Simson Line

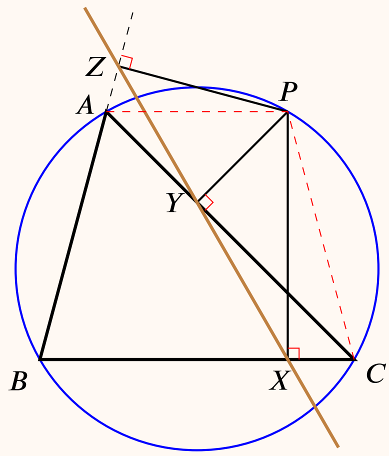
Zhiqin Lu[1], zlu@uci.edu, Shiyi Lyu[2], slyu6@uci.edu

(last updated: June 18, 2022)

Let $\triangle ABC$ be a fixed triangle, and let P be any point. Let the orthogonal projections of P to the sides BC, CA, AB be X, Y, Z, respectively. Then $\triangle XYZ$ is called the *pedal triangle*. The famous "Simson Line" result states that when P is on the circumcircle, then the pedal triangle is degenerated, namely, X, Y, Z are collinear.

Theorem 1. (Simson Line)

Let P be an arbitrary point, and let X, Y, Z be the projections of P to the lines BC, CA and AB, respectively. Then X, Y, Z are collinear if and only if P lies on the circumcircle of $\triangle ABC$.



External Link. *The above line is called the Simson Line of the point P, named after Robert Simson (October 14, 1687 – October 1, 1768), who was a Scottish mathematician and professor of mathematics at the University of Glasgow. However, by Mackay, the line was in fact first discovered by Wallace, (1768–1843). See Wikipedia for further information.*

Proof. In order to prove that X, Y, Z are collinear, we need to prove that

$$\angle AYZ = \angle XYC. \tag{1}$$

We connect AP and PC. Since $PZ \perp ZA$, $PY \perp AY$, then Z, P, Y, A are concyclic. Therefore $\angle AYZ = \angle APZ = 90° - \angle ZAP$. Similarly, since $PY \perp YC$, $PX \perp XC$, then P, C, X, Y are concyclic, and therefore $\angle XYC = \angle XPC = 90° - \angle PCX$.

Finally, since P, A, B, C are concyclic, $\angle PCX = \angle ZAP$. We therefore proved

[1,2]The authors thank Stephanie Wang for her careful reading and many suggestions.

(1).

Next we prove that P is on the circumcircle, if X, Y, Z are collinear. We need to prove that

$$\angle APC + \angle B = 180^\circ \tag{2}$$

in this case.

We essentially reverse the above proof. Since $PZ \perp AB$ and $PY \perp AC$, then Z, A, Y, P are concyclic, which implies $\angle AZY = \angle APY$. Similarly, since P, Y, X, C are concyclic, we have $\angle YPX = \angle YCX$ and $\angle XPC = \angle XYC$.

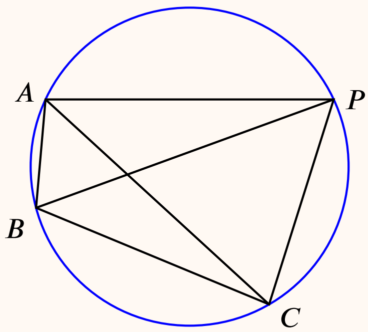
With these, we get $\angle APC = \angle APY + \angle YPC = \angle AZY + 180^\circ - \angle YXC = 180^\circ - \angle B$. We therefore proved (2) and hence the theorem. ■

As an application, we prove that the Theorem 1 implies the Ptolemy's Theorem (See Wikipedia or Topic 10).

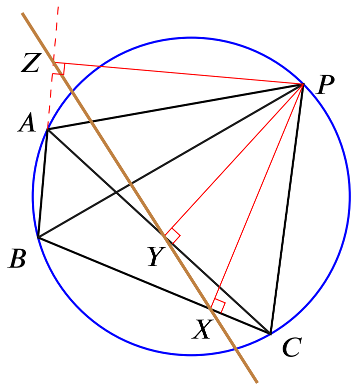
Theorem 2. (Ptolemy's Theorem)

For a cyclic quadrilateral (that is, a quadrilateral inscribed in a circle), the product of the diagonals equals the sum of the products of the opposite sides. In the following picture, we have

$$AC \cdot BP = AB \cdot CP + AP \cdot BC.$$



Proof. Here we use Theorem 1. In the following picture, let $PZ \perp AB$, $PY \perp CA$ and $PX \perp BC$.



By the law of sines, we have

$$ZY = AP \cdot \sin \angle ZPY = AP \cdot \sin \angle BAC = \frac{AP \cdot BC}{2R},$$

where R is the radius of circumcircle. Similarly, we have

$$YX = \frac{CP \cdot AB}{2R}, \quad ZX = \frac{AC \cdot BP}{2R}.$$

Since X, Y, Z are collinear, $ZY + YX = ZX$. Therefore

$$\frac{AC \cdot BP}{2R} = \frac{AP \cdot BC}{2R} + \frac{CP \cdot AB}{2R},$$

which implies the Ptolemy Theorem.

Remark If $ABCD$ is not concyclic, then X, Y, Z are not collinear in general. However, the triangle inequality

$$ZY + YX \geq ZX,$$

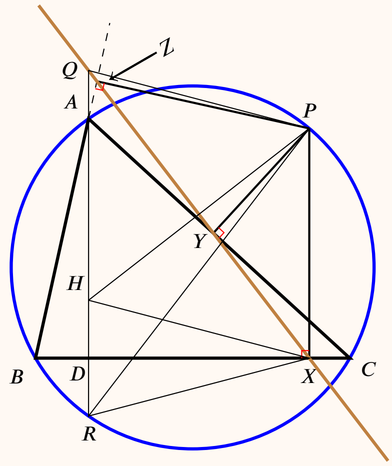
which implies the Ptolemy Inequality

$$AC \cdot BP \leq AB \cdot CP + AP \cdot BC.$$

One of the remarkable feature of the Simson line is the following

Theorem 3

The Simson line of a point bisects the segment joining that point to the orthocenter. In the following picture, let P a point on the circumcircle of $\triangle ABC$ and let XYZ be the Simson line of P. Let H be the orthocenter of $\triangle ABC$. Then the Simson line bisects the line segment PH.



Proof. Let Q be the intersection of the height AH with the Simson line XYZ and let R be the intersection of that to the circumcircle. We shall prove that the quadrilateral $PQHX$ is a parallelogram and therefore the diagonal XQ bisects to the other diagonal, the line segment PH. Since $HD = DR$[a], $\triangle XHR$ is an isosceles triangle. Thus it suffices to prove that $PQRX$ is an isosceles trapezoid. To prove that, we observe that $\angle XQD = 90° - \angle QXD$. Since $PZBX$ is concyclic, we must have $\angle QXD = \angle QXB = \angle ZPB = 90° - \angle ZBP = 90° - \angle QRP$. Combining the above two equations, we get $\angle XQD = \angle QRP$ and hence $PQRX$ is an isosceles trapezoid. ∎

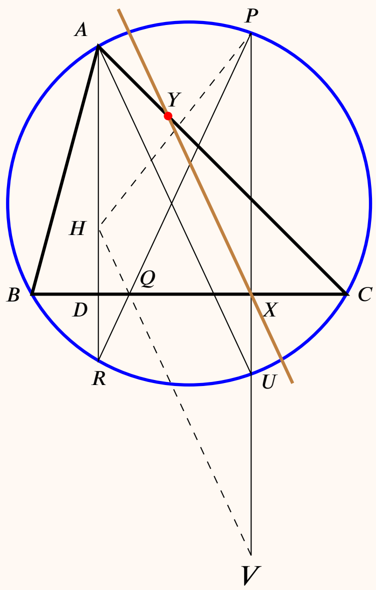
[a]because $\triangle BHR$ is isosceles.

Remark In fact, in the above theorem, the midpoint of PH is on the *nine-point circle* (See Wikipedia and Topic 13), because the orthocenter H is the *homothetic center* (see Wikipedia).

The fact that the Simson line bisects PH yields the following interesting result.

Lemma 1

In the following picture, let XY be the Simson line with respect to the point P. Let U be the intersection of PX to the circumcircle. Then the Simson line XY of P is parallel to AU.



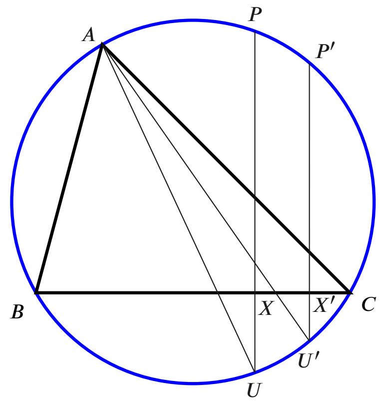
Proof. We define the symmetric point V of P with respect to BC, that is, $PX = XV$. By the above theorem, XY bisects PH. Therefore XY is the mid-segment of $\triangle PHV$ with respect to HV. As a result, $AU \parallel HV$. Since $PX = XV$, $\triangle QPV$ is an isosceles triangle. Thus $\angle V = \angle RPU$. But $\angle RPU = \angle RAU = \angle AUP$, concluding $\angle V = \angle AUP$. Therefore $AU \parallel HV$ and the lemma is proved.

Using the above lemma, we get

Theorem 4

The angle between the Simson lines of two points P and P' on the circumcircles is half of the angular measure of the arc PP'.

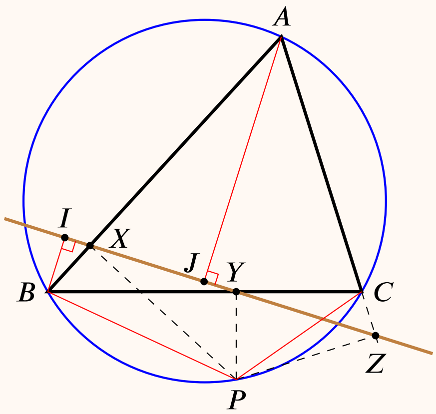
Proof. In the following picture, $PU \perp BC$, $PU' \perp BC$. By the above lemma, we know that AU, AU' are parallel to the Simson lines of P, P', respectively. The angle $\angle UAU'$ of these two lines is equal to half of the arc length of UU'. By symmetry, the arc length of UU' is equal to that of PP'. This proves the theorem.



At last, an interesting property of the Simson Line is the following:

Theorem 5

In the following picture, let point I, J be the pedal points of B, A to the Simson line, respectively. Then the line segment $IJ = YZ$. In other words, the projection of one side of the triangle to the Simson Line is equal to the length of the Simson Line between the other two sides.



Proof. In the above picture, $BXYP$ is concyclic. Therefore $\angle PBY = \angle PXY$, and because $PX \perp AB$ and $AJ \perp XY$, we then get $\angle PXY = \angle XAJ$.

By the law of sines, we have

$$AB \cdot \sin\angle PBY = AX \cdot \sin\angle XAJ + XB \cdot \sin\angle IBX = XJ + IX = IJ.$$

Since $PYCZ$ concyclic, we then have

$$YZ = PC \cdot \sin\angle ACB = 2R \cdot \sin\angle PBY \cdot \sin\angle ACB = AB \cdot \angle PBY = IJ,$$

where R is the radius of circumcircle of $\triangle ABC$.