Viviani's Theorem

Yufei Ren[1], yufeir@uci.edu

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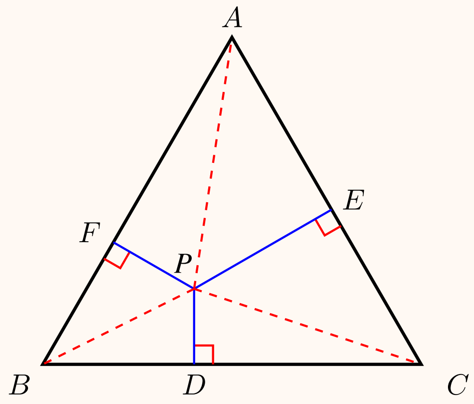
1 Introduction

The Viviani's Theorem states that the sum of the distances from any interior point to the sides of an equilateral triangle equals the length of the triangle's altitude. The Theorem is named for *Vincenzo Viviani* (1622–1703), a pupil of both *Galileo* and *Torricelli*.

2 The Theorem

Theorem 1. (Viviani's Theorem)

Let P be an interior point of the following equilateral triangle $\triangle ABC$. Then the sum of distances from P to the sides BC, CA and AB equals the length of the triangle's altitude.



Proof: Let a be the side length of $\triangle ABC$, and let PD, PE, PF be the perpendicular lines to the sides BC, CA and AB, respectively. Then the area of $\triangle BPC$, $\triangle CPA$, $\triangle APB$ are equal to

$$\frac{a \cdot PD}{2}, \quad \frac{a \cdot PE}{2}, \quad \frac{a \cdot PF}{2},$$

respectively. Since

$$S_{\triangle ABC} = S_{\triangle BPC} + S_{\triangle CPA} + S_{\triangle APB},$$

we have

$$\frac{\sqrt{3}}{4}a^2 = S_{\triangle ABC} = \frac{a \cdot PD}{2} + \frac{a \cdot PE}{2} + \frac{a \cdot PF}{2}.$$

[1]The author thanks Dr. Zhiqin Lu for his help.

We thus conclude that

$$PD + PE + PF = \frac{\sqrt{3}}{2}a,$$

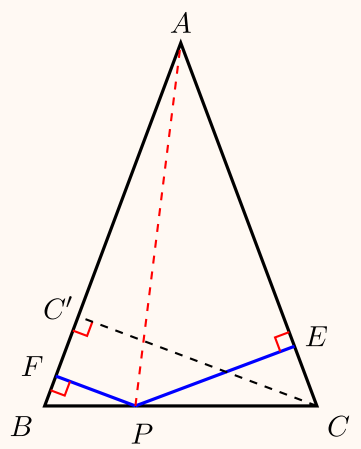
where the right side of the above equation is the altitude of $\triangle ABC$. ∎

3 Generalization

There are two directions that the Viviani's Theorem can be generalized. The first one is the following.

Theorem 2. (Viviani's Theorem on Isosceles Triangle)

Let $\triangle ABC$ be an isosceles triangle; $AB = AC$. Let P be a point on BC. Then the sum of distances from P to the two legs is equal to the altitude of the side AB.



Proof: In the above picture, assume that $CC' \perp AB$. Let $a = AB = AC$ and let $h = CC'$. If P is a point on the base BC, then the area of $\triangle ABC$, $\triangle APC$, and $\triangle APB$ are equal to

$$\frac{h \cdot a}{2}, \quad \frac{PE \cdot a}{2}, \quad \frac{PF \cdot a}{2}$$

respectively. Because $S_{\triangle ABC} = S_{\triangle APC} + S_{\triangle APB}$, we have

$$\frac{h \cdot a}{2} = \frac{PE \cdot a}{2} + \frac{PF \cdot a}{2}.$$

Thus we have

$$PE + PF = h,$$

completing the proof of the theorem. ∎

Before stating the second generalization, we first introduce the concept of signed distance.

Definition 1. (Signed Distance)

On a Cartesian coordinate system, let the equation of a line L to be $ax + by + c = 0$, and let $P = (x_0, y_0)$ be a point. Then the signed distance of the point P to L is given by

$$\frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}}.$$

Notice that the signed distance depends not only on the point and the line, but also depends on the orientation of the line: both $ax + by + c = 0$ and $-ax - by - c = 0$ represent the same line, but the corresponding signed distances differ by a negative sign.

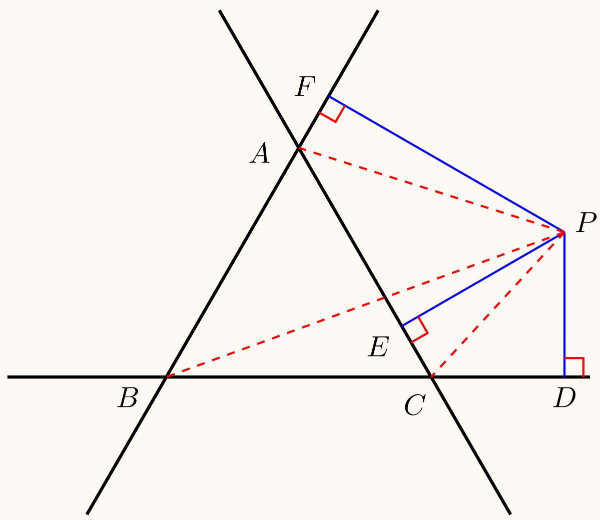
Using the concept of singed distance, we are able to prove the Viviani's Theorem when a point is outside of the equilateral triangle.

Let $\triangle ABC$ be a fixed triangle. We define the orientations of the lines BC, CA and AB in such a way that the signed distances of A to BC; B to CA; and C to AB, respectively, are all positive.

Theorem 3. (Viviani's Theorem on Signed Distance)

Using the above orientations, then the sum of the signed distances from any point to the sides of an equilateral triangle equals the length of the triangle's altitude.

In particular, if P is in the interior of the $\triangle ABC$, then the theorem is reduced to the original Viviani's Theorem (Theorem 1).



Proof: In the above picture, let $\triangle ABC$ be an equilateral triangle, and let a be the length of the sides. Let P be any point, and let $\tilde{x}, \tilde{y}, \tilde{z}$ be the signed distances to BC, CA, AB, respectively.

We observe that $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ are linear functions of P, so is $\tilde{x} + \tilde{y} + \tilde{z}$. When P is an interior point of $\triangle ABC$, we have

$$\tilde{x} + \tilde{y} + \tilde{z} = \frac{\sqrt{3}}{2}a,$$

=-== by Theorem 1. Therefore such a linear function $\tilde{x}+\tilde{y}+\tilde{z}$ must be identically equal to the constant $\frac{\sqrt{3}}{2}a$.

The theorem can also be proved by separating different cases. For example, if P locates as showed in the above picture, then $\tilde{x}, \tilde{z}$ are positive but $\tilde{y}$ is negative. Thus

$$PD=\tilde{x}, \quad PE=-\tilde{y}, \quad PF=\tilde{z}.$$

Since

$$S_{\triangle ABC}=S_{\triangle BPC}-S_{\triangle CPA}+S_{\triangle APB},$$

from the above picture, we have

$$PD-PE+PF=\frac{\sqrt{3}}{2}a,$$

and hence

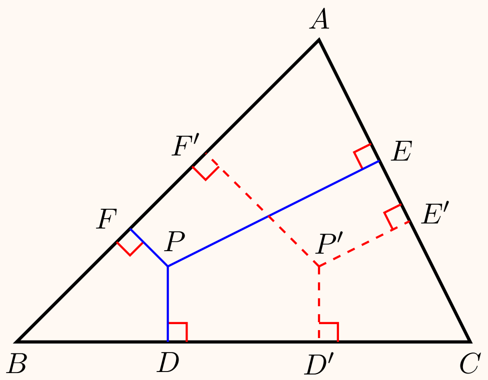
$$\tilde{x}+\tilde{y}+\tilde{z}=\frac{\sqrt{3}}{2}a.$$

4 The Converse Viviani's Theorem

The converse of the Viviani's Theorem holds.

Theorem 4. (The Converse Viviani's Theorem)

Let P be a point inside a fixed $\triangle ABC$, and let PD, PE, PF be the distances from P to BC, CA and AB, respectively. If $PD+PE+PF$ is a constant, then $\triangle ABC$ is equilateral.



Proof: Let $a=BC$, $b=CA$ and $c=AB$, and let $\beta=S_{\triangle ABC}$ be the area of $\triangle ABC$. Let $x=PD$, $y=PE$ and $z=PF$. Assume that

$$\alpha=PD+PE+PF=x+y+z.$$

Since

$$S_{\triangle ABC}=S_{\triangle BPC}+S_{\triangle APC}+S_{\triangle APB},$$

we have

$$a \cdot x + b \cdot y + c \cdot z = 2\beta.$$

In summary, we have the following system of linear equations of three variables

$$x + y + z = \alpha,$$

$$a \cdot x + b \cdot y + c \cdot z = 2\beta.$$

If $\triangle ABC$ is not equilateral, then without loss of generality, we may assume $a \neq b$. As a result, we have

$$x = \frac{2\beta - b\alpha}{a - b} + \frac{(b-c)z}{a-b},$$
$$y = \frac{2\beta - a\alpha}{b - a} + \frac{(a-c)z}{b-a}.$$

Let P' be another point in $\triangle ABC$ that has same distance to the side BC as P, that is $PD = P'D' = z$. Then by the above calculation, $P'D' = x = PD$ and $P'E' = y = PE$. But this is obviously a contradiction because it would have implied $P = P'$. ■

External Link. *For further reading, we refer to Viviani's Theorem From Wikipedia. An alternative proof of the Converse Viviani's Theorem can be found here.*