Napoleon's Theorem

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We start with an AIME exam problem.

Problem 1. (AIME 2017-I-15)

The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths $2\sqrt{3},\ 5$, *and* $\sqrt{37}$, *as shown, is* $\frac{m\sqrt{p}}{n}$, *where* $m,\ n$, *and* p *are positive integers,* m *and* n *are relatively prime, and* p *is not divisible by the square of any prime. Find* $m+n+p$.

Solution: The following solution is from AoPS, where you can find other solutions.

Let's start by proving a lemma: If x, y satisfy $px+qy=1$, then the minimal value of $\sqrt{x^2+y^2}$ is $\frac{1}{\sqrt{p^2+q^2}}$.

Proof: Recall that the distance between the point (x_0, y_0) and the line $px+qy+r=0$ is given by $\frac{|px_0+qy_0+r|}{\sqrt{p^2+q^2}}$. In particular, the distance between the origin and any point (x,y) on the line $px+qy=1$ is at least $\frac{1}{\sqrt{p^2+q^2}}$.

Let the vertices of the right triangle be $(0,0)$, $(5,0)$, $(0,2\sqrt{3})$, and let $(a,0)$, $(0,b)$ be the two vertices of the equilateral triangle on the legs of the right triangle. Then, the third vertex of the equilateral triangle is $\left(\frac{a+\sqrt{3}b}{2}, \frac{\sqrt{3}a+b}{2}\right)$. This point must lie on the hypotenuse $\frac{x}{5}+\frac{y}{2\sqrt{3}}=1$, i.e. a, b must satisfy

$$\frac{a+\sqrt{3}b}{10}+\frac{\sqrt{3}a+b}{4\sqrt{3}}=1,$$

which can be simplified to

$$\frac{7}{20}a+\frac{11\sqrt{3}}{60}b=1.$$

By the lemma, the minimal value of $\sqrt{a^2+b^2}$ is

$$\frac{1}{\sqrt{\left(\frac{7}{20}\right)^2+\left(\frac{11\sqrt{3}}{60}\right)^2}}=\frac{10\sqrt{3}}{\sqrt{67}},$$

so the minimal area of the equilateral triangle is

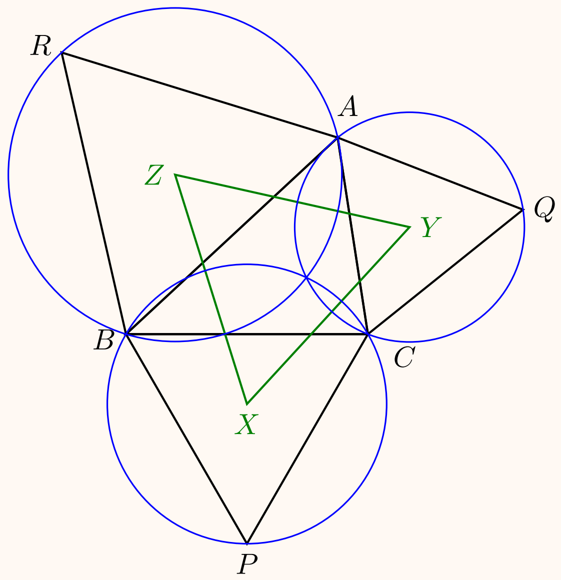
$$\frac{\sqrt{3}}{4}\cdot\left(\frac{10\sqrt{3}}{\sqrt{67}}\right)^2=\frac{\sqrt{3}}{4}\cdot\frac{300}{67}=\frac{75\sqrt{3}}{67},$$

and hence the answer is $75+3+67=145$. ■

The above problem leads to the more general question. How to determine the minimal inscribed equilateral triangle of a given triangle? To answer this question, we first need to introduce the Napoleon Theorem.

Theorem 1. (Napoleon's Theorem)

In the following, $\triangle BCP$, $\triangle CAQ$, and $\triangle ABR$ are equilateral triangle. Let X, Y, Z be the centers of $\triangle BCP$, $\triangle CAQ$, and $\triangle ABR$ respectively. Then $\triangle XYZ$ is equilateral.



Remark The theorem is often attributed to the French emperor Napoleon Bonaparte. See Wikipedia for details.

Proof: The best way to prove this theorem is to use complex numbers. Assume that A, B, C correspond to complex numbers a, b, c. We can determine the complex number of point P as follows. Let $\sigma = e^{\pi i/3}$. Then $(b-c)\sigma + c$ is the complex number of P. The complex number of the center X is

$$\frac{1}{3}(b + c + (b-c)\sigma + c) = \frac{1}{3}((1+\sigma)b + (2-\sigma)c).$$

By symmetry, Y corresponds to $\frac{1}{3}((1+\sigma)c + (2-\sigma)a)$, and Z corresponds to $\frac{1}{3}((1+\sigma)a + (2-\sigma)b)$. We compute

$$Z - X = \frac{1}{3}((1+\sigma)a + (1-2\sigma)b - (2-\sigma)c),$$

and

$$Y - X = \frac{1}{3}((2-\sigma)a - (1+\sigma)b - (1-2\sigma)c).$$

We need to prove that $|Z - X| = |Y - X|$. In order to do that, we assume that $a = 0$. Then we have

$$|Z - X| = |Y - X| = \frac{\sqrt{3}}{3}|b - \sigma c|.$$

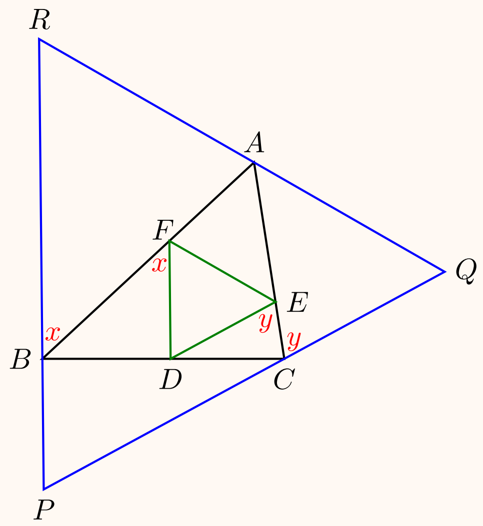
This completes the proof of the theorem. ■

Interestingly, in order to find the minimal inscribed equilateral triangle, we first need to find the maximum circumscribed equilateral triangle. The following result showed the relation of the two equilateral triangles.

Theorem

Assume that $\triangle DEF$ is an inscribed equilateral triangle. Assume also that $DE \parallel PQ$, $EF \parallel QR$, and $FD \parallel RP$. Then

$$DE \cdot PQ = \frac{4}{\sqrt{3}} \text{Area}(\triangle ABC).$$



Proof: By the assumption, then $\triangle PQR$ is also an equilateral triangle. Let $\angle A = \alpha$, $\angle B = \beta$, and $\angle C = \gamma$. Let R be the radius of the circumscribed circle of $\triangle ABC$. Let $DE = a$, $QR = b$. By law of sines, we have

$$b = RA + AQ = \frac{AB}{\sin 60^\circ} \cdot \sin x + \frac{AC}{\sin 60^\circ} \cdot \sin y = \frac{4R}{\sqrt{3}}(\sin x \sin \gamma + \sin y \sin \beta).$$

Similarly,

$$BC = BD + DC = \frac{a}{\sin \beta} \cdot \sin x + \frac{a}{\sin \gamma} \cdot \sin y.$$

Thus

$$a = \frac{BC}{\dfrac{\sin x}{\sin \beta} + \dfrac{\sin y}{\sin \gamma}} = \frac{2R \sin \alpha}{\dfrac{\sin x}{\sin \beta} + \dfrac{\sin y}{\sin \gamma}}.$$

Therefore, we have

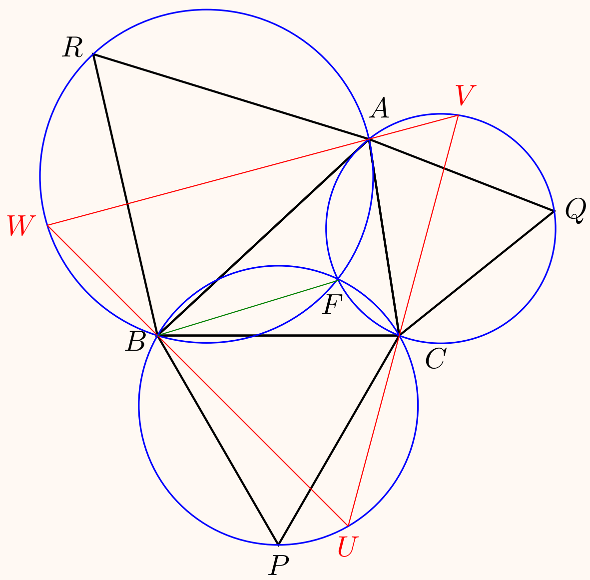
$$ab = \frac{8R^2}{\sqrt{3}} \sin \alpha \sin \beta \sin \gamma.$$

∎

From the above theorem, we need to find the maximum circumscribed equilateral triangle. In the following theorem, we characterize all these triangles.

Theorem

In the following picture, starting from a point U on the circumscribed circle of $\triangle BCP$. Connecting UB intersecting at W, and UC intersecting on V. Then U, A, W are collinear. Moreover, $\triangle UVW$ is equilateral.



Proof: It is easy to see that $\angle U = \angle V = \angle W = 60°$. Then we must $\angle VAW = 180°$. Thus U, A, W are collinear. ∎

Finally, we are able to solve the minimal inscribed equilateral triangle problem. In the following picture, $WU = 2\,RS \leq 2O_1O_2$. Thus WU will be maximized when $WU \perp AB$. Thus when $WU \perp AB$, we get the maximum circumscribed equilateral triangle, and hence the minimal inscribed triangle can be located as well.