General Fermat Point Theorem

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CATALOGUE

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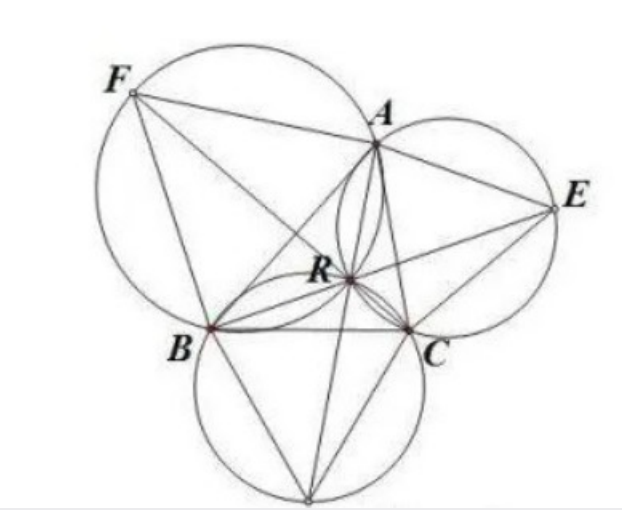
01

Background

01 Background

Speculation proposal

The Fermat's point problem was initially proposed by the French mathematician Pierre de Fermat in a letter to the Italian mathematician Evangelista Torricelli, who invented the barometer. Torricelli was the first to solve this problem, while the 19th-century mathematician Steiner rediscovered it and systematically extended its applications. As a result, this point is also known as the Torricelli point or Steiner point, and related problems are referred to as the Fermat-Torricelli-Steiner problems. The resolution of this problem greatly contributed to the development of unified mathematics and holds a significant milestone in the history of modern mathematics.



01 Background

French lawyer and mathematician Pierre de Fermat. The 1670 version of Diophantine's "Arithmetic", which includes the Fermat conjecture, is known as his "Last Theorem" (in the red box).

01 Background

Guessing content

①The "Fermat point" refers to a point located inside a triangle that minimizes the sum of its distances to the three vertices of the triangle.

②If a triangle △ABC is given, the sum of distances from the Fermat point P of the triangle to its three vertices A, B, and C is smaller than the sum of distances from any other point within the triangle to the same vertices.

③For every given triangle, there is only one unique point that satisfies the condition of being the Fermat point.

01 Background

Introduction to Theorem

If all three interior angles of a triangle are less than 120°, then the three line segments connecting the vertices of the triangle to the Fermat point divide the angles at the Fermat point equally. In other words, the angles formed by these line segments with the sides of the triangle are all equal to 120°. Therefore, the Fermat point of a triangle is also known as the triangle's isogonic center or the equilateral centroid.

If a triangle has an interior angle that is greater than or equal to 120°, then the vertex corresponding to that obtuse angle is the point with the minimum sum of distances. In other words, the vertex of the obtuse angle is the Fermat point of the triangle in such cases.

02

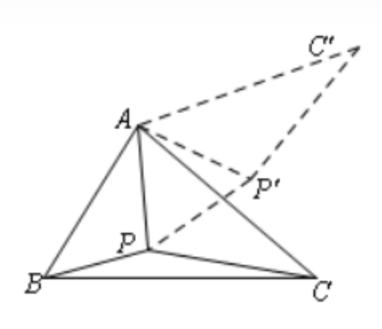
Proof process

02 Proof process

Proof of Fermat's Last Theorem

(1).When the angles of triangle ABC are each less than or equal to 120° , as shown in the diagram below.

As shown in the diagram, rotate triangle APC counterclockwise around point A by 60° to obtain triangle AP'C', then connect PP'.



Then triangle APP' is an equilateral triangle.AP= PP′， P′C′=PC.

So PA+PB+PC= PP′+ PB+ P′C′

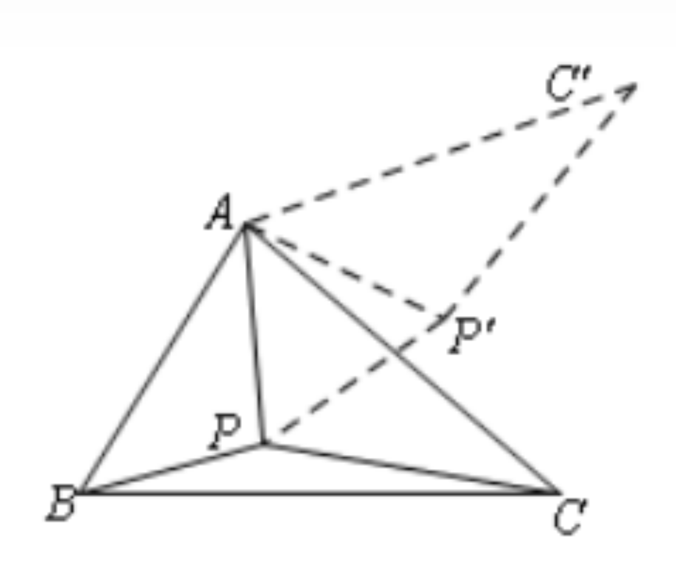
Point C' can be seen as the fixed point obtained by rotating line segment AC counterclockwise around point A by 60° . Since BC' is of fixed length, when points B, P, P', and C' are collinear, PA + PB + PC is minimized.

02 Proof process

Then $\angle BPA=180^\circ-\angle APP'=180^\circ-60^\circ=120^\circ$,

$\angle APC=\angle AP'C'=180^\circ-\angle AP'P=180^\circ-60^\circ=120^\circ$,

$\angle BPC=360^\circ-\angle BPA-\angle APC=360^\circ-120^\circ-120^\circ=120^\circ$,



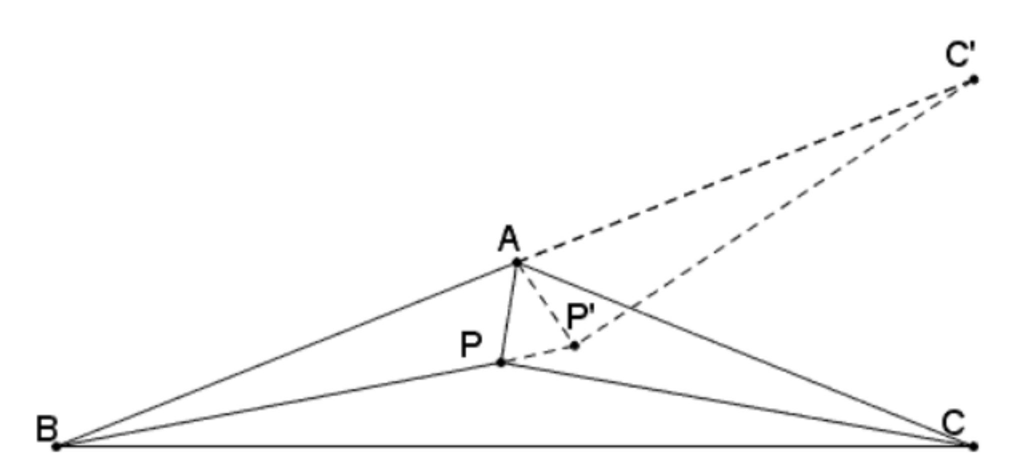
Therefore, when each interior angle is less than 120°, the desired point P forms angles of 120° with each side of the triangle. To locate P, construct arcs of 120° on sides AB and BC. The intersection of these two arcs within the triangle is the point P.

02 Proof process

(2).When triangle ABC has an interior angle that exceeds 120°, as shown in the diagram below.

As shown in the diagram, extend BA to C' such that AC = AC'. Construct angle C'AP' equal to angle CAP and ensure that AP' = AP and PC' = PC. (In other words, we have essentially rotated triangle APC about point A as the center.)

Then $\triangle APC \cong \triangle AP'C'$,



$\because \angle BAC \geq 120^\circ$,

$\therefore \angle PAP' = 180^\circ - \angle BAP - \angle C'AP' = 180^\circ - \angle BAP - \angle CAP = 180^\circ - \angle BAC \leq 60^\circ$

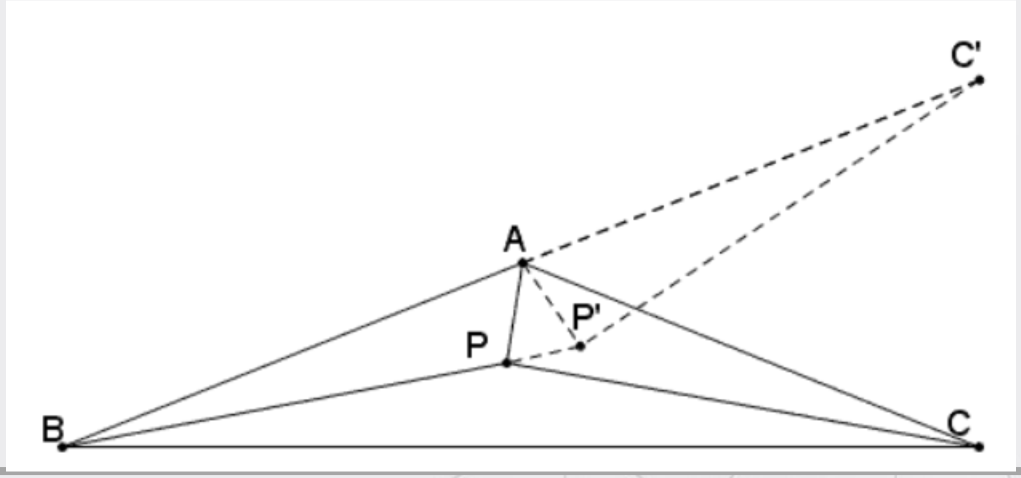
02 Proof process

∴In the isosceles triangle PAP', we have $AP \geq PP'$,

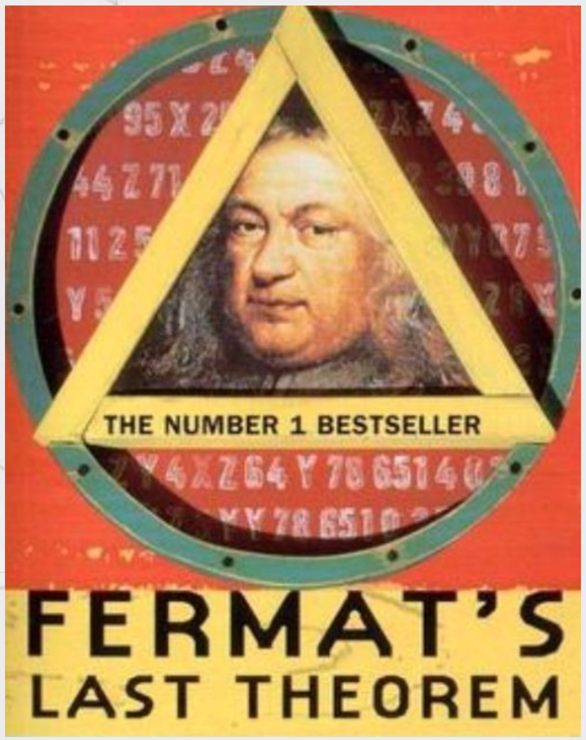
∴$PA+PB+PC \geq PP'+PB+PC' > BC'=AB+AC$

Therefore, A is the Fermat point.

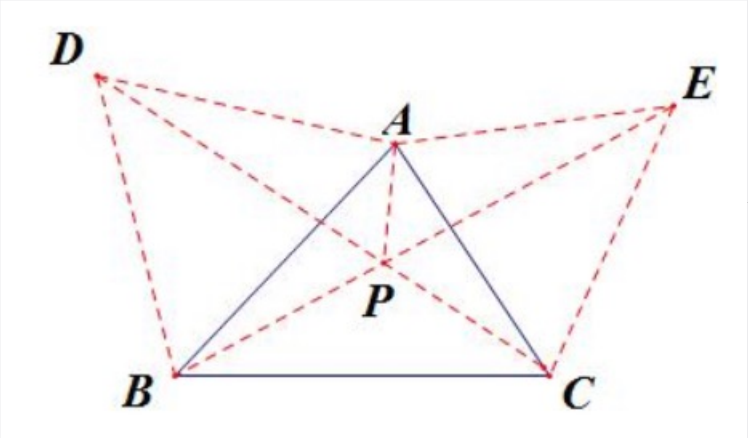
Therefore, when there is an interior angle greater than or equal to 120°, the desired point P is located at the vertex of the obtuse angle.



02 Proof process



When triangle ABC is a triangle with all interior angles less than 120° , construct equilateral triangles △ABD and △ACE outward from sides AB and AC respectively. Then, connect DC and BE. These two lines will intersect at a point, denoted as point P. This point P is the desired Fermat point.



03

Practical Application

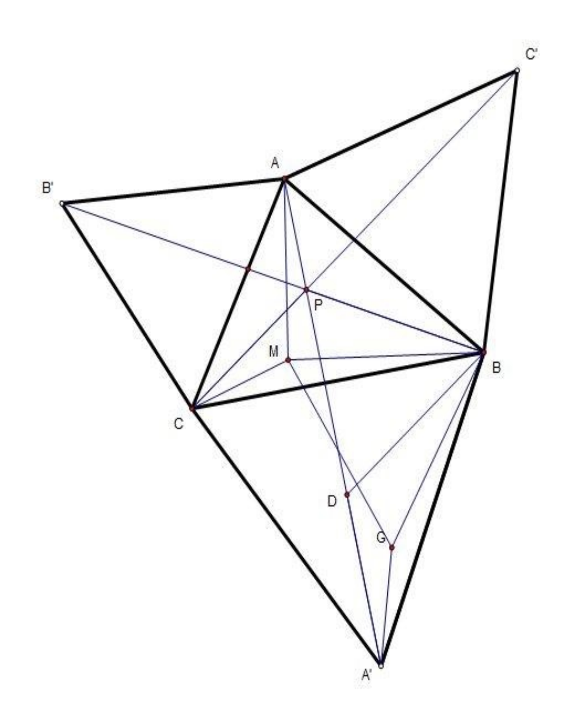
03 Practical Application

Fermat point of a quadrilateral

(1) In a convex quadrilateral ABCD, the Fermat point is the intersection point of the diagonals AC and BD, denoted as point P.

(2) In a concave quadrilateral ABCD, the Fermat point is the concave vertex D (P).

When a triangle has an interior angle that is greater than or equal to 120°, the Fermat point coincides with the vertex of that interior angle. However, when all three interior angles of the triangle are less than 120°, the Fermat point is the point that forms angles of 120° with each pair of lines connecting the Fermat point and the three vertices of the triangle.



03 Practical Application

Another more concise proof is as follows: Let O be the point on the shortest path between the three vertices. Draw a circle with center A and radius AO, and consider this circle as a mirror. It is evident that point O should be the reflection point of the ray originating from B and reaching C after reflection from the mirror (If not, let the reflection point be O'. Then we would have $BO' + CO' < BO + CO$, but $AO' = AO$, which would imply $AO' + BO' + CO' < AO + BO + CO$).

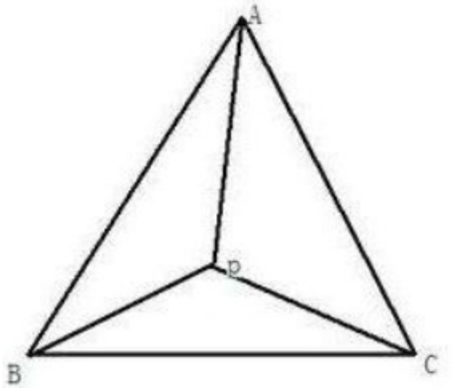
03 Practical Application

Relevant calculations

$$BC = c\frac{PB}{PC} = t = \frac{1+\tan B}{1+\tan C}$$

$$PB = \frac{tc}{\sqrt{t^2+t+1}} \sim\sim\sim\sim PC = \frac{c}{\sqrt{t^2+t+1}}$$

$$Y = PA + PB + PC \sim\sim\sim Y = \sqrt{\frac{a^2+b^2+c^2+4\sqrt{3}S}{2}}$$



c is the correlation coefficient.

t is the comparative coefficient.

The specific value of them depends on the analysis of the specific triangle.

04

Conclusion

04 Conclusion

The conclusion regarding the Fermat point of a triangle is as follows:

1.When a triangle has an interior angle greater than or equal to 120°, the Fermat point coincides with the vertex of that interior angle.

2.When all three interior angles of a triangle are less than 120°, the Fermat point is the point that forms angles of 120° with each pair of lines connecting the Fermat point and the three vertices of the triangle.

When all three interior angles of a triangle are less than 120°, the Fermat point is the point that forms angles of 120° with each pair of lines connecting the Fermat point and the three vertices of the triangle.

THANK YOU

Informant：