Taylor Circle

Jimena Razo[1], razoji@uci.edu

(last updated: June 18, 2022)

1 Introduction

Named after *Henry Martyn Taylor*[2], the *Taylor Circle* is a circle created by six concyclic points on a triangle. Taylor is well known for having transcribed many important scientific and mathematical works into Braille after he became blind in 1894.

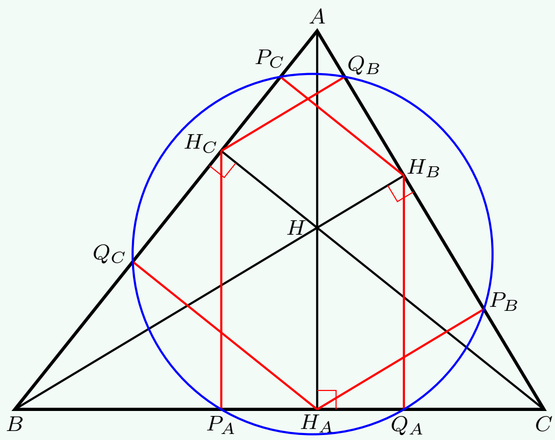
2 Definition of the Taylor Circle

Definition 1. (Taylor Circle)

Let $\triangle ABC$ be the following triangle and let H be its orthocenter, which is the concurrent point of the three altitudes AH_A, BH_B, and CH_C.

Let $P_A, P_B, P_C, Q_A, Q_B, Q_C$ be the corresponding projections of H_A, H_B, H_C to the triangle's three sides.

Then these six points $P_A, P_B, P_C, Q_A, Q_B, Q_C$ are concyclic, creating the circle called the Taylor Circle.



This definition leaves the question: How do we know that these six points are concyclic? We shall prove this below.

[1]The author thanks Dr. Zhiqin Lu for his help, and Stephanie Wang for her careful reading and many suggestions.

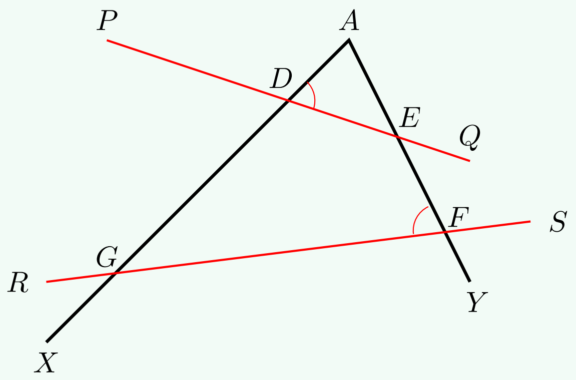
[2]He is not *Brook Taylor*, who is well-known for the *Taylor Theorem* or *Taylor series*.

3 Anti-parallel Lines

Parallelism is one of the fundamental concepts in Euclidean geometry. In relation to that, we have an interesting concept called *anti-parallel lines*. This concept is very important in triangle geometry and we will be using it for our proof.

Definition 2. (Anti-Parallel Line)

Anti-parallel lines must be defined with respect to a fixed reference angle. In the following picture, let $\angle XAY$ be our fixed angle. Lines PQ, FG are considered anti-parallel lines, if $\angle ADE = \angle AFG$.



From the above diagram, we know:

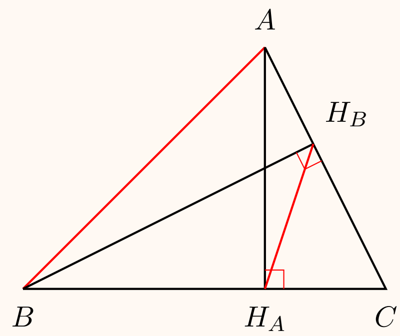
Theorem 1. (First Property of Anti-parallel Lines)

PQ and RS are anti-parallel lines if and only if D, G, F, E are concyclic.

For the rest of the article, we shall use the following property of anti-parallel lines repeatedly.

Corollary 1

In the following picture, let AH_A be the altitude over BC, and BH_B be the altitude over CA. Then the line H_AH_B is anti-parallel to the third side AB.



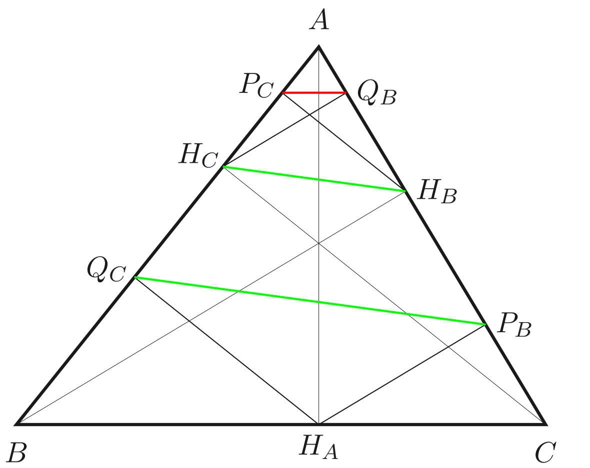
Solution: Since $\angle AH_BB = \angle AH_AB = 90^\circ$, A, B, H_A, H_B are concyclic. Therefore by Theorem 1, H_AH_B and AB are anti-parallel. ■

Theorem 2. (Transitivity Properties)

We have the following transitivity results pertaining to parallel and anti-parallel lines. Let L_1, L_2, L_3 be three lines. Then

1. *If L_1 is parallel to L_2, and L_2 is parallel to L_3, then L_1 is parallel to L_3;*
2. *If L_1 is parallel to L_2, and L_2 is anti-parallel to L_3, then L_1 is anti-parallel to L_3;*
3. *If L_1 is anti-parallel to L_2, and L_2 is parallel to L_3, then L_1 is anti-parallel to L_3;*
4. *If L_1 is anti-parallel to L_2, and L_2 is anti-parallel to L_3, then L_1 is parallel to L_3.*

Proof of the Taylor Circle. We first prove that P_B, Q_B, P_C, Q_C are concyclic.



Since $H_AQ_C \perp AB$ and $H_AP_B \perp AC$, we know A, Q_C, H_A, P_B are concyclic. Therefore we have $\angle H_AQ_CP_B = \angle H_AAC$. Furthermore $\angle BQ_CP_B + \angle C = 90^\circ + \angle H_AQ_CP_B + \angle C = 180^\circ$. As a result, Q_C, B, C, P_B are concyclic, and hence P_BQ_C is anti-parallel to BC. On the other hand, by Corollary 1, H_BH_C is anti-parallel to BC, and P_BQ_B is anti-parallel to H_BH_C. Using Theorem 2,P_CQ_B is anti-parallel to P_BQ_C. Therefore P_B, Q_B, P_C, Q_C are concyclic.

By the same reason, P_C, Q_C, P_A, Q_A and P_A, Q_A, P_B, Q_B are concyclic.

By *Davis' Theorem* (see Topic 28), we conclude that the six points

$$P_A, P_B, P_C, Q_A, Q_B, Q_C$$

are concyclic.

■

4 Further Information

The Taylor Circle belongs to the *Tucker Circles* family. In relation to the points on the Taylor circle, the hexagon $Q_A P_B Q_C P_A Q_B P_C$ is called the *Tucker's Hexagon*. In the following Tucker Hexagon, the three black lines are parallel to the corresponding sides, while the three red lines are anti-parallel to the corresponding three sides of the triangle. For more details of Tucker Circles, see Wolfram Math World or Topic 29.