Five Centers

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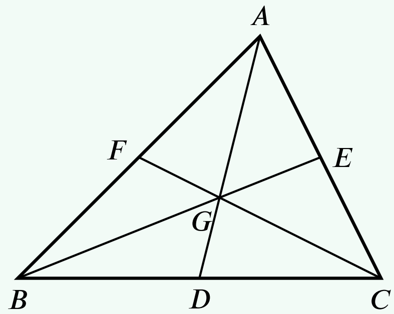
There are five important special centers of a triangle: *centroid*, *incenter*, *excenter(s)*, *circumcenter*, and *orthocenter*. These centers play important rules in Euclidean geometry.

1 Introduction of the Centers

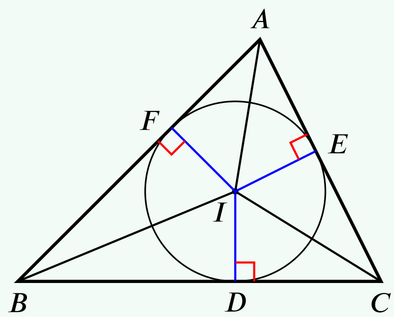
Definition 1. (Centroid)

In triangle $\triangle ABC$, let AD, BE, CF be the medians on the respective sides. Then AD, BE, CF are concurrent to the point G, which is called the centroid, or center of gravity, of the triangle.



Definition 2. (Incenter)

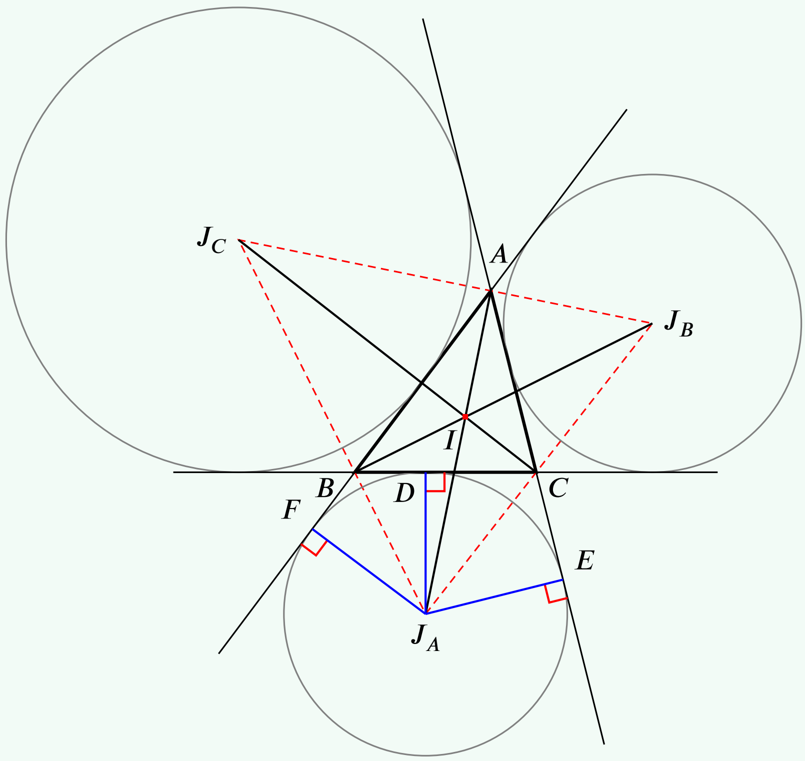
In triangle $\triangle ABC$, the center I of the inscribed circle is called incenter. Let AI, BI, and CI be the angle bisectors of the corresponding angles $\angle A, \angle B$, and $\angle C$. Then these angle bisectors are concurrent at I.



[1,2]The authors thank Stephanie Wang for her careful reading and many suggestions.

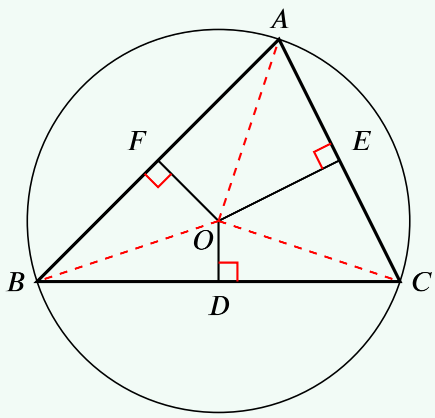
Definition 3. (Excenter)

In triangle $\triangle ABC$, the center of an escribed circle or excircle is called excenter. There are three excenters J_A, J_B, J_C of $\triangle ABC$, corresponding to the vertices A, B, C.
In the following picture, the circle J_A is tangent to the line segment BC and the extended line segments AB, CA; circle J_B is tangent to CA and the extended line segments of AB, BC; and circle J_C is tangent to line segment AB and the extended line segments CA and BC. Moreover, AJ_A is the angle bisector of $\angle A$, and both BJ_A, CJ_A are the corresponding external angle bisectors (of the angles $\angle CBF$ and $\angle BCE$).



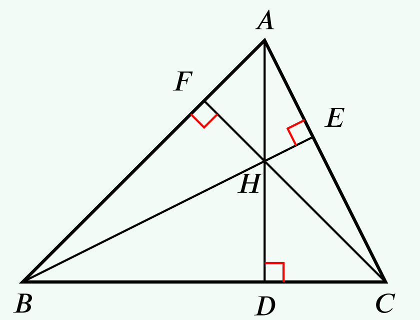
Definition 4. (Circumcenter)

In $\triangle ABC$, the center O of the circumscribed circle is called circumcenter. Let OD, OE, OF be the perpendicular bisectors of three sides BC, CA, AB, respectively. Then these lines are concurrent at O.



Definition 5. (Orthocenter)

In $\triangle ABC$, let AD, BE, CF be the heights on BC, CA, AB, respectively. Then they are concurrent at a point H, which is called the orthocenter of a triangle.

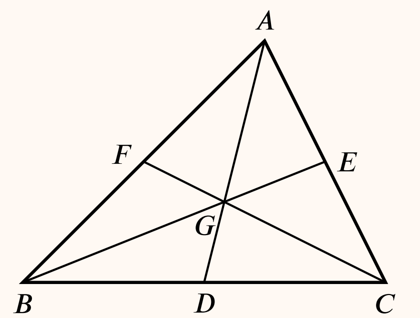


2 The Proofs of Concurrency

In this section, we give classical proofs of the concurrency of the lines.

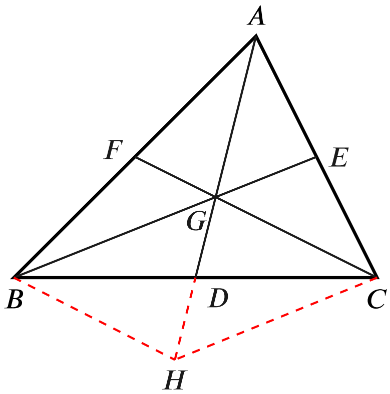
Theorem 1. (Centroid)

In triangle $\triangle ABC$, let AD, BE, CF be the medians on the respective sides. Then they are concurrent.



Proof: Assume that E is the midpoint of CA; F is the midpoint of AB; and BE and CF intersect at point G. Extend AG to intersect BC at D. We then need to prove that D is the midpoint of BC.

In the following picture, we extend line segment AD to H so that $AG = GH$, and then connect BH and CH.

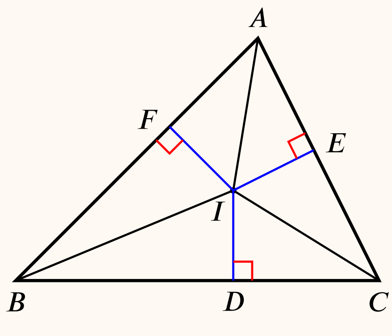


Since $AG = GH$ and $AE = EC$, so that EG is the mid-segment of $\triangle ACH$. In particular, $EG \parallel CH$. Similarly, $FG \parallel BH$. Therefore $BGCH$ is a parallelogram. Since GH and BC are the diagonals of $\square BGCH$, we have $BD = DC$, hence D is the midpoint of BC.

■

Theorem 2. (Incenter)

In triangle $\triangle ABC$, assume that AI is the angel bisector of $\angle A$; BI is the angle bisector of $\angle B$, and CI is the angel bisector of $\angle C$. Then AI, BI, CI are concurrent.



Proof: Assume that the angle bisectors AI and BI of $\angle A$ and $\angle B$ intersect at the point I, we need to prove that CI must be the angle bisector of $\angle C$.

Taking the perpendicular lines $IE \perp CA$, $IF \perp AB$, and $ID \perp BC$. Since AI is the angle bisector of $\angle A$, we must have $IE = IF$. Similarly, we have $IF = ID$. Thus $ID = IE$. From this, we conclude that CI is the angle bisector of $\angle C$.

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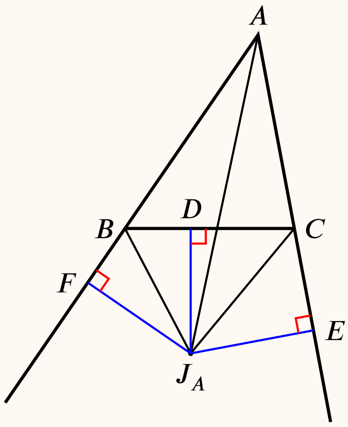
Theorem 3. (Excenter)

In $\triangle ABC$, assume that AJ_A is the angle bisector of $\angle A$; and assume that BJ_A, CJ_A are the angle bisectors of the exterior angles $\angle FBC$, $\angle ECB$, respectively. Then these three angle bisectors are concurrent.



Proof: The proof is similar to that of Theorem 2.

We assume that angle bisectors of $\angle FBC$ and $\angle ECD$ intersect at J_A. We need to prove that AJ_A is the angle bisector of $\angle A$.

Let D, E, F be the projections of J_A to BC, CA, AB, respectively. By the assumption that BJ_A is the angle bisector of $\angle FBC$, we know that $FJ_A = DJ_A$. Similarly, we have $EJ_A = DJ_A$. Thus $EJ_A = FJ_A$, and therefore, AJ_A is the angle bisector of $\angle A$.

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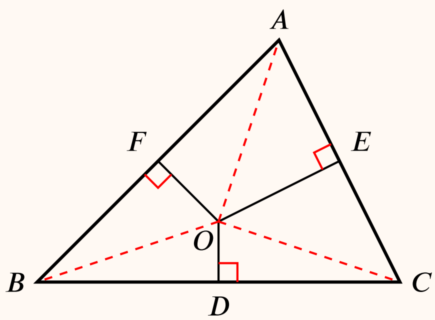
Theorem 4. (Circumcenter)

In triangle $\triangle ABC$, assume that OD is the perpendicular bisector of BC; OE is the perpendicular bisector of CA; and OF is the perpendicular bisector of AB. Then OD, OE, OF are concurrent.



Proof: Since OD is the perpendicular bisector of BC, then $\triangle OBC$ is an isosceles triangle; so $OB = OC$. Similarly, $\triangle OAC$ is an isosceles triangle; then $OA = OC$. These imply that $OB = OA$ and hence $\triangle OAB$ is an isosceles triangle. Since $OF \perp AB$, it must be the perpendicular bisector of AB.

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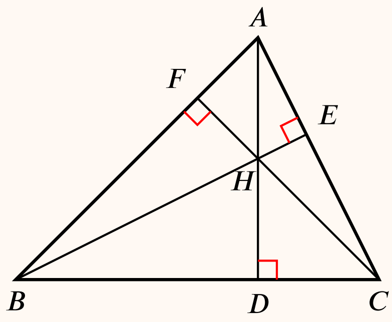
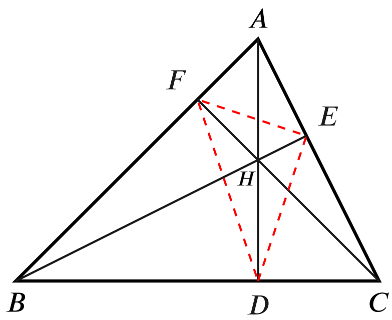
Theorem 5. (Orthocenter)

The three heights of a triangle are concurrent. The point is called the orthocenter of a triangle.



There are numerous different proofs of the theorem that the three heights of a triangle are concurrent. Here we give two proofs using Theorem 2 and 4.

Proof Using Incenter: We need to prove that H is the incenter of $\triangle DEF$.



Since $\angle AFC = \angle ADC = 90°$, $AFDC$ is concyclic, and therefore $\angle FDB = \angle A$. Similarly, since $ABDE$ is con-cyclic, $\angle CDE = \angle A$. As a result, $\angle FDB = \angle CDE$. Since $AD \perp BC$, we get $\angle FDA = 90° - \angle FDB = 90° - \angle EDC = \angle EDA$. Therefore HD is the angle bisector of $\angle EDF$. By the same argument, HE is the angle bisector of $\angle DEF$ and HF is the angle bisector of $\angle DFE$. By Theorem 2, these three lines must be concurrent.

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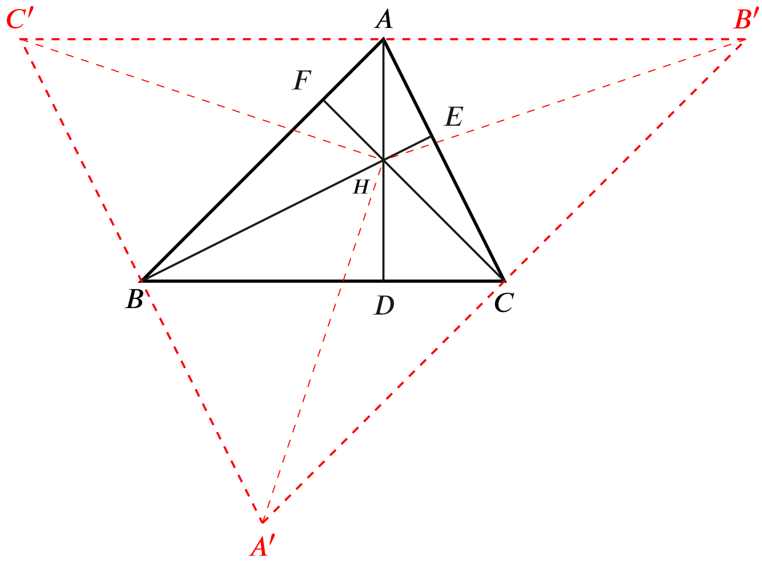
Proof Using Circumcenter: We draw lines $A'B', B'C', C'A'$ to be parallel to AB, BC, CA, respectively. We shall prove that H is the circumcenter of $\triangle A'B'C'$.



Since $BC \parallel C'B'$, and $AB \parallel B'C$, $ABCB'$ is a parallelogram. Similarly, $ACBC'$ is also a parallelogram. Thus $C'A = BC = AB'$, and HA is the perpendicular bisector of $B'C'$. Similarly, HC is the perpendicular bisector of $A'B'$, and HB is the perpendicular bisector of $C'A'$. By Theorem 4, HA, HB, HC are concurrent at H. This proves that the three heights are concurrent. ■

3 The Unified Treatments

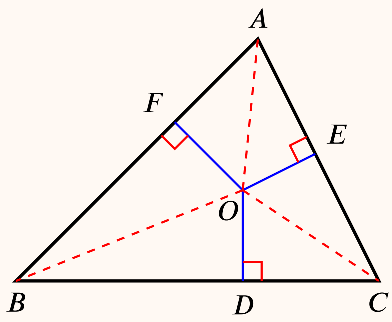
In the last section, we prove several sets of special lines of a triangle are concurrent, namely we prove that in a triangle, the three medians, the three angle bisectors (and one angle bisector with the other two exterior angle bisectors), the three side perpendicular bisectors, and the three heights are concurrent respectively.

We are able to give somewhat unified treatments to prove the concurrency of these lines. We begin with the following result.

Theorem 6. (Carnot's Theorem)

In the following triangle $\triangle ABC$, assume that $OD \perp BC$, $OE \perp CA$ and $OF \perp AB$. Then OD, OE and OF are concurrent if and only if

$$BD^2 - DC^2 + CE^2 - EA^2 + AF^2 - FB^2 = 0. \tag{1}$$



See Wikipedia for details of the Carnot's Theorem.

Proof: We assume that OD, OE and OF are concurrent at O. By the Pythagorean Theorem, we have the following three equations

$$BD^2 - DC^2 = OB^2 - OC^2, \tag{2}$$

$$CE^2 - EA^2 = OC^2 - OA^2, \tag{3}$$

$$AF^2 - FB^2 = OA^2 - OB^2. \tag{4}$$

Adding the above three equations, we get (1).

Conversely, if (1) is valid. Assume that OD, OE interact at O. Then using the Pythagorean Theorem, (2), (3) are valid. Thus combining with (1), we have

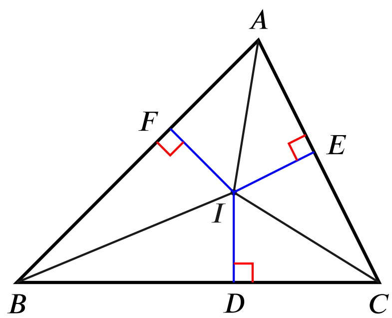
$$AF^2 - FB^2 = OA^2 - OB^2,$$

hence by using the Pythagorean Theorem again, $OF \perp AB$.

Remark The theorem is still valid even when O is outside the triangle.

Unified Treatment 1: We can use the above result to prove several theorems in the last section.

Proof of Theorem 2. In the below picture,



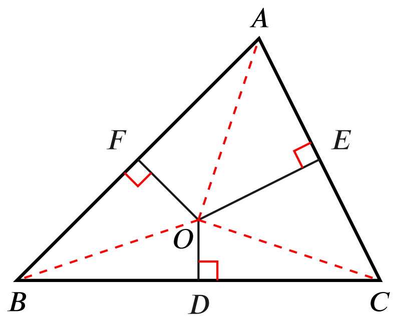
we have $BD = BF$, $CD = CE$, and $AE = AF$. Thus

$$BD^2 - DC^2 + CE^2 - EA^2 + AF^2 - FB^2 = 0,$$

and hence ID, IE, IF and AI, BI, CI are concurrent.

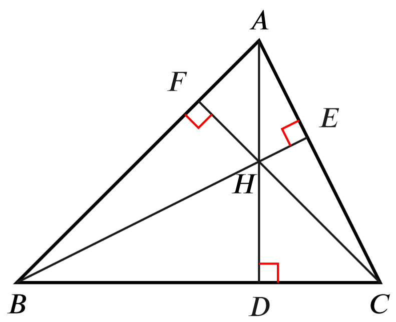
Theorem 3 can be proved using a similar argument.

Proof of Theorem 4. In the following picture,



we have $BD = DC$, $CE = EA$, and $AF = FB$. So by Theorem 6, OD, OE, OF are concurrent.

Proof of Theorem 5. In the following picture,



we have $BD^2 - DC^2 = AB^2 - CA^2$. Similarly, we have

$$CE^2 - EA^2 = BC^2 - AB^2, \qquad AF^2 - FB^2 = CA^2 - BC^2.$$

Thus by Theorem 6, AD, BE, CF are concurrent.

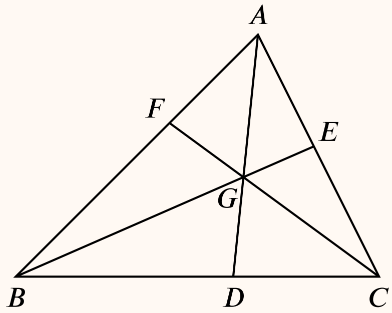
■

The second systematic method to prove the concurrency of lines is to use the Ceva Theorem.

Theorem 7. (Ceva's Theorem)

In the following picture, the lines AD, BE, CF are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$



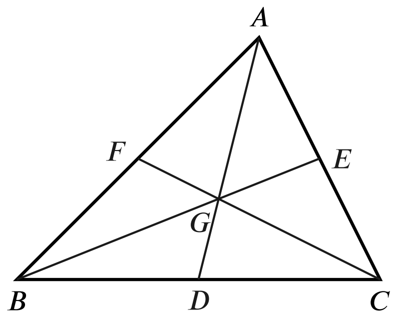
Conversely, if the above equation is valid, then AD, BE, CF are concurrent[a].

[a] The lines AF, BD, CE are concurrent or parallel. For the sake of simplicity, we often omit to state the latter case.

For a proof of the Ceva Theorem, see Wikipedia.

Unified Treatment 2: We can use Ceva Theorem to prove several theorems in the last section.

Proof of Theorem 1. In the following picture,

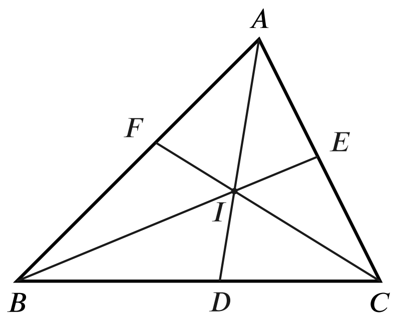


we have

$$BD = DC, \quad CE = EA, \quad AF = FB.$$

Thus by Ceva Theorem, we conclude that AD, BE, CF are concurrent.

Proof of Theorem 2. In the following picture,



AD, BE, CF are the angle bisectors intersect on BC, CA, AB at D, E, F, respectively.

By the Angle Bisector Theorem, we have

$$\frac{BD}{DC}=\frac{AB}{CA}, \quad \frac{CE}{EA}=\frac{BC}{AB}, \quad \frac{AF}{FB}=\frac{CA}{BC}.$$

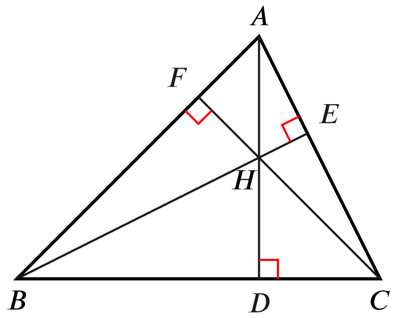
Therefore, we have

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=\frac{AB}{AC}\cdot\frac{BC}{BA}\cdot\frac{AC}{BC}=1.$$

By the Ceva Theorem, AD, BE and CF are concurrent.

The proof of Theorem 3 is similar.

Proof of Theorem 5. Using a little trigonometry, we get



$$\frac{BD}{DC}=\frac{AB\cdot\cos B}{CA\cdot\cos C},$$
$$\frac{CE}{EA}=\frac{BC\cdot\cos C}{AB\cdot\cos A},$$
$$\frac{AF}{FB}=\frac{CA\cdot\cos A}{BC\cdot\cos B}.$$

Therefore

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=\frac{AB\cdot BC\cdot CA\cdot\cos B\cdot\cos C\cdot\cos A}{CA\cdot AB\cdot BC\cdot\cos C\cdot\cos A\cdot\cos B}=1.$$

Using the Ceva Theorem, we conclude that AD, BE, CF are concurrent.

4 Triangle Centers

The centroid, incenter, excenters, circumcenter, and orthocenters are the top five centers in the more than 40,000 triangle centers in *Clack Kimberling*'s web page of Triangle Centers.

According to the above web page, the incenter is the first center $X(1)$; the centroid is $X(2)$; the circumcenter is $X(3)$; and the orthocenter is $X(4)$. All of them play important roles in triangle geometry.

But there are other useful triangle centers. We end this article by quoting the introduction in the Kimberling's web page.

> **Long ago**, someone drew a triangle and three segments across it. Each segment started at a vertex and stopped at the midpoint of the opposite side. The segments

> met in a point. The person was impressed and repeated the experiment on a different shape of triangle. Again the segments met in a point. The person drew yet a third triangle, very carefully, with the same result. He told his friends. To their surprise and delight, the coincidence worked for them, too. Word spread, and the magic of the three segments was regarded as the work of a higher power.
>
> Centuries passed, and someone proved that the three medians do indeed concur in a point, now called the *centroid*. The ancients found other points, too, now called the *incenter*, *circumcenter*, and *orthocenter*. More centuries passed, more special points were discovered ...

Would you like to dicover yet one more triangle center?