Topic 40 Steiner-Lehmus' Theorem

Math 199C
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Thanks Dr. Zhiqin Lu for his help

1. General Introduction

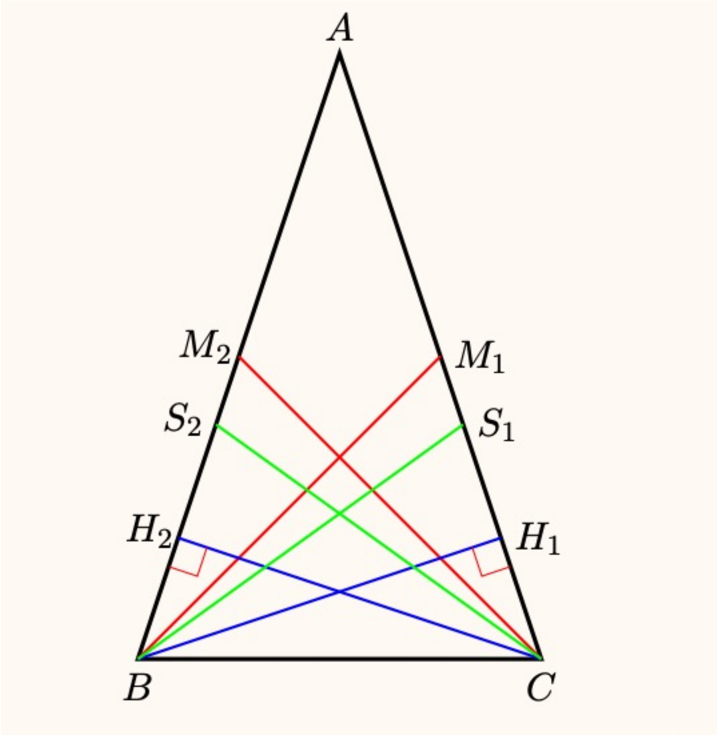
2. Steiner-Lehmus' Theorem

3. Gergonne Cevian

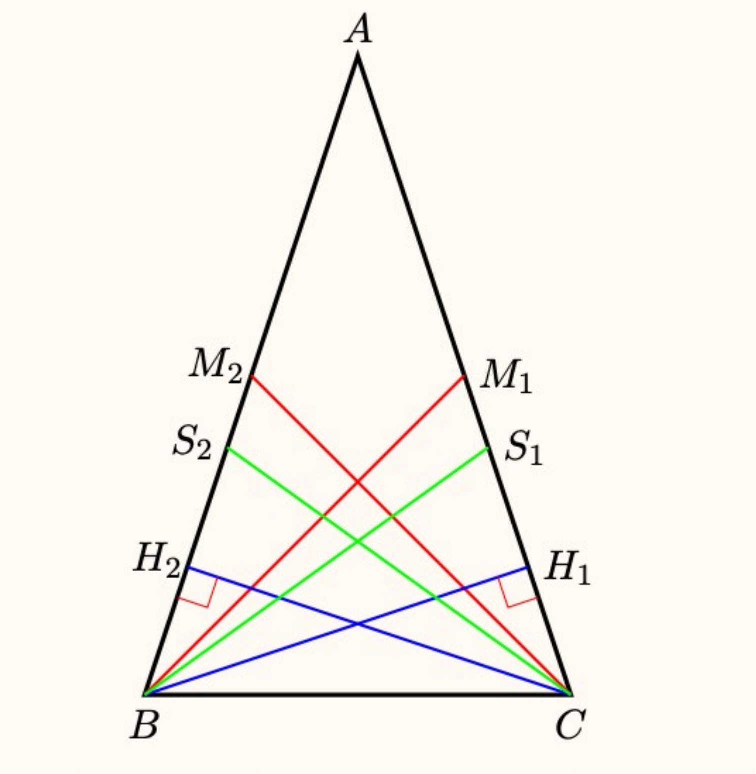
4. Extensions& Applications

1. General Introduction

- **Theorem**
- **On an isosceles triangle, two medians, heights, and angle bisectors are equal.**
- Let BH1, CH2 be heights; BM1, CM2 be medians, and BS1, CS2 be angle bisectors on sides AC, AB, respectively.
- Then BH1 = CH2, BM1 = CM2, BS1 = CS2.



1. General Introduction



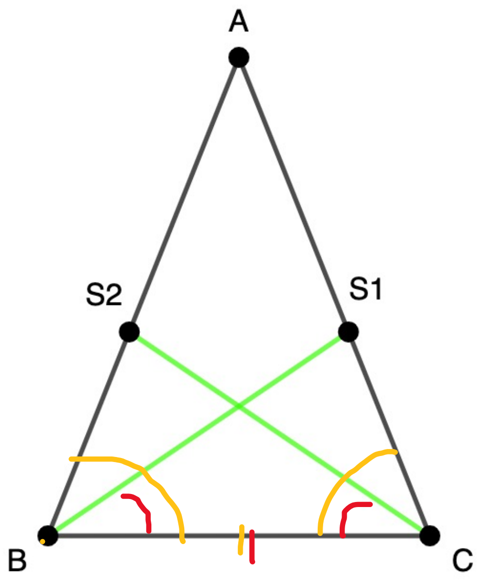
Proof. Since $AB = AC$, we have $\angle B = \angle C$.

Since $\angle CH_1B = \angle BH_2C = 90°$, $\angle C = \angle B$, and BC is the common side, we have $\triangle CH_1B \cong \triangle BH_2C$. Therefore $BH_1 = CH_2$.

Since we $CM_1 = BM_2 = \frac{1}{2}AB$, $\angle C = \angle B$, and BC is the common side, we have $\triangle CM_1B \cong \triangle BM_2C$. Therefore $BM_1 = CM_2$.

Cited from Tongtong Li

1. General Introduction



Since $\angle S_1BC = \angle S_2CB = \frac{1}{2}\angle B$, $\angle C = \angle B$, and BC is the common side, we have $\triangle CS_2B \cong \triangle BS_1C$. Therefore $BS_1 = CS_2$. ■

Easy!

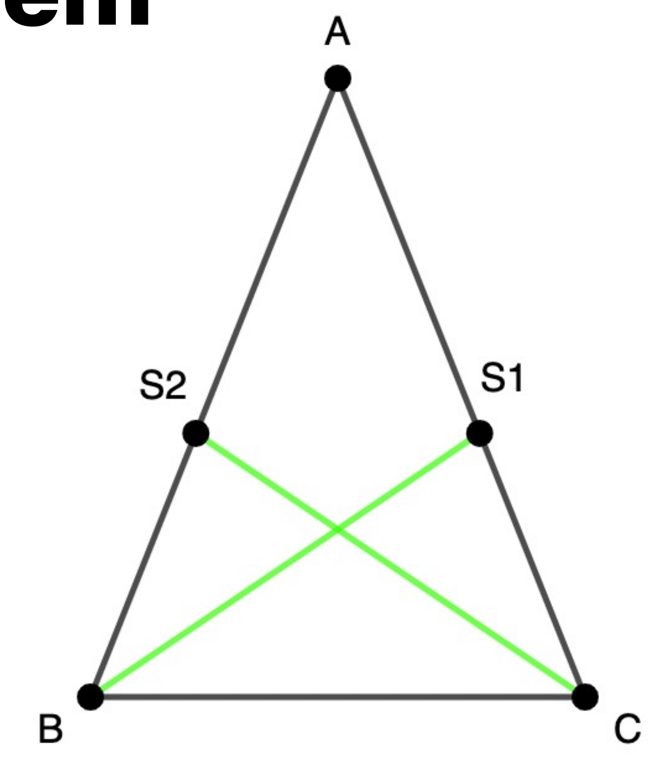
Cited from Tongtong Li

Are the converse theorems still true?

YES!!!

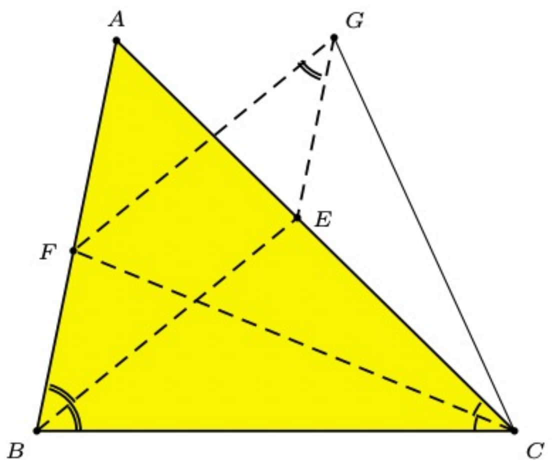
2. Steiner-Lehmus' Theorem

- In triangle $\triangle ABC$, let BS1 and CS2 be angle bisectors.
- Assume that BS1 = CS2.
- Then $\triangle ABC$ is an isosceles triangle.



Cited from K.R.S.Sastry

2. Steiner-Lehmus' Theorem

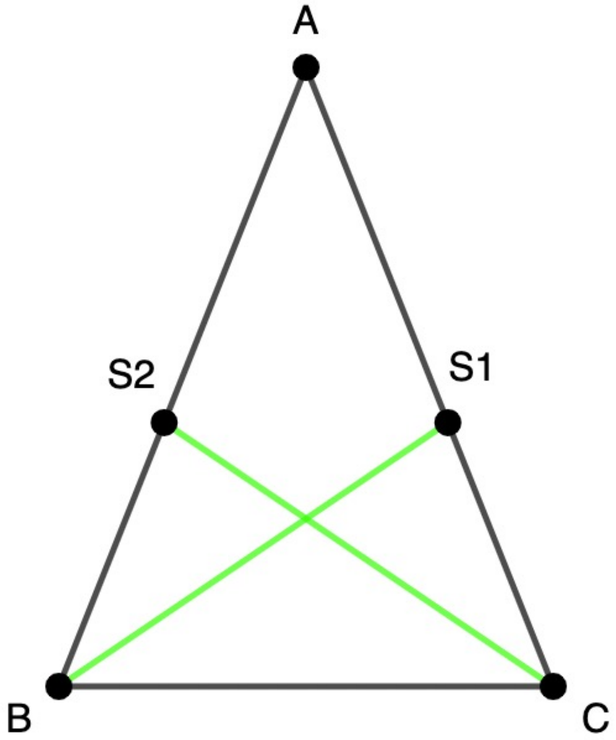


Complex!

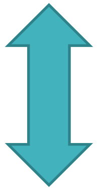
Different Proofs of Steiner-Lehmus' Theorem

- Proof Using Trigonometry
- Proof by Direct Computation
- Proof by Contrapositive
- Basic Geometric Proof
- …

2. Steiner-Lehmus' Theorem



Given AB=AC
Then BS1=CS2



Given BS1=CS2
Then AB=AC

Easy

Hard

3. Gergonne Cevian

The lines (cevians) joining the vertices of a triangle ABC to the tangent points D, E, and F of the inscribed circle are concurrent at point G called the Gergonne Point.

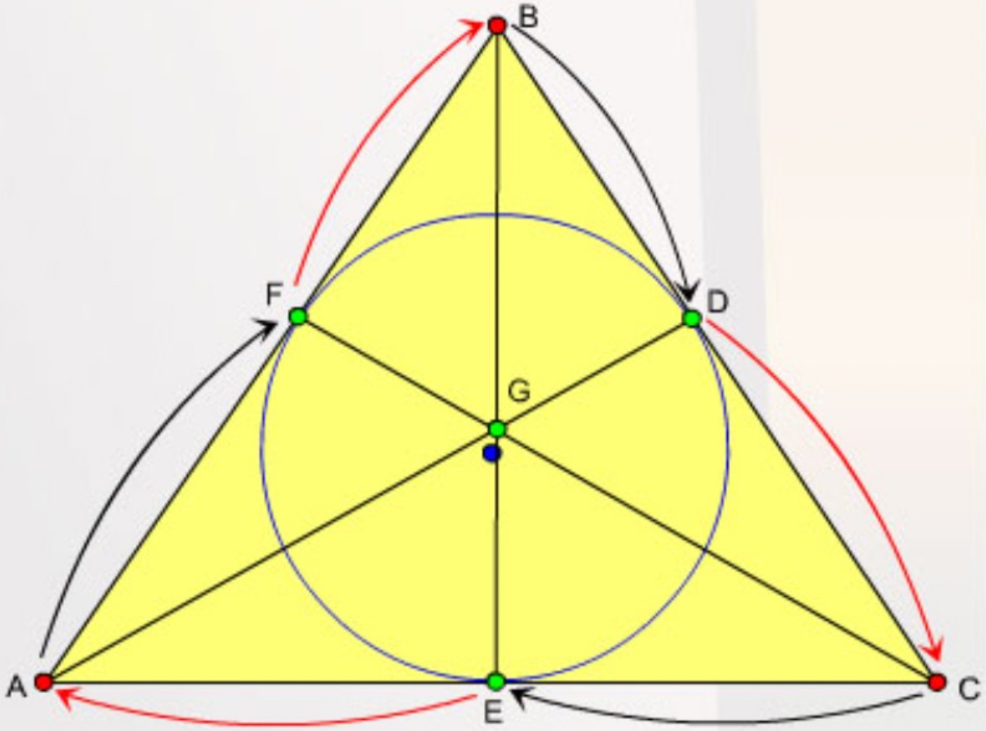
Proof:

1. AF = EA (two tangent segments theorem)
2. BD = FB (two tangent segments theorem)
3. CE = DC (two tangent segments theorem)
4. Multiplying (1) x (2) x (3):

   $AF \cdot BD \cdot CE = EA \cdot FB \cdot DC$

   $$\therefore \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

   Then, by Ceva's Theorem AD, BE and CF are concurrent.

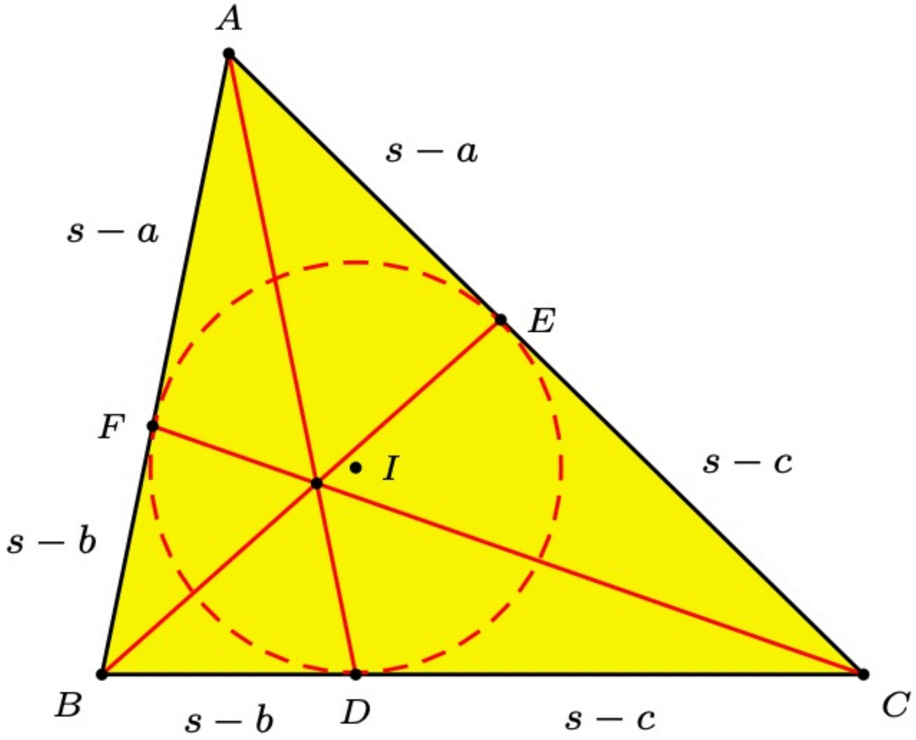


Gergonne Point G

3. Gergonne Cevian

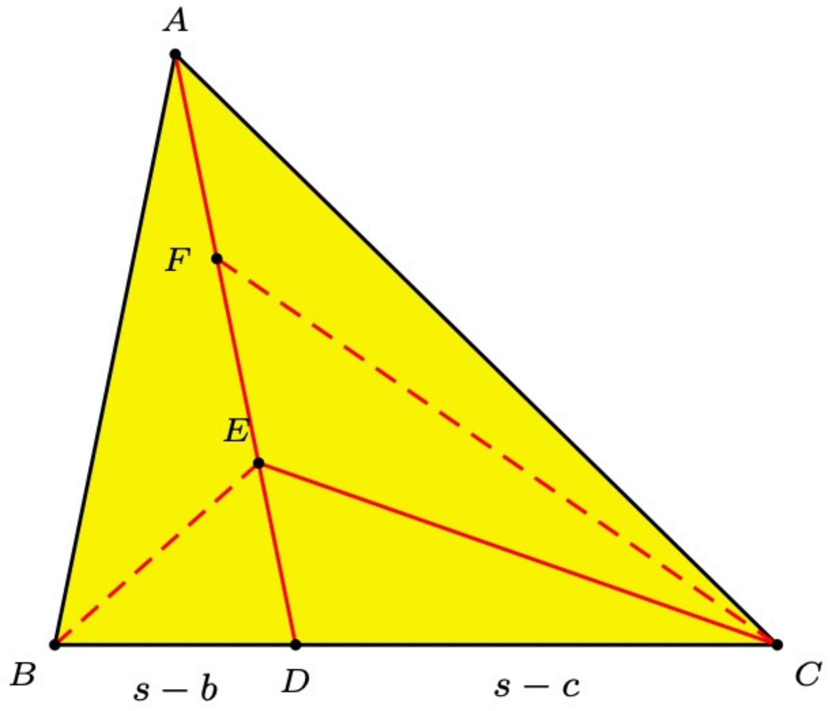
- **Theorem:** If two Gergonne cevians of a triangle are equal, then the triangle is isosceles.

If BE=CF
Then AB=AC



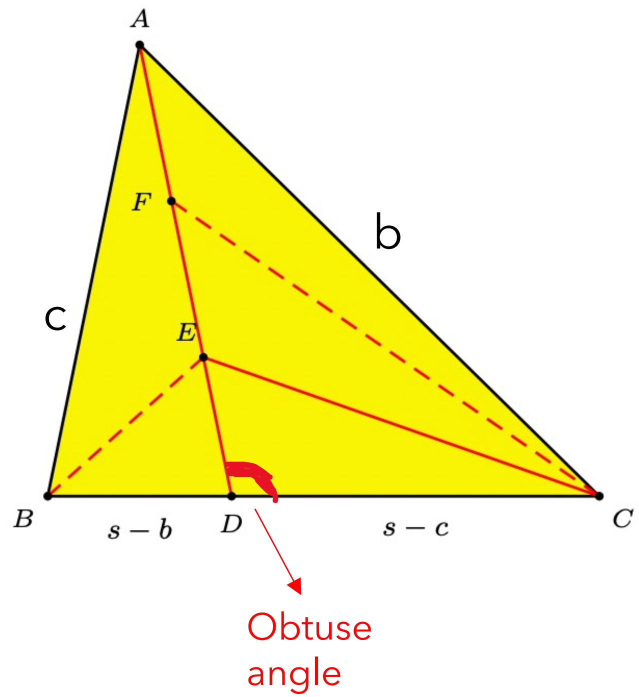
4. Extensions & Applications

- **Theorem:** The internal angle bisectors of the angles ABC and ACB of triangle ABC meet the Gergonne cevian AD at E and F respectively.
- If BE = CF, then triangle ABC is isosceles.



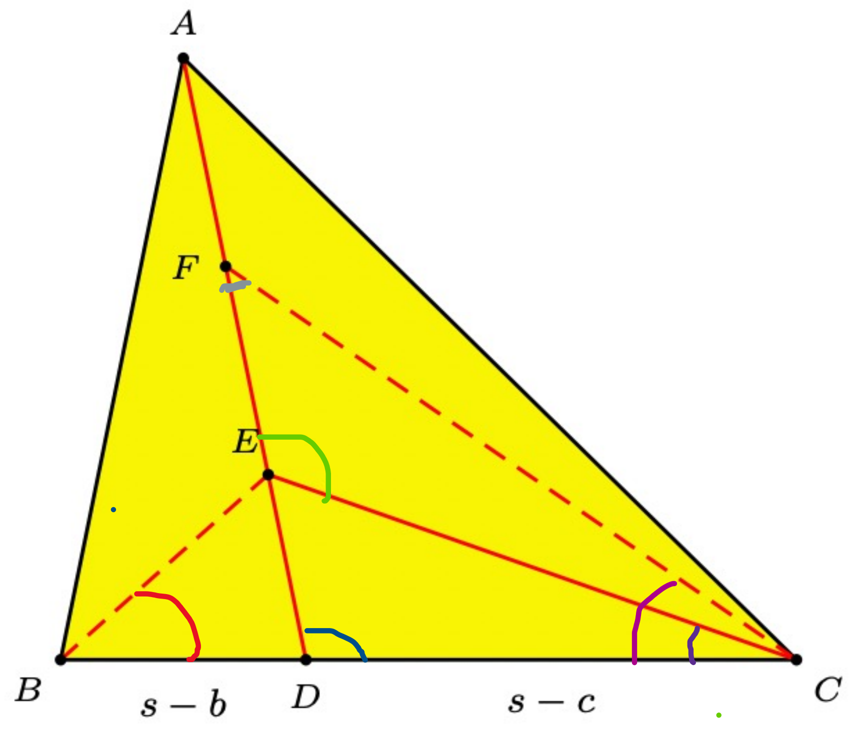
4. Extensions & Applications

- Proof: We refer to Figure.
- If AB not equal to AC, let AB < AC.
- Hence b>c, s – b < s – c
- E lies below F on AD.
- A simple calculation with the help of the angle bisector theorem shows that the Gergonne cevian AD lies to the left of the cevian that bisects ∠BAC and hence that ∠ADC is obtuse

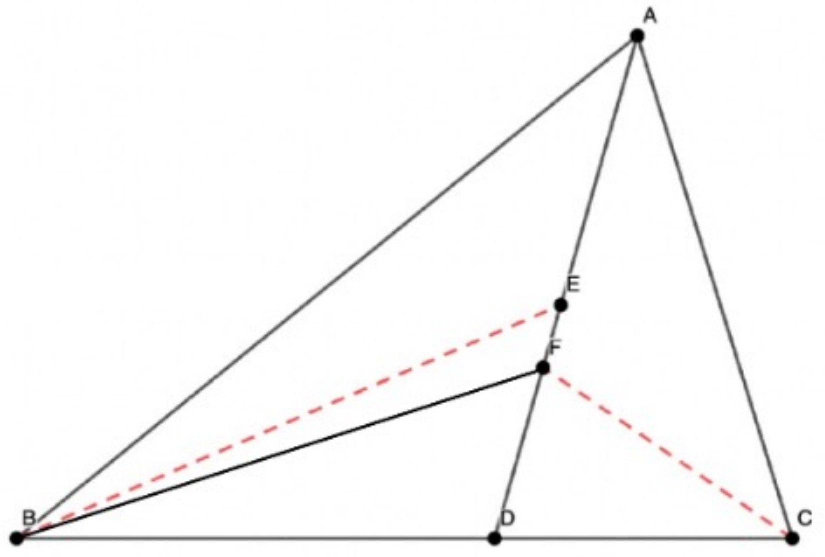


4. Extensions & Applications

+ $\angle ABC > \angle ACB \Rightarrow \angle EBC > \angle FCD > \angle ECB$.
+ Therefore, $CE > BE$ or $CE > CF$ (1) (because $BE = CF$).
+ However, $\angle ADC = \angle EDC > \pi/2$
+ Hence $\angle FEC = \angle EDC + \angle ECD > \pi/2$ and $\angle EFC < \pi/2$
+ $\Rightarrow$ $CE < CF$, contradicting (1).



4. Extensions & Applications



+ Likewise, the assumption AB > AC also leads to a contradiction. This means that triangle ABC must be isosceles.

- By assumption, $\angle ABC < \angle ACB \Rightarrow \angle FCD > \angle EBD > \angle FBD$. Therefore, $BF > CF$ or $BF > BE$. Because $BE = CF(6)$.
- However, $\angle ADB = \angle EDB > \pi/2$ as mentioned above. Hence $\angle EFB = \angle FDB + \angle FBD > \pi/2$ and $\angle FEB < \pi/2 \rightarrow BE > BF$, contradicting (6).

4. Extensions & Applications
How about the external angle bisector?

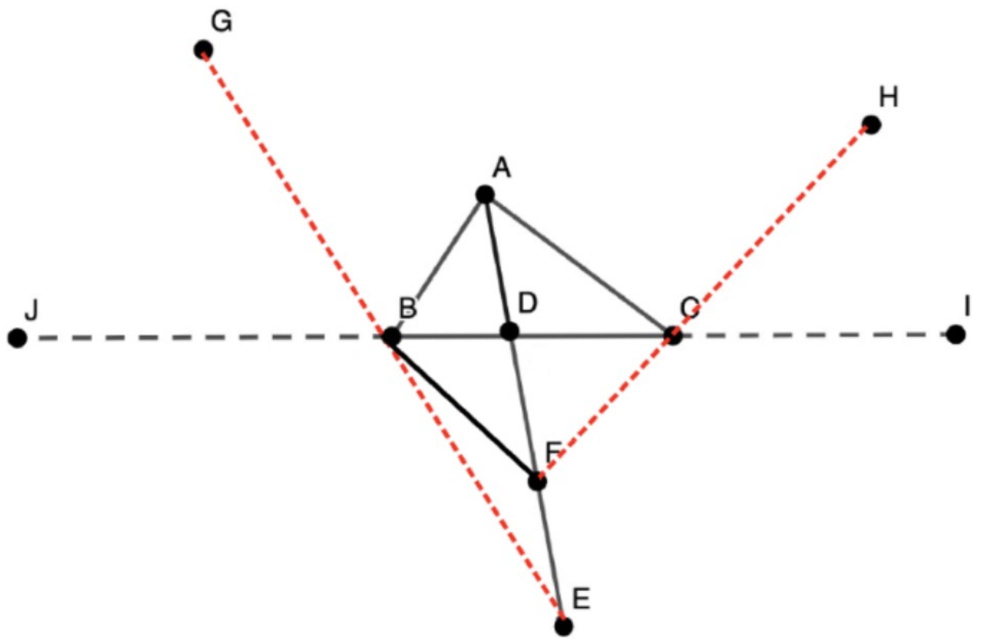
4. Extensions & Applications

How about the external angle bisector?

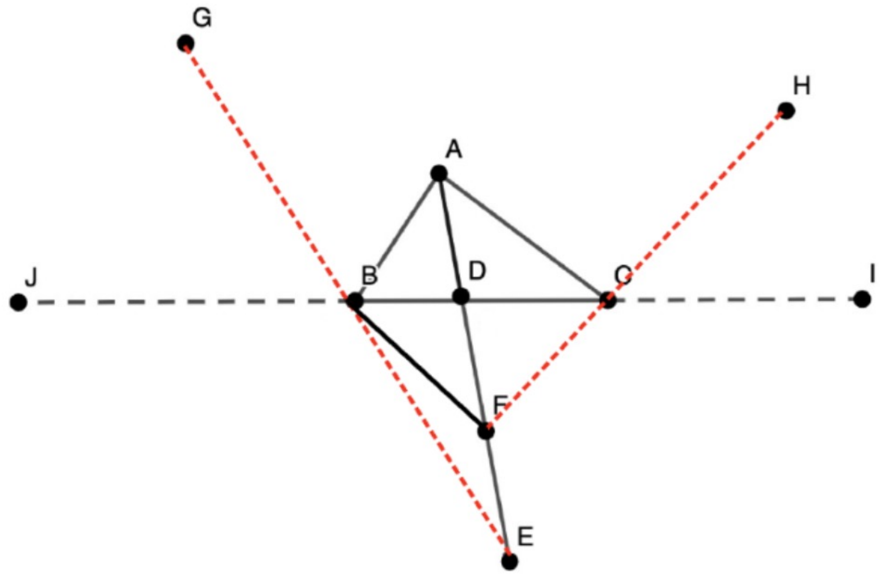
- **Collaroy**
- The external angle bisectors of ∠ABC and ∠ACB meet the extension of the Gergonne cevian AD at the points E and F respectively. If BE = CF, triangle ABC is isosceles.

4. Extensions & Applications

How about the external angle bisector?



If AB not equal to AC
let AB < AC



Proof. We refer to the figure below. If $AB \neq AC$, let $AB < AC$. Hence $b > c$, $s - b < s - c$ and E lies below F on the extension of AD. A simple calculation with the help of angle bisector theorem shows that the Gergonne cevian, which is the extension of line AD lies to the left of the cevian that bisects the external angle of ∠BAC and hence that ∠ADC is obtuse. □

- By assumption, $\angle ABC > \angle ACB \Rightarrow \angle ABJ < \angle ACI$. Thus, $\angle GBJ = \angle CBE < \angle HCI = \angle BCF$. Hence, $\angle DCF > \angle DBE > \angle DBF \rightarrow BF > CF$ or $BF > BE$ because $CF = BE$ (7).
- However, $\angle ADC = \angle BDF > \pi/2$. $\angle EFB = \angle BDF + \angle DBF > \pi/2$ and $\angle E < \Pi/2$. Hence, $\angle EFB > \angle E \rightarrow BE > BF$, contradicting (7).

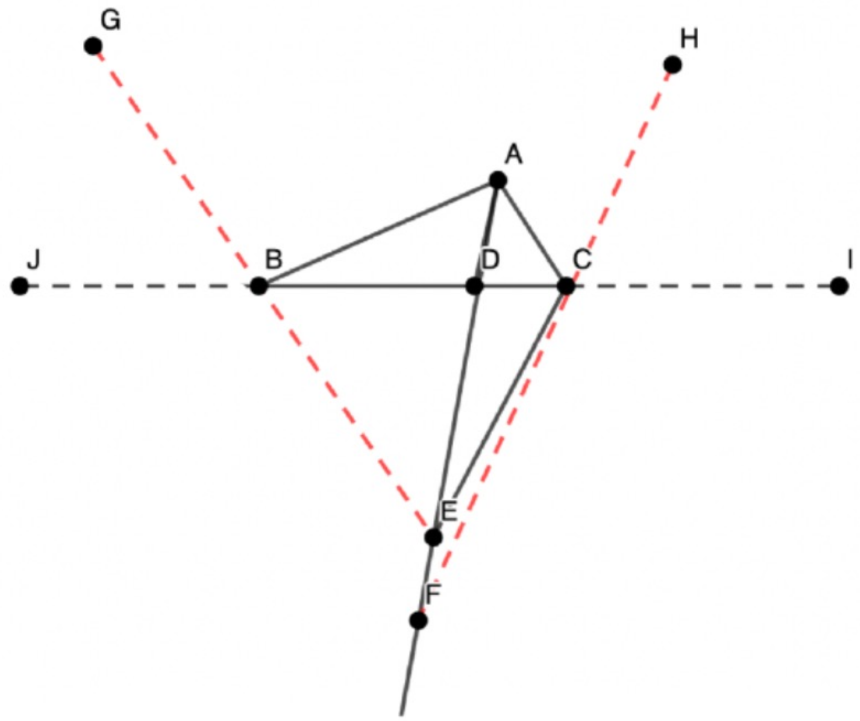
4. Extensions & Applications

How about the external angle bisector?

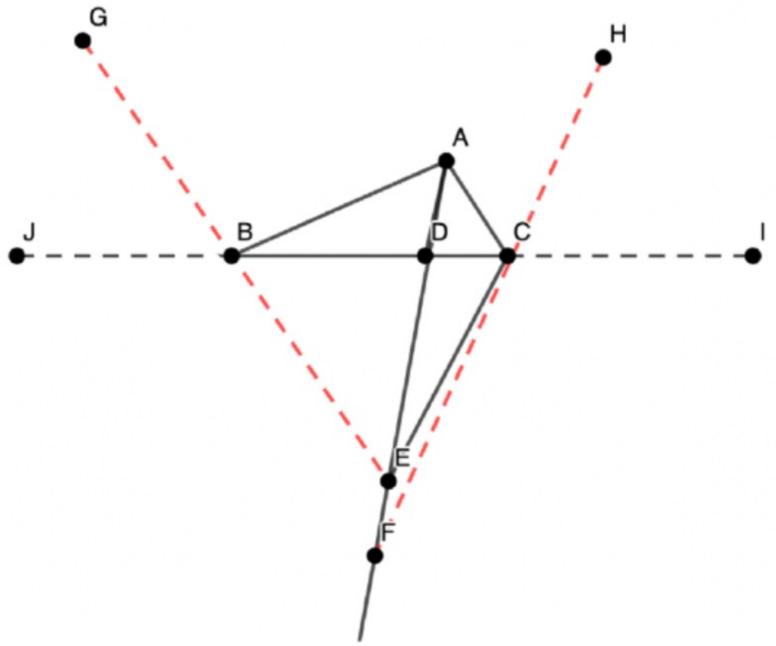
4. Extensions & Applications
How about the external angle bisector?

If AB not equal to AC

let AB > AC



4. Extensions & Applications
How about the external angle bisector?



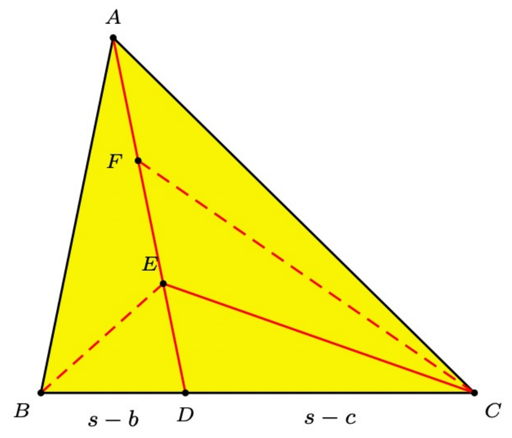
Proof. Likewise, we refer to the figure below. The assumption $AB > AC$,hence $c > b$. $s - b > s - c$ and F lies below E on the extension line of AD. Similarly, with the help of the angle bisector theorem shows that the Gergonne cevian, which is the extension of line AD lies to the right of the cevian that bisects the external angle of $\angle ACB$ and hence that $\angle ADB$ is obtuse. □

- By assumption, $\angle ABC < \angle ACB \Rightarrow \angle ACI < \angle ABJ$. Thus, $\angle HCI = \angle DCF < \angle GBJ = \angle DBE$. Hence, $\angle DBE > \angle DCF > \angle DCE \rightarrow BE < CE$ or $CF < CE$ because $CF = BE$ (8).

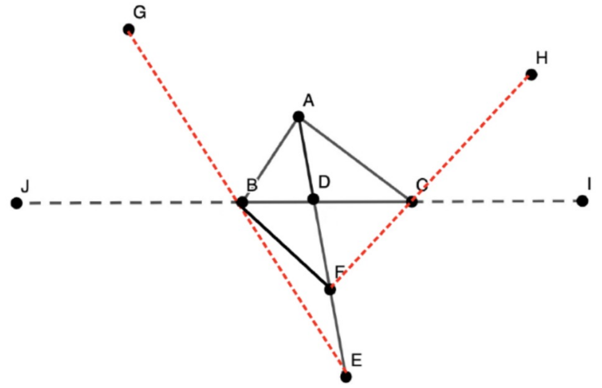
- However, $\angle ADB = \angle CDE > \pi/2$. $\angle FEC = \angle CDE + \angle ECD > \pi/2$ and $\angle F < \Pi/2$. Hence, $\angle FEC > \angle F \rightarrow CF > CE$, contradicting (8).

4. Extensions & Applications

Other Applications



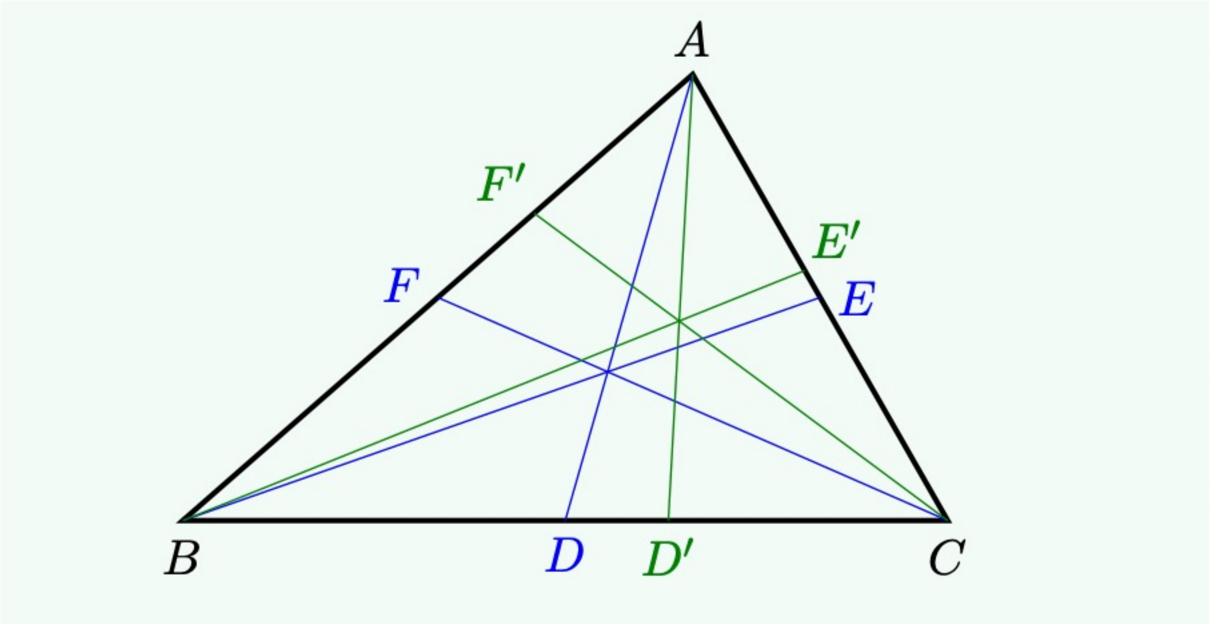
+ If BE<CF
+ Then AB<AC?



+ If BE>CF
+ Then AB>AC?

4. Extensions & Applications

Other Applications



If BE′=CF′

AB=AC?

Symmedians

+ AD, BE, CF are the medians, and AD′ , BE′ , CF′ are the symmedians.
+ Their corresponding intersections are centroid and symmedian point.

(Please refer to Topic 16 for more information)

THANK YOU