Pascal's and Brianchon's Theorems

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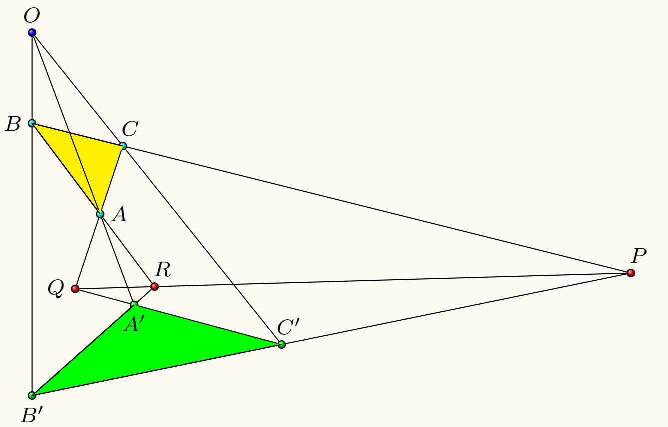
1 Desargues' Theorem

The Pascal's and Brainchon's Theorems are two famous "dual" theorems in Projective Geometry. Let's first introduce a related classical theorem.

Theorem 1. (Desargues' Theorem)

We consider triangles $\triangle ABC$ and $\triangle A'B'C'$. Assume that lines BC, $B'C'$ intersect at P, CA and $C'A'$ intersect at Q, and AB, $A'B'$ intersect at R. Then P, Q, R are collinear if and only if AA', BB' and CC' are concurrent.



The line PQR is called the *axis of perspectivity*, and point O is called the *center of perspectivity*. The theorem is known as *Perspective Principle* in painting. See Wikipedia for details.

Proof: Here we provide a proof using Menelaus' Theorem. On $\triangle OB'C'$, since B, C, P are collinear, we must have
$$\frac{OB}{BB'}\cdot\frac{B'P}{PC'}\cdot\frac{C'C}{CO}=1.$$
On $\triangle OC'A'$, since C, A, Q are collinear, we have
$$\frac{C'C}{CO}\cdot\frac{OA}{AA'}\cdot\frac{A'Q}{QC'}=1.$$
On $\triangle OA'B'$, since A, B, R are collinear, we have
$$\frac{OB}{BB'}\cdot\frac{B'R}{RA'}\cdot\frac{A'A}{AO}=1.$$

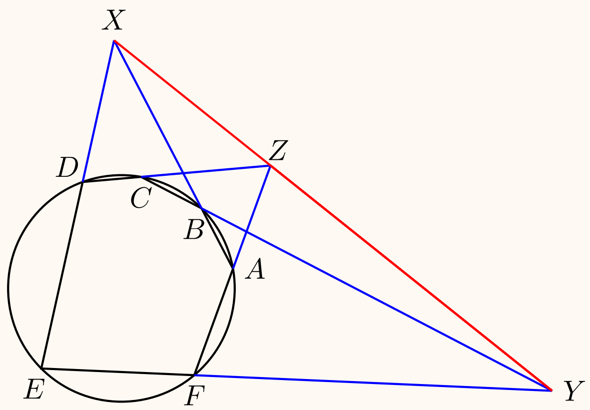
[1]The first author thanks Stephanie Wang for her careful reading and many comments.

From the above three equations, we get
$$\frac{A'Q}{Q'C} \cdot \frac{C'P}{PB'} \cdot \frac{B'R}{RA'} = 1.$$
Therefore, by using Menelaus' Theorem on $\triangle A'B'C'$, we conclude that P, Q, R are collinear. ■

2 Pascal's Theorem

Theorem 2. (Pascal's Theorem)

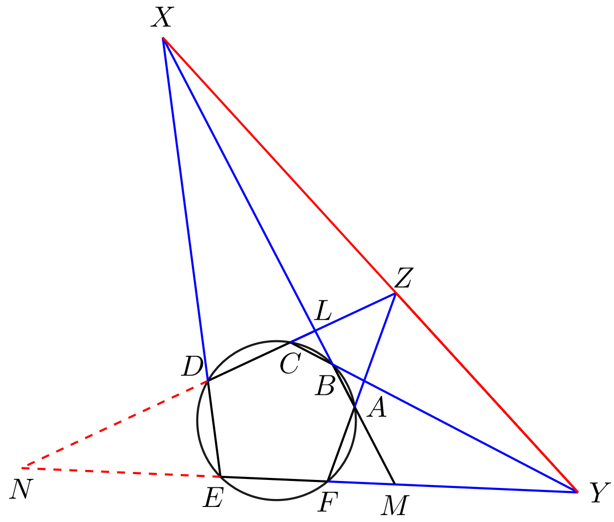
The hexagon $ABCDEF$ is inscribed to a circle. Assume that AB, DE intersects at X; BC, EF intersects at Y; and CD, FA intersects at Z. Then X, Y, Z are collinear.



Remark The above is called *Pascal's Theorem*, which was discovered by the French mathematician *Blaise Pascal*, when he was 16 years old. The theorem can be generalized to the case of conic section (see Wikipedia). When the conic section is degenerated to two lines, it is also called the *Pappus' Hexagon Theorem*, as shown below. It was believed that Euclid knew this theorem before Pappus.

We first give a purely Euclidean Geometry proof.

First proof: As in the graph drawn below, let AB and CD intersect at L, BA and EF intersect at M, CD and FE intersect at N.



On $\triangle LMN$, since C, B, Y are collinear, by applying Menelaus' Theorem we obtain

$$\frac{LB}{BM} \cdot \frac{MY}{YN} \cdot \frac{NC}{CL} = 1.$$

Similarly, since F, A, Z are collinear, we obtain

$$\frac{LA}{AM} \cdot \frac{MF}{FN} \cdot \frac{NZ}{ZL} = 1,$$

and since E, D, X are collinear, we also get

$$\frac{ND}{DL} \cdot \frac{LX}{XM} \cdot \frac{ME}{EN} = 1.$$

In the circle $ABCDEF$, by using the Power of Point Theorem, we will get

$$LA \cdot LB = LD \cdot LC,$$

$$NC \cdot ND = NE \cdot NF,$$

$$MA \cdot MB = MF \cdot ME.$$

Combining the above six equations, we obtain that

$$\frac{LX}{XM} \cdot \frac{MY}{YN} \cdot \frac{NZ}{ZL} = 1$$

Thus, by the inverse of Menelaus' Theorem we conclude that X, Y, Z are collinear.

We are able to use algebraic method to prove Pascal's Theorem as well. However, it is surprising that the algebra behind the theorem is about the factorization of cubic polynomials.

Second proof: We assume the circle is the unit circle. Let the equations for

$$AB, BC, CD, DE, EF, FA$$

be $\ell_1, \ell_2, \cdots, \ell_6$. These functions ℓ_j are linear functions. As a result, we consider two cubic polynomials $\ell_1\ell_3\ell_5$ and $\ell_2\ell_4\ell_6$. Obviously, these two polynomials pass the nine points $A, B, C, D, E, F, X, Y, Z$.

We choose a general point P in the circle. Choose a number λ such that

$$(\ell_1\ell_3\ell_5 + \lambda\ell_2\ell_4\ell_6)(P) = 0.$$

Here is a fundamental question: in general, if a cubic curve doesn't vanish identically on the unit circle, then what is the maximum number of intersections? The answer is six, and we shall prove it.

We can use complex numbers to write any cubic polynomials as

$$f(z) = Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3 + Ez^2 + Fz\bar{z} + G\bar{z}^2 + Hz + I\bar{z} + J = 0.$$

Let $z = e^{i\theta}$ be a point on the unit disk, then $\bar{z} = 1/z$. If we multiply the above equation by z^3 on both sides, we get a degree six polynomial of single variables. In general, such a 6-degree polynomial has at most six roots. Since $f(z)$ vanishes on seven points A, B, C, D, E, F, P on the unit circle, it must be vanishing identically on the circle. As a result, we can factorize it as

$$f(z) = (|z|^2 - 1)\ell(z),$$

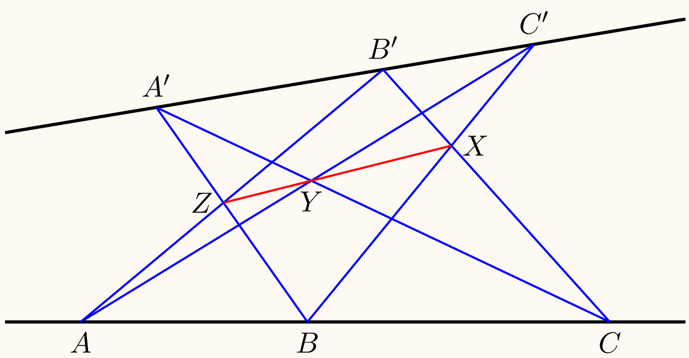
where, by the degree consideration, $\ell(z)$ must be linear. Since ℓ passes X, Y, Z, we conclude that X, Y, Z are collinear.

■

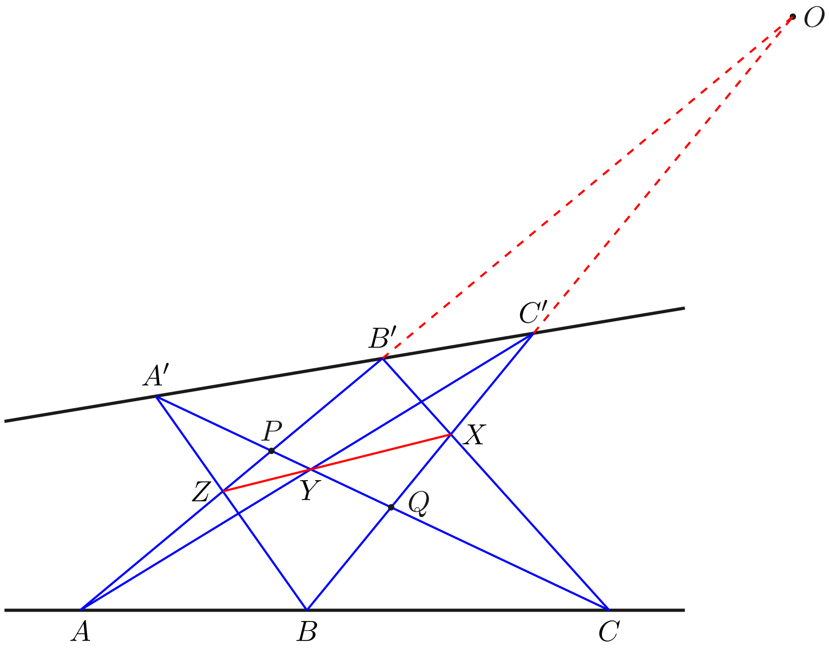
Pascal's Theorem can be generalized to the case of conic curves, and above algebraic proof can be used to prove the general case. When a conic curve is degenerated to two lines, the conic Pascal's Theorem is also know as the *Pappus' Theorem*.

Theorem 3. (Pappus' Theorem)

In the following picture, the Hexagon $BC'AB'CA'$ is inscribed on the two black lines. Assume that $BC', B'C$ intersect at X; $CA', A'C$ intersect at Y, and $AB', B'A$ intersect at Z. Then X, Y, Z are collinear.



Proof of the Pappus' Theorem: As in the graph drawn below, let AB' and BC' intersect at O, AB' and $A'C$ intersect at P, $A'C$ and BC' intersect at Q.



On $\triangle OPQ$, since B', X, C are collinear, by applying Menelaus' Theorem we obtain

$$\frac{QX}{XO} \cdot \frac{OB'}{B'P} \cdot \frac{PC}{CQ} = 1.$$

Similarly, since C', Y, A are collinear, we obtain

$$\frac{QC'}{C'O} \cdot \frac{OA}{AP} \cdot \frac{PY}{YQ} = 1.$$

Since A', Z, B are collinear, we obtain

$$\frac{QB}{BO} \cdot \frac{OZ}{ZP} \cdot \frac{PA'}{A'Q} = 1.$$

Since A', B', C' are collinear, we obtain

$$\frac{QA'}{A'P} \cdot \frac{PB'}{B'O} \cdot \frac{OC'}{C'Q} = 1.$$

Since A, B, C are collinear, we obtain

$$\frac{QC}{CP} \cdot \frac{PA}{AO} \cdot \frac{OB}{BQ} = 1.$$

Multiplying the above five equations, we obtain

$$\frac{QX}{XO} \cdot \frac{OZ}{ZP} \cdot \frac{PY}{YQ} = 1.$$

Thus, by using the Menelaus' Theorem one more time we conclude that X, Y, Z are collinear.

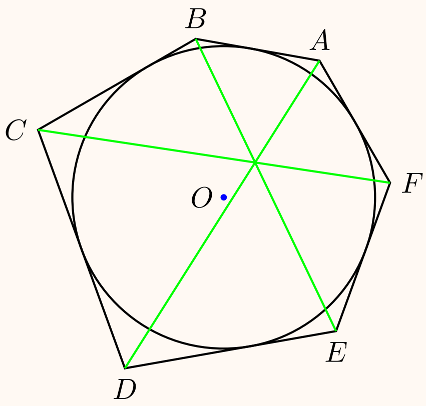
■

3 Briancho's Theorem

In this section, we introduce the *Brianchon's Theorem* on a circumscribed hexagon.

Theorem 4. (Brianchon's Theorem)

The Hexagon $ABCDEF$ is circumscribed on a circle. Then AD, BE, and CF are concurrent.

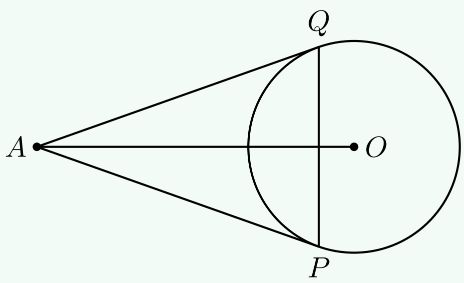


Remark Pascal's Theorem and Brainchon's Theorem are two famous "dual" theorems (see Wikipedia). Using the following concepts of *Pole and Polar*, we can prove that Pascal's and Brianchon's Theorems are equivalent.

Definition. (Pole and Polar)

Let O be the unit circle. The the pair (A, PQ) is called the pair of pole and polar, where A is the pole, and PQ is the polar, if AP and AQ are the tangent lines to the circle. From analytic geometry point of view, there is a nice relationship between the pole and the polar. Let (x_0, y_0) be the coordinates of A. Then the equation of PQ is given by

$$x_0x + y_0y - 1 = 0.$$



Proof of the Brianchon Theorem (using pole and polar): See the graph below. Let the coordinates of A_i be (x_i, y_i) for $1 \le i \le 6$. Then the equations for B_6B_1 is

$$\ell_1(x, y) = x_1x + y_1y - 1.$$

Similarly, the equations for B_iB_{i+1} for $1 \le i \le 5$ are

$$\ell_i(x, y) = x_ix + y_iy - 1.$$

Using Pascal's Theorem, there is a number λ such that

$$\ell_1\ell_3\ell_5 + \lambda\ell_2\ell_4\ell_6 = C(x^2 + y^2 - 1)(px + qy - 1),$$

where C is a constant. We claim (p, q) is on the lines A_1A_4, A_2A_5, and A_3A_6.

In order to prove this, let $P = (p_1, q_1)$ be the intersection of B_1B_6 and B_3B_4. Then we have

$$x_1p_1 + y_1q_1 - 1 = 0, \qquad x_4p_1 + y_4q_1 - 1 = 0.$$

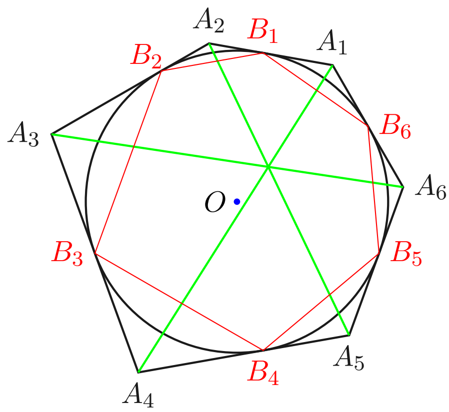
Moreover, we have

$$pp_1 + qq_1 - 1 = 0.$$

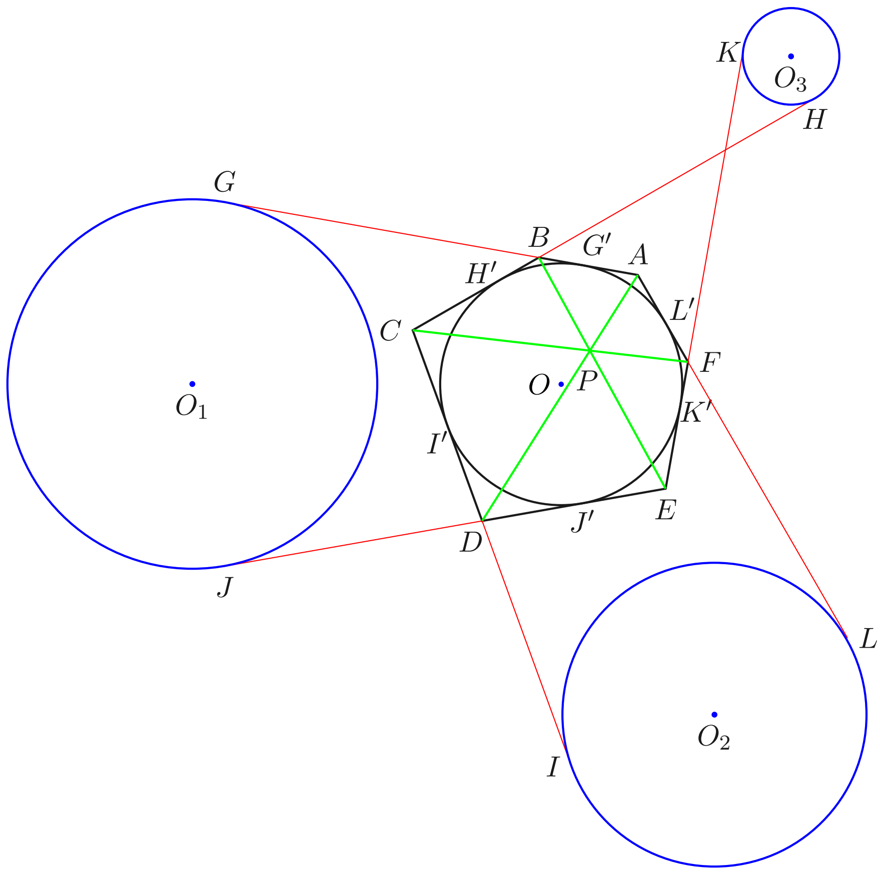
Thus the three points A_1, A_4 and (p, q) are on the line

$$p_1x + q_1y - 1 = 0.$$

This completes the proof.



A Euclidean Geometry proof: We shall use the Monge's Theorem of radical axes to prove the result.



In the above theorem, let G', H', I', J', K', L' be the tangent points of the lines AB, BC, CD, DE, FE and FA respectively. Define G, H, I, J, K, L such that

$$GG' = HH' = II' = JJ' = KK' = LL'.$$

Define circles O_1, O_2 and O_3 such that GG', JJ' are tangent lines to O_1, II', LL' are tangent lines to O_2, and KK', HH' are tangent to O_3. It is well known that AD is the radical axis of O_1, O_2; BE is the radical axis of O_3, O_1; and CF is the radical axis of O_2, O_3. By Monge's Theorem, AD, BE and CF are concurrent.

External Link. *Here is the Wikipedia link to the Brianchon's Theorem.*