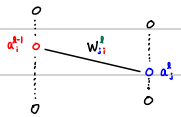
Layer $l-1$ l



$a^l = \sigma(w^l a^{l-1} + b^l)$, a^l, b^l : $n_l \times 1$ 행렬, w^l : $n_{l-1} \times n_l$ 행렬

a^l의 j번째 값 : $a_j^l = \sigma\left(\sum_{i=1}^{n_{l-1}} w_{ji}^l a_i^{l-1} + b_j^l\right)$

$\sigma(\cdot)$: 활성화 함수 (activation function)

C : 기준함수 (criterion function) +) 손실함수 Loss function 으로 일반적으로 많이

[Input 과 Feedforward]

- Input : a^1
- $z^l = w^l \cdot a^{l-1} + b^l$ and $a^l = \sigma(z^l)$ where $l = 2, 3, \cdots, L$

[Output error 와 Back propagate the error]

- Output error : $\delta^l = \frac{\partial C}{\partial z^l}$ 이라 하자.

l이 $L-1, L-2, \cdots, 2$ 에서 $\delta^l = \left((w^{l+1})^T \delta^{l+1}\right) \odot \sigma'(z^l)$ 을 만족한다.

pf) $\delta^{l-1} = \frac{\partial C}{\partial z^{l-1}} = \left(\frac{\partial C}{\partial z_1^{l-1}}, \frac{\partial C}{\partial z_2^{l-1}}, \cdots, \frac{\partial C}{\partial z_{n_{l-1}}^{l-1}}\right)^T$

$= \left(\frac{\partial C}{\partial a_1^{l-1}}\frac{\partial a_1^{l-1}}{\partial z_1^{l-1}}, \frac{\partial C}{\partial a_2^{l-1}}\frac{\partial a_2^{l-1}}{\partial z_2^{l-1}}, \cdots\right)^T$

$= \left(\frac{\partial C}{\partial a_1^{l-1}}, \frac{\partial C}{\partial a_2^{l-1}}, \cdots\right)^T \odot \sigma'(z^{l-1})$

$= \left((w^l)^T \delta^l\right) \odot \sigma'(z^{l-1})$ ···· by ①

① i 가 $1, 2, \cdots, n_{l-1}$ 일 때

$\frac{\partial C}{\partial a_i^{l-1}} = \sum_{j=1}^{n_l} \frac{\partial C}{\partial a_j^l} \cdot \frac{\partial a_j^l}{\partial a_i^{l-1}}$ ··· by ②

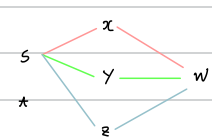
$= \sum_{j=1}^{n_l} \frac{\partial C}{\partial a_j^l} \cdot \frac{\partial a_j^l}{\partial z_j^l} \cdot \frac{\partial z_j^l}{\partial a_i^{l-1}} = \sum_{j=1}^{n_l} \delta_j^l \cdot w_{ji}^l = \left[(w^l)^T \delta^l\right]_i$

[Weight update]

- $\frac{\partial C}{\partial w_{ji}^l} = \frac{\partial C}{\partial z_j^l} \cdot \frac{\partial z_j^l}{\partial w_{ji}^l} = \delta_j^l a_i^{l-1} \Rightarrow \frac{\partial C}{\partial w^l} = \delta^l (a^{l-1})^T$
- $\frac{\partial C}{\partial b_j^l} = \frac{\partial C}{\partial z_j^l} \cdot \frac{\partial z_j^l}{\partial b_j^l} = \delta_j^l \Rightarrow \frac{\partial C}{\partial b^l} = \delta^l$

② Chain rule : 연쇄법칙

case) $w = f(x, y, z)$, $x = p(s,t)$, $y = q(s,t)$, $z = r(s,t)$



$\frac{dw}{ds} = \frac{dw}{dx} \cdot \frac{dx}{ds} + \frac{dw}{dy} \cdot \frac{dy}{ds} + \frac{dw}{dz} \cdot \frac{dz}{ds}$

[문제 1]

input : a^1과 target : t 가 $a^1 = \begin{pmatrix}1\\1\end{pmatrix}$, $t = \begin{pmatrix}2\\3\end{pmatrix}$ 이고

$W^2 = \begin{pmatrix}1 & -2\\2 & 4\\-3 & 1\end{pmatrix}$, $b^2 = \begin{pmatrix}1\\2\\3\end{pmatrix}$

$W^3 = \begin{pmatrix}1 & 2 & -3\\2 & -1 & 3\end{pmatrix}$, $b^3 = \begin{pmatrix}2\\1\end{pmatrix}$, activation : $\sigma(x) = x$

$\begin{pmatrix}a_1^3\\a_2^3\end{pmatrix} = \sigma\left(W^3 \cdot \sigma\left(W^2 \cdot a^1 + b^2\right) + b^3\right)$ 일때

Loss $= \frac{1}{2}\left[(a_1^3 - t_1)^2 + (a_2^3 - t_2)^2\right]$ 에 대하여 $lr = 0.01$로

weight update를 한 결과를 구해보자.

Input	Hidden	Output	target
	0		
0		0 a_1^3	0 t_1
	0		
0		0 a_2^3	0 t_2
	0		

[forward]

Layer : l forward $\xrightarrow{Wa+b}$ z^l $\xrightarrow{\sigma(z)}$ a^l

1 $\quad \begin{pmatrix}1\\1\end{pmatrix}$

2 $\quad W^2a^1 + b^2 \longrightarrow \begin{pmatrix}0\\8\\1\end{pmatrix} \longrightarrow \begin{pmatrix}0\\8\\1\end{pmatrix}$

3 $\quad W^3a^2 + b^3 \longrightarrow \begin{pmatrix}15\\-4\end{pmatrix} \longrightarrow \begin{pmatrix}15\\-4\end{pmatrix}$

[Backward]

Layer : l backward $\longrightarrow$ δ^l

3 $\quad \frac{\partial L}{\partial z^3} \longrightarrow \begin{pmatrix}13\\-7\end{pmatrix}$

2 $\quad \left((W^3)^T \cdot \delta^3\right) \odot \sigma'(z^2) \longrightarrow \begin{pmatrix}-1\\33\\-60\end{pmatrix}$

[Weight update]

$\frac{\partial L}{\partial W^3} = \begin{pmatrix}13\\-7\end{pmatrix}(0\ 8\ 1) = \begin{pmatrix}0 & 104 & 13\\0 & -56 & -7\end{pmatrix}$, $\frac{\partial L}{\partial b^3} = \begin{pmatrix}13\\-7\end{pmatrix}$

$W^3 - 0.01\frac{\partial L}{\partial W^3} = \begin{pmatrix}1 & 0.96 & -3.13\\2 & -0.44 & 3.07\end{pmatrix}$

$b^3 - 0.01\frac{\partial L}{\partial b^3} = \begin{pmatrix}1.87\\1.07\end{pmatrix}$

$\frac{\partial L}{\partial W^2} = \begin{pmatrix}-1\\33\\-60\end{pmatrix}(1\ 1) = \begin{pmatrix}-1 & -1\\33 & 33\\-60 & -60\end{pmatrix}$, $\frac{\partial L}{\partial b^2} = \begin{pmatrix}-1\\33\\60\end{pmatrix}$

$W^2 - 0.01\frac{\partial L}{\partial W^2} = \begin{pmatrix}1.01 & -1.99\\1.67 & 3.67\\-2.4 & 1.6\end{pmatrix}$

$b^2 - 0.01\frac{\partial L}{\partial b^2} = \begin{pmatrix}1.01\\1.67\\3.6\end{pmatrix}$

[문제 2]

input : a^1과 target : t 가 $a^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $t = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ 이고

$W^2 = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$, $b^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$W^3 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$, $b^3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, activation : $\sigma(x) = x^2$

$\begin{pmatrix} a_1^3 \\ a_2^3 \end{pmatrix} = \sigma(W^3 \cdot \sigma(W^2 \cdot a^1 + b^2) + b^3)$ 일때

$Loss = \frac{1}{2}\left[(a_1^3 - t_1)^2 + (a_2^3 - t_2)^2\right]$ 에 대하여 $lr = 0.01$일

Weight update를 한 결과를 구해보자.

Input	Hidden	Output	target
0	0	0 a_1^3	0 t_1
0	0	0 a_2^3	0 t_2

[forward]

Layer : l forward $\xrightarrow{Wa+b}$ z^l $\xrightarrow{\sigma(z)}$ a^l

1		$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
2	$W^2a^1 + b^2$	$\begin{pmatrix} -1 \\ 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 4 \end{pmatrix}$
3	$W^3a^2 + b^3$	$\begin{pmatrix} -3 \\ -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

[Backward]

Layer : l backward $\longrightarrow$ δ^l

3	$\frac{\partial L}{\partial z^3}$	$\longrightarrow \begin{pmatrix} -42 \\ -4 \end{pmatrix}$
2	$((W^3)^T \cdot \delta^3) \odot \sigma'(z^2)$	$\longrightarrow \begin{pmatrix} 100 \\ 184 \end{pmatrix}$

$\frac{\partial L}{\partial z^3} = \frac{\partial L}{\partial a^3} \cdot \frac{\partial a^3}{\partial z^3} = \begin{pmatrix} a_1^3 - t_1 \\ a_2^3 - t_2 \end{pmatrix} \odot \begin{pmatrix} 2 \cdot z_1^3 \\ 2 \cdot z_2^3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} \odot \begin{pmatrix} -6 \\ -4 \end{pmatrix} = \begin{pmatrix} -42 \\ -4 \end{pmatrix}$

$((W^3)^T \cdot \delta^3) \odot \sigma'(z^2) = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -42 \\ -4 \end{pmatrix} \odot \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 100 \\ 184 \end{pmatrix}$

[Weight update]

$\frac{\partial L}{\partial W^3} = \begin{pmatrix} -42 \\ -4 \end{pmatrix}(1 \quad 4) = \begin{pmatrix} -42 & -168 \\ -4 & -16 \end{pmatrix}$, $\frac{\partial L}{\partial b^3} = \begin{pmatrix} -42 \\ -4 \end{pmatrix}$

$W^3 - 0.01\frac{\partial L}{\partial W^3} = \begin{pmatrix} 1.42 & 0.68 \\ 2.04 & -0.84 \end{pmatrix}$

$b^3 - 0.01\frac{\partial L}{\partial b^3} = \begin{pmatrix} 0.42 \\ 0.04 \end{pmatrix}$

$\frac{\partial L}{\partial W^2} = \begin{pmatrix} 100 \\ 184 \end{pmatrix}(1 \quad 1) = \begin{pmatrix} 100 & 100 \\ 184 & 184 \end{pmatrix}$, $\frac{\partial L}{\partial b^2} = \begin{pmatrix} 100 \\ 184 \end{pmatrix}$

$W^2 - 0.01\frac{\partial L}{\partial W^2} = \begin{pmatrix} 0 & -3 \\ -0.84 & -0.84 \end{pmatrix}$

$b^2 - 0.01\frac{\partial L}{\partial b^2} = \begin{pmatrix} -1 \\ -1.84 \end{pmatrix}$