Symmedian Point

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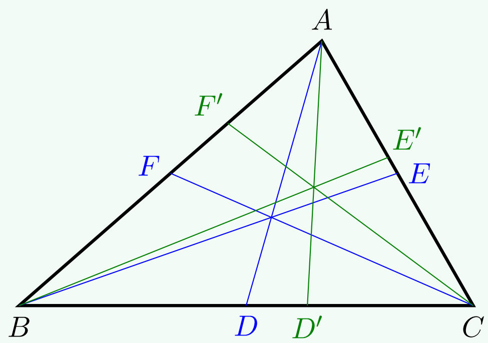
1 Introduction

Symmedian Point is the intersection of three *Symmedians* of a triangle. *Symmedians* are lines that are isogonal to a triangle's medians. *Émile Lemoine*, a French mathematician, proved the existence of symmedian point in 1873. Therefore, symmedian point is also called *Lemoine Point* (in England and France). It is also known as *Grebe Point* (in Germany).

We begin with the definitions of symmedians and symmedian point.

Definition 1. (Symmedians)

Symmedians are lines that are isogonal (Topic 7) to the medians. In the following $\triangle ABC$, let AD, BE, CF be medians on their respective sides. The lines AD', BE', CF' are isogonal to AD, BE, CF, respectively, which means that $\angle D'AC = \angle BAD, \angle E'BA = \angle CBE, \angle F'CA = \angle FCB$. These lines are called $\triangle ABC$'s Symmedians.



Definition 2. (Symmedian Point)

Symmedian Point[a] is the intersection of the three Symmedians. It is the isogonal conjugate point (see Topic 7) of centroid. In the following picture, AD, BE, CF are the medians, and AD', BE', CF' are the symmedians. Their corresponding intersections are centroid and symmedian point.

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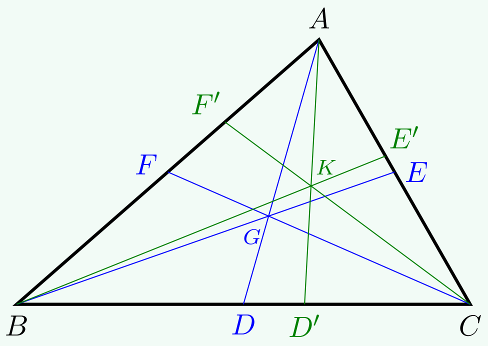


[a]X(6): Symmedian Point is the 6th point in the Encyclopedia of Triangle Centers.

Theorem 1

Three symmedians of a triangle are concurrent.

Proof: By Ceva's theorem, we know that the lines AD', BE', CF' are concurrent if and only if

$$\frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = 1.$$

By definition, symmedians are lines that are isogonal to the corresponding medians of a triangle. Thus by Theorem 2 in Topic 7,

$$\frac{BD}{DC} \cdot \frac{BD'}{D'C} = \left(\frac{AB}{CA}\right)^2$$

which implies

$$\frac{BD'}{D'C} = \left(\frac{AB}{CA}\right)^2$$

Similarly, we have

$$\frac{CE'}{E'A} = \left(\frac{BC}{AB}\right)^2, \qquad \frac{AF'}{F'B} = \left(\frac{CA}{BC}\right)^2.$$

Thus we have

$$\frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = \left(\frac{AB}{CA}\right)^2 \cdot \left(\frac{BC}{AB}\right)^2 \cdot \left(\frac{CA}{BC}\right)^2 = 1.$$

Hence, AD', BE', CF' are concurrent at Symmedian Point K. ■

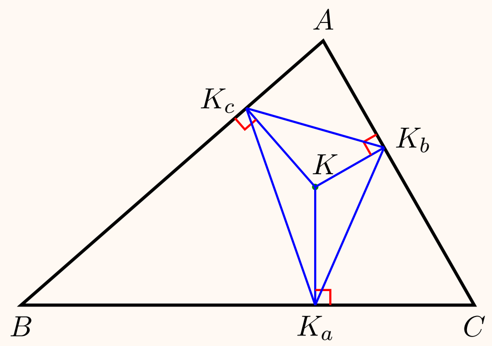
2 Properties of Symmedian Point

We start by computing the trilinear coordinates (Topic 37) of symmedian point. For $\triangle ABC$, let $BC = a, CA = b$ and $AB = c$. Since the barycentric coordinates of the centroid G of $\triangle ABC$ are $(1, 1, 1)$, the trilinear coordinates of G are $(1/a, 1/b, 1/c)$. By Theorem 5 of Topic 7, we conclude that the trilinear coordinates of K are (a, b, c). Consequently, the barycentric coordinates of K are (a^2, b^2, c^2).

Using the trinlinear coordinates of K, we are able to prove the following two results.

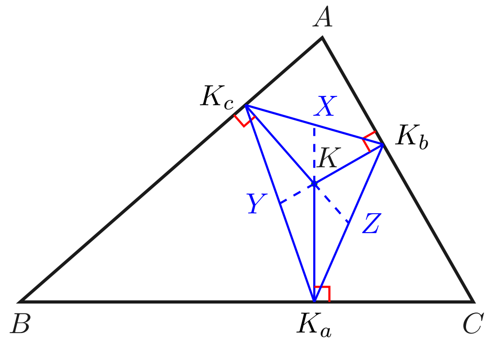
Theorem 2

Let K be the symmedian point of $\triangle ABC$. We draw lines KK_a, KK_b, KK_c perpendicular to BC, CA, AB, respectively. [a] *Then K is the centroid of $\triangle K_aK_bK_c$.*



[a] $\triangle K_aK_bK_c$ is called the pedal triangle of $\triangle ABC$ with respect to K.

Proof: Let K_aK intersect K_cK_b at X; K_bK intersect K_cK_a at Y; and K_cK intersect K_aK_b at Z.



We prove that X is the midpoint of K_bK_c. By law of sines, we have

$$\frac{XK_c}{\sin\angle XKK_c} = \frac{KK_c}{\sin\angle KXK_c}, \quad \frac{XK_b}{\sin\angle XKK_b} = \frac{KK_b}{\sin\angle KXK_b}.$$

Since $\sin\angle KXK_c = \sin\angle KXK_b$, we have

$$\frac{XK_c}{XK_b} = \frac{KK_c}{KK_b} \cdot \frac{\sin(\angle XKK_c)}{\sin(\angle XKK_b)}.$$

Since the trilinear coordinates of K are (a, b, c), we have

$$\frac{KK_c}{KK_b} = \frac{c}{b}.$$

Since KK_a, KK_b, KK_c are perpendicular to BC, CA, AB, respectively, we have

$$\angle XKK_c = \angle B, \quad \angle XKK_b = \angle C.$$

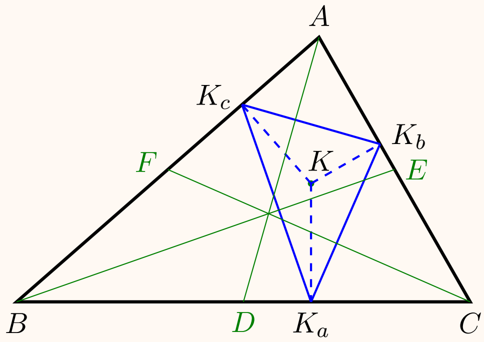
Therefore, using the law of sines again, we have

$$\frac{XK_c}{XK_b} = \frac{c}{b} \cdot \frac{\sin(\angle B)}{\sin(\angle C)} = 1,$$

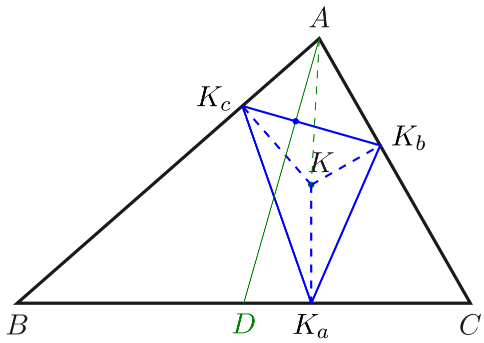
hence X is the midpoint of K_bK_c. Similarly, Y, Z are the midpoints of K_cK_a, K_aK_b, respectively. As a result, K is the centroid of $\triangle K_aK_bK_c$. ∎

Corollary 1

In $\triangle ABC$, let AD, BE, CF be medians over BC, CA and AB, respectively. Then $AD \perp K_bK_c$, $BE \perp K_cK_a$ and $CF \perp K_aK_b$.



Proof: We only need to prove $AD \perp K_bK_c$. The other two relations are identical to prove.



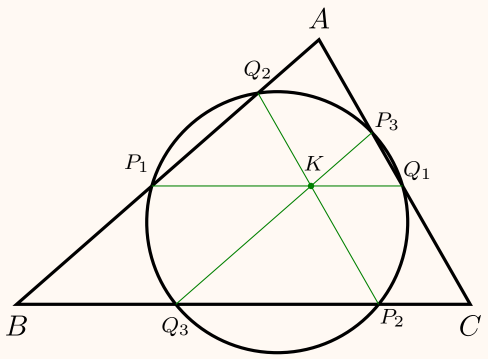
Since $\angle AK_cK = \angle AK_bK = 90°$, four points A, K_c, K, K_b are concyclic. Therefore $\angle AK_cK_b = \angle AKK_b$. On the other hand, since AD and AK are isogonal lines, we have $\angle BAD = \angle CAK$. Thus $\angle AK_cK_b + \angle BAD = \angle AKK_b + \angle CAK = 90°$, and hence $AD \perp K_bK_c$. ■

3 Lemoine Circles

Symmedian point is closely related to the First and Second Lemoine Circles (Topic 17), both of which belong to the family of Tucker Circles (Topic 29).

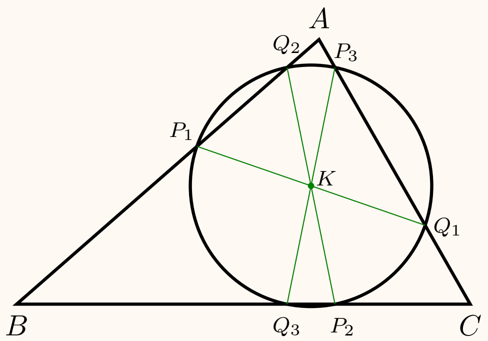
Theorem 3. (First Lemoine Circle)

Let K be the Symmedian Point of $\triangle ABC$. Let $P_1Q_1 \parallel BC$; $P_2Q_2 \parallel CA$; and $P_3Q_3 \parallel AB$. Then six points $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are concyclic. The circle is called the First Lemoine Circle.



Theorem 4. (Second Lemoine Circle)

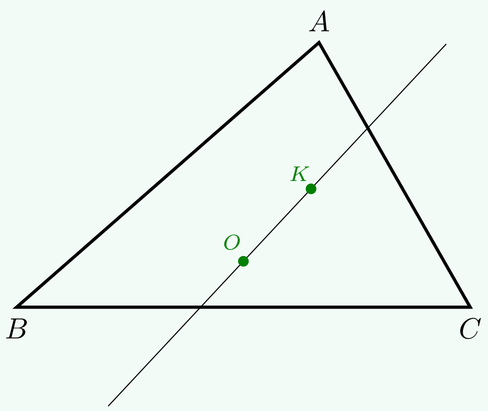
Let K be the Symmedian Point of $\triangle ABC$. Let P_1Q_1 be an antiparallel line of BC; P_2Q_2 be an antiparallel line of CA; and P_3Q_3 be an antiparallel line of AB. Then six points $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are concyclic. The circle is called the Second Lemoine Circle.



4 Brocard Axis

Definition 3. (Brocard Axis)

Brocard Axis is the line passing through a triangle's symmedian point K and circumcenter O[a]. OK is called the Brocard Diameter.



[a]Circumcenter is the center of a triangle's circumcircle. See Topic 3.

Theorem 5

The Symmedian Point, the Circumcenter, the First and the Second Isodynamic Points (Topic 33), and Brocard midpoint (Topic 25) all lie along the Brocard axis[a].

[a]The Brocard Midpoint has the trilinear coordinates

$$(a(b^2+c^2), b(c^2+a^2), c(a^2+b^2)).$$

The *First Isodynamic Point* has the trilinear coordinates

$$(\sin(\alpha+\pi/3), \sin(\beta+\pi/3), \sin(\gamma+\pi/3)),$$

and the *Second Isodynamic Point* has the trilinear coordinates

$$(\sin(\alpha-\pi/3), \sin(\beta-\pi/3), \sin(\gamma-\pi/3)).$$

Here α, β, γ are the three angles of triangle $\triangle ABC$.

Proof: To prove that these five points are collinear, we have to take groups of three at a time and prove them collinear individually.

First, we show that the symmedian point and the isodynamic points are collinear. To do that, we need to show the determinant of the following matrix is zero:

$$\det \begin{bmatrix} \sin(\alpha+\pi/3) & \sin(\beta+\pi/3) & \sin(\gamma+\pi/3) \\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3) \\ \sin\alpha & \sin\beta & \sin\gamma \end{bmatrix}.$$

Using the sum to product identity for sin, we add the second equation to the first to get

$$\det \begin{bmatrix} 2\sin\alpha\cdot\cos\frac{\pi}{3} & 2\sin\beta\cdot\cos\frac{\pi}{3} & 2\sin\gamma\cdot\cos\frac{\pi}{3} \\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3) \\ \sin\alpha & \sin\beta & \sin\gamma \end{bmatrix}.$$

Then, divide the first row by $2\cos\frac{\pi}{3}$ to get

$$\det \begin{bmatrix} \sin\alpha & \sin\beta & \sin\gamma \\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3) \\ \sin\alpha & \sin\beta & \sin\gamma \end{bmatrix}.$$

Since two rows are the same, the determinant of this matrix is 0. As such, the symmedian point is collinear with the first and second isodynamic points.

Next, we show that the two isodynamic points are collinear with the circumcenter.

$$\det\begin{bmatrix}\sin(\alpha+\pi/3) & \sin(\beta+\pi/3) & \sin(\gamma+\pi/3)\\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3)\\ \cos\alpha & \cos\beta & \cos\gamma\end{bmatrix}.$$

Now, we use the sum to product identity and subtract the second row from the first to get

$$\det\begin{bmatrix}2\cos\alpha\cdot\sin\frac{\pi}{3} & 2\cos\beta\cdot\sin\frac{\pi}{3} & 2\cos\gamma\cdot\sin\frac{\pi}{3}\\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3)\\ \cos\alpha & \cos\beta & \cos\gamma\end{bmatrix}.$$

Dividing both sides of the first row by $2\sin\frac{\pi}{3}$ gives

$$\det\begin{bmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3)\\ \cos\alpha & \cos\beta & \cos\gamma\end{bmatrix}.$$

Again, since two rows of the matrix are the same, the determinant is 0, which shows that the two isodynamic points are collinear with the circumcenter.

Since the isodynamic points are collinear with the symmedian point (see Problem 7) and the isodynamic points are also collinear with the circumcenter, these 4 points are all collinear.

Finally, we need to prove that the Brocard Midpoint is collinear with the symmedian point and the circumcenter.

$$\det\begin{bmatrix}\sin\alpha & \sin\beta & \sin\gamma\\ \cos\alpha & \cos\beta & \cos\gamma\\ a(b^2+c^2) & b(c^2+a^2) & c(a^2+b^2)\end{bmatrix}.$$

Applying the law of cosines to the second row, applying the law of sines to the first row and multiplying $2abc$ gives

$$\det\begin{bmatrix}a & b & c\\ a(b^2+c^2-a^2) & b(c^2+a^2-b^2) & c(a^2+b^2-c^2)\\ a(b^2+c^2) & b(c^2+a^2) & c(a^2+b^2)\end{bmatrix}.$$

Now, we multiply the third row by 2 and subtract row 2 from it to get

$$\det\begin{bmatrix}a & b & c\\ a(b^2+c^2-a^2) & b(c^2+a^2-b^2) & c(a^2+b^2-c^2)\\ a(a^2+b^2+c^2) & b(a^2+b^2+c^2) & c(a^2+b^2+c^2)\end{bmatrix}.$$

Divide the third row by $a^2 + b^2 + c^2$ to get

$$\det \begin{bmatrix} a & b & c \\ a(b^2+c^2-a^2) & b(c^2+a^2-b^2) & c(a^2+b^2-c^2) \\ a & b & c \end{bmatrix}.$$

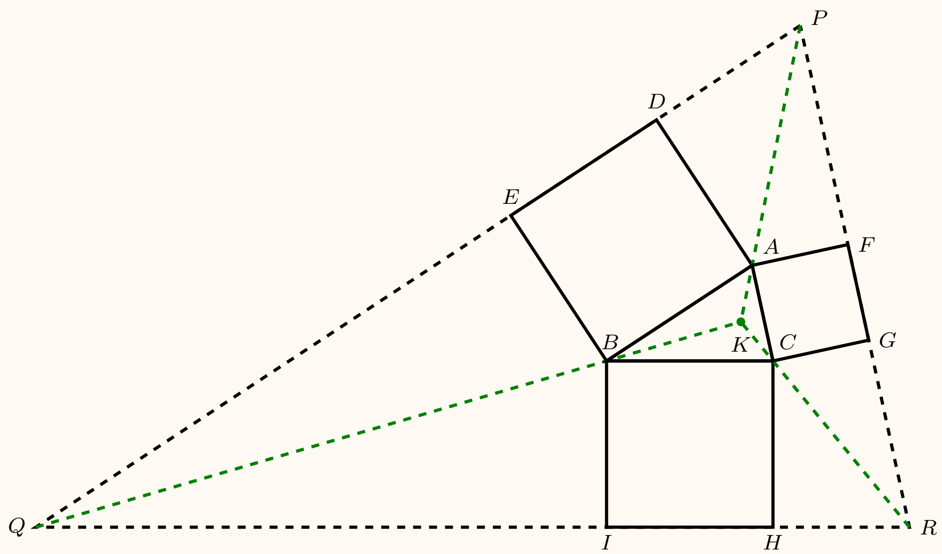
Since the first and the third row are the same, the determinant is 0, which shows that the symmedian point, the circumcenter, and the Brocard midpoint are collinear.

Thus, the symmedian point, the circumcenter, the Brocard midpoint, and the first and second isodynamic points are collinear.

5 Grebe Construction Method

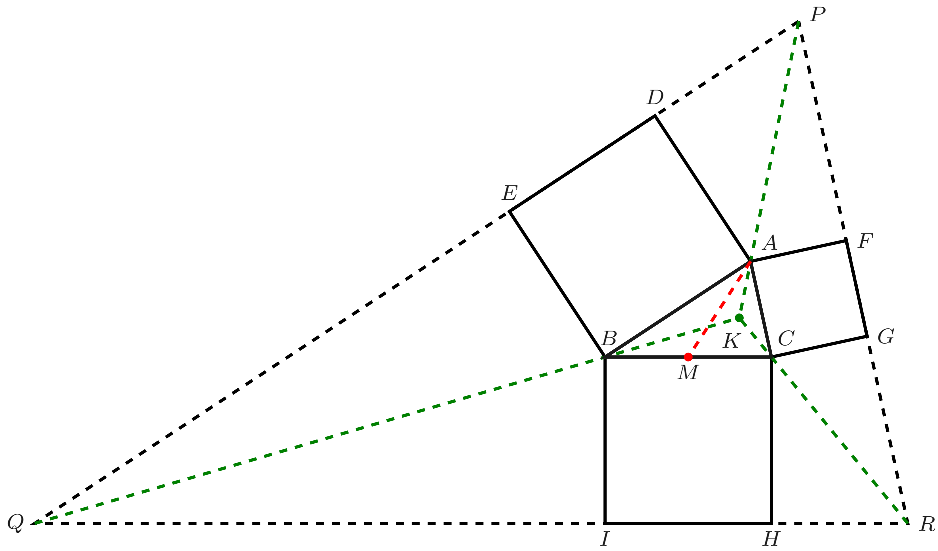
Theorem 6. (Grebe Construction Method)

In the following picture, $\square CBIH$, $\square ACGF$, and $\square ADEB$ are three squares with sides of BC, CA, and AB of $\triangle ABC$ respectively. Extend lines IH, GF, and DE to form $\triangle PQR$. Then lines PA, QB, RC are concurrent at K, the symmedian point of $\triangle ABC$[a].



[a]This method is called the *Grebe Construction Method* of Symmedian Point.

Proof: We only need to prove that line PA passes the Symmedian Point K. In the following picture, let AM be the median over BC.



We let $BC = a, CA = b$ and $AB = c$. Since $AB \parallel PQ$, we have $\sin\angle BAK = \sin DPA = AD/AP = c/AP$. By using the same method, we have $\sin\angle CAK = b/AP$. As a result, we have

$$\frac{\sin\angle BAK}{\sin\angle CAK} = \frac{c}{b}.$$

On the other hand, by law of sines, we have

$$\frac{BM}{\sin\angle MAB} = \frac{AB}{\sin\angle AMB}, \quad \frac{CM}{\sin\angle CAM} = \frac{AC}{\sin\angle AMC}.$$

Since $\sin\angle AMB = \sin\angle AMC$ and $BM = CM$, we have

$$\frac{\sin\angle MAB}{\sin\angle CAM} = \frac{CA}{AB} = \frac{b}{c}.$$

Therefore,

$$\frac{\sin\angle BAK}{\sin\angle CAK} = \frac{\sin\angle CAM}{\sin\angle MAB}.$$

Note that $\angle BAK + \angle CAK = \angle CAM + \angle MAB = \angle BAC$. Let $\angle BAC = \alpha$; $\angle BAK = x$; and $\angle CAM = y$. Then the above equation can be written as

$$\frac{\sin x}{\sin(\alpha - x)} = \frac{\sin y}{\sin(\alpha - y)}.$$

Taking cross product, we obtain

$$\sin x \sin(\alpha - y) = \sin y \sin(\alpha - x) = 0.$$

Expanding $\sin(\alpha - x)$ and $\sin(\alpha - y)$, we obtain

$$\sin x(\sin\alpha\cos y - \cos\alpha\sin y) = \sin y(\sin\alpha\cos x - \cos\alpha\sin x),$$

which, after sinplification, is equivalent to

$$\sin\alpha(\sin x\cos y - \cos x\sin y) = 0.$$

Thus

$$\sin\alpha\sin(x - y) = 0,$$

and hence

$$\angle BAK = x = y = \angle CAM.$$

Therefore PA is the isogonal line to the median AM, and therefore it must pass the Symmedian Point K of $\triangle ABC$. $\blacksquare$