Lemoine Line

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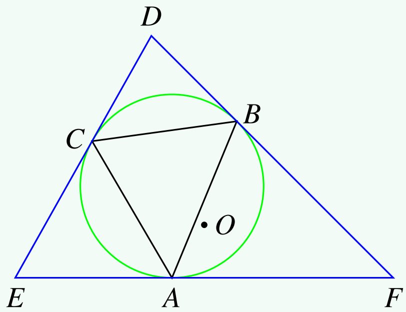
(last updated: June 3, 2023)

Definition 1

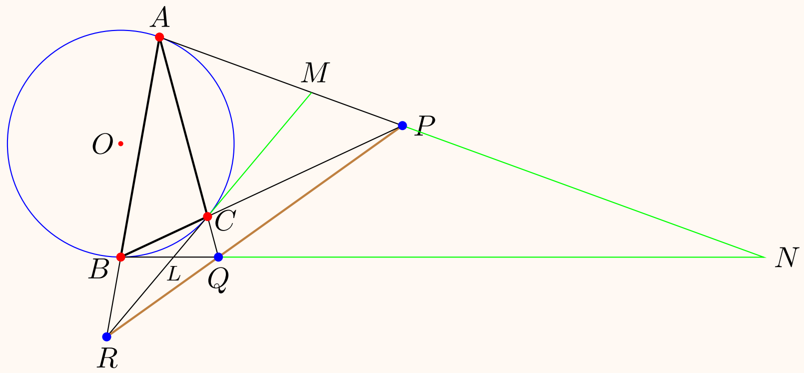
Given a triangle $\triangle ABC$ and its circumcircle O, let EF, FD and DE be the tangent lines of the circle O at points A, B and C, respectively. Then $\triangle DEF$ is called the tangential triangle of $\triangle ABC$.



It is well-known that the lines DA, EB and FC are concurrent, and the intersection is called the Gergonne point of the triangle $\triangle DEF$ (see Topic 8). Alternatively, we have the following result

Theorem 1. (Lemoine Line)

Let $\triangle ABC$ be inscribed in circle O. The tangent lines AP, BQ, CR forms the tangent triangle of $\triangle ABC$. Then P, Q, R are collinear. This line is called the Lemoine Line, or the Lemoine Axis.



Proof: Since $\angle CAP = \angle B$ and $\angle CPA = \angle APB$ (By Alternate Segment Theo-

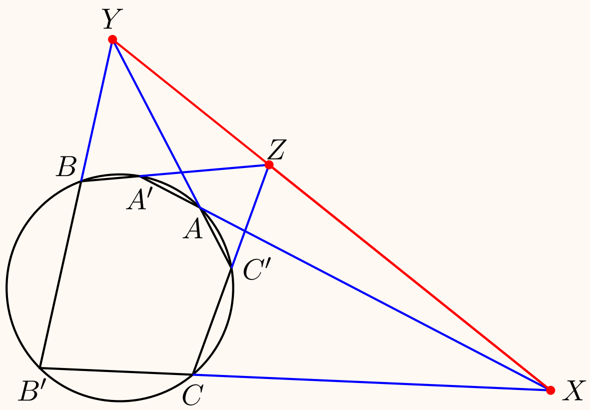
[1]The author thanks Dr. Zhiqin Lu for his help and Stephanie Wang for her careful reading and many comments.

rem), we have $\triangle PCA \backsim \triangle PAB$. Thus
$$\frac{BP}{BA} = \frac{PA}{AC}, \quad \frac{PC}{AC} = \frac{AP}{AB}.$$
As a result,
$$\frac{BP}{PC} = \frac{AB^2}{CA^2}.$$
Similarly, we have
$$\frac{AR}{RB} = \frac{CA^2}{BC^2}, \quad \frac{CQ}{QA} = \frac{BC^2}{AB^2}.$$
Therefore
$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = \frac{AB^2}{CA^2} \cdot \frac{BC^2}{AB^2} \cdot \frac{CA^2}{BC^2} = 1.$$
By Menelaus' Theorem, P, Q, R are collinear.

The above theorem about the Lemoine line is a limiting case of the following Pascal's theorem.

Theorem 2. (Pascal's Theorem)

Let Hexagon $AA'BB'CC'$ be inscribed in a circle. Let AA' and $B'C$ intersect at X; BB' and $C'A$ intersect at Y; and CC' and $A'B$ intersect at Z. Then X, Y, Z are collinear.



If A' is sufficiently close to A, then the secant line AA' becomes the tangent line of the circle at A. Similarly, if B' is sufficiently close to B and C' is sufficiently close to C, then the Pascal's line is reduced to the Lemoine's line.