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The Morley's Miracle

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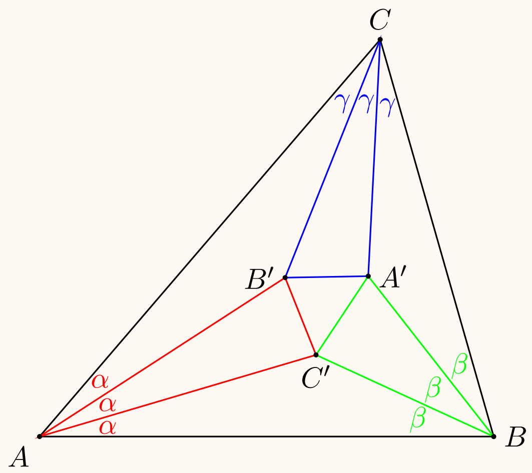
The Morley's Theorem states that in any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle. The theorem was discovered by Frank Morley in 1899.

Theorem 1. (Morley's Miracle)

In the following picture, the red, green, and blue lines are the angle trisectors of the corresponding angles. Then $\Delta A'B'C'$ is an equilateral triangle.



There are a lot different proofs of the Morley's Theorem. The following trigonometry method is one of the simplest.

Solution: Let R be the circumradius of the ΔABC. Then

$$AB = 2R\sin C = 2R\sin 3\gamma.$$

Using the law of sines, we have

$$AC' = \frac{AB}{\sin(\alpha+\beta)}\cdot\sin\beta = 2R\frac{\sin 3\gamma}{\sin(\alpha+\beta)}\sin\beta = 2R\frac{\sin 3(\alpha+\beta)}{\sin(\alpha+\beta)}\sin\beta.$$

Using the triple angle formula $\sin 3x = 3\sin x - 4\sin^3 x$, we get

$$AC' = 2R\frac{\sin 3(\alpha+\beta)}{\sin(\alpha+\beta)}\sin\beta = 2R(3-4\sin^2(\alpha+\beta))\sin\beta.$$

[2]The author thanks Stephanie Wang for her careful reading of the paper.

The key is to simplify the above expression. We have

$$3-4\sin^2(\alpha+\beta)=3-2(1+\cos 2(\alpha+\beta))$$
$$=2(\cos\frac{\pi}{3}-\cos 2(\alpha+\beta))=4\cos(\frac{\pi}{6}+\alpha+\beta)\cos(\alpha+\beta-\frac{\pi}{6}).$$

Note that $\alpha+\beta+\gamma=\pi/3$, we have

$$3-4\sin^2(\alpha+\beta)=2\sin\gamma\cos(\frac{\pi}{6}-\gamma).$$

Thus

$$AC'=8R\sin\beta\sin\gamma\cos(\frac{\pi}{6}-\gamma).$$

Using the same method, we have

$$AB'=8R\sin\beta\sin\gamma\cos(\frac{\pi}{6}-\beta).$$

Let $\sigma=8R\sin\beta\sin\gamma$. Then by law of cosines, we have

$$(B'C')^2=\sigma^2(\cos^2(\frac{\pi}{6}-\gamma)+\cos^2(\frac{\pi}{6}-\beta)-2\cos(\frac{\pi}{6}-\gamma)\cos(\frac{\pi}{6}-\beta)\cos\alpha).$$

We have

$$\sigma^2(\cos^2(\frac{\pi}{6}-\gamma)+\cos^2(\frac{\pi}{6}-\beta)-2\cos(\frac{\pi}{6}-\gamma)\cos(\frac{\pi}{6}-\beta)\cos\alpha)$$
$$=\frac{1+\cos(\frac{\pi}{3}-2\gamma)}{2}+\frac{1+\cos(\frac{\pi}{3}-2\beta)}{2}-(\cos(\frac{\pi}{3}-\gamma-\beta)+\cos(\beta-\gamma))\cos\alpha$$
$$=1+\frac{1}{2}(\cos(\frac{\pi}{3}-2\gamma)+\cos(\frac{\pi}{3}-2\beta))-\cos^2\alpha-\cos(\beta-\gamma)\cos\alpha$$
$$=\sin^2\alpha.$$

Thus

$$B'C'=8R\sin\alpha\sin\beta\sin\gamma.$$

By symmetry, $C'A'=A'B'=B'C'=8R\sin\alpha\sin\beta\sin\gamma$.

Remark Alternatively, we are able to use the law of sines only to prove the same result. We observe that

$$\frac{AB'}{AC'}=\frac{\cos(\frac{\pi}{6}-\beta)}{\cos(\frac{\pi}{6}-\gamma)}=\frac{\sin(\frac{\pi}{3}+\beta)}{\sin(\frac{\pi}{3}+\gamma)}.$$

Thus we have $\angle B'C'A=\frac{\pi}{3}+\beta$. Similarly, $\angle BC'A'=\frac{\pi}{3}+\alpha$. Thus

$$\angle A'B'C'=2\pi-(\frac{\pi}{3}+\beta)-(\frac{\pi}{3}+\gamma)-(\pi-\alpha-\beta)=\frac{\pi}{3}.$$

In the following, we use the uniqueness method to give a plane geometric proof.

Solution: If we draw the triangle $\triangle ABC$, and then draw the angle trisectors to construct points A', B', C', we shall note that points A', B', C' are *uniquely* determined by verties A, B, C.

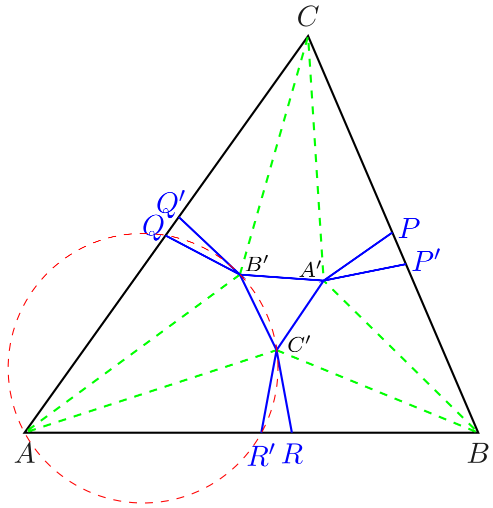
We reverse the procedure. We start from an equilateral $\triangle A'B'C'$. Let $\angle A+$

$\angle B + \angle C = 180°$ be three angles given. Let $\angle A = 3\alpha$, $\angle B = 3\beta$, and $\angle C = 3\gamma$. Then $\alpha + \beta + \gamma = \pi/3$. In the picture below, let $\triangle A'B'C'$ be the equilateral triangle with side length 1. We construct points P, Q, R, P', Q', R' such that

$$\angle B'C'R' = \angle QB'C' = \pi - 2\alpha,$$
$$\angle C'A'P' = \angle RC'A' = \pi - 2\beta,$$
$$\angle A'B'Q' = \angle PA'B' = \pi - 2\gamma,$$

and

$$A'P = A'P' = B'Q = B'Q' = C'R = C'R' = 1.$$



Connecting PP', QQ' and RR' we shall form $\triangle ABC$.

We need to prove

$$\angle A = 3\alpha, \qquad \angle B = 3\beta, \qquad \angle C = 3\gamma,$$

and $AB', AC', BC', BA', CA', CB'$ are the corresponding angle trisectors.

We compute

$$\angle RC'R' = 2\pi - (\pi - 2\alpha) - (\pi - 2\beta) - \frac{\pi}{3} = \frac{\pi}{3} - 2\gamma$$

Since $\triangle C'R'R$ is isosceles, we conclude that $\angle C'R'R = \frac{\pi}{3} + \gamma$. Similarly, $\angle CQB' = \frac{\pi}{3} + \beta$. Thus, from the fact that the summation of the angles of the pentagon $AQB'C'R'$ is 3π, we conclude that $\angle A = 3\alpha$.

By the construction, the quadrilateral $QB'C'R'$ is an isosceles trapezoid. Thus it must be concyclic. Point A must be on the circle because if A is outside the circle, we must have $\angle B'AB < 2\alpha$, and $\angle CAB' < \alpha$; and if A is inside the circle, we have the reverse inequalities, which is a contradiction to the fact that $\angle A = 3\alpha$. Thus the green lines are angle trisectors and this completes the proof.

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External Link. *Further readings on Morley's Theorem. A group theoretic proof of Morley's Theorem by the Fields Medalist A. Connes: A new proof of Morley's theorem.*