Pappus' Area Theorem

Dawei Liu[1], daweil7@uci.edu

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1 Introduction

The Pappus' Area Theorem was discovered by the Greek mathematician *Pappus of Alexandria*. The theorem describes the relationship between the areas of three parallelograms attached to three sides of an arbitrary triangle. The Pappus' Area Theorem is one of generalizations of the Pythagorean Theorem.

2 The Pythagorean Theorem

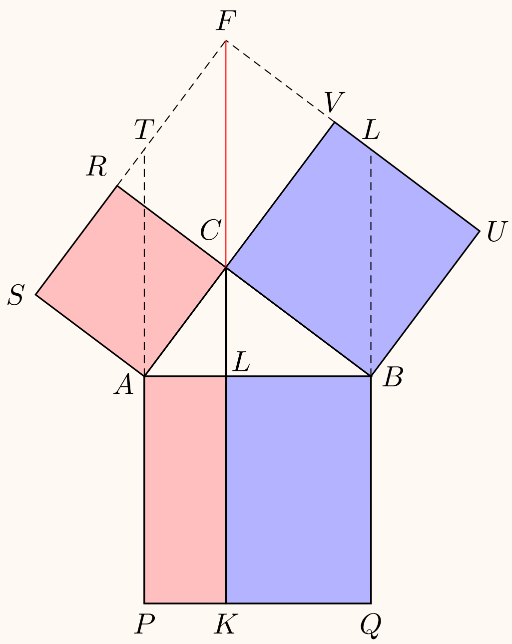
Theorem 1. (Pythagorean Theorem)

In the following picture, ΔABC is a right triangle ($\angle ACB = 90°$) and CRSA, BUVC and APQB are three squares attached to the triangle sides CA, BC and AB, respectively. Then

$$AC^2 + BC^2 = AB^2$$

or, we can interpret the theorem in terms of areas.

$$S_{APQB} = S_{CRSA} + S_{BUVC}.$$



[1]The author thanks Dr. Zhiqin Lu for his help.

Proof: From the above graph, $RSAC$ and $BUVC$ are two squares attached to the right triangle sides AC and BC. Assume that the extended square sides SR and UV intersect at F.

Draw the the line FC and its extension intersects AB at L and PQ at K. We claim that $FC \perp AB$: by the construction, $\triangle ABC \cong CFR$. Thus $\angle CAB + \angle ACL = \angle RCF + \angle ACL = 90°$, and hence $\angle ALC = 90°$.

So we have $FC \parallel TA$. Since $TF \parallel AC$, $FTAC$ is a parallelogram, and hence

$$S_{CRSA} = S_{FTAC}.$$

On the other hand, we have $FC = BA = LK$. Thus

$$S_{FTAC} = S_{APKL}.$$

We therefore conclude that

$$S_{CRSA} = S_{APKL}.$$

Using the same method, we get

$$S_{BUVC} = S_{LKQB}.$$

We then get

$$S_{APQB} = S_{APKL} + S_{BUVC} = S_{CRSA} + S_{LKQB},$$

completing the proof of the theorem. ■

External Link. *Here is the Pythagorean Theorem in Wikipedia.*

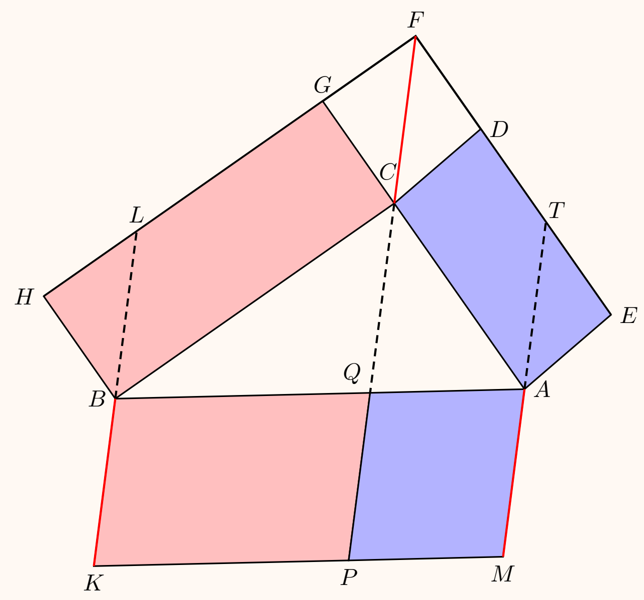
3 Pappus' Area Theorem

In the Pappus' Area Theorem, the triangle in question is not necessarily a right triangle.

Theorem 2. (Pappus' Area Theorem)

In the following picture, $\triangle ABC$ is an arbitrary triangle. Let $HBCG$ and $EACD$ be two arbitrary parallelograms attached to the triangle sides BC and AC, respectively. The extension of sides HG and ED intersect at F. Assume that the extension of FC intersects BA at Q and KM at P. Assume that $BKMA$ is the paralogram such that $BK \parallel FC \parallel AM$ and $BK = FC = AM$. Then

$$S_{BKMA} = S_{HBCG} + S_{AEDC}.$$



Proof: From the above graph, $HBCG$ and $AEDC$ are two parallelograms attached to the triangle sides BC and AC. Assume that the extension of sides KB and MA intersect HG and ED at L and T, respectively.

Refer to the proof of Pythagorean Theorem above, we have

$$S_{HBCG} = S_{LBCF}, \quad S_{EACD} = S_{TACF}.$$

By the same reason,

$$S_{LBCF} = S_{BKPQ}, \quad S_{TACF} = S_{QPMA}.$$

Thus, we get

$$S_{HBCG} = S_{BKPQ}, \quad S_{EACD} = S_{QPMA}.$$

We then get

$$S_{BKMA} = S_{HBCG} + S_{AEDC},$$

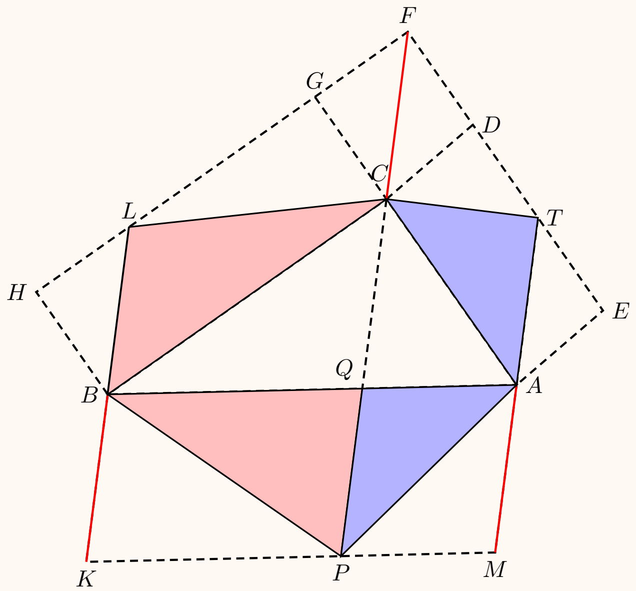
completing the proof of the theorem.

We also have the alternative version of the Pappus' Theorem replacing parallogram by triangle.

Theorem 3. (Alternative Version of the Pappus' Area Theorem)

In the following picture, $\triangle ABC$ is an arbitrary triangle, $HBCG$ and $EACD$ are the two arbitrary parallelograms attached to the triangle sides BC and AC. $\triangle BLC$ and $\triangle CTA$ are two triangles inscribed in them. The extension of sides HG and ED *intersects at F. Assume that $BKMA$ is the parallelogram such that $BK \parallel FC \parallel AM$ and $BK = FC = AM$ and $\triangle BPA$ inscribed in it. Then*

$$S_{BPA} = S_{BLC} + S_{CTA}$$



Note that the areas of the regions enclosed by the triangles are half the areas of the regions enclosed by the parallelograms. So the proof follows from that of the Pappus' Theorem.

Pappus' area theorem generalizes the Pythagorean theorem twofold. Firstly, it works with any triangle, not just right triangle. Secondly, it uses a parallelogram instead of a square. In a right triangle, two parallelograms attached to the right side create a rectangle with an area equal to the third side. If the two parallelograms are squares, then the rectangle on the third side is also a square.

External Link. *For further reading, we refer to Pappus' Area Theorem From Wikipedia.*

Remark Notice that Pappus' Area Theorem is different from the *Pappus' Theorem*, also known as the *Pappus' Hexagon Theorem*. See Wikipedia or Theorem 3 of Topic 06 for details.