Differential and Integral Calculus : Recitations

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1 Instructor Information

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Part I
Sequences and Series

1 Sequences

Recitation 1 − Exercise 1.
Prove:
$$\lim_{n\to\infty} \frac{2n^2+n+1}{n^2+3} = 2$$

Recitation 1 − Solution 1.
Let
$$\varepsilon > 0$$

$$
\begin{aligned}
\left| \frac{2n^2+n+1}{n^2+3} - 2 \right| &= \left| \frac{2n^2+n+1-2n^2-6}{n^2+3} \right| \\
&= \left| \frac{n-5}{n^2+3} \right| \\
&\leq \left| \frac{n-5}{n^2} \right| \\
&\leq \frac{1}{n} \\
&< \varepsilon
\end{aligned}
$$

Therefore, let $N = \left[\dfrac{1}{\varepsilon} \right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.

Therefore, $\lim_{n\to\infty} \dfrac{2n^2+n+1}{n^2+3} = 2$. $\qquad\qquad\square$

Recitation 1 − Exercise 2.
Prove
$$\lim_{n\to\infty} \frac{n^3+\sin n+n}{2n^4} = 0$$

Recitation 1 – Solution 2.

Let $\varepsilon > 0$

$$\left| \frac{n^3 + \sin n + n}{2n^4} \right| \leq \left| \frac{n^3 + 1 + n}{2n^4} \right|$$

$$\leq \left| \frac{3n^3}{2n^4} \right| = \frac{3}{2} \cdot \frac{1}{n} < \varepsilon$$

Therefore, let $N = \left[\dfrac{3}{2\varepsilon} \right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.

Therefore, $\displaystyle\lim_{n \to \infty} \frac{n^3 + \sin n + n}{2n^4} = 0$ $\qquad\qquad\qquad\qquad\qquad$ $\square$

Recitation 1 – Exercise 3.

Calculate $\sqrt[3]{n^3 + 3n} - n$.

Recitation 1 – Solution 3.

$$a^n - b^n = (a - b) \cdot (a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Therefore, let

$$a = \sqrt[3]{n^3 + 3n}$$
$$b = \sqrt[3]{n^3}$$

$$a - b = \frac{a^3 - b^3}{a^2 + ab + b^2}$$

$$\therefore \sqrt[3]{n^3 + 3n} - n = \frac{n^3 + 3n - n^3}{(n^3 + 3n)^{2/3} + (n^3 + 3n)^{1/3}n + n^2}$$

$$= \frac{3}{\left(\dfrac{n^3 + 3n}{n^{3/2}} \right)^{2/3} + \left(\dfrac{n^3 + 3n}{n^3} \right)^{1/3n} + n}$$

Therefore, the limit is 0.

Recitation 1 – Exercise 4.

Prove

$$\lim_{n \to \infty} \frac{n!}{n^n} = 0$$

Recitation 1 − Solution 4.

$$0 \leq \frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\ldots\frac{n}{n} \leq \frac{1}{n}$$

Therefore, by the Sandwich Theorem, $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.

Recitation 1 − Exercise 5.

Let $a_1 = 3$, $a_{n+1} = 1 + \sqrt{6 + a_n}$. Prove that a_n converges and find its limit.

Recitation 1 − Solution 5.

If possible, let $\lim\limits_{n\to\infty} a_n = l$.

$$a_{n+1} = 1 + \sqrt{6 + a_n}$$

Taking the limit on both sides,

$$l = 1 + \sqrt{6 + l}$$
$$\therefore l - 1 = \sqrt{6 + l}$$
$$\therefore l = \frac{3 \pm \sqrt{29}}{2}$$

As $a_n \geq 0$, $l = \dfrac{3 + \sqrt{29}}{2}$.

$$a_2 = 1 + \sqrt{6 + a_1}$$
$$= 1 + \sqrt{6 + 3}$$
$$= 4$$
$$\therefore a_2 > a_1$$

If possible, let $a_n \geq a_{n-1}$.
Therefore,

$$a_{n+1} = 1 + \sqrt{6 + a_n}$$
$$\geq 1 + \sqrt{6 + a_{n+1}} = a_n$$

Therefore by induction, $\{a_n\}$ is monotonically increasing.

$$a_1 = 3$$
$$\therefore a_1 \leq 5$$

If possible, let $a_n \leq 5$.
Therefore,

$$a_{n+1} = 1 + \sqrt{6 + a_n} \leq q + \sqrt{11} \leq 5$$

Therefore by induction, $\{a_n\}$ is bounded from above by 5.

1.1 Limit of a Function by Heine

Definition 1.

$$\lim_{x \to x_0} f(x) = l$$

if for every sequence x_n, such that $\lim_{n \to \infty} x_n = x_0$,

$$\lim_{n \to \infty} f(x_n) = l$$

Theorem 1. *If f is continuous at x_0 and $x_n \to x_0$, then*

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right) = f_{x_0}$$

Recitation 2 − Exercise 1.
Calculate $\lim_{n \to \infty} \sqrt[n]{n}$.

Recitation 2 − Solution 1.
Let

$$f(x) = x^{1/x}$$

Therefore,

$$\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} e^{\frac{\overset{0}{\cancel{\ln x}}}{x}}$$
$$= 1$$

1.2 Sub-sequences

Recitation 2 – Exercise 2.

Find all partial limits and $\overline{\lim}$ and $\underline{\lim}$ of

$$a_n = \left(\cos \frac{\pi n}{4} \right)^n$$

Recitation 2 – Solution 2.

Let $k, z \in \mathbb{Z}$

$$\cos \frac{\pi n}{4} = \cos \frac{\pi(n+k)}{4}$$

$$\therefore \frac{\pi n}{4} = \frac{\pi(n+k)}{4} + 2\pi z$$

$$\therefore \pi n = \pi(n+k) + 8\pi z$$

$$\therefore k = 8z$$

Therefore,

$$a_{8k} = \left(\cos \frac{\pi \cdot 8k}{4} \right)^{8k}$$

$$= (\cos(2\pi k))^{8k}$$

$$= 1$$

$$a_{8k+1} = \left(\cos \frac{\pi \cdot (8k+1)}{4} \right)^{8k+1}$$

$$= \left(\cos \frac{\pi}{4} \right)^{8k+1}$$

$$= \left(\frac{\sqrt{2}}{2} \right)^{8k+1}$$

$$a_{8k+2} = \left(\cos \frac{\pi \cdot (8k+2)}{4} \right)^{8k+2}$$

$$= \left(\cos \frac{\pi}{2} \right)^{8k+2}$$

Therefore,

$$\lim_{k \to \infty} a_{8k} = 1$$

$$\lim_{k \to \infty} a_{8k+1} = \lim_{k \to \infty} \left(\frac{\sqrt{2}}{2} \right)^{8k+1}$$

$$= 0$$

Similarly,

$$\lim_{k\to\infty} a_{8k+2} = 0$$

$$\lim_{k\to\infty} a_{8k+3} = 0$$

$$\lim_{k\to\infty} a_{8k+4} = \lim_{k\to\infty} (-1)^{8k+4}$$
$$= 1$$

$$\lim_{k\to\infty} a_{8k+5} = 0$$

$$\lim_{k\to\infty} a_{8k+6} = 0$$

$$\lim_{k\to\infty} a_{8k+7} = 0$$

Therefore, $\{a_n\}$ has two partial limits, 0 and 1.

$$\overline{\lim} a_n = 1$$
$$\underline{\lim} a_n = 0$$

2 Series

Definition 2 (Convergence of a series). Let $\{a_n\}$ be a sequence. Let S_n be a sequence of partial sums of a_n, s.t.

$$S_n = \sum_{k=1}^{n} a_k$$

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to l if

$$\lim_{n\to\infty} S_n = l$$

that is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n$$

Recitation 2 − Exercise 3.

Does $\sum_{k=0}^{\infty} q^k$ where $-1 < q < 1$ converge?

Recitation 2 – Solution 3.

$$\sum_{k=0}^{\infty} q^k = \lim_{n \to \infty} \sum_{k=0}^{n} q^k$$
$$= \lim_{n \to \infty} \frac{1 - q^{n+1}}{1 - q}$$
$$= \frac{1}{1 - q}$$

Therefore, the series converges.

Recitation 2 – Exercise 4.

Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converge?

Recitation 2 – Solution 4.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$
$$= \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$
$$= \lim_{n \to \infty} \left(1 - \frac{1}{n+1} \right)$$
$$= 1$$

Recitation 2 – Exercise 5.

Does $\displaystyle\sum_{k=1}^{\infty} \left(1 + \frac{1}{k} \right)^k$ converge?

Recitation 2 – Solution 5.

$$\lim_{k \to \infty} \left(1 + \frac{1}{k} \right)^k = e$$
$$\therefore \lim_{k \to \infty} \left(1 + \frac{1}{k} \right)^k \neq 0$$

Therefore, the necessary condition is nt satisfied. Hence, the series does not converge.

2.1 Comparison Tests for Positive Series

Theorem 2 (First Comparison Test)**.** *If $a_n \geq 0$, $b_n \geq 0$, and $a_n \leq b_n$, then*

1. *If $\sum b_n$ converges, then $\sum a_n$ converges.*

2. *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Theorem 3 (Second Comparison Test)**.** *If $a_n \geq 0$, $b_n \geq 0$ and*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = l$$

where $0 < l < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge simultaneously.

Recitation 3 – Exercise 1.
Suppose the sequence a_n satisfies the condition

$$a_{n+1} - a_n > \frac{1}{n}$$

$\forall n \in \mathbb{N}$.
Prove that $\lim_{n \to \infty} a_n = \infty$.

Recitation 3 – Solution 1.

$$\begin{aligned}
a_{n+1} &= a_{n+1} - a_n + a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_2 - a_1 + a_1 \\
&= \sum_{k=1}^{n} (a_{k+1} - a_k) + a_1 \\
&\geq \sum_{k=1}^{n} \frac{1}{k} + a_1
\end{aligned}$$

As the harmonic series diverges, $\sum\limits_{k=1}^{n} \frac{1}{k} + a_1$ diverges.

Therefore, by the First Comparison Test, $\sum\limits_{k=1}^{\infty} (a_{k+1} - a_k)$ diverges.

Recitation 3 – Exercise 2.
Check the convergence of $\sum\limits_{n=2}^{\infty} \dfrac{n + \sin n}{n^3 + \cos \pi n}$.

Recitation 3 − Solution 2.

The series is non-negative. Therefore, the comparison tests are applicable.

$$\frac{n + \sin n}{n^3 + \cos \pi n} \leq \frac{n + 1}{n^3 - 1}$$

$$\therefore \frac{n + \sin n}{n^3 + \cos \pi n} \leq \frac{2n}{n^3 - \dfrac{n^3}{2}} \qquad\qquad \leq \frac{4}{n^2}$$

Therefore, by the First Comparison Test, as $\dfrac{4}{n^2}$ converges, $\sum_{n=2}^{\infty} \dfrac{n + \sin n}{n^3 + \cos \pi n}$ also converges.

Recitation 3 − Exercise 3.

Let $a_n \geq 0$ and suppose that $\sum a_n$ converges. Prove that $\sum a_n^2$ converges. Is it true without the assumption $a_n \geq 0$?

Recitation 3 − Solution 3.

As $\sum a_n$ converges, $\lim\limits_{n \to \infty} a_n = 0$.

Therefore, $\exists N \in \mathbb{N}$, such that $\forall n > N$, $a_n < 1$.

Therefore, $\forall n > N$, $a_n^2 \leq a_n$. Hence, as $\sum\limits_{n=N+1}^{\infty} a_n$ converges, $\sum\limits_{n=N+1}^{\infty} a_n^2$ also converges. Hence, $\sum\limits_{n=1}^{\infty} a_n$ also converges.

This is not true without the assumption $a_n \geq 0$, as the argument $a_n^2 \leq a_n$ does not hold.

Recitation 3 − Exercise 4.

For which α does $\sum \left(\sqrt{n+1} - \sqrt{n} \right)^{\alpha/2}$ converge?

Recitation 3 − Solution 4.

$$\sum \left(\sqrt{n+1} - \sqrt{n} \right)^{\alpha/2} = \sum \left(\frac{n + 1 - n}{\sqrt{n+1} + \sqrt{n}} \right)^{\alpha/2}$$

$$= \sum \left(\frac{1}{\sqrt{n+1} - \sqrt{n}} \right)^{\alpha/2}$$

The series is positive. Therefore, the comparison tests are applicable.

Comparing with $\left(\dfrac{1}{\sqrt{n}}\right)^{\alpha/2}$,

$$\frac{\left(\dfrac{1}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2}}{\left(\dfrac{1}{\sqrt{n}}\right)^{\alpha/2}} = \left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2}$$

$$\therefore \lim_{n\to\infty} \left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2} = \left(\frac{1}{2}\right)^{\alpha/2}$$

$\sum \dfrac{1}{n^{\alpha/2}}$ converges if and only if $\dfrac{\alpha}{4} > 1$, i.e. if an inly if $\alpha > 4$.

By the Second Comparison Test, $\sum \dfrac{1}{n^{\alpha/4}}$ and the series converge or diverge simultaneously.

Therefore, the series converges for $\alpha > 4$.

Recitation 3 – Exercise 5.

Check the convergence of $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$.

Recitation 3 – Solution 5.

$\forall n \in \mathbb{N},\ \sin\dfrac{1}{n} \geq 0$

$$\lim_{n\to\infty} \frac{\sin\dfrac{1}{n}}{\dfrac{1}{n}} = 1$$

Therefore, by Second Comparison Test, $\sum \dfrac{1}{n}$ and $\sum \sin\dfrac{1}{n}$ diverge simultaneously.

2.2 d'Alembert Criteria (Ratio Test)

Definition 3 (Absolute and conditional convergence). The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

Theorem 4. *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 5 (d'Alembert Criteria (Ratio Test)).   *1. If*

$$\lim_{n\to\infty} \left| \frac{a_{n-1}}{a_n} \right| = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\lim_{n\to\infty} \left| \frac{a_{n-1}}{a_n} \right| = L > 1$$

(including $L = \infty$), then $\sum a_n$ converges diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 − Exercise 6.

Check the convergence of $\sum \dfrac{(-1)^n \cdot n^{1000}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$.

Recitation 3 − Solution 6.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n \cdot n^{1000}}{1 \cdots (2n-1)} \right| = \sum_{n=1}^{\infty} \frac{n^{1000}}{1 \cdots (2n-1)}$$

Therefore, by the d'Alembert Criteria (Ratio Test),

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^{1000}}{1 \cdots (2n+1)}}{\dfrac{n^{1000}}{1 \cdots (2n-1)}}$$

$$= \left(\frac{n+1}{n} \right)^{1000} \cdot \frac{1}{2n+1}$$

$$\therefore \lim_{nto\infty} \left(\frac{n+1}{n} \right)^{1000} \cdot \frac{1}{2n+1} = 0$$

$$\therefore \left(\frac{n+1}{n} \right)^{1000} \cdot \frac{1}{2n+1} < 1$$

Therefore, by the d'Alembert Criteria (Ratio Test), the series converges absolutely, and hence converges.

2.3   Cauchy Criteria (Cauchy Root Test)

Theorem 6 (Cauchy Criteria (Cauchy Root Test)).   *1. If*

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

(including $L = \infty$), then $\sum a_n$ diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 − Exercise 7.

Check the convergence of $\sum \left(1 - \dfrac{2}{n} \right)^{n^2}$.

Recitation 3 − Solution 7.

$$\sqrt[n]{\left(1 - \frac{2}{n} \right)^{n^2}} = \left(1 - \frac{2}{n} \right)^{n}$$

$$\therefore \lim_{n \to \infty} \left(1 - \frac{2}{n} \right)^{n} = e^{-2}$$

$$\therefore \lim_{n \to \infty} \left(1 - \frac{2}{n} \right)^{n} < 1$$

Therefore, by the Cauchy Criteria (Cauchy Root Test), $\sum \left(1 - \dfrac{2}{n} \right)^{n^2}$ converges.

2.4   Leibniz's Criteria

Definition 4 (Alternating series). The series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

Theorem 7 (Leibniz's Criteria for Convergence). *If an alternating series $\sum (-1)^{n-1} a_n$ with $a_n > 0$ satisfies*

1. $a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.

2. $\lim\limits_{n\to\infty} a_n = 0$

then the series $(-1)^{n-1}a_n$ converges.

Recitation 3 – Exercise 8.

Prove or disprove: There exists $\{a_n\}$, such that $\sum a_n$ converges and $\sum(1 + a_n)a_n$ diverges.

Recitation 3 – Solution 8.

Let $a_n = \dfrac{(-1)^n}{\sqrt{n}}$.

Therefore, by Leibniz's Criteria for Convergence, $\sum \dfrac{(-1)^n}{\sqrt{n}}$ converges.

$$\sum(1 + a_n)a_n = \sum\left(1 + \frac{(-1)^n}{\sqrt{n}}\right)\frac{(-1)^n}{\sqrt{n}}$$
$$= \sum\left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right)$$

Therefore, as $\sum \dfrac{1}{n}$ diverges, and $\sum \dfrac{(-1)^n}{\sqrt{n}}$ converges, $\sum\left(\dfrac{1}{n} + \dfrac{(-1)^n}{\sqrt{n}}\right)$ diverges.

2.5 Integral Test

Theorem 8 (Integral Test). *If $f(x) : [1, \infty) \to [0, \infty)$ is monotonically decreasing. Then, $\sum\limits_{n=1}^{\infty} f(n)$ and $\int\limits_{1}^{\infty} f(x)\,\mathrm{d}x$ converge or diverge simultaneously.*

Recitation 3 – Exercise 9.

Check the convergence of $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$

Recitation 3 – Solution 9.

Let

$$f(x) = \frac{1}{x \ln x}$$

$f(x)$ is monotonically decreasing. Therefore, the Integral Test is applicable. Therefore,

$$\int_{2}^{\infty} \frac{1}{x \ln x} \, \mathrm{d}x = \int_{\ln 2}^{\infty} \frac{1}{y} \, \mathrm{d}y$$
$$= \ln y |_{\ln 2}^{\infty}$$
$$= \infty$$

Therefore, by the integral test, $\sum \dfrac{1}{n \ln n}$ diverges.

Recitation 4 − Exercise 1.
Let $d_n \geq 0$ and suppose

$$\sum_{n=0}^{\infty} d_n = \infty$$

Prove that

$$\sum_{n=0}^{\infty} \frac{d_n}{1 + d_n} = \infty$$

Recitation 4 − Solution 1.
If possible, let d_n be a bounded sequence. Then there exists M, such that $d_n \leq M, \forall n \in \mathbb{N}$.
Therefore,

$$\frac{d_n}{1 + d_n} \geq \frac{d_n}{1 + M}$$

Therefore, by the Second Comparison Test, as $\sum d_n$ diverges, $\sum \dfrac{d_n}{1 + d_n}$ also diverges.

If d_n is not bounded, then there is a subsequence d_{n_k} which diverges. Therefore,

$$\frac{d_{n_k}}{1 + d_{n_k}} = \frac{1}{\dfrac{1}{d_{n_k}} + 1}$$
$$\therefore \lim_{k \to \infty} \frac{d_{n_k}}{1 + d_{n_k}} = 1$$

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Therefore,

$$\lim_{n \to \infty} \frac{d_n}{1 + d_n} \neq 0$$

Therefore, the necessary condition for convergence is not fulfilled. Therefore, the series converges.

Recitation 4 – Exercise 2.

Let

$$d_n = \begin{cases} 1 & ; & n = k^2, k \in \mathbb{N} \\ 0 & ; & n \neq k^2, k \in \mathbb{N} \end{cases}$$

Does $\sum \dfrac{d_n}{1 + n \cdot d_n}$ diverge?

Recitation 4 – Solution 2.

$$d_n = \begin{cases} 1 & ; & n = k^2, k \in \mathbb{N} \\ 0 & ; & n \neq k^2, k \in \mathbb{N} \end{cases}$$

$$\therefore \frac{d_n}{1 + n \cdot d_n} = \begin{cases} \dfrac{1}{1 + k^2} & ; & n = k^2, k \in \mathbb{N} \\ 0 & ; & n \neq k^2, k \in \mathbb{N} \end{cases}$$

As $\dfrac{1}{1 + k^2} \leq \dfrac{1}{k^2}$ and as $\dfrac{1}{k^2}$ converges, $\sum \dfrac{1}{1 + k^2}$ also converges.

Recitation 4 – Exercise 3.

Let a_n be a sequence such that $|a_{n+1} - a_n| \leq b_{n+1}$ for all $n \in \mathbb{N}$ where $\sum b_k$ converges. Prove that $\{a_n\}$ converges.

Recitation 4 – Solution 3.

Let $\varepsilon > 0$.

$$|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} + \cdots - a_n|$$
$$\leq \sum_{k=n+1}^{m} |a_k - a_{k-1}|$$
$$\leq \sum_{k=n+1}^{m} b_k$$

Therefore, as $\sum b_n$ converges, the series satisfies the Cauchy Criteria (Cauchy Root Test). Therefore, there exists N, such that $\forall m > n > N, \left| \sum\limits_{k=n+1}^{m} b_k \right| < \varepsilon$. Therefore, for $m > n > N$,

$$|a_m - a_n| \leq \sum_{k=n+1}^{m} b_n < \varepsilon$$

2.6 Power Series

Definition 5 (Power series). A power series around x_0 is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $\{a_n\}$ is a sequence of real numbers.

Theorem 9 (Abel's Theorem). *For every power series $\sum a_n (x - x_0)^n$, there exists $R \in [0, \infty]$, such that for all x satisfying $|x - x_0| < R$, the series converges and for all x satisfying $|x - x_0| > R$ the series diverges.*

Theorem 10 (Cauchy's Formula for Radius of Convergence).

$$R = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$$

Theorem 11 (Hadamard's Formula for Radius of Convergence). *If $\lim\limits_{n\to\infty} \left| \dfrac{a_n}{a_{n+1}} \right|$ exists, then*

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R$$

Recitation 4 − Exercise 4.

Find the domain of convergence of $\sum\limits_{n=1}^{\infty} \dfrac{(2x - 4)^n}{n}$.

Recitation 4 – Solution 4.

$$\sum_{n=1}^{\infty} \frac{(2x-4)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{n}$$

Therefore, by Cauchy's Formula for Radius of Convergence,

$$R = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$$

$$= \frac{1}{\lim_{n\to\infty} \sqrt[n]{\dfrac{2^n}{n}}}$$

$$= \frac{1}{\lim_{n\to\infty} \dfrac{2}{\sqrt[n]{n}}}$$

$$= \frac{1}{2}$$

Therefore, the series converges for

$$|x-2| < \frac{1}{2}$$

and diverges for

$$|x-2| > \frac{1}{2}$$

If $x = \dfrac{5}{2}$,

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{5}{2} - 2\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

Therefore, the series diverges.

If $x = \dfrac{3}{2}$,

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{3}{2} - 2\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

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Therefore, by Leibniz's Criteria for Convergence, the series converges.
Therefore, the domain of convergence is $\left[\dfrac{3}{2}, \dfrac{5}{2}\right)$.

Recitation 4 − Exercise 5.
Find the radius of convergence of $\sum_{n=0}^{\infty} n! x^{n!}$.

Recitation 4 − Solution 5.

$$\frac{1}{\sqrt[n]{a_n}} = x + x + 2x^2 + 6x^6 + 24x^{24} + \dots$$

Therefore,

$$a_n = \begin{cases} n & ; & n = k^2 \\ 0 & ; & n \neq k^2 \end{cases}$$

Therefore,

$$\begin{aligned}
R &= \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{a_n}} \\
&= \frac{1}{\lim\limits_{k\to\infty} \sqrt[k!]{k!}} \\
&= 1
\end{aligned}$$