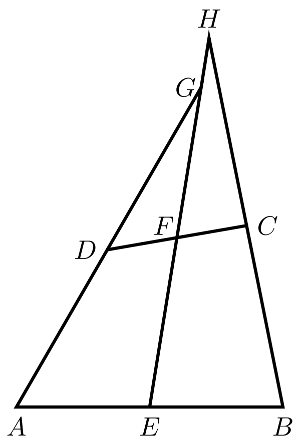
THE USAGE OF SPECIAL TECHNIQUES

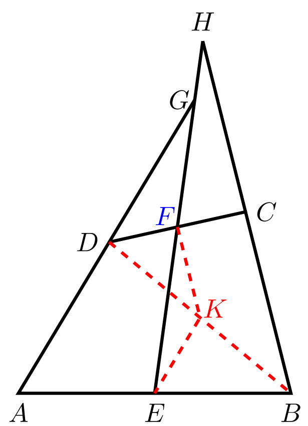
ZHIQIN LU

In this article, we use an example in the classical book of Chunfang Xu (许莼舫) [1] to show that different methods can be used in plane geometry. I added two new methods for this problem.

Problem. *In the quadrilateral $ABCD$, assume that $AD = BC$. Assume that E, F are the midpoints of AB and CD, respectively. Prove $\angle AGE = \angle H$.*

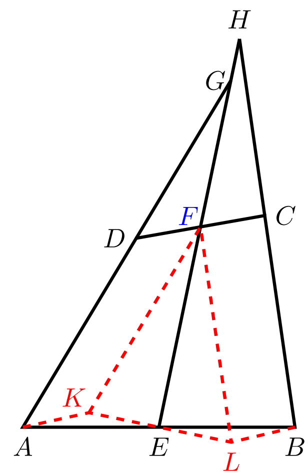


Solution: (Construction of Isosceles Triangle.) Connect DB and let K be the midpoint of DB. Connect KE, KF. We can prove that $\triangle KEF$ is an isosceles triangle. Thus $\angle H = \angle EFK = \angle FEK = \angle AGE$.



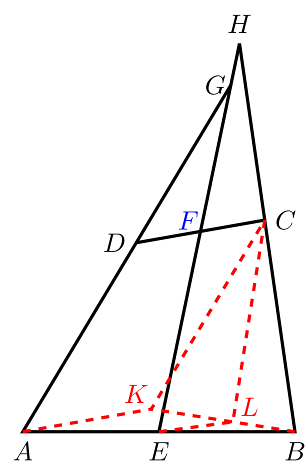
$\square$

Solution: (Translation Method (1).) We define points K and L such that $DFKA$ and $FCBL$ are parallelograms. Thus $AKBL$ is also a parallelogram. In particular, K, E, L are collinear, and $KE = LE$.



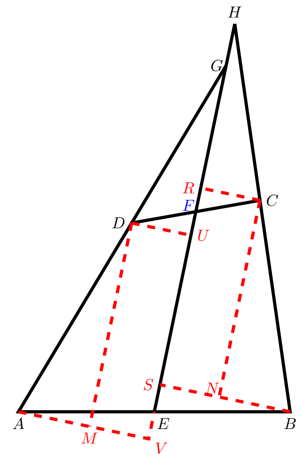
Since $\triangle FKL$ is an isosceles triangle, we must have $\angle AGE = \angle KFL = \angle EFL = \angle EHB$. □

Solution: **(Translation Method (2).)** In this method, we let $DCKA$ be a parallelogram. Let L be the mid poin tof KB. connect EL and LC.



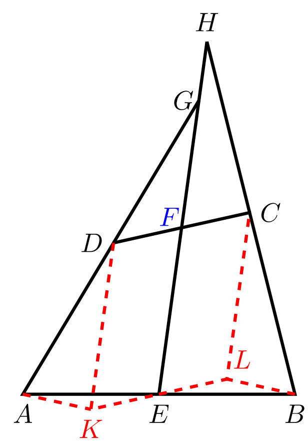
We then can prove that $FCLE$ is a parallelogram. Thus $CL \parallel HE$. This completes the proof. □

Solution: **(Translation Method (3).)** We make rectangles $RCNS$ and $UDMV$. Using that, we can prove $\triangle ADM \cong \triangle BCN$.



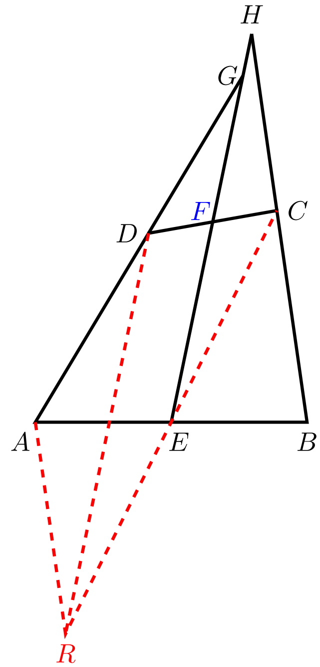
□

Solution: **(Translation Method (4).)** We can construct an parallelogram $DCLK$. Then we can prove $\triangle ADK \cong \triangle BCL$.



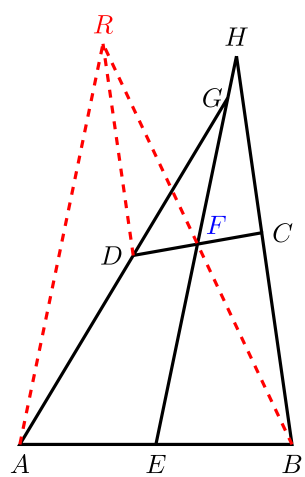
□

Solution: **(Rotation Method (1).)** In the following graph, if $ARBC$ is a parallelogram, then we can prove $\triangle ADR$ is isosceles.



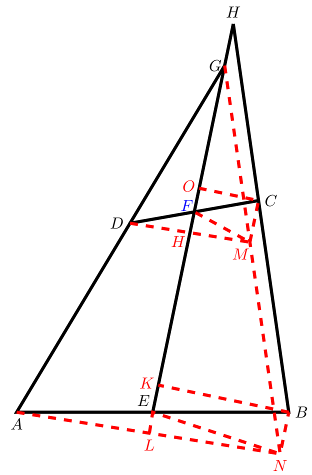
□

Solution: **(Rotation Method (2).)** Making $RHBD$ is a parallelogram. Then we can prove $\triangle DAR$ is isosceles triangle.



□

Solution: **(Flipping Method.)** This method might be complicated, but one can flipping $\triangle GAE$ into $\triangle GNE$ and prove that the trapezoids $COKG$ and $MHLN$ are congruent.



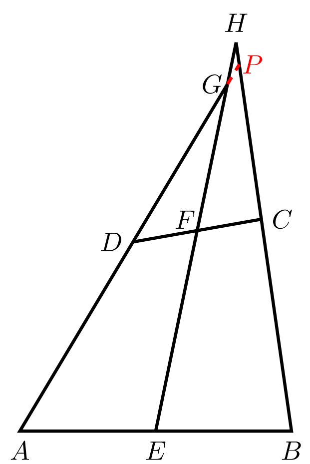
$\square$

Solution: (Z. Lu's Proof Using Menelaus Theorem.) Considering the line HE intersects with $\triangle PDC$, we have

$$\frac{PG}{GD} \cdot \frac{DF}{FC} \cdot \frac{CH}{HP} = 1.$$

Since $DF = FC$, we have

$$\frac{PG}{GD} \cdot \frac{CH}{HP} = 1.$$



Similarly, considering the line HE intersects with $\triangle PAB$, we have

$$\frac{PG}{GA} \cdot \frac{BH}{HP} = 1.$$

Thus we have

$$\frac{CH}{GD} = \frac{BH}{GA}.$$

Assume $AD = BC = a$. We then have

$$\frac{CH}{GD} = \frac{CH + a}{GD + a}.$$

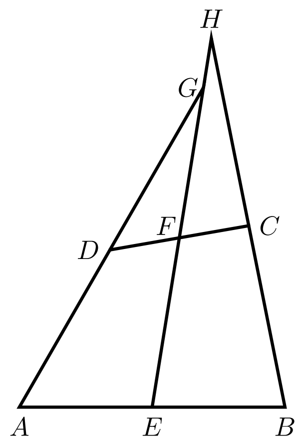
Thus we have $CH = GD$ which implies $PG = HP$. Therefore $\triangle PHG$ is isosceles. Thus

$$\angle EHB = \angle PGH = \angle AGE.$$

□

Solution: **(Z. Lu's Proof Using Vector Algebra.)** Let A, B, C, D be represented by $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$, and $\vec{\mathbf{d}}$. Then vectors $\vec{GA}$, $\vec{HE}$, and $\vec{HB}$ are

$$\begin{aligned}
\vec{\mathbf{v}}_1 &= \vec{\mathbf{a}} - \vec{\mathbf{d}}, \\
\vec{\mathbf{v}}_2 &= (\vec{\mathbf{a}} - \vec{\mathbf{d}})/2 + (\vec{\mathbf{b}} - \vec{\mathbf{c}})/2, \\
\vec{\mathbf{v}}_3 &= \vec{\mathbf{b}} - \vec{\mathbf{c}}.
\end{aligned}$$



By assumption, $\vec{\mathbf{v}}_1 = \vec{\mathbf{v}}_3$. Thus

$$\cos\angle AGE = \frac{\langle \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\rangle}{\|\vec{\mathbf{v}}_1\|\cdot\|\vec{\mathbf{v}}_2\|} = \frac{\langle \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\rangle}{\|\vec{\mathbf{v}}_3\|\cdot\|\vec{\mathbf{v}}_2\|} = \cos\angle H.$$

Thus $\angle AGE = \angle H$.

□

Using coordinate geometry and complex numbers, one would get similar proofs as above.

References

[1] 许莼舫, 许莼舫初等几何四种, 中国青年出版社, 1978.

Department of Mathematics, University of California, Irvine, Irvine, CA 92697

Email address: zlu@uci.edu