DIFFERENTIAL AND INTEGRAL CALCULUS
ASSIGNMENT 6

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Exercise 1.
Check the pointwise and uniform convergence of the following sequences of functions in the given domain

(1) $f_n(x) = \frac{x^n}{3+2x^n}$ in $[0, 2]$.
(2) $f_n(x) = x^{n+1}e^{-nx}$ in $[0, \pi]$.
(3) $f_n(x) = \frac{n^n}{n^n x - (n+1)^n}$.
(4) $f_n(x) = \begin{cases} \frac{x+nx+1}{n} & ; \quad n-1 < x < n \\ x & ; \quad \text{otherwise} \end{cases}$ in $[0, \infty)$.
(5) $f_n(x) = n \ln\left(1 + \frac{1}{nx^2}\right)$ in $[0, 2]$.
(6) $f_n(x) = \frac{nx\sin(nx)}{n+x^4}$ in $[0, 1]$.

Solution 1.
(1)

$$\lim_{n\to\infty} \frac{x^n}{3+2x^n} = \lim_{n\to\infty} \frac{1}{\frac{3}{x^n}+2}$$

Therefore,

$$\lim_{n\to\infty} \frac{x^n}{3+2x^n} = \begin{cases} 0 & ; \quad 0 \le x < 1 \\ \frac{1}{5} & ; \quad x = 1 \\ \frac{1}{2} & ; \quad 1 < x \le 2 \end{cases}$$

Therefore, $f_n(x)$ converges pointwise to

$$f(x) = \begin{cases} 0 & ; \quad 0 \le x < 1 \\ \frac{1}{5} & ; \quad x = 1 \\ \frac{1}{2} & ; \quad 1 < x \le 2 \end{cases}$$

As $f(x)$ is continuous but all $f_n(x)$ are, the convergence is not uniform.

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(2)

$$\begin{aligned}\lim_{n\to\infty} x^{n+1}e^{-nx} &= \lim_{n\to\infty} \frac{x^{n+1}}{e^{nx}}\\ &= \lim_{n\to\infty} \frac{x^{n+1}\ln x}{xe^{nx}}\\ &= \frac{\ln x}{x}\lim_{n\to\infty} \frac{x^{n+1}}{e^{nx}}\\ \therefore \lim_{n\to\infty} x^{n+1}e^{-nx} &= 0\end{aligned}$$

Therefore, $f_n(x)$ converges pointwise to $f(x) = 0$.

$$\sup_{[0,\pi]} |f_n(x) - f(x)| = \sup_{[0,\pi]} \left|x^{n+1}e^{-nx}\right|$$

As the function is continuous and the interval is closed, by the Weierstrass theorem, the function has a maximum. Therefore,

$$\sup_{[0,\pi]} |f_n(x) - f(x)| = \max_{[0,\pi]} \left|x^{n+1}e^{-nx}\right|$$

Differentiating to find the maximum,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{n+1}e^{-nx}\right) = e^{-nx}x^n(-nx+n+1)$$

Therefore,

$$\begin{aligned} & e^{-nx}x^n(-nx+n+1) = 0\\ \iff & x^n(-nx+n+1) = 0\end{aligned}$$

$\iff$

$$x = 0 \qquad \text{or} \qquad x = \frac{n+1}{n}$$

Therefore, the values of the functions at the critical points and the end points are

$$\begin{aligned} f_n(0) &= 0\\ f_n\left(\frac{n+1}{n}\right) &= \left(\frac{n+1}{n}\right)^{n+1} e^{-n-1}\\ f_n(\pi) &= \pi^{n+1}e^{-n\pi}\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\sup_{[0,\pi]} |f_n - f(x)| &= \lim_{[0,\pi]}\max_{[0,\pi]} x^{n+1}e^{-nx}\\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} e^{-n-1}\\ &= 0\end{aligned}$$

Therefore, the convergence is uniform.

(3)

$$\begin{aligned}\lim_{n\to\infty}\frac{n^n}{n^n x-(n+1)^n} &= \lim_{n\to\infty}\frac{1}{x-\left(\frac{n+1}{n}\right)^n}\\ &= \lim_{n\to\infty}\frac{1}{x-\left(1+\frac{1}{n}\right)^n}\\ &= \frac{1}{x-e}\end{aligned}$$

Therefore, as the function $\frac{1}{x-e}$ does not exist for $x=e$, $f_n(x)$ does not converge pointwise.
Hence there is also no uniform convergence

(4)

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & ; \quad n-1<x<n \\ x & ; \quad \text{otherwise}\end{cases}$$

Therefore, $f_n(x)$ converges pointwise to $f(x)=x$.

$$\sup_{[0,\infty)}|f_n(x)-f(x)| = \begin{cases} \frac{x(1+n)+1}{n} & ; \quad n-1<x<n \\ x & ; \quad \text{otherwise}\end{cases}$$

Therefore,

$$\lim_{n\to\infty}\sup_{[0,\infty)}|f_n(x)-f(x)| = \infty$$

Therefore, the convergence is not uniform.

(5)

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(1+\frac{1}{nx^2}\right) &= \lim_{n\to\infty}\ln\left(1+\frac{1}{nx^2}\right)^n\\ &= \lim_{n\to\infty}\ln\left(\left(1+\frac{1}{nx^2}\right)^{nx^2}\right)^{\frac{1}{x^2}}\\ &= \ln e^{\frac{1}{x^2}}\\ &= \frac{1}{x^2}\end{aligned}$$

Therefore, as the function $\frac{1}{x^2}$ does not exist for $x=0$, $f_n(x)$ does not converge pointwise.
Hence there is also no uniform convergence

(6)

$$\begin{aligned}\lim_{n\to\infty}\frac{nx\sin(nx)}{n+x^4} &= \lim_{n\to\infty}\frac{x\sin(nx)}{1+\frac{x^4}{n}}\\ &= \lim_{n\to\infty} x\sin(nx)\end{aligned}$$

Therefore, the limit does not exist.
Hence, $f_n(x)$ does converges neither pointwise nor uniformly.

Exercise 2.
Prove or disprove

(1) Let $\{f_n(x)\}$ be a sequence of functions defined in the interval $[a, b]$. Then f_n converges to the constant zero function $f(x) = 0$ if and only if the sequence $|f_n(x)|$ converges to the constant zero function.
(2) Let $\sum\limits_{n=1}^{\infty} a_n$ be a positive convergent series and let $\{f_n(x)\}$ be a sequence of functions defined in the interval $[a, b]$ satisfying $|f_n(x) - f_{n-1}(x)| \leq a_n$ for every $x \in [a, b]$. Then the sequence f_n converges uniformly in $[a, b]$ (Hint: Cauchy's criterion for uniform convergence).
(3) Let $f(x)$ be defined for every $x > 0$ and assume that $\lim\limits_{n\to\infty} f(x) = 0$. For every $x > 0$ define $f_n(x) = f(nx)$.
   (a) f_n converges pointwise in $(0, \infty)$.
   (b) f_n converges uniformly in $(0, \infty)$.

Solution 2.
(1) f_n converges to f if and only if

$$\lim_{n\to\infty} \sup_{[a,b]} |f_n(x) - f(x)| = 0$$

$$\iff \lim_{n\to\infty} \sup_{[a,b]} |f_n(x) - 0| = 0 \iff \lim_{n\to\infty} \sup_{[a,b]} \big||f_n(x)| - 0\big| = 0$$

if and only if $|f_n(x)|$ converges to the constant function 0. □

(3)

$$\begin{aligned}\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} f(nx) \\ &= 0\end{aligned}$$

Therefore, $f_n(x)$ converges uniformly to the constant zero function.

$$\begin{aligned}\sup_{(0,\infty)} |f_n(x) - f(x)| &= \sup_{(0,\infty)} |0 - 0| \\ &= 0\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \sup_{(0,\infty)} |f_n(x) - f(x)| = 0$$

Therefore, the convergence is uniform.

Exercise 3.
Let $\{f_n\}$ be a sequence of continuous functions that converge uniformly to f in $[a, b]$. Prove that if $x_n \to x_0$ then $f_n(x_n) \to f(x_0)$.

Solution 3.

As $f_n(x)$ converges uniformly to $f(x)$, it also converges pointwise. Therefore,

$$\begin{aligned}\lim_{n\to\infty} f_n(x_n) &= f(x_n)\\ \therefore \lim_{x_n\to x_0}\lim_{n\to\infty} f_n(x_n) &= \lim_{n\to\infty} f_n(x_0)\\ &= f(x_0)\end{aligned}$$