TWENTY DEGREE PROBLEMS

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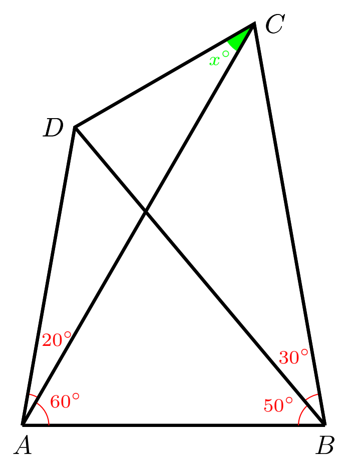
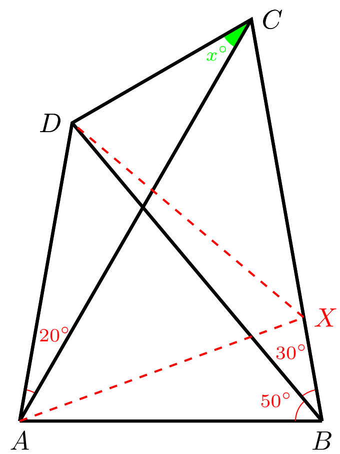
In Euclidean Geometry, there are problems related to 20° degree angle. The best way to solve these problems is to use trigonometry. If we would like to use purely geometric method, the key is to construct an equilateral triangle. See Mind Your Decisions.

We start with the famous "world's hardest geometry problem".

1. The world's hardest geometry problem. In the following digram, assume that $\angle DAC = 20^\circ, \angle CAB = 60^\circ, \angle ABD = 50^\circ$, and $\angle CBD = 30^\circ$. Prove that $\angle ACD = 30^\circ$.



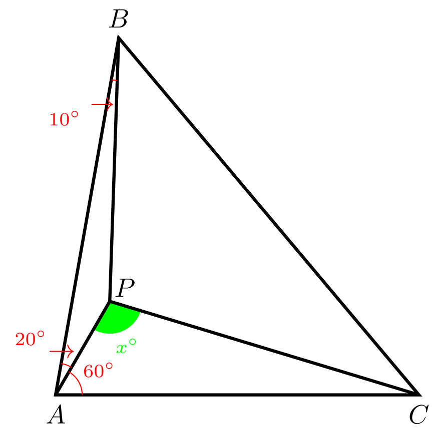
Solution: Let X be a point on BC such that $\angle XAB = 20^\circ$. Connect XD. By assumption and construction, both ΔABD, ΔABX, and ΔXAC are isosceles triangles. Thus $DA = BA = XA$. Since $\angle DAX = 60^\circ$. Thus ΔADX is an equilateral triangle. Thus $DX = CX$, and ΔXDC is an isosceles triangle. Thus $\angle DCX = 70^\circ$. Thus $x^\circ = 30^\circ$.



C
x°
D
X
20°
30°
50°
A
B

□

2. Variation of the problem. In the following diagram, $AB = AC$. Assume that $\angle PAC = 60°, \angle BAP = 20°$, and $\angle PBA = 10°$. Prove that $\angle APC = 100°$.

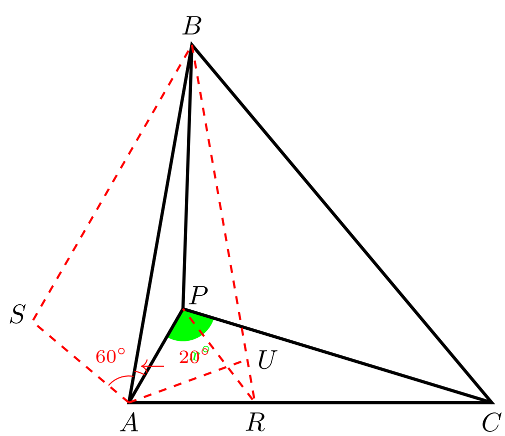


Solution: We rotate ΔPAC counter clock-wisely of degree $80°$ with respect to A. The new triangle is ΔSAB, which is obviously congruent to ΔPAC.

Draw BR with $\angle PBR = 10°$ intersects AC to R. Connect PR.

Let U be a point on BR such that $\angle UAR = 20°$.

By construction, we can prove that ΔPAR is an equilateral triangle. Thus ΔSAB is congruent to ΔUAB. Thus $\angle SBA = \angle ABR = 20°$. Thus $x° = \angle S = 100°$.



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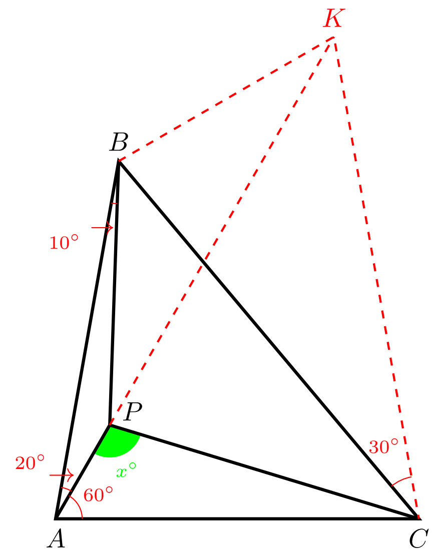
What is the relationship between Problem 1 and Problem 2?

Solution: We draw KC such that $\angle KCB = 30°$ intersecting with CP at K. Connect KB, KP.

It is not hard to compute that $\angle AKC = 40° = \angle PBC$. Thus K, BP, C are con-cycle. As a result, $\angle BCP = \angle BKP = 30°$ by Problem 1.

So $\angle PCA = 20°$ and $x = 100°$.



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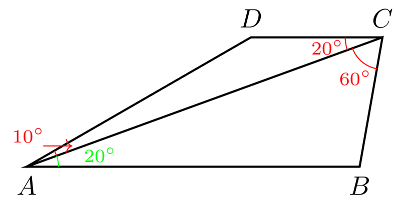
3. Another variation of the problem. In the quadrilateral $ABCD$, $DC = CB$. Assume that $\angle DAC = 10°$, $\angle ACD = 20°$, and $\angle ACB = 60°$. Prove that $\angle CAB = 20°$.



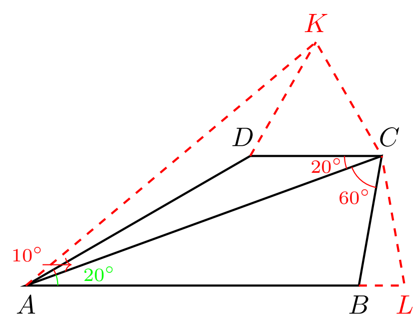
Solution: Draw AK such that $\angle KAD = 10°$ and $KA = CA$. Connect KD and KC.

Extend AB to L such that $AL = AC$.

By the construction, we conclude that ΔADK is congruent to ΔADC. Then $KD = CD$.

On the other hand, ΔACK is congruent to ΔACL. Thus $KC = LC$.

Thus $DC = CB = CL = CK$. So ΔKDC is an equilateral triangle. So $\angle KDC = 60°$. It is not difficult to compute that $\angle DCA = 20°$. This completes the proof.



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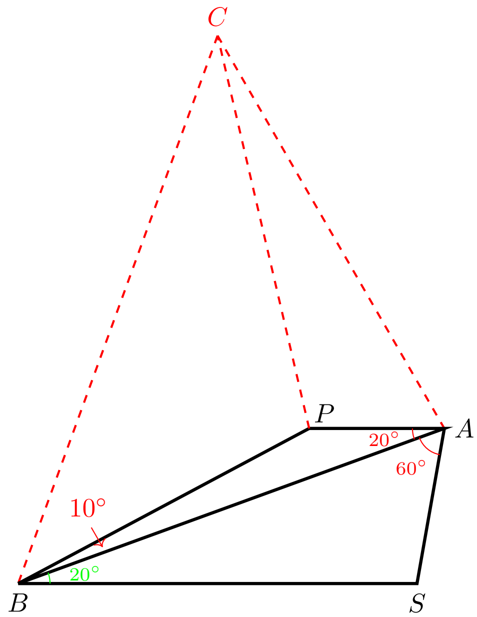
What is the relationship between Problem 3 and Problem 2?

Solution: The relation between Problem 3 and Problem 2 can be showed as follows.



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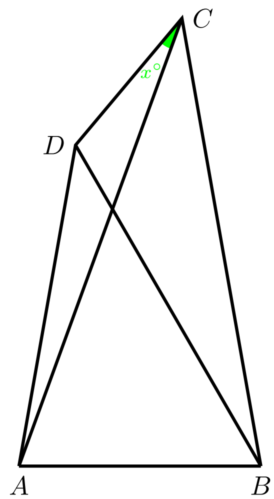
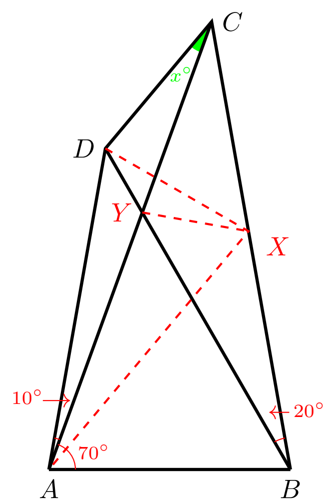
4. In the following digram, assume that $\angle DAC = 20°, \angle CAB = 60°, \angle ABD = 50°$, and $\angle CBD = 30°$. Prove that $\angle ACD = 20°$.



Solution: Let $\angle XAB = 50°$. Connect XY, XD.

Since $\angle AXB = 50° = \angle AYB$, X, Y, A, B are con-cyclic. Thus $\angle XYB = \angle XAB = 50°$.

By the result of Problem 1, we have $\angle XDY = 30° = \angle ACB$. Thus C, D, Y, X are con-cyclic. As a result, $x = \angle DXY = \angle XYB - \angle XDY = 50° - 30° = 20°$.



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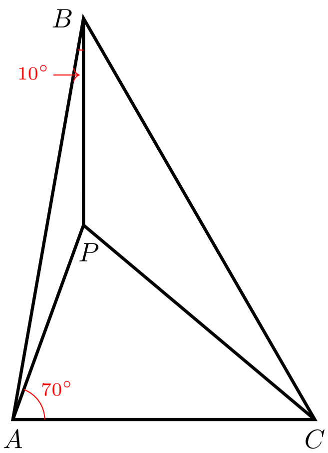
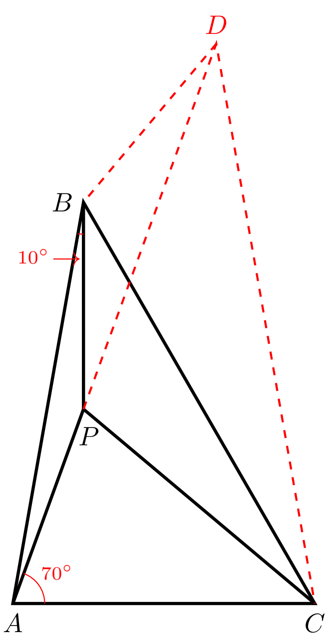
5. In the following triangle, assume that $\angle A = 80°$; $\angle B = 40°$, and $\angle C = 60°$. Assume that $\angle ABP = \angle BAP = 10°$. Prove that $\angle APC = 70°$.



Solution: We extend AP to D such that $\angle DCA = 80°$. Connect DB.

By the assumption, we have $\angle ADC = 30° = \angle PBC$. Thus D, B, P, C are con-cyclic. Thus by the result in Problem 4, we have $\angle BCP = \angle BDP = 20°$. Thus $\angle PCA = 40°$. So $\angle APC = 70°$.



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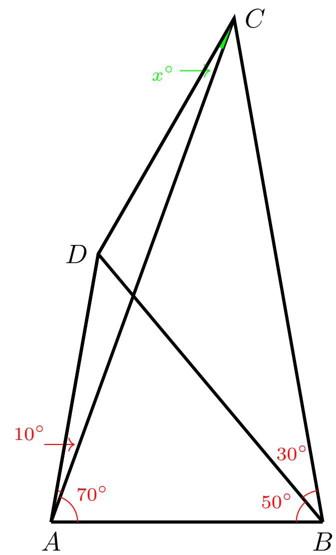
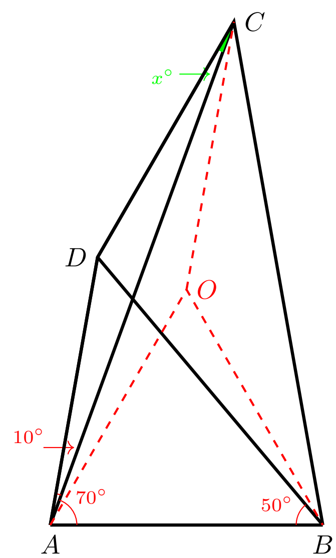
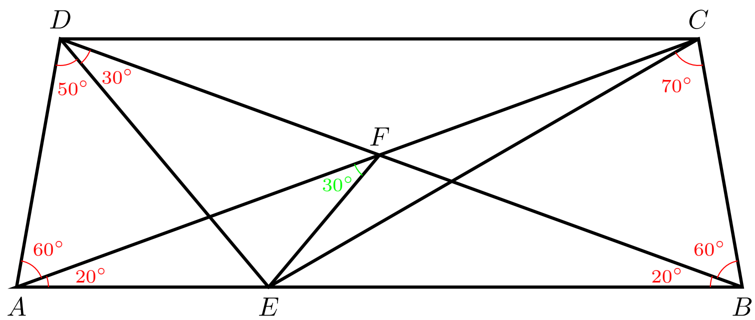
6. In the following digram, assume that $\angle DAC = 10°, \angle CAB = 70°, \angle ABD = 50°$, and $\angle CBD = 30°$. Prove that $\angle ACD = 10°$.

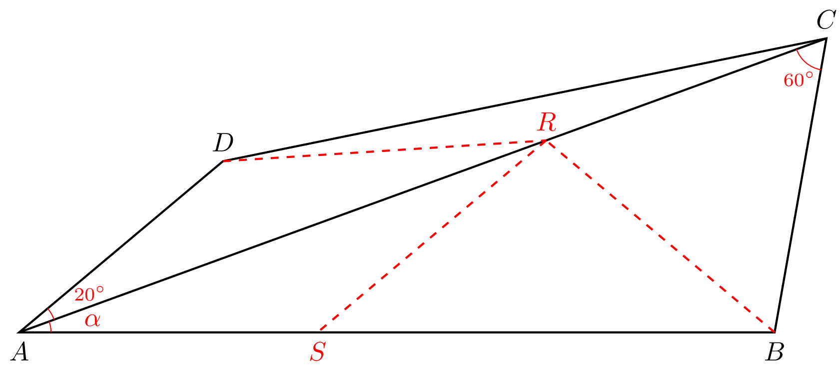




Solution:

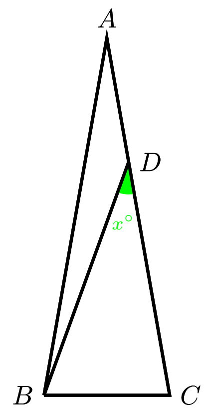
□


D
C
50°
30°
70°
F
30°
60°
60°
20°
20°
A
E
B

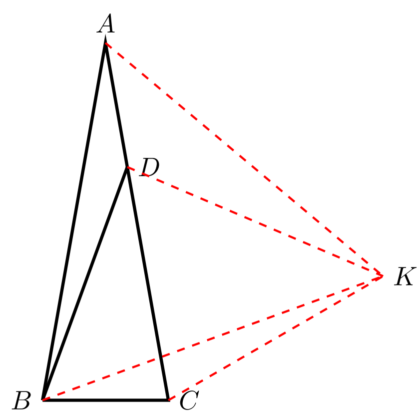


C
60°
D
R
20°
α
A
S
B

7. Assume that ΔABC is an isosceles trangle, $AB = AC$. Let D be a point on AC such that $AD = BC$. Let $\angle A = 20°$. Prove that $\angle BDC = 30°$.



Solution: Draw a line BK such that $\angle KBC = 20°$ and $BK = AB$. Connect KA, KD, KC.
Since $\angle A = 20°$ and since ΔABC is an isosceles triangle, we know that $\angle ABC = 80°$. therefore $\angle ABK = 60°$ by the construction.



Since $AD = BC$, $BK = AB$, $AK = BK$, and $\angle A = \angle KBC = 20°$, we have

$$\Delta ABD \cong \Delta KBC.$$

So in particular, $\angle KCB = \angle BDA$. Let's now compute $\angle KCB$.
Since $\angle KAB = 60°$, we have $\angle KAC = 40°$. Since $AK = AB = AC$, ΔACK is an isosceles triangle, thus $\angle ACK = 70°$. Therefore $\angle KCB = 70° + 80° = 150°$. So $\angle BDA = \angle KCB = 150°$, and therefore $\angle BDC = 30°$.

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Solution: By Law of Sine we have

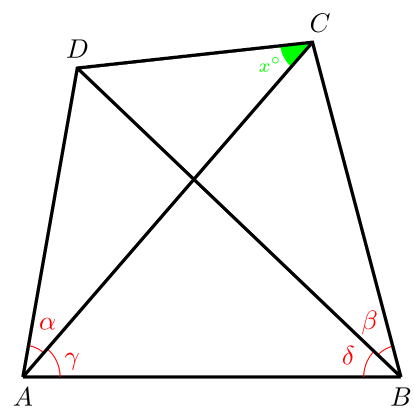
$$\frac{AC}{\sin(x+\alpha)} = \frac{AD}{\sin x}, \qquad \frac{AD}{\sin\delta} = \frac{AB}{\sin(\alpha+\gamma+\delta)}, \qquad \frac{AC}{\sin(\beta+\delta)} = \frac{AB}{\sin(\gamma+\beta+\delta)}.$$

We therefore have

$$\frac{\sin(\alpha+x)}{\sin x} = \frac{\sin(\beta+\delta)\sin(\alpha+\gamma+\delta)}{\sin\delta\sin(\gamma+\beta+\delta)}.$$

We therefore have

$$\cot x = \frac{\sin(\beta+\delta)\sin(\alpha+\gamma+\delta)}{\sin\delta\sin(\gamma+\beta+\delta)\sin\alpha} - \cot\alpha$$



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Solution: By Law of Sine we have

$$\frac{AC}{\sin(x+\alpha)} = \frac{AD}{\sin x}, \qquad \frac{AD}{\sin(80-\beta)} = \frac{AB}{\sin(160-\beta)}, \qquad \frac{AC}{\sin(80)} = \frac{AB}{\sin(160-\alpha)}.$$

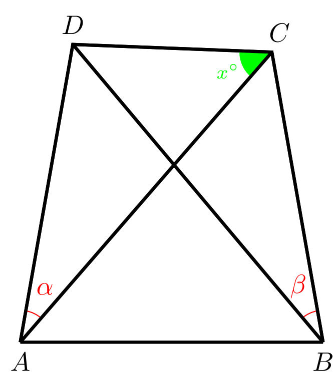
We therefore have

$$\frac{\sin(\alpha+x)}{\sin x} = \frac{\sin 80 \sin(160-\beta)}{\sin(160-\alpha)\sin(80-\beta)}.$$

We therefore have

$$\cot x = \frac{\sin 80 \sin(160-\beta)}{\sin(160-\alpha)\sin(80-\beta)\sin\alpha} - \cot\alpha$$

$$= \frac{\sin 80 \sin(20+\beta)}{\sin(20+\alpha)\cos(10+\beta)\sin\alpha} - \cot\alpha$$



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