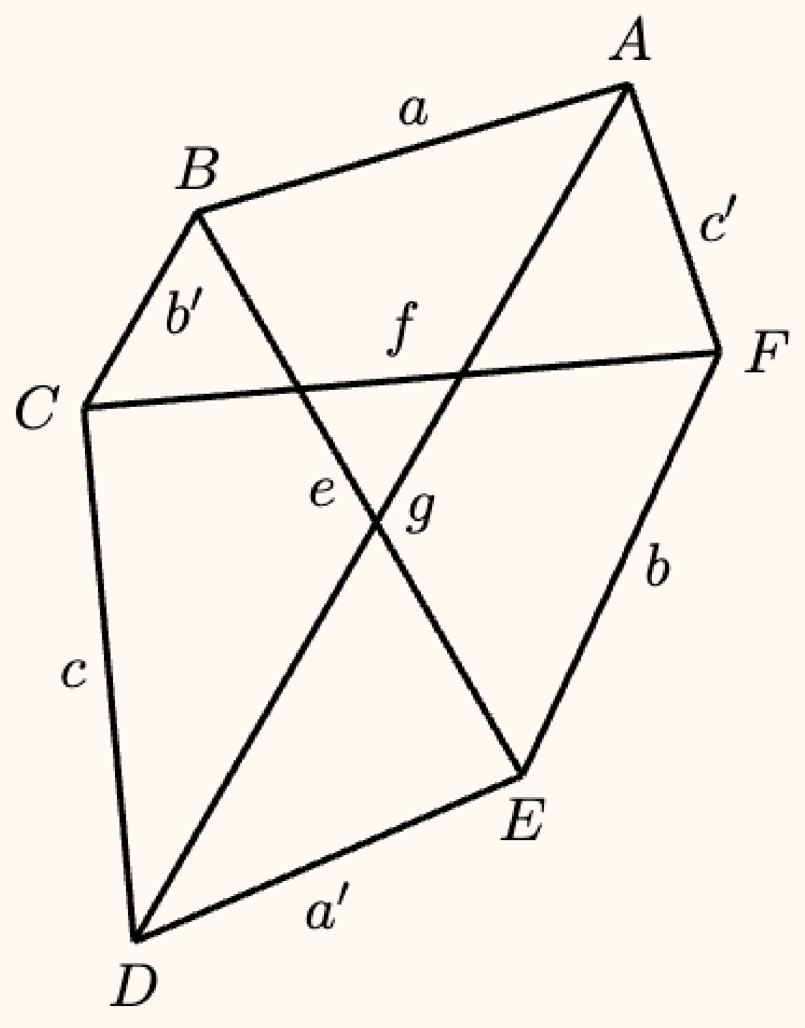
Maybe named: Songhan's Inequality

Theorem 2. Fuhrmann's Inequality

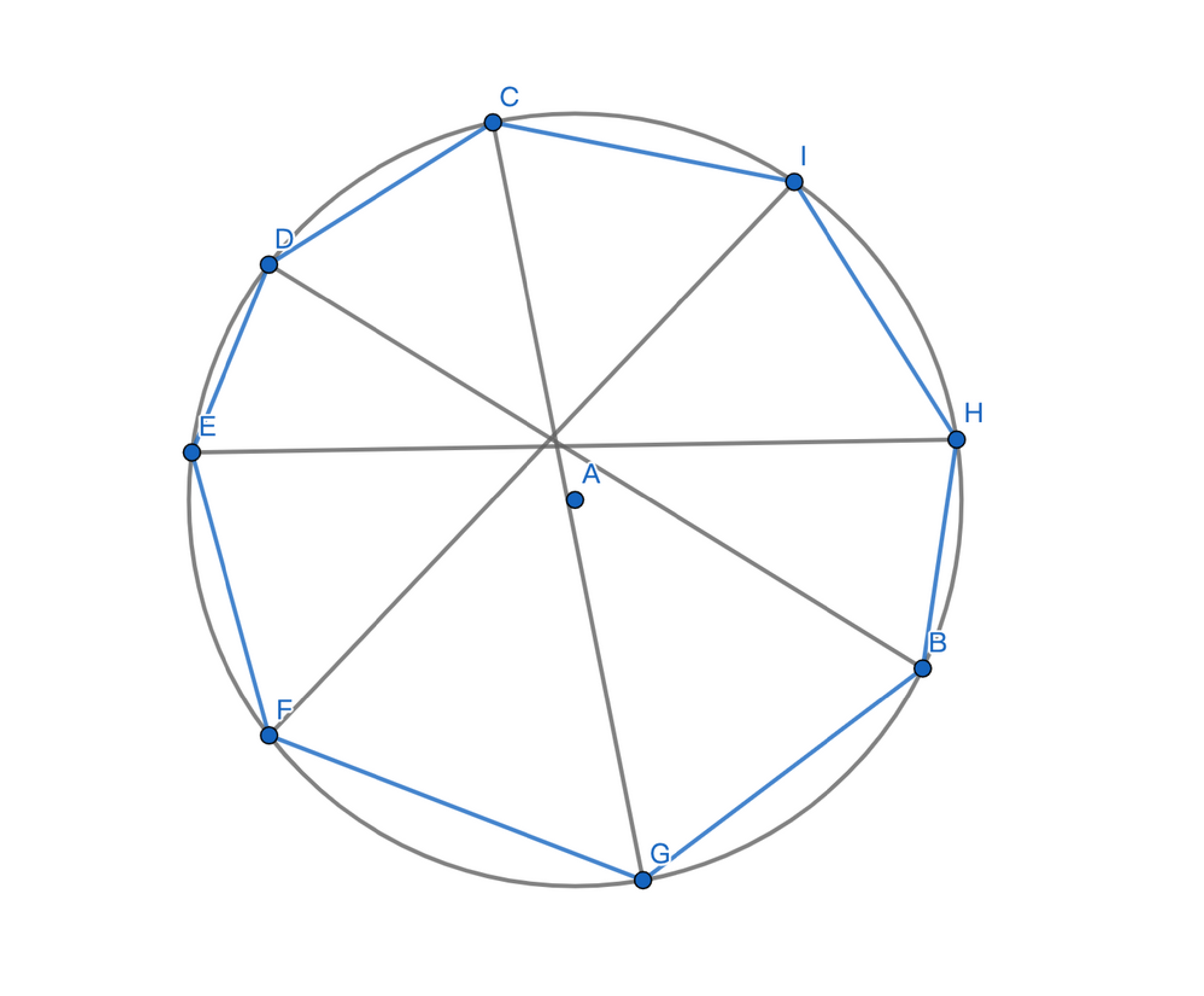
Let $ABCDEF$ be any hexagon on the R^n. Let a, b', c, a', b, c' be the side lengths of AB, BC, CD, DE, EF, FA, respectively. Let e, f, g be the lengths of the main diagonals AD, BE, CF, respectively. Then

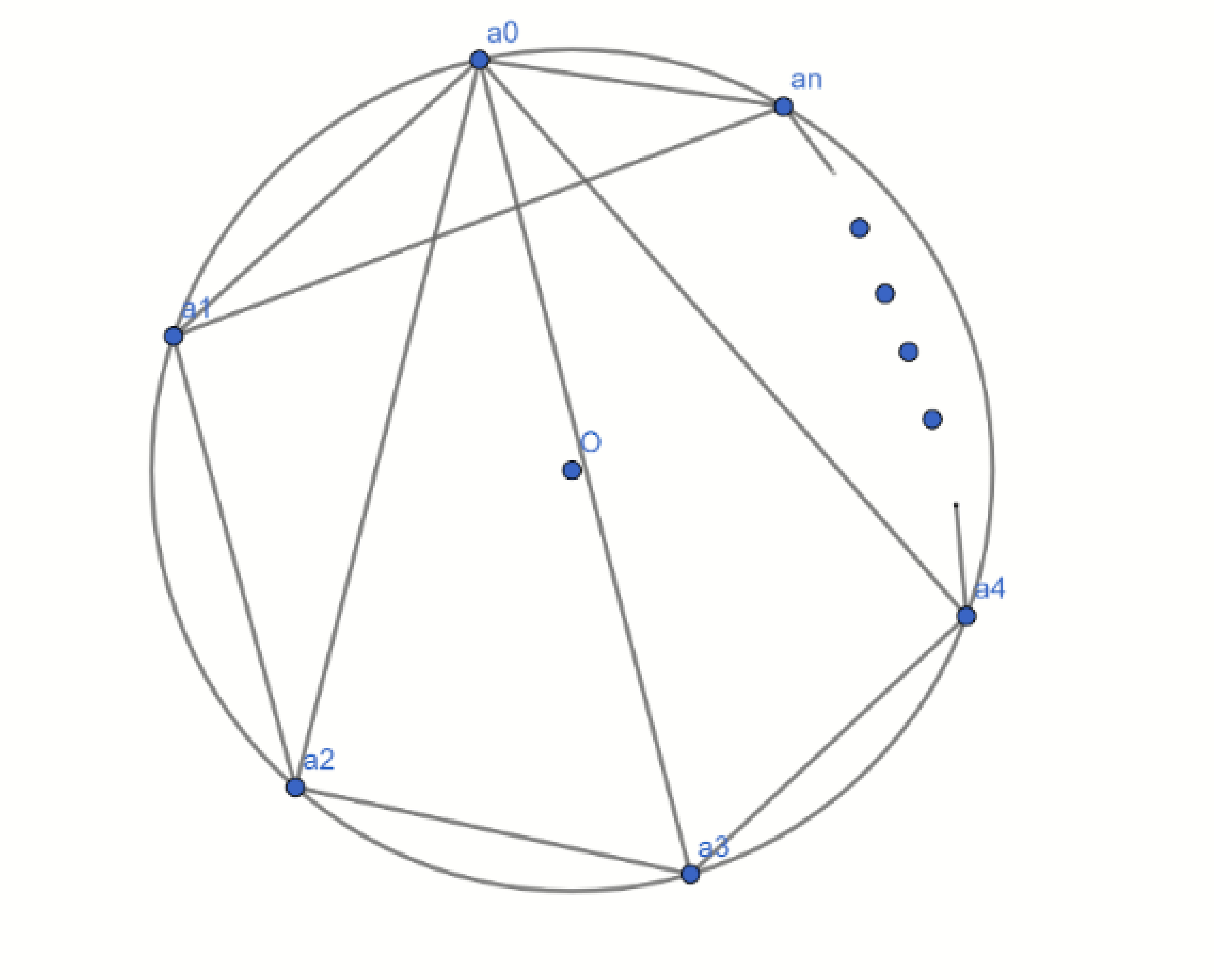
$$efg \leq aa'e + bb'f + cc'g + abc + a'b'c'.$$

And equality will be achieved when the hexagon is cyclic and convex.

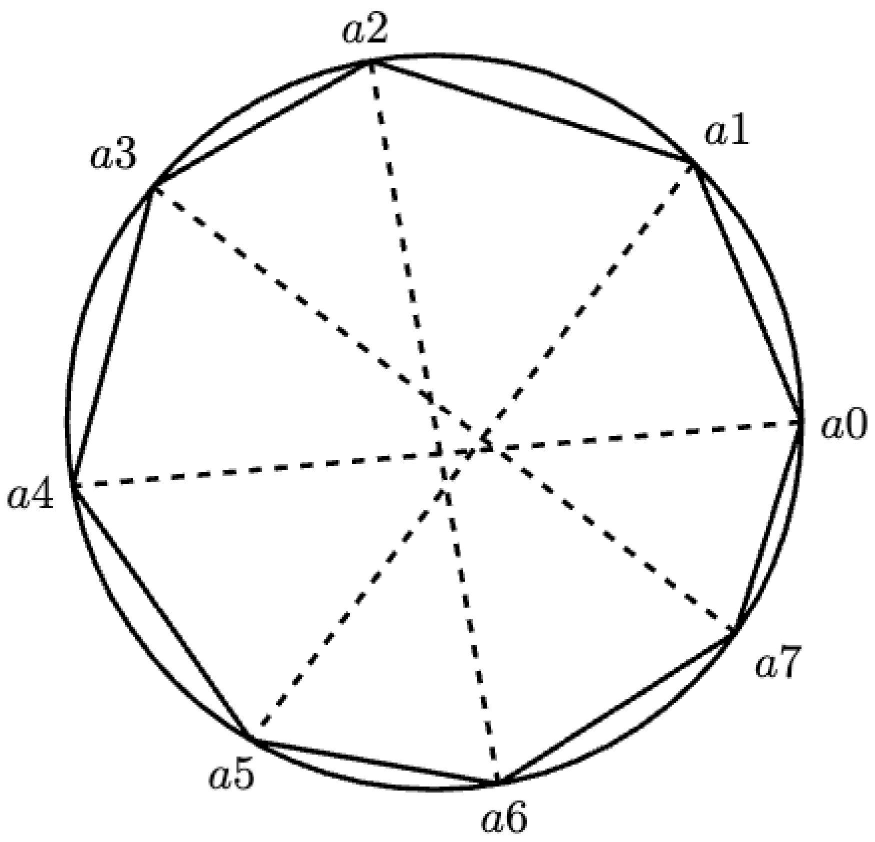


Extension 2: how if it is Octagon/Decagon/ Polygon?





Proof is so interesting!



For the cyclic octagon, we have the following extension of Fuhrmann's Theorem. Suppose that $a_i, i \in \{0, ..., 7\}$ is the vertexes of the octagon, and we suppose $\{i\}$ is a group with $+$. Then,

$$\prod_{i=0}^{3} a_i a_{i+4} = \sum_{i=0}^{7} a_i a_{i+1} \cdot a_{i-1} a_{i+2} \cdot a_{i+3} a_{i+4} \cdot a_{i-2} * a_{i-3} + \sum_{i=0}^{3} a_i a_{i+1} \cdot a_{i+2} a_{i-2} \cdot a_{i+3} a_{i-1} \cdot a_{i+4} a_{i-3} \tag{5}$$

Thanks