Monge's Theorem

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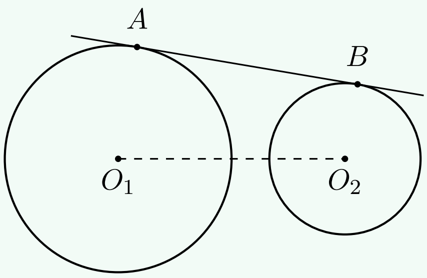
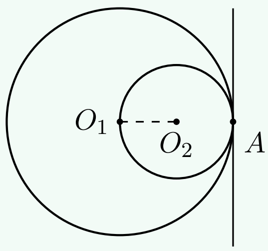
The Monge's Theorem is developed by the French mathematician *Gaspard Monge*, who is known as the inventor of descriptive geometry and the father of differential geometry. This theorem shows the relationship between the pairs of common external tangents that three circles generate. Before looking at the theorem itself, let's first introduce the concept of a common external tangent.

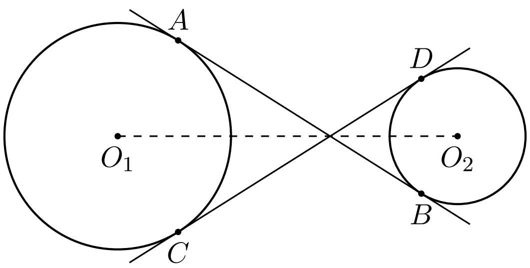
Definition 1. (Common External Tangents)

A common external tangent is a line that is tangent to two circles and it does not intersect the line connecting the two centers. See the two pictures below.





This special type of tangent is to be distinguished from a *common internal tangent*, where the tangent line crosses with the line joining the two centers.



Notice that two circles can have at most two common external tangents. If the circles are not overlapping and have different radii, then their common external tangents intersect at one point. If they have the same radii, then the two common external tangents are parallel to each other. In the Monge's Theorem, we look at three circles with distinct radii that generate three pairs of common external tangents.

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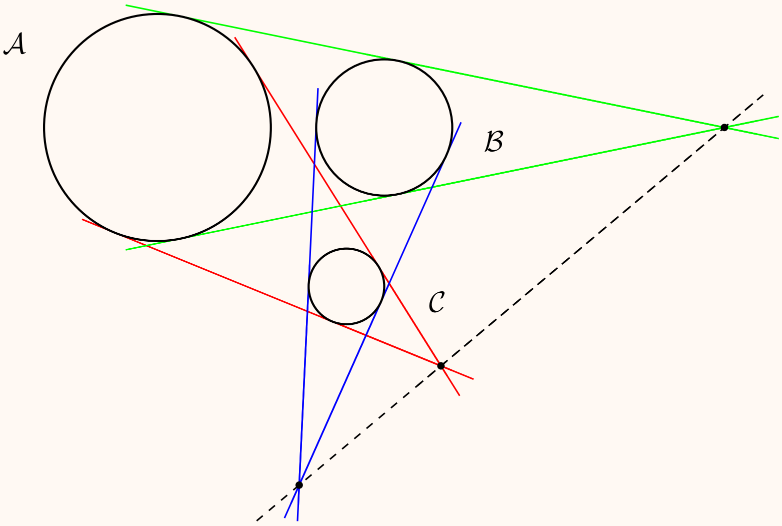
Theorem 1. (Monge's Theorem)

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be non-overlapping circles with different radii. For each pair of circles, draw their common external tangents. Then, the points of intersection of those tangent lines are collinear.

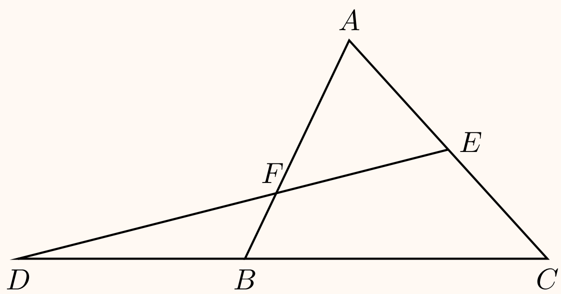


One of the simplest ways to prove the Monge's Theorem is to use the Menelaus' Theorem, which is a topic discussed previously on this website. You can read more on the theorem and its proof from Wikipedia, or Topic 02.

Theorem 2. (Menelaus' Theorem)

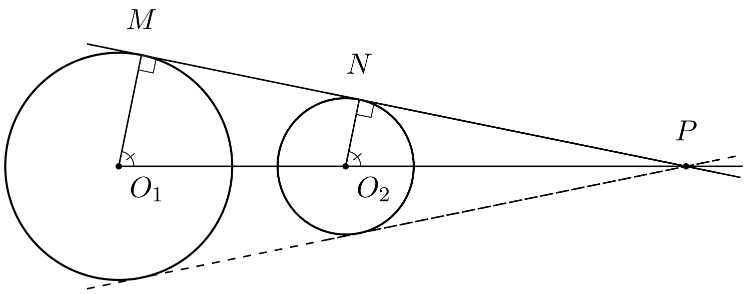
In the following $\triangle ABC$, D, E, F are points on BC, CA, and AB, respectively. Assume that D, E, F are collinear. Then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$



Conversely, if the above equation is valid, then the points D, E, F are collinear.

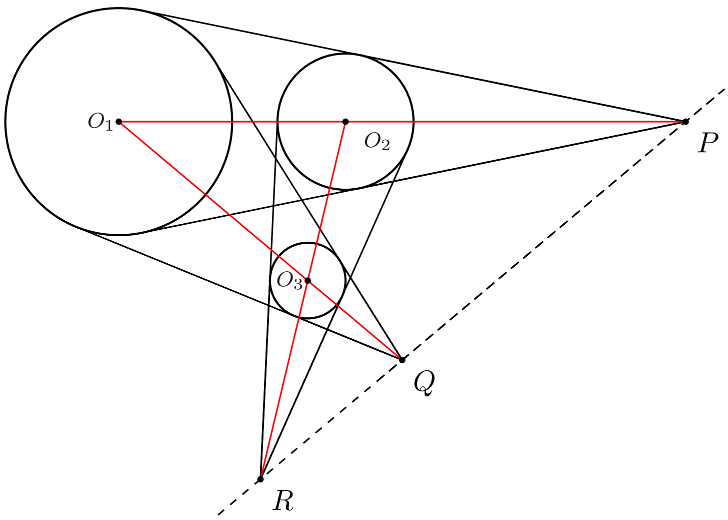
Proof of the Monge's Theorem. Consider two circles centered at O_1, O_2 with radii $r_1 \neq r_2$, respectively. Their common external tangents intersect at P, generating two triangles $\triangle PMO_1$ and $\triangle PNO_2$.



Notice that $\triangle PMO_1 \sim \triangle PNO_2$, so we have:

$$\frac{O_1P}{O_2P} = \frac{O_1M}{O_2N} = \frac{r_1}{r_2}.$$

Now, taking into account three circles centered at O_1, O_2, O_3 with radii r_1, r_2, r_3, respectively. Their common external tangents intersect at three points P, Q, R:



By Menelaus' Theorem, P, R, Q are collinear if and only if

$$\frac{O_1P}{PO_2} \cdot \frac{O_2R}{RO_3} \cdot \frac{O_3Q}{QO_1} = 1.$$

Since we know from above that

$$\frac{O_1P}{PO_2} = \frac{r_1}{r_2}, \qquad \frac{O_2R}{RO_3} = \frac{r_2}{r_3}, \qquad \frac{O_3Q}{QO_1} = \frac{r_3}{r_1},$$

we have

$$\frac{O_1P}{PO_2} \cdot \frac{O_2R}{RO_3} \cdot \frac{O_3Q}{QO_1} = \frac{r_1}{r_2} \cdot \frac{r_2}{r_3} \cdot \frac{r_3}{r_1} = 1$$

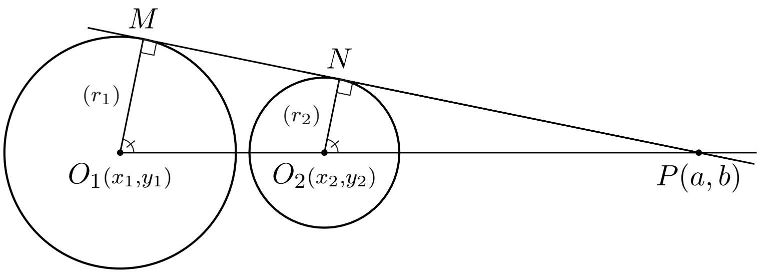
which proves the Monge's Theorem.

We are also able to prove the Monge's Theorem using analytic geometry.

Proof of the Monge's Theorem (using analytical geometry). Using algebra, we can prove that three points are collinear if the lines connecting each pair of points have the same slope. Imagine that the three circles lie on a plane. Let P, Q, R be the points of intersections of common external tangent lines. So P, Q, R can be represented by pairs of coordinates:

$$P = (a, b), \qquad Q = (c, d), \qquad R = (e, f).$$

To find the coordinates of each point, we look at each pair of circles separately:



Similar to the proof above, we use the fact that $\triangle PMO_1 \sim \triangle PNO_2$, which means:

$$\frac{O_1P}{O_2P} = \frac{r_1}{r_2}.$$

Now let's break down the equation to its coordinates:

$$\begin{aligned} \frac{a - x_1}{a - x_2} &= \frac{r_1}{r_2}, & \frac{b - y_1}{b - y_2} &= \frac{r_1}{r_2}, \\ r_2(a - x_1) &= r_1(a - x_2), & r_2(b - y_1) &= r_1(b - y_2), \\ r_2a - r_2x_1 &= r_1a - r_1x_2, & r_2b - r_2y_1 &= r_1b - r_1y_2, \\ r_1x_2 - r_2x_1 &= a(r_1 - r_2). & r_1y_2 - r_2y_1 &= b(r_1 - r_2). \end{aligned}$$

Thus we have

$$\begin{aligned} P &= (a, b) \\ &= \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right) \end{aligned}$$

By symmetry, we can conclude the coordinates Q, R to be as followed:

$$\begin{aligned} Q &= \left(\frac{r_3x_1 - r_1x_3}{r_3 - r_1}, \frac{r_3y_1 - r_1y_3}{r_3 - r_1}\right), \\ R &= \left(\frac{r_2x_3 - r_3x_2}{r_2 - r_3}, \frac{r_2y_3 - r_3y_2}{r_2 - r_3}\right). \end{aligned}$$

Finally, we compute the slope between $\overline{PQ}$ and the slope between $\overline{PR}$. Since they share the common point P, if they have the same slope, then P, Q, R lie on the same line. After applying the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

we obtain

$$\text{Slope of } \overline{PQ} = \frac{r_1(y_3 - y_2) + r_2(y_1 - y_3) + r_3(y_2 - y_1)}{r_1(x_3 - x_2) + r_2(x_1 - x_3) + r_3(x_2 - x_1)},$$

$$\text{Slope of } \overline{PR} = \frac{r_1(y_3 - y_2) + r_2(y_1 - y_3) + r_3(y_2 - y_1)}{r_1(x_3 - x_2) + r_2(x_1 - x_3) + r_3(x_2 - x_1)}.$$

This proves that P, Q, R are collinear, which implies the Monge's Theorem. ■

What happens if two of the circles have the same radius? That pair of circles would have two common external tangents parallel to each other. As a result, there would only be two intersection points created by the common external tangents. According to the Monge's Theorem, those two points would create a line parallel to the tangents that intersect at infinity.