Isogonal Conjugate and Isotomic Conjugate Points

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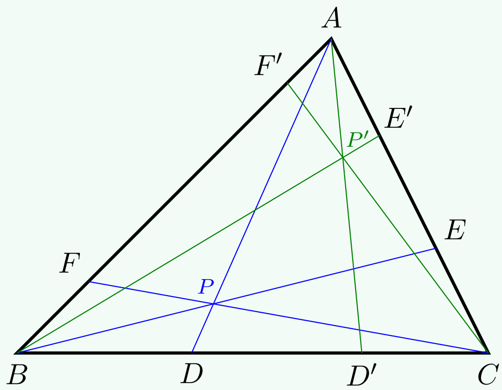
1 Introduction

The *isogonal conjugate points* and *isotomic conjugate points* are the two important concepts in triangle geometry.

Definition 1. (Isogonal Conjugate Points)

Let P be any point. Assume that AP intersects BC at D; BP intersects CA at E; and CP intersects AB at F. The line AD' is called the isogonal conjugate line of AD, if $\angle CAD' = \angle BAD$. Let BE' and CF' be the corresponding isogonal conjugate lines similarly defined. Then AD', BE', CF' are concurrent at a point P', which is called the isogonal conjugate point of P.

Isogonal points are reflexive, that is, if P' is the isogonal conjugate point of P, then P is the isogonal conjugate point of P'.



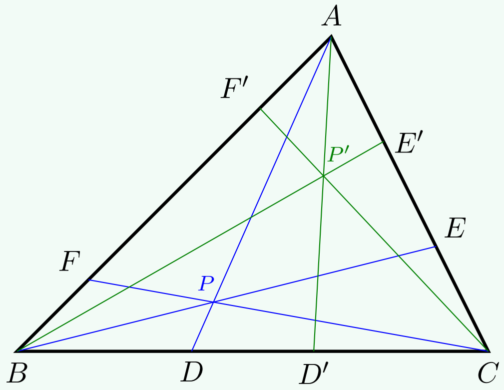
There are a lot of examples of isogonal points in triangle. The isogonal conjugate point of the incenter and the excenters are themselves. The orthocenter and circumcenter are a pair of isogonal conjugate points (see § 3 for details).

Similar to the concept of isogonal conjugate points, we have the following

Definition 2. (Isotomic Conjugate Points)

Let P be any point. Assume that AP intersects BC at D; BP intersects CA at E; and CP intersects AB at F. The line AD' is called the isotomic conjugate line of AD, if $BD = D'C$. Let BE' and CF' be the corresponding isotomic conjugate lines similarly defined. Then AD', BE', CF' are concurrent at P'. P' is called the isotomic conjugate *point of P.*

Isotomic points are reflexive, that is, if P' is the isotomic conjugate point of P, then P is the isotomic conjugate point of P'.



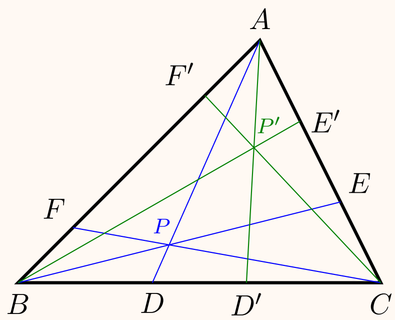
The isotomic conjugate point of the centroid is itself. More examples of isotomic conjugate points can be found in §3.

2 Existence and Basic Properties of the Conjugate Points

We will prove that the isogonal or isotomic conjugate lines are concurrent using the Ceva's Theorem. We begin with the following theorems.

Theorem 1

Assume that AD, BE, CF are concurrent at P. Then their isotomic conjugate lines AD', BE', CF' are concurrent at a point P'.



Proof By definition of the isotomic conjugate lines, we have

$$\frac{BD'}{D'C} = \frac{DC}{BD} = \left(\frac{BD}{DC}\right)^{-1}.$$

Similarly, we have

$$\frac{CE'}{E'A} = \left(\frac{CE}{EA}\right)^{-1}, \qquad \frac{AF'}{F'B} = \left(\frac{AF}{FB}\right)^{-1}.$$

Thus we have

$$\frac{BD'}{D'C}\cdot\frac{CE'}{E'A}\cdot\frac{AF'}{F'B}=\left(\frac{BD}{DC}\right)^{-1}\cdot\left(\frac{CE}{EA}\right)^{-1}\cdot\left(\frac{AF}{FB}\right)^{-1}=1,$$

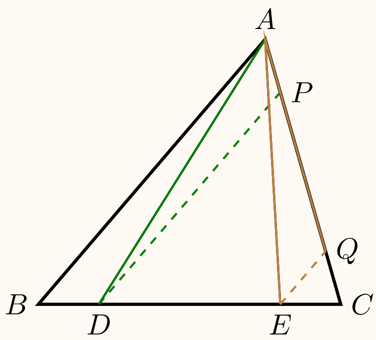
and hence AD', BE', CF' are concurrent. Here we used the Ceva's Theorem and its converse. ■

To use the same method to prove the existence of the isogonal conjugate point, we need the following generalization of the Angel Bisector Theorem, which is interesting by itself.

Theorem 2

In the following picture, assume that $\angle BAD=\angle EAC$. Prove that

$$\frac{BD}{DC}\cdot\frac{BE}{EC}=\left(\frac{AB}{AC}\right)^2.$$



In particular, if AD is the angle bisector of $\angle A$, then the theorem is reduced to the Angel Bisector Theorem.

Proof The easiest way to prove the result is to use the Law of Sines. But in what follows, we provide a pure geometric proof.

Draw $DP \parallel EQ \parallel BA$ interesting on AC on P and Q, respectively. We have $\angle ADP=\angle BAD=\angle EAQ$ and $\angle DAP=\angle BAE=\angle AEQ$. Thus $\triangle ADP\sim\triangle EAQ$. As a result,

$$\frac{AP}{EQ}=\frac{DP}{AQ}.$$

We therefore have

$$\frac{BD}{DC}\cdot\frac{BE}{EC}=\frac{AP\cdot AQ}{DC\cdot EC}=\frac{EQ\cdot DP}{DC\cdot EC}.$$

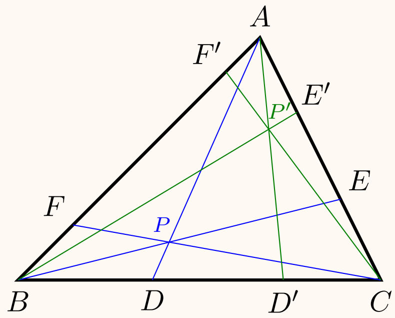
But

$$\frac{EQ}{EC}=\frac{AB}{AC},\qquad\frac{DP}{DC}=\frac{AB}{AC}.$$

This completes the proof. ■

Theorem 3

Assume that AD, BE, CF are concurrent at P. Then their isogonal conjugate lines AD', BE', CF' are concurrent at a point P'.



Proof By Theorem 2,

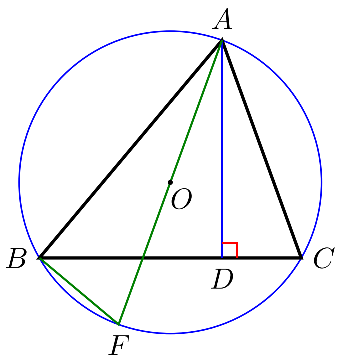
$$\frac{BD}{DC}\cdot\frac{BD'}{D'C}=\left(\frac{AB}{AC}\right)^2,\quad \frac{CE}{EA}\cdot\frac{CE'}{E'A}=\left(\frac{BC}{CA}\right)^2,\quad \frac{AF}{FB}\cdot\frac{AF'}{F'B}=\left(\frac{CA}{BC}\right)^2.$$

The result then follows from the Ceva's Theorem, similar to the proof of the previous theorem. ■

Example 1 (Typical Isogonal Lines) In the following picture, AF is a diameter of the circle (where O is the circumcenter). $AD \perp BC$. Then since $\angle BFA = \angle BCA$, we have

$$\angle BAF = \angle DAC.$$

Thus AD and AF are isogonal lines.



Based on the above Example 1, we can give the second proof of the above theorem using the Carnot's Theorem.

Second Proof In the following picture, let X, Y, Z be the projections of P to BC, CA, AB, respectively. The $\triangle XYZ$ is called the *pedal triangle* (see here for more details of pedal triangles).

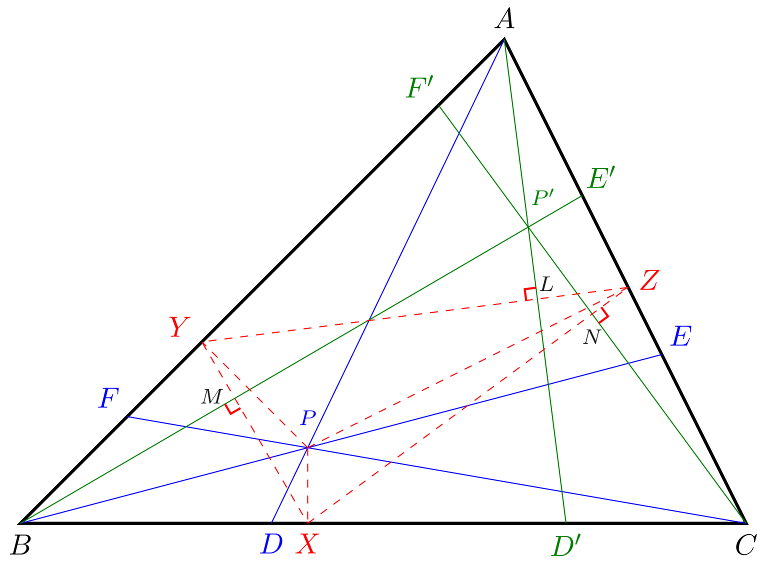
The key observation here is that the isogonal conjugate lines AD', BE', CF' are perpendicular to the corresponding sides of the pedal triangle $\triangle XYZ$. This can be proved using the following argument: since $PY \perp AB, PZ \perp AC$, $AYPZ$ is concyclic. Thus

$$\angle LAZ + \angle AZY = \angle YAP + \angle APY = 90^\circ.$$

Thus $AD' \perp ZY$. Similarly, $BE' \perp XY$, and $CF' \perp AB$.

By the Carnot's Theorem (see also Topic 35), we know that the three green lines AD', BE', CF' are concurrent if

$$XL^2 - LZ^2 + ZN^2 - NX^2 + XM^2 - MY^2 = 0. \tag{1}$$



However, we have

$$YL^2 - LZ^2 = AY^2 - AZ^2,$$
$$ZN^2 - NX^2 = CZ^2 - CX^2,$$
$$XM^2 - MY^2 = BX^2 - BY^2.$$

Therefore, Equation (1) is valid if and only if

$$AY^2 - BY^2 + BD^2 - CD^2 + CE^2 - AE^2 = 0,$$

but this follows from the Carnot's Theorem again and the fact that PX, PY, PZ are concurrent.

3 Examples of Isogonal Conjugate and Isotomic Conjugate Points

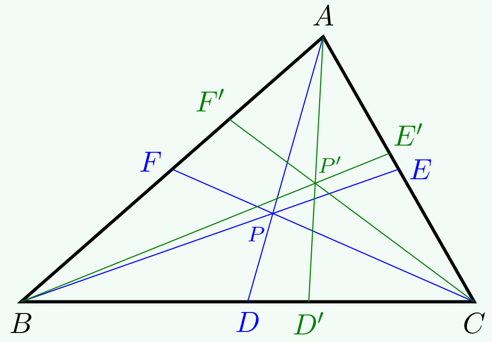
As we mentioned before, incenter and excenters are self isogonal, and centroid is self isotomic. In this section, we provide more examples.

By Example 1, we know that the orthocenter and the circumcenter are a pair of isogonal conjugate points. The orthocenter and the circumcenter also give the prototype in the second proof of Theorem 3.

The second example of isogonal conjugate points is showing below.

Definition 3. (Centroid and Symmedian Point)

The isogonal conjugate point of centroid is called the symmedian point. In the following $\triangle ABC$, P is the centroid and P' is the symmedian point.



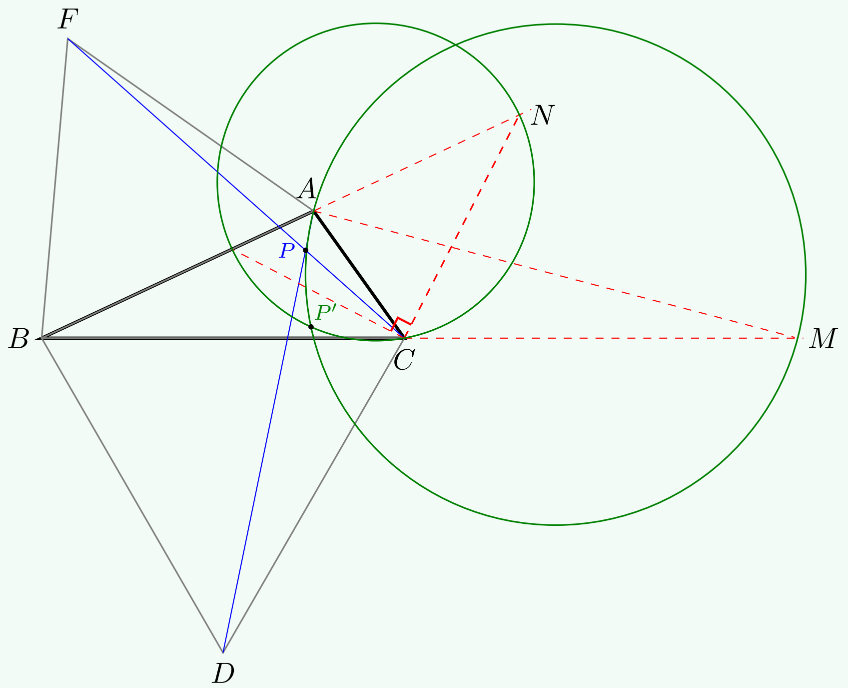
Symmedian point received a lot of attention in triangle geometry. For more details of Symmedian point, see Wikipedia or Topic 16.

A more exotic pair of isogonal point is given by the *First Isogonal Center* and the *Isodynamic Point*.

Definition 4. (First Isogonal Center and First Isodynamic Point)

In $\triangle ABC$, we make three (outside) equilateral triangles by the three sides. The circumcircle of these three triangles must be concurrent at the point called the first isogonal center. In the following picture, we only draw two triangles $\triangle BFA$ and $\triangle BDC$, they are the equilateral triangles of AB and BC, then the intersection of AD and CF is the first isogonal center P.

Isodynamic point is the points of the intersection of three circles whose diameter are the point where the bisectors of the internal and external angles of $\triangle ABC$ intersect (extensions) on the opposite side. Usually there are two isodynamic points, the one P', in the following picture, is the isogoanl conjugate point of P, which is called the first isodynamic point.



Additionally, the first isodynamic point also satisfy the relationship:

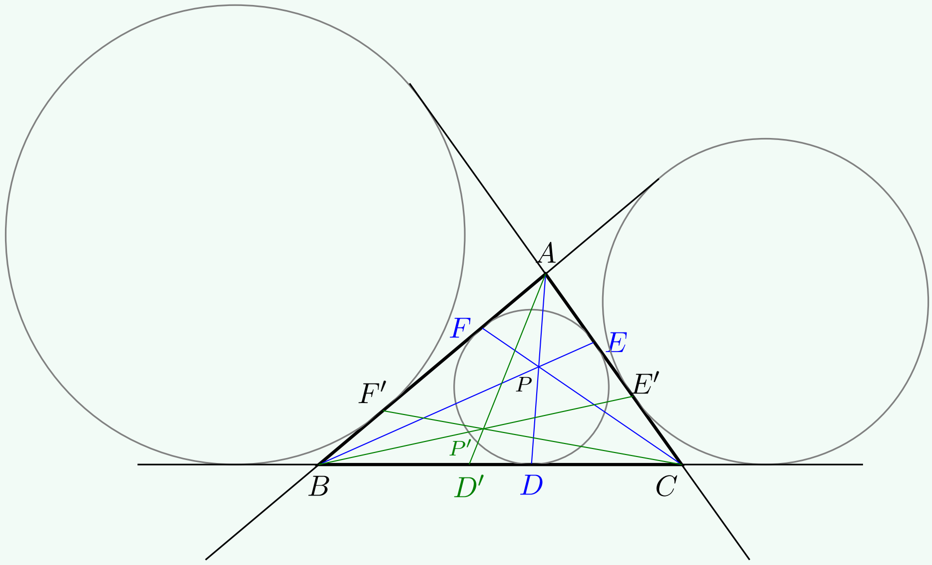
$$P'A \cdot BC = P'B \cdot CA = P'C \cdot AB.$$

For the second isodynamic point, which is the other intersection of three circles. And for more information about second isodynamic point , see Topic 33.

The typical example of isotomic conjugate points are given by the *Gergonne Point* and the *Nagel Point*.

Definition 5. (Gergonne Point and Nagel Point)

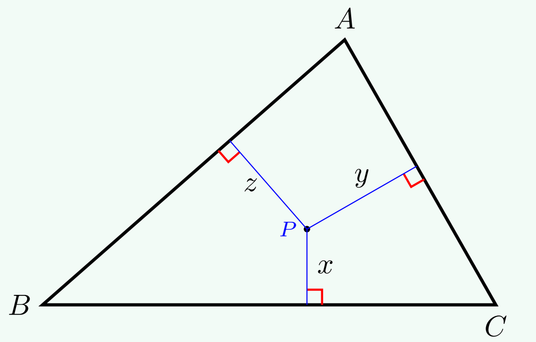
In the following $\triangle ABC$, let O be the incircle, which is tangent to BC at D; CA at E; AB at F. AD, BE, CF must concurrent at point P, which is the gergonne point. Let J_A, J_B and J_C be the external tangent circles, J_A is tangent to BC at D'; J_B is tangent to AC at E'; J_C is tangent to AB at F'. AD', BE' and CF' must concurrent at point P', which is the nagel point, and P' is the isotomic conjugate point of P.



4 Basic Properties

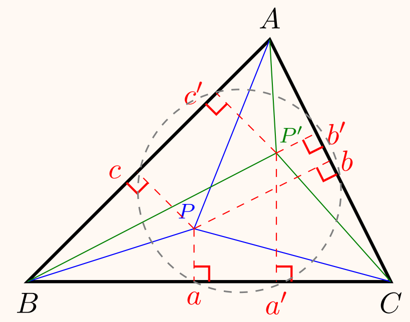
Definition 6. (Trilinear Coordinates)

Given a triangle $\triangle ABC$, we can define a useful coordinate system called trilinear coordinate system to the Euclidean plane – with respect to the triangle. Let P be a point inside $\triangle ABC$, then the trilinear coordinates of P is given by the ratios of its distances to the three sides. In the following picture, let P be a point inside $\triangle ABC$ and let x, y, z be the distances of P to the sides BC, CA, AB, respectively. Then the trilinear coordinates of P is (x, y, z), or $x : y : z$. For more information, see Topic 37.



Theorem 4

In the following $\triangle ABC$, P' is the isogonal conjugate point of P, and then the trilinear coordinates of P and P' inversely proportional to each other. Point a, a', b, b', c, c' are con-cyclic.



Proof. In $\triangle ABC$, $\triangle PaC \sim \triangle P'b'C$ and $\triangle PbC \sim \triangle P'a'C$. By this, We have

$$\frac{Pa}{P'b'} = \frac{PC}{P'C}, \quad \frac{Pb}{P'a'} = \frac{PC}{P'C}.$$

Thus

$$\frac{Pa}{P'b'} = \frac{Pb}{P'a'},$$

so that

$$Pa : Pb = \frac{1}{P'a'} : \frac{1}{P'b'}.$$

Similarly, we will get

$$Pb : Pc = \frac{1}{P'b'} : \frac{1}{P'c'}.$$

Therefore,

$$Pa : Pb : Pc = \frac{1}{P'a'} : \frac{1}{P'b'} : \frac{1}{P'c'}.$$

Or

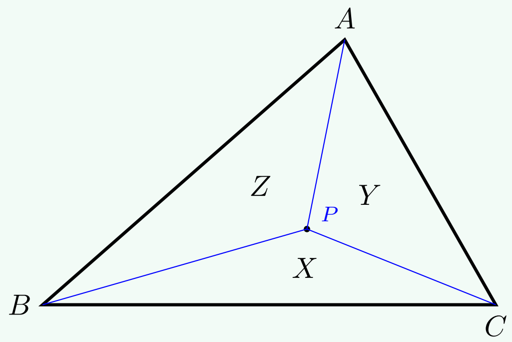
$$Pa \cdot P'a' = Pb \cdot P'b' = Pc \cdot P'c'.$$

Finally, we get that if the coordinate of P is $(x : y : z)$, then the coordinate of P' is $(x' : y' : z')$. ■

With the theorem of trilinear coordinates, there is an extended theorem called Barycentric Coordinate.

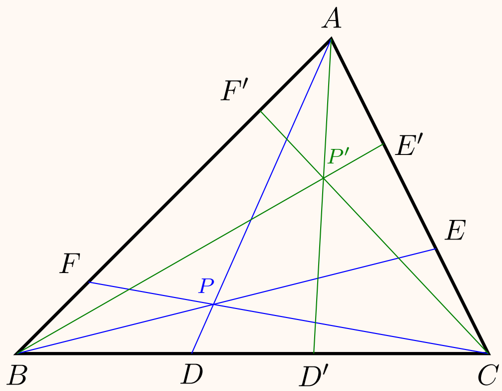
Definition 7. (Barycentric Coordinate System)

Barycentric coordinates can be used to express the position of any point located on the triangle with three scalars. The location of this point includes any position inside the triangle, any position on any of the three edges of the triangles, or any one of the three triangle's vertices themselves. To compute the position of this point using barycentric coordinates we use the following equation $P = YA + ZB + XC$, where A, B and C are the vertices of a triangle and Y, Z, and X (the barycentric coordinates), three real numbers (scalars) such that $Y + Z + X = 1$ (barycentric coordinates are normalized).



Theorem 5

The barycentric coordinate of P and P' are inversely proportional.



Proof. We use $|\triangle XYZ|$ to denote the area of $\triangle XYZ$. We then have

$$\frac{|\triangle EBC|}{|\triangle EBA|} = \frac{|\triangle EPC|}{|\triangle EPA|} = \frac{CE}{EA}.$$

Thus, we have

$$\frac{|\triangle PBC|}{|\triangle PAB|} = \frac{CE}{EA}.$$

Similarly, we have

$$\frac{|\triangle P'BC|}{|\triangle P'AB|} = \frac{CE'}{E'A}.$$

Since $CE = AE'$, we have

$$\frac{CE}{EA} = \left(\frac{CE'}{E'A}\right)^{-1}.$$

Therefore, we have

$$\frac{|\triangle PBC|}{|\triangle PAB|} = \frac{(|\triangle P'BC|)^{-1}}{(|\triangle P'AB|)^{-1}}.$$

Applying the above argument to the other sides of $\triangle ABC$, we get

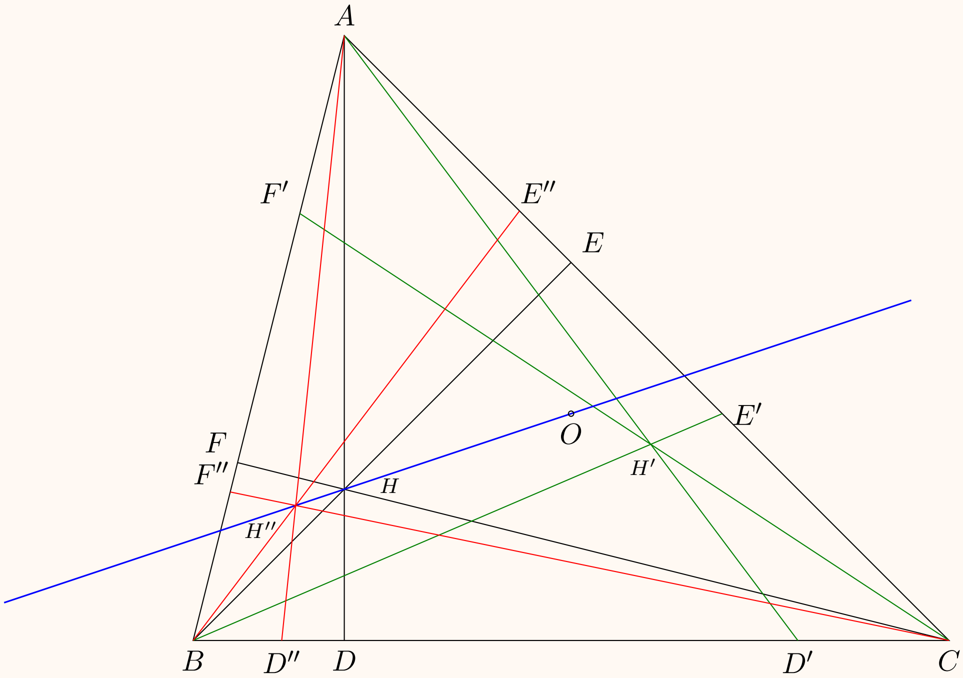
$$|\triangle PBC| : |\triangle PCA| : |\triangle PAB| = (|\triangle P'BC|)^{-1} : (|\triangle P'CA|)^{-1} : (|\triangle P'AB|)^{-1}$$

and the result is proved. ∎

5 An Intrigue Example

Theorem 6

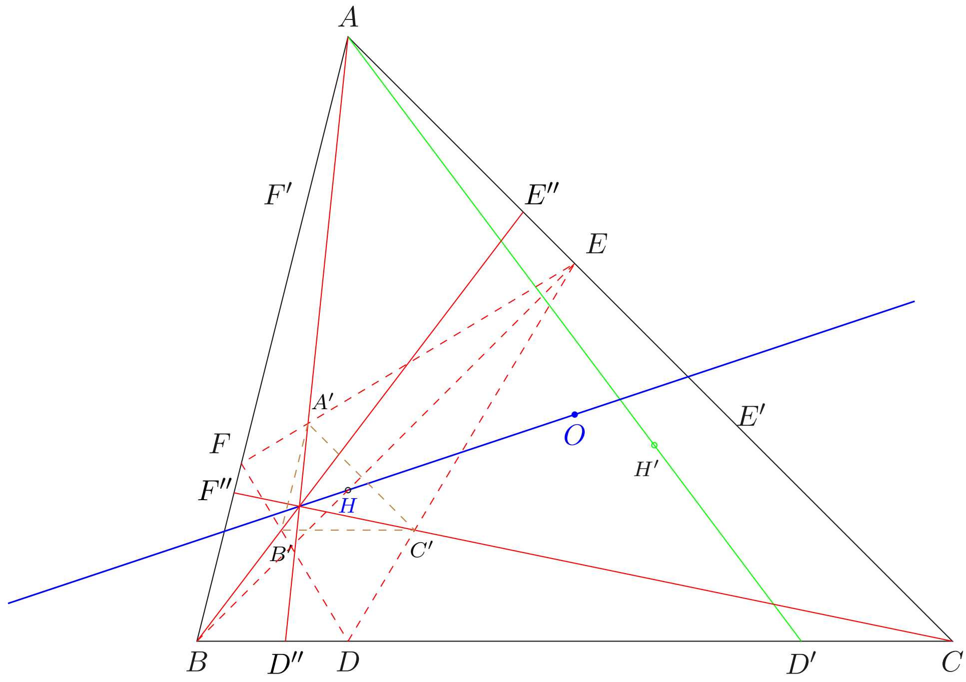
The isogonal conjugate point of the isotomic conjugate point of the orthocenter of a triangle is on its Euler line.



In the above picture, AD, BE, CF are the heights of the triangle $\triangle ABC$ and H is the orthocenter. The green lines AD', BE', CF' are the isotomic lines of AD, BE, CF, respectively and they are concurrent at H', the isotomic conjugate point of H. The red lines AD'', BE'', CF'' are the isogonal lines of AD', BE', CF', respectively and they are concurrent at H'', the isogonal conjugate point of H'. The blue line is the Euler line of the triangle, where O is the enter of the circumscribed circle. The theorem asserts that O, H, H'' are collinear.

Proof: Let AD'' and EF intersect at A', BE'' and DF intersect at B', and CF'' and DE intersect at C'. In order to prove the theorem, we need to prove

1. $\triangle ABC$ is homothetic to $\triangle A'B'C'$, with the homothetic center H''.
2. H is the circumcenter of $\triangle A'B'C'$.



The key observation is that $\triangle AA'E \sim \triangle AD'B$: we have $\angle AEA' = \angle ABD'$ and $\angle A'AE = D'AB$. By this, we have

$$\frac{E'A}{BD'} = \frac{AE}{AB} = \cos A.$$

Thus $E'A = BD' \cos A = DC \cos A = AC \cos C \cos A$.

Similarly, $EC' = AC \cos C \cos A = EA'$. Since $\angle FEB = \angle BED$, we know that BE perpendicular bisects $A'C'$. Thus H is the circumcenter of $\triangle A'B'C'$. Moreover, $A'B' \parallel AB$. Thus $\triangle ABC$ and $\triangle A'B'C'$ are homothetic.

Therefore, O, H, H'' are collinear.

Using the trilinear and barycentric coordinate systems, we are able to give the above result an algebraic proof.

Second Proof From Wikipedia or Topic 37, we know that the trilinear coordinates for H is $\sec A : \sec B : \sec C$. Therefore, the barycentric of H is $\tan A : \tan B : \tan C$; its isotomic conjugate point H' has the barycentric coordinates $\cot A : \cot B : \cot C$, which is equvalent to its trilinear coordinates

$$\frac{\cos A}{\sin^2 A} : \frac{\cos B}{\sin^2 B} : \frac{\cos C}{\sin^2 C}.$$

As a result, the trilinear coordinates of its isogonal conjugate point H'' has the trinlinear coordinates

$$\frac{\sin^2 A}{\cos A} : \frac{\sin^2 B}{\cos B} : \frac{\sin^2 C}{\cos C}.$$

Since the circumcenter O has the trilinear coordinates $\cos A : \cos B : \cos C$, we are able to prove that H, O, H'' are collinear by the obvious equation

$$\det \begin{bmatrix} \sec A & \sec B & \sec C \\ \cos A & \cos B & \cos C \\ \frac{\sin^2 A}{\cos A} & \frac{\sin^2 B}{\cos B} & \frac{\sin^2 C}{\cos C} \end{bmatrix} = 0.$$