Dual Triangles

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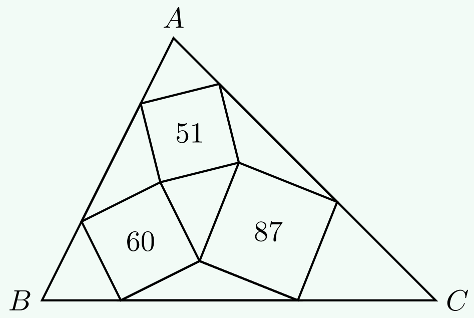
We begin with the following fun problem:

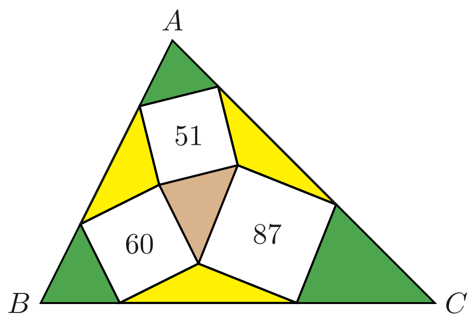
Problem 1

In the following picture, assume that the area of the three squares are 51, 60, and 87, respectively. Then what is the area of $\triangle ABC$*?*



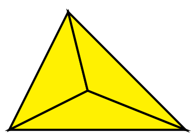
Solution: The first observation is that the area of the brown triangle is equal of the area of three yellow triangles: this is because the area formula is $S = \frac{1}{2}ab\sin C$, and two triangles are having a pair of supplementary angles.

If we cut the three yellow triangles out, they can be resembled to the yellow triangle which is similar to $\triangle ABC$. Similarly, we can do the same thing for green triangle.

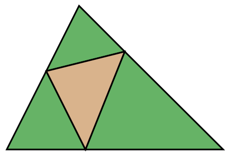


Now assume that A is the area of triangle $\triangle ABC$, S is the area of the brown triangle. The sides of the triangle are $\sqrt{51}$, $\sqrt{60}$, and $\sqrt{87}$. By the Heron formula, the area of the triangle $S = 27$. Then the sum of the area of the brown and green triangles would be

$$A - 51 - 60 - 87 - 3S = A - 279.$$



Since all these triangle are similar, we have
$$\sqrt{\frac{3S}{A}}+\sqrt{\frac{A-279}{A}}=1.$$
So $A=400$.



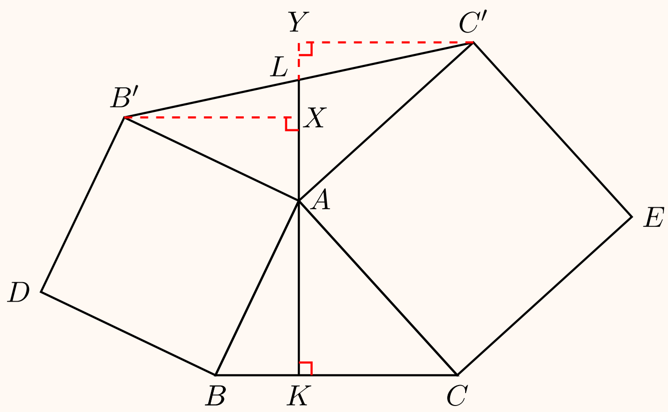
Let's study the problem using the "standard' method, which will lead the concepts *dual triangles*.

We say triangles $\triangle ABC$ and $\triangle AB'C'$ are *dual* triangles, if the two corresponding sides are the same, and the angles made by the two sides are supplementary.

Since $\angle BAC = 180^\circ - \angle B'AC'$, the area of $\triangle ABC$ and $\triangle AB'C'$ are the same. Moreover, we have the following

Theorem 1

In the following picture, $AB'DB$ and $ACEC'$ are squares. Let AK be a height over BC, then AL is the median of $\triangle AB'C'$. Moreover, we have
$$AL=\frac{1}{2}BC.$$



So the median of one triangle is the height of its dual triangle, and vice versa.

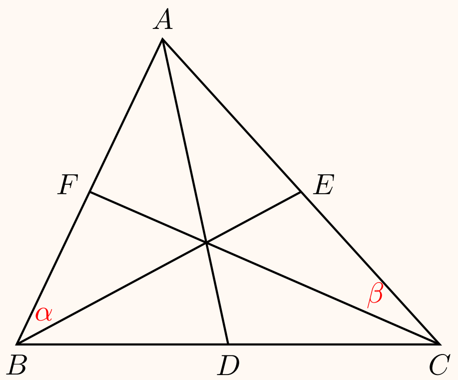
Proof: Obviously, we would see that $\triangle AB'X$ is congruent to $\triangle BAK$ and $\triangle AC'Y$ is congruent to $\triangle CAK$. Thus $B'K = AK = C'Y$, and hence $B'L = LC'$. ■

The following theorem seems to be unexpected, as in the conclusion, the left side is not symmetric with the three sides, but the right side is.

Theorem 2

In the following picture, assume that $BC = a, CA = b$ and $AB = c$. Let $\triangle$ be the area of the triangle $\triangle ABC$. Then

$$\cot\alpha + \cot\beta = \frac{a^2+b^2+c^2}{2\Delta}.$$



In particular,

$$\cot\alpha + \cot\beta = \cot\angle BCF + \cot\angle BAD = \cot\angle CAD + \cot\angle CBE.$$

Proof: By the Apollonius Theorem, we have

$$BE^2 = \frac{1}{2}(a^2+c^2) - \frac{1}{8}c^2.$$

Thus

$$\cos\alpha = \frac{c^2+BE^2-AE^2}{2\cdot BE\cdot c}, \qquad \sin\alpha = \frac{\triangle}{c\cdot BE}.$$

Thus,

$$\cot\alpha = \frac{2a^2-b^2+5c^2}{8\Delta}.$$

Therefore,

$$\cot\alpha + \cot\beta = \frac{a^2+b^2+c^2}{2\Delta}.$$

External Link. *The above theorem is related to the Apollonius Theorem and Stewart Theorem. See Apollonius Theorem and the more general Stewart Theorem on the Wikipedia.*

Now we can compute the area of $\triangle ABC$ using trigonometry. Let $DE = x$. Let $\angle ORP = \alpha$, and let $\angle OQP = \beta$. By the law of sines, we have

$$BD = \frac{b}{\sin\angle B}\cdot\sin\angle BID = OP\cdot\cot\alpha.$$

Similary, we have

$$EC = OP\cdot\cot\beta.$$

Using the above two propositions, we get

$$BC = BD + DE + EC = (1 + \frac{a^2+b^2+c^2}{6S})x,$$

where S is the area of $\triangle PQR$. Thus the area of $\triangle ABC$ is

$$\frac{(a^2+b^2+c^2+6S)^2}{12S}.$$