Differential and Integral Calculus : Recitations

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1 Instructor Information

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Part I
Sequences and Series

1 Sequences

Recitation 1 – Exercise 1.
Prove:

$$\lim_{n\to\infty} \frac{2n^2+n+1}{n^2+3} = 2$$

Recitation 1 – Solution 1.
Let

$$\varepsilon > 0$$

$$\begin{aligned}
\left|\frac{2n^2+n+1}{n^2+3} - 2\right| &= \left|\frac{2n^2+n+1-2n^2-6}{n^2+3}\right| \\
&= \left|\frac{n-5}{n^2+3}\right| \\
&\leq \left|\frac{n-5}{n^2}\right| \\
&\leq \frac{1}{n} \\
&< \varepsilon
\end{aligned}$$

Therefore, let $N = \left[\frac{1}{\varepsilon}\right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.
Therefore, $\lim_{n\to\infty} \frac{2n^2+n+1}{n^2+3} = 2$. □

Recitation 1 – Exercise 2.
Prove

$$\lim_{n\to\infty} \frac{n^3+\sin n+n}{2n^4} = 0$$

Recitation 1 – Solution 2.
Let $\varepsilon > 0$

$$\begin{aligned} \left|\frac{n^3+\sin n+n}{2n^4}\right| &\le \left|\frac{n^3+1+n}{2n^4}\right| \\ &\le \left|\frac{3n^3}{2n^4}\right| = \frac{3}{2}\cdot\frac{1}{n} < \varepsilon \end{aligned}$$

Therefore, let $N = \left[\frac{3}{2\varepsilon}\right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.

Therefore, $\lim_{n\to\infty} \frac{n^3+\sin n+n}{2n^4} = 0$ □

Recitation 1 – Exercise 3.
Calculate $\sqrt[3]{n^3+3n} - n$.

Recitation 1 – Solution 3.

$$a^n - b^n = (a-b)\cdot(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Therefore, let

$$\begin{aligned} a &= \sqrt[3]{n^3+3n} \\ b &= \sqrt[3]{n^3} \end{aligned}$$

$$\begin{aligned} a-b &= \frac{a^3-b^3}{a^2+ab+b^2} \\ \therefore \sqrt[3]{n^3+3n} - n &= \frac{n^3+3n-n^3}{(n^3+3n)^{2/3} + (n^3+3n)^{1/3}n + n^2} \\ &= \frac{3}{\left(\frac{n^3+3n}{n^{3/2}}\right)^{2/3} + \left(\frac{n^3+3n}{n^3}\right)^{1/3n} + n} \end{aligned}$$

Therefore, the limit is 0.

Recitation 1 – Exercise 4.
Prove

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Recitation 1 – Solution 4.

$$0 \le \frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\dots\frac{n}{n} \le \frac{1}{n}$$

Therefore, by the Sandwich Theorem, $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0$.

Recitation 1 – Exercise 5.

Let $a_1 = 3$, $a_{n+1} = 1 + \sqrt{6 + a_n}$. Prove that a_n converges and find its limit.

Recitation 1 – Solution 5.

If possible, let $\lim\limits_{n\to\infty} a_n = l$.

$$a_{n+1} = 1 + \sqrt{6 + a_n}$$

Taking the limit on both sides,

$$\begin{aligned} l &= 1 + \sqrt{6 + l} \\ \therefore l - 1 &= \sqrt{6 + l} \\ \therefore l &= \frac{3 \pm \sqrt{29}}{2} \end{aligned}$$

As $a_n \ge 0$, $l = \frac{3 + \sqrt{29}}{2}$.

$$\begin{aligned} a_2 &= 1 + \sqrt{6 + a_1} \\ &= 1 + \sqrt{6 + 3} \\ &= 4 \\ \therefore a_2 &> a_1 \end{aligned}$$

If possible, let $a_n \ge a_{n-1}$.
Therefore,

$$\begin{aligned} a_{n+1} &= 1 + \sqrt{6 + a_n} \\ &\ge 1 + \sqrt{6 + a_{n+1}} = a_n \end{aligned}$$

Therefore by induction, $\{a_n\}$ is monotonically increasing.

$$\begin{aligned} a_1 &= 3 \\ \therefore a_1 &\le 5 \end{aligned}$$

If possible, let $a_n \leq 5$.
Therefore,

$$a_{n+1} = 1 + \sqrt{6 + a_n} \leq q + \sqrt{11} \leq 5$$

Therefore by induction, $\{a_n\}$ is bounded from above by 5.

1.1 Limit of a Function by Heine

Definition 1.

$$\lim_{x \to x_0} f(x) = l$$

if for every sequence x_n, such that $\lim_{n \to \infty} x_n = x_0$,

$$\lim_{n \to \infty} f(x_n) = l$$

Theorem 1. *If f is continuous at x_0 and $x_n \to x_0$, then*

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right) = f_{x_0}$$

Recitation 2 – Exercise 1.
Calculate $\lim_{n \to \infty} \sqrt[n]{n}$.

Recitation 2 – Solution 1.
Let

$$f(x) = x^{1/x}$$

Therefore,

$$\begin{aligned} \lim_{x \to \infty} x^{1/x} &= \lim_{x \to \infty} e^{\cancelto{0}{\frac{\ln x}{x}}} \\ &= 1 \end{aligned}$$

1.2 Sub-sequences

Recitation 2 – Exercise 2.
Find all partial limits and $\overline{\lim}$ and $\underline{\lim}$ of

$$a_n = \left(\cos\frac{\pi n}{4}\right)^n$$

Recitation 2 – Solution 2.
Let $k, z \in \mathbb{Z}$

$$\begin{aligned}
\cos\frac{\pi n}{4} &= \cos\frac{\pi(n+k)}{4} \\
\therefore \frac{\pi n}{4} &= \frac{\pi(n+k)}{4} + 2\pi z \\
\therefore \pi n &= \pi(n+k) + 8\pi z \\
\therefore k &= 8z
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_{8k} &= \left(\cos\frac{\pi \cdot 8k}{4}\right)^{8k} \\
&= (\cos(2\pi k))^{8k} \\
&= 1 \\
a_{8k+1} &= \left(\cos\frac{\pi \cdot (8k+1)}{4}\right)^{8k+1} \\
&= \left(\cos\frac{\pi}{4}\right)^{8k+1} \\
&= \left(\frac{\sqrt{2}}{2}\right)^{8k+1} \\
a_{8k+2} &= \left(\cos\frac{\pi \cdot (8k+2)}{4}\right)^{8k+2} \\
&= \left(\cos\frac{\pi}{2}\right)^{8k+2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{k\to\infty} a_{8k} &= 1 \\
\lim_{k\to\infty} a_{8k+1} &= \lim_{k\to\infty}\left(\frac{\sqrt{2}}{2}\right)^{8k+1} \\
&= 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{k\to\infty} a_{8k+2} &= 0\\
\lim_{k\to\infty} a_{8k+3} &= 0\\
\lim_{k\to\infty} a_{8k+4} &= \lim_{k\to\infty} (-1)^{8k+4}\\
&= 1\\
\lim_{k\to\infty} a_{8k+5} &= 0\\
\lim_{k\to\infty} a_{8k+6} &= 0\\
\lim_{k\to\infty} a_{8k+7} &= 0
\end{aligned}$$

Therefore, $\{a_n\}$ has two partial limits, 0 and 1.

$$\overline{\lim} a_n = 1$$
$$\underline{\lim} a_n = 0$$

2 Series

Definition 2 (Convergence of a series)**.** Let $\{a_n\}$ be a sequence. Let S_n be a sequence of partial sums of a_n, s.t.

$$S_n = \sum_{k=1}^{n} a_k$$

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to l if

$$\lim_{n\to\infty} S_n = l$$

that is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n$$

Recitation 2 – Exercise 3.

Does $\displaystyle\sum_{k=0}^{\infty} q^k$ where $-1 < q < 1$ converge?

Recitation 2 – Solution 3.

$$\begin{aligned}
\sum_{k=0}^{\infty} q^k &= \lim_{n\to\infty} \sum_{k=0}^{n} q^k \\
&= \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q} \\
&= \frac{1}{1-q}
\end{aligned}$$

Therefore, the series converges.

Recitation 2 – Exercise 4.

Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converge?

Recitation 2 – Solution 4.

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k(k+1)} &= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \\
&= 1
\end{aligned}$$

Recitation 2 – Exercise 5.

Does $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$ converge?

Recitation 2 – Solution 5.

$$\begin{aligned}
&\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e \\
\therefore\ &\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k \neq 0
\end{aligned}$$

Therefore, the necessary condition is nt satisfied. Hence, the series does not converge.

2.1 Comparison Tests for Positive Series

Theorem 2 (First Comparison Test). *If* $a_n \geq 0$, $b_n \geq 0$, *and* $a_n \leq b_n$, *then*

1. *If* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
2. *If* $\sum a_n$ *diverges, then* $\sum b_n$ *diverges.*

Theorem 3 (Second Comparison Test). *If* $a_n \geq 0$, $b_n \geq 0$ *and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l$$

where $0 < l < \infty$, *then* $\sum a_n$ *and* $\sum b_n$ *converge or diverge simultaneously.*