Quadrilateral Area Formulas

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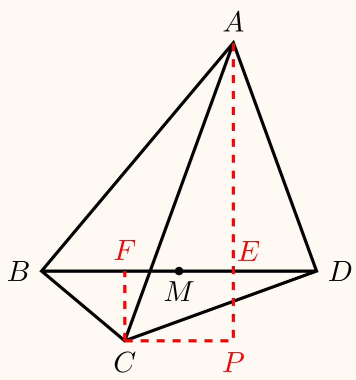
There is no simple area formula for a general quadrilateral. This is partially due to the *instability of quadrilateral*, that is, unlike the case of triangle, four sides do not completely determine a unique quadrilateral.

In the following *Bretschneider's Formula*, the area is represented by the four sides and the two diagonals of a quadrilateral.

Theorem 1. (Bretschneider Formula)

In the following convex quadrilateral $ABCD$. Assume that $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, and $BD = f$. Then the area S of $ABCD$ is given by

$$S = \frac{1}{4}\sqrt{4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2}$$



Proof. Draw $AP \perp BD$ and $CP \perp AP$. The key computation should be AP, because

$$S = \frac{1}{2}BD \cdot AP. \tag{1}$$

Let M be the midpoint of BD. By the Pythagorean Theorem, we have

$$AB^2 - AD^2 = BE^2 - ED^2 = (BE + ED)(BE - ED).$$

Thus

$$a^2 - d^2 = f(BE - ED) = f(BM + ME - (MD - ME)) = 2f \cdot ME.$$

Similarly, we have

$$c^2 - b^2 = 2f \cdot FM.$$

[1]The author thanks Stephanie Wang for her careful reading and many suggestions.

Therefore, we have

$$a^2 - b^2 + c^2 - d^2 = 2f \cdot EF.$$

Thus

$$AP = \sqrt{AC^2 - EF^2} = \sqrt{e^2 - \frac{1}{4f^2}(a^2 - b^2 + c^2 - d^2)^2}.$$

The Bretschneider's Formula follows from the above equation and (1). ■

An algebraic variant of Bretschneider's Formula is the following:

Corollary 1. (Coolidge Formula)

Let $p = (a+b+c+d)/2$ be the semiperimeter of the quadrilateral. Then the area S is given by

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d) - \frac{1}{4}(ac+bd-ef)(ac+bd-ef)}.$$

Proof. We have the following algebraic identity

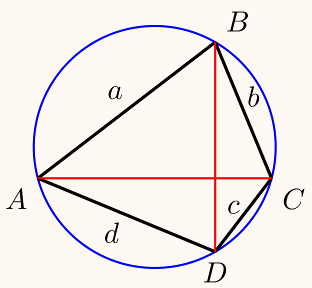
$$\begin{aligned}
&4(ac+bd)^2 - (a^2 - b^2 + c^2 - d^2)^2 \\
&= (2ac + 2bd + a^2 - b^2 + c^2 - d^2)(2ac + 2bd - a^2 + b^2 - c^2 + d^2) \\
&= ((a+c)^2 - (b-d)^2)((b+d)^2 - (a-c)^2) \\
&= (a+c+b-d)(a+c-b+d)(b+d+a-c)(B+d-a+c) \\
&= 16(p-a)(p-b)(p-c)(p-d).
\end{aligned}$$

The corollary then follows from Bretschneider's Formula. ■

A quadrilateral is called *cyclic*, if the four vertices all lie on a single circle.

Corollary 2. (Brahmagupta's formula)

Assume that $ABCD$ is a convex cyclic quadrilateral with sides a, b, c, d, respectively.



Then the area of $ABCD$ is given by

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)},$$

where $p = (a+b+c+d)/2$ is the semiperimeter.

Proof. Using the Ptolemy's Theorem (see Wikipeida or Topic 10), we have

$$ef = ad + bd.$$

Therefore the formula follows from the Coolidge formula (Corollary 1). ■

Corollary 3. (Heron's Formula)

Assume that the three sides of a triangle are a, b, c, respectively. Then the area of the triangle is given by

$$S = \sqrt{p(p-a)(p-b)(p-c)},$$

where p is the semiperimeter.

Proof. We take $d = 0$ in Brahmagupta's formula. When one side of a quadrilateral is degenerate to zero, the quadrilateral is degenerate to a triangle, and the Heron's formula follows. ■

A polygon is called *tangential*, or *circumscribed*, if it is a convex polygon that contains an inscribed circle.

For the rest of this article, we introduce area formulas for tangential quadrilaterals and general tangential polygons.

Theorem 2

Let $P = A_1 \cdots A_n$ be a tangential n-polygon with r being the radius of the inscribed circle, where A_i are the vertexes for $1 \le i \le n$. Let $a_i = A_iA_{i+1}$ for $1 \le i \le n-1$ and $a_n = A_nA_1$ be the side lengths of the polygon. Then the area of the polygon is given by

$$S = \frac{1}{2}(\sum_{i=1}^{n} a_i)r.$$

The proof is obvious and left to the reader.

Before applying the above theorem into quadrilateral, we first study the following example.

Problem

A circle is inscribed in quadrilateral $ABCD$, tangent to $\overline{AB}$ at P and to $\overline{CD}$ at Q. Given that $AP = 19$, $PB = 26$, $CQ = 37$, and $QD = 23$, find the square of the radius of the circle.

This is the Problem No. 10 in the 2000 AIME (II).

Solution. Call the center of the circle O. By drawing the lines from O tangent to the sides and from O to the vertices of the quadrilateral, four pairs of congruent right triangles are formed. Thus

$$\angle AOP + \angle POB + \angle COQ + \angle QOD = 180,$$

or

$$(\arctan(\tfrac{19}{r}) + \arctan(\tfrac{26}{r})) + (\arctan(\tfrac{37}{r}) + \arctan(\tfrac{23}{r})) = 180^\circ.$$

Take the tangent of both sides and use the identity for $\tan(A+B)$ to get

$$\tan(\arctan(\tfrac{19}{r}) + \arctan(\tfrac{26}{r})) + \tan(\arctan(\tfrac{37}{r}) + \arctan(\tfrac{23}{r})) = 0.$$

Use the identity for $\tan(A+B)$ again to get

$$\frac{\frac{45}{r}}{1 - 19\cdot\frac{26}{r^2}} + \frac{\frac{60}{r}}{1 - 37\cdot\frac{23}{r^2}} = 0.$$

Solving the above equation, we get $r^2 = 647$.

In the above solution, we used the formula

$$\alpha + \beta = \arctan\left(\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\right).$$

It turns out that we have the following generalization of the above formula. Let $x_1, \cdots, x_n$ be real numbers. As is well-known,

$$\tan(\arctan x_1 + \arctan x_2) = \frac{x_1 + x_2}{1 - x_1 x_2}.$$

Thus we have

$$\tan(\arctan x_1 + \arctan x_2 + \arctan x_3) = \frac{\tan(\arctan x_1 + \arctan x_2) + x_3}{1 - \tan(\arctan x_1 + \arctan x_2)x_3}.$$

Using the formula again, we get

$$\tan(\arctan x_1 + \arctan x_2 + \arctan x_3) = \frac{\frac{x_1+x_2}{1-x_1x_2} + x_3}{1 - \frac{x_1+x_2}{1-x_1x_2}x_3}$$
$$= \frac{x_1 + x_2 + x_3 - x_1x_2x_3}{1 - (x_1x_2 + x_2x_3 + x_3x_1)}.$$

In general, we define the *elementary symmetric polynomials* by

$$\sigma_0 = 1$$
$$\sigma_1 = x_1 + \cdots + x_n = \sum_i x_i$$
$$\sigma_2 = \sum_{i<j} x_i x_j$$
$$\cdots\cdots$$
$$\sigma_k = \sum_{j_1<j_2<\cdots<j_k} x_{j_1}\cdots x_{j_k}$$
$$\cdots\cdots$$
$$\sigma_n = x_1 \cdots x_n$$

and $\sigma_r = 0$ if $r > n$. Then we have the following result.

Theorem 3

Using the above notations, we have

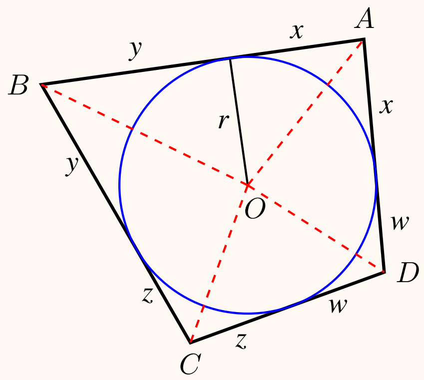
$$\tan\left(\sum_i \arctan x_i\right) = \frac{\sigma_1 - \sigma_3 + \sigma_5 - \cdots}{1 - \sigma_2 + \sigma_4 - \cdots}.$$

The theorem can be proved using the mathematical induction. We omit the proof here.

Theorem 4

In the following quadrilateral $ABCD$ inscribed by the circle of radius r, we assume that the lengths of the tangents are x, y, z, w, respectively. Then the area S is given by

$$S = \sqrt{(x+y+z+w)(xyz+yzw+zwx+wxy)}.$$



Proof. We connect OA, OB, OC, OD. Then

$$S = S_{\triangle OAB} + S_{\triangle OBC} + S_{\triangle OCD} + S_{\triangle ODA} = (x+y+z+w)r.$$

In order to express r in terms of a, b, c, d, we observe that

$$\angle OAB + \angle OBC + \angle OCD + \angle ODA = 360°.$$

Thus we have

$$\arctan\frac{x}{r} + \arctan\frac{y}{r} + \arctan\frac{z}{r} + \arctan\frac{w}{r} = 180°.$$

By Theorem 3, we have

$$0 = \sigma_1 - \sigma_3,$$

where $\sigma_1 = (x+y+z+w)/r$, and $\sigma_3 = (xyz+yzw+zwx+wxy)/r^3$. Thus we solve

$$r = \sqrt{\frac{xyz+yzw+zwx+wxy}{x+y+z+w}}.$$

The formula then follows from Theorem 2. ∎

A quadrilateral is called *bicentric*, if it is both tangential and cyclic.

Corollary 4

Let a, b, c, d be the side lengths of a bicentric quadrilateral. Then its area is given by

$$S = \sqrt{abcd}.$$

Proof. We use the picture in Theorem 4, where the lengths of tangent lines from the vertexes are x, y, z, w. Then the semiperimeter p is given by

$$p = x + y + z + w.$$

By definition, $a = x + y$, $b = y + z$, $c = z + w$, and $d = w + x$. Then we have $p - a = z + w = c$ and similarly, $p - b = d$, $p - c = a$, and $p - d = b$. Thus the corollary follows from Brahmagupta's formula (Corollary 2). ∎