

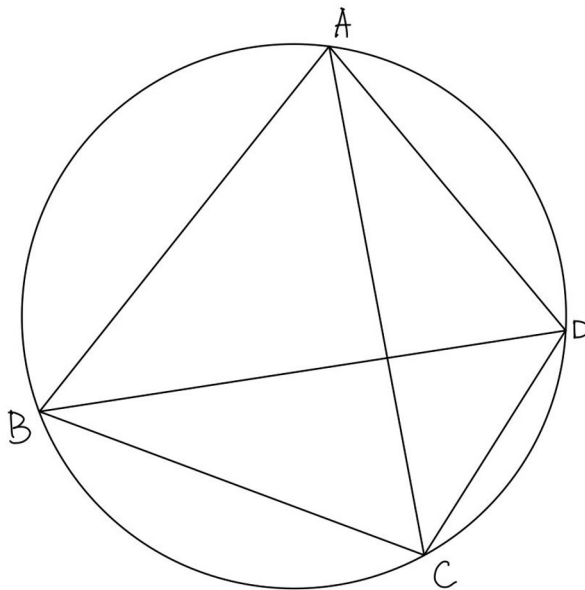
The Ptolemy's Theorem and Kelvin Transform

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Ptolemy's Theorem

Let ABCD be a cyclic quadrilateral, then:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$



Proof 1

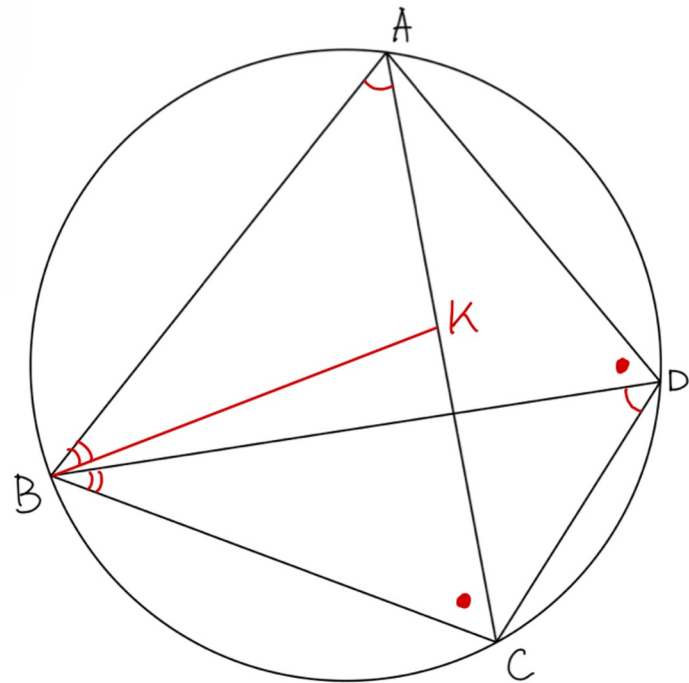
Define point K on AC, $\angle ABK = \angle DBC$

A, B, C, D are concyclic $\rightarrow \angle BAK = \angle BDC$

Thus $\triangle ABK$ is similar to $\triangle DBC$

$$\frac{AB}{BD} = \frac{AK}{CD}$$

$$AB \cdot CD = BD \cdot AK$$

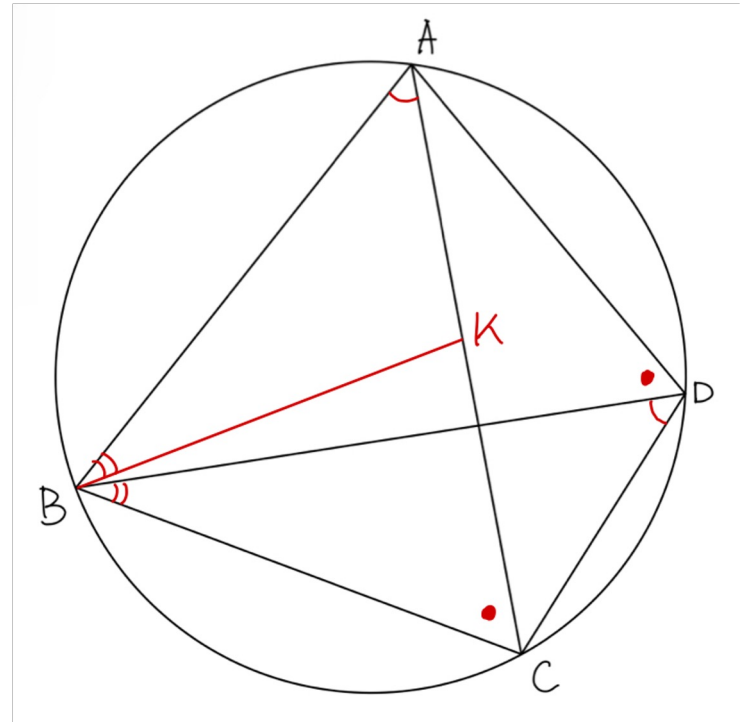


Proof 1

$\triangle KBC$ is similar to $\triangle ABD$

$$\frac{BC}{BD} = \frac{KC}{AD}$$

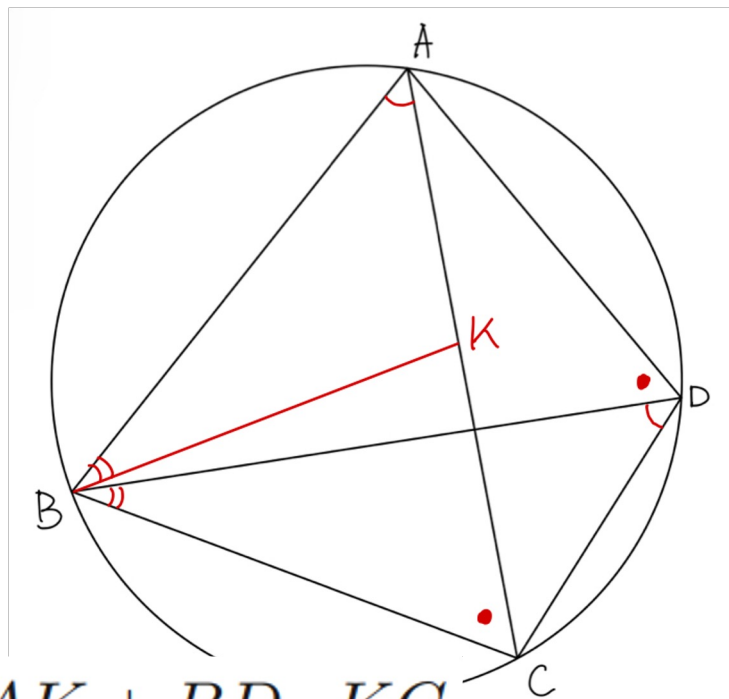
$$AD \cdot BC = BD \cdot KC$$



Proof 1

$$AB \cdot CD = BD \cdot AK$$

$$AD \cdot BC = BD \cdot KC$$



$$\begin{aligned} AB \cdot CD + AD \cdot BC &= BD \cdot AK + BD \cdot KC \\ &= BD \cdot AC \end{aligned}$$

Proof 2

For concyclic triangles:

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$

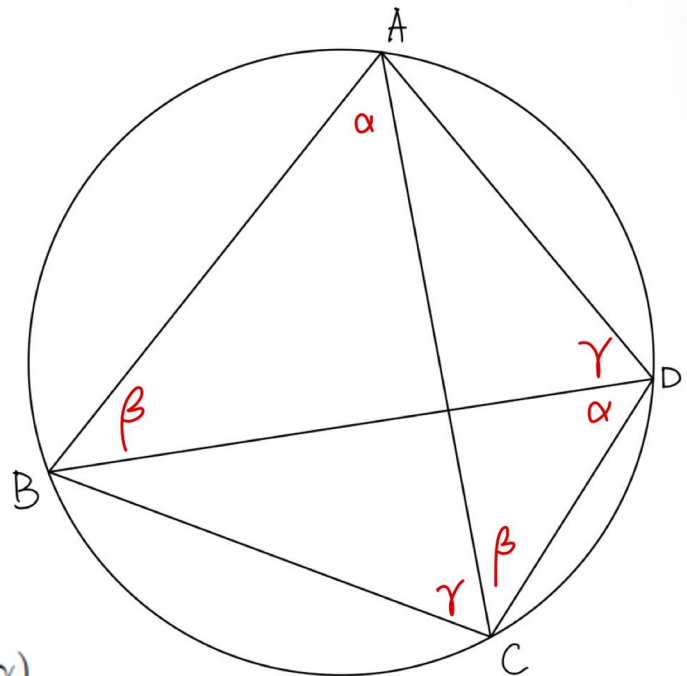
So

$$AB = 2R \sin(\gamma), \quad BC = 2R \sin(\alpha),$$

$$CD = 2R \sin(180 - \alpha - \beta - \gamma) = 2R \sin(\alpha + \beta + \gamma),$$

$$DA = 2R \sin(\beta).$$

$$AC = 2R \sin(\alpha + \gamma), \quad BD = 2R \sin(\beta + \gamma).$$



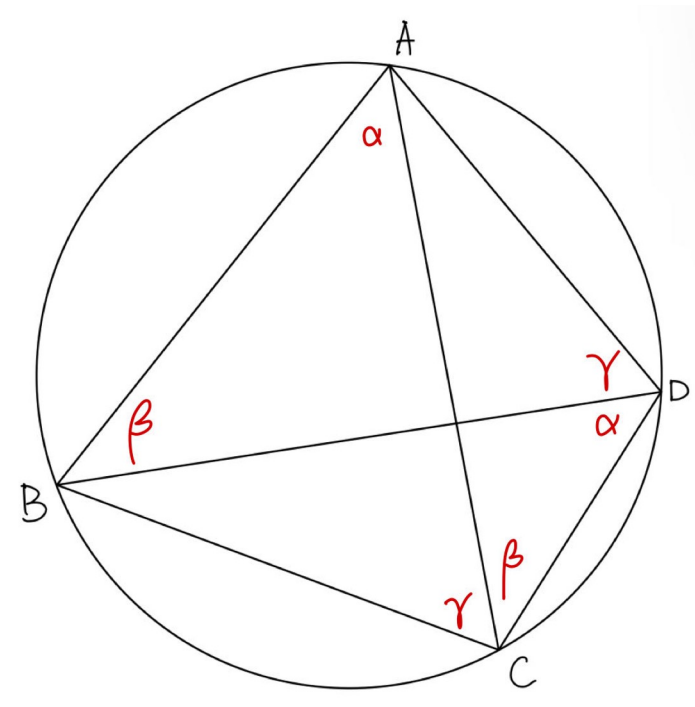
Proof 2

For concyclic triangles:

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$

So

$$\begin{aligned} & \sin(\alpha + \gamma) \cdot \sin(\beta + \gamma) \\ &= \sin(\alpha) \cdot \sin(\beta) + \sin(\gamma) \cdot \sin(\alpha + \beta + \gamma) \end{aligned}$$



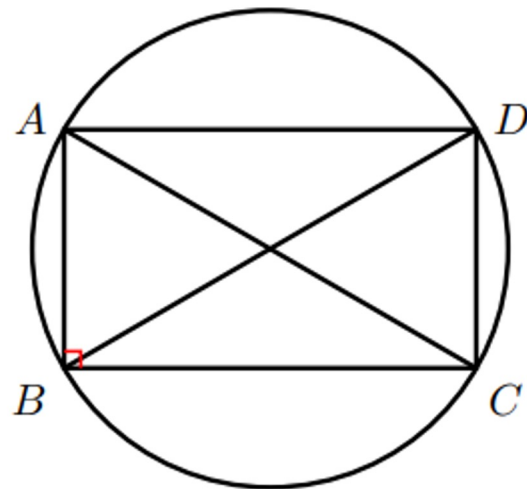
Pythagorean Theorem

Let $\triangle ABC$ be a right triangle. $\angle ABC = 90^\circ$, then:

$$AB^2 + BC^2 = AC^2$$

By the Ptolemy Theorem:

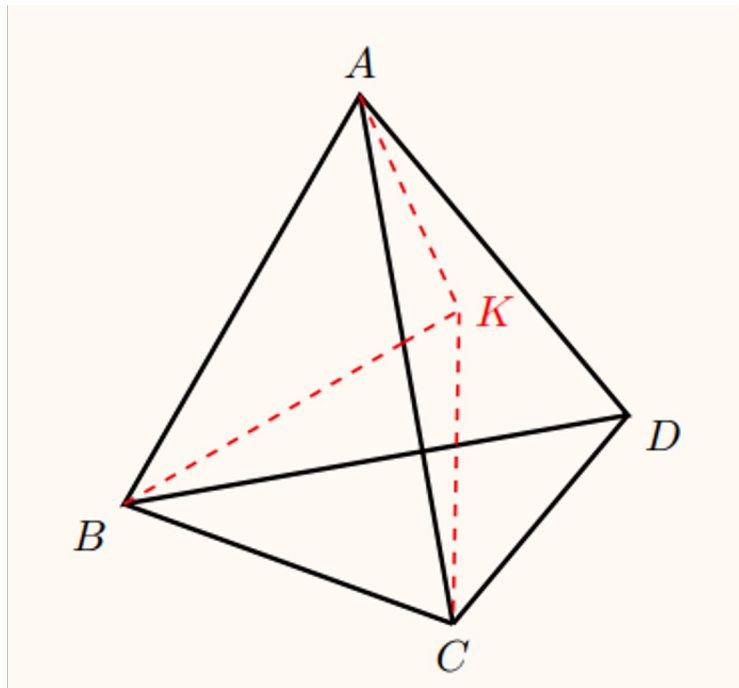
$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$



Ptolemy Inequality

Let ABCD be a quadrilateral, then:

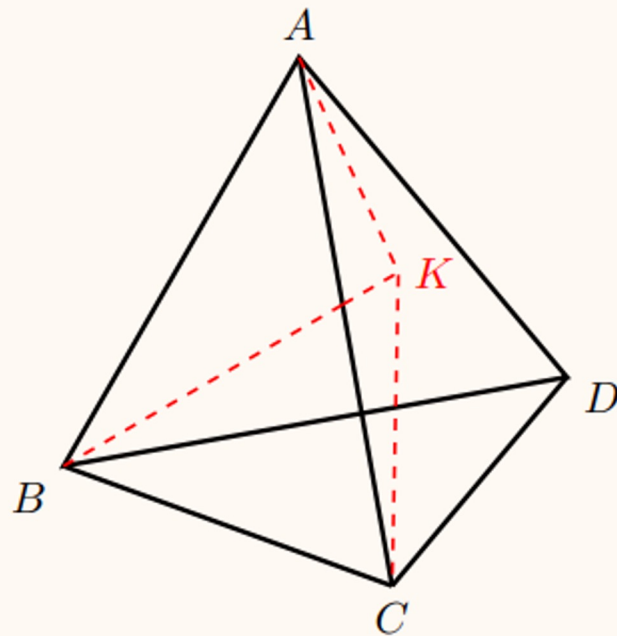
$$AC \cdot BD \leq AB \cdot CD + AD \cdot BC$$



Ptolemy Inequality

$$\angle ABK = \angle DBC$$

$$\left| \frac{AB}{DB} = \frac{BK}{BC} \right|$$

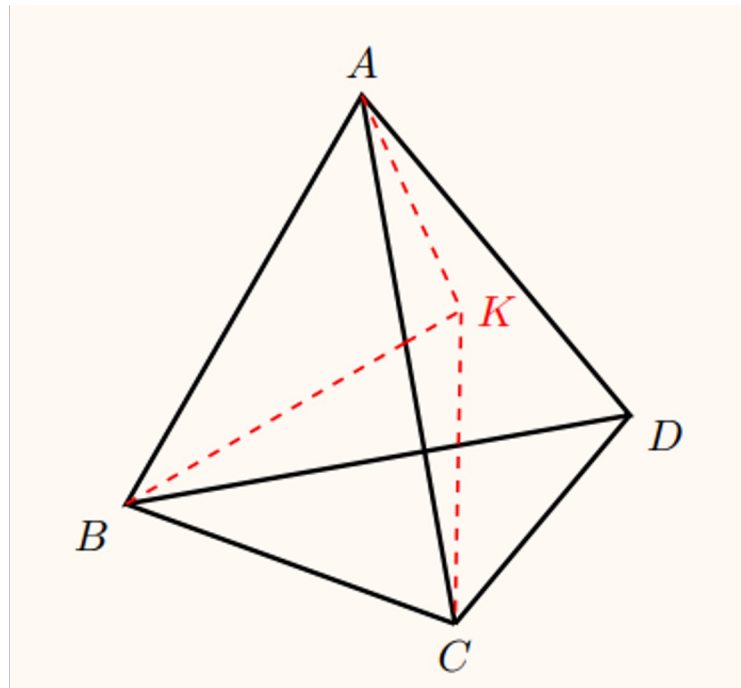


$$\frac{AB}{BK} = \frac{DB}{BC}, \quad \frac{AK}{CD} = \frac{AB}{BD}, \quad \frac{AD}{KC} = \frac{DB}{BC}.$$

Ptolemy Inequality

$$AB \cdot CD = BD \cdot AK,$$

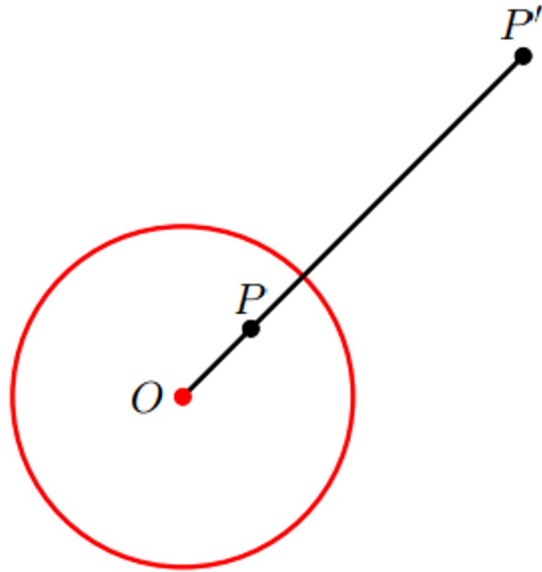
$$AD \cdot BC = BD \cdot KC.$$



$$AB \cdot CD + AD \cdot BC = BD \cdot (AK + KC) \geq BD \cdot AC.$$

Kelvin Transform

$$OP \cdot OP' = r^2.$$

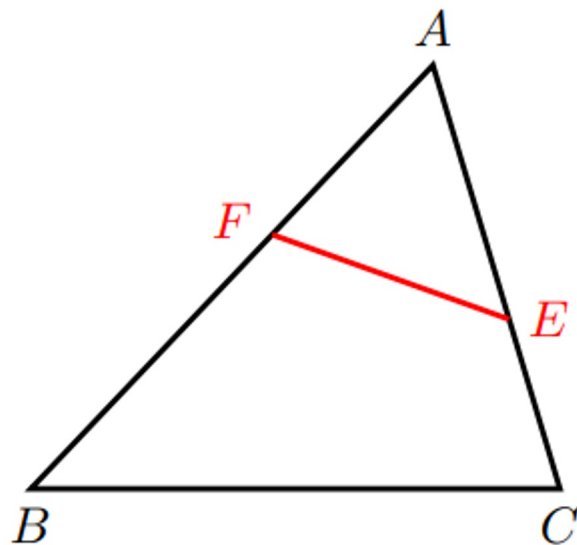


Preparation

$$AE \cdot AC = 1, \quad AF \cdot AB = 1.$$

$$AE \cdot AC = AF \cdot AB,$$

$$\frac{AE}{AB} = \frac{AF}{AC}.$$



Line EF is called an *anti-parallel* line.

$$BC = \frac{AB}{AE} \cdot EF.$$

$$BC = \frac{EF}{AE \cdot AF}.$$

Proof 3

$$AB \cdot AB' = AC \cdot AC' = AD \cdot AD' = 1.$$

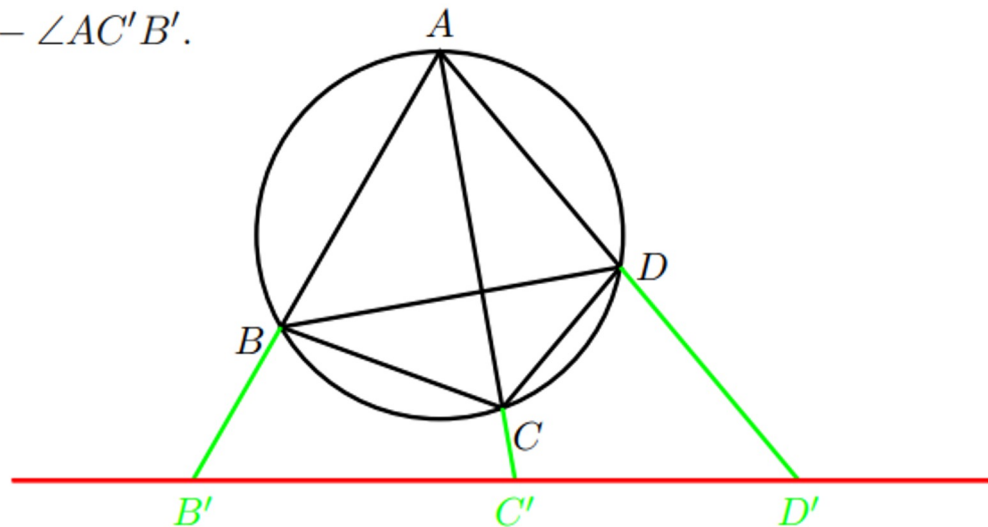
$$\angle AC'B' = \angle ABC = 180^\circ - \angle ADC = 180^\circ - \angle AC'D'.$$

$$B'C' = \frac{BC}{AB \cdot AC}$$

$$B'D' = \frac{BD}{AB \cdot AD}$$

$$C'D' = \frac{CD}{AC \cdot AD}$$

$$\frac{BD}{AB \cdot AD} = \frac{BC}{AB \cdot AC} + \frac{CD}{AC \cdot AD}$$



The background is a solid pink color. In the top right corner, there is a decorative pattern of overlapping geometric shapes, including triangles and squares, in various shades of pink and magenta.

The End