

Physics 2

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Contents

1	Lecturer Information	2
2	Required Reading	2
3	Additional Reading	2
I	Sequences and Series	3
1	Sequences	3
1.1	Limit of a Sequence	5

1 Lecturer Information

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2 Required Reading

Protter and Morrey: *A first Course in Real Analysis*, UTM Series, Springer-Verlag, 1991

3 Additional Reading

Thomas and Finney, *Calculus and Analytic Geometry*, 9th edition, Addison-Wesley, 1996

Part I

Sequences and Series

1 Sequences

Definition 1 (Sequence). A sequence of real numbers is a set of numbers which are written in some order. There are infinitely many terms in a sequence. It is denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Example 1. $1, \frac{1}{2}, \frac{1}{3}, \dots$ is called the harmonic sequence.

$$a_n = \frac{1}{n}$$

Example 2. $1, -\frac{1}{2}, \frac{1}{3}, \dots$ is called the alternating harmonic sequence.

$$a_n = (-1)^{n+1} \frac{1}{n}$$

Example 3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$

$$a_n = \frac{n}{n+1}$$

Example 4. $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \dots$

$$a_n = \frac{n+1}{3^n}$$

Example 5. The Fibonacci sequence is given by

$$f_n = \begin{cases} 1 & ; \quad n = 1, 2 \\ f_{n-1} + f_{n-2} & ; \quad n \geq 3 \end{cases}$$

Example 6. A geometric sequence is given by

$$a_n = a_1 q^{n-1}$$

where q is called the common ratio.

Example 7. A geometric sequence is given by

$$a_n = a_1 + d(n - 1)$$

where d is called the common difference.

Definition 2 (Equal sequences). Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equal if $a_n = b_n, \forall n \in \mathbb{N}$.

Definition 3 (Sequences bounded from above). $\{a_n\}$ is said to be bounded from above if $\exists M \in \mathbb{R}$, s.t. $a_n \leq M, \forall n \in \mathbb{N}$. Each such M is called an upper bound of $\{a_n\}$.

Definition 4 (Sequences bounded from below). $\{a_n\}$ is said to be bounded from below if $\exists m \in \mathbb{R}$, s.t. $a_n \geq m, \forall n \in \mathbb{N}$. Each such m is called a lower bound of $\{a_n\}$.

Definition 5. $\{a_n\}$ is said to be bounded if it is bounded from below and bounded from above.

Example 8. The sequence $a_n = n^2 + 2$ is not bounded from above but is bounded from below, by all $m \leq 3$.

Example 9. $\left\{ \frac{2n - 1}{3n} \right\}$ is bounded.

$$m = 0 \leq \frac{2n - 1}{3n} \leq \frac{2n}{3n} = \frac{2}{3} = M$$

Definition 6 (Monotonic increasing sequence). A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \leq a_{n+1}, \forall n \geq n_0$.

Definition 7 (Monotonic decreasing sequence). A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \geq a_{n+1}, \forall n \geq n_0$.

Definition 8 (Strongly increasing sequence). A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n < a_{n+1}, \forall n \geq n_0$.

Definition 9 (Strongly decreasing sequence). A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n > a_{n+1}, \forall n \geq n_0$.

Example 10. The sequence $\left\{\frac{n^2}{2^n}\right\}$ is strongly decreasing. However, this is not evident by observing the first few terms. $\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \dots$

$$\begin{aligned}
 & a_n > a_{n+1} \\
 \iff & \frac{n^2}{2^n} > \frac{(n+1)^2}{2^{n+1}} \\
 \iff & 2n^2 > (n+1)^2 \\
 \iff & \sqrt{2}n > n+1 \\
 \iff & n(\sqrt{2}-1) > 1 \\
 \iff & n > \frac{1}{\sqrt{2}-1} \\
 \iff & n > 3
 \end{aligned}$$

Exercise 1.

Is $a_n = (-1)^n$ monotonic?

Solution 1.

The sequence $-1, 1, -1, 1, \dots$ is not monotonic.

1.1 Limit of a Sequence

Definition 10. Let $\{a_n\}$ be a given sequence. A number L is said to be the limit of the sequence if $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$, s.t. $|a_n - L| < \varepsilon, \forall n \geq n_0$. That is, there are infinitely many terms inside the interval and a finite number of terms outside it.

Example 11. The sequence $\left\{\frac{1}{n}\right\}$ tends to 0, i.e. for any open interval $(-\varepsilon, \varepsilon)$, there are finite number of terms of the sequence outside the interval, and therefore there are infinitely many terms inside the interval.

Exercise 2.

Prove

$$\lim_{n \rightarrow \infty} \frac{n+2}{2n-1} = \frac{1}{2}$$

Exercise 3.

Prove that 2 is not a limit of $\left\{ \frac{3n+1}{n} \right\}$.

Theorem 1. *If a sequence $\{a_n\}$ has a limit L then the limit is unique.*

Proof. If possible let there exist two limits L_1 and L_2 . Therefore, $\forall \varepsilon > 0$, there exist a finite number of terms in the interval $(L_1 - \varepsilon, L_1 + \varepsilon)$. Therefore, there exist a finite number of terms in the interval $(L_2 - \varepsilon, L_2 + \varepsilon)$. This contradicts the definition of a limit. Therefore, the limit is unique. \square

Theorem 2. *If a sequence $\{a_n\}$ has limit L , then the sequence is bounded.*

Theorem 3. *Let*

$$\begin{aligned}\lim_{n \rightarrow \infty} a_n &= a \\ \lim_{n \rightarrow \infty} b_n &= b\end{aligned}$$

and let c be a constant. Then,

$$\begin{aligned}\lim c &= c \\ \lim(ca_n) &= c \lim a_n \\ \lim(a_n \pm b_n) &= \lim a_n \pm \lim b_n \\ \lim(a_n b_n) &= \lim a_n \lim b_n \\ \lim\left(\frac{a_n}{b_n}\right) &= \frac{\lim a_n}{\lim b_n} \quad (\text{if } \lim b_n \neq 0)\end{aligned}$$

Theorem 4. *Let $\{b_n\}$ be bounded and let $\lim a_n = 0$. Then,*

$$\lim(a_n b_n) = 0$$

Theorem 5 (Sandwich Theorem). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences. If*

$$\lim a_n = \lim b_n = \lim c_n = L$$

and $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n \leq b_n \leq c_n$. Then,

$$\lim b_n = L$$

Exercise 4.

Calculate $\lim_{n \rightarrow \infty} \sqrt[n]{2^n + 3^n}$

Solution 4.

$$\begin{aligned}\sqrt[n]{3^n} &\leq \sqrt[n]{2^n + 3^n} \leq \sqrt[n]{3^n + 3^n} = \sqrt[n]{2 \cdot 3^n} = \sqrt[n]{2} \cdot \sqrt[n]{3^n} \\ \therefore 3 &\leq \sqrt[n]{2^n + 3^n} \leq 3 \sqrt[n]{2}\end{aligned}$$

Therefore, by the Sandwich Theorem, $\lim_{n \rightarrow \infty} \sqrt[n]{2^n + 3^n} = 3$.

Theorem 6. *Any monotonically increasing sequence which is bounded from above converges. Similarly, any monotonically decreasing sequence which is bounded from below converges.*

Exercise 5.

Prove that there exists a limit for $a_n = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}_{n \text{ times}}$ and find it.

Solution 5.

$$a_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = a_2$$

If possible, let

$$\begin{aligned}a_{n-1} &< a_n \\ \therefore \sqrt{2 + a_{n-1}} &< \sqrt{2 + a_n} \\ \therefore a_n &< a_{n+1}\end{aligned}$$

Hence, by induction, $\{a_n\}$ is monotonically increasing.

$$a_1 = \sqrt{2} \leq 2$$

If possible, let

$$\begin{aligned}a_n &\leq 2 \therefore \sqrt{2 + a_n} \leq \sqrt{2 + 2} \\ \therefore a_{n+1} &\leq 2\end{aligned}$$

Hence, by induction, $\{a_n\}$ is bounded from above by 2. Therefore, by , $\{a_n\}$ converges.