Harmonic Quadrilateral

Ruoying Deng[1], ruoyind1@uci.edu

(last updated: June 15, 2022)

1 Introduction

In this article, we introduce a special kind of quadrilateral called *harmonic quadrilateral* and prove some of its basic properties. Harmonic quadrilateral has many applications in Projective Geometry and Conic curves. But we only confine ourselves to the plane geometry properties of harmonic quadrilateral.

Definition 1

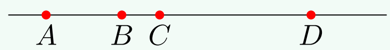
Harmonic quadrilateral is a concyclic (convex) quadrilateral whose products of the length of the opposite side are equal.

2 Property of Harmonic Quadrilateral

The terminology harmonic quadrilateral is closely related to the *harmonic division* of a line.

Definition 2. (Harmonic Division)

Let L be a line. Let A, B, C, D be four points on L. We assume that B is inside the segment line AC, and D is to the right of C. If $AB \cdot CD = AD \cdot BC$, then these four points are called a harmonic division of L.

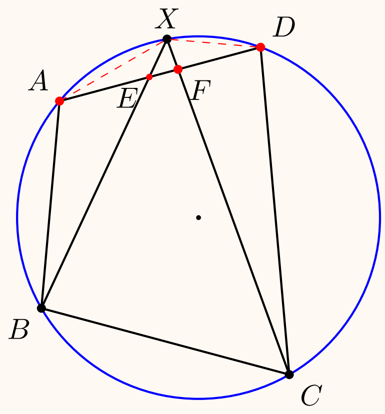


The following is an essential property of harmonic quadrilateral.

Theorem 1

Let $ABCD$ be a concyclic quadrilateral. Let X be a point on the arc $\overset{\frown}{AD}$. Assume that XB intersects AD at E; and XC intersects AD at F. Then $ABCD$ is a harmonic quadrilateral if and only if A, E, F, D forms a harmonic division of the line AD.

[1]The author thanks Dr. Zhiqin Lu for his help.

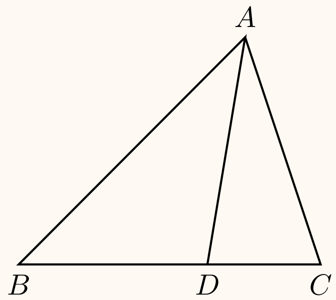


Before proving the theorem, we need the following lemma which, in wikipedia, is called generalized angle bisector theorem:

Lemma 1. (Generalized Angle Bisector Theorem)

Let D be a point on the side BC of $\triangle ABC$. Then

$$\frac{BD}{DC} = \frac{AB \sin \angle BAD}{AC \sin \angle DAC}.$$



Proof. This follows from applying law of sines on both $\triangle ABD$ and $\triangle ADC$. Let $\angle ADB = x$. Then we have

$$\frac{BD}{\sin \angle BAD} = \frac{AB}{\sin x}, \quad \frac{DC}{\sin \angle DAC} = \frac{AC}{\sin(180° - x)}.$$

The above equations imply the lemma. ∎

Remark If AD is the angle bisector of $\angle A$, then the lemma is reduced to the Angle Bisector Theorem.

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

Now we turn to the proof of Theorem 1.

Proof of Theorem 1. By the generalized angle bisector theorem, we have

$$\frac{AE}{EF} = \frac{AX}{XF} \cdot \frac{\sin \angle AXE}{\sin \angle EXF}.$$

Using the law of sines, we get

$$\frac{AE}{EF} = \frac{AX}{XF} \cdot \frac{AB}{BC}.$$

Similarly[a], we have

$$\frac{AD}{FD} = \frac{AX}{XF} \cdot \frac{\sin \angle AXD}{\sin \angle FXD} = \frac{AX}{XF} \cdot \frac{AD}{CD}.$$

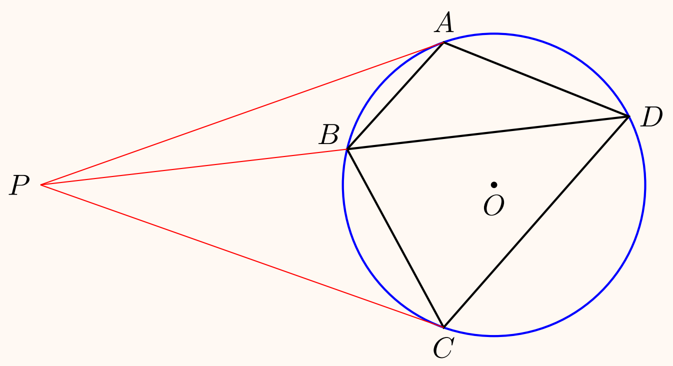
Comparing the above two equations, we conclude that A, E, F, D forms a harmonic division, if and only if $AB \cdot CDE = AD \cdot BC$, which implies that A, E, F, D forms a harmonic division if and only if $ABCD$ is a harmonic quadrilateral. ∎

[a] Note that D is on the extended line of the segment AF, but the lemma still applies.

The following theorem is another characterization of harmonic qudrilaterl.

Theorem 2

Let $ABCD$ be a concyclic quadrilateral. Let ℓ_1 and ℓ_2 be the tangent lines of the circle at A and C, respectively. Then $ABCD$ is a harmonic quadrilateral if and only if ℓ_1, ℓ_2 and BD are concurrent or parallel.



Proof. We first assume that the lines PA, PB and PC are concurrent. Then since

$$\triangle PAB \backsim \triangle PDA, \quad \triangle PCB \backsim \triangle PDC,$$

We get

$$\frac{PA}{PD} = \frac{AB}{AD}, \quad \frac{PC}{PD} = \frac{BC}{CD}.$$

Since PA, PC are tangent lines, we have $PA = PC$. Then we have

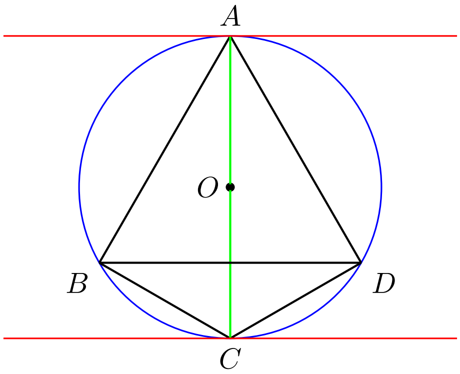
$$\frac{AB}{AD} = \frac{BC}{CD},$$

and thus $ABCD$ is a harmonic quadrilateral.

Conversely, we assume that PA, PC are the tangent lines, and let PD intersects the Circle O at B'. Then $AB'CD$ is a harmonic quadrilateral by the above argument. Since $ABCD$ is a quadrilateral by assumption, we get $B' = B$ by uniqueness and this completes the proof of the theorem.

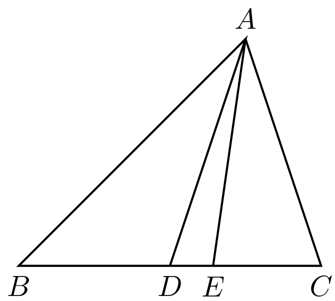
∎

Remark It is possible that the lines ℓ_1, ℓ_2, and BD are parallel, as showed in the following picture. In this case, $ABCD$ is a concyclic kite, where $AC \perp BD$ is a diameter of the circle O.



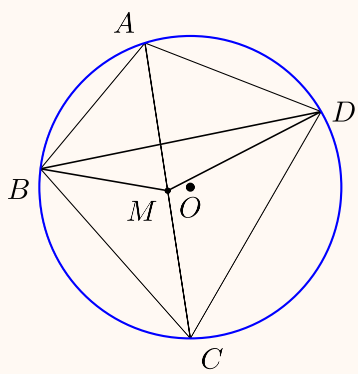
Harmonic quadrilateral is related to the symmedian line. In the following $\triangle ABC$, let D be the midpoint of BC. The line AD is called the *median* of $\triangle ABC$ on BC. Let AE be the *isogonal line* of AD, that is, $\angle EAC = \angle BAD$. Then line AE is called the *symmedian* of $\triangle ABC$ on BC.

For more details of symmedian, see Wikipedia or Topic 16. For isogonal conjugate line, see Topic 7.



Lemma 2

Let M be the midpoint of the diagonal AC of a harmonic quadrilateral $ABCD$. Then $\angle ABM = \angle DBC$, and $\angle BMC = \angle DMC$.



Proof. Since $ABCD$ is a harmonic quadrilateral, we have

$$AB \cdot CD = AD \cdot BC.$$

By Ptolemy's Theorem (See Wikipedia or Topic 10) , we know that

$$AB \cdot CD = \frac{1}{2}(AB \cdot CD + AD \cdot BC) = \frac{1}{2}BD \cdot AC = AM \cdot BD.$$

Thus we have

$$\frac{CD}{AM} = \frac{BD}{AB}.$$

Since

$$\angle BDC = \angle BAM,$$

we conclude that

$$\triangle BDC \backsim \triangle BAM.$$

Therefore we have $\angle ABM = \angle DBC$. Moreover,

$$\angle BMC = 180^\circ - \angle AMB = 180^\circ - \angle DCB.$$

By the same method, we conclude that

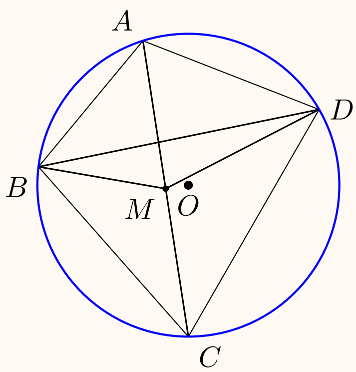
$$\angle DMC = 180^\circ - \angle DCB,$$

which completes the proof. ∎

With the above lemma, we give the following third characterization of harmonic quadrilateral.

Theorem 3

Let $ABCD$ be a concyclic quadrilateral. Then it is a harmonic quadrilateral if and only if the diagonal BD is the symmedian of $\triangle ABC$ on AC.



Proof. The only if part easily follows the above lemma. Conversely, if BD is the symmedian, then $\angle DBC = \angle ABM$. Since $\angle BDC = \angle BAM$, we conclude that $\triangle BDC \backsim \triangle BAM$. As a result, we have

$$\frac{AB}{BD} = \frac{AM}{DC},$$

or

$$AB \cdot CD = \frac{1}{2}BD \cdot AC.$$

Similarly, we can prove $\triangle BMC \backsim \triangle ABD$, and hence

$$BC \cdot AD = \frac{1}{2} AC \cdot BD.$$

Thus we have

$$AB \cdot CD = BC \cdot AD,$$

and $ABCD$ is a harmonic quadrilateral.

■

3 Harmonic Quadrilateral and Kelvin Transform

In this section, we shall prove that a concyclic quadrilateral is harmonic if and only of it is the Kelvin transform of square.

For details of Kelvin Transform, see Wikipedia or Topic 10. For our application, we shall use the following

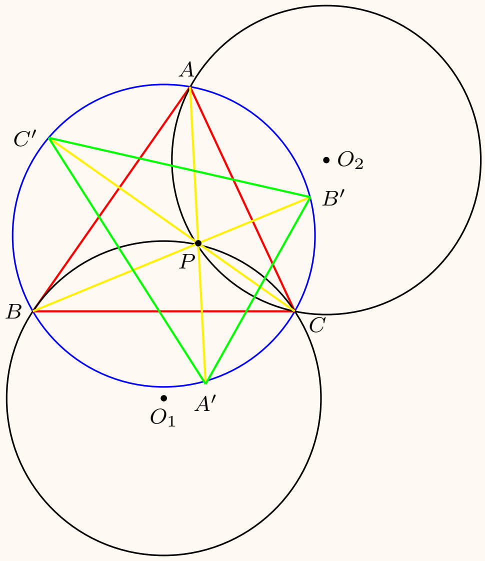
Definition 3. (Kelvin Transform)

Let P be a point on the plane, and let k be a non-zero constant. Let A be a point on the plane other than P. We determine A' such that it is a point on the line PA satisfying $PA \cdot PA' = k$. Here our convention is that if $k > 0$, then A' is on the ray PA, and if $k < 0$, then P is between A and A'. We call the mapping from A to A' the Kelvin Transform with respect to the center P and radius $\sqrt{|k|}$.

Before proving the main result of this section, we prove the following theorem which is of interest by itself.

Theorem 4

Let Circle O be the circumcircle of $\triangle ABC$. Then given any positive numbers α, β, γ with $\alpha + \beta + \gamma = 180°$, there is a point P such that if AP, BP, CP intersect to the circle at A', B', C', respectively, then $\angle A' = \alpha$, $\angle B' = \beta$, and $\angle C' = \gamma$. In other words, the Kelvin transform of $\triangle ABC$ with respect to P is the triangle $\triangle A'B'C'$ with the prescribed three interior angles α, β, γ.



Remark If $\triangle A'B'C'$ is an equilateral triangle, then P is called the *First Isodynamic Point*. See Topic 33 for details.

Remark By the Intersecting Chords Theorem, we have

$$PA \cdot PA' = PB \cdot PB' = PC \cdot PC' \stackrel{\text{def}}{=\!=} r^2,$$

where r^2 is the negative of the power of P to Circle O.

Proof. Let O_1 be the circle passing B, C such that the inscribed angle of the arc BC (counterclockwise) is $\angle A + \alpha$; let O_2 be the circle passing C, A such that the inscribed angle of the arc CA (counterclockwise) is $\angle B + \beta$.

Assume that these two circles intersect at a point P (other than C). We shall prove that P satisfies the requirement of the theorem.

Considering the (concave) quadrilateral $A'B'PC'$, we know

$$\angle PC'A' + \angle PB'A' + \angle A' + 360^\circ - \angle B'PC' = 360^\circ.$$

Since $\angle B'PC' = \angle BPC = \angle A + \alpha$, $\angle PC'A' = \angle PAC$, and $\angle PB'A' = \angle PAB$, we get $\angle A' = \alpha$.

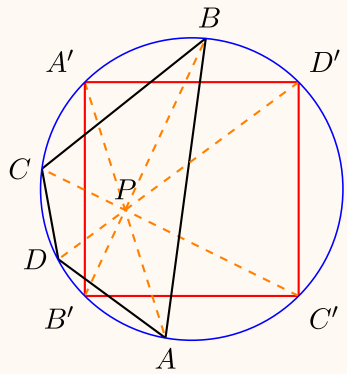
Similarly, we get $\angle B' = \beta$. Since $\gamma = 180^\circ - \alpha - \beta$, we get $\angle C' = \gamma$, which completes the proof of the theorem. ■

The above theorem implies, essentially, that there is a Kelvin Transform between any two triangles inscribed in the same circle (up to a rotation of one of the triangles; see the proof of Corollary 1 below).

The following theorem is a remarkable one characterizing harmonic quadrilaterals.

Theorem 5

A quadrilateral is harmonic if and only if it is the Kelvin Tranform of a square.



Proof. First, we assume that $A'B'C'D'$ is a square inscribed in the circle. Let P be a point inside the circle, and let PA', PB', PC', PD' intersect the circle at A, B, C, D. We shall prove that $ABCD$ is harmonic.

By the Intersecting Chords Theorem, $\triangle APB \backsim \triangle B'PA'$. Thus

$$\frac{AB}{A'B'} = \frac{PA}{PB'}.$$

Similarly, we have

$$\frac{CD}{C'D'} = \frac{PC}{PD'}, \quad \frac{BC}{B'C'} = \frac{PC}{PB'}, \quad \frac{AD}{A'D'} = \frac{PA}{PD'}.$$

So

$$\frac{AB \cdot CD}{A'B' \cdot C'D'} = \frac{PA \cdot PC}{PB' \cdot PD'} = \frac{BC \cdot AD}{B'C' \cdot A'D'}. \tag{1}$$

Since $A'B'C'D'$ is a square, we have $A'B' = C'D' = BC' = A'D'$ and hence

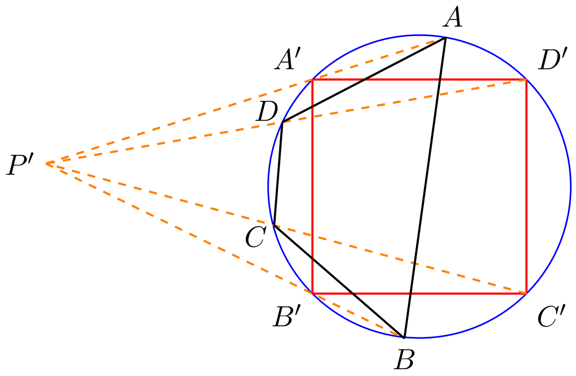
$$AB \cdot CD = BC \cdot AD.$$

Conversely, if $ABCD$ is harmonic, then by Theorem 4, there is a point P such that the Kelvin Transform of $\triangle ABC$ is an isosceles right triangle $\triangle A'B'C'$. Let D' be the intersection of DP to the circle. Then by (1), we have

$$A'B' \cdot C'D' = A'D' \cdot B'C'.$$

Since $\triangle A'B'C'$ is the isosceles right triangle, from the above we conclude that $A'B'C'D'$ is a square. This completes the proof. ■

Remark The Kelvin Transform center is not unique. There is a point P' outside the circle through which the Kelvin Transform of $ABCD$ is also $A'B'C'D'$ (See picture below). We left the proof to the reader.



Corollary 1

Let $ABCD$ and $EFGH$ be two harmonic quadrilaterals inscribed in the same circle. Then there is a point P such that the Kelvin Transform centered at P maps to a quadrilateral $E'F'G'H'$ which is a rotation of $EFGH$.

Proof. Suitable Kelvin Transforms map both $ABCD$ and $EFGH$ to squares, and two squares inscribed in a circle differ by a rotation. The corollary follows from the fact that the set of Kelvin Transforms is a group. ∎