DIFFERENTIAL AND INTEGRAL CALCULUS
ASSIGNMENT 5

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Exercise 1.
Find radius of convergence and domain of convergence of the following power series.

(1) $\sum\limits_{n=0}^{\infty} \frac{x^n}{n+2}$
(2) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n x^n}{n^{\frac{1}{3}}}$
(3) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n x^n}{n 2^n}$
(4) $\sum\limits_{n=1}^{\infty} \frac{(x+1)^n}{n(n+1)}$
(5) $\sum\limits_{n=0}^{\infty} \frac{n!(x-\pi)^n}{10^n}$
(6) $\sum\limits_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$
(7) $\sum\limits_{n=0}^{\infty} \left(\frac{n}{2}\right)^n (x+6)^n$
(8) $\sum\limits_{n=1}^{\infty} \frac{n x^n}{(2n-1)!}$
(9) $\sum\limits_{n=0}^{\infty} \frac{n! x^n}{(2n)!}$
(10) $\sum\limits_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$
(11) $\sum\limits_{n=1}^{\infty} (\ln n) x^n$

Solution 1.
(1)

$$\begin{aligned}
R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{n+2}{n+3} \right| \\
&= 1
\end{aligned}$$

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If $x = 1$, the series is $\sum \frac{1}{n+2}$ which diverges.
If $x = -1$, the series is $\sum \frac{(-1)^n}{n+2}$ which converges by Leibniz's criteria.
Therefore, the domain of convergence is $[-1, 1)$.

(2)

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(-1)^n (n+1)^{\frac{1}{3}}}{(-1)^{n+1} n^{\frac{1}{3}}} \right| \\ &= \lim_{n\to\infty} \left| -\frac{n^{\frac{1}{3}}}{(n+1)^{\frac{1}{3}}} \right| \\ &= 1 \end{aligned}$$

If $x = 1$, the series is $\sum \frac{(-1)^n}{n^{\frac{1}{3}}}$ which converges by Leibniz's criteria.
If $x = -1$, the series is $\sum \frac{(-1)^n(-1)^n}{n^{\frac{1}{3}}}$ which diverges.
Therefore, the domain of convergence is $(-1, 1]$.

(3)

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(-1)^n (n+1) 2^{n+1}}{(-1)^{n+1} n 2^n} \right| \\ &= \lim_{n\to\infty} \frac{2(n+1)}{n} \\ &= 2 \end{aligned}$$

If $x = 2$, the series is $\sum \frac{(-1)^n 2^n}{n 2^n} = \sum \frac{(-1)^n}{n}$ which converges by Leibniz's criteria.
If $x = -2$, the series is $\sum \frac{(-1)^n(-2)^n}{n2^n} = \sum \frac{1}{n}$ which diverges. Therefore, the domain of convergence is $(-2, 2]$.

(4)

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)(n+2)}{n(n+1)} \right| \\ &= 1 \end{aligned}$$

If $x = 1 - 1 = 0$, the series if $\sum \frac{1}{n(n+1)}$ which converges.
If $x = -1 - 1 = -2$, the series if $\sum \frac{(-1)^n}{n(n+1)}$ which converges.
Therefore, the domain of convergence is $[0, -2]$.

(5)

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{n!10^{n+1}}{(n+1)!10^n}\right| \\ &= \lim_{n\to\infty}\frac{10}{n+1} \\ &= 0 \end{aligned}$$

Therefore, the domain of convergence is $\{\pi\}$.

(6)

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{n^n}\right| \\ &= \infty \end{aligned}$$

Therefore, the domain of convergence is $\mathbb{R}$.

(7)

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{n^n2^{n+1}}{(n+1)^{n+1}2^n}\right| \\ &= \lim_{n\to\infty}\frac{2n^n}{(n+1)^{n+1}} \\ &= \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n\frac{1}{n+1} \\ &= 0 \end{aligned}$$

Therefore, the domain of convergence is $\{-6\}$.

(8)

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{n(2n+1)!}{(n+1)(2n-1)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{n(2n)(2n+1)}{(n+1)}\right| \\ &= \infty \end{aligned}$$

Therefore, the domain of convergence is $\mathbb{R}$.

(9)

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{n!(2n+2)!}{(n+1)!(2n)!} \right| \\ &= \lim_{n\to\infty} \left| \frac{(2n+1)(2n+2)}{n+1} \right| \\ &= \infty \end{aligned}$$

Therefore, the domain of convergence is $\mathbb{R}$.

(10) Let

$$3n = m$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

Therefore,

$$\begin{aligned} R &= \lim_{m\to\infty} \left| \frac{a_m}{a_{m+1}} \right| \\ &= \lim_{m\to\infty} \left| \frac{(m+1)!}{m!} \right| \\ &= \lim_{m\to\infty} m+1 \\ &= \infty \end{aligned}$$

Therefore, the domain of convergence is $\mathbb{R}$.

(11)

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{m\to\infty} \left| \frac{\ln n}{\ln(n+1)} \right| \\ &= \lim_{m\to\infty} \log_{n+1} n \\ &= 1 \end{aligned}$$

If $x = 1$, the series is $\sum \ln n$ which diverges.
If $x = -1$, the series is $\sum (-1)^n \ln n$ which diverges.
Therefore, the domain of convergence is $(-1, 1)$.

Exercise 2.

Calculate the sum of the following power series inside their radius of convergence, i.e. write these sums as an elementary function.

(1) $\sum_{n=0}^{\infty} n^2 x^{n-1}$

Solution 2.

(1) Let

$$f(x) = \sum_{n=0}^{\infty} n^2 x^{n-1}$$

Therefore,

$$\begin{aligned}\int f(x)\,\mathrm{d}x &= \sum_{n=0}^{\infty} n x^n \\ &= x \sum_{n=0}^{\infty} n x^{n-1}\end{aligned}$$

Let

$$g(x) = \sum_{n=0}^{\infty} n x^{n-1}$$

Therefore,

$$\begin{aligned}\int g(x)\,\mathrm{d}x &= \sum_{n=0}^{\infty} x^n \\ &= \frac{x}{1-x} \\ \therefore g(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1-x}\right) \\ &= \frac{1}{(1-x)^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\int f(x)\,\mathrm{d}x &= xg(x) \\ &= \frac{x}{(1-x)^2} \\ \therefore f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{(1-x)^2}\right) \\ &= \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2}\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} n^2 x^{n-1} = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2}$$

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Exercise 3.
Find the Taylor's series of the following functions.

(1) $f(x) = \cos x$ around 0.
(2) $f(x) = \cos x$ around 2π.
(3) $f(x) = e^x$ around 0.
(4) $f(x) = e^{-x}$ around 0.
(5) $f(x) = \ln(1 + x)$ around 0.

Solution 3.
(1)

$$\begin{aligned}
\frac{\mathrm{d} \cos x}{\mathrm{d}x} &= -\sin x \\
\frac{\mathrm{d}^2 \cos x}{\mathrm{d}x^2} &= -\cos x \\
\frac{\mathrm{d}^3 \cos x}{\mathrm{d}x^3} &= \sin x \\
\frac{\mathrm{d}^4 \cos x}{\mathrm{d}x^4} &= \cos x \\
&\vdots
\end{aligned}$$

Therefore, the Taylor series of $\cos x$ around 0 is

$$\begin{aligned}
f(x) &= \frac{\cos 0}{0!}x^0 + \frac{-\sin 0}{1!}x^1 + \dots \\
&= x^0 - \frac{x^2}{2!} + \dots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{\mathrm{d} \cos x}{\mathrm{d}x} &= -\sin x \\
\frac{\mathrm{d}^2 \cos x}{\mathrm{d}x^2} &= -\cos x \\
\frac{\mathrm{d}^3 \cos x}{\mathrm{d}x^3} &= \sin x \\
\frac{\mathrm{d}^4 \cos x}{\mathrm{d}x^4} &= \cos x \\
&\vdots
\end{aligned}$$

Therefore, the Taylor series of $\cos x$ around 0 is

$$\begin{aligned} f(x) &= \frac{\cos 2\pi}{0!}(x-2\pi)^0 + \frac{-\sin 2\pi}{1!}(x-2\pi)^1 + \dots \\ &= (x-2\pi)^0 - \frac{(x-2\pi)^2}{2!} + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (x-2\pi)^{2n}}{(2n)!} \end{aligned}$$

(3)

$$\begin{aligned} \frac{\mathrm{d}e^x}{\mathrm{d}x} &= e^x \\ \frac{\mathrm{d}^2e^x}{\mathrm{d}x^2} &= e^x \\ \frac{\mathrm{d}^3e^x}{\mathrm{d}x^3} &= e^x \\ \frac{\mathrm{d}^4e^x}{\mathrm{d}x^4} &= e^x \\ &\vdots \end{aligned}$$

Therefore, the Taylor series of e^x around 0 is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{e^0}{n!}(x-0)^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

(4)

$$\begin{aligned} \frac{\mathrm{d}e^{-x}}{\mathrm{d}x} &= -e^x \\ \frac{\mathrm{d}^2e^{-x}}{\mathrm{d}x^2} &= e^x \\ \frac{\mathrm{d}^3e^{-x}}{\mathrm{d}x^3} &= -e^x \\ \frac{\mathrm{d}^4e^{-x}}{\mathrm{d}x^4} &= e^x \\ &\vdots \end{aligned}$$

Therefore, the Taylor series of e^x around 0 is

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n e^0}{n!}(x-0)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}
\end{aligned}
$$

(5)

$$
\begin{aligned}
\frac{\mathrm{d} \ln(1+x)}{\mathrm{d}x} &= \frac{1}{1+x} \\
\frac{\mathrm{d}^2 \ln(1+x)}{\mathrm{d}x^2} &= -\frac{1}{(1+x)^2} \\
\frac{\mathrm{d}^3 \ln(1+x)}{\mathrm{d}x^3} &= \frac{2}{(1+x)^3} \\
\frac{\mathrm{d}^4 \ln(1+x)}{\mathrm{d}x^4} &= -\frac{6}{(1+x)^4} \\
\frac{\mathrm{d}^5 \ln(1+x)}{\mathrm{d}x^5} &= \frac{24}{(1+x)^5} \\
&\vdots
\end{aligned}
$$

Therefore, the Taylor series of $\ln(1+x)$ around 0 is

$$
\begin{aligned}
f(x) &= \ln(1+0) + \sum_{n=0}^{\infty} \frac{\frac{(-1)^n n!}{1+0}}{n!}(x-0)^n \\
&= \ln 1 + \sum_{n=0}^{\infty} (-1)^n x^n \\
&= \sum_{n=0}^{\infty} (-x)^n
\end{aligned}
$$