DIFFERENTIAL AND INTEGRAL CALCULUS
ASSIGNMENT 8

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Exercise 1.
Calculate the following integrals

(1) $\iint\limits_D y \,\mathrm{d}x \,\mathrm{d}y$ where D is the region bounded by the curves $y^2 = 4+4x$, $y^2 = 4-4x$, $y = 0$.
Hint: Use the change of variables $u = 4x$, $v = y^2$

(2) $\iint\limits_D y \,\mathrm{d}x \,\mathrm{d}y$ where D is the region bounded by the curves $y = x$, $y = 3x$, $xy = 1$, $xy = 3$.
Hint: Use the change of variables $x = \frac{u}{v}$, $y = v$.

(3) $\iint\limits_D e^{\frac{x+y}{x-y}} \,\mathrm{d}x \,\mathrm{d}y$ where D is the region in the fourth quadrant bounded by the lines $y = 0$, $x = 0$, $y = x-1$, $y = x-2$.

(4) $\int\limits_0^2 \int\limits_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \,\mathrm{d}y \,\mathrm{d}x$.

Solution 1.

(1) The curves $y^2 = 4+4x$ and $y^2 = 4-4x$ intersect at $(0,-2)$ and $(0,2)$.
Therefore, $y \in [0,2]$.
Let

$$u = 4x$$
$$v = y^2$$

Therefore,

$$\begin{aligned} y^2 &= 4+4x & &\to & v &= 4+u \\ y^2 &= 4-4x & &\to & v &= 4-u \\ y &= 0 & &\to & u &= 0 \end{aligned}$$

Therefore, as $y \in [0,2]$, $v \in [0,4]$.
Therefore, the domain $D = \{(x,y) | -\frac{y^2-4}{4} \le x \le \frac{y^2-4}{4}, 0 \le y \le 2\}$ is transformed to $\Delta = \{(u,v) | v-4 \le u \le 4-v, 0 \le v \le 4\}$.

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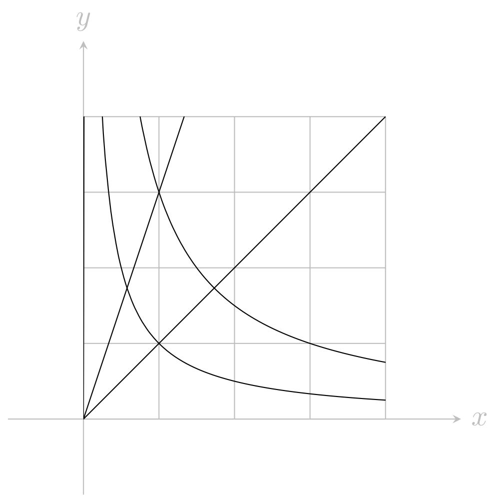
The Jacobian is

$$
\begin{aligned}
J &= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \\
\therefore \frac{1}{J} &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\
&= \begin{vmatrix} 4 & 0 \\ 0 & 2y \end{vmatrix} \\
&= 8y \\
\therefore J &= \frac{1}{8y}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\iint\limits_D y \, \mathrm{d}x \, \mathrm{d}y &= \iint\limits_\Delta y|J| \, \mathrm{d}u \, \mathrm{d}v \\
&= \int\limits_0^4 \int\limits_{v-4}^{4-v} y \frac{1}{|8y|} \, \mathrm{d}u \, \mathrm{d}v \\
&= \int\limits_0^4 \int\limits_{v-4}^{4-v} \frac{1}{8} \, \mathrm{d}u \, \mathrm{d}v \\
&= \frac{1}{8} \int\limits_0^4 (4 - v - v + 4) \, \mathrm{d}v \\
&= \frac{1}{8} \int\limits_0^4 (8 - 2v) \, \mathrm{d}v \\
&= \frac{1}{8} \left(8 \cdot 4 - 4^2 \right) \\
&= 2
\end{aligned}
$$



(2) Let

$$x = \frac{u}{v}$$
$$y = v$$

Therefore,

$$\begin{aligned}
y &= x & &\rightarrow & \frac{u}{v} &= v \\
\therefore y &= x & &\rightarrow & u &= v^2 \\
y &= 3x & &\rightarrow & \frac{u}{v} &= 3v \\
\therefore y &= x & &\rightarrow & u &= 3v^2 \\
xy &= 1 & &\rightarrow & u &= 1 \\
xy &= 3 & &\rightarrow & u &= 3
\end{aligned}$$

Therefore, the Jacobian is

$$\begin{aligned}
J &= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} \\
&= \frac{1}{v}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\iint\limits_D y \, \mathrm{d}x \, \mathrm{d}y &= \iint\limits_\Delta v|J| \, \mathrm{d}u \, \mathrm{d}v \\
&= \iint\limits_\Delta v \cdot \frac{1}{v} \, \mathrm{d}u \, \mathrm{d}v \\
&= \int\limits_1^3 \int\limits_{\sqrt{\frac{u}{3}}}^{\sqrt{u}} \mathrm{d}v \, \mathrm{d}u \\
&= \int\limits_1^3 \sqrt{\frac{u}{3}} - \sqrt{u} \, \mathrm{d}u \\
&= \left(\frac{1}{\sqrt{3}} - 1\right) \int\limits_1^3 \sqrt{u} \, \mathrm{d}u \\
&= \left(\frac{1}{\sqrt{3}} - 1\right) \frac{2}{3} x^{\frac{3}{2}} \bigg|_1^3 \\
&= \left(\frac{1}{\sqrt{3}} - 1\right) \left(\frac{2}{3} 3\sqrt{3} - \frac{2}{3}\right) \\
&= \left(\frac{1}{\sqrt{3}} - 1\right) \left(2\sqrt{3} - \frac{2}{3}\right)
\end{aligned}$$

(3) Let

$$u = x + y$$
$$v = x - y$$

Therefore,

$$\begin{aligned} x &= 0 && \to && u = -v \\ y &= 0 && \to && u = v \\ y &= x - 1 && \to && v = 1 \\ y &= x - 2 && \to && v = 2 \end{aligned}$$

Therefore, the Jacobian is

$$\begin{aligned} J &= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \\ \therefore \frac{1}{J} &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ &= -2 \end{aligned}$$

Therefore,

$$\begin{aligned} \iint\limits_D e^{\frac{x+y}{x-y}} \, \mathrm{d}x \, \mathrm{d}y &= \iint\limits_\Delta e^{\frac{u}{v}} |J| \, \mathrm{d}u \, \mathrm{d}v \\ &= 2 \int\limits_1^2 \int\limits_{-v}^{v} e^{\frac{u}{v}} \, \mathrm{d}u \, \mathrm{d}v \\ &= 2 \int\limits_1^2 v e^{\frac{u}{v}} \Big|_{-v}^{v} \\ &= 2 \int\limits_1^2 \left(v e^{\frac{v}{v}} - v e^{\frac{-v}{v}} \right) \mathrm{d}v \\ &= 2 \int\limits_1^2 \left(v e - \frac{v}{e} \right) \mathrm{d}v \\ &= 2 \left(e - \frac{1}{e} \right) \int\limits_1^2 v \, \mathrm{d}v \\ &= 2 \left(e - \frac{1}{e} \right) \frac{3}{2} \\ &= 3 \left(e - \frac{1}{e} \right) \end{aligned}$$

(4)

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, \mathrm{d}y \, \mathrm{d}x$$

Therefore,

$$\begin{aligned} &0 \le x \le 2 \\ &0 \le y \le \sqrt{2x-x^2} \end{aligned}$$

Therefore, the boundary is

$$\begin{aligned} y &= \sqrt{2x-x^2} \\ \therefore y^2 &= 2x-x^2 \\ \therefore x^2+y^2-2x &= 0 \\ \therefore (x-1)^2+y^2 &= 1 \end{aligned}$$

Therefore, the boundary is a circle with radius 1, centred at $(1,0)$.
Therefore, let

$$\begin{aligned} x-1 &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, \mathrm{d}y \, \mathrm{d}x &= \int_0^1 \int_0^{2\pi} r \cdot r \, \mathrm{d}\theta \, \mathrm{d}r \\ &= 2\pi \int_0^1 r^2 \, \mathrm{d}r \\ &= 2\pi \cdot \frac{1}{3} \\ &= \frac{2\pi}{3} \end{aligned}$$

Exercise 2.
Calculate the area bounded by the curves $x = qy$, $x = py$, $xy = b^2$, $xy = a^2$, where $q > p > 0$, $b > a > 0$.

Solution 2.
The boundaries are,

$$
\begin{aligned}
x &= qy \\
\therefore \frac{x}{y} &= q \\
x &= py \\
\therefore \frac{x}{y} &= p \\
xy &= b^2 \\
xy &= a^2
\end{aligned}
$$

Therefore, let

$$
\begin{aligned}
\frac{x}{y} &= u \\
xy &= v
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{x}{y} &= q && \rightarrow && u = q \\
\frac{x}{y} &= p && \rightarrow && u = p \\
xy &= b^2 && \rightarrow && v = b^2 \\
xy &= a^2 && \rightarrow && v = a^2
\end{aligned}
$$

Therefore, the Jacobian is

$$
\begin{aligned}
J &= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \\
\therefore \frac{1}{J} &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ y & x \end{vmatrix} \\
&= \frac{x}{y} + \frac{x}{y} \\
&= 2\frac{x}{y} \\
\therefore J &= 2\frac{y}{x}
\end{aligned}
$$

Therefore,

$$\begin{aligned}\left|\iint\limits_{D} \mathrm{d}x\,\mathrm{d}y\right| &= \left|\iint\limits_{\Delta} |J|\,\mathrm{d}u\,\mathrm{d}v\right| \\ &= \left|\int\limits_{p}^{q}\int\limits_{a^2}^{b^2} \left|2\frac{y}{x}\right| \mathrm{d}u\,\mathrm{d}v\right| \\ &= \left|\int\limits_{p}^{q} \frac{1}{u}\,\mathrm{d}u \int\limits_{a^2}^{b^2} \mathrm{d}v\right| \\ &= \left|\left(\ln\frac{q}{p}\right)\left(b^2 - a^2\right)\right|\end{aligned}$$

Exercise 3.

The change of variables

$$\begin{aligned} x &= u + 2v + 1 \\ y &= 2u + v + 1 \end{aligned}$$

maps the unit circle $u^2 + v^2 \le 1$ to a region D in the x-y plane.

(1) Find the area of D.

(2) Let R be some region in the u-v plane and let D be its image in the x-y plane under the above change of variables. Prove that $\mathrm{Area}(D) = 3\mathrm{Area}(R)$.

(3) Let

$$\begin{aligned} x &= \varphi(u, v) \\ y &= \psi(u, v) \end{aligned}$$

be some other change of variables from the u-v plane into the x-y plane, where φ, ψ are one-to-one, C^1 functions for which there exist $M > 0$, such that for every point in the u-v plane, the following inequalities are satisfied

$$\begin{aligned} |\varphi_u| &\le M \\ |\varphi_v| &\le M \\ |\psi_u| &\le M \\ |\psi_v| &\le M \end{aligned}$$

Let R be a region in the u-v plane and let D be its image under the above change of variables. Prove that $\mathrm{Area}(D) \le 2M^2 \cdot \mathrm{Area}(R)$.

Solution 3.

(1)

$$\begin{aligned} x &= u + 2v + 1 \\ y &= 2u + v + 1 \end{aligned}$$

Therefore, the Jacobian is

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= -3$$

$$\begin{aligned}
\iint\limits_D \mathrm{d}x\,\mathrm{d}y &= \iint\limits_\Delta |J|\,\mathrm{d}u\,\mathrm{d}v \\
&= \iint\limits_\Delta 3\,\mathrm{d}u\,\mathrm{d}v \\
&= \int\limits_{-1}^{1} \int\limits_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 3\,\mathrm{d}v\,\mathrm{d}u \\
&= 3\int\limits_{-1}^{1} \left(\sqrt{1-u^2} + \sqrt{1-u^2}\right) \mathrm{d}u \\
&= 6\int\limits_{-1}^{1} \sqrt{1-u^2}\,\mathrm{d}u \\
&= 6\left(\frac{1}{2}\left(\sqrt{1-u^2}u + \sin^{-1}(u)\right)\right)\Bigg|_{-1}^{1} \\
&= 3\left(\sin^{-1} 1 - \sin^{-1} -1\right) \\
&= 3\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \\
&= 3\pi
\end{aligned}$$

(2)

$$\begin{aligned}
x &= u + 2v + 1 \\
y &= 2u + v + 1
\end{aligned}$$

Therefore, the Jacobian is

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= -3$$

$$\begin{aligned}
\text{Area}(D) &= \iint\limits_D \mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_R |J|\,\mathrm{d}u\,\mathrm{d}v \\
&= |J| \iint\limits_R \mathrm{d}u\,\mathrm{d}v \\
&= 3 \iint\limits_R \mathrm{d}u\,\mathrm{d}v \\
&= 3 \cdot \text{Area}(R)
\end{aligned}$$

□

(3)

$$x = \varphi(u, v)$$
$$y = \psi(u, v)$$

Therefore, the Jacobian is

$$\begin{aligned}
J &= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \\
&= \begin{vmatrix} \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{vmatrix} \\
&= \varphi_u \psi_v - \varphi_v \psi_u
\end{aligned}$$

Therefore, as $|\varphi_u| \leq M$, $|\varphi_v| \leq M$ $|\psi_u| \leq M$, $|\psi_v| \leq M$,

$$J \leq M^2 + M^2$$
$$\therefore J \leq 2M^2$$

Therefore,

$$\begin{aligned}
\text{Area}(D) &= \iint\limits_D \mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_R |J|\,\mathrm{d}u\,\mathrm{d}v \\
&= |J| \iint\limits_R \mathrm{d}u\,\mathrm{d}v \\
&= |J|\text{Area}(R)
\end{aligned}$$

$$\therefore \text{Area}(D) \leq |2M^2|\text{Area}(R)$$
$$\therefore \text{Area}(D) \leq 2M^2\text{Area}(R)$$

□