

Quadrilateral Area Formulas

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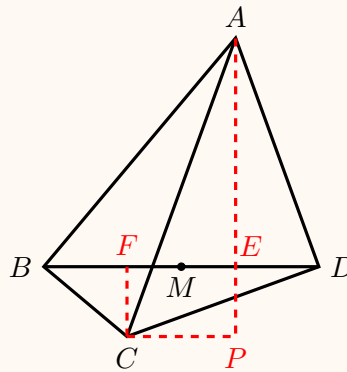
There is no simple area formula for a general quadrilateral. This is partially due to the *instability of quadrilateral*, that is, unlike the case of triangle, four sides do not completely determine a unique quadrilateral.

In the following *Bretschneider Formula*, the area is represented by the four sides and the two diagonals of a quadrilateral.

Theorem 1. (Bretschneider Formula)

In the following convex quadrilateral $ABCD$. Assume that $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, and $BD = f$. Then the area S of $ABCD$ is given by

$$S = \frac{1}{4} \sqrt{4e^2 f^2 - (a^2 - b^2 + c^2 - d^2)^2}$$



Proof. Draw $AP \perp BD$ and $CP \perp AP$. The key computation should be AP , because

$$S = \frac{1}{2} BD \cdot AP. \quad (1)$$

Let M be the midpoint of BD . By the Pythagorean Theorem, we have

$$AB^2 - AD^2 = BE^2 - ED^2 = (BE + ED)(BE - ED).$$

Thus

$$a^2 - d^2 = f(BE - ED) = f(BM + ME - (MD - ME)) = 2f \cdot ME.$$

Similarly, we have

$$c^2 - b^2 = 2f \cdot FM.$$

We therefore have

$$a^2 - b^2 + c^2 - d^2 = 2f \cdot EF.$$

Thus

$$AP = \sqrt{AC^2 - EF^2} = \sqrt{e^2 - \frac{1}{4f^2}(a^2 - b^2 + c^2 - d^2)^2}.$$

The Bretschneider Formula follows from the above equation and (1). ■

An algebraic variant of the Bretschneider Formula is the following

Corollary 0.1. (Coolidge Formula)

Let $p = (a + b + c + d)/2$ be the semi-perimeter of the quadrilateral. Then the area S is given by

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d) - \frac{1}{4}(ac+bd-ef)(ac+bd-ef)}.$$

Proof. We have the following algebraic identity

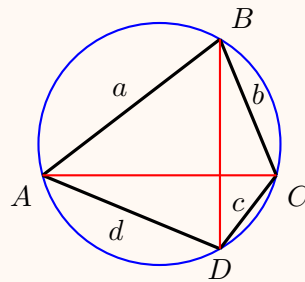
$$\begin{aligned} & 4(ac+bd)^2 - (a^2 - b^2 + c^2 - d^2)^2 \\ &= (2ac + 2bd + a^2 - b^2 + c^2 - d^2)(2ac + 2bd - a^2 + b^2 - c^2 + d^2) \\ &= ((a+c)^2 - (b-d)^2)((b+d)^2 - (a-c)^2) \\ &= (a+c+b-d)(a+c-b+d)(b+d+a-c)(b+d-a+c) \\ &= 16(p-a)(p-b)(p-c)(p-d). \end{aligned}$$

The corollary then follows from the Bretschneider Formula. ■

A quadrilateral is called *cyclic*, if the four vertices all lie on a single circle.

Corollary 0.2. (Brahmagupta formula)

Assume that $ABCD$ is a convex cyclic quadrilateral with sides a, b, c, d , respectively.



Then the area of $ABCD$ is given by

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)},$$

where $p = (a + b + c + d)/2$ is the semi-perimeter.

Proof. Using the Ptolemy's Theorem (see [Wikipedia](#) or [Topic 10](#)), we have

$$ef = ad + bd.$$

Therefore the formula follows from the Coolidge formula (Corollary 0.1). ■

Corollary 0.3. (Heron Formula)

Assume that the three sides of a triangle are a, b, c , respectively. Then the area of the triangle is given by

$$S = \sqrt{p(p-a)(p-b)(p-c)},$$

where p is the semi-perimeter.

Proof. We take $d = 0$ in the above corollary. ■

A polygon is called *tangential*, or *circumscribed*, if it is a convex polygon that contains an inscribed circle.

For the rest of this article, we introduce area formulas for tangential quadrilateral and general tangential polygon.

Theorem 2

Let $P = A_1 \cdots A_n$ be a tangential n -polygon with r being the radius of the inscribed circle, where A_i are the vertexes for $1 \leq i \leq n$. Let $a_i = A_i A_{i+1}$ for $1 \leq i \leq n-1$ and $a_n = A_n A_1$ be the side lengths of the polygon. Then the area of the polygon is given by

$$S = \frac{1}{2} \left(\sum_{i=1}^n a_i \right) r.$$

The proof is obvious and left to the reader.

Before applying the above theorem into quadrilateral, we first study the following example.

Problem

A circle is inscribed in quadrilateral $ABCD$, tangent to \overline{AB} at P and to \overline{CD} at Q . Given that $AP = 19$, $PB = 26$, $CQ = 37$, and $QD = 23$, find the square of the radius of the circle.

This is the **Problem No. 10** in the 2000 AIME (II).

Solution. Call the center of the circle O . By drawing the lines from O tangent to the sides and from O to the vertices of the quadrilateral, four pairs of congruent right triangles are formed. Thus

$$\angle AOP + \angle POB + \angle COQ + \angle QOD = 180,$$

or

$$\left(\arctan\left(\frac{19}{r}\right) + \arctan\left(\frac{26}{r}\right) \right) + \left(\arctan\left(\frac{37}{r}\right) + \arctan\left(\frac{23}{r}\right) \right) = 180^\circ.$$

Take the tan of both sides and use the identity for $\tan(A + B)$ to get

$$\tan(\arctan(\frac{19}{r}) + \arctan(\frac{26}{r})) + \tan(\arctan(\frac{37}{r}) + \arctan(\frac{23}{r})) = 0.$$

Use the identity for $\tan(A + B)$ again to get

$$\frac{\frac{45}{r}}{1 - 19 \cdot \frac{26}{r^2}} + \frac{\frac{60}{r}}{1 - 37 \cdot \frac{23}{r^2}} = 0.$$

Solving the above equation, we get $r^2 = 647$. ■

In the above solution, we used the formula

$$\alpha + \beta = \arctan\left(\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}\right).$$

It turns out that we have the following generalization of the above formula. Let x_1, \dots, x_n be real numbers. As is well-known,

$$\tan(\arctan x_1 + \arctan x_2) = \frac{x_1 + x_2}{1 - x_1 x_2}.$$

Thus we have

$$\tan(\arctan x_1 + \arctan x_2 + \arctan x_3) = \frac{\tan(\arctan x_1 + \arctan x_2) + x_3}{1 - \tan(\arctan x_1 + \arctan x_2)x_3}.$$

Using the formula again, we get

$$\begin{aligned} \tan(\arctan x_1 + \arctan x_2 + \arctan x_3) &= \frac{\frac{x_1 + x_2}{1 - x_1 x_2} + x_3}{1 - \frac{x_1 + x_2}{1 - x_1 x_2} x_3} \\ &= \frac{x_1 + x_2 + x_3 - x_1 x_2 x_3}{1 - (x_1 x_2 + x_2 x_3 + x_3 x_1)}. \end{aligned}$$

In general, we define the *elementary symmetric polynomials* by

$$\begin{aligned} \sigma_0 &= 1 \\ \sigma_1 &= x_1 + \dots + x_n = \sum_i x_i \\ \sigma_2 &= \sum_{i < j} x_i x_j \\ &\dots\dots\dots \\ \sigma_k &= \sum_{j_1 < j_2 < \dots < j_k} x_{j_1} \dots x_{j_k} \\ &\dots\dots\dots \\ \sigma_n &= x_1 \dots x_n \end{aligned}$$

and $\sigma_r = 0$ if $r > n$. Then we have

Theorem 3

Using the above notations, we have

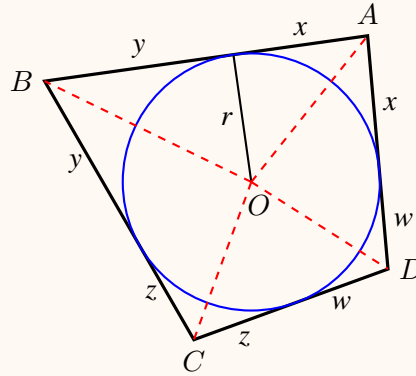
$$\tan\left(\sum_i \arctan x_i\right) = \frac{\sigma_1 - \sigma_3 + \sigma_5 - \dots}{1 - \sigma_2 + \sigma_4 - \dots}.$$

The theorem can be proved using the mathematical induction, we omit the proof here.

Theorem 4

In the following quadrilateral $ABCD$ inscribed by the circle of radius r , we assume that the lengths of the tangents are x, y, z, w , respectively as shown. Then the area S is given by

$$S = \sqrt{(x + y + z + w)(xyz + yzw + zwx + wxy)}.$$



Proof. We connect OA, OB, OC, OD . Then

$$S = S_{\triangle OAB} + S_{\triangle OBC} + S_{\triangle OCD} + S_{\triangle ODA} = (x + y + z + w)r.$$

In order to express r in terms of a, b, c, d , we observe that

$$\angle OAB + \angle OBC + \angle OCD + \angle ODA = 360^\circ.$$

Thus we have

$$\arctan \frac{x}{r} + \arctan \frac{y}{r} + \arctan \frac{z}{r} + \arctan \frac{w}{r} = 180^\circ.$$

By Theorem 3, we have

$$0 = \sigma_1 - \sigma_3,$$

where $\sigma_1 = (x + y + z + w)/r$, and $\sigma_3 = (xyz + yzw + zwx + wxy)/r^3$. Thus we solve

$$r = \sqrt{\frac{xyz + yzw + zwx + wxy}{x + y + z + w}}.$$

The formula then follows from Theorem 2. ■

A quadrilateral is called *bicentric*, if it is both tangential and cyclic.

Corollary 0.4

Let a, b, c, d be the side lengths of a bicentric quadrilateral. Then its area is given by

$$S = \sqrt{abcd}.$$

Proof. We use the picture in Theorem 4, where the lengths of tangent lines from the vertexes are x, y, z, w . Then the semi-perimeter p is given by

$$p = x + y + z + w.$$

By definition, $a = x + y$, $b = y + z$, $c = z + w$, and $d = w + x$. Then we have $p - a = z + w = c$ and similarly, $p - b = d$, $p - c = a$, and $p - d = b$. Thus the corollary follows from the Brahmagupta formula (Corollary 0.2). ■