Topic 15 Complete Quadrilateral and Complete Quadrangle

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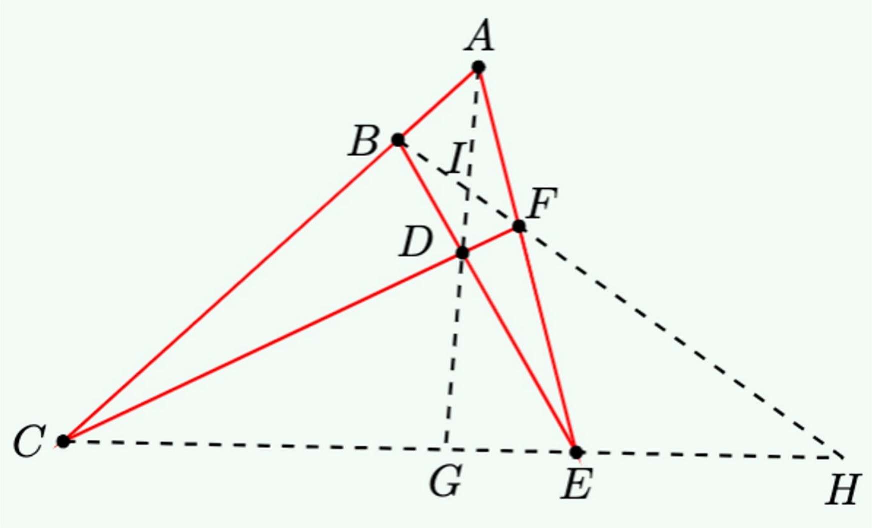
What do Quadrilateral and Quadrangle mean?

- The root **quad** comes from Latin, where it has the meaning **"four, fourth."**

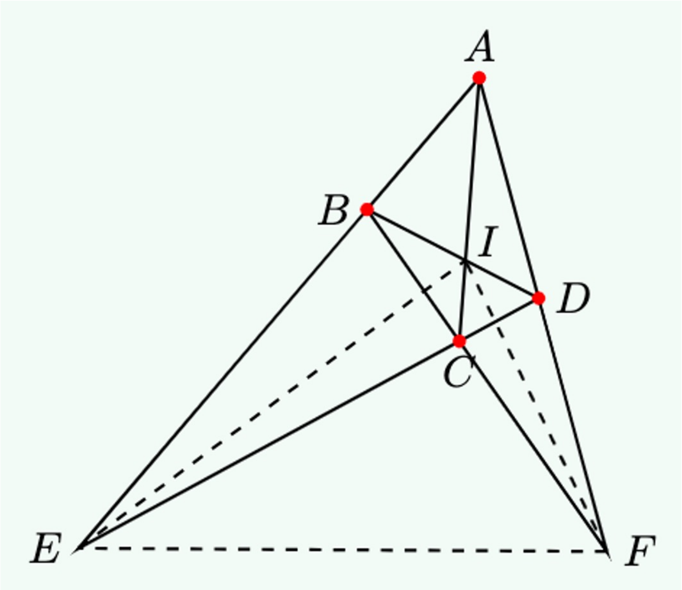
- The adjective **lateral** comes from the Latin word lateralis, which means **“belonging to the side”**

Definition

A **Complete Quadrilateral** is a system of **four lines**, with no three concurrent, and the six points of intersection of these lines.



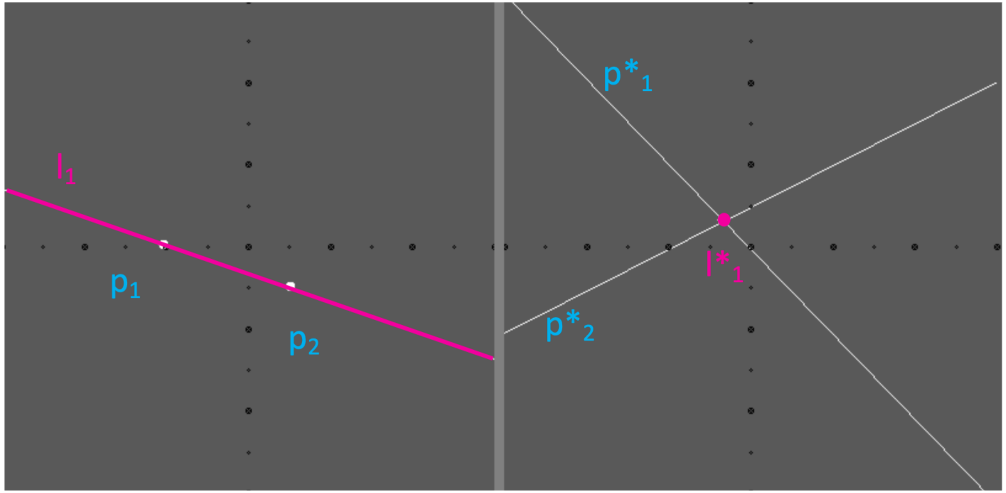
A **Complete Quadrangle** is a set of **four points**, with no three collinear, and the six lines which join them.



Duality Principle

Definition 3. (Duality Principle)

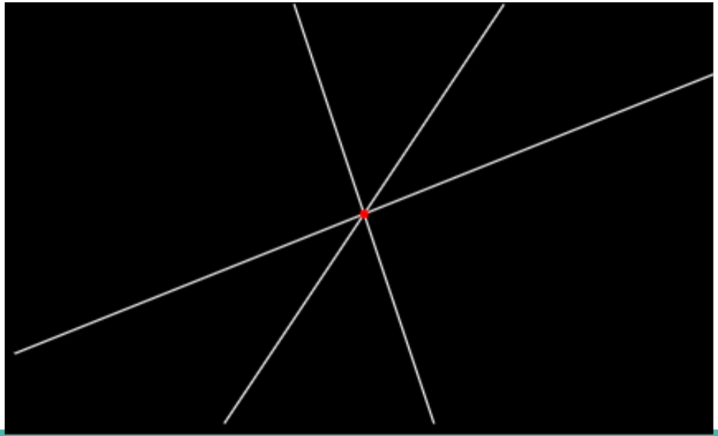
All the propositions in projective geometry occur in dual pairs, which have the property that, starting from either proposition of a pair, the other can be immediately inferred by interchanging the parts played by the words "point" and "line."



Example: Collinear Points & Concurrent Lines (1)

Property:

Concurrency is dual of Collinearity



Suppose we have three colinear points A, B, C, and their coordinates are

$$A = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, B = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}, C = \begin{bmatrix} a_3 \\ b_3 \end{bmatrix}$$

The dual lines of points A, B, C are:

$$l_1 : a_1 x + b_1 y = 1$$

$$l_2 : a_2 x + b_2 y = 1$$

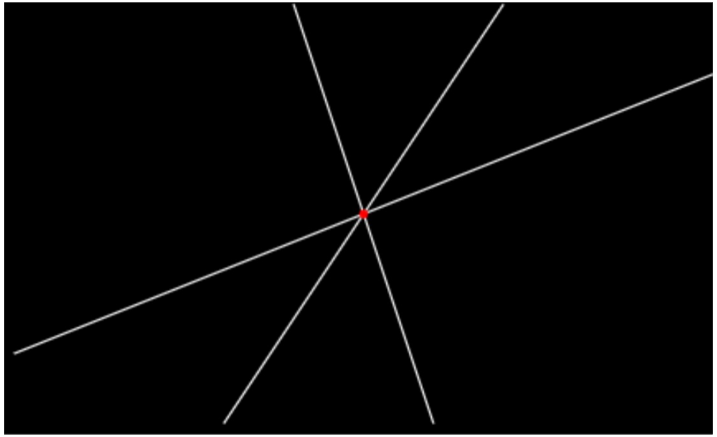
$$l_3 : a_3 x + b_3 y = 1$$

Suppose $\frac{AC}{CB} = \lambda$. Then, by Section Formula, we have

$$a_3 = \frac{1}{1+\lambda} a_1 + \frac{\lambda}{1+\lambda} a_2$$

$$b_3 = \frac{1}{1+\lambda} b_1 + \frac{\lambda}{1+\lambda} b_2$$

Example: Collinear Points & Concurrent Lines (2)



We want to show that l_1, l_2, l_3 are concurrent. We can construct the following matrix:

$$\begin{bmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{bmatrix}$$

That is,

$$\begin{bmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ \frac{1}{1+\lambda}a_1 + \frac{\lambda}{1+\lambda}a_2 & \frac{1}{1+\lambda}b_1 + \frac{\lambda}{1+\lambda}b_2 & 1 \end{bmatrix}$$

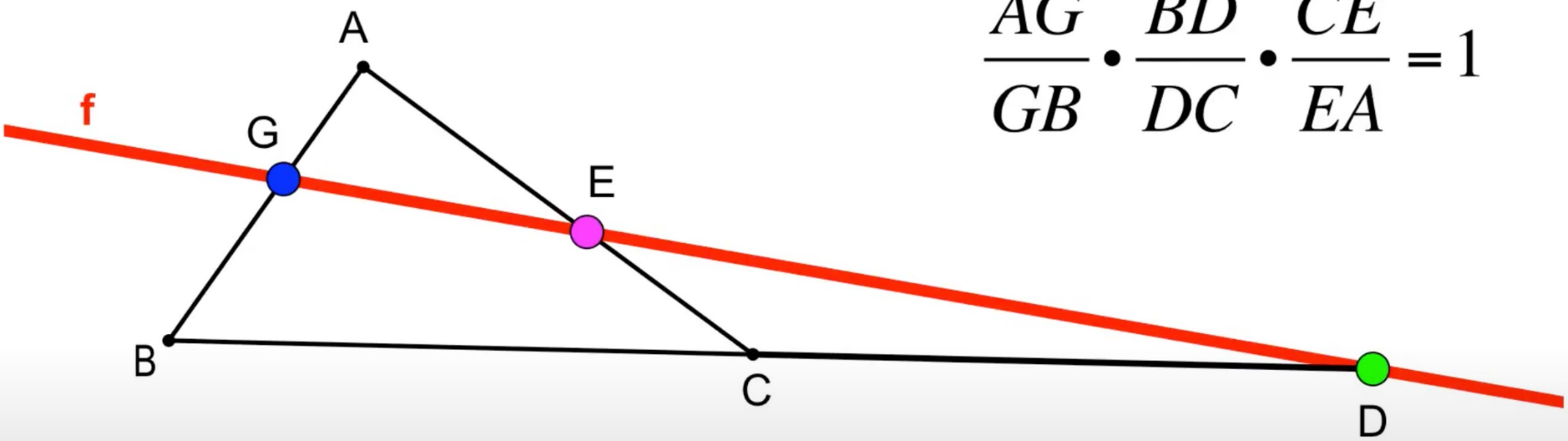
We want to show that the determinant of the 3×3 matrix is equal to 0 and the rank of the augmented matrix should be less than or equal to 2. By observing the above matrix, we could find that the third row = the first row $\times$ $\frac{1}{1+\lambda}$ + the second row $\times$ $\frac{\lambda}{1+\lambda}$. Thus, the third row is dependent of the first and second row. We concludes that the determinant of the matrix is equal to 0. l_1, l_2, l_3 are concurrent

Menelaus Theorem & Ceva's Theorem (1)

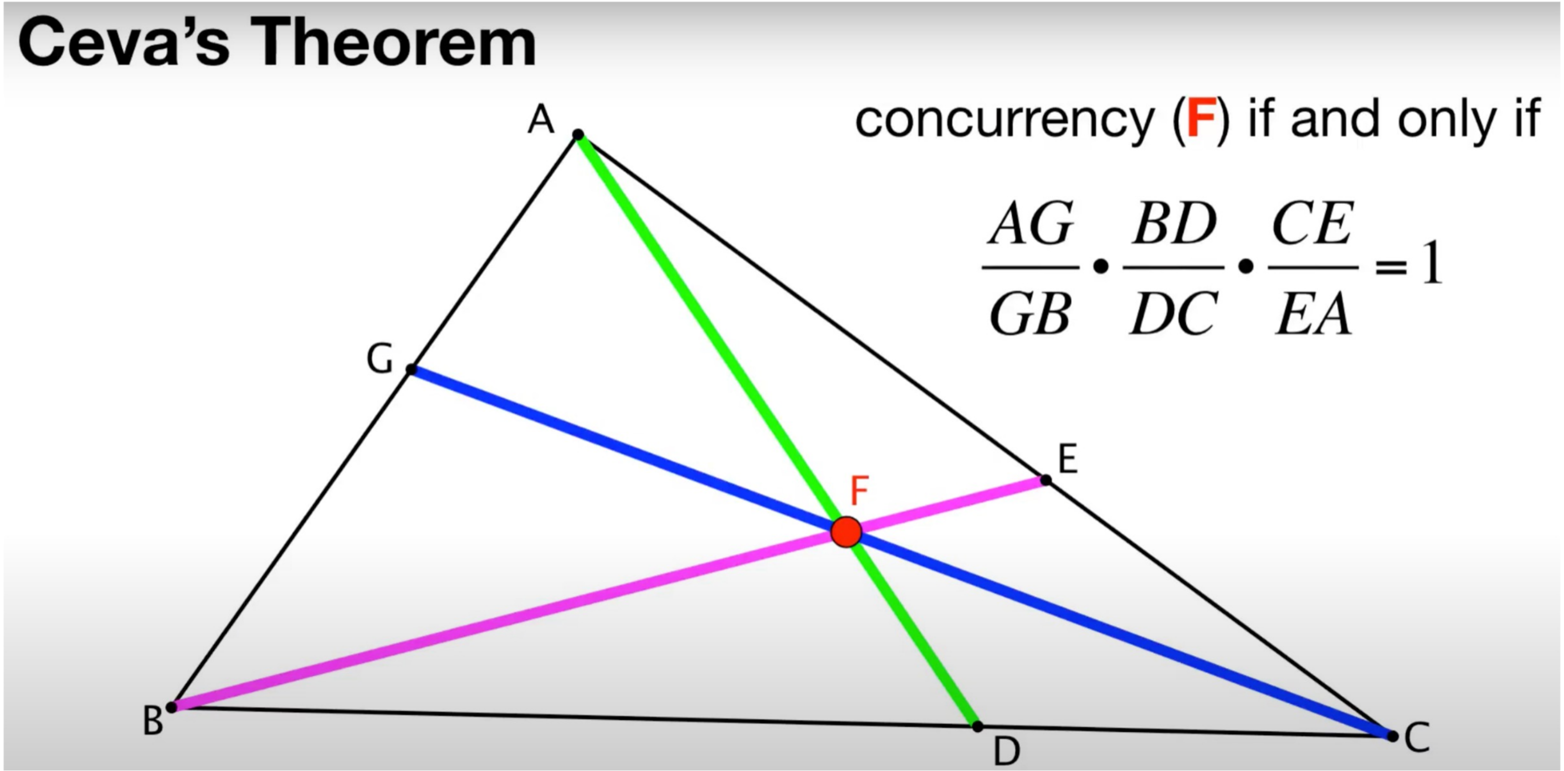
Menelaus' Theorem

collinearity (f) if and only if

$$\frac{AG}{GB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$



Menelaus Theorem & Ceva’s Theorem (2)



Cross-ratio and Harmonicity

Cross-ratio of points

Definition 4. (Cross-ratio)

Let ℓ be a line and A, B, C, and D are four points which lie in this order on it.

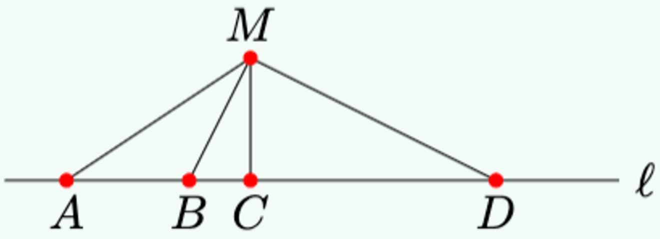
We define the cross-ratio *of A, B, C, D by*

$$(A, B; C, D) = \frac{AC}{BC} \cdot \frac{BD}{AD}.$$

Cross-ratio of lines

Alternatively, if four lines MA, MB, MC and MD are concurrent to a point M outside line ℓ, then we define the cross-ratio of lines MA, MB, MC, MD [a] *by*

$$(MA, MB; MC, MD) = \frac{\sin\angle AMC}{\sin\angle BMC} \cdot \frac{\sin\angle BMD}{\sin\angle AMD}.$$



Theorem 2

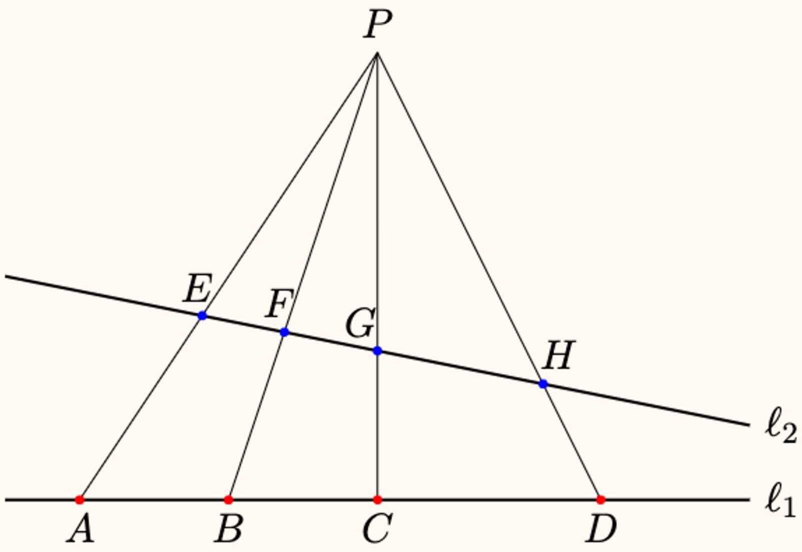
The cross-ratio of concurrent lines is equal to the cross-ratio of the corresponding four points, that is, in the above picture, we have

$$(A, B; C, D) = (MA, MB; MC, MD).$$

Theorem 3

Let P be a point outside line ℓ_1. Let PA, PB, PC, PD intersect with another line ℓ_2 at E, F, G, H, respectively. Then the cross-ratios of the two groups of points are the same

$$(A, C; B, D) = (E, G; F, H).$$



Definition 5. (Harmonic Division)

The four-point A, C, B, D is called a harmonic range of points [a] *if and only if*

$$(A, B; C, D) = 1.$$

That pencil MA, MB, MC, MD is harmonic or called harmonic pencil of lines if and only if

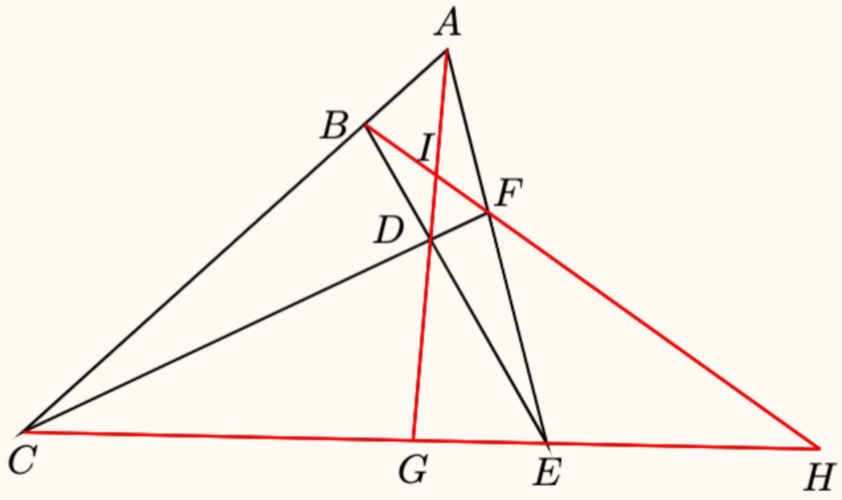
$$(MA, MB; MC, MD) = 1.$$

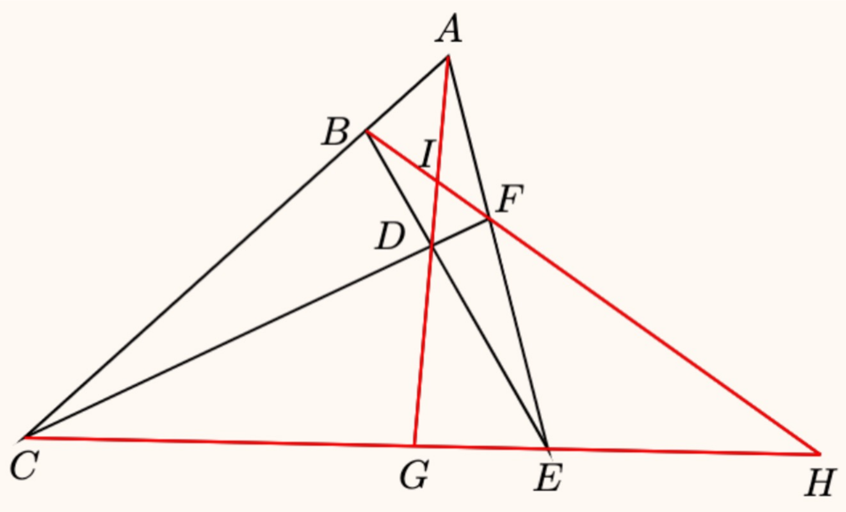
Harmonicity in Complete Quadrilateral and Complete Quadrangle

Theorem 5. (Harmonicity in Complete Quadrilateral)

Let $ABCDEF$ be a complete quadrilateral. Let G, H be the intersection of the diagonals AD and BF to CE, respectively. Then G, H harmonically divide the diagonal CE. Similarly, H, I harmonically divide the diagonal BF, and G, I harmonically divide AD. In short, we can conclude that any two diagonals harmonic divides the third diagonals.

1. *C, E, G, H are the harmonic range of points;*
2. *B, F, I, H are the harmonic range of points;*
3. *A, D, I, G are the harmonic range of points.*





Proof: By applying Menelaus' Theorem to $\triangle ACE$, we have

$$\frac{AB}{BC} \cdot \frac{CH}{HE} \cdot \frac{EF}{FA} = 1.$$

By using Ceva's Theorem in $\triangle ACE$, we get,

$$\frac{AB}{BC} \cdot \frac{CG}{GE} \cdot \frac{EF}{FA} = 1.$$

Comparing the above two equations, we get

$$\frac{CH}{HE} = \frac{CG}{GE},$$

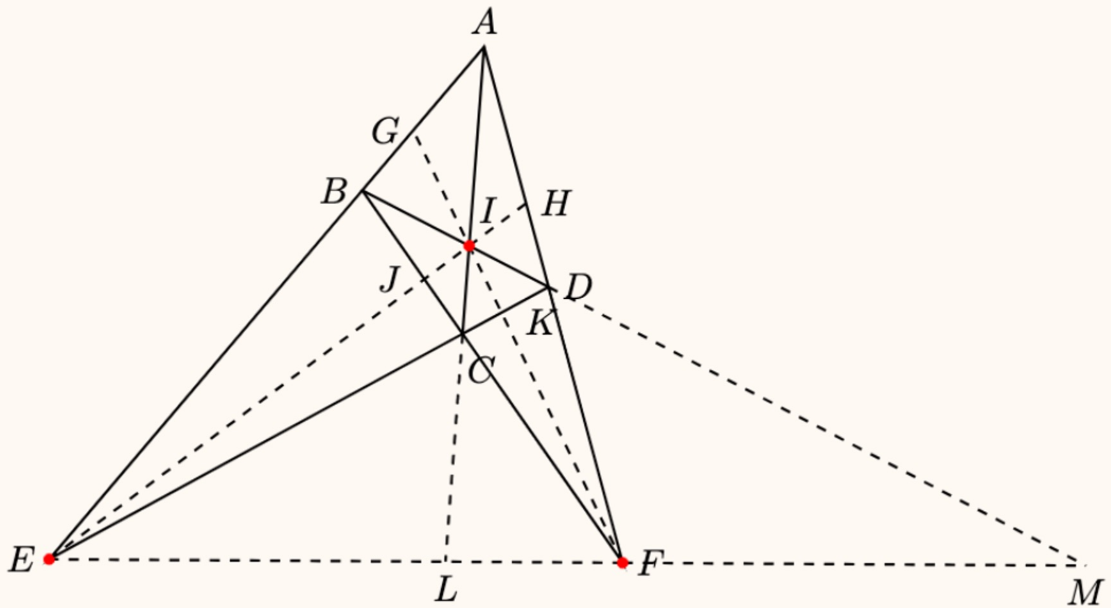
and hence B, I, F, H are harmonic range of points.

The other two assertions follow by the same method.

Theorem 6. (Harmonic Pencil of Lines in Complete Quadrangle)

Let G be the intersection point of AB and IF, H be the intersection point of AD and IE, L be the intersection point of AC and EF, and M be the intersection point of BD and EF. There are three set of harmonic pencil of lines in the complete quadrangle $AECF$:

1. EA, EH, ED, EF *are harmonic pencil of lines;*
2. FA, FG, FB, FE *are harmonic pencil of lines;*
3. IE, IL, IF, IM *are harmonic pencil of lines.*



Thank you for watching :)