Steiner-Lehmus Theorem

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History

- The theorem was first mentioned in 1840 in a letter by C. L. Lehmus to C. Sturm, in which he asked for a purely geometric proof. Sturm passed the request on to other mathematicians and Steiner was among the first to provide a solution, so that’s why the theorem is called Steiner-Lehmus Theorem.

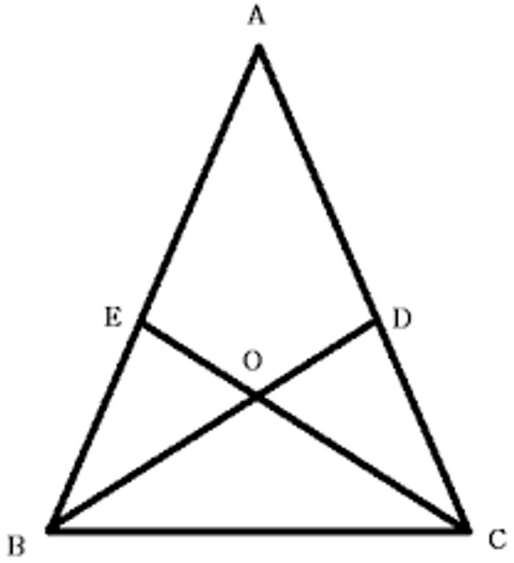
What is Steiner-Lehmus Theorem?

- Every triangle with two angle bisectors of equal lengths is isosceles.

In other words,

If ∠ABD= ∠DBC, ∠ACE= ∠ECB and BD=CE, then

∠B= ∠C ⇒ △ABC is isosceles.

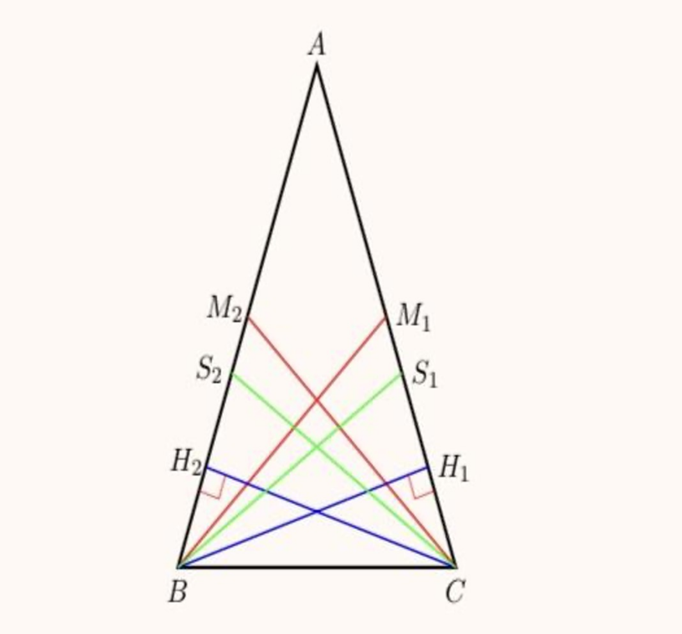


Theorem 1

Let △ABC be an isosceles triangle with AB = AC.

Let BH1, CH2 be heights; BM1, CM2 be medians, and BS1, CS2 be angle bisectors on sides AC, AB, respectively.

Then BH1 = CH2, BM1 = CM2, BS1 = CS2.



Proof:

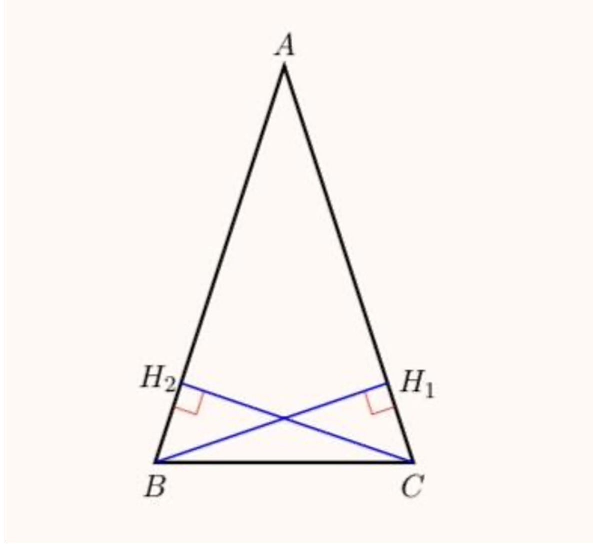
— Since AB = AC, we have ∠B = ∠C.

— Since ∠CH1B = ∠BH2C = 90◦ , ∠C = ∠B, and BC is the common side, we have △CH1B ~= △BH2C. Therefore BH1 = CH2.

— Since CM1 = BM2 = 1/2AB, ∠C = ∠B, and BC is the common side, we have △CM1B ~= △BM2C. Therefore BM1 = CM2.

— Since ∠S1BC = ∠S2CB = 1/2∠B, ∠C = ∠B, and BC is the common side, we have △CS2B ~= △BS1C. Therefore BS1 = CS2

Theorem 2



In triangle $\triangle ABC$, let BH1 and CH2 be heights. Assume that BH1 = CH2. Then $\triangle ABC$ is an isosceles triangle.

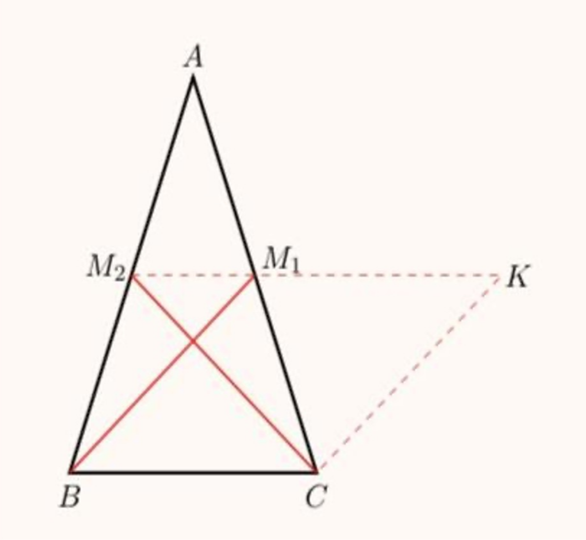
Proof: Since BH1 = CH2, $\angle BH1C = \angle CH2B = 90^\circ$, and BC is the common side. Then $\triangle BH1C \cong \triangle CH2B$. Thus $\angle C = \angle B$ and hence $\triangle ABC$ is isosceles.

Theorem 3

In triangle △ABC, let BM1 and CM2 be medians. Assume that BM1 = CM2. Then △ABC is an isosceles triangle.

Proof: We draw M2M1 and extend it to K such that M1K = BC. Then since M2M1 is the midline, we have M2M1 ∥ BC, and M1K = BC, then M1BCK is a parallelogram. Thus CM2 = BM1 = CK and hence △CKM2 is an isosceles triangle.

As a result, we have ∠M1BC = ∠K = ∠KM2C = ∠M2CB. Thus △BM1C ~= △CM2B since BM1 = CM2 and BC is a common side. Therefore ∠C = ∠B and △ABC is an isosceles triangle.



Thank you!