# Trilinear Coordinate System

Natasha Xiao[1], jxiao12@uci.edu

(last updated: March 21, 2023)

## 1  Introduction

Given a fixed triangle, we can define a coordinate system called *trilinear coordinate system* to the Euclidean plane – with respect to the triangle. Such a coordinate system is useful in triangle geometry because it adapts the shape of the given triangle.

Trilinear coordinate system was introduced by *Plücker* in 1835. The point on the Euclidean plane is represented by homogeneous coordinates $(x : y : z)$ or $(x, y, z)$, where the latter has different meaning than the homogeneous coordinates in projective geometry.

Before introducing the trilinear coordinate system, we first need to introduce the concept of *signed distance*, which is a generalization of the distance from a point to a line.

> **Definition 1. (Signed Distance)**
>
> *Let*
> $$px + qy + r = 0$$
> *represent a line $L$ in the Euclidean plane. Let $P = (x_0, y_0)$ be a point. Then the signed distance of $P$ to the line $L$ is defined to be*
> $$\mathrm{dist}(P, L) = \frac{px_0 + qy_0 + r}{\sqrt{p^2 + q^2}}.$$

We make two remarks on the signed distance. First, $|\mathrm{dist}(P, L)|$ is the distance of the point to the line by analytic geometry. Second, the signed distance depends not only on the point and the line, but also on the *orientation* of the line. Both $ax + by + c = 0$ and $-ax - by - c = 0$ represent the same line, but the corresponding signed distances differ by a negative sign.

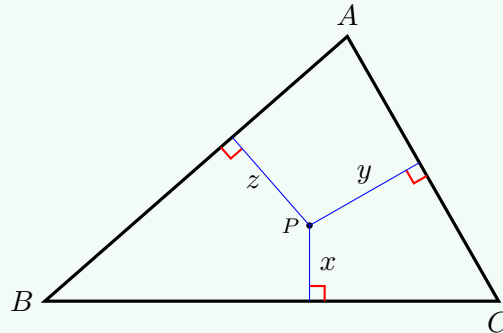> **Definition 2. (Trilinear Coordinate System)**
>
> *Let $\triangle ABC$ be a fixed triangle. We shall choose the orientations of the lines $BC, CA, AB$ so that any point in the interior of the triangle would have positive signed distances to all three sides.*
>
> *Let $P$ be any point in the plane. The trilinear coordinates of $P$ are the ratios of its signed distances to the three sides $BC, CA, AB$.*
>
> *In the following picture, $P$ is an interior point of $\triangle ABC$. Let $(x, y, z)$ be the distances*

*(=signed distances) of $P$ to $BC, CA, AB$, respectively. Then, the trilinear coordinates of $P$ is given by $(x, y, z)$, or $(x : y : z)$.*



## 2  Basic Properties of the Trilinear Coordinate System

The definition in the last section leaves a theoretical question whether trilinear coordinates are well-defined or not.

**Theorem 1**

*Trilinear coordinate system is indeed a coordinate system. For homogeneous coordinates $(x, y, z)$ and $(x_1, y_1, z_1)$, if there is a non-zero number $k$ such that $x = kx_1, y = ky_1, z = kz_1$, then $(x, y, z)$ and $(x_1, y_1, z_1)$ represent the same point.*

> **Proof.**  We use the picture in Definition 2. Let $a, b, c$ be the lengths of $BC, CA, AB$, respectively. Let $\tilde{x}, \tilde{y}, \tilde{z}$ be the signed distances to $BC, CA, AB$, respectively. We claim that
> $$a\tilde{x} + b\tilde{y} + c\tilde{z} = 2\Delta, \tag{1}$$
> where $\Delta$ is the area of $\triangle ABC$.
>
> To prove the above identity, we observe that the signed distances $\tilde{x}, \tilde{y}, \tilde{z}$ are linear functions of $P$, hence so is $a\tilde{x} + b\tilde{y} + c\tilde{z}$. When $P$ is an interior point of $\triangle ABC$, we then have
> $$a\tilde{x} + b\tilde{y} + c\tilde{z} = 2S_{\triangle PBC} + 2S_{\triangle PCA} + 2S_{\triangle PAB} = 2\Delta.$$
> Therefore, such a linear function must be equal to the constant $2\Delta$. This proves the claim.
>
> By (1), we conclude that if $(x, y, z) = k(\tilde{x}, \tilde{y}, \tilde{z})$, then $k$ must be positive. Thus $(x, y, z)$ and $(x_1, y_1, z_1)$ represent the same point.
>
> ■

**Remark** It is impossible for all three trilinear coordinates to be non-positive.
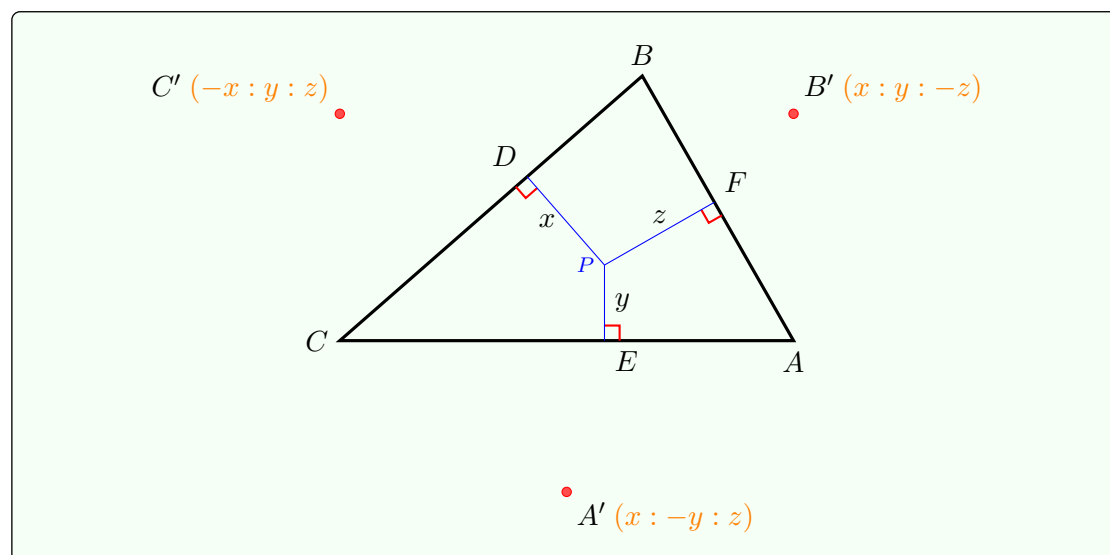
- The coordinates of vertices $A$, $B$, and $C$ of the triangle are $(1 : 0 : 0), (0 : 1 : 0)$, and $(0 : 0 : 1)$, respectively.
- The **positive** and **negative** signs of the *trilinear coordinate* components can be determined according to the following rules:
    - The component $x$ of a point $P$ is positive if it is on the same side of the edge $BC$ as vertex $A$, and negative when $P$ and vertex $A$ are on opposite sides of $BC$.
    - The component $y$ of a point $P$ is positive if it is on the same side of the edge $CA$ as vertex $B$, and negative when $P$ and vertex $B$ are on opposite sides of $CA$.
    - The component $z$ of a point $P$ is positive if it is on the same side of the edge $AB$ as vertex $C$, and negative when $P$ and vertex $C$ are on opposite sides of $AB$.

⤢ **External Link.** *Here is the introduction of the Trilinear Coordinate System in Wikipedia.*

## 3   The Trilinear Coordinates of the Five Triangle Centers

> **Theorem 2**
>
> *Given a triangle $\triangle ABC$ with three vertices $A$, $B$, and $C$, each vertex corresponds to its opposite side length $a$, $b$ and $c$, respectively.*
>
> *The trilinear coordinates of the five triangle centers:*
> - *The trilinear coordinates for the incenter is $(1 : 1 : 1)$; and three excenters are $(-1 : 1 : 1), (1 : -1 : 1)$, and $(1 : 1 : -1)$, respectively.*
> - *The trilinear coordinates for the centroid is $(\csc(A) : \csc(B) : \csc(C))$ or $(a^{-1} : b^{-1} : c^{-1})$.*
> - *The trilinear coordinates for the circumcenter is $(\cos(A) : \cos(B) : \cos(C))$.*
> - *The trilinear coordinates for the orthocenter is $(\sec(A) : \sec(B) : \sec(C))$.*

**Proof.** It is well known that the (signed) distances of the incenter to the three sides are equal. Therefore the trilinear coordinates of the incenter is $(1, 1, 1)$.

Similarly, the distances of any excenters to the three sides are equal as well. However, one of the signed distances is negative. Thus the coordinates of the excenters are $(-1 : 1 : 1), (1 : -1 : 1)$, and $(1 : 1 : -1)$, respectively.

Let $G$ be the centroid of $\triangle ABC$ in the following picture. Let $x, y, z$ be the distances to the three sides $BC, CA$ and $AB$, respectively. By the property of the centroid, the area of $\triangle BGC$ = the area of $\triangle AGC$ = the area of $\triangle AGB$, which implies $ax = by = cz$.
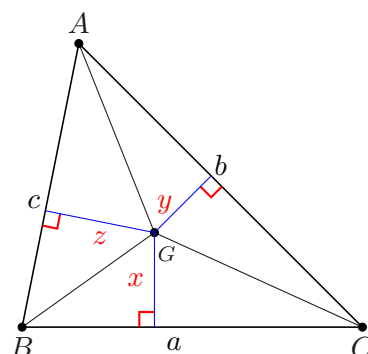


Therefore, the trilinear coordinate for the centroid of $\triangle ABC$ is

$$(x : y : z) = (a^{-1} : b^{-1} : c^{-1}) = (\csc(A) : \csc(B) : \csc(C)).$$

Now we turn to the circumcenter. In the following picture, let $O$ be the circumcenter.
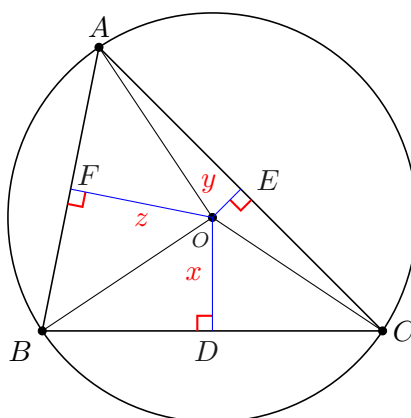


By definition, the circumcenter is the center of the circumcircle, which is the circle passing through the three vertices $A, B$, and $C$. Since $\angle BAC$ is an inscribed angle, we have

$$\angle A = \frac{1}{2}\angle BOC = \angle BOD.$$
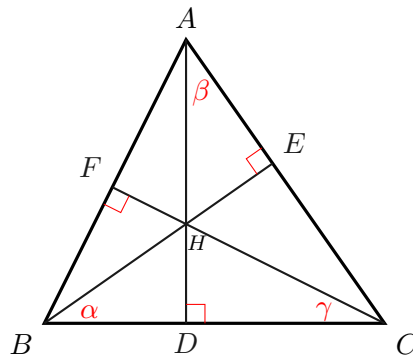
Therefore,

$$x = R\cos A,$$

where $R$ is the radius of the circumcircle. Similarly, we have

$$y = R\cos B, \quad z = R\cos C.$$

Thus, the *trilinear coordinates* of the circumcenter of $\triangle ABC$ are

$$(\cos A : \cos B : \cos C).$$

    Finally, we turn to the orthocenter. In the following picture, let $H$ be the orthocenter of $\triangle ABC$.



Let $\angle EBC = \alpha$, $\angle CAD = \beta$ and $\angle FCB = \gamma$. Then

$$\begin{aligned}
HD &= BD\tan\alpha = BD\cot C \\
&= AB\cos B\cot C = \frac{2R\cos B\cos C\cos A}{\cos A} \\
&= 2R\cos B\cos C.
\end{aligned}$$

Similarly, we have

$$HE = 2R\cos A\cos C, \quad FH = 2R\cos A\cos B$$

Therefore, the *trilinear coordinates* of $H$ are

$$(R\cos B\cos C : R\cos A\cos C : R\cos A\cos B) = (\sec A : \sec B : \sec C).$$

                                                             ■

## 4   Converting Between Cartesian and Trilinear Coordinates

    In this section, we prove a transformation formula between *Trilinear coordinates* and *Cartesian coordinates*.

> **Theorem 3**
>
> *Let $a, b, c$ be the side lengths of $\triangle ABC$, respectively, and let $(x : y : z)$ be the trilinear coordinates of a point $P$. Let $\vec{A}, \vec{B}, \vec{C}$ be the vector representations of vertexes $A, B, C$,*
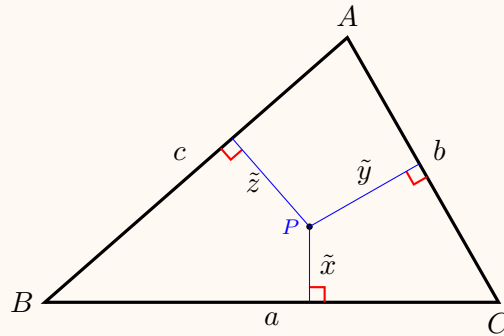
*respectively. Then we have*

$$\vec{P} = \frac{ax}{ax + by + cz}\vec{A} + \frac{by}{ax + by + cz}\vec{B} + \frac{cz}{ax + by + cz}\vec{C},$$

*where $\vec{P}$ is the vector representation of $P$. Conversely, let*

$$\vec{P} = k_1\vec{A} + k_2\vec{B} + k_3\vec{C},$$

*where $k_1, k_2, k_3$ are real numbers such that $k_1 + k_2 + k_3 = 1$. Then the trilinear coordinates of $P$ are given by*

$$(a^{-1}k_1 : b^{-1}k_2, c^{-1}k_3).$$



**Proof.**   We make the following observation regarding to the signed distance. Let $\ell$ be the equation of a line $L$ and let $P$ be a point. Then by Definition 1, the signed distance of $P$ to the line is given by $c(\ell)\ell(\vec{P})$, where $c(\ell)$ is a constant depending only on $\ell$.

Let $\tilde{x}, \tilde{y}, \tilde{z}$ be the signed distances of $P$ to $BC, CA$ and $AB$, respectively, and let $\ell_{BC} = 0$ be the equations of the line $BC$. Then

$$\tilde{x} = c(\ell_{BC})\ell_{BC}(\vec{P}).$$

Writing $\vec{P} = k_1\vec{A} + k_2\vec{B} + k_3\vec{C}$, with $k_1 + k_2 + k_3 = 1$, we have

$$\tilde{x} = k_1 c(\ell_{BC})\ell_{BC}(\vec{A})$$

because $\ell_{BC}(\vec{B}) = \ell_{BC}(\vec{C}) = 0$. Let $h_1$ be the height of $\triangle ABC$ over the side $BC$. Then $h_1 = c(\ell_{BC})\ell_{BC}(\vec{A})$ and hence $\tilde{x} = k_1 h_1$. Similarly, we have $\tilde{y} = k_2 h_2$ and $\tilde{z} = k_3 h_3$, where $h_2, h_3$ are the heights over $CA, AB$, respectively.

Let $(x : y : z)$ be the trilinear coordinates of $P$. Then

$$\frac{ax}{ax + by + cz} = \frac{a\tilde{x}}{a\tilde{x} + b\tilde{y} + c\tilde{z}}.$$

Since $a\tilde{x} + b\tilde{y} + c\tilde{z} = 2\Delta$ by (1), we have

$$\frac{ax}{ax + by + cz} = k_1.$$

Similarly, we have

$$\frac{by}{ax + by + cz} = k_2, \quad \frac{cz}{ax + by + cz} = k_3.$$

This proves the first part of the theorem. For the second part, we just need to observe that

$$h_1 : h_2 : h_3 = a^{-1} : b^{-1} : c^{-1}.$$

Thus

$$(\tilde{x} : \tilde{y} : \tilde{z}) = (k_1 h_1 : k_2 h_2 : k_3 h_3) = (a^{-1} k_1 : b^{-1} k_2, c^{-1} k_3).$$

∎

# 5 Example

**Theorem 4. (Euler Line)**

*In any non-equilateral triangle, the orthocenter, the centroid, and the circumcenter are collinear. The line is called the Euler Line.*

**Proof.** We first remark that three points $P_i$ ($i = 1, 2, 3$) with trilinear coordinates $(x_i : y_i : z_i)$ ($i = 1, 2, 3$), respectively, are collinear if and only if

$$\det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = 0.$$

By Theorem 2, in order to prove the theorem, we only need to prove

$$\det \begin{bmatrix} \sec(A) & \sec(B) & \sec(C) \\ \csc(A) & \csc(B) & \csc(C) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix} = 0.$$

Multiplying the 1st row by $(\cos A \cos B \cos C)$ and the 2nd row by $(\sin A \sin B \sin C)$, we get

$$\sigma \stackrel{def}{=} \det \begin{bmatrix} \sin(B)\sin(C) & \sin(C)\sin(A) & \sin(A)\sin(B) \\ \cos(B)\cos(C) & \cos(C)\cos(A) & \cos(A)\cos(B) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix}.$$

Subtracting the first row by the second row, we get

$$\sigma = \begin{bmatrix} -\cos(B+C) & -\cos(C+A) & -\cos(A+B) \\ \cos(B)\cos(C) & \cos(C)\cos(A) & \cos(A)\cos(B) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix}.$$

Since $\angle A + \angle B + \angle C = 180°$, we have

$$\cos(A) = -\cos(B+C), \cos(B) = -\cos(C+A), \cos(C) = -\cos(A+B).$$

Thus

$$\sigma = \begin{bmatrix} \cos(A) & \cos(B) & \cos(C) \\ \cos(B)\cos(C) & \cos(C)\cos(A) & \cos(A)\cos(B) \\ \cos(A) & \cos(B) & \cos(C) \end{bmatrix} = 0,$$

completing the proof of the theorem.                                                ■

# 6  Barycentric Coordinate System

There is a similar coordinate system, called

> **Definition 3. (Barycentric Coordinate System)**
>
> *Let $\triangle ABC$ be a fixed triangle. We shall choose the orientations of the lines $BC, CA, AB$ so that any point in the interior of the triangle would have positive signed distances to all three sides.*
>
> *The signed area of $\triangle PBC$ is defined as $\frac{1}{2}BC \cdot x$, where $x$ is the signed distance from $P$ to $BC$. The singed are of $\triangle PCA$ and $\triangle PAB$ can be defined in a similar way.*
>
> *Let $P$ be any point in the plane. The barycentric coordinates of $P$ are the ratios of the signed areas of $\triangle PBC, \triangle PCA$ and $\triangle PAB$.*
>
> *In the following picture, $P$ is an interior point of $\triangle ABC$. Let $BC = a, CA = b$ and $AB = c$. Let $(x, y, z)$ be the distances (=signed distances) of $P$ to $BC, CA, AB$, respectively. Then, the barycentric coordinates of $P$ is given by $(ax, by, cz)$, or $(ax : by : cz)$.*
>
> 
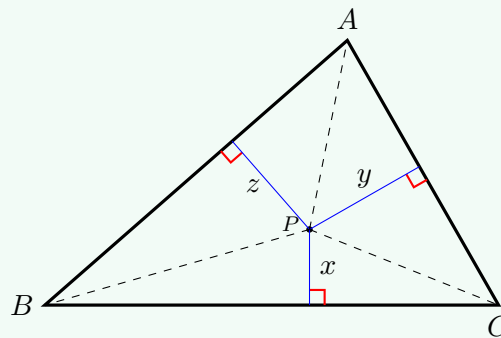
Using the abvoe definition, the folloowing theorem is strictforward.

> **Theorem 5**
>
> *Let $(x, y, z)$ be the trilinear coordinates of a point $P$ with respect to a triangle $\triangle ABC$ with side lengths $a, b, c$, respectively. Then the brarycentric coordinates of $P$ is given by $(ax, by, cz)$.*