Routh's Theorem

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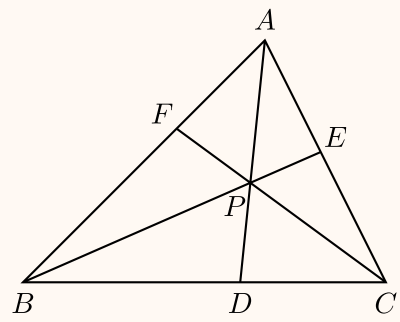
Routh's Theorem is a generalization of the Ceva's Theorem.

Theorem 1. (Ceva's Theorem)

In the following triangle $\triangle ABC$, let

$$\frac{BD}{DC} = x, \quad \frac{CE}{EA} = y, \quad \frac{AF}{FB} = z.$$



Then, AD, BE, and CF are concurrent if and only if

$$xyz = 1.$$

For details of the Ceva's Theorem, see Wikipedia, or Topic 02 .

We define the following:

Definition 1. (Cevian)

A Cevian is a line segment which joins a vertex of a triangle with a point on the opposite side (or its extension). For example, in the above picture, AD, BE and CF are Cevians. The condition for three general Cevians from the three vertices of a triangle to concur is known as Ceva's theorem.

For a fixed triangle, if the three Cevians are not concurrent, then what is the area of the triangle formed by the pairwise intersections of them? *The Routh's Theorem* provides the answer to this question. If the area is zero, then the triangle is degenerated to a point, and therefore, the theorem is reduced to the Ceva's Theorem. The Routh's Theorem was discovered by *Edward John Routh* in 1896.

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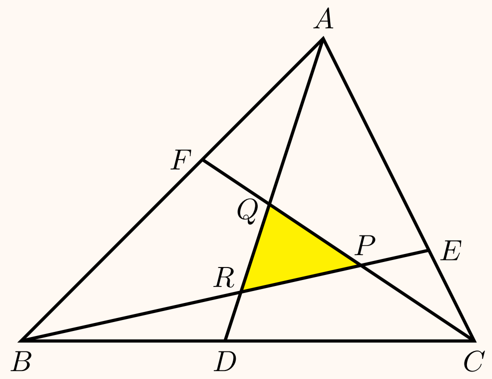
Theorem 2. (Routh's Theorem)

In the following $\triangle ABC$, let D, E, F be points on BC, CA, and AB, respectively. Assume that

$$\frac{BD}{DC} = x, \quad \frac{CE}{AE} = y, \quad \frac{AF}{FB} = z.$$

Then, the area of $\triangle PRQ$ formed by the Cevians AD, BE, and CF is the area of $\triangle ABC$ times

$$\frac{(xyz-1)^2}{(zx+z+1)(xy+x+1)(yz+y+1)}.$$



In particular, if $xyz = 1$, then the three Cevians AD, BE, and CF are concurrent, and the theorem is reduced to the Ceva's Theorem.

Proof: Assume that the area of $\triangle ABC$ is 1. For $\triangle ABD$ and line FQC, using the Menelaus's theorem, we obtain

$$\frac{AF}{FB} \cdot \frac{BC}{CD} \cdot \frac{DQ}{QA} = 1.$$

Then, we can get

$$\frac{QA}{DQ} = \frac{AF}{FB} \cdot \frac{BC}{CD} = z(x+1).$$

As a result, we have

$$\frac{AQ}{AD} = \frac{z(x+1)}{1+z(x+1)}.$$

Thus, the area of $\triangle CQA$ is

$$S_{\triangle CQA} = \frac{AQ}{AD} \cdot \frac{DC}{BC} \cdot S_{\triangle ABC} = \frac{z}{zx+z+1}.$$

Similarly, we get the area of $\triangle ARB$ and $\triangle BPC$

$$S_{\triangle ARB} = \frac{x}{xy+x+1}.$$

$$S_{\triangle BPC} = \frac{y}{yz+y+1}.$$

Therefore,

$$\begin{aligned} S_{\triangle PQR} &= S_{\triangle ABC} - S_{\triangle CQA} - S_{\triangle ARB} - S_{\triangle BPC} \\ &= 1 - \frac{z}{zx+z+1} - \frac{x}{xy+x+1} - \frac{y}{yz+y+1} \\ &= \frac{(xyz-1)^2}{(zx+z+1)(xy+x+1)(yz+y+1)}. \end{aligned}$$

The theorem is proved. ■

Remark The identity above can be proved by a tedious computation. Here we provide another proof. Let

$$a = xy + x + 1,$$

$$b = yz + y + 1,$$

$$c = zx + z + 1.$$

Let

$$f(x, y, z) = abc - zab - xbc - yca - (xyz - 1)^2.$$

We need to prove that $f \equiv 0$. Without loss of generality, we assume that x, y, z are distinct. Note that for fixed y, z, $f(x, y, z)$ is a quadratic polynomial of x. Thus we only need to prove the identity for three different values of x.

First, we assume that $x = 0$. Then $a = 1$; $b = yz + y + 1$; and $c = z + 1$. Thus we have

$$f(0, y, z) = (yz + y + 1)(z + 1) - z(yz + y + 1) - y(z + 1) - 1 = 0.$$

Next, we assume that $x = -1/(y + 1)$. Then $a = 0$. From this we conclude that

$$0 = xyz + zx + z$$

and therefore $c = 1 - xyz$. Similarly, we have $bx = xyz - 1$. Thus

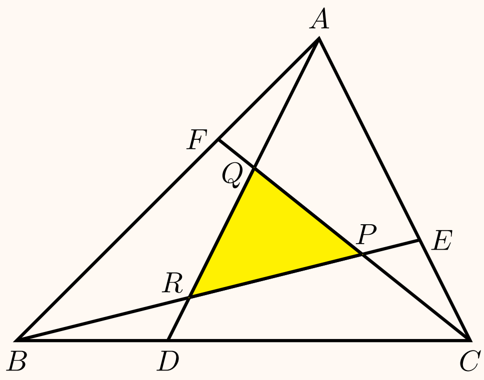
$$f(-1/(y + 1), y, z) = -xbc - (xyz - 1)^2 = 0.$$

By symmetry, we have $f(-1/(z + 1), y, z) = 0$. This proves the identity.

A special case of the Routh's Theorem is the following One-seventh area triangle problem.

Theorem 3. (One-seventh Area Triangle)

If $x = y = z = 1/2$ in the above theorem, then the Routh's Theorem is reduced to the popular one-seventh area triangle problem.



The area of $\triangle PRQ$ is one-seventh of the area of $\triangle ABC$.

This problem was introduced in the famous book of *Hugo Steinhaus*, *Mathematical Snapshots (the 1982 version), page 9*.