Differential and Integral Calculus

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1 Lecturer Information

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2 Required Reading

Protter and Morrey: *A first Course in Real Analysis*, UTM Series, Springer-Verlag, 1991

3 Additional Reading

Thomas and Finney, *Calculus and Analytic Geometry*, 9th edition, Addison-Wesley, 1996

Part I
Sequences and Series

1 Sequences

Definition 1 (Sequence)**.** A sequence of real numbers is a set of numbers which are written in some order. There are infinitely many terms in a sequence. It is denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Example 1. $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is called the harmonic sequence.

$$a_n = \frac{1}{n}$$

Example 2. $1, -\frac{1}{2}, \frac{1}{3}, \ldots$ is called the alternating harmonic sequence.

$$a_n = (-1)^{n+1}\frac{1}{n}$$

Example 3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$$a_n = \frac{n}{n+1}$$

Example 4. $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \ldots$

$$a_n = \frac{n+1}{3^n}$$

Example 5. The Fibonacci sequence is given by

$$f_n = \begin{cases} 1 & ; \quad n = 1, 2 \\ f_{n-1} + f_{n-2} & ; \quad n \geq 3 \end{cases}$$

Example 6. A geometric sequence is given by

$$a_n = a_1 q^{n-1}$$

where q is called the common ratio.

Example 7. A geometric sequence is given by

$$a_n = a_1 + d(n-1)$$

where d is called the common difference.

Definition 2 (Equal sequences)**.** Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equal if $a_n = b_n$, $\forall n \in \mathbb{N}$.

Definition 3 (Sequences bounded from above)**.** $\{a_n\}$ is said to be bounded from above if $\exists M \in \mathbb{R}$, s.t. $a_n \leq M$, $\forall n \in \mathbb{N}$. Each such M is called an upper bound of $\{a_n\}$.

Definition 4 (Sequences bounded from below)**.** $\{a_n\}$ is said to be bounded from below if $\exists m \in \mathbb{R}$, s.t. $a_n \geq M$, $\forall n \in \mathbb{N}$. Each such M is called an lower bound of $\{a_n\}$.

Definition 5. $\{a_n\}$ is said to be bounded if it is bounded from below and bounded from above.

Example 8. The sequence $a_n = n^2 + 2$ is not bounded from above but is bounded from below, by all $m \leq 3$.

Example 9. $\left\{\dfrac{2n-1}{3n}\right\}$ is bounded.

$$m = 0 \leq \frac{2n-1}{3n} \leq \frac{2n}{3n} = \frac{2}{3} = M$$

Definition 6 (Monotonic increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \leq a_{n+1}$, $\forall n \geq n_0$.

Definition 7 (Monotonic decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \geq a_{n+1}$, $\forall n \geq n_0$.

Definition 8 (Strongly increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n < a_{n+1}$, $\forall n \geq n_0$.

Definition 9 (Strongly decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n > a_{n+1}$, $\forall n \geq n_0$.

Example 10. The sequence $\left\{\dfrac{n^2}{2^n}\right\}$ is strongly decreasing. However, this is not evident by observing the first few terms. $\dfrac{1}{2}, 1, \dfrac{9}{8}, 1, \dfrac{25}{32}, \ldots$

$$\begin{aligned} a_n &> a_{n+1} \\ \iff \frac{n^2}{2^n} &> \frac{(n+1)^2}{2^{n+1}} \\ \iff 2n^2 &> (n+1)^2 \\ \iff \sqrt{2}n &> n+1 \\ \iff n(\sqrt{2}-1) &> 1 \\ \iff n &> \frac{1}{\sqrt{2}-1} \\ \iff n &> 3 \end{aligned}$$

Exercise 1.
Is $a_n = (-1)^n$ monotonic?

Solution 1.
The sequence $-1, 1, -1, 1, \ldots$ is not monotonic.

1.1 Limit of a Sequence

Definition 10. Let $\{a_n\}$ be a given sequence. A number L is said to be the limit of the sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $|a_n - L| < \varepsilon$, $\forall n \geq n_0$. That is, there are infinitely many terms inside the interval and a finite number of terms outside it.

Example 11. The sequence $\{\frac{1}{n}\}$ tends to 0, i.e. for any open interval $(-\varepsilon, \varepsilon)$, there are finite number of terms of the sequence outside the interval, and therefore there are infinitely many terms inside the interval.

Exercise 2.
Prove

$$\lim_{n\to\infty} \frac{n+2}{2n-1} = \frac{1}{2}$$

Exercise 3.

Prove that 2 is not a limit of $\left\{\dfrac{3n+1}{n}\right\}$.

Theorem 1. *If a sequence $\{a_n\}$ has a limit L then the limit is unique.*

Proof. If possible let there exist two limits L_1 and L_2. Therefore, $\forall \varepsilon > 0$, there exist a finite number of terms in the interval $(L_1 - \varepsilon, L_1 + \varepsilon)$. Therefore, there exist a finite number of terms in the interval $(L_2 - \varepsilon, L_2 + \varepsilon)$. This contradicts the definition of a limit. Therefore, the limit is unique. □

Theorem 2. *If a sequence $\{a_n\}$ has limit L, then the sequence is bounded.*

Theorem 3. *Let*

$$\lim_{n\to\infty} a_n = a$$
$$\lim_{n\to\infty} b_n = b$$

and let c be a constant. Then,

$$\begin{aligned}
\lim c &= c \\
\lim(ca_n) &= c \lim a_n \\
\lim(a_n \pm b_n) &= \lim a_n \pm \lim b_n \\
\lim(a_n b_n) &= \lim a_n \lim b_n \\
\lim\left(\frac{a_n}{b_n}\right) &= \frac{\lim a_n}{\lim b_n} \quad (\text{ if } \lim b \neq 0)
\end{aligned}$$

Theorem 4. *Let $\{b_n\}$ be bounded and let $\lim a_n = 0$. Then,*

$$\lim(a_n b_n) = 0$$

Theorem 5 (Sandwich Theorem)**.** *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences. If*

$$\lim a_n = \lim b_n = L$$

and $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n \leq b_n \leq c_n$. Then,

$$\lim b_n = L$$

Exercise 4.
Calculate $\lim_{n\to\infty} \sqrt[n]{2^n+3^n}$

Solution 4.

$$\sqrt[n]{3^n} \leq \sqrt[n]{2^n+3^n} \leq \sqrt[n]{3^n+3^n} = \sqrt[3]{2\cdot 3^n}$$
$$\therefore 3 \leq \sqrt[n]{2^n+3^n} \leq 3\sqrt[n]{2}$$

Therefore, by the Sandwich Theorem, $\lim_{n\to\infty} \sqrt[n]{2^n+3^n} = 3$.

Theorem 6. *Any monotonically increasing sequence which is bounded from above converges. Similarly, any monotonically decreasing sequence which is bounded from below converges.*

Exercise 5.

Prove that there exists a limit for $a_n = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}}_{n \text{ times}}$ and find it.

Solution 5.

$$a_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = a_2$$

If possible, let

$$a_{n-1} < a_n$$
$$\therefore \sqrt{2+a_{n-1}} < \sqrt{2+a_n}$$
$$\therefore a_n < a_{n+1}$$

Hence, by induction, $\{a_n\}$ is monotonically increasing.

$$a_1 = \sqrt{2} \leq 2$$

If possible, let

$$a_n \leq 2 \therefore \sqrt{2+a_n} \qquad \leq \sqrt{2+2}$$
$$\therefore a_{n+1} \leq 2$$

Hence, by induction, $\{a_n\}$ is bounded from above by 2. Therefore, by , $\{a_n\}$ converges.