Complete Quadrilateral and Complete Quadrangle

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(last updated: May 20, 2023)

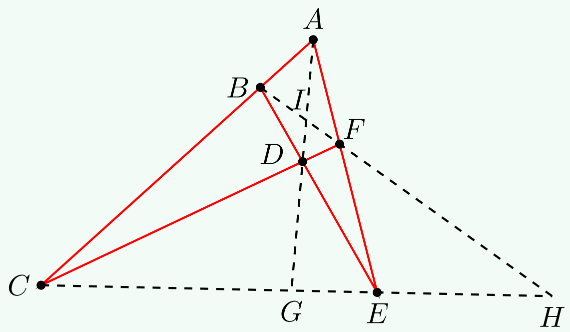
1 Introduction

Complete quadrilateral and *Complete quadrangle* are a pair of projective dual configurations in Euclidean geometry. They have some interesting properties which we would discuss in this article.

We begin by giving definitions of those two objects.

Definition 1. (Complete Quadrilateral)

A complete quadrilateral is a system of four lines, no three of which pass through the same point, and the six points of intersection of these lines.

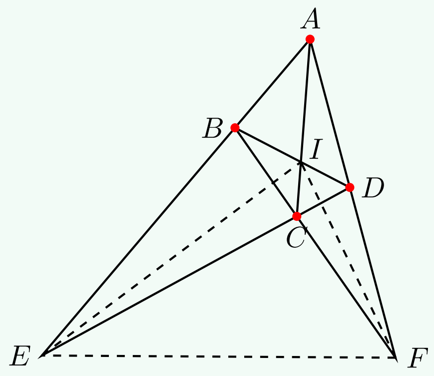


In the above picture, the linese AB, BD, DF and FA are called sides; the points A, B, C, D, E, F are called vertices; the lines AD, BF, and CE are called diagonals, and $\triangle IGH$ is called diagonal triangle.

Definition 2. (Complete Quadrangle)

A Complete Quadrangle is a set of four points, no three collinear, and the six lines which join them.

[1]The authors thank Dr. Zhiqin Lu for his help.



In the above picture, A, B, C, D are called vertices, the lines AB and CD, AC and BD, AD and BC are called pairs of opposite sides, and $\triangle IEF$ are called diagonal triangle.

Complete quadrilateral and Complete quadrangle are dual to each other in the sense of projective geometry.

Definition 3. (Duality Principle)

All the propositions in projective geometry occur in dual pairs, which have the property that, starting from either proposition of a pair, the other can be immediately inferred by interchanging the parts played by the words "point" and "line."

Complete quadrilateral and Complete quadrangle are dual to each other by exchanging “line” and “point”. As a result, each projective theorem about a complete quadrilateral has its corresponding dual theorem with respect to a complete quadrangle, and vice versa.

2 Cross-Ratio

Cross-ratio is a very important concept in projective geometry.

Definition 4. (Cross-ratio)

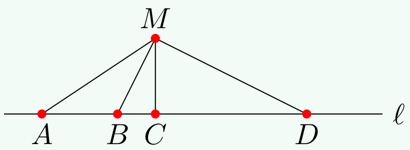
Let ℓ be a line and A, B, C, and D are four points which lie in this order on it.

We define the cross-ratio of A, B, C, D by

$$(A, B; C, D) = \frac{AC}{BC} \cdot \frac{BD}{AD}.$$

Alternatively, if four lines MA, MB, MC and MD are concurrent to a point M outside *line ℓ, then we define the cross-ratio of lines MA, MB, MC, MD* [a] *by*

$$(MA, MB; MC, MD) = \frac{\sin\angle AMC}{\sin\angle BMC} \cdot \frac{\sin\angle BMD}{\sin\angle AMD}.$$



[a] It should be noted that the cross-ratio is independent to the line ℓ. See Theorem 1 of Topic 36.

We have

Theorem 1

The cross-ratio of concurrent lines is equal to the cross-ratio of the corresponding four points, that is, in the above picture, we have

$$(A, B; C, D) = (MA, MB; MC, MD).$$

Proof: Since A, B, C, D are collinear points and MA, MB, MC, MD are concurrent lines,

$$\frac{S_{\triangle AMC}}{S_{\triangle BMC}} \cdot \frac{S_{\triangle BMD}}{S_{\triangle AMD}} = \frac{AC}{BC} \cdot \frac{AD}{AD} = (A, B; C, D).$$

By the law of sines, the area of a triangle can be expressed as $S_{\triangle AMC} = \frac{1}{2} \cdot MA \cdot MB \cdot \sin\angle AMC$. Thus,

$$\begin{aligned}
&\frac{S_{\triangle AMC}}{S_{\triangle BMC}} \cdot \frac{S_{\triangle BMD}}{S_{\triangle AMD}} \\
&= \frac{\frac{1}{2}MA \cdot MB \cdot \sin\angle AMC}{\frac{1}{2}MB \cdot MC \cdot \sin\angle BMC} \cdot \frac{\frac{1}{2}MB \cdot MD \cdot \sin\angle BMD}{\frac{1}{2}MA \cdot MD \cdot \sin\angle AMD} \\
&= \frac{\sin\angle AMC}{\sin\angle BMC} \cdot \frac{\sin\angle BMD}{\sin\angle AMD} = (MA, MB; MC, MD).
\end{aligned}$$

Therefore, we conclude that

$$(A, B; C, D) = (MA, MB; MC, MD).$$

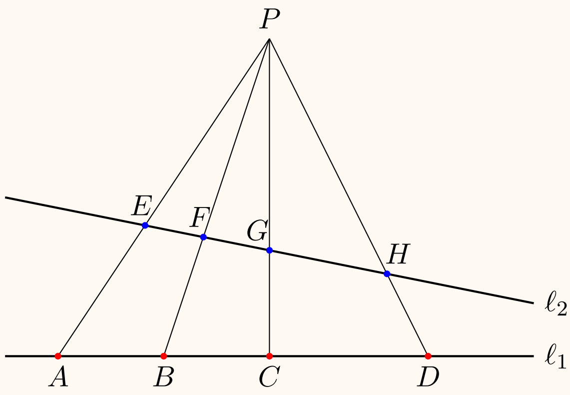
■

The following invariance property of cross-ratio is essential in our discussion of complete quadrilateral and complete quadrangle. It essentially follows from the above theorem.

Theorem 2

Let P be a point outside line ℓ_1. Let PA, PB, PC, PD intersect with another line ℓ_2 at E, F, G, H, respectively. Then the cross-ratios of the two groups of points are the same

$$(A, B; C, D) = (E, F; G, H).$$



Proof: By Definition 4 and Theorem 1,

$$\begin{aligned}
&(A, B; C, D) \\
&= (PA, PB; PC, PD) \\
&= \frac{\sin \angle APC}{\sin \angle BPC} \cdot \frac{\sin \angle BPD}{\sin \angle APD} \\
&= (PE, PF; PG, PH) \\
&= (E, F; G, H).
\end{aligned}$$

Definition 5. (Harmonic Division)

The four-point A, B, C, D is called a harmonic range of points [a] *if and only if*

$$(A, B; C, D) = 1.$$

That pencil MA, MB, MC, MD is harmonic or called harmonic pencil of lines if and only if

$$(MA, MB; MC, MD) = 1.$$

[a] For more information and properties of Harmonic properties please refer to Topic 24.

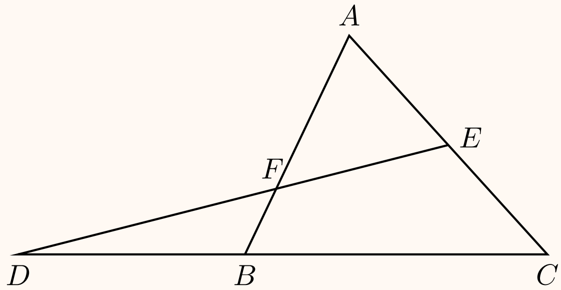
3 Harmonicity in Complete Quadrilateral

Menelaus' Theorem and Ceva's Theorem are fundamental theorems in our paper. For details, see Theorem 1 and Theorem 3 of Topic 2.

Theorem 3. (Menelaus' Theorem)

In the following $\triangle ABC$, D, E, F are points on BC, CA, and AB, respectively. Assume that D, E, F are collinear. Then

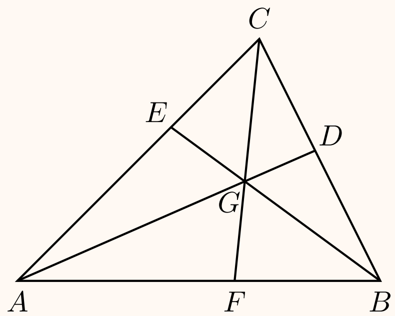
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$



Theorem 4. (Ceva's Theorem)

In the following $\triangle ABC$, the lines AD, BE, CF are concurrent. Then

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

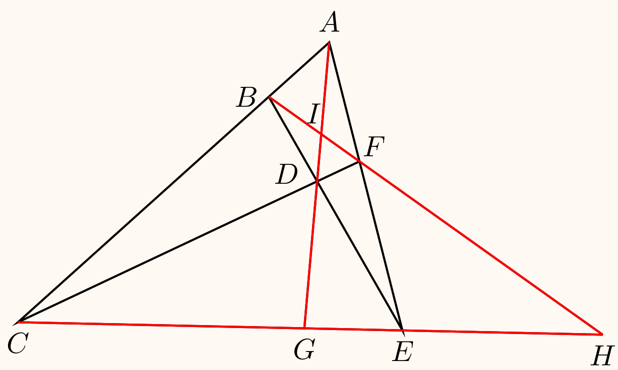


The following theorem is essential to complete quadrilateral.

Theorem 5. (Harmonicity in Complete Quadrilateral)

Let $ABCDEF$ be a complete quadrilateral. Let G, H be the intersection of the diagonals AD and BF to CE, respectively. Then G, H harmonically divide the diagonal CE. Similarly, H, I harmonically divide the diagonal BF, and G, I harmonically divide AD. In short, we can conclude that any two diagonals harmonic divides the third diagonals.

1. *C, G, E, H are the harmonic range of points;*
2. *B, I, F, H are the harmonic range of points;*
3. *A, I, D, G are the harmonic range of points.*



Proof: By applying Menelaus' Theorem to $\triangle ACE$, we have

$$\frac{AB}{BC} \cdot \frac{CH}{HE} \cdot \frac{EF}{FA} = 1.$$

By using Ceva's Theorem in $\triangle ACE$, we get,

$$\frac{AB}{BC} \cdot \frac{CG}{GE} \cdot \frac{EF}{FA} = 1.$$

Comparing the above two equations, we get

$$\frac{CH}{HE} = \frac{CG}{GE},$$

and hence C, G, E, H are harmonic range of points.

The other two assertions follow by the same method. ∎

Remark If $BF \parallel CE$, then we say point H will meet at infinity and we have

$$\frac{BI}{IF} = \frac{CG}{GE}$$

and the Theorem 5 is still valid.

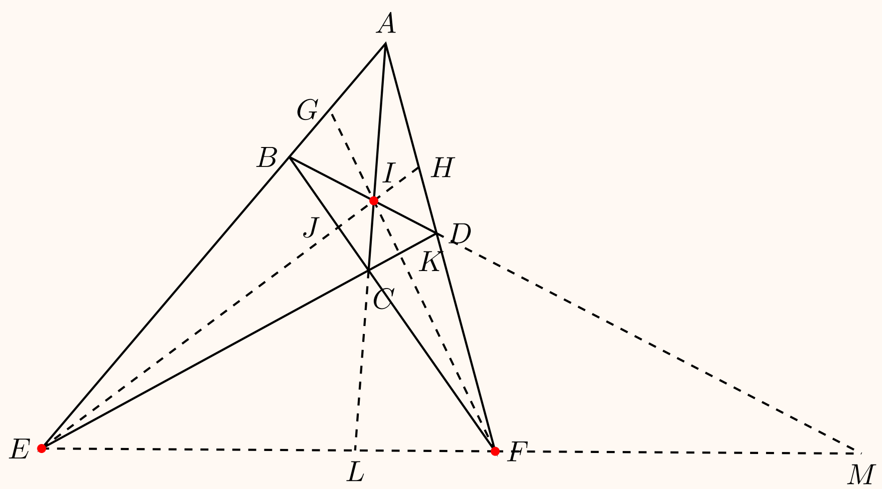
4 Harmonicity in Complete Quadrangle

As the dual figure of complete quadrilateral, complete quadrangle has its corresponding harmonic properties. In the section, we discuss harmonic pencil of the complete quadrangle.

Theorem 6. (Harmonic Pencil of Lines in Complete Quadrangle)

Let G be the intersection point of AB and IF, H be the intersection point of AD and IE, L be the intersection point of AC and EF, and M be the intersection point of BD and EF. There are three set of harmonic pencil of lines in the complete quadrangle $AECF$:

1. *EA, EH, ED, EF are harmonic pencil of lines;*
2. *FA, FG, FB, FE are harmonic pencil of lines;*
3. *IE, IL, IF, IM are harmonic pencil of lines.*



Proof: This theorem is the dual theorem of Theorem 5.

By applying Duality principle to Theorem 5, we can get EA, EH, ED, EF and FA, FG, FB, FE are two pairs of harmonic pencil of lines. That is,

$$(EA, EH; ED, EF) = 1$$

$$(FA, FG, FB, FE) = 1$$

By Theorem 2, we get

$$(E, B; G, A) = (E, J; I, H) = 1$$

$$(F, K; I, G) = (F, D; H, A) = 1$$

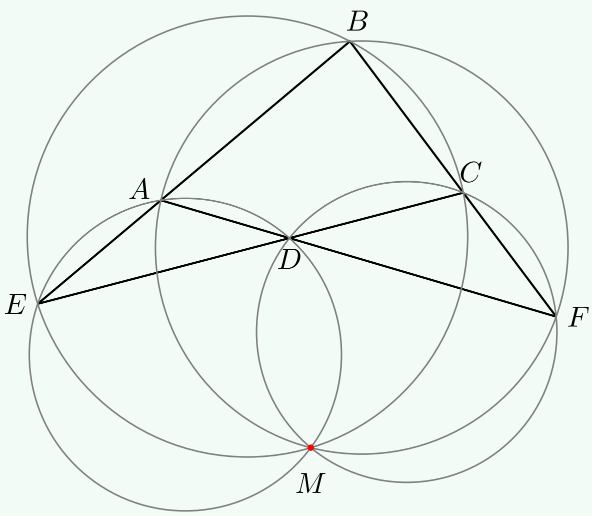
Thus, we get the four harmonic pencil of point.

5 Theorems Related to Complete Quadrilateral

Despite the harmonicity properties, complete quadrilateral has some other interesting properties. The following are theorems related to Miqual Point, Simson Line, and Newton Line.

Definition 6. (Miquel Point)

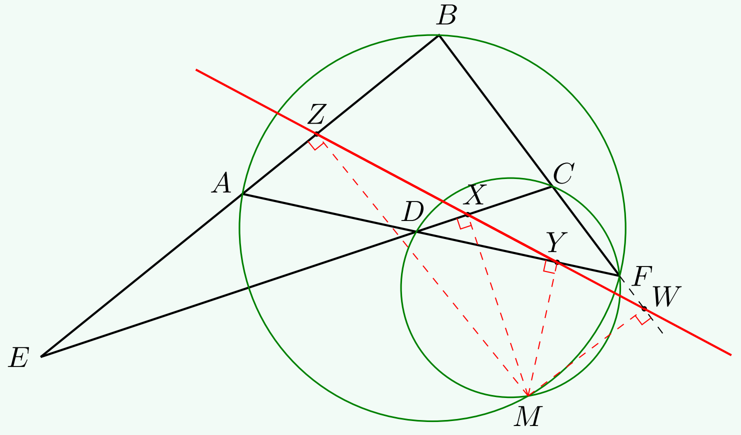
A complete quadrilateral contains four triangles. Their circumcircles are concurrent, and the concurrent point is called the Miquel Point [a] of the complete quadrilateral.



[a] For more information and proof of Miquel Point, please refer to the Topic 20.

Definition 7. (Simson Line)

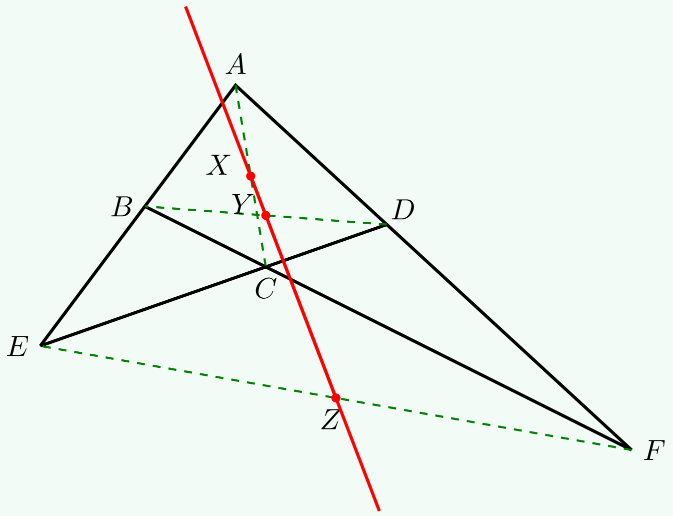
Let M be the Miquel Point of complete quadrilateral $ABCDEF$. Then the pedal points of M to each side of the quadrilateral are collinear. The line is called the Simson Line [a] of the complete quadrilateral.



[a]For more information and proof of the Simson Line please refer to the Topic 20.

Definition 8. (Newton Line)

Let $ABCDEF$ be a complete quadrilateral. Let X, Y, and Z be the midpoints of the diagonals AC, BD and EF, respectively. Then X, Y, and Z are collinear and this line is called the Newton Line.



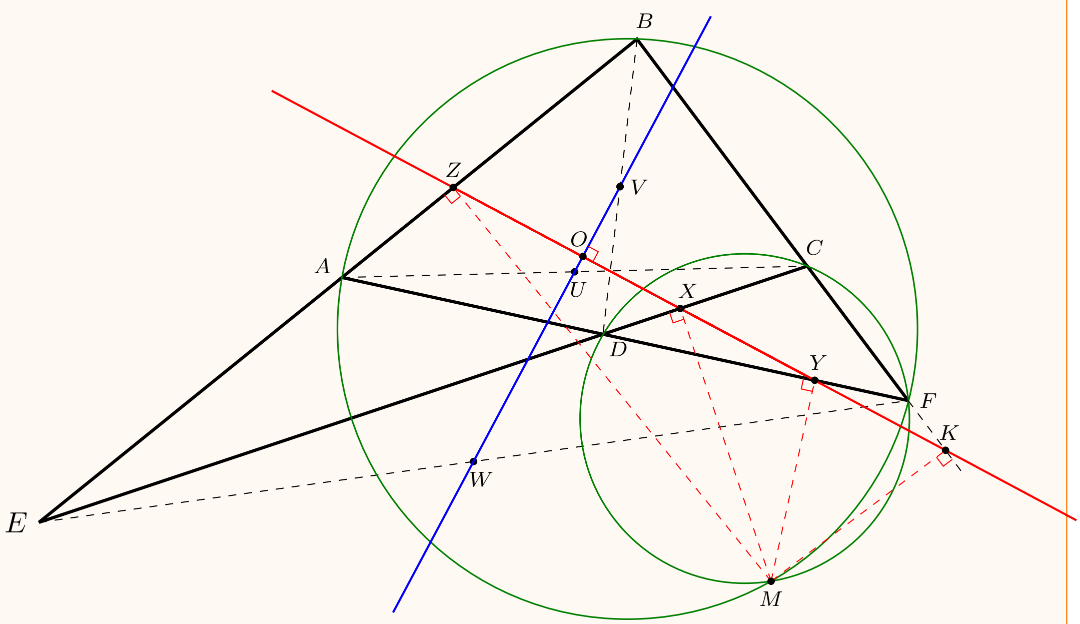
The midpoints of the three diagonals of a complete quadrilateral lie on the Newton Line.[a]

[a]For more information and proof of Newton Line please refer to the Topic 26.

The following result about the relation of the Simson line and the Newton line is interesting. For a proof, see Topic 20.

Theorem 7

The Newton Line and the Simson Line of a complete quadrilateral are perpendicular.[a]



[a] The red line is the Simson Line, and the blue line UV is Newton Line, where U, V, W are the midpoints of AC, BD and EF, respectively.