Topic 2

On Ceva’s and Menelaus’s Theorems

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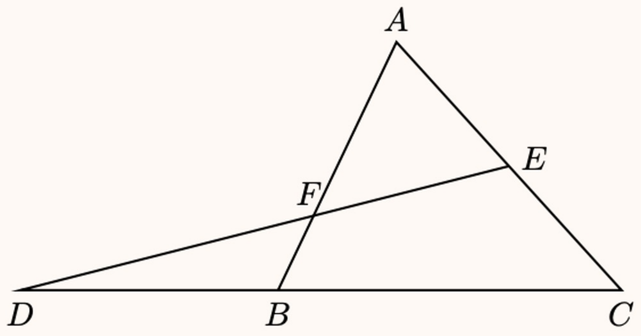
Menelaus' Theorem

The Menelaus Theorem, named for *Menelaus of Alexandria*, is a very powerful theorem in proving collinearity of points on the Euclidean plane.

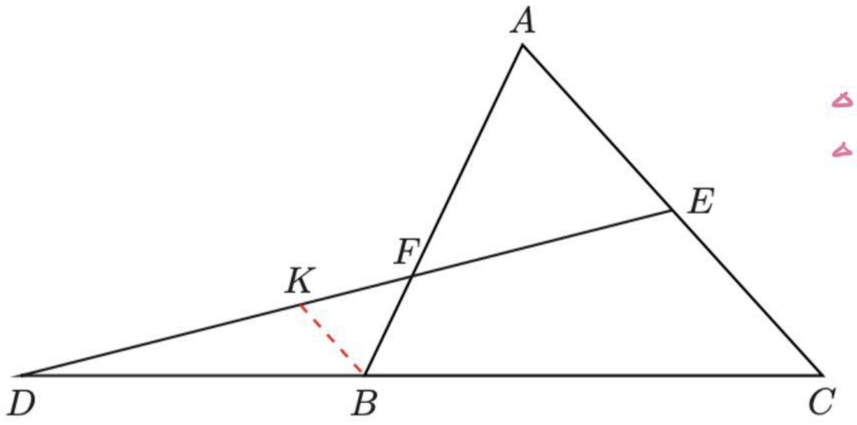
Theorem 1. (Menelaus' Theorem)

In the following $\triangle ABC$, D, E, F are points on BC, CA, and AB, respectively. Assume that D, E, F are collinear. Then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$



Conversely, if the above equation is valid, then the points D, E, F are collinear.



$BK \parallel CA$
$\triangle AFE \sim \triangle FBK$
$\triangle BDK \sim \triangle CDE$

Proof: In the picture below, we draw BK parallel to CA. The idea is to represent all three ratios as ratios on line DE. First, we have

$\frac{BK}{EA} = \frac{KF}{FE} \Leftarrow \triangle AFE \sim \triangle FBK \Rightarrow$

$$\frac{AF}{FB} = \frac{FE}{KF}, \qquad \frac{BD}{DC} = \frac{DK}{DE}$$

$\Leftarrow \triangle BDK \sim \triangle CDE \Rightarrow \frac{CE}{BK} = \frac{DE}{DK}$

We also have

$$\frac{CE}{EA} = \frac{CE}{BK} \cdot \frac{BK}{EA} = \frac{DE}{DK} \cdot \frac{KF}{FE}.$$

$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA}$
$= \frac{FE}{KF} \cdot \frac{DK}{DE} \cdot \frac{DE}{DK} \cdot \frac{KF}{FE} = 1$

Putting the above all together, we prove the theorem.

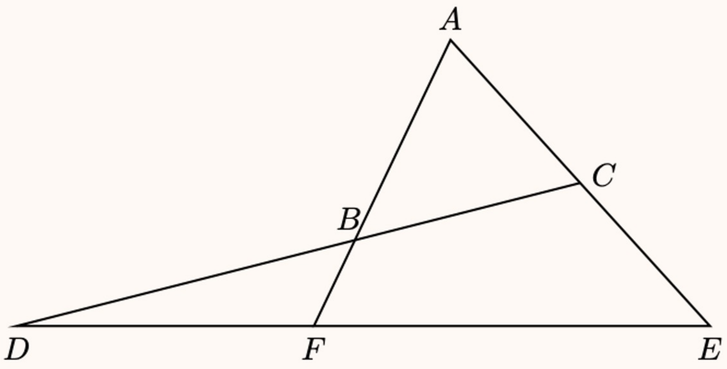
External Menelaus' Theorem

There is a variant of the Menelaus Theorem when the line DEF intersect the extended lines of the three sides of the triangle. The theorem is called the *External Menelaus Theorem*.

Theorem 2. (External Menelaus' Theorem)

In the following $\triangle ABC$, D, E, F are points on the extended lines of BC, CA, and AB, respectively. Assume that D, E, F are collinear. Then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$



Conversely, if the above equation is valid, then the points D, E, F are collinear.

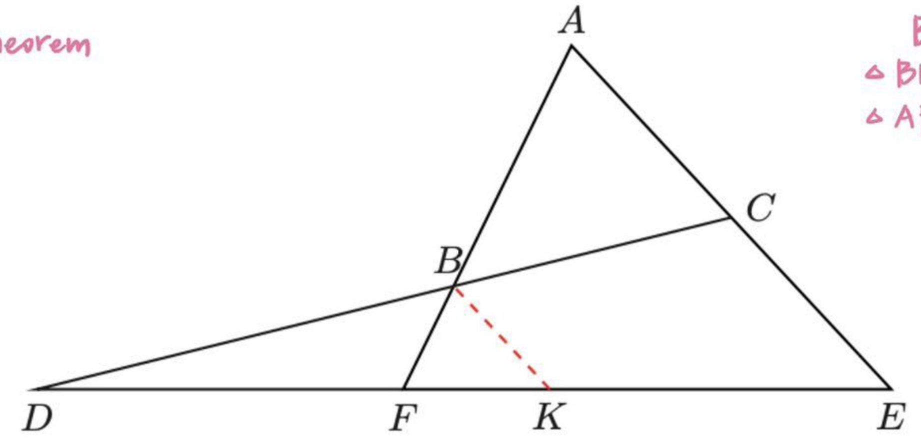
External Menelaus Theorem

$\rightarrow \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$

$BK \parallel AE$

$\triangle BDK \sim \triangle CDE$

$\triangle AFE \sim \triangle BFK$



Proof: We draw BK parallel to AE. By triangle similarity we have

$$\triangle BDK \sim \triangle CDE \rightarrow \quad \frac{BD}{DC} = \frac{BK}{CE},$$

and

$$\triangle AFE \sim \triangle BFK \rightarrow \quad \frac{AF}{FB} = \frac{AE}{BK}.$$

$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA}$

$= \frac{AE}{CE} \cdot \frac{CE}{EA}$

$= 1$

Thus after multiple the above two equations together we will obtain

$$\frac{BD}{DC} \cdot \frac{AF}{FB} = \frac{AE}{CE}, = \frac{BK}{CE} \cdot \frac{AE}{BK}$$

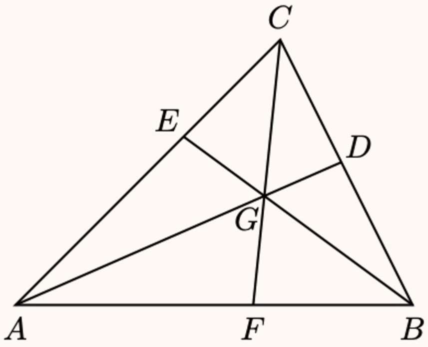
which implies the external Menelaus Theorem.

Ceva's Theorem

Theorem 3. (Ceva's Theorem)

In the following picture, the lines AD, BE, CF are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$



Conversely, if the above equation is valid, then AD, BE, CF are concurrent[a].

Dual to the Menelaus Theorem, the following Ceva Theorem, attributed to Giovanni Ceva, an Italian hydraulic engineer and mathematician, is very powerful in proving the concurrency of lines.

Ceva's Theorem

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

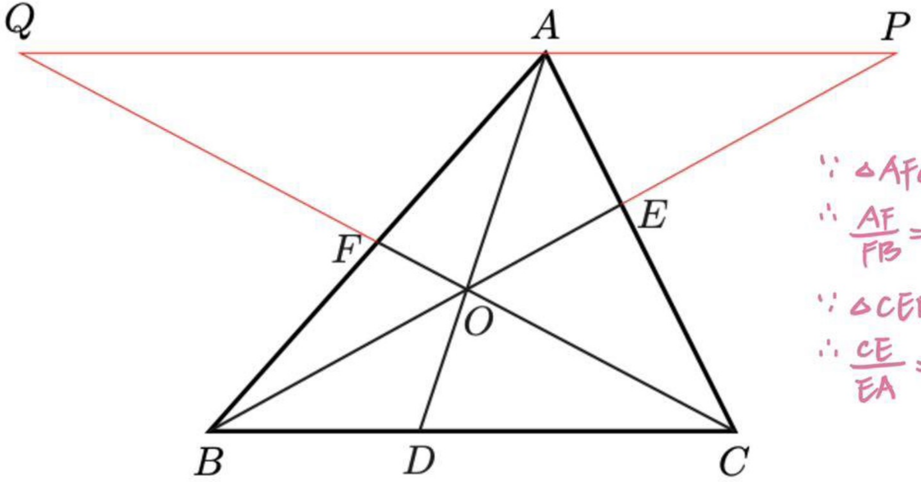
$QP \parallel BC$

$\triangle AOQ \sim \triangle DOC$

$\rightarrow \frac{QA}{DC} = \frac{AO}{OD}$

$\triangle BOD \sim \triangle POA$

$\rightarrow \frac{BD}{AP} = \frac{OD}{AO}$



$QP \parallel BC$

$\triangle AFQ \sim \triangle BFC$

$\triangle CEB \sim \triangle AEP$

$\because \triangle AFQ \sim \triangle BFC$

$\therefore \frac{AF}{FB} = \frac{QA}{BC}$

$\because \triangle CEB \sim \triangle AEP$

$\therefore \frac{CE}{EA} = \frac{BC}{AP}$

Proof: The Ceva Theorem can be proved using similar triangle properties. In the following picture, we have

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{QA}{BC} \cdot \frac{BD}{DC} \cdot \frac{BC}{AP} = \frac{QA}{DC} \cdot \frac{BD}{AP}.$$

We thus have

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{QA}{DC} \cdot \frac{BD}{AP} = \frac{AO}{OD} \cdot \frac{OD}{AO} = 1.$$

Example 1 In the following picture, E, F are points on AC and AB, respectively. DE, DF intersect on the line LK at K and L, respectively. Assume that $LK \parallel BC$. Prove that $AK = AL$.

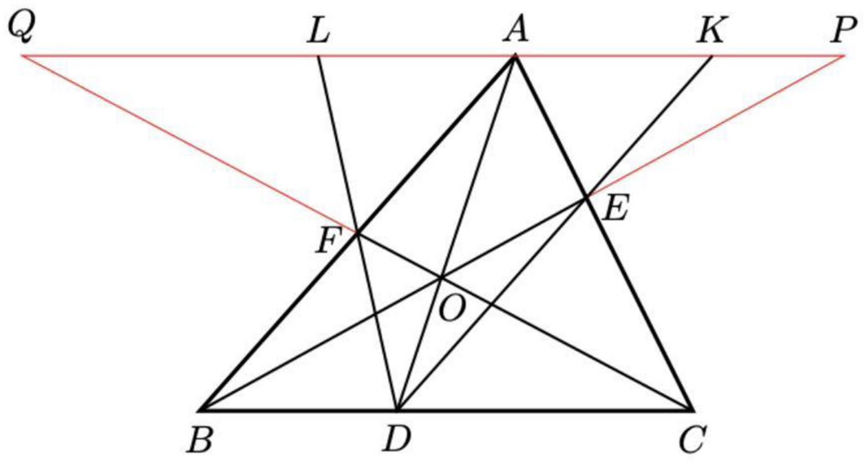
LK // BC

$\triangle ALF \sim \triangle BDF \rightarrow \frac{AL}{BD} = \frac{AF}{FB}$

$\triangle AKE \sim \triangle CDE \rightarrow \frac{AK}{DC} = \frac{AE}{EC}$

Ceva's Theorem

$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$



Proof: Completing the above picture by the red lines. Since

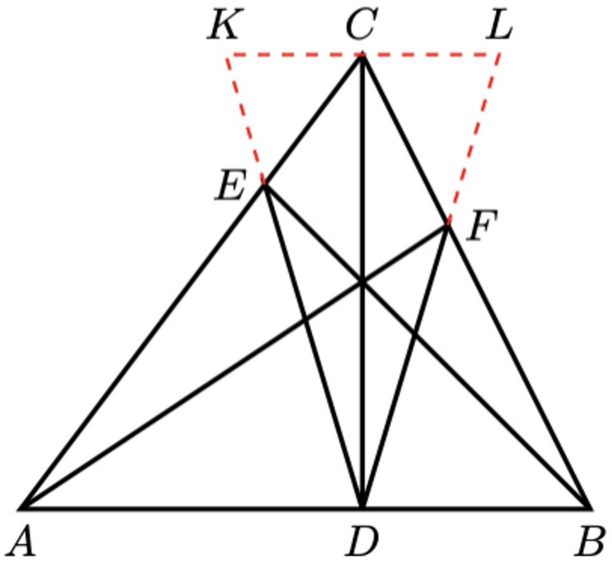
$$AL = \frac{AF}{FB} \cdot BD, \qquad AK = \frac{AE}{EC} \cdot DC,$$

by the Ceva Theorem, we have

$$\frac{AL}{AK} = 1, = \frac{AF \cdot BD \cdot EC}{FB \cdot AE \cdot DC}$$

that is, $AL = AK$.

Example 2 In the following picture, assume that $CD \perp AB$, then $\angle EDC = \angle FDC$.



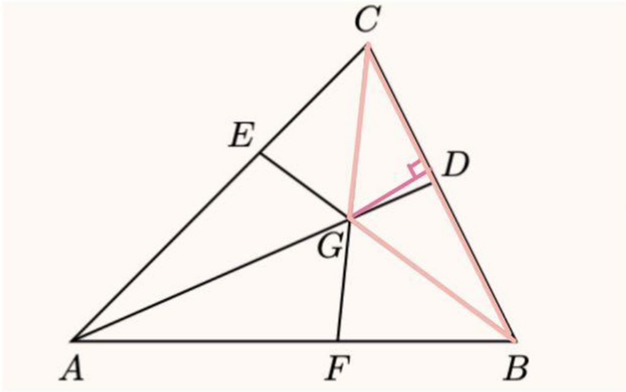
Proof: Using the result of the above example, we conclude that $\triangle DKL$ is isosceles, which implies our result.

Proof of Ceva's Theorem Using Area Method: We use the pictture in Theorem 3 to give an alternate proof to show that if AD, BE,CF are concurrent, then

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Actually it is easy to see that

$$\frac{BD}{DC} = \frac{[\triangle BAG]}{[\triangle CAG]},$$



$$\frac{CE}{EA} = \frac{[\triangle CBG]}{[\triangle ABG]},$$

and

$$\frac{AF}{FB} = \frac{[\triangle ACG]}{[\triangle BCG]}.$$

Multiplying the above equations together, we get

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1,$$

which finishes the proof.

It is also possible to prove the Menelaus' Theorem using the similar area method as well.

Proof of Menelaus Theorem Using Area Method: In the following picture, we have

$\triangle APF \sim \triangle BQF$

$\rightarrow \frac{AF}{FB} = \frac{AP}{BQ}$

$$\frac{AF}{FB} = \frac{AP}{BQ} = \frac{[\triangle DAE]}{[\triangle DBE]}.$$

$[\triangle DAE] = DE \cdot AP \cdot \frac{1}{2}$

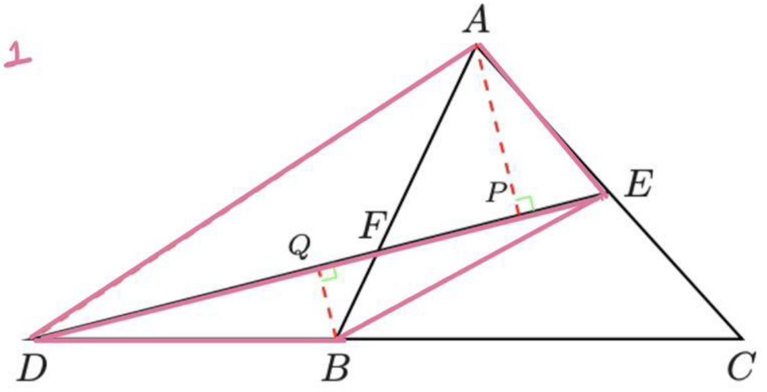
$[\triangle DBE] = DE \cdot BQ \cdot \frac{1}{2}$

Thus we have

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{[\triangle DAE]}{[\triangle DBE]} \cdot \frac{[\triangle DBE]}{[\triangle DCE]} \cdot \frac{[\triangle DCE]}{[\triangle DAE]} = 1.$$

Menelaus Theorem

$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$

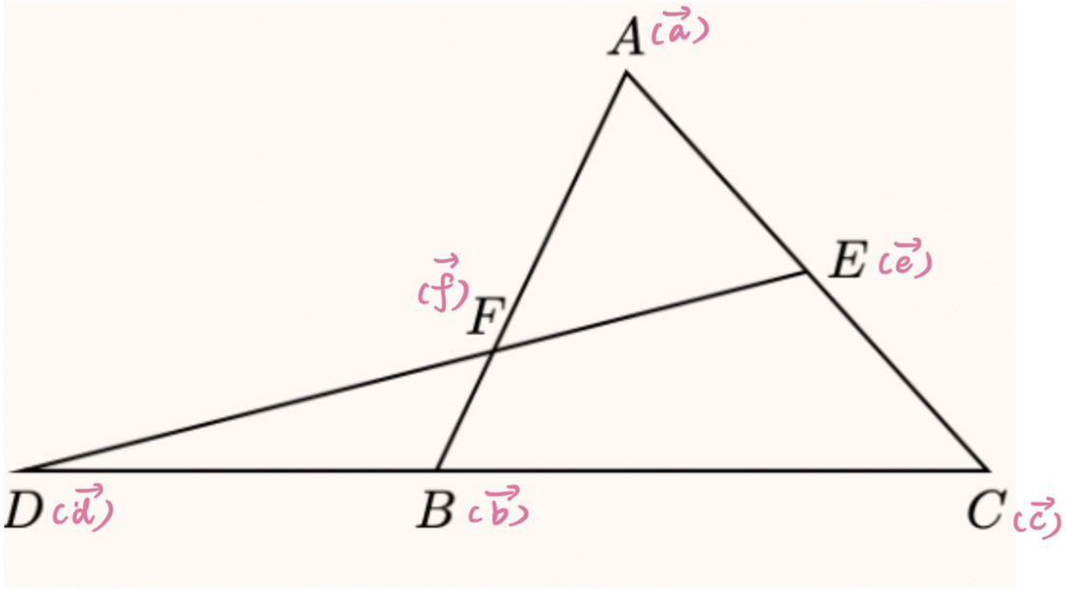


Algebraic Proof of Menelaus Theorem

In this section, we seek the algebra behind both Ceva and Menelaus Theorems using vector algebra. We start with the Menelaus Theorem. Let A, B, C be represented by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, respectively. Assume that

$$\frac{BD}{DC} = \lambda, \qquad \frac{CE}{EA} = \mu, \qquad \frac{AF}{FB} = \nu.$$

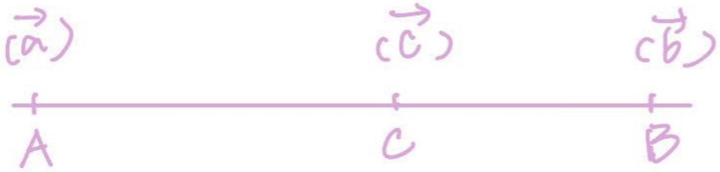
Let D, E, F be represented by vectors $\mathbf{d}, \mathbf{e}, \mathbf{f}$.

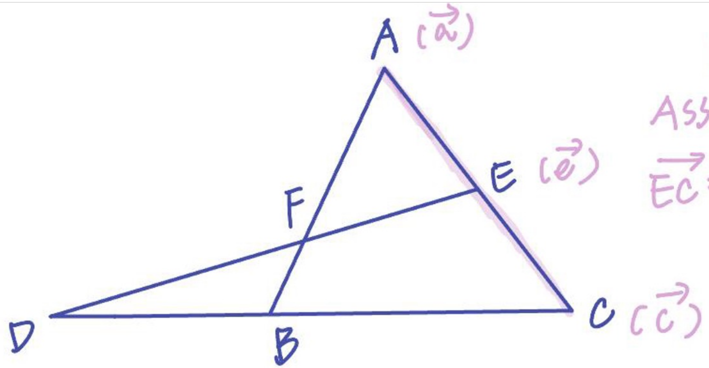


The Definite Ratio Point Formula
（定比分点公式）

The definite ratio point formula is the basic formula of geometry. It states that If we get a line AB, C is a point on the line AB and between the point A and point B. When we define A, B, C as vector a, b, c, vector c can be represented in the form of vector a and vector b, where the coefficients c1 and c2 should be added up to 1. In this case, both c1 and c2 should be greater or equal than 0 and less or equal than 1. However, when point C is located on the extended line AB(outside the point A or point B), the c1 and c2 can be negative but still have a sum equal to 1.

$\vec{c} = c_1\vec{a} + c_2\vec{b}$ $\quad$ $c_1 + c_2 = 1$





$\vec{EC} = \vec{e} - \vec{c} \quad \vec{EA} = \vec{e} - \vec{a}$

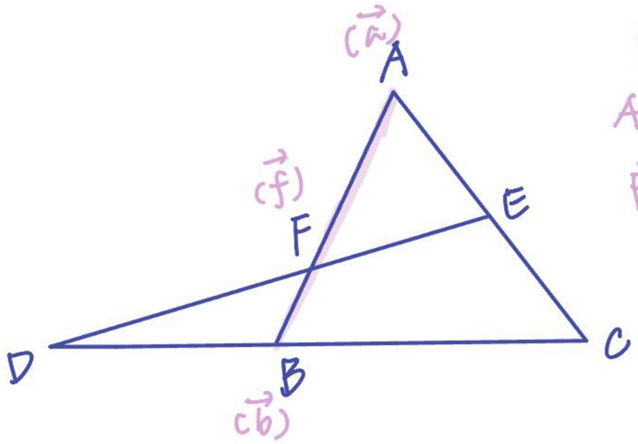
Assume $\vec{e} = c_1\vec{a} + c_2\vec{c} \quad c_1 + c_2 = 1$

$$\begin{aligned}\vec{EC} = \vec{e} - \vec{c} &= c_1\vec{a} + c_2\vec{c} - \vec{c} \\ &= c_1\vec{a} - (\vec{c} - c_2\vec{c}) \\ &= c_1\vec{a} - c_1\vec{c} \\ &= c_1(\vec{a} - \vec{c})\end{aligned}$$

$$\begin{aligned}\vec{EA} = \vec{e} - \vec{a} &= c_1\vec{a} + c_2\vec{c} - \vec{a} \\ &= c_2\vec{c} - (\vec{a} - c_1\vec{a}) \\ &= c_2\vec{c} - c_2\vec{a} \\ &= c_2(\vec{c} - \vec{a})\end{aligned}$$

$$\frac{CE}{EA} = \mu \quad \mu = \frac{|\vec{EC}|}{|\vec{EA}|} = \frac{c_1|\vec{a} - \vec{c}|}{c_2|\vec{c} - \vec{a}|} \Rightarrow \frac{c_1}{c_2} = \mu \quad c_1 + c_2 = 1$$

$$c_1 = \frac{\mu}{1+\mu} \quad c_2 = \frac{1}{1+\mu} \quad \vec{e} = \frac{\mu}{1+\mu}\vec{a} + \frac{1}{1+\mu}c$$



$\overrightarrow{FA} = \vec{f} - \vec{a} \quad \overrightarrow{FB} = \vec{f} - \vec{b}$

Assume $\vec{f} = c_3\vec{a} + c_4\vec{b} \quad c_3 + c_4 = 1$

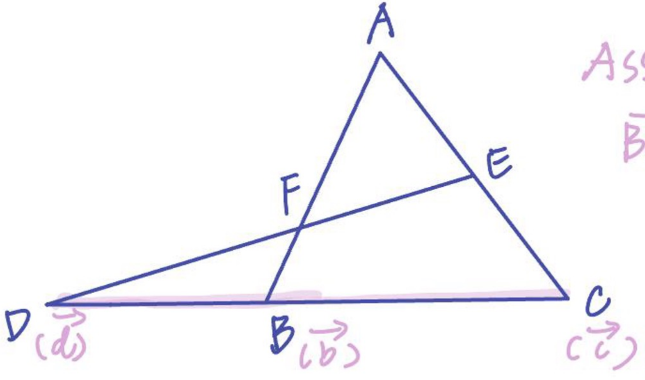
$$\begin{aligned}
\overrightarrow{FA} = \vec{f} - \vec{a} &= c_3\vec{a} + c_4\vec{b} - \vec{a} \\
&= c_4\vec{b} - (\vec{a} - c_3\vec{a}) \\
&= c_4\vec{b} - (1 - c_3)\vec{a} \\
&= c_4\vec{b} - c_4\vec{a} \\
&= c_4(\vec{b} - \vec{a})
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{FB} = \vec{f} - \vec{b} &= c_3\vec{a} + c_4\vec{b} - \vec{b} \\
&= c_3\vec{a} - (1 - c_4)\vec{b} \\
&= c_3\vec{a} - c_3\vec{b} \\
&= c_3(\vec{a} - \vec{b})
\end{aligned}$$

$$\frac{AF}{FB} = \nu.$$

$\nu = \dfrac{|\overrightarrow{FA}|}{|\overrightarrow{FB}|} = \dfrac{c_4|\vec{b} - \vec{a}|}{c_3|\vec{a} - \vec{b}|} \Rightarrow \dfrac{c_4}{c_3} = \nu \quad c_4 + c_3 = 1$

$c_4 = \dfrac{\nu}{1+\nu} \quad c_3 = \dfrac{1}{1+\nu} \qquad \vec{f} = \dfrac{1}{1+\nu}\vec{a} + \dfrac{\nu}{1+\nu} b$



$\overrightarrow{BD} = \vec{b} - \vec{d} \quad \overrightarrow{DC} = \vec{c} - \vec{d}$

Assume $\vec{b} = c_5\vec{d} + c_6\vec{c} \quad c_5 + c_6 = 1$

$\overrightarrow{BD} = \vec{b} - \vec{d} = c_5\vec{d} + c_6\vec{c} - \vec{d}$

$= c_6\vec{c} - (1 - c_5)\vec{d}$

$= c_6\vec{c} - c_6\vec{d}$

$= c_6(\vec{c} - \vec{d})$

$\overrightarrow{DC} = \vec{c} - \vec{d}$

$$\frac{BD}{DC} = \lambda,$$

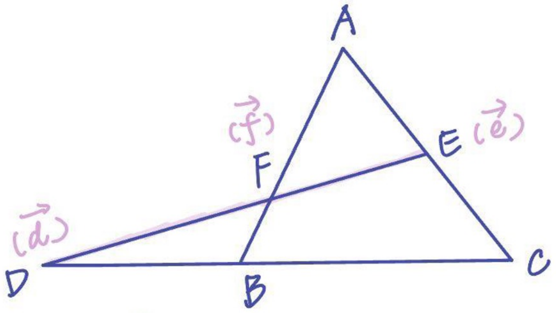
$$\lambda = \frac{|\overrightarrow{BD}|}{|\overrightarrow{DC}|} = \frac{c_6|\vec{c} - \vec{d}|}{|\vec{c} - \vec{d}|} = c_6 = \lambda \quad c_5 = 1 - \lambda$$

$\vec{b} = (1-\lambda)\vec{d} + \lambda\vec{c}$

$(1-\lambda)\vec{d} = \vec{b} - \lambda\vec{c}$

$\vec{d} = \frac{1}{1-\lambda}\vec{b} - \frac{\lambda}{1-\lambda}\vec{c}$

$\vec{d} = \frac{1}{1-\lambda}\vec{b} + \frac{-\lambda}{1-\lambda}\vec{c}$



Assume D, E, F are collinear

d can be written as a linear combination of e, f

$\vec d = c_7 \vec e + c_8 \vec f$ where $c_7 + c_8 = 1$

$\vec d = c_7 \vec e + c_8 \vec f$

$\vec e = \frac{1}{1+\mu}\vec c + \frac{\mu}{1+\mu}\vec a$

$\vec f = \frac{1}{1+v}\vec a + \frac{v}{1+v}\vec b$

$\vec d = \frac{1}{1-\lambda}\vec b + \frac{-\lambda}{1-\lambda}\vec c$

$\frac{1}{1-\lambda}\vec b + \frac{-\lambda}{1-\lambda}\vec c = \frac{c_7}{1+\mu}\vec c + \frac{c_7\mu}{1+\mu}\vec a + \frac{c_8}{1+v}\vec a + \frac{c_8 v}{1+v}\vec b$

$\frac{1}{1-\lambda}\vec b + \frac{-\lambda}{1-\lambda}\vec c = \frac{c_8 v}{1+v}\vec b + \left(\frac{c_7\mu}{1+\mu} + \frac{c_8}{1+v}\right)\vec a + \frac{c_7}{1+\mu}\vec c$

$\frac{1}{1-\lambda} = \frac{c_8 v}{1+v}$

$\frac{-\lambda}{1-\lambda} = \frac{c_7}{1+\mu}$ $\Rightarrow$ $c_7 = \frac{-\lambda(1+\mu)}{1-\lambda}$ where $c_7 + c_8 = 1$

$c_8 = \frac{1+v}{(1-\lambda)v}$

$\frac{c_7\mu}{1+\mu} + \frac{c_8}{1+v} = 0$

$\frac{-\lambda(1+\mu)}{1-\lambda} + \frac{1+v}{(1-\lambda)v} = 1$ $\Rightarrow$ only and if only when $v\lambda\mu = 1$ $c_7 + c_8 = 1$

Otherwise D, E, F are not collinear.

$-v\lambda(1+\mu) + 1 + v = (1-\lambda)v$

$-v\lambda - v\lambda\mu + 1 + v - v + v\lambda = 0$

$v\lambda\mu = 1$

3 The Algebra Behind the Ceva's and Menelaus' Theorems

In this section, we seek the algebra behind both Ceva and Menelaus Theorems using vector algebra. We start with the Menelaus Theorem. Let A, B, C be represented by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, respectively. Assume that

$$\frac{BD}{DC} = \lambda, \qquad \frac{CE}{EA} = \mu, \qquad \frac{AF}{FB} = \nu.$$

Let D, E, F be represented by vectors $\mathbf{d}, \mathbf{e}, \mathbf{f}$. Then we have

$$\mathbf{e} = \frac{1}{1+\mu}\mathbf{c} + \frac{\mu}{1+\mu}\mathbf{a}, \qquad \mathbf{f} = \frac{1}{1+\nu}\mathbf{a} + \frac{\nu}{1+\nu}\mathbf{b}.$$

The vector $\mathbf{d}$ is a little special. Since D is on the extended line segment BC, we have

$$\mathbf{d} = \frac{1}{1-\lambda}\mathbf{b} + \frac{-\lambda}{1-\lambda}\mathbf{c}$$

Assume that D, E, F are collinear. Then we can write one vector, say $\mathbf{d}$, as a linear combination of $\mathbf{e}, \mathbf{f}$, that is, we can have

$$\mathbf{d} = c_1\mathbf{e} + c_2\mathbf{f},$$

which is

$$\frac{1}{1-\lambda}\mathbf{b} + \frac{-\lambda}{1-\lambda}\mathbf{c} = \frac{c_1}{1+\mu}\mathbf{c} + \frac{c_1\mu}{1+\mu}\mathbf{a} + \frac{c_2}{1+\nu}\mathbf{a} + \frac{c_2\nu}{1+\nu}\mathbf{b}.$$

From the above equation, we get

$$\frac{1}{1-\lambda} = \frac{c_2\nu}{1+\nu}, \quad \frac{-\lambda}{1-\lambda} = \frac{c_1}{1+\mu}, \quad \frac{c_1\mu}{1+\mu} + \frac{c_2}{1+\nu} = 0.$$

From the above first two equations, we have

$$c_1 = \frac{-\lambda(1+\mu)}{1-\lambda}, \quad c_2 = \frac{1+\nu}{(1-\lambda)\nu}.$$

Substituting these into the third equation and simplifying, we get $\lambda\mu\nu = 1$. This completes the proof of the Menelaus Theorem.

Conclusion

External Link. *By the above discussion, we know that even though the Menelaus and Ceva Theorems appear to be different, the algebra foundation behind both are the same. In Projective Geometry, we call such pair of theorems dual theorems.*

Thank you for watching this presentation！ 💗