Steiner – Lehmus Theorem

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1 Introduction

On an isosceles triangle, two medians, heights, and angle bisectors are equal. Are the converse theorems still true?

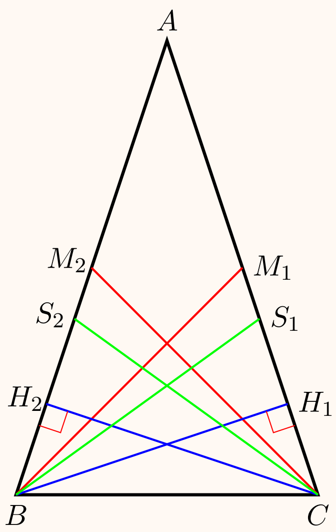
It is not hard to prove (and we shall do that below) that when two heights or two medians are equal, then the triangle is isosceles. However, the same question for the angle bisectors is surprisingly difficult to solve using basic geometry method.

In a letter to C. Sturm in 1840, C. L. Lehmus demanded such a basic geometric proof. Sturm circulated the request toward other mathematicians, and Jakob Steiner was first to respond. The theorem has been a popular topic in elementary geometry since then, with publications on it appearing on a regular basis.

Theorem 1

Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Let BH_1, CH_2 be heights; BM_1, CM_2 be medians, and BS_1, CS_2 be angle bisectors on sides AC, AB, respectively. Then

$$BH_1 = CH_2, \quad BM_1 = CM_2, \quad BS_1 = CS_2.$$



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Proof. Since $AB = AC$, we have $\angle B = \angle C$.

Since $\angle CH_1B = \angle BH_2C = 90°$, $\angle C = \angle B$, and BC is the common side, we have $\triangle CH_1B \cong \triangle BH_2C$. Therefore $BH_1 = CH_2$.

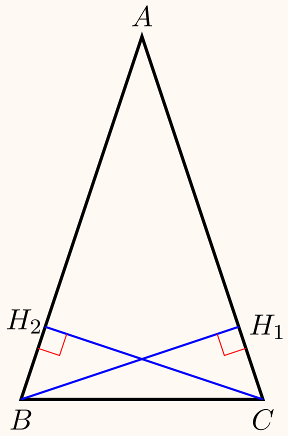
Since $CM_1 = BM_2 = \frac{1}{2}AB$, $\angle C = \angle B$, and BC is the common side, we have $\triangle CM_1B \cong \triangle BM_2C$. Therefore $BM_1 = CM_2$.

Since $\angle S_1BC = \angle S_2CB = \frac{1}{2}\angle B$, $\angle C = \angle B$, and BC is the common side, we have $\triangle CS_2B \cong \triangle BS_1C$. Therefore $BS_1 = CS_2$. ■

Conversely, we have the following

Theorem 2

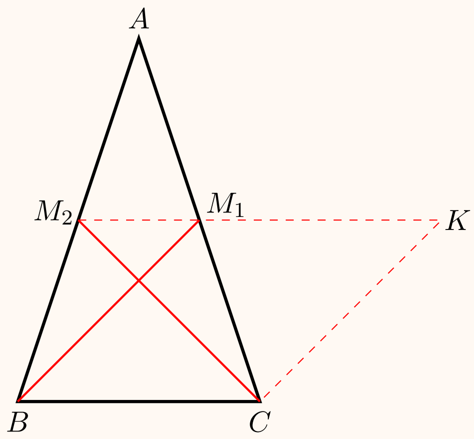
In triangle $\triangle ABC$, let BH_1 and CH_2 be heights. Assume that $BH_1 = CH_2$. Then $\triangle ABC$ is an isosceles triangle.



Proof. Since $BH_1 = CH_2$, $\angle BH_1C = \angle CH_2B = 90°$, and BC is the common side. Then $\triangle BH_1C \cong \triangle CH_2B$. Thus $\angle C = \angle B$ and hence $\triangle ABC$ is isosceles. ■

Theorem 3

In triangle $\triangle ABC$, let BM_1 and CM_2 be medians. Assume that $BM_1 = CM_2$. Then $\triangle ABC$ is an isosceles triangle.



Proof. We wish to prove that $\triangle BM_1C \cong \triangle CM_2B$. But this can not be done directly. We draw M_2M_1 and extend it to K such that $M_1K = BC$. Then since M_2M_1 is the midline, we have $M_2M_1 \parallel BC$, and $M_1K = BC$, then M_1BCK is a parallelogram. Thus $CM_2 = BM_1 = CK$ and hence $\triangle CKM_2$ is an isosceles triangle. As a result, we have

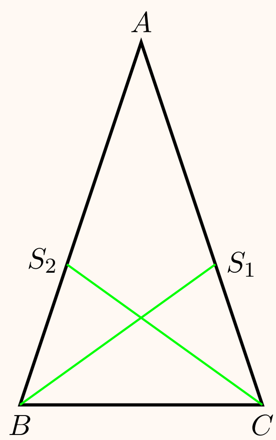
$$\angle M_1BC = \angle K = \angle KM_2C = \angle M_2CB.$$

Thus $\triangle BM_1C \cong \triangle CM_2B$ since $BM_1 = CM_2$ and BC is a common side. Therefore $\angle C = \angle B$ and $\triangle ABC$ is an isosceles triangle. ∎

Contrary to the cases of heights and medians, the following theorem about angle bisectors is surprisingly difficult to prove.

Theorem 4. (Steiner-Lehmus' Theorem)

In triangle $\triangle ABC$, let BS_1 and CS_2 be angle bisectors. Assume that $BS_1 = CS_2$. Then $\triangle ABC$ is an isosceles triangle.



We shall prove this theorem using different methods, the most difficult of which is using the basic geometry method.

2 Different Proofs of Steiner-Lehmus' Theorem

Proof Using Trigonometry. By law of sines, we have

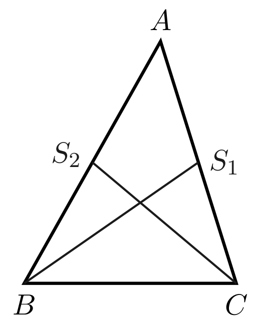
$$\frac{BS_1}{\sin C} = \frac{BC}{\sin(180^\circ - C - B/2)}, \quad \frac{CS_2}{\sin B} = \frac{BC}{\sin(180^\circ - B - C/2)}.$$

Since $BS_1 = CS_2$, using the above equation, we must have

$$\frac{\sin C}{\sin(C + B/2)} = \frac{\sin B}{\sin(B + C/2)}.$$

Thus

$$\sin C \cdot \sin(B + C/2) = \sin B \cdot \sin(C + B/2).$$



We use the double angle formula to get

$$\sin\frac{C}{2}\cos\frac{C}{2}\sin(B + \frac{C}{2}) = \sin\frac{B}{2}\cos\frac{B}{2}\sin(C + \frac{B}{2}).$$

Using the product-to-sum formula, we have

$$\sin\frac{C}{2}(\sin(B + C) + \sin B) = \sin\frac{B}{2}(\sin(B + C) + \sin C).$$

Using $\sin(B + C) = \sin A$, from the above, we get

$$\sin A\,(\sin\frac{C}{2} - \sin\frac{B}{2}) = \sin\frac{B}{2}\sin C - \sin\frac{C}{2}\sin B$$
$$= 2\sin\frac{B}{2}\sin\frac{C}{2}(\cos\frac{C}{2} - \cos\frac{B}{2}).$$

If $B \neq C$, then $(\sin\frac{C}{2} - \sin\frac{B}{2})$ and $(\cos\frac{C}{2} - \cos\frac{B}{2})$ are of opposite sign, which is impossible. Therefore $B = C$ and $\triangle ABC$ is an isosceles triangle. ∎

We provide another proof which is similar to the above one by direct computation. We first need a lemma.

Lemma 1

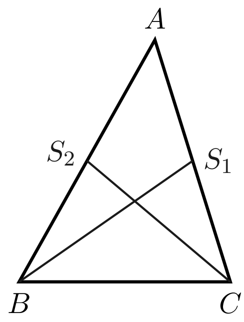
Let AD be the angle besector of $\angle A$ of $\triangle ABC$. Then

$$AD = \frac{2AB \cdot AC \cdot \cos A/2}{AB + AC}.$$

This is a well-known fact in geometry so we omit the proof.

Proof by Direct Computation. Let $BC = a, CA = b, AB = c$. Then by the above lemma, we know that

$$BS_1 = \frac{2ca \cdot \cos B/2}{a+c}, \quad CS_2 = \frac{2ba \cdot \cos C/2}{a+b}.$$



Let $BS_1 = CS_2 = k$. Then we have

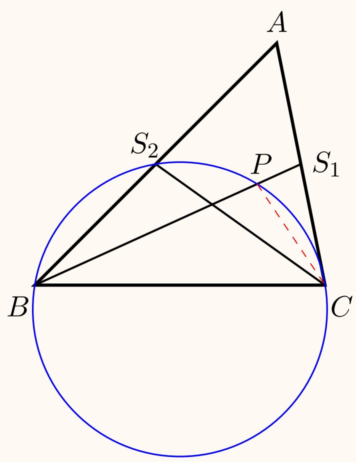
$$\cos\frac{B}{2} - \cos\frac{C}{2} = k\left(\frac{a+c}{2ac} - \frac{a+b}{2ab}\right) = \frac{k}{2}\left(\frac{1}{c} - \frac{1}{b}\right).$$

We can use the similar argument as in the previous proof: if $B > C$, then the above left side is negative but the right side is positive; on the other hand, if $B < C$, then the above left side is positive but the right side is negative. Therefore we have $b = c$, and the triangle is isosceles. ■

Our second proof depends on the following result.

Theorem 5

Assume that $AB > AC$ in the following $\triangle ABC$. Let BS_1, CS_2 be the angle bisectors of $\angle B, \angle C$, respectively. Then $BS_1 > CS_2$.



Proof. Let the circumcircle of $\triangle S_2BC$ intersect BS_1 at P. Since $AB > AC$, we have $\angle C > \angle B$. Thus from the fact that $\angle PCS_2 = \angle ABS_2 = \frac{1}{2}\angle B < \frac{1}{2}\angle C =$

$\angle ACS_2$, we know that P is *between* B and S_1. Thus $BS_1 > BP$. Moreover, since

$$\angle PCB = \angle PCS_2 + \angle S_2CB = \angle ABS_1 + \angle S_2CB > \angle B,$$

we have $BP > CS_2$. Therefore $BS_1 > CS_2$. ■

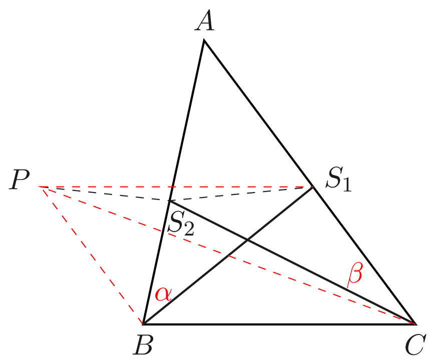
Using the above theorem, we can easily provide another proof of the Steiner-Lehums' Theorem.

Proof by Contrapositive. By the above theorem, if $AB > AC$, then $BS_1 > CS_2$; and if $AB < AC$, then $BS_1 < CS_2$. In either cases, we get contradictions. Therefore $AB = AC$, and the triangle is isosceles. ■

Finally, we give a direct and elementary geometric proof of the Steiner-Lehmus Theorem.

Basic Geometric Proof. In the below picture, let $\angle ABS_1 = \angle S_1BC = \alpha$ and let $\angle ACS_2 = \angle S_2CB = \beta$.

We draw PS_1 so that $\angle PS_1B = \beta$ and $PS_1 = BC$. Since $BS_1 = S_2C$ by assumption, we have $\triangle PS_1B \cong BCS_2$.



We can get two conclusions from the the congruency of two triangles. First, we would have

$$\angle PBS_1 = \angle BS_2C.$$

Second, we have

$$PB = BS_2.$$

From the first conclusion, we have

$$\angle PBC = \angle PBS_1 + \angle S_1BC = \angle BS_2C + \angle S_1BC = 180° - \alpha - \beta.$$

Thus

$$\angle PBC = \angle CS_1P. \tag{1}$$

We consider triangles $\triangle PBC$ and $\triangle CS_1P$. By assumption, we have (1); $PS_1 = CB$, and $BC = CB$ is the common side. In general, this doesn't imply that these two triangles are congruent. However, we note that $\angle PBC = 180° - \alpha - \beta =$

$90^\circ + \angle A/2 > 90^\circ$ is an obtuse angle. Thus $\triangle PBC \cong \triangle CS_1P$ which implies $PB = CS_1$ and $PBCS_1$ is a parallelogram. Therefore $PS_1 \parallel BC$ and hence $\alpha = \beta$, completing the proof of the theorem. ■

Remark Note that in the above picture, P, S_2, S_1 are actually collinear.