Fuhrmann's Theorem

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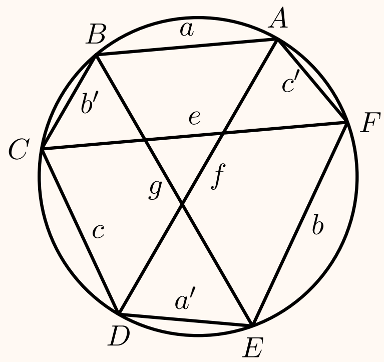
(last updated: November 21, 2023)

Fuhrmann's Theorem describes a relation between the six sides and the three diagonals of a cyclic hexagon. It was discovered by *Wilhelm Fuhrmann*, a German mathematician.

Theorem 1. (Fuhrmann's Theorem)

Let $ABCDEF$ be a convex concyclic hexagon. Let a, b', c, a', b, c' be the side lengths of AB, BC, CD, DE, EF, FA, respectively. Let e, f, g be the lengths of the main diagonals CF, AD, BE, respectively.



Then

$$efg = aa'e + bb'f + cc'g + abc + a'b'c'.$$

Example 1 Let $ABCDEF$ be a regular Hexagon with side length a. It is well-known that the lengths of the main diagonals are $2a$. That is

$$a = b = c = a' = b' = c', \quad e = f = g = 2a$$

One can easily verify Fuhrmann's Theorem in this special case.

Example 2 Assume that the Hexagon $ABCDEF$ is degenerated to a quadrilateral. For example, assume that $a' = b' = 0$. Then $g = c$. In this case, the Fuhrmann's Theorem is reduced to

$$efc = cc'g + abc$$

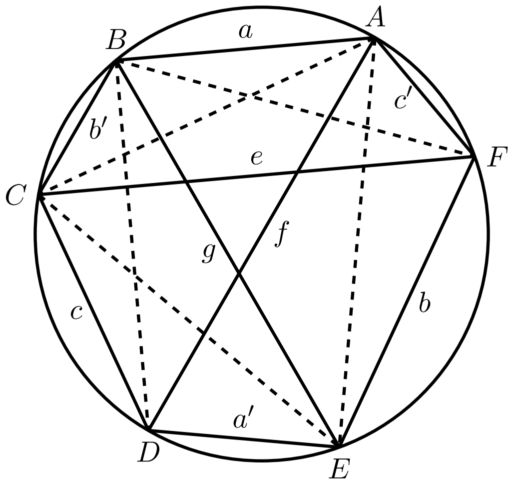
which is equivalent to

$$ef = c'g + ab.$$

Thus, in the degenerated case, the Fuhrmann's Theorem is reduced to the Ptolemy's Theorem.

[1]The author thanks Dr. Zhiqin Lu for his help.

Proof of Fuhrmann's Theorem. We connect BD, AE and AC, CE in the following picture.



Then we apply the Ptolemy's Theorem (Topic 10) on the convex concyclic quadrilaterals $ABDE$, $BCDE$, $ACEF$, $ABCD$ respectively to get

$$\begin{cases} fg = BD \cdot AE + AA' & (\alpha) \\ BD \cdot CE = cg + a'b' & (\beta) \\ AE \cdot e = CE \cdot c' + AC \cdot b & (\gamma) \\ AC \cdot BD = b'f + ac & (\delta) \end{cases}.$$

Using equations (α) and (γ), we get

$$efg = BD \cdot CE \cdot c' + BD \cdot AC \cdot b + aa'e.$$

Substituting Equations (β) and (δ) into the above, we get

$$efg = aa'e + bb'f + cc'g + abc + a'b'c'. \tag{1}$$

This completes the proof of the theorem. ■

Like in the case of Ptolemy's Theorem, Fuhrmann's Theorem is also closely related to the Kelvin Transform.

Definition 1. (Kelvin Transform)

Let n be a postive integer. The Kelvin transform[a] is a geometric transform involving a fixed point O and a constant $r > 0$. Given a point P in the n-dimensional Euclidean space $\mathbb{R}^n$, the Kelvin transform maps it to point $P' \in \mathbb{R}^n$ such that

$$OP \cdot OP' = r^2.$$

[a] See Topic 10 for details the transform on the Euclidean plane.

We can use the vector notations to represent a Kelvin transform. For the sake of simplicity, we take O to be the origin of $\mathbb{R}^n$, and take $r = 1$. Let $x \neq 0$ be a vector in $\mathbb{R}^n$. Then the Kelvin transform $K(x)$ can be represented by

$$K(x) = \frac{x}{\|x\|^2}.$$

Proposition 1. (Kelvin Equality)

Let $x, y \in \mathbb{R}^n$ be non-zero vectors in $\mathbb{R}^n$. Then we have

$$\|K(x) - K(y)\| = \left\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\| = \frac{\|x - y\|}{\|x\| \, \|y\|}.$$

Proof. Since

$$\left\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\| = \left\| \frac{x \|y\|^2 - y \|x\|^2}{\|x\|^2 \|y\|^2} \right\|,$$

we have

$$\begin{aligned}\left(\left\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\| \right)^2 &= \frac{\left\| x \|y\|^2 - y \|x\|^2 \right\|^2}{\|x\|^4 \|y\|^4} \\ &= \frac{\|x\|^2 \|y\|^4 + \|x\|^4 \|y\|^2 - 2(x \cdot y) \|x\|^2 \|y\|^2}{\|x\|^4 \|y\|^4}.\end{aligned}$$

On the other hand, we have

$$\left(\frac{\|x - y\|}{\|x\| \, \|y\|} \right)^2 = \frac{\|x\|^2 \|y\|^4 - 2(x \cdot y) \|x\|^2 \|y\|^2 + \|x\|^4 \|y\|^2}{\|x\|^4 \|y\|^4}.$$

The proposition thus follows. ∎

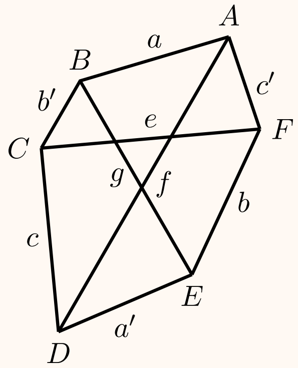
Using the Kelvin Equality, we can prove the following

Theorem 2. (Fuhrmann's Inequality)

Let n be a positive integer. Let $ABCDEF$ be any hexagon on the $\mathbb{R}^n$. Let a, b', c, a', b, c' be the side lengths of AB, BC, CD, DE, EF, FA, respectively. Let e, f, g be the lengths of the main diagonals AD, BE, CF, respectively. Then

$$efg \leq aa'e + bb'f + cc'g + abc + a'b'c'.$$

Equality will be achieved when the hexagon is planar, cyclic and convex.



When the above hexagon is degenerated to a quadrilateral, then we have the following inequality. See Theorem 2 of Topic 10 for details.

Corollary 1. (Ptolemy's Inequality)

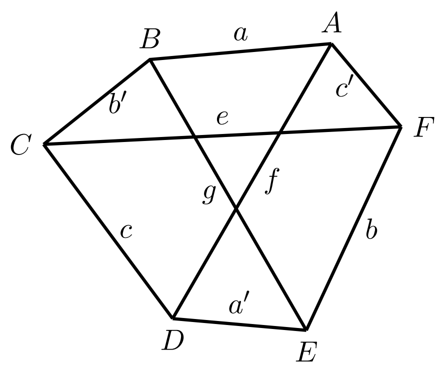
Let A, B, C, D be four points in $\mathbb{R}^n$. Then

$$AC \cdot BD \leq AB \cdot CD + BC \cdot DA$$

Equality holds under the special circumstances when the quadrilateral is planar, cyclic and convex.

It is interesting that we can use the Ptolemy's Inequality to prove the Fuhrmann's Inequality. The proof of similar to the proof of the Fuhtmann's Theorem.

Proof of Fuhrmann's Inequality. For any hexagon $ABCDEF$ in $\mathbb{R}^n$, we apply Ptolemy's Inequality on the quadrilaterals $ABDE$, $BCDE$, $ACEF$, $ABCD$ respectively. Then we get



$$\begin{cases} fg \leq BD \cdot AE + AA' & (\alpha) \\ BD \cdot CE \leq cg + a'b' & (\beta) \\ AE \cdot e \leq CE \cdot c' + AC \cdot b & (\gamma) \\ AC \cdot BD \leq b'f + ac & (\delta) \end{cases}.$$

By using these four equations just as the proof of the Fuhrmann's Theorem, we obtain our desired result:

$$efg \leq aa'e + bb'f + cc'g + abc + a'b'c' \tag{2}$$

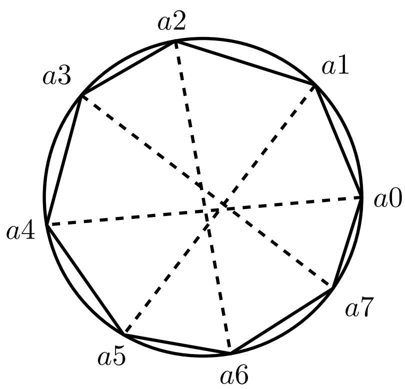
If equality holds in Fuhrmann's Inequality, then equality must hold in Ptolemy's Inequalities for the four quadrilaterals simultaneously. This implies that the quadrilaterals $ABDE$, $BCDE$, $ACEF$, and $ABCD$ are planar, cyclic and convex. Therefore, the hexagon $ABCDEF$ is also planar, cyclic and convex. This completes the proof. ■

For the rest of this article, we shall extend both the Ptolemy and Fuhrmann Theorems to $2n$-gons.

Definition 2. (Sub-main Diagonal)

Let $a_i, i \in \{0, \cdots, 2n-1\}$ be a $2n$-polygon, where $\{a_0, \cdots, a_{2n-1}\}$ are the vertices. For the sake of simplicity, we define $a_k = a_\ell$ for $0 \leq \ell \leq 2n-1$ if $k = \ell + 2nm$ for some $m \in \mathbb{Z}$. We define $a_i a_{i+n}$ to be the main diagonals and we define the sub-main diagonals to be $a_i a_{i+1+n}$ or $a_i a_{i-1+n}$ for all i.

Example 3 In the picture, we show a cyclic octagon.



We extend Fuhrmann's Theorem to this cyclic octagon. Suppose $a_i, i \in 0, ..., 7$ are the vertices of the octagon, and we assume i is a group with addition. Then, we have the following equation relating the product of the main diagonals and the products of the main diagonals, side length,s and sub-main diagonals.:

$$\prod_{i=0}^{3} a_i a_{i+4} = \sum_{i=0}^{7} a_i a_{i+1} \cdot a_{i+2} a_{i+7} \cdot a_{i+3} a_{i+4} \cdot a_{i+5} a_{i+6} + \sum_{i=0}^{3} a_i a_{i+1} \cdot a_{i+4} a_{i+5} \cdot a_{i+2} a_{i+6} \cdot a_{i+3} a_{i+7} \tag{3}$$

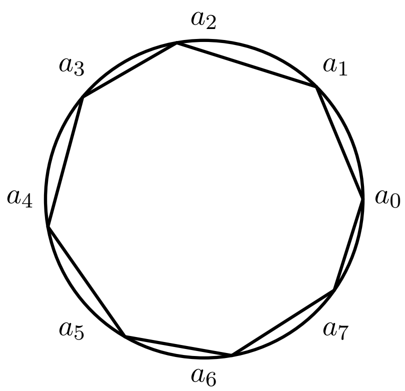
This example demonstrates an attractive property of the products of the main diagonals. To prove this, we need to use Ptolemy's Theorem and Fuhrmann's Theorem.

Proof. For a cyclic hexagon $a_3a_2a_1a_7a_6a_5$ we can apply Fuhrmann's Theorem so we will get

$$\begin{aligned} a_3a_7 \cdot a_2a_6 \cdot a_1a_5 = a_1a_7 \cdot a_2a_6 \cdot a_3a_5 + a_1a_2 \cdot a_3a_7 \cdot a_5a_6 \\ + a_2a_3 \cdot a_1a_5 \cdot a_6a_7 + a_1a_7 \cdot a_2a_3 \cdot a_5a_6 \\ + a_1a_2 \cdot a_3a_5 \cdot a_6a_7 \end{aligned}$$

multiplying a_0a_4 on both sides and we can apply Ptolemy theorem to the following quadrilaterals one by one: $a_0a_2a_4a_6, a_0a_3a_4a_5, a_0a_1a_4a_7$ and $a_2a_3a_4a_5, a_1a_0a_7a_6$, also $a_2a_1a_0a_7, a_3a_4a_5a_6$ lastly $a_0a_3a_4a_7, a_0a_1a_4a_5$
we will get our desired result. ■

We can justify the statement with a regular octagon.



If the side length is 1, the length of the main diagonal:

$$L = \sqrt{4 + 2\sqrt{2}}$$

The length of the sub-main diagonal:

$$S = 1 + \sqrt{2}$$

Then we will get the following:

$$L^4 = 8 * S^2 + 4 * L^2$$

which is the octagon theorem.
For the same reasons, we can also extend Fuhrmann's Inequality to the octagon. To achieve this, we just need to substitute all the $=$ to $\leq$.

In order to generalize Fuhrmann's Theorem in 2-n concyclic polygon, we define the xth Element of the polygon as follows:

Definition 3. (x-th Element)

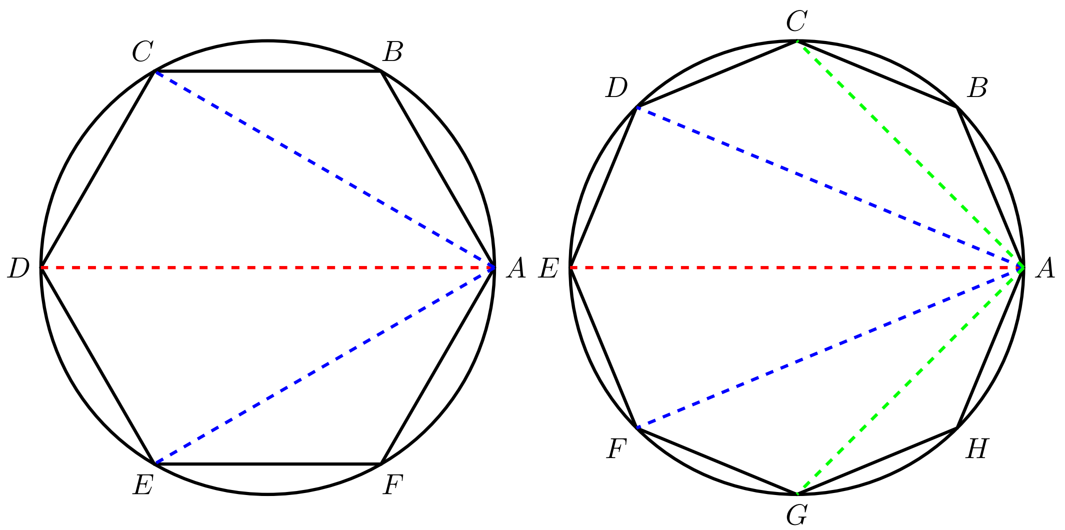
In a $2n$-sided polygon, given a vertex V, we can construct line segments to non-adjacent vertices in a specific order. The main diagonal, which connects V to the vertex exactly n steps away in either direction, is defined as the 1st element.

For $x > 1$, the x-th element is defined as the line segments connecting V to the vertices $n-(x-1)$ and $n+(x-1)$ steps away from V, if these vertices exist. This definition essentially creates an order of 'sub-main' diagonals, with each xth element being the set of diagonals one step further from the main diagonal than the $(x-1)$th element.

Note: This definition uses a zero-based index for the vertices, meaning the first vertex in the sequence is V_0.

Example 4 Consider a hexagon labeled clockwise as A, B, C, D, E, and F and an octagon labeled as A, B, C, D, E, F, G, H. If we fix one point, for example, A, we have:

- For the hexagon:
  - The 1st element is AD.
  - The 2nd elements are AC and AE.
- For the octagon:
  - The 1st element is AE.
  - The 2nd elements are AD and AF.
  - The 3rd elements are AC and AG.



The diagram shows the 1st element (AD) in red and the 2nd element (AC, AE) in blue for the hexagon with the fixed point at A.

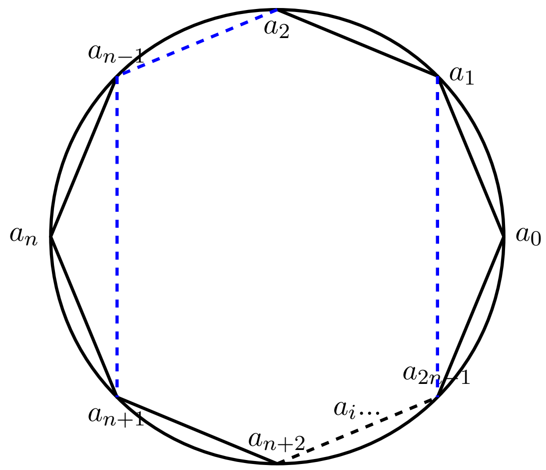
Upon observing quadrilaterals, hexagons, and octagons, we noted interesting relationships between their main diagonals and the properties of their elements. For quadrilaterals, the product of main diagonals only concerns their sides. For hexagons, this product is tied to the sides and their 1st elements. Moving onto octagons, we observed a more complex interplay: their sides, 1st and 2nd elements all influence the product of main diagonals.

This led us to question: is this a universal principle? For a 2n polygon, does the product of its main diagonals consistently relate to its sides and 1st element up to the (n-2)th element? That's the inquiry we aim to explore further in this study.

Theorem 3. ($2n$-gon Fuhrmann's Theorem

In any polygon of 2n sides, the product of its main diagonals exhibits a meaningful relation with the set of its elements from the main diagonal (1st element) through to the (n-2)th element and their sides.

Proof. We can prove it by induction. Suppose we have a main diagonal equation for 2n-2 cyclic polygons, and we can construct 2n-2 cyclic polygons in the 2n polygons like the following.



There we have $a_{2n-1}a_1a_2 \ldots a_{n-1}a_{n+1} \ldots$ as a 2n-2 polygon, which has a close form of a multiple main diagonal equation. So on the left side of the equation, is the multiple of main diagonals. In this case, we multiply a_0a_n on both sides of the equation.

And we can apply Ptolemy's Theorem of quadrilateral $a_{2n-1}a_0a_1a_n$. That is because there is one term on the right side consisting of side a_1a_3 and a_4a_6 and we need to simplify them because they are the least element.

And then apply Ptolemy's theorem to quadrilateral $a_1a_{n-1}a_na_{n+1}$ and $a_{2n+1}a_{n-1}a_na_{n+1}$. Then we will eliminate the a_1a_3 and a_4a_6 on the right side of the equation.

■

Remark In our recent proof for any 2n concyclic polygon, it's established that the product of the main diagonals can be expressed through a combination of the main diagonals up to the $(n-2)$th element, paired with their corresponding sides. It's intriguing to note that this result can typically be segmented into two distinct components: one that correlates the main diagonals with their respective sides, and the other involving the remaining elements. Notably, the former appears to adhere to a closed-form representation, suggesting an inherent structural pattern to the relations within 2n concyclic polygons.

Theorem 4. (Closed Form of First Components of Main Diagonal Relation)

For any concyclic $2n$ polygon, we have the following,

$$\prod_{i=0}^{n-1} a_i a_{i+n} = \sum_{i=0}^{n} a_i a_{i+1} * a_{i+n} a_{i+n+1} \prod_{j=i+2}^{n-1+i} a_j a_{j+n} + F(x) \tag{4}$$

for some $F(x)$. That is, we always have a close form of the first components of the main diagonal relation.

Example 5 Let $ABCDEF$ be a convex concyclic hexagon. Let a, b', c, a', b, c' be the side lengths of AB, BC, CD, DE, EF, FA, respectively. Let e, f, g be the lengths of the main diagonals CF, AD, BE, respectively, then

$$efg = aa'e + bb'f + cc'g + F(x)$$

for some $F(x)$.

Example 6 Let $a_i, i \in 0, ..., 7$ be the vertices of a concyclic octagon, and we assume i is a group with addition. Then,

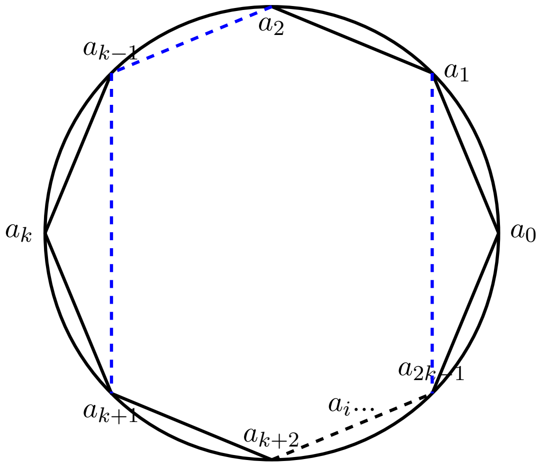
$$\prod_{i=0}^{3} a_i a_{i+4} = \sum_{i=0}^{3} a_i a_{i+1} \cdot a_{i+4} a_{i+5} \cdot a_{i+2} a_{i+6} \cdot a_{i+3} a_{i+7} + F(x)$$

for some $F(x)$.

Proof of Theorem 4. We can proceed with our proof by induction. It is true when $n = 3$(Hexagon). Suppose it is true when $n = k - 1$. We want to show it is true when $n = k$. For a $2k - 2$ polygon, we have the following relation for random $a_i s$:

$$\prod_{i=1}^{k-1} a_i a_{i+k} = \sum_{i=1}^{k-1} a_i a_{i+1} * a_{i+k-1} a_{i+k} \prod_{j=i+2}^{k-2+i} a_j a_{j+k-1} + G(x)$$

We construct a $2k$ concyclic polygon by the following. Accordingly, we let the true vertex substitute the equation. For convenience, we assume all random a_i s represent true vertex.



Multiply a_0a_k to both side of the equation, we can have

$$\prod_{i=0}^{k-1} a_ia_{i+k} = (\sum_{i=1}^{k-1} a_ia_{i+1} * a_{i+k-1}a_{i+k} \prod_{j=i+2}^{k-2+i} a_ja_{j+k-1}) * a_0a_k + G(x) * a_0a_k$$

We want to show the equation is equal to the equation 64. We can subtract both sides to prove the equation on the right-hand side equals 0. That is, we want to show

$$a_0a_1 \cdot a_ka_{k+1} \cdot \prod_{i=2}^{k-1} a_ia_{i+k} + a_{2k-1}a_0 \cdot a_{k-1}a_k \cdot \prod_{i=1}^{k-2} a_ia_{i+k} + F(x)$$
$$= a_1a_{2k-1} \cdot a_{k-1}a_{k+1} \cdot a_0a_k \cdot \prod_{i=2}^{k-2} a_ia_{i+k} + G(x) \cdot a_0a_k.$$

We apply the Ptolemy Theorem to Quadrilateral $a_0a_{k-1}a_ka_{k+1}$ on the right-hand side to get two new terms. Then apply the Ptolemy Theorem to Quadrilateral $a_0a_1a_{k+1}a_{2k-1}$ and $a_0a_1a_{k-1}a_{2k-1}$ Then we will make the RHS like following

$$a_0a_1 \cdot a_ka_{k+1} \cdot \prod_{i=2}^{k-1} a_ia_{i+k} + a_{2k-1}a_0 \cdot a_{k-1}a_k \cdot \prod_{i=1}^{k-2} a_ia_{i+k} + H(x).$$

Then we can set $H(x) = F(x)$ to get our desired result. ∎