Projective Harmonic Conjugate

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1 Introduction

Projective Harmonic Conjugate is a very useful concept in triangle geometry and projective geometry. In this short article, we introduce the concept, prove some of its basic properties, and provide some applications.

Definition 1

Let A, B, C, D be four consecutive points on the number line from left to right. The Cross-ratio $(A, C; B, D)$ is defined by

$$(A, C; B, D) = \frac{BC}{AB} \cdot \frac{AD}{DC}.$$

If $(A, C; B, D) = 1$, then we call these four points Projective Harmonic Conjugate points.

The following proposition justifies the terminology "harmonic conjugate".

Proposition 1

Assume that $(A, C; B, D) = 1$. Then

$$\frac{2}{AC} = \frac{1}{AB} + \frac{1}{AD}.$$

In other words, AC is the harmonic mean of AB and AD.

2 Basic Properties

The most important property of cross-ratio is its projective invariance.

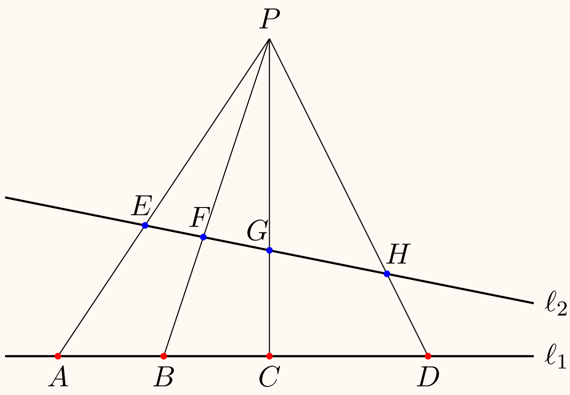
Theorem 1

Let P be a point outside line ℓ_1. Let PA, PB, PC, PD intersect with another line ℓ_2 at E, F, G, H, respectively. Then the cross-ratios of the two groups of points are the same

$$(A, C; B, D) = (E, G; F, H).$$

In particular, A, B, C, D are projective harmonic conjugate points if and only if E, F, G, H are projective harmonic conjugate points.

[1]The author thanks Dr. Zhiqin Lu for his help.

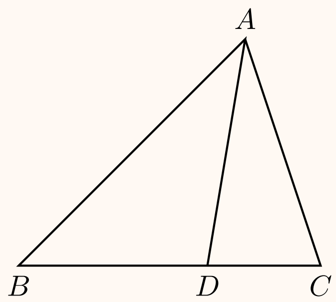


In order to prove the theorem, we need to prove a so-called *Generalized Angle Bisector Theorem*, which can be stated as follows.

Theorem 2. (Generalized Angle Bisector Theorem)

Let D be a point on line BC. Then

$$\frac{BD}{DC} = \frac{AB \cdot \sin \angle BAD}{AC \cdot \sin \angle DAC}.$$



In particular, if $\angle BAD = \angle DAC$, in which case AD is the angel bisector of $\angle A$, then the Angle Bisector Theorem states that

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

Proof. We use the area method. It is well known that

$$\frac{S_{\triangle BAD}}{S_{\triangle ADC}} = \frac{BD}{DC}.$$

The theorem follows from the fact that

$$S_{\triangle BAD} = \frac{1}{2} AB \cdot AD \cdot \sin \angle BAD,$$
$$S_{\triangle DAC} = \frac{1}{2} AD \cdot AC \cdot \sin \angle DAC.$$

∎

We use the above result to prove Theorem 1.

Proof of Theorem 1. Using the Generalized Angle Bisector Theorem, we have

$$\frac{BC}{AB} = \frac{PC \cdot \sin \angle BPC}{PA \cdot \sin \angle APB}, \quad \frac{AD}{DC} = \frac{PA \cdot \sin \angle APD}{PC \cdot \sin \angle DPC}.$$

Thus

$$(A, C; B, D) = \frac{BC}{AB} \cdot \frac{AD}{DC} = \frac{\sin \angle BPC \cdot \sin \angle APD}{\sin \angle APB \cdot \sin \angle DPC}. \tag{1}$$

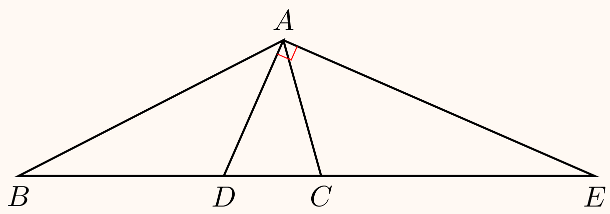
Therefore the cross-ratio depends only on the rays PA, PB, PC, PD, and is independent to lines ℓ_1 and ℓ_2. ■

Definition 2

In the above theorem, the lines PA, PB, PC, PD are called a Projective Harmonic Conjugate Pencil.

Theorem 3

Assume that B, D, C, E are projective harmonic conjugate points. Moreover, assume that $AD \perp AE$. Then AD is the angle bisector of $\angle BAC$, and AE is the angle bisector of the exterior angle of A.



Proof. The easiest way to prove this theorem is to use trigonometry. Assume that $\angle BAD = \alpha$, and $\angle DAC = \beta$. Then by assumption, $\angle CAE = 90^\circ - \beta$, and $\angle BAE = 90 + \alpha$. Thus by (1), we have

$$1 = (B, C; D, E) = \frac{\sin \beta \cdot \sin(90^\circ + \alpha)}{\sin \alpha \cdot \sin(90^\circ - \beta)} = \tan \beta \cdot \cot \alpha.$$

Therefore $\alpha = \beta$, as anticipated. ■

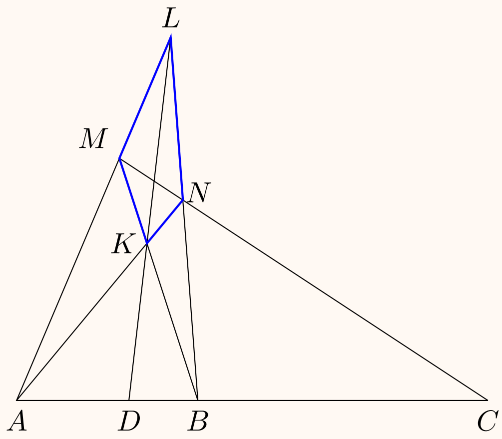
3 Applications

If A, B, C, D are projective harmonic conjugate points, then we say that B, D *harmonically divide* line segment AC. Obviously, if B, D harmonically divide AC, then A, C harmonically divide BD.

In a Topic 15, two diagonals harmonically divide the third one in the following sense.

Theorem 4

Let $LMKN$ be a quadrilateral. Assume LM and NK intersect at A, and LN and MK intersect at B so that $LMKNAB$ is a complete quadrilateral. The diagonals LK and MN intersect the third diagonal AB at D, C, respectively. Then A, D, B, C are *projective harmonic conjugate points.*



Proof. We consider $\triangle LAB$. Since M, N, C are collinear, by Menelaus' Theorem, we have

$$\frac{LM}{MA} \cdot \frac{AC}{CB} \cdot \frac{BN}{NL} = 1.$$

Since LD, AN, BM are concurrent, by Ceva's Theorem, we have

$$\frac{LM}{MA} \cdot \frac{AD}{DB} \cdot \frac{BN}{NL} = 1.$$

Comparing the above two equations, we have

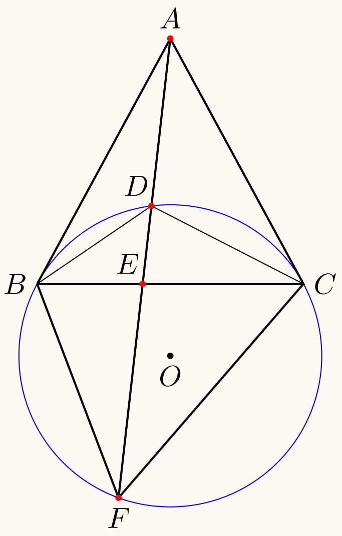
$$\frac{AC}{CB} = \frac{AD}{DB},$$

and hence the cross ratio $(A, B; D, C) = 1$. This completes the proof of the theorem.

Another example of harmonic conjugate points is related to Topic 24.

Theorem 5

Let A be a point outside Circle O. Let AB, AC be tangent lines to the circle and let AF be a secant line intersecting the circle at D and F. Assume that AF and BC intersect at E. Then A, D, E, F are projective harmonic conjugate points.



Proof. $BFCD$ is a harmonic quadrilateral by Theorem 2 of Topic 24. Therefore, we have

$$BF \cdot DC = BD \cdot FC.$$

Using the law of sines, the above equation can be transformed into

$$2R \sin \angle ECF \cdot 2R \sin \angle DFC = 2R \sin \angle DCE \cdot 2R \sin \angle FBC.$$

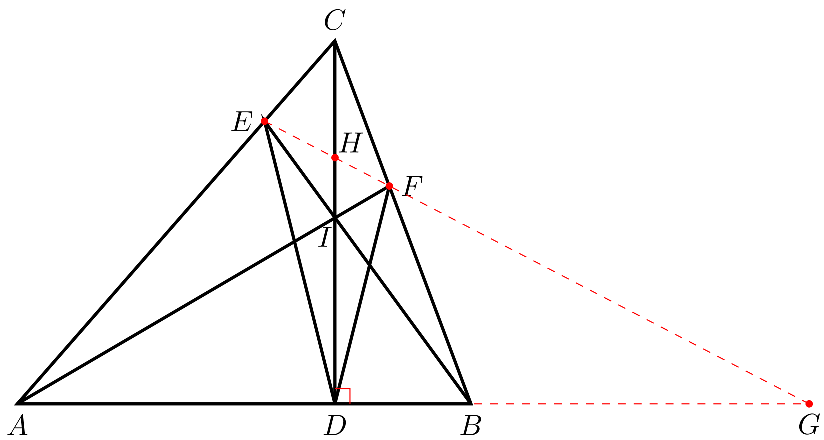
Since $\angle DFC = \angle ACD$ and $\angle FBC = 180^\circ - \angle ACF$, by (1), we have

$$(A, E; D, F) = \frac{\sin \angle DCE \cdot \sin \angle ACF}{\sin \angle ACD \cdot \sin \angle ECF} = 1,$$

completing the proof of the theorem. ∎

As an application of Theorem 3, we prove the following result.

Example 1 In the following picture, assume that $CD \perp AB$, and let CD, BE, and AF be concurrent at I. Then $\angle EDC = \angle FDC$.



Proof. We consider the complete quadrilateral $CEIFAB$. By Theorem 4, E, H, F, G are projective harmonic conjugate points. Since $CD \perp AB$, by Theorem 3, DC is the angle bisector of $\angle EDF$. ■

4 Further Discussions on Cross-ratio

Let A, B, C, D be collinear. By (1), it is not hard to prove the following formula:

$$(A, C; B, D) = \frac{S_{\triangle PBC} \cdot S_{\triangle PAD}}{S_{\triangle PAB} \cdot S_{\triangle PDC}}.$$

Using vector cross product, we have

$$(A, C; B, D) = \frac{\|\overrightarrow{PB} \times \overrightarrow{PC}\| \cdot \|\overrightarrow{PA} \times \overrightarrow{PD}\|}{\|\overrightarrow{PA} \times \overrightarrow{PB}\| \cdot \|\overrightarrow{PD} \times \overrightarrow{PC}\|}.$$

We can extend the notation of cross-ratio to include negative numbers by the following: let $\vec{\mathbf{n}}$ be the normal vector of the Euclidean plane. Note that, for example, $\frac{1}{2}(\overrightarrow{PB} \times \overrightarrow{PC}) \cdot \vec{\mathbf{n}}$ represents the *signed* area of $\triangle PBC$. Then we can define

$$(A, C; B, D) = \frac{((\overrightarrow{PB} \times \overrightarrow{PC}) \cdot \vec{\mathbf{n}}) \cdot ((\overrightarrow{PA} \times \overrightarrow{PD}) \cdot \vec{\mathbf{n}})}{((\overrightarrow{PA} \times \overrightarrow{PB}) \cdot \vec{\mathbf{n}}) \cdot ((\overrightarrow{PD} \times \overrightarrow{PC}) \cdot \vec{\mathbf{n}})}.$$

By abusing of notation, we use P, A, B, C, D to represent their homogeneous coordinates as well. Then we can write

$$(A, C; B, D) = \frac{[P, B, C] \cdot [P, A, D]}{[P, A, B] \cdot [P, D, C]},$$

where $[\cdot, \cdot, \cdot]$ represents the determinant of the 3×3 matrix using the homogeneous coordinates. Using this more general definition, four points are projective harmonic conjugate points if and only if its cross-ratio is equal to -1.