# Topic 3
# Five Centers

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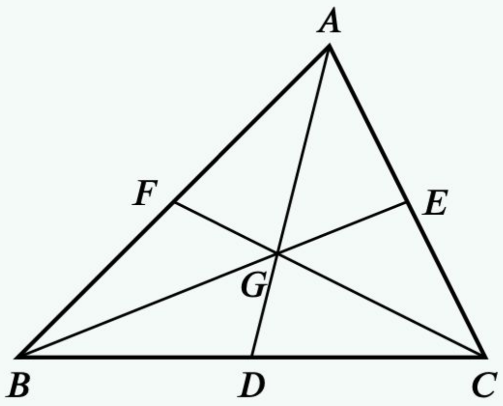
# Introduction of the Centers & Proofs of Concurrency

- Centroid
- Incenter
- Excenter
- Circumcenter
- Orthocenter

# Centroid

**Definition 1. (Centroid)**

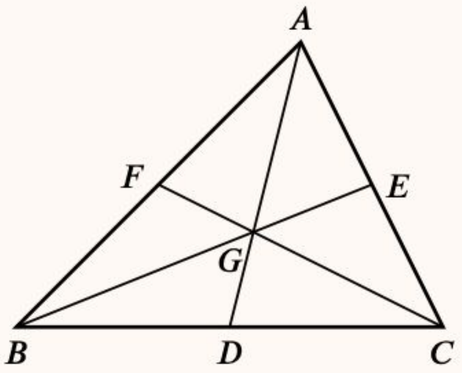
*In triangle $\triangle ABC$, let $AD, BE, CF$ be the medians on the respective sides. Then $AD, BE, CF$ are concurrent to the point $G$, which is called the centroid, or center of gravity, of the triangle.*



# Centroid

**Theorem 1. (Centroid)**

*In triangle $\triangle ABC$, let $AD, BE, CF$ be the medians on the respective sides. Then they are concurrent.*

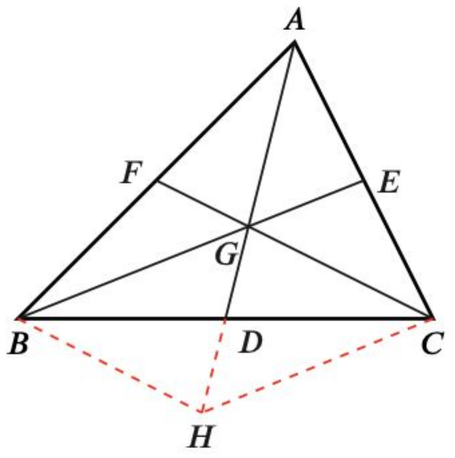


# Centroid

**Proof:** Assume that $E$ is the midpoint of $CA$; $F$ is the midpoint of $AB$; and $BE$ and $CF$ intersect at point $G$. Extend $AG$ to intersect $BC$ at $D$. We then need to prove that $D$ is the midpoint of $BC$.

In the following picture, we extend line segment $AD$ to $H$ so that $AG = GH$, and then connect $BH$ and $CH$.

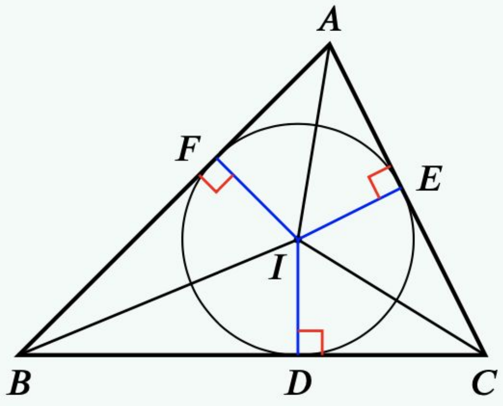
Since $AG = GH$ and $AE = EC$, so that $EG$ is the mid-segment of $\triangle ACH$. In particular, $EG \parallel CH$. Similarly, $FG \parallel BH$. Therefore $BGCH$ is a parallelogram. Since $GH$ and $BC$ are the diagonals of $\square BGCH$, we have $BD = DC$, hence $D$ is the midpoint of $BC$.



# Incenter

**Definition 2. (Incenter)**

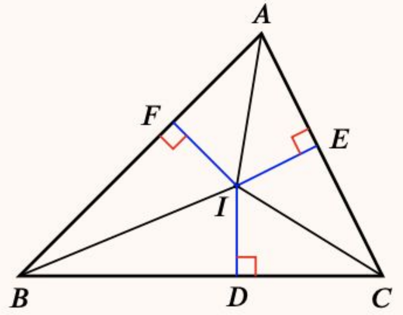
*In triangle $\triangle ABC$, the center $I$ of the inscribed circleis called incenter. Let $AI$, $BI$, and $CI$ be the angle bisectors of the corresponding angles $\angle A, \angle B$, and $\angle C$. Then these angle bisectors are concurrent at $I$.*



# Incenter

**Theorem 2. (Incenter)**

*In triangle $\triangle ABC$, assume that $AI$ is the angel bisector of $\angle A$; $BI$ is the angle bisector of $\angle B$, and $CI$ is the angel bisector of $\angle C$. Then $AI, BI, CI$ are concurrent.*




**Proof:** Assume that the angle bisectors $AI$ and $BI$ of $\angle A$ and $\angle B$ intersect at the point I, we need to prove that $CI$ must be the angle bisector of $\angle C$.

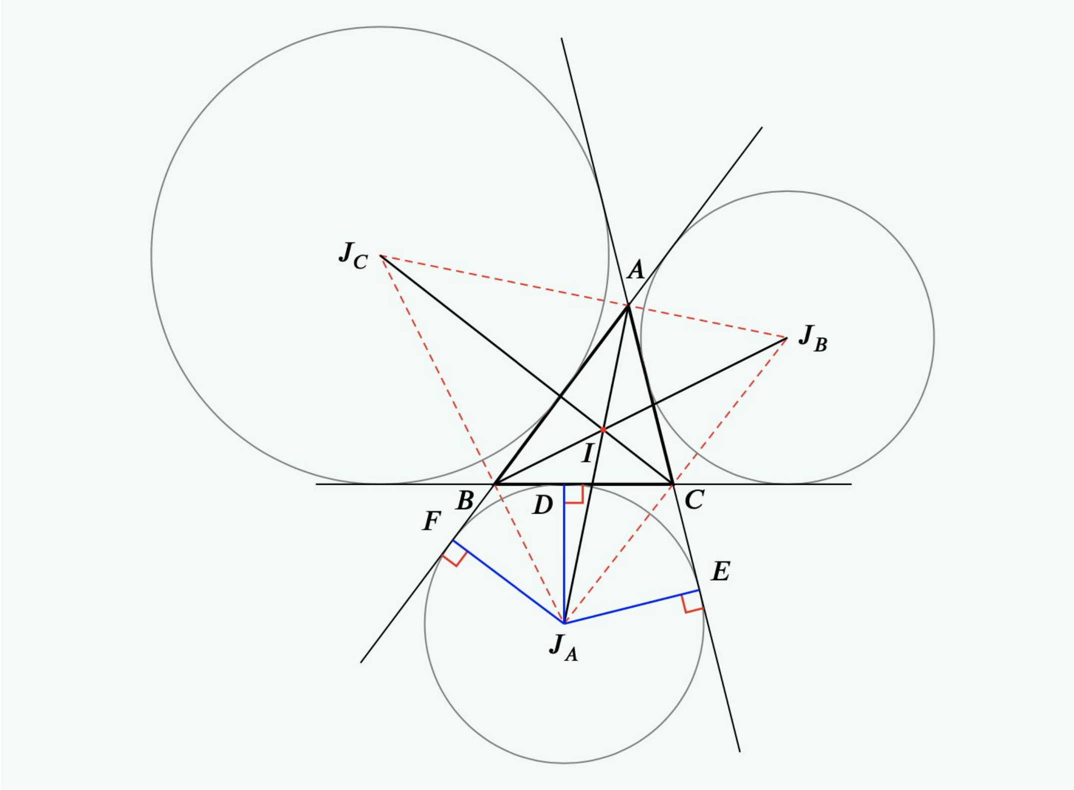
Taking the perpendicular lines $IE \perp CA$, $IF \perp AB$, and $ID \perp BC$. Since $AI$ is the angle bisector of $\angle A$, we must have $IE = IF$. Similarly, we have $IF = ID$. Thus $ID = IE$. From this, we conclude that $CI$ is the angle bisector of $\angle C$.

# Excenter

**Definition 3. (Excenter)**

*In triangle $\triangle ABC$, the center of an escribed circle (=excircle) is called excenter. There are three excenters $J_A, J_B, J_C$ of $\triangle ABC$, corresponding to the vertexes $A, B, C$. In the following picture, the circle $J_A$ is tangent to the line segment $BC$ and the extended line segments $AB, CA$; circle $J_B$ is tangent to $CA$ and the extended line segments of $AB, BC$; and circle $J_C$ are tangent to line segment $AB$ and the extended line segements $CA$ and $BC$. Moreover, $AJ_A$ is the angle bisector of $\angle A$, and both $BJ_A$, $CJ_A$ are the corresponding external angle bisectors (of the angles $\angle CBF$ and $\angle BCE$).*

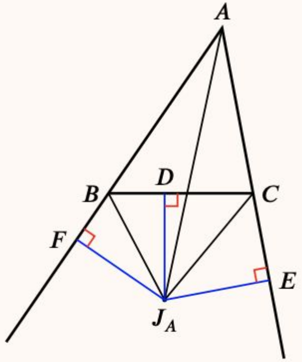
# Excenter



# Excenter

**Theorem 3. (Excenter)**

*In $\triangle ABC$, assume that $AJ_A$ is the angle bisector of $\angle A$; and assume that $BJ_A, CJ_A$ are the angle bisectors of the exterior angles $\angle FBC$, $\angle ECB$, respectively. Then these three angle bisectors are concurrent.*




**Proof:** The proof is similar to that of Theorem 2.

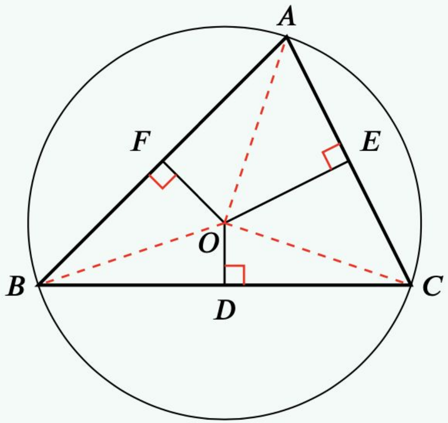
We assume that angle bisectors of $\angle FBC$ and $\angle ECD$ intersect at $J_A$. We need to prove that $AJ_A$ is the angle bisector of $\angle A$.

Let $D, E, F$ be the projections of $J_A$ to $BC, CA, AB$, respectively. By the assumption that $BJ_A$ is the angle bisector of $\angle FBC$, we know that $FJ_A = DJ_A$. Similarly, we have $EJ_A = DJ_A$. Thus $EJ_A = FJ_A$, and therefore, $AJ_A$ is the angle bisector of $\angle A$.

# Circumcenter

**Definition 4. (Circumcenter)**

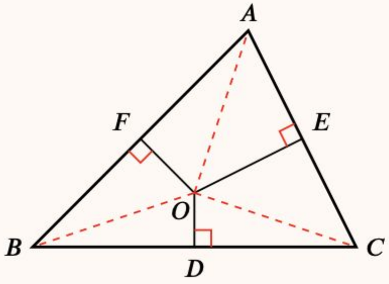
*In $\triangle ABC$, the center $O$ of the circumscribed circle is called circumcenter. Let $OD, OE, OF$ be the perpendicular bisectors of three sides $BC, CA, AB$, respectively. Then these lines are concurrent at $O$.*



# Circumcenter

**Theorem 4. (Circumcenter)**

*In triangle $\triangle ABC$, assume that $OD$ is the perpendicular bisector of $BC$; $OE$ is the perpendicular bisector of $CA$; and $OF$ is the perpendicular bisector of $AB$. Then $OD, OE, OF$ are concurrent.*

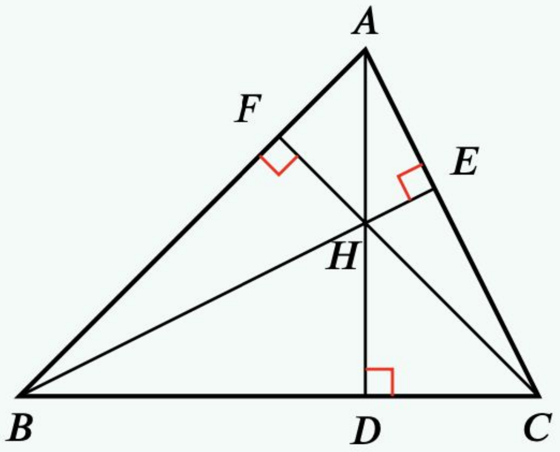



**Proof:** Since $OD$ is the perpendicular bisector of $BC$, then $\triangle OBC$ is an isosceles triangle; so $OB = OC$. Similarly, $\triangle OAC$ is an isosceles triangle; then $OA = OC$. These imply that $OB = OA$ and hence $\triangle OAB$ is an isosceles triangle. Since $OF \perp AB$, it must be the perpendicular bisector of $AB$.

# Orthocenter

**Definition 5. (Orthocenter)**

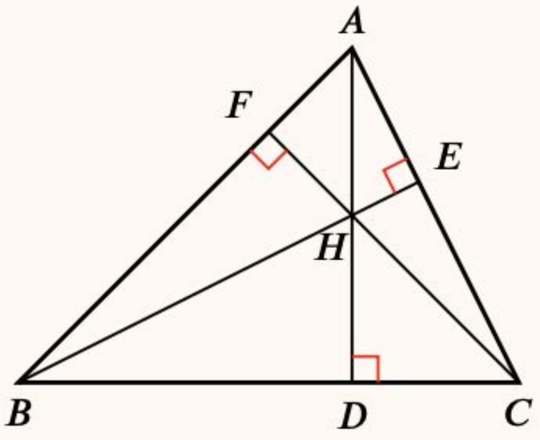
*In $\triangle ABC$, let $AD, BE, CF$ be the heights on $BC, CA, AB$, respectively. Then they are concurrent at a point $H$, which is called the orthocenter of a triangle.*



# Orthocenter

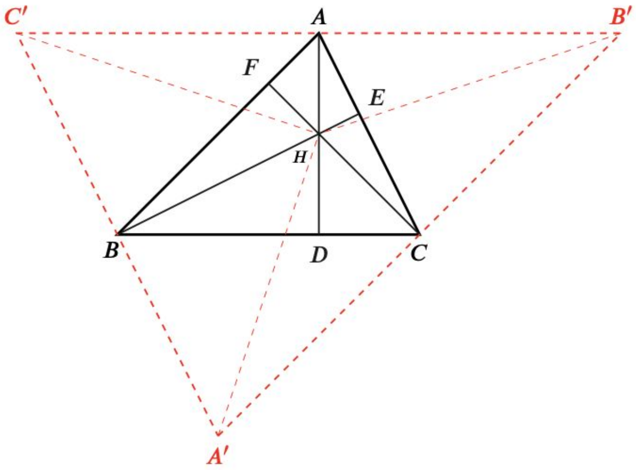
**Theorem 5. (Orthocenter)**

*The three heights of a triangle are concurrent. The point is called the orthocenter of a triangle.*



# Using Circumcenter to Proof

**Proof Using Circumcenter:** We draw lines $A'B', B'C', C'A'$ to be parallel to $AB, BC, CA$, respectively. We shall prove that $H$ is the circumcenter of $\triangle A'B'C'$.




Since $BC \parallel C'B'$, and $AB \parallel B'C$, $ABCB'$ is a parallelogram. Similarly, $ACBC'$ is also a parallelogram. Thus $C'A = BC = AB'$, and $HA$ is the perpendicular bisector of $B'C'$. Similarly, $HC$ is the perpendicular bisector of $A'B'$, and $HB$ is the perpendicular bisector of $C'A'$. By Theorem 4, $HA, HB, HC$ are concurrent at $H$. This proves that the three heights are concurrent.

# Thank you for Watching!