

Differential and Integral Calculus

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1 Lecturer Information

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2 Required Reading

Protter and Morrey: *A first Course in Real Analysis*, UTM Series, Springer-Verlag, 1991

3 Additional Reading

Thomas and Finney, *Calculus and Analytic Geometry*, 9th edition, Addison-Wesley, 1996

Part I

Sequences and Series

1 Sequences

Definition 1 (Sequence). A sequence of real numbers is a set of numbers which are written in some order. There are infinitely many terms in a sequence. It is denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Example 1. $1, \frac{1}{2}, \frac{1}{3}, \dots$ is called the harmonic sequence.

$$a_n = \frac{1}{n}$$

Example 2. $1, -\frac{1}{2}, \frac{1}{3}, \dots$ is called the alternating harmonic sequence.

$$a_n = (-1)^{n+1} \frac{1}{n}$$

Example 3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$

$$a_n = \frac{n}{n+1}$$

Example 4. $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \dots$

$$a_n = \frac{n+1}{3^n}$$

Example 5. The Fibonacci sequence is given by

$$f_n = \begin{cases} 1 & ; \quad n = 1, 2 \\ f_{n-1} + f_{n-2} & ; \quad n \geq 3 \end{cases}$$

Example 6. A geometric sequence is given by

$$a_n = a_1 q^{n-1}$$

where q is called the common ratio.

Example 7. A geometric sequence is given by

$$a_n = a_1 + d(n - 1)$$

where d is called the common difference.

Definition 2 (Equal sequences). Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equal if $a_n = b_n, \forall n \in \mathbb{N}$.

Definition 3 (Sequences bounded from above). $\{a_n\}$ is said to be bounded from above if $\exists M \in \mathbb{R}$, s.t. $a_n \leq M, \forall n \in \mathbb{N}$. Each such M is called an upper bound of $\{a_n\}$.

Definition 4 (Sequences bounded from below). $\{a_n\}$ is said to be bounded from below if $\exists m \in \mathbb{R}$, s.t. $a_n \geq m, \forall n \in \mathbb{N}$. Each such m is called a lower bound of $\{a_n\}$.

Definition 5. $\{a_n\}$ is said to be bounded if it is bounded from below and bounded from above.

Example 8. The sequence $a_n = n^2 + 2$ is not bounded from above but is bounded from below, by all $m \leq 3$.

Example 9. $\left\{ \frac{2n-1}{3n} \right\}$ is bounded.

$$m = 0 \leq \frac{2n-1}{3n} \leq \frac{2n}{3n} = \frac{2}{3} = M$$

Definition 6 (Monotonic increasing sequence). A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \leq a_{n+1}, \forall n \geq n_0$.

Definition 7 (Monotonic decreasing sequence). A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \geq a_{n+1}, \forall n \geq n_0$.

Definition 8 (Strongly increasing sequence). A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n < a_{n+1}, \forall n \geq n_0$.

Definition 9 (Strongly decreasing sequence). A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n > a_{n+1}, \forall n \geq n_0$.

Example 10. The sequence $\left\{\frac{n^2}{2^n}\right\}$ is strongly decreasing. However, this is not evident by observing the first few terms. $\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \dots$

$$\begin{aligned}
& a_n > a_{n+1} \\
\iff & \frac{n^2}{2^n} > \frac{(n+1)^2}{2^{n+1}} \\
\iff & 2n^2 > (n+1)^2 \\
\iff & \sqrt{2}n > n+1 \\
\iff & n(\sqrt{2}-1) > 1 \\
\iff & n > \frac{1}{\sqrt{2}-1} \\
\iff & n > 3
\end{aligned}$$

Exercise 1.

Is $a_n = (-1)^n$ monotonic?

Solution 1.

The sequence $-1, 1, -1, 1, \dots$ is not monotonic.

1.1 Limit of a Sequence

Definition 10. Let $\{a_n\}$ be a given sequence. A number L is said to be the limit of the sequence if $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$, s.t. $|a_n - L| < \varepsilon, \forall n \geq n_0$. That is, there are infinitely many terms inside the interval and a finite number of terms outside it.

Example 11. The sequence $\left\{\frac{1}{n}\right\}$ tends to 0, i.e. for any open interval $(-\varepsilon, \varepsilon)$, there are finite number of terms of the sequence outside the interval, and therefore there are infinitely many terms inside the interval.

Exercise 2.

Prove

$$\lim_{n \rightarrow \infty} \frac{n+2}{2n-1} = \frac{1}{2}$$

Solution 2.

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$

Exercise 3.

Prove that 2 is not a limit of $\left\{ \frac{3n+1}{n} \right\}$.

Solution 3.

If possible, let

$$\lim_{n \rightarrow \infty} \frac{3n+1}{n} = 2$$

Then, $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$, s.t. $\left| \frac{3n+1}{n} - 2 \right| < \varepsilon, \forall n \geq n_0$. However,

$$\left| \frac{3n+1}{n} - 2 \right| = 1 + \frac{1}{n} > 1$$

This is a contradiction for $\varepsilon = \frac{1}{2}$. Therefore, 2 is not a limit.

Theorem 1. *If a sequence $\{a_n\}$ has a limit L then the limit is unique.*

Proof. If possible let there exist two limits L_1 and L_2 . Therefore, $\forall \varepsilon > 0$, there exist a finite number of terms in the interval $(L_1 - \varepsilon, L_1 + \varepsilon)$. Therefore, there exist a finite number of terms in the interval $(L_2 - \varepsilon, L_2 + \varepsilon)$. This contradicts the definition of a limit. Therefore, the limit is unique. \square

Theorem 2. *If a sequence $\{a_n\}$ has limit L , then the sequence is bounded.*

Theorem 3. *Let*

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= a \\ \lim_{n \rightarrow \infty} b_n &= b \end{aligned}$$

and let c be a constant. Then,

$$\begin{aligned} \lim c &= c \\ \lim(ca_n) &= c \lim a_n \\ \lim(a_n \pm b_n) &= \lim a_n \pm \lim b_n \\ \lim(a_n b_n) &= \lim a_n \lim b_n \\ \lim\left(\frac{a_n}{b_n}\right) &= \frac{\lim a_n}{\lim b_n} \quad (\text{if } \lim b_n \neq 0) \end{aligned}$$

Theorem 4. Let $\{b_n\}$ be bounded and let $\lim a_n = 0$. Then,

$$\lim(a_nb_n) = 0$$

Theorem 5 (Sandwich Theorem). Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences. If

$$\lim a_n = \lim b_n = L$$

and $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n \leq b_n \leq c_n$. Then,

$$\lim b_n = L$$

Exercise 4.

Calculate $\lim_{n \rightarrow \infty} \sqrt[n]{2^n + 3^n}$

Solution 4.

$$\begin{aligned} \sqrt[n]{3^n} &\leq \sqrt[n]{2^n + 3^n} \leq \sqrt[n]{3^n + 3^n} = \sqrt[n]{2 \cdot 3^n} \\ \therefore 3 &\leq \sqrt[n]{2^n + 3^n} \leq 3\sqrt[n]{2} \end{aligned}$$

Therefore, by the Sandwich Theorem, $\lim_{n \rightarrow \infty} \sqrt[n]{2^n + 3^n} = 3$.

Theorem 6. Any monotonically increasing sequence which is bounded from above converges. Similarly, any monotonically decreasing sequence which is bounded from below converges.

Exercise 5.

Prove that there exists a limit for $a_n = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}_{n \text{ times}}$ and find it.

Solution 5.

$$a_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = a_2$$

If possible, let

$$\begin{aligned} a_{n-1} &< a_n \\ \therefore \sqrt{2 + a_{n-1}} &< \sqrt{2 + a_n} \\ \therefore a_n &< a_{n+1} \end{aligned}$$

Hence, by induction, $\{a_n\}$ is monotonically increasing.

$$a_1 = \sqrt{2} \leq 2$$

If possible, let

$$\begin{aligned} a_n &\leq 2 \therefore \sqrt{2 + a_n} && \leq \sqrt{2 + 2} \\ \therefore a_{n+1} &\leq 2 \end{aligned}$$

Hence, by induction, $\{a_n\}$ is bounded from above by 2. Therefore, by , $\{a_n\}$ converges.

Definition 11 (Limit in a wide sense). The sequence $\{a_n\}$ is said to converge to $+\infty$ if $\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0, a_n > M$.

The sequence $\{a_n\}$ is said to converge to $-\infty$ if $\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0, a_n < M$.

1.2 Sub-sequences

Definition 12 (Sub-sequence). Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_k\}_{k=1}^{\infty}$ be a strongly increasing sequence of natural numbers. Let $\{b_k\}_{k=1}^{\infty}$ be a sequence such that $b_k = a_{n_k}$. Then $\{b_k\}_{k=1}^{\infty}$ is called a sub-sequence of $\{a_n\}_{n=1}^{\infty}$.

Example 12.

$$a_n = \frac{1}{n}$$

If we choose $n_k = k^2$,

$$b_k = a_{n_k} = a_{k^2} = \frac{1}{k^2}$$

Therefore,

$$\{b_k\} = 1, \frac{1}{4}, \frac{1}{9}, \dots$$

Theorem 7. *If the sequence $\{a_n\}$ converges to L in a wide sense, i.e. L can be infinite, then any sub-sequence of $\{a_n\}$ converges to the same limit L .*

Definition 13 (Partial limit). A real number a , which may be infinite, is called a partial limit of the sequence $\{a_n\}$ if there exists a sub-sequence of $\{a_n\}$ which converges to a .

Example 13. Let

$$a_n = (-1)^n$$

Therefore, $\nexists \lim_{n \rightarrow \infty} a_n$. Let

$$b_k = a_{n_k} = a_{2k-1}$$

Therefore,

$$\begin{aligned} \{b_k\} &= -1, -1, -1, \dots \\ \therefore \lim_{k \rightarrow \infty} b_k &= -1 \end{aligned}$$

Therefore, -1 is a partial limit of $\{a_n\}$.

Theorem 8 (Bolzano-Weierstrass Theorem). *For any bounded sequence there exists a subsequence which is convergent, s.t. there exists at least one partial limit.*

Definition 14 (Upper partial limit). The largest partial limit of a sequence is called the upper partial limit. It is denoted by $\overline{\lim} a_n$ or $\limsup a_n$.

Definition 15 (Lower partial limit). The smallest partial limit of a sequence is called the lower partial limit. It is denoted by $\underline{\lim} a_n$ or $\liminf a_n$.

Theorem 9. *If the sequence $\{a_n\}$ is bounded and*

$$\overline{\lim} a_n = \underline{\lim} a_n = a$$

then

$$\exists \lim a_n = a$$

1.3 Cauchy Characterisation of Convergence

Definition 16. A sequence $\{a_n\}$ is called a Cauchy sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall m, n \geq n_0$, $|a_n - a_m| < \varepsilon$.

Theorem 10 (Cauchy Characterisation of Convergence). *A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.*

Proof. Let

$$\lim_{n \rightarrow \infty} a_n = L$$

Then $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$, $|a_n - L| < \frac{\varepsilon}{2}$. Therefore if $n \geq n_0$ and $m \geq n_0$, then

$$\begin{aligned} |a_n - a_m| &= |a_n - L + L - a_m| \\ &\leq |a_n - L| + |L - a_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ \therefore |a_n - a_m| &= \varepsilon \end{aligned}$$

Similarly, the converse can be proved by Theorem 9. \square

Theorem 11 (Another Formulation of the Cauchy Characterisation Theorem). *The sequence $\{a_n\}$ converges if and only if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$.*

Exercise 6.

Prove that the sequence

$$a_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

is convergent.

Solution 6.

$$\begin{aligned} |a_{n+p} - a_n| &= \left| \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(n+p)^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right) \right| \\ &= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(n+p-1)(n+p)} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n} - \frac{1}{\cancel{n+1}} + \frac{1}{\cancel{n+1}} + \cdots + \frac{1}{\cancel{n+p-1}} - \frac{1}{n+p} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n} - \frac{1}{n+p} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n} \end{aligned}$$

Therefore, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$, where $n_0 > \frac{1}{\varepsilon}$. \square

Exercise 7.

Prove that the sequence

$$a_n = \frac{1}{1} + \frac{1}{n} + \cdots + \frac{1}{n}$$

diverges.

Solution 7.

If possible, let the sequence converge. Then, by the Cauchy Characterisation of Convergence, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$. Therefore,

$$\begin{aligned} |a_{n+p} - a_n| &= \left| \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+p} - \left(\frac{1}{n} + \cdots + \frac{1}{n} \right) \right| \\ &= \frac{1}{n+1} + \cdots + \frac{1}{n+p} \\ &\geq p \cdot \frac{1}{n+p} \\ \therefore |a_{n+p} - a_n| &> \frac{p}{n+p} \end{aligned}$$

If $n = p$,

$$\frac{p}{n+p} = \frac{1}{2}$$

This contradicts the result obtained from the Cauchy Characterisation of Convergence, for $\varepsilon = \frac{1}{4}$.

Therefore, the sequence diverges.

2 Series

Definition 17 (Series). Given a sequence $\{a_n\}$, the sum $a_1 + \cdots + a_n + \cdots$ is called an infinite series or series. It is denoted as $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

Definition 18 (Partial sum). The partial sum of the series $\sum a_n$ is defined as

$$S_i = a_1 + \cdots + a_i$$

Definition 19 (Convergent and divergent series). If the sequence $\{S_n\}_{n=1}^{\infty}$ converges, then the series is called convergent. Otherwise, the series is called divergent.

Definition 20 (Sum of a series). If the sequence $\{S_n\}_{n=1}^{\infty}$ converges to $S \neq \pm\infty$, the number S is called the sum of the series.

$$\sum_{n=1}^{\infty} a_n = S$$

Example 14.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

Therefore,

$$S_1 = \frac{1}{2} \tag{1}$$

$$S_2 = \frac{1}{2} + \frac{1}{2^2} \tag{2}$$

$$\vdots S_n = \frac{1}{2} + \cdots + \frac{1}{2^n} \tag{3}$$

$$= \frac{a_1(1 - q^n)}{1 - q} \tag{4}$$

$$= \frac{1/2(1 - 1/2^n)}{1 - 1/2} \tag{5}$$

$$= 1 - \frac{1}{2^n} \tag{6}$$

$$\lim_{n \rightarrow \infty} S_n = 1 \tag{7}$$

Therefore, the series converges.

$$S = \sum_{n=1}^{\infty} = 1$$

Theorem 12. A geometric series $\sum_{n=1}^{\infty} a_1 q^{n-1}$, $a_1 \neq 0$ converges if $|q| < 1$ and then,

$$S = \sum_{n=1}^{\infty} a_1 q^{n-1} = \frac{a_1}{1 - q}$$

Definition 21 (p -series). The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the p -series.

Theorem 13. *The p -series converges for $p > 1$ and diverges for $p \leq 1$.*

Theorem 14. *If $\sum a_n$ converges, then*

$$\lim_{n \rightarrow \infty} a_n = 0$$

Proof.

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ \therefore \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} \\ &= S - S \\ &= 0 \end{aligned}$$

□

Theorem 15. *If $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n \pm b_n)$ and $\sum ca_n$, where c is a constant, also converge. Also,*

$$\begin{aligned} \sum(a_n \pm b_n) &= \sum a_n \pm \sum b_n \\ \sum(ca_n) &= c \sum a_n \end{aligned}$$

2.1 Convergence Criteria

2.1.1 Leibniz's Criteria

Definition 22 (Alternating series). The series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

Theorem 16 (Leibniz's Criteria for Convergence). *If an alternating series $\sum (-1)^{n-1} a_n$ with $a_n > 0$ satisfies*

1. $a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.

2. $\lim_{n \rightarrow \infty} a_n = 0$

then the series $\sum (-1)^{n-1} a_n$ converges.

Proof. Consider the even partial sums of the series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$.

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$$

As $\{a_n\}$ is monotonically increasing, all brackets are non-negative. Therefore,

$$S_{2m+2} \geq S_{2m}$$

Therefore, $\{S_{2m}\}$ is increasing.

Also,

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$$

All brackets and a_{2m} are non-negative. Therefore,

$$S_{2m} \leq a_1$$

Therefore, $\{S_{2m}\}$ is bounded from above by a_1 . Hence,

$$\exists \lim_{m \rightarrow \infty} S_{2m} = S$$

For S_{2m+1} ,

$$\begin{aligned} S_{2m+1} &= S_{2m} + a_{2m+1} \\ \therefore \lim_{m \rightarrow \infty} S_{2m+1} &= \lim_{m \rightarrow \infty} S_{2m} + \lim_{m \rightarrow \infty} a_{2m+1} \\ &= S + 0 \\ &= S \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} S_n = S$$

□

Example 15. The alternating harmonic series $\sum \frac{(-1)^{n-1}}{n}$ converges as $a_n = \frac{1}{n} > 0$, a_n decreases and $\lim a_n = 0$.

2.1.2 Comparison Test

Theorem 17 (Comparison Test for Convergence). Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$.

1. If $a_n \leq b_n$, $\forall n \geq n_0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $a_n \geq b_n$, $\forall n \geq n_0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 18 (Another Formulation of the Comparison Test for Convergence). Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$ and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = a > 0$$

where a is a finite number. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

2.1.3 d'Alembert Criteria (Ratio Test)

Definition 23 (Absolute and conditional convergence). The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

Example 16. The series $\sum \frac{(-1)^{n-1}}{n^2}$ converges absolutely, as $\sum \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum \frac{1}{n^2}$ converges.

Example 17. The series $\sum \frac{(-1)^{n-1}}{n}$ converges conditionally, as it converges, but $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$ diverges.

Theorem 19. *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 20 (d'Alembert Criteria (Ratio Test)). 1. If

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n-1}}{a_n} \right| = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n-1}}{a_n} \right| = L > 1$$

(including $L = \infty$), then $\sum a_n$ converges diverges.

3. If $L = 1$, the test does not apply.

2.1.4 Cauchy Criteria (Cauchy Root Test)

Theorem 21 (Cauchy Criteria (Cauchy Root Test)). 1. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

(including $L = \infty$), then $\sum a_n$ diverges.

3. If $L = 1$, the test does not apply.

2.1.5 Integral Test

Theorem 22 (Integral Test for Series Convergence). *Let $f(x)$ be a continuous, non-negative, monotonic decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x) dx$ converges.*

Exercise 8.

Does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge or diverge?

Solution 8.

Let

$$f(x) = \frac{1}{x^p}$$

with $p > 0$.

Therefore, $f(x)$ is continuous, non-negative and monotonic decreasing on $[1, \infty)$. Therefore, the Integral Test for Series Convergence is applicable.

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^p} dx$$

If $p \neq 1$,

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} &= \lim_{t \rightarrow \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^t \\ &= \lim_{t \rightarrow \infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} \right) \\ &= \frac{1}{p-1} \end{aligned}$$

If $p = 1$,

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} &= \lim_{t \rightarrow \infty} \ln x \Big|_1^t \\ &= \infty \end{aligned}$$

Therefore, the series converges for $p > 1$ and diverges for $p \leq 1$.

Theorem 23. *If the series $\sum a_n$ absolutely converges and the series $\sum b_n$ is obtained from $\sum a_n$ by changing the order of the terms in $\sum a_n$ then $\sum b_n$ also absolutely converges and $\sum b_n = \sum a_n$.*

Theorem 24. *If a series converges then the series with brackets without changing the order of terms also converges. That is, if $\sum a_n$ converges, then any series of the form $(a_1 + a_2) + (a_3 + a_4 + a_5) + a_6 + \dots$ also converges.*

Theorem 25. *If a series with brackets converges and the terms in the brackets have the same sign, then the series without brackets also converges.*

3 Power Series

Definition 24 (Power series). The series $\sum_{n=0}^{\infty} a_n(x - c)^n$ is called a power series.

Theorem 26 (Cauchy-Hadamard Theorem). *For any power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ there exists the limit, which may be infinity,*

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$$

and the series converges for $|x - c| < R$ and diverges for $|x - c| > R$. The end points of the interval, i.e. $x = c - R$ and $x = c + R$ must be separately checked for series convergence.

Definition 25 (Radius of convergence and convergence interval). The number R is called the radius of convergence and the interval $|x - c| < R$ is called the convergence interval of the series. The point c is called the centre of the convergence interval.

Theorem 27. *If $\exists \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$, which may be infinite, then,*

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Series	Centre of convergence interval	Radius of convergence	Convergence interval
$\sum_{n=0}^{\infty} x^n$			
$\sum_{n=0}^{\infty} n!x^n$			
$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$			
$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$			

Theorem 28 (Stirling's Approximation). *For $n \rightarrow \infty$,*

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$