Topic 4
Erdős–Mordell Inequality

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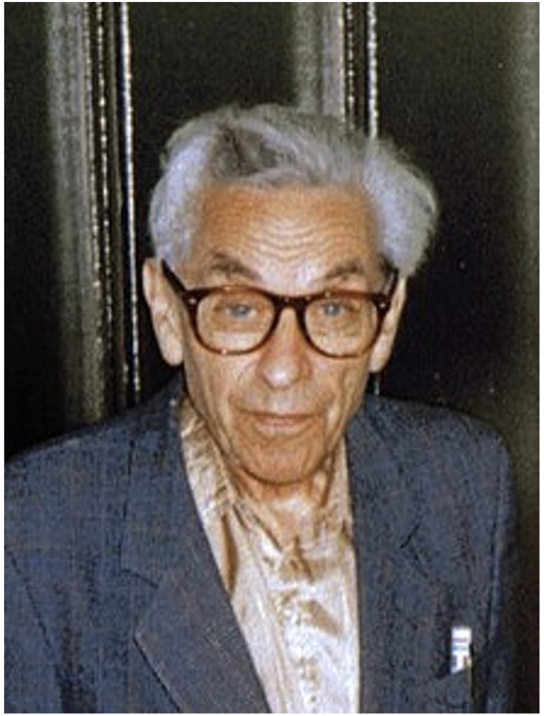
Erdős–Mordell Inequality

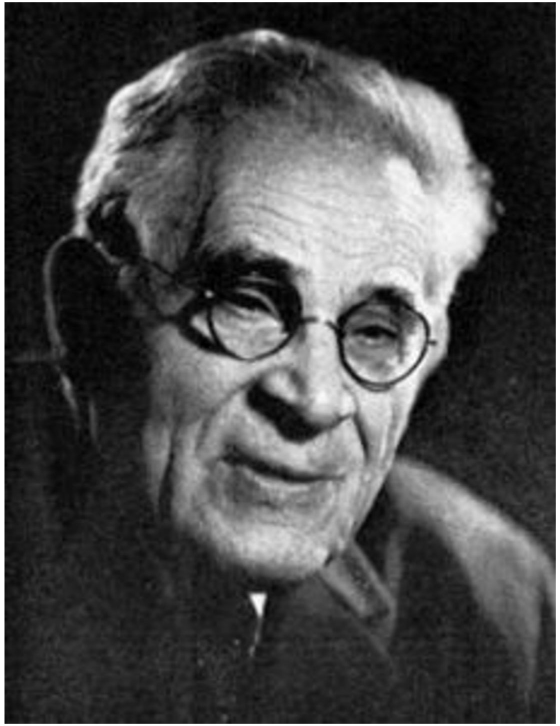
- Named after Paul Erdős and Louis Mordell
- Erdős (1935) posted the problem of proving the inequality
- A proof was provided two years later by L. J. Mordell and D. F. Barrow (1937)

Paul Erdős



- Erdős published around 1,500 mathematical papers during his lifetime, a figure that remains unsurpassed.
- He was one of the most prolific mathematicians and producers of mathematical conjectures of the 20th century.
- He was known both for his social practice of mathematics, working with more than 500 collaborators, and for his eccentric lifestyle; Time magazine called him "The Oddball's Oddball".
- He devoted his waking hours to mathematics, even into his later years—indeed, his death came only hours after he solved a geometry problem at a conference in Warsaw.

Louis Mordell



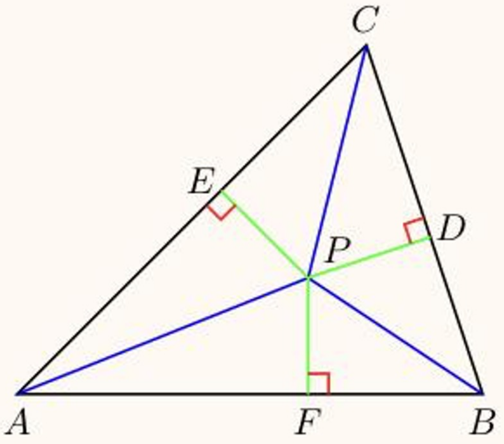
- He was an American-born British mathematician, known for pioneering research in number theory. He was born in Philadelphia, United States, in a Jewish family of Lithuanian（立陶宛）extraction.

Erdős–Mordell Inequality

Theorem 1. (Erdős-Mordell Inequality)

Let P be a point inside triangle $\triangle ABC$. Let PD, PE, PF to be orthogonal to AB, BC, CA respectively. Then

$$PA + PB + PC \geq 2(PD + PE + PF).$$



Methods in Proofs

Law of sines and cosines:

Law of sines

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

Inequality of arithmetic and geometric means(AM–GM inequality):

For any nonnegative real numbers x1,…,xn:

$$A_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$G_n = \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$$

$$A_n \geq G_n$$

- The arithmetic mean of a list of non-negative real numbers is greater than or equal to the geometric mean of the same list.(equal iff every number in the list is the same)

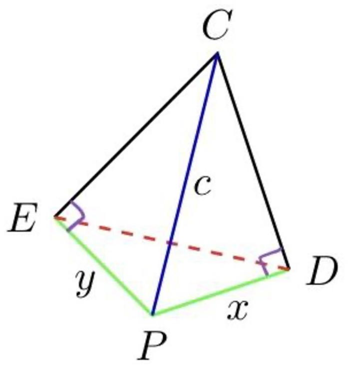
The simplest case:
For two non-negative numbers a and b,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

First Proof

In the above right picture, since $\angle CEP = \angle CDP = 90^\circ$, we can use the law of cosines to obtain

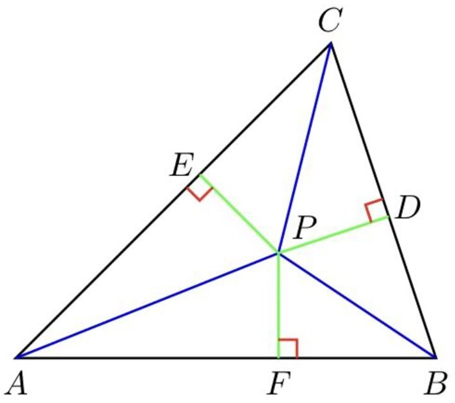
$$ED^2 = x^2 + y^2 - 2xy\cos\angle EPD.$$



Since $\angle EPD = 180^\circ - \angle C = \angle A + \angle B$, we have[a]

Using the Lagrange method of completing square, we have

$$ED^2 = x^2 + y^2 - 2xy\cos(A+B) = (x\sin B + y\sin A)^2 + (x\cos B - y\cos B)^2.$$

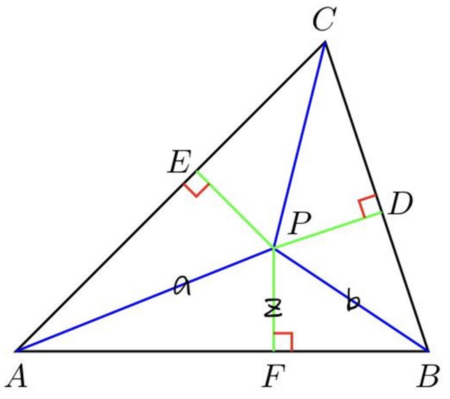


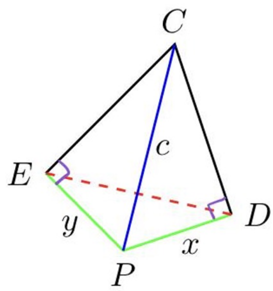
We therefore have

$$ED \geq x\sin B + y\sin A.$$

First Proof

$$ED \geq x \sin B + y \sin A.$$





By law of sines, we have $ED = c \sin C$. Thus we get the inequality

$$c \geq x \frac{\sin B}{\sin C} + y \frac{\sin A}{\sin C}.$$

Similarly, if $PA = a$, $PB = b$, and $PF = z$, then we have

$$a \geq z \frac{\sin B}{\sin A} + y \frac{\sin C}{\sin A}, \qquad b \geq x \frac{\sin C}{\sin B} + z \frac{\sin A}{\sin B}.$$

Therefore, we have

$$a + b + c \geq x \left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B} \right) + y \left(\frac{\sin A}{\sin C} + \frac{\sin C}{\sin A} \right) + z \left(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B} \right).$$

By the Arithmetic-Geometric Inequality, we have

$$\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B} \geq 2, \qquad \frac{\sin A}{\sin C} + \frac{\sin C}{\sin A} \geq 2, \qquad \frac{\sin B}{\sin A} + \frac{\sin A}{\sin B} \geq 2.$$

Then we have

$$a + b + c \geq 2(x + y + z).$$

$$\frac{\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B}}{2} \geqslant \sqrt{\frac{\cancel{\sin B}}{\cancel{\sin C}} \cdot \frac{\cancel{\sin C}}{\cancel{\sin B}}}$$

$$\frac{\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B}}{2} \geqslant 1$$

$$\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B} \geqslant 2$$

Second proof

Second Proof We shall use area method to prove the inequality.

Assume that $PA = a, PB = b, PC = c, PD = x, PE = y, PF = z$. Moreover, assume that $BC = p, CA = q$ and $AB = r$. Then since $CP + PF$ is no less than the height of the triangle over AB, we have

$$r(c + z) \geq qy + px + rz,$$

which is simplified to $rc + \cancel{rz} \geq qy + px + \cancel{rz}$

$$rc \geq qy + px.$$

Note that the above inequality is true for any point P in the cone of $\angle ACB$, even when P is outside of the triangle. So if we fix the triangle $\triangle ABC$ and fix the length $PC = c$, we can allow x, y to vary. Switching x, y, we get

$$rc \geq qx + py,$$

which is

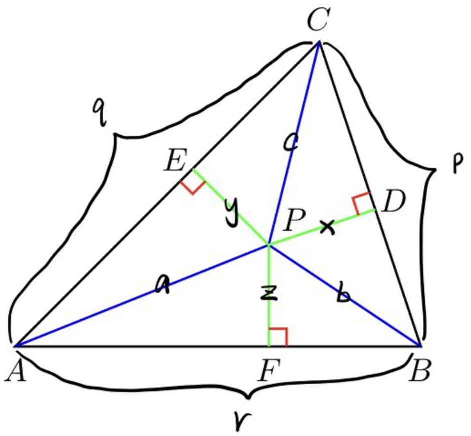
$$c \geq x\frac{q}{r} + y\frac{p}{r}.$$

Similarly, we have

$$b \geq x\frac{r}{q} + z\frac{p}{q}, \qquad a \geq z\frac{q}{p} + y\frac{q}{p}.$$

Therefore

$$a + b + c \geq x\left(\frac{r}{q} + \frac{q}{r}\right) + y\left(\frac{p}{r} + \frac{r}{p}\right) + z\left(\frac{p}{q} + \frac{q}{p}\right) \geq 2(x + y + z).$$



Similarly,

$qb \geq rz + px$

switching x and z,

$qb \geq rx + pz$

$b \geq x\frac{r}{q} + z\frac{p}{q}$

$pa \geq qy + rz$

switching y and z,

$pa \geq qz + ry$

$a \geq z\frac{q}{p} + r\frac{y}{p}$

By the Arithmetic – Geometric Inequality,

$$\frac{\frac{r}{q} + \frac{q}{r}}{2} \geq \sqrt{\frac{\cancel{r}}{\cancel{q}} \times \frac{\cancel{q}}{\cancel{r}}}$$

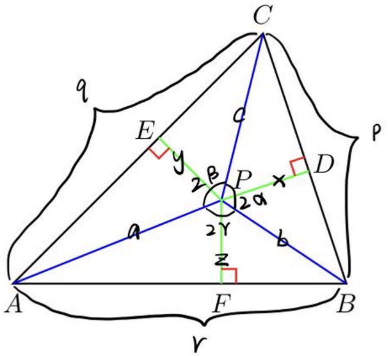
$$\frac{r}{q} + \frac{q}{r} \geq 2$$

Third Proof

Third Proof Here we provide the most important proof using algebra.

As before, we assume that $PA = a, PB = b, PC = c, PD = x, PE = y, PF = z$, $BC = p, CA = q$, and $AB = r$. Moreover, we assume that $\angle BPC = 2\alpha, \angle CPA = 2\beta$ and $\angle APB = 2\gamma$. We claim that

$$z \leq \sqrt{ab}\cos\gamma.$$



$$r^2 \geq 4ab\sin^2\gamma$$

$$r \geq 2\sqrt{ab}\sin\gamma$$

$$\frac{1}{2}ab \cdot \sin 2\gamma = \frac{1}{2} r z$$

$$z = \frac{ab \cdot \sin 2\gamma}{r}$$

$$\Rightarrow \sin 2\gamma = 2\sin\gamma \cdot \cos\gamma$$

$$z = \frac{ab \cdot 2\sin\gamma \cdot \cos\gamma}{r}$$

$$\leq \frac{\overset{\sqrt{ab}}{\cancel{ab}} \cdot \cancel{2}\cancel{\sin\gamma} \cdot \cos\gamma}{\cancel{2}\,\cancel{\sqrt{ab}}\,\cancel{\sin\gamma}}$$

Thus, $z \leq \sqrt{ab}\cos\gamma$

To prove this, we use the law of cosines to obtain

$$r^2 = a^2+b^2-2ab\cos 2\gamma = (a-b)^2+2ab(1-\cos 2\gamma) \geq 2ab(1-\cos 2\gamma) = 4ab\sin^2\gamma.$$

Thus

$$z = \frac{ab\sin 2\gamma}{r} \leq \sqrt{ab}\cos\gamma.$$

Thus the Erdős-Mordell inequality can be strengthened to the following algebraic inequality

$$a+b+c \geq 2\sqrt{ab}\cos\gamma + 2\sqrt{bc}\cos\alpha + 2\sqrt{ca}\cos\beta.$$

Thank you for Watching:)