The Path Following Control of a Unicycle Based on the Chained Form of a Kinematic Model Derived with Respect to the Serret-Frenet Frame

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***Abstract*— In the paper the path following problem for a wheeled mobile robot of (2,0) type is considered. The Serret-Frenet frame is used to describe the robot relative to the desired path. The way how to transform obtained equations describing kinematic model derived with respect to the Serret-Frenet frame into the chained form is introduced. The kinematic control algorithm which is a combination of Astolfi algorithm – used for systems in the chained form – and Samson algorithm – dedicated to the unicycle – is proposed and compared to the Morin-Samson algorithm. Simulation results are shown to verify the effectiveness of the proposed method.**

I. Introduction

Many researchers have brought their attention to the control problems of wheeled mobile robots in recent years. The problems addressed in the literature can be clasified into three groups:

- point stabilization – the robot should be stabilized at a given target point,
- trajectory tracking – the vehicle has to track a time parametrized curve,
- path following – the robot has to follow a path which is parametrized by curvilinear distance from a fixed point.

The path following of the mobile platforms with nonholonomic constraints can be realized e.g. by Pomet theorem [1], what was presented in [2], or using algorithms dedicated to particular types of the platforms. The important advantage of Pomet algorithm is that it can be applied to many nonholonomic systems, however it has also two disadvantages – very slow convergence and the occurrence of the chattering phenomenon. The example of the algorithm dedicated to a certain type of mobile robots is Samson algorithm [3] which guarantees asymptotic path following for a unicycle – the wheeled mobile robot of (2,0) type.

The approach presented in [4] bases on controlling explicitly the rate of progression of a "virtual target" to be tracked along the path. Such an approach overcomes the restriction imposed on the initial position of the robot in the case when the position of the virtual target is simply defined by the orthogonal projection of the actual vehicle on that path.

The method proposed in [5] solves the problem of the bounded path curvature. It neither requires the computation of a projection of the robot position on the path, nor does it need to consider a moving virtual target to be tracked.

The another idea of developing control strategy in a path following task is to transform kinematic equations of the robot derived with respect to the Serret-Frenet frame into a canonical form called the chained form and then propose an algorithm for such a system. Morin and Samson applied that solution and introduced the algorithm which may be used e.g. for the path following of the unicycle, [6]. The same idea to derive the kinematic equations of a car-like robot in terms of path coordinates and then transform them into the chained form was presented in [7].

In this paper the new control strategy for the unicyle which is a combination of Astolfi algorithm [8] – designed for the stabilization of the systems in the chained form – and Samson algorithm is proposed.

II. Description of a robot relative to a given path

In this paper we restrict our considerations to the nonholonomic mobile platform of (2,0) type. We have taken an assumption that the wheeled mobile platform should move without slippage of its wheels. It is equivalent to an assumption that the momentary velocity at the contact point between each wheel and the motion plane is equal to zero.

The wheeled mobile robot of (2,0) class with a path that has to be followed was depicted in Fig. 1. The kinematic model of such a robot can be described by the equations

$$\left\{ \begin{array}{rcl} \dot{x} & = & v\cos\theta, \\ \dot{y} & = & v\sin\theta, \\ \dot{\theta} & = & \omega, \end{array} \right. \tag{1}$$

where (x, y) represents coordinates of the center of the mass of the robot relative to the inertial frame X_0Y_0 and θ is a robot's orientation. Symbols v and ω denote linear and angular velocities, respectively.

A. Representation in a Serret-Frenet frame

The path P is characterized by a curvature $\kappa(s)$, which is the inversion of the radius of the circle tangent to the path at a point characterized by the parameter s. Let us consider

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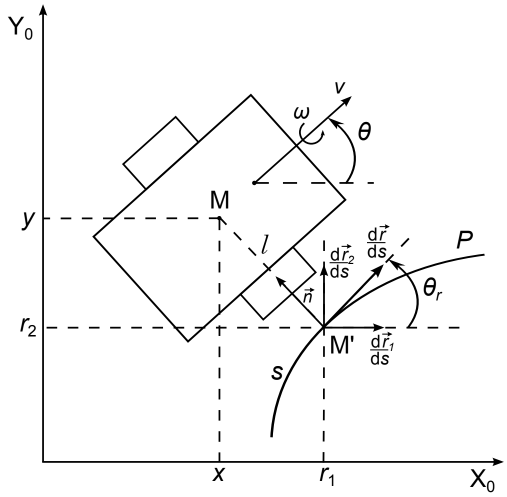

Fig. 1. The parameters of the unicycle and Serret-Frenet frame definition

a moving point M and the associated Serret–Frenet frame defined on the curve P by the normal and tangent unit vectors $\overrightarrow{n}$ and $\frac{d\overrightarrow{r}}{ds}$. The point M is the robot's guidance point located in the middle of the wheel axle of the vehicle and M' is the orthogonal projection of the point M on the path P.

The point M' exists and is uniquely defined if the following conditions are satisfied, see e.g. [9]:

- The curvature $\kappa(s)$ is not bigger than $1/r_{min} > 0$.
- The distance between the path P and the point M is smaller than r_{min}.

The coordinates of the point M relative to the Serret–Frenet frame are $(0, l)$ and relative to the basic frame X_0Y_0 are equal to (x, y), where l is the distance between M and M'. The curvilinear abscissa of M' is equal to s, where s is a distance along the path from some arbitrarily chosen point. The desired orientation of the platform satisfies the equation

$$\frac{\mathrm{d}\theta_r}{\mathrm{d}t} = \pm\kappa(s)\dot{s}. \tag{2}$$

If we want to express the position of the point M not in coordinates (x, y) relative to inertial frame, but relative to the given path P, we should use certain geometric relationships,

$$\dot{l} = (-\sin\theta_r \quad \cos\theta_r)\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}, \tag{3}$$

$$\dot{s} = \frac{(\cos\theta_r \quad \sin\theta_r)}{1 \mp \kappa(s)l}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}, \tag{4}$$

where $\dot{x}$ and $\dot{y}$ can be expressed e.q. by the nonholonomic constraints of a mobile platform. In addition, we determine the orientation error

$$\tilde{\theta} = \theta - \theta_r \tag{5}$$

and its derivative

$$\dot{\tilde{\theta}} = \dot{\theta} - \dot{\theta}_r = \dot{\theta} \mp \kappa(s)\dot{s}. \tag{6}$$

The restrictions imposed on the desired path cause that inequality $|l\ \kappa(s)| < 1$ must hold.

For the unicycle the kinematic model expressed with respect to the Serret-Frenet frame is given by the following system of equations

$$\left\{\begin{array}{lcl} \dot{l} & = & v\sin\tilde{\theta}, \\ \dot{s} & = & \frac{v\cos\tilde{\theta}}{1\mp\kappa(s)l}, \\ \dot{\tilde{\theta}} & = & \omega \mp \frac{\kappa(s)v\cos\tilde{\theta}}{1\mp\kappa(s)l}. \end{array}\right. \tag{7}$$

B. Transformation of the kinematic model into the chained form

The kinematic equations of some of the mobile robots can be transformed into the chained form via a change of state and control variables, e.g. the equations of the unicycle and of the kinematic car. Such an transformation can be generalized to the kinematics models expressed with respect to the Serret-Frenet frame.

Suppose that $z_1 = s$. The time derivative of z_1 is equal

$$\dot{z}_1 = \dot{s} = \frac{v\ \cos\tilde{\theta}}{1 \mp \kappa(s)l} = u_1.$$

Let now assume that $z_3 = l$. We calculate $\dot{z}_3$

$$\dot{z}_3 = \dot{l} = v\sin\tilde{\theta} = u_1\tan\tilde{\theta}(1 \mp \kappa(s)l).$$

Hence $z_2 = \tan\tilde{\theta}(1 \mp \kappa(s)l)$. Then we determine $\dot{z}_2$

$$\dot{z}_2 = (\mp\dot{c}(s)l \mp \kappa(s)u_1z_2)\cdot\tan\tilde{\theta} + \frac{1 \mp \kappa(s)l}{\cos^2\tilde{\theta}}(\omega + u_1\kappa(s)) = u_2.$$

The model described by the equations (7) can be transformed into the three-dimensional chained system

$$\begin{array}{lcl} \dot{z}_1 & = & u_1, \\ \dot{z}_2 & = & u_2, \\ \dot{z}_3 & = & z_2u_1 \end{array} \tag{8}$$

using the change of coordinates

$$\begin{array}{lcl} z_1 & = & s, \\ z_2 & = & (1 \mp \kappa(s)l)\tan\tilde{\theta}, \\ z_3 & = & l \end{array} \tag{9}$$

and control variables

$$\begin{array}{lcl} u_1 & = & \frac{v\cos\tilde{\theta}}{1\mp\kappa(s)l}, \\ u_2 & = & (\mp\dot{c}(s)l \mp \kappa(s)u_1z_2)\cdot\tan\tilde{\theta} + \frac{1\mp\kappa(s)l}{\cos^2\tilde{\theta}}(\omega + u_1\kappa(s)). \end{array} \tag{10}$$

It is worth to mention that the presented transformation is local ($|\tilde{\theta}| < \frac{\pi}{2}$).

III. Path following for a wheeled mobile robot of (2,0) class

There are different approaches to formulation of a path following task. The robot's motion can finish on stopping on a path or it may be continuous, cyclical. In this paper the latter approach is adopted. Let us assume that a direction of a movement along the desired curve is opposite to the clockwise direction, this means that

$$\frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}t} = \kappa(s)\dot{s}.$$

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Hence the equations describing the distance and the orientation errors have a below form

$$\begin{array}{rcl} \dot{l} & = & v \sin\tilde{\theta}, \\ \dot{\tilde{\theta}} & = & \omega - \frac{\kappa(s) v \cos\tilde{\theta}}{1-\kappa(s) l}. \end{array} \quad (11)$$

A. *Morin-Samson algorithm*

The objective of Morin and Samson work was to design a control law which allows the vehicle to follow the path in a stable manner, independently of the sign of the longitudinal velocity.

1) First proposal: For the kinematic equations expressed with respect to a Serret-Frenet frame which are transformed into the 3-dimensional chained system (8), the following control law is proposed

$$\begin{array}{rcl} u_1 & = & const, \\ u_2 & = & -u_1 k_3 z_3 - |u_1| k_2 z_2. \end{array} \quad (12)$$

What is more, if the following inequality holds

$$z_3^2(0) + \frac{1}{k_3} z_2^2(0) < \frac{1}{\kappa_{max}}$$

with $\kappa_{max} = max_s |\kappa(s)|$, then the constraint $|l\ \kappa(s)| < 1$ is satisfied along any solution to the controlled system.

2) Second proposal: Morin and Samson noticed that from a practical point of view it can be useful to complement the control action with an integral term. Let us define a new variable z_0 by

$$\dot{z}_0 = u_1 z_3, \quad z_0(0) = 0.$$

The control (12) can be modified as follows

$$\begin{array}{rcl} u_1 & = & const, \\ u_2 & = & -|u_1| k_0 z_0 - u_1 k_3 z_3 - |u_1| k_2 z_2 = \\ & = & -|u_1| k_0 \int_0^t u_1 z_3 - u_1 k_3 z_3 - |u_1| k_2 z_2. \end{array} \quad (13)$$

If the below conditions are satisfied:

1) the polynomial $s^3 + k_3 s^2 + k_2 s + k_0$ is Hurwitz stable,
2) the initial conditions verify $z_3^2(0) + \frac{1}{k_3 - \frac{k_0}{k_2}} z_2^2(0) < \frac{1}{\kappa_{max}}$,

then the constraint $|l\ \kappa(s)| < 1$ is satisfied along any solution to the controlled system.

B. *The proposed kinematic controller*

The proposed control strategy is a combination of two algorithms. At the beginning Astolfi algorithm is used to bring the variables s, l and $\tilde{\theta}$ to zero. After that the control is switched to Samson algorithm which ensures that the robot moves along the desired path.

1) Astolfi algorithm: The control law introduced by Astolfi belongs to the algorithms which realize the point stabilization of nonholonomic system using discontinuous static feedback from the state. This algorithm is dedicated to the chained systems with only two control inputs

$$\begin{array}{rcl} \dot{z}_1 & = & u_1, \\ \dot{z}_2 & = & u_2, \\ \dot{z}_3 & = & z_2 u_1, \\ & \vdots & \\ \dot{z}_n & = & z_{n-1} u_1. \end{array} \quad (14)$$

The control law for the system (14) proposed by Astolfi is given below

$$u = \begin{pmatrix} -k z_1 \\ p_2 z_2 + p_3 \frac{z_3}{z_1} + \ldots + p_{n-1} \frac{z_{n-1}}{z_1^{n-3}} + p_n \frac{z_n}{z_1^{n-2}} \end{pmatrix}. \quad (15)$$

The parameters p_i should be chosen in such a way that all eigenvalues of the matrix Λ

$$\Lambda = \begin{bmatrix} p_2 & p_3 & p_4 & \ldots & p_{n-1} & p_n \\ -k & k & 0 & \ldots & 0 & 0 \\ 0 & -k & 2k & \ldots & 0 & 0 \\ 0 & 0 & -k & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & (n-3)k & 0 \\ 0 & 0 & 0 & \ldots & -k & (n-2)k \end{bmatrix} \quad (16)$$

have negative real part. This control algorithm is well-conditioned only if the assumption $z_1 \neq 0$ holds.

2) Samson algorithm: For the path following errors for the mobile platfom of (2,0) type expressed by the equations

$$\dot{l} = v \sin\tilde{\theta}, \quad (17)$$

$$\dot{\tilde{\theta}} = u, \quad (18)$$

where $u = \omega - \frac{\kappa(s)\cos\tilde{\theta}}{1-\kappa(s) l} v$ is a new control for the second equation, Samson kinematic controller is equal to

$$v = const, \qquad u = -k_2 l v \frac{\sin\tilde{\theta}}{\tilde{\theta}} - k_3 \tilde{\theta}, \qquad k_2, k_3 > 0. \quad (19)$$

3) The combination of Astolfi algorithm and Samson algorithm: The proposed control strategy may be described by the following pseudocode

if $|\tilde{\theta}| < \epsilon_{\tilde{\theta}}$
    use Samson algorithm for the system (7),
else
    transform (7) *into the chained form* (8),
    use Astolfi algorithm for (8),

where $\epsilon_{\tilde{\theta}}$ is a small positive number, e.g. 0.0001.

Note that Astolfi algorithm requires that $z_1 \neq 0$. In the considered case $z_1 = s$, so it has to be ensured that $s(0) \neq 0$. In practice it is enough if $s(0) = \epsilon_s$, where ϵ_s a small positive number.

IV. SIMULATION RESULTS

The simulations were carried out to illustrate the behaviour of the wheeled mobile robot of (2,0) type with the presented controllers. The initial position of the platform was equal to $(x_0, y_0, \theta_0) = (1.5, 0, \frac{\pi}{3})$ and the desired path was the circle described by the equations

$$\begin{array}{rcl} x(s) & = & R\cos\left(\frac{s}{R}\right), \\ y(s) & = & R\sin\left(\frac{s}{R}\right), \end{array}$$

where $R = 2$ m. Parameters of the kinematic controllers were set to below values:

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- Morin-Samson algorithm, first proposal: $k_2 = 10$, $k_3 = 100$, $u_1 = 1$,
- Morin-Samson algorithm, second proposal: $k_0 = 1$, $k_2 = k_3 = 3$, $u_1 = 1$,
- Samson algorithm: $k_2 = k_3 = 1$, $v = 1$,
- Astolfi algorithm: $k = 1$, $p_2 = -21$, $p_3 = 100$.

The results of the simulations have been presented in Fig. 2-7.

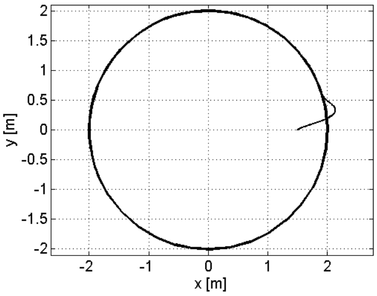
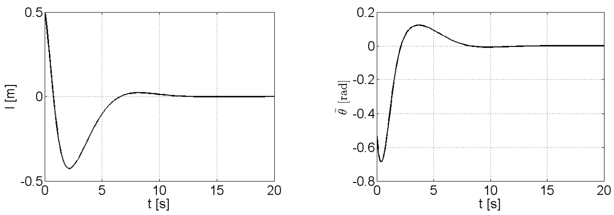


Fig. 2. The path following for the unicycle, XY plot (control algorithm – Morin-Samson first proposal)



(a) (b)

Fig. 3. The path following errors (control algorithm – Morin-Samson first proposal): (a) the distance error l, (b) the orientation error $\tilde{\theta}$

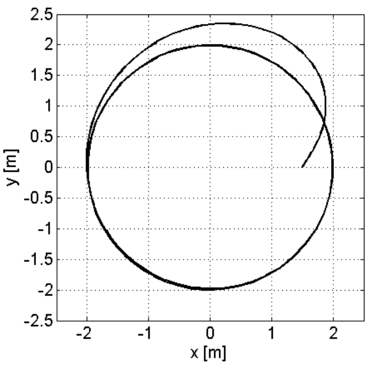
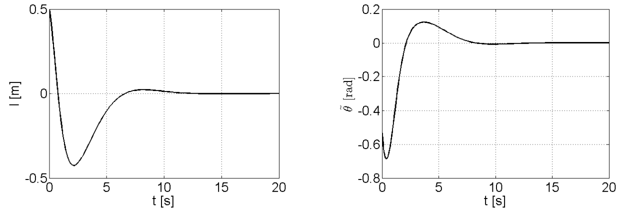


Fig. 4. The path following for the unicycle, XY plot (control algorithm – Morin-Samson second proposal)

V. Conclusion

When comparing Morin-Samson algorithm with the proposed algorithm being a combination of Astolfi and Samson algorithms, one can observe that path following errors could converge to zero quicker in the latter algorithm. At the beginning of the control process the robot reaches the start point of



(a) (b)

Fig. 5. The path following errors (control algorithm – Morin-Samson second proposal): (a) the distance error l, (b) the orientation error $\tilde{\theta}$

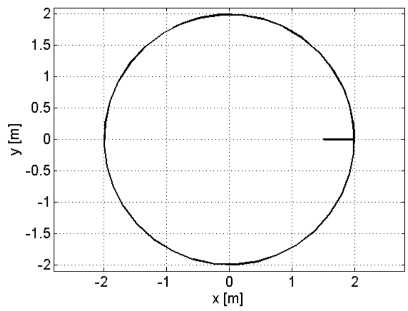


Fig. 6. The path following for the unicycle, XY plot (control algorithm – the combination of Astolfi and Samson algorithms)

the desired quickly due to the usage of Astolfi algorithm and later continues to move along the path according to Samson algorithm.

The presented algorithm – combination of Astolfi and Samson algorithms – could be implemented in practice for the robot moving in an environment without obstacles. The control signal produced by a kinematic controller can be quite efficiently scaled in order to decrease magnitude of velocities. An extension of this work could be dealing with obstacle avoidance in a static and dynamic environment.

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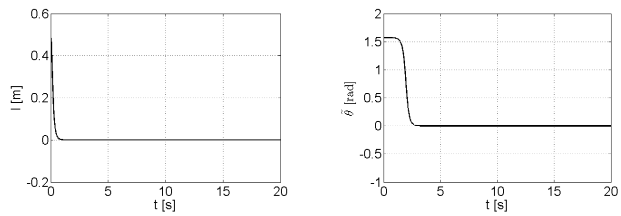

(a) (b)

Fig. 7. The path following errors (control algorithm – the combination of Astolfi and Samson algorithms): (a) the distance error l, (b) the orientation error $\tilde{\theta}$

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