2.1.5. *Integral Test.*

Theorem 21 (Integral Test for Series Convergence)**.** *Let $f(x)$ be a continuous, non-negative, monotonic decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int\limits_{1}^{\infty} f(x)\,\mathrm{d}x$ converges.*

Theorem 22. *If the series $\sum a_n$ absolutely converges and the series $\sum b_n$ is obtained from $\sum a_n$ by changing the order of the terms in $\sum a_n$ then $\sum b_n$ also absolutely converges and $\sum b_n = \sum a_n$.*

Theorem 23. *If a series converges then the series with brackets without changing the order of terms also converges. That is, if $\sum a_n$ converges, then any series of the form $(a_1 + a_2) + (a_3 + a_4 + a_5) + a_6 + \dots$ also converges.*

Theorem 24. *If a series with brackets converges and the terms in the brackets have the same sign, then the series without brackets also converges.*

3. Power Series

Definition 16 (Power series)**.** The series $\sum\limits_{n=0}^{\infty} a_n (x-c)^n$ is called a power series.

Theorem 25 (Cauchy-Hadamard Theorem)**.** *For any power series $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ there exists the limit, which may be infinity, $R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}$ and the series converges for $|x - c| < R$ and diverges for $|x - c| > R$. The end points of the interval, i.e. $x = c - R$ and $x = c + R$ must be separately checked for series convergence.*

Definition 17 (Radius of convergence and convergence interval)**.** The number R is called the radius of convergence and the interval $|x-c| < R$ is called the convergence interval of the series. The point c is called the centre of the convergence interval.

Theorem 26. *If $\exists \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, which may be infinite, then, $R = \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$.*

Theorem 27 (Stirling's Approximation)**.** *For $n \to \infty$, $n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$.*

Exercise 6.

Find the domain of convergence of $\sum\limits_{n=1}^{\infty} \dfrac{(2x-4)^n}{n}$.

Solution 6.

$$\sum_{n=1}^{\infty} \frac{(2x-4)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n (x-2)^n}{n}$$

Therefore, by Cauchy's Formula for Radius of Convergence,

$$\begin{aligned} R &= \frac{1}{\lim \sqrt[n]{|a_n|}} \\ &= \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\frac{2^n}{n}}} \\ &= \frac{1}{\lim\limits_{n\to\infty} \frac{2}{\sqrt[n]{n}}} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, the series converges for

$$|x-2| < \frac{1}{2}$$

and diverges for

$$|x-2| > \frac{1}{2}$$

If $x = \frac{5}{2}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{5}{2} - 2\right)^n \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \end{aligned}$$

Therefore, the series diverges.
If $x = \frac{3}{2}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{3}{2} - 2\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \end{aligned}$$

Therefore, by Leibniz's Criteria for Convergence, the series converges.

Therefore, the domain of convergence is $\left[\frac{3}{2}, \frac{5}{2}\right)$.

Exercise 7.

Find the radius of convergence of $\sum\limits_{n=0}^{\infty} n! x^{n!}$.

Solution 7.

$$\frac{1}{\lim \sqrt[n]{a_n}} = x + x + 2x^2 + 6x^6 + 24x^{24} + \dots$$

Therefore,

$$a_n = \begin{cases} n & ; \quad n = k^2 \\ 0 & ; \quad n \neq k^2 \end{cases}$$

Therefore,

$$\begin{aligned} R &= \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{a_n}} \\ &= \frac{1}{\lim\limits_{k\to\infty} \sqrt[k!]{k!}} \\ &= 1 \end{aligned}$$

DIFFERENTIAL AND INTEGRAL CALCULUS : COMPENDIUM

AAKASH JOG

1. Sequences

Definition 1 (Sequences bounded from above)**.** $\{a_n\}$ is said to be bounded from above if $\exists M \in \mathbb{R}$, s.t. $a_n \leq M$, $\forall n \in \mathbb{N}$. Each such M is called an upper bound of $\{a_n\}$.

Definition 2 (Sequences bounded from below)**.** $\{a_n\}$ is said to be bounded from below if $\exists m \in \mathbb{R}$, s.t. $a_n \geq M$, $\forall n \in \mathbb{N}$. Each such M is called an lower bound of $\{a_n\}$.

Definition 3. $\{a_n\}$ is said to be bounded if it is bounded from below and bounded from above.

Definition 4 (Monotonic increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \leq a_{n+1}$, $\forall n \geq n_0$.

Definition 5 (Monotonic decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \geq a_{n+1}$, $\forall n \geq n_0$.

Definition 6 (Strongly increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n < a_{n+1}$, $\forall n \geq n_0$.

Definition 7 (Strongly decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n > a_{n+1}$, $\forall n \geq n_0$.

Example 1. The sequence $\left\{\frac{n^2}{2^n}\right\}$ is strongly decreasing. However, this is not evident by observing the first few terms. $\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \dots$

$$\begin{aligned} & a_n > a_{n+1} \\ \iff & \frac{n^2}{2^n} > \frac{(n+1)^2}{2^{n+1}} \\ \iff & 2n^2 > (n+1)^2 \\ \iff & \sqrt{2}n > n+1 \\ \iff & n(\sqrt{2}-1) > 1 \\ \iff & n > \frac{1}{\sqrt{2}-1} \\ \iff & n > 3 \end{aligned}$$

1.1. Limit of a Sequence.

Definition 8 (Limit of a sequence)**.** Let $\{a_n\}$ be a given sequence. A number L is said to be the limit of the sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $|a_n - L| < \varepsilon$, $\forall n \geq n_0$. That is, there are infinitely many terms inside the interval and a finite number of terms outside it.

Exercise 1.
Prove that 2 is not a limit of $\left\{\frac{3n+1}{n}\right\}$.

Solution 1.
If possible, let

$$\lim_{n\to\infty} \frac{3n+1}{n} = 2$$

Then, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\left|\frac{3n+1}{n} - 2\right| < \varepsilon$, $\forall n \geq n_0$.
However,

$$\left|\frac{3n+1}{n} - 2\right| = 1 + \frac{1}{n} > 1$$

This is a contradiction for $\varepsilon = \frac{1}{2}$. Therefore, 2 is not a limit.

Theorem 1. *If a sequence $\{a_n\}$ has a limit L then the limit is unique.*

Theorem 2. *If a sequence $\{a_n\}$ has limit L, then the sequence is bounded.*

Theorem 3. *Let*

$$\begin{aligned} \lim_{n\to\infty} a_n &= a \\ \lim_{n\to\infty} b_n &= b \end{aligned}$$

and let c be a constant. Then,

$$\begin{aligned} \lim c &= c \\ \lim(ca_n) &= c \lim a_n \\ \lim(a_n \pm b_n) &= \lim a_n \pm \lim b_n \\ \lim(a_n b_n) &= \lim a_n \lim b_n \\ \lim\left(\frac{a_n}{b_n}\right) &= \frac{\lim a_n}{\lim b_n} \quad (\text{ if } \lim b \neq 0) \end{aligned}$$

Theorem 4. *Let $\{b_n\}$ be bounded and let $\lim a_n = 0$. Then,*

$$\lim(a_n b_n) = 0$$

Theorem 5 (Sandwich Theorem)**.** *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences. If*

$$\lim a_n = \lim b_n = L$$

and $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n \leq b_n \leq c_n$. Then,

$$\lim b_n = L$$

Exercise 2.
Calculate $\lim\limits_{n\to\infty} \sqrt[n]{2^n + 3^n}$

Date: Wednesday 29th April, 2015.

Solution 2.

$$\sqrt[n]{3^n} \leq \sqrt[n]{2^n+3^n} \leq \sqrt[n]{3^n+3^n} = \sqrt[3]{2 \cdot 3^n}$$
$$\therefore 3 \leq \sqrt[n]{2^n+3^n} \leq 3\sqrt[n]{2}$$

Therefore, by the Sandwich Theorem, $\lim\limits_{n\to\infty} \sqrt[n]{2^n+3^n} = 3$.

Theorem 6. *Any monotonically increasing sequence which is bounded from above converges. Similarly, any monotonically decreasing sequence which is bounded from below converges.*

Exercise 3.
Prove that there exists a limit for $a_n = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}}_{n \text{ times}}$ and find it.

Solution 3.

$$a_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = a_2$$

If possible, let

$$a_{n-1} < a_n$$
$$\therefore \sqrt{2+a_{n-1}} < \sqrt{2+a_n}$$
$$\therefore a_n < a_{n+1}$$

Hence, by induction, $\{a_n\}$ is monotonically increasing.

$$a_1 = \sqrt{2} \leq 2$$

If possible, let

$$a_n \leq 2$$
$$\therefore \sqrt{2+a_n} \leq \sqrt{2+2}$$
$$\therefore a_{n+1} \leq 2$$

Hence, by induction, $\{a_n\}$ is bounded from above by 2. Therefore, by , $\{a_n\}$ converges.

1.2. Sub-sequences.

Definition 9 (Sub-sequence)**.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_k\}_{k=1}^{\infty}$ be a strongly increasing sequence of natural numbers. Let $\{b_k\}_{k=1}^{\infty}$ be a sequence such that $b_k = a_{n_k}$. Then $\{b_k\}_{k=1}^{\infty}$ is called a sub-sequence of $\{a_n\}_{n=1}^{\infty}$.

Theorem 7. *If the sequence $\{a_n\}$ converges to L in a wide sense, i.e. L can be infinite, then any sub-sequence of $\{a_n\}$ converges to the same limit L.*

Definition 10 (Partial limit)**.** A real number a, which may be infinite, is called a partial limit of the sequence $\{a_n\}$ is there exists a sub-sequence of $\{a_n\}$ which converges to a.

Theorem 8 (Bolzano-Weierstrass Theorem)**.** *For any bounded sequence there exists a subsequence which is convergent, s.t. there exists at least one partial limit.*

Definition 11 (Upper partial limit)**.** The largest partial limit of a sequence is called the upper partial limit. It is denoted by $\overline{\lim} a_n$ or $\limsup a_n$.

Definition 12 (Lower partial limit)**.** The smallest partial limit of a sequence is called the upper partial limit. It is denoted by $\underline{\lim} a_n$ or $\liminf a_n$.

Theorem 9. *If the sequence $\{a_n\}$ is bounded and $\overline{\lim} a_n = \underline{\lim} a_n = a$ then $\exists \lim a_n = a$.*

1.3. Cauchy Characterisation of Convergence.

Definition 13. A sequence $\{a_n\}$ is called a Cauchy sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall m, n \geq n_0$, $|a_n - a_m| < \varepsilon$.

Theorem 10 (Cauchy Characterisation of Convergence)**.** *A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.*

Theorem 11 (Another Formulation of the Cauchy Characterisation Theorem)**.** *The sequence $\{a_n\}$ converges if and only if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$.*

Exercise 4.
Prove that the sequence $a_n = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}$ is convergent.

Solution 4.

$$|a_{n+p} - a_n| = \left|\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{(n+p)^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}\right)\right|$$
$$= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(n+p)^2}$$
$$\therefore |a_{n+p} - a_n| < \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(n+p-1)(n+p)}$$
$$\therefore |a_{n+p} - a_n| < \frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} + \dots + \cancel{\frac{1}{n+p-1}} - \frac{1}{n+p}$$
$$\therefore |a_{n+p} - a_n| < \frac{1}{n} - \frac{1}{n+p}$$
$$\therefore |a_{n+p} - a_n| < \frac{1}{n}$$

Therefore, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$, where $n_0 > \frac{1}{\varepsilon}$. □

Exercise 5.
Prove that the sequence

$$a_n = \frac{1}{1} + \frac{1}{n} + \dots + \frac{1}{n}$$

diverges.

Solution 5.
If possible, let the sequence converge. Then, by the Cauchy Characterisation of Convergence, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$.
Therefore,

$$|a_{n+p} - a_n| = \left|\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n+p} - \left(\frac{1}{n} + \dots + \frac{1}{n}\right)\right|$$
$$= \frac{1}{n+1} + \dots + \frac{1}{n+p}$$
$$\geq p \cdot \frac{1}{n+p}$$
$$\therefore |a_{n+p} - a_n| > \frac{p}{n+p}$$

If $n = p$,

$$\frac{p}{n+p} = \frac{1}{2}$$

This contradicts the result obtained from the Cauchy Characterisation of Convergence, for $\varepsilon = \frac{1}{4}$.
Therefore, the sequence diverges.

2. Series

Definition 14 (p-series)**.** The series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ is called the p-series.

Theorem 12. *The p-series converges for $p > 1$ and diverges for $p \leq 1$.*

Theorem 13. *If $\sum a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$, but the converse is not true.*

Theorem 14. *If $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n \pm b_n)$ and $\sum c a_n$, where c is a constant, also converge. Also,*

$$\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$$
$$\sum(c a_n) = c \sum a_n$$

2.1. Convergence Criteria.

2.1.1. *Leibniz's Criteria.*

Theorem 15 (Leibniz's Criteria for Convergence)**.** *If an alternating series $\sum(-1)^{n-1} a_n$ with $a_n > 0$ satisfies*

(1) $a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.
(2) $\lim\limits_{n\to\infty} a_n = 0$

then the series $(-1)^{n-1} a_n$ converges.

Example 2. The alternating harmonic series $\sum \frac{(-1)^{n-1}}{n}$ converges as $a_n = \frac{1}{n} > 0$, a_n decreases and $\lim a_n = 0$.

2.1.2. *Comparison Test.*

Theorem 16 (First Comparison Test for Convergence)**.** *Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$.*

(1) If $a_n \leq b_n$, $\forall n \geq n_0$ and $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.
(2) If $a_n \geq b_n$, $\forall n \geq n_0$ and $\sum\limits_{n=1}^{\infty} b_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Theorem 17 (Another Formulation of the Comparison Test for Convergence)**.** *Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$, then if $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ exists, then $\sum a_n$ and $\sum b_n$ converge or diverge simultaneously.*

2.1.3. *d'Alembert Criteria (Ratio Test).*

Definition 15 (Absolute and conditional convergence)**.** The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

Theorem 18. *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 19 (d'Alembert Criteria (Ratio Test))**.**

(1) If $\lim\limits_{n\to\infty} \left|\frac{a_{n-1}}{a_n}\right| = L < 1$ then $\sum a_n$ converges absolutely.
(2) If $\lim\limits_{n\to\infty} \left|\frac{a_{n-1}}{a_n}\right| = L > 1$(including $L = \infty$), then $\sum a_n$ converges diverges.
(3) If $L = 1$, the test does not apply.

2.1.4. *Cauchy Criteria (Cauchy Root Test).*

Theorem 20 (Cauchy Criteria (Cauchy Root Test))**.**

(1) If $\overline{\lim} \sqrt[n]{|a_n|} = L < 1$ then $\sum a_n$ converges absolutely.
(2) If $\overline{\lim} \sqrt[n]{|a_n|} = L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
(3) If $L = 1$, the test does not apply.