A NEW PROOF OF MORLEY’S THEOREM

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The Morley's Theorem

- *"The three points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle."*

- Alain Connes proved it using a short calculation involving the group of affine transformations on a complex line.
- At first, Connes looked at gi as plane isometries with the intention to express in their terms the threefold symmetry of the equilateral triangle. This attempt failed and then he established the impossibility of such a presentation. The success came with the interpretation of gi's as affine transformations of the (complex) line. Then, with rotations centered at each vertex of the original triangle with an angle 2/3 the corresponding interior angle. The vertices of the Morley triangle are then fixed points of compositions of 2 consecutive such rotations.

AFFINE GROUP

Affine Group: Let k be a commutative field and G the affine group over k, the group of mappings $g_{a,b}(x) = ax + b$ where $a \neq 0$ and $x \in k$. Thus, G can be the group of 2×2 invertible matrices g:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

Define a group homomorphism $\delta : G \to k^*$, where k^* is the of non-zero elements of k, define $T = \operatorname{Ker} \delta$ is the group of translations, and $\ker \delta$ is the group of translations of k. For $a \in k$, $a \neq 0$, and $b \in k$, let $g \in G$ be defined as:

$$\delta(g) = a \in k^*$$

For each $g \in G$, we define an affine transformation: $g(x) = ax + b$ for all $a \in k$, $b \in k$, $x \in k$, and $a \neq 1$. Then, there is a unique fixed point:

$$\operatorname{fix}(g) = \frac{b}{1-a}$$

Connes's Theorem:

Let $g_1, g_2, g_3 \in G$ *be such that* g_1g_2, g_2g_3, g_3g_1, *and* $g_1g_2g_3$ *are not translations. Let* $j = \delta(g_1g_2g_3)$. *The following two conditions are equivalent:*

1. $g_1^3 g_2^3 g_3^3 = 1$.
2. $j^3 = 1$ *and* $A + jB + j^2C = 0$, *where* $A = \text{fix}(g_1g_2)$, $B = \text{fix}(g_2g_3)$, *and* $C = \text{fix}(g_3g_1)$.

Proof

$$g_i = \begin{bmatrix} a_i & b_i \\ 0 & 1 \end{bmatrix}.$$

Solution: To prove this equivalence, let $g_i = a_i x + b$, $i = 1, 2, 3$. We notice that the equality $g_1^3 g_2^3 g_3^3 = 1$ is equivalent to $\delta(g_1^3 g_2^3 g_3^3) = 1$, and $b = 0$, where b is the translational part of $g_1^3 g_2^3 g_3^3$. The first condition is exactly $j^3 = 1$. By hypothesis $j \neq 1$, then we can write the expression for b:

$$b = (a_1^2 + a_1 + 1)b_1 + a_1^3(a_2^2 + a_2 + 1)b_2 + (a_1 a_2)^3(a_3^2 + a_3 + 1)b_3$$

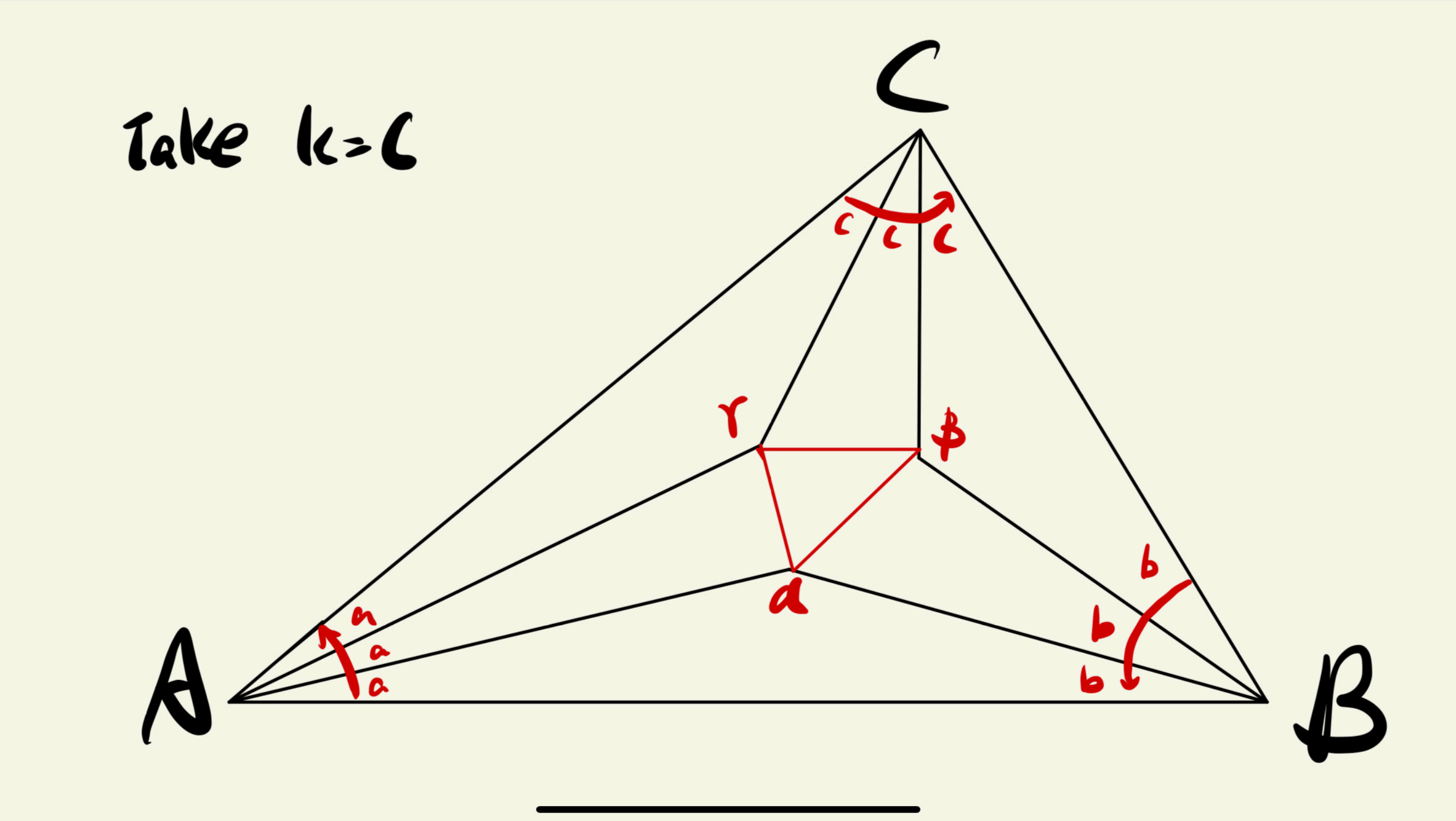
Using $j = a_1 a_2 a_3$, we can rewrite the expression as:

$$b = -j a_1^2 a_2 (a_1 - j)(a_2 - j)(a_3 - j)(A + jB + j^2 C)$$

where the fixed points have been expressed explicitly as:

$$A = \frac{a_1 b_2 + b_1}{1 - a_1 a_2}, \quad B = \frac{a_2 b_3 + b_2}{1 - a_2 a_3}, \quad C = \frac{a_3 b_1 + b_3}{1 - a_3 a_1}.$$

The factor $(a_1 - j)$ cannot be zero because $a_1 - j = a_1(1 - a_2 a_3)$ and 23 is not a translation, which implies $a_2 a_3 \neq 1$. The same occurs for the factors $(a_2 - j)$ and $(a_3 - j)$. Therefore, $b = 0$ is equivalent to $A + jB + j^2 C = 0$.

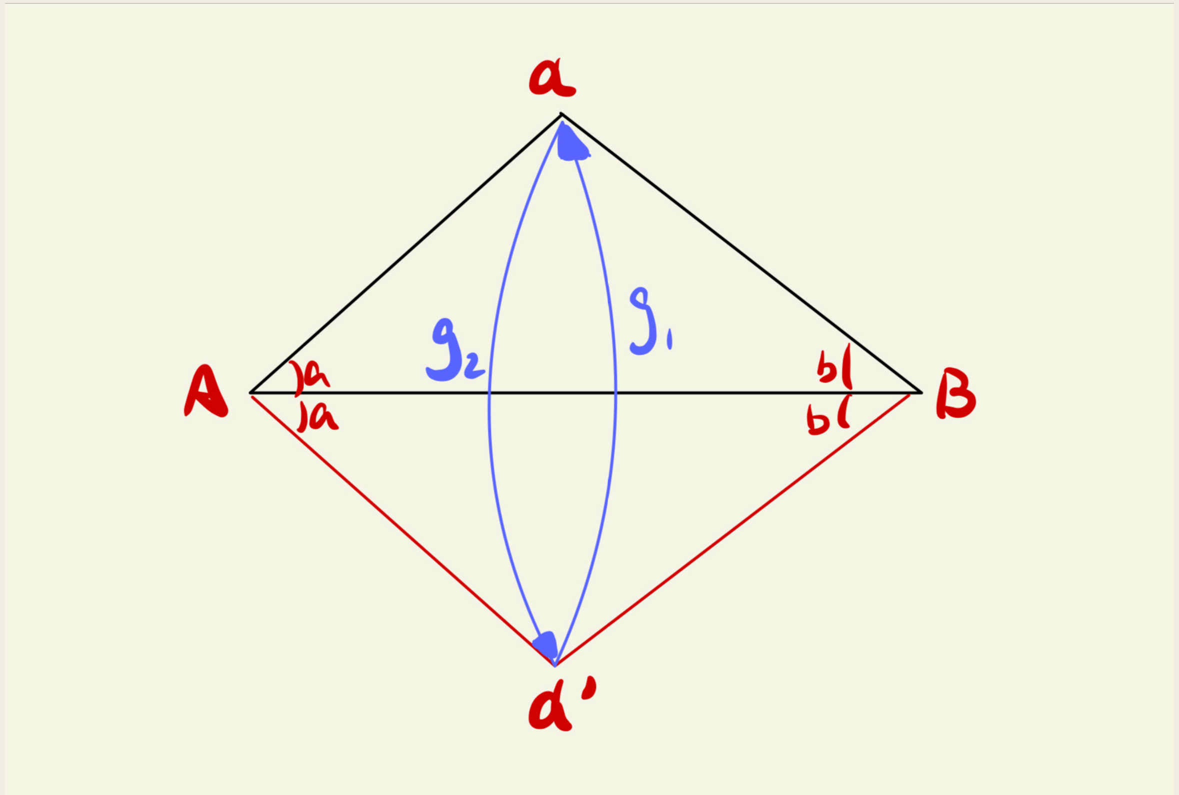


Corollary. Morley's theorem. We take $k = \mathbb{C}$ and define g_1, g_2, g_3 as follows:

g_1 *is the rotation with center A and angle* $2a$ *where* $3a = \widehat{BAC}$

g_2 *is the rotation with center B and angle* $2b$ *where* $3b = \widehat{CBA}$

g_3 *is the rotation with center C and angle* $2c$ *where* $3c = \widehat{ACB}$



Consider the point alpha, which is the intersection of the trisection of angles A and B closest to the side AB. The rotation g_2 transforms alpha to alpha', and the rotation g_1 transforms alpha' back to alpha. Therefore, alpha is a fixed point of g_1, g_2.

Similarly for the other two triangles that beta is a fixed point of g_2 g_3 and gamma is a fixed point of g_3, g_1 are the intersection transactors.

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- This establishes the condition of Connes's theorem, which implies j^3 = 1 (where j is the cube root of unity, j = e^(i2π/3)), and α + jβ + j^2γ = 0. This equation is a classic characterization of an equilateral triangle.

Transformation

- $\alpha + j\beta + j^2\gamma = 0$.
- Here is the transformation:
- Since we know that $j^3 = 1$, we can express j as $j = -1 - j^2$.
- Substituting this into the equation, we have:
- $\alpha + (-1 - j^2)\beta + j^2 \gamma = 0$
- $\alpha - \beta - j^2\beta + j^2 \gamma = 0$
- $\alpha - \beta = -j^2(\gamma - \beta)$
- $\alpha - \beta = e^{(i\pi/3)}(\gamma - \beta)$
- Therefore, the vector $\beta\alpha$ is obtained by rotation of angle $\pi/3$ from vector $\beta\gamma$. There is a same angle $\pi/3$ occurs in the two other cases, which proves that the triangle $\triangle\alpha\beta\gamma$ is equilateral.

References

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Thanks for watching