Brahmagupta Theorem

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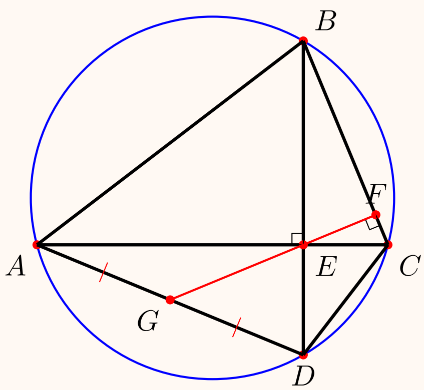
(last updated: June 18, 2022)

Brahmagupta's Theorem is named after the Indian mathematician and astronomer *Brahmagupta*. India is known as the birthplace of the number "0," but what many are not aware of is that it was Brahmagupta who first came up for the rules for computation with number "0". Not only was his work a big contribution in the branch of mathematics, his research in astronomy has been claimed to be light-years ahead of it's time.

In this article, we introduce a Euclidean geometry theorem discovered by Brahmagupta.

Theorem 1. (Brahmagupta's Theorem)

A cyclic, orthodiagonal quadrilateral is an quarilateral inscribed in a circle with perpendicular diagonals. Then the line, passing through the intersection of diagonals, which is a perpendicular to one side of the quadrilateral will bisect the opposite side.



Proof: In the above picture, we assume that $ABCD$ is an inscribed quadrilateral of the circle. Assume that the diagonals $AC \perp BD$. Let $EF \perp BC$. EF intersects AD at G. We shall prove that $AG = GD$.

We claim that $AG = GE$. This is because as inscribed angles over the same arc, $\angle DAC = \angle DBC$. On the other hand, both $\angle DBC$ and $\angle FEC$ are complement angles[a] of $\angle BCE$. So they are equal. Therefore we have

$$\angle GAE = \angle DBC = \angle FEC = \angle GEA.$$

As a result, $\triangle GAE$ is isosceles and hence $AG = GE$. By the same reason, $GE = GD$. Therefore $AG = GD$, completing the proof. ∎

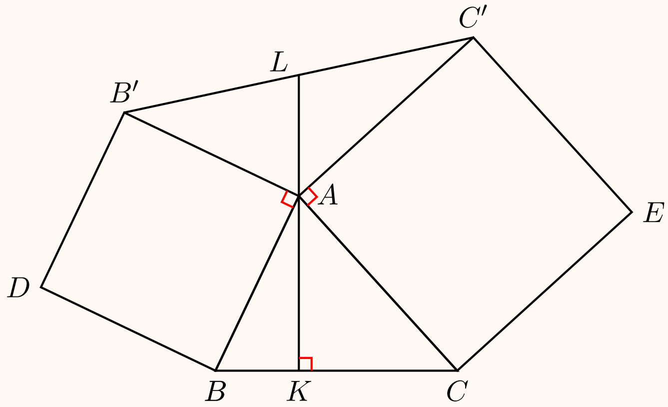
[a] Meaning that $\angle DBC + \angle BCE = 90° = \angle FEC + \angle BCE$.

[1] The author thanks Dr. Zhiqin Lu for his help.

The phenomena that the height of one triangle extends to the median of another triangle appears in other situations. For example, in Topic 11, a similar result is proved for dual triangles.

Theorem 2

In the following picture, $AB'DB$ and $ACEC'$ are squares. Let AK be a height over BC, then AL is the median of $\triangle AB'C'$.



So the median of one triangle is the height of its dual triangle, and vice versa.

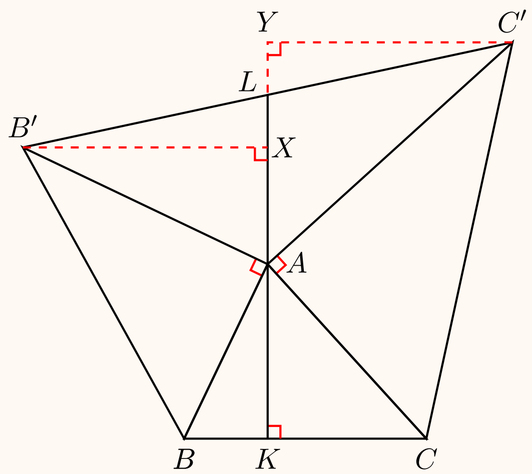
In this article, we shall prove the following result, which provides the common generalization of the above two theorems.

Theorem 3. (Generalized Brahmagupta's Theorem)

In the following picture, assume that $\triangle ABB'$ and $\triangle ACC'$ are right triangles. Asumme that

$$AB' \cdot AC = AB \cdot AC'. \tag{1}$$

Then the height AK of $\triangle ABC$ extends to a median AL of $\triangle AB'C'$.



Proof: We observe that $\angle AC'Y = 90° - \angle YAC' = \angle CAK$. Since both $\triangle AC'Y$ and $\triangle CAK$ are right triangles, they are similar by the AA similarity postulate. Therefore we have

$$\frac{C'Y}{AK} = \frac{C'A}{AC}.$$

Similarly, $\triangle B'AX \sim \triangle BAK$ and as a result,

$$\frac{B'X}{AK} = \frac{B'A}{AB}.$$

Using the above two equations and the assumption (1), we conclude that

$$C'Y = B'X.$$

Since $B'X = C'Y$, $\angle B'LX = \angle C'LY$ (vertical angles) and $\angle B'XL = \angle C'YL$, we can claim that $\triangle B'LX \cong \triangle C'LY$ using the AAS congruence postulate. Therefore $B'L = C'L$ and the theorem is proved.

■

By the Intersecting Chords Theorem, $AE \cdot EC = BE \cdot ED$ in the picture of Theorem 1. Therefore Theorem 1 follows from Theorem 3. On the other hand, since $AB' = AB.AC' = AC$ in Theorem 2, the theorem also follows from Theorem 3.

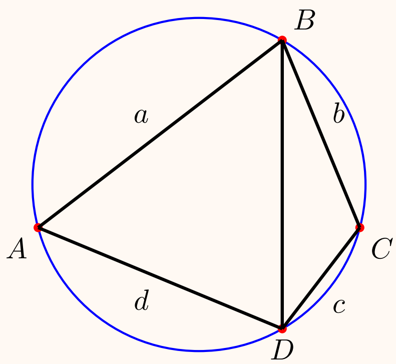
Brahmagupta probably is more well-known by his area formula of cyclic quadrilateral.

Theorem 4. (Brahmagupta's formula)

Assume that $ABCD$ is an inscribed quadrilateral of the circle. Let

$$p = \frac{a+b+c+d}{2}$$

be the semi-perimeter. Then



$$\text{Area}(ABCD) = \sqrt{(p-a)(p-b)(p-c)(p-d)}.$$

Proof: We provide a proof using trigonometry. Let S denote the area of the cyclic-quadrilateral below. Then

$$S = \text{Area}(\triangle ADB) + \text{Area}(\triangle CBD) = \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin C.$$

Since $ABCD$ is cyclic, $\angle BAD = 180° - \angle DCB$. Therefore, $\sin A = \sin C$ and we can write S as:

$$S = \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin A = \frac{1}{2}(ad+bc)\sin A.$$

We therefore conclude that

$$4S^2 = (ad+bc)^2\sin^2 A = (ad+bc)^2(1-\cos^2 A).$$

In order to compute $\cos A$, we use the laws of cosines to show that

$$d^2 + a^2 - 2ad\cos A = BD^2 = b^2 + c^2 - 2bc\cos C.$$

Since $\cos C = -\cos A$, this gives

$$(ad+bc)\cos A = \frac{d^2+a^2-b^2-c^2}{2}.$$

Now, substituting this in the area equation we get:

$$4S^2 = (ad+bc)^2 - \frac{1}{4}(d^2+a^2-b^2-c^2)^2,$$

and hence

$$16S^2 = (a+b+c-d)(a+b-c+d)(a-b+c+d)(b+c+d-a).$$

Using the semi-perimeter, we have

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)},$$

and the theorem is proved.

More generally, we have the following Coolidge's formula.

Theorem 5. (Coolidge's Formula)

Let a, b, c, d be the side lengths of the four sides of a convex quadrilateral. Let e, f be the lengths of the diagonals. Then the area of the quadrilateral is given by

$$\sqrt{(p-a)(p-b)(p-c)(p-d) - \frac{1}{4}(ac+bd+ef)(ac+bd-ef)}.$$

When the quadrilateral is cyclic, then by the Ptolemy Theorem, $ef = ac + bd$, and then the Coolidge's formula is reduced to the Brahmagupta's formula. For more details of the Coolidge's formula, see J. L. Coolidge, *A Historically Interesting Formula for the Area of a Quadrilateral*, The American Mathematical Monthly Vol. 46, No. 6 (Jun. - Jul., 1939), pp. 345-347.