Davis' Theorem

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(last updated: June 18, 2022)

1 History

R. F. Davis is an amateur mathematician in the late 19th – early 20th century in Euclidean geometry. He published papers in mathematical journals like *Mathematical Gazette* between 1900 and 1906.

The *Davis' Theorem*, name after him, is sometimes mistyped as Davies' Theorem, where the name came from a British mathematician named Thomas Stephens Davies.

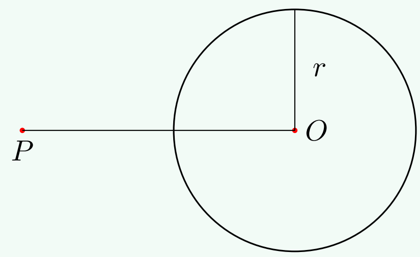
2 Circle Power Theorem

The Circle Power Theorem is a theorem in plane geometry, which is the unification of the *Intersecting Chord Theorem*, the *Secant Theorem* and the *Tangent-secant Theorem*.

Definition 1. (Power to Circle)

Let O be a fixed circle with radius r, and let P be a point. The Power of the point P to the cicle is defined to be

$$OP^2 - r^2.$$



Theorem 1. (Circle Power Theorem)

Assume AB and CD are two chords of $\odot O$ of radius r, and AB, CD intersect at P. Then

- **Case 1:** *If P is inside the circle, then*

$$PA \cdot PC = PD \cdot PB = r^2 - OP^2,$$

which is the negative of the power of P to the circle;

- **Case 2:** *If P is outside the circle, then*

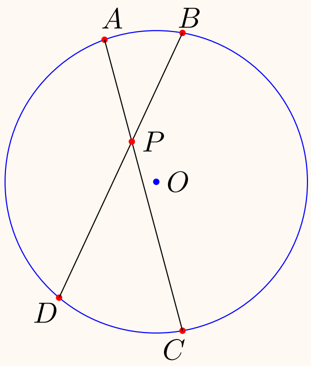
$$PA \cdot PC = PB \cdot PD = OP^2 - r^2,$$

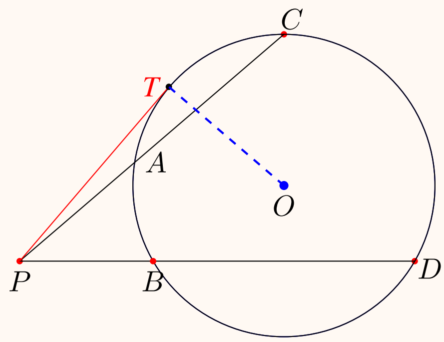
[1]The author thanks Dr. Zhiqin Lu for his help.

which is the power of P to the circle. In addition, if PT is a tangent line, then

$$PT^2 = OP^2 - r^2,$$

which means that the power of a point to the circle is equal to the square of the length of the tangent line.



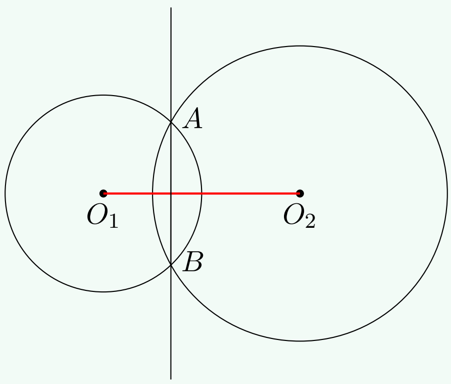


3 Davis' Theorem

Davis' Theorem is an important theorem regarding to concyclic points. Before introducing the Davis' Theorem, we define *Radical Axis*.

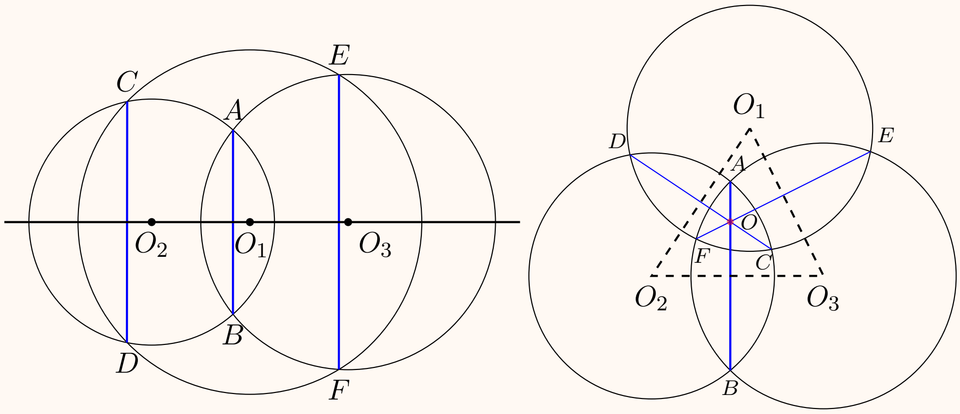
Definition 2. (Radical Axis)

Given two circles $\odot O_1$ and $\odot O_2$ which are not concentric, the set of points of equal power to the two circles is a straight line, and is called the radical axis of the two circles.
In particular, if $\odot O_1$ and $\odot O_2$ are intersecting, then common chord AB is the radical axis.



Theorem 2. (Root Heart Theorem)

Let $\odot O_1$, $\odot O_2$, and $\odot O_3$ be three circles, none of the two are concentric. Then the radical axes of $\odot O_1O_2$, $\odot O_2O_3$, and $\odot O_3O_1$ are concurrent or parallel.



Proof: Let the equations of the three circles $\odot O_1$,$\odot O_2$, and $\odot O_3$ be

$$\begin{aligned}\mathbf{C_1}: &\quad (x-a_1)^2+(y-b_1)^2-(r_1)^2=0,\\ \mathbf{C_2}: &\quad (x-a_2)^2+(y-b_2)^2-(r_2)^2=0,\\ \mathbf{C_3}: &\quad (x-a_3)^2+(y-b_3)^2-(r_3)^2=0,\end{aligned}$$

respectively. Note that for any point (x, y), the number

$$(x-a_i)^2+(y-b_i)^2-(r_i)^2$$

for $i = 1, 2, 3$ are the powers of the point to the circles $\odot O_1$, $\odot O_2$, and $\odot O_3$, respectively. As a result, the equation

$$(x-a_1)^2+(y-b_1)^2-(r_1)^2=(x-a_2)^2+(y-b_2)^2-(r_2)^2$$

is the radical axis equation of $\odot O_1$,$\odot O_2$, which can be abbreviated as $\mathbf{C_1}-\mathbf{C_2}=0$. Similarly, the radical axis equations of $\odot O_2$,$\odot O_3$ and $\odot O_3$,$\odot O_1$ can be represented by $\mathbf{C_2}-\mathbf{C_3}=0$ and $\mathbf{C_3}-\mathbf{C_1}=0$, respectively.

By subtracting each two equations, the equations of the radical axes are

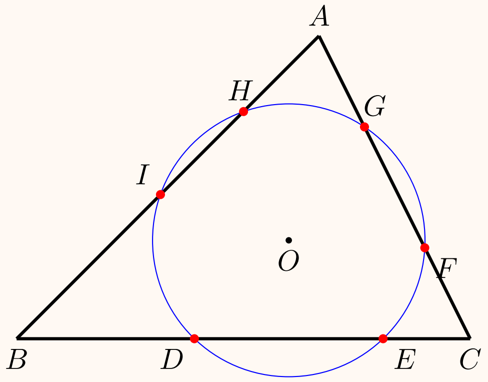
$$\begin{aligned}2(a_2-a_1)x+2(b_2-b_1)y+f_1-f_2&=0,\\ 2(a_3-a_2)x+2(b_3-b_2)y+f_2-f_3&=0,\\ 2(a_1-a_3)x+2(b_1-b_3)y+f_3-f_1&=0.\end{aligned}$$

The linear equations of these three radical axes are *linearly dependent*, in particular, the summation of the first two equations gives the third equation and hence the radical axes are either concurrent or parallel. ∎

Now we prove the main theorem of this article.

Theorem 3. (Davis' Theorem)

In the following picture, let D, E be points on BC; F, G be points on CA; and H, I be points on AB. If the quadrilaterals $DEFG$, $FGHI$, and $HIDE$ are concyclic, then the hexagon $DEFGHI$ is concyclic.



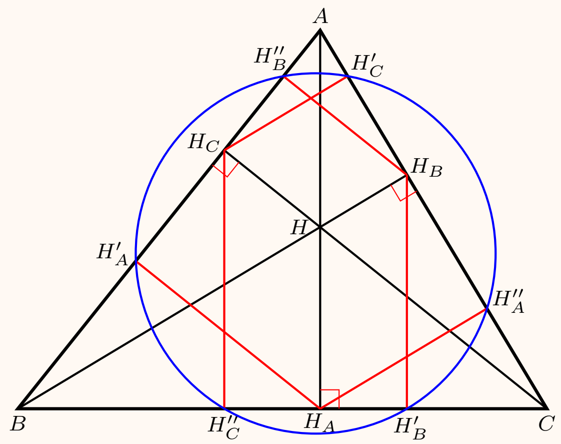
Proof: Let $\odot O_1$, $\odot O_2$, and $\odot O_3$ be the circumcircles of the quadrilaterals $DEFG$, $FGHI$, and $HIDE$ respectively. Then BC, CA, and AB are the radical axes of $\odot O_3, \odot O_1$, $\odot O_1, \odot O_2$, and $\odot O_2, \odot O_3$, respectively. Since BC, CA, and AB form a triangle, so they are neither concurrent nor parallel. By the Root Heart Theorem, $\odot O_1, \odot O_2, \odot O_3$ must be concentric, and hence the hexagon $DEFGHI$ is concyclic. ∎

4 Taylor Circle

Davis' Theorem has a lot of applications. As an example, we prove the following *Taylor's Theorem*.

Theorem 4. (Taylor Theorem)

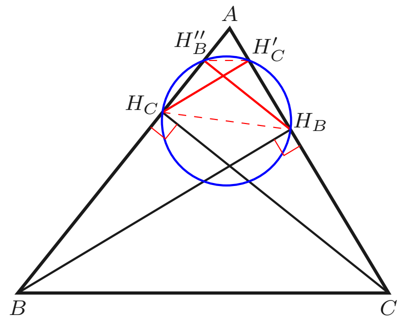
Let points H_A, H_B, and H_C be the feet of each altitude of a triangle $\triangle ABC$. Let H'_A, H''_A, H'_B, H''_B, H'_C, and H''_C be the projection of points H_A, H_B, and H_C to each side of the triangle as shown in the picture. Then the six points H'_A, H''_A, H'_B, H''_B, H'_C,H''_C are concyclic, and the circle is called the Taylor Circle.



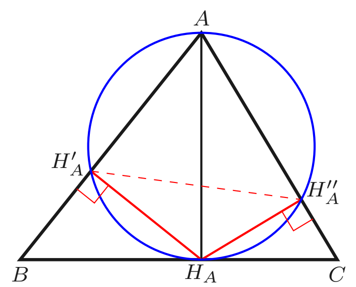
Proof: By Davis' Theorem, we only need to prove that H''_A, H'_C, H''_B, H'_A are concyclic.

We use the picture below. Since H_B and H_C are the attitude of $\triangle ABC$, then B, C, H_B, H_C are concyclic. Therefore, $\angle AH_CH_B = \angle C$.

Similarly, H_B, H'_C, H''_B, H_C are concyclic, $\angle AH_CH_B = \angle AH'_CH''_B$. Thus, $H''_BH'_C \parallel BC$.



Since point H'_A and H''_A are the projections of point H_A to AB, AC, respectively, A, H'_A, H_A, H''_A are concyclic. Therefore $\angle H''_AH_AC = H_AAC$. Thus, $\angle AH'_AH''_A = \angle AH_AH''_A = 90° - \angle H''_AH_AC = 90° - \angle H_AAC = \angle C$.



Summarizing the above, we get $\angle AH'_CH''_B = \angle AH'_AH''_A$. Thus, the quadrilateral

$Q_A P_B P_C Q_C$ are concylic, and hence this completes the proof of the theorem. ∎

External Link. *For more details of Taylor Circle, please refer to Topic 30.*