Differential and Integral Calculus
Review Session

Aakash Jog

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Exercise 1.
Does the sequence of functions

$$f_n(x) = \frac{x^2}{x^2 + (nx-1)^2}$$

converge uniformly on $[0,1]$?

Solution 1.
$\forall x \in [0,1]$,

$$\lim_{n\to\infty} \frac{x^2}{x^2 + (nx-1)^2} = 0$$

Therefore, $f_n(x)$ converges pointwise to $f(x) = 0$ in $[0,1]$.

$$\lim_{n\to\infty} \sup_{[0,1]} |f_n(x) - f(x)| = \lim_{n-\infty} \sup_{[0,1]} \frac{x^2}{x^2 + (nx-1)^2}$$

As the function is continuous on the interval,

$$\sup_{[0,1]} \frac{x^2}{x^2 + (nx-1)^2} = \max_{[0,1]} \frac{x^2}{x^2 + (nx-1)^2}$$

Therefore, differentiating to find critical points,

$$\begin{aligned}
{f_n}'(x) &= \frac{2x\left(x^2 + (nx-1)^2\right) - x^2\left(2x + 2(nx-1)n\right)}{\left(x^2 + (nx-1)^2\right)^2} \\
&= \frac{2x(nx-1)^2 - 2nx^2(nx-1)}{\left(x^2 + (nx-1)^2\right)^2} \\
&= \frac{2x(nx-1)(nx-1-nx)}{\left(x^2 + (nx-1)^2\right)^2} \\
&= \frac{-2x(nx-1)}{\left(x^2 + (nx-1)^2\right)^2} \\
&= \frac{2x(1-nx)}{\left(x^2 + (nx-1)^2\right)^2}
\end{aligned}$$

Therefore, the function has critical points at $x = 0$ and $x = \frac{1}{n}$.
Therefore, checking the critical points and the end-points,

$$\begin{aligned}
f_n(0) &= 0 \\
f_n(1) &= \frac{1}{1 + (n-1)^2} \\
f_n\left(\frac{1}{n}\right) &= \frac{\frac{1}{n^2}}{\frac{1}{n^2} + 0} \\
&= 1
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{[0,1]} \frac{x^2}{x^2 + (nx-1)^2} &= \max_{[0,1]} \frac{x^2}{x^2 + (nx-1)^2} \\
&= 1
\end{aligned}$$

Therefore,

$$\lim_{n \to \infty} \sup_{[0,1]} \frac{x^2}{x^2 + (nx-1)^2} = 1$$

Therefore, as $\lim\limits_{n \to \infty} |f_n(x) - f(x)| \neq 0$, the sequence does not converge uniformly on $[0, 1]$.

Exercise 2.
Given $a, b > 0$, calculate the double integral $\iint\limits_D xy \, \mathrm{d}A$, where D is a quarter of the ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, for $x \geq 0$, $y \geq 0$.

Solution 2.
Let

$$\begin{aligned}
x &= ar\cos\theta \\
y &= br\sin\theta
\end{aligned}$$

Therefore,

$$\begin{aligned}
J &= \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} \\
&= \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} \\
&= abr
\end{aligned}$$

Therefore,

$$\begin{aligned}
\iint\limits_D xy\,\mathrm{d}A &= \int\limits_0^1 \int\limits_0^{\frac{\pi}{2}} ar\cos\theta \cdot br\sin\theta \cdot abr\,\mathrm{d}\theta\,\mathrm{d}r \\
&= a^2b^2 \int\limits_0^1 r^3\,\mathrm{d}r \int\limits_0^{\frac{\pi}{2}} \cos\theta\sin\theta\,\mathrm{d}\theta \\
&= \frac{1}{8}a^2b^2
\end{aligned}$$

Exercise 3.

Check if the following series converge conditionally, converge absolutely, or diverge.

1. $\sum\limits_{n=1}^{\infty}(-1)^n \sin^3 \frac{1}{\sqrt{n}}$

2. $\sum\limits_{n=1}^{\infty} \frac{n!}{n^{\frac{n}{2}}}$

Solution 3.

1.

$$\begin{aligned}
\left|(-1)^n \sin^3 \frac{1}{\sqrt{n}}\right| &= \sin^3 \frac{1}{\sqrt{n}} \\
&\approx \left(\frac{1}{\sqrt{n}}\right)^3 \\
&= \frac{1}{n^{\frac{3}{2}}}
\end{aligned}$$

Therefore, as $\sum \frac{1}{n^{\frac{3}{2}}}$ converges, the series converges absolutely.

2.

$$a_n = \frac{n!}{n^{\frac{n}{2}}}$$

Therefore,

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\frac{n!}{n^{\frac{n}{2}}}}{\frac{(n+1)!}{(n+1)^{\frac{n+1}{2}}}} \\
&= \frac{(n+1)n^{\frac{n}{2}}}{(n+1)^{\frac{1}{2}}(n+1)^{\frac{n}{2}}} \\
&= \sqrt{n+1}\frac{1}{\left(\frac{n+1}{n}\right)^{\frac{n}{2}}} \\
&= \sqrt{n+1}\frac{1}{\left(\left(1+\frac{1}{n}\right)^n\right)^{\frac{1}{2}}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\sqrt{n+1}\frac{1}{\left(\left(1+\frac{1}{n}\right)^n\right)^{\frac{1}{2}}} \\
&= \infty \\
&> 1
\end{aligned}$$

Therefore, by the ratio test, the series diverges.

Exercise 4.
Calculate the volume of the body which is bounded by a cone $z = \sqrt{x^2+y^2}$ and the cylinder $x^2+y^2 = x$ with $z \geq 0$.

Solution 4.

$$\begin{aligned}
x^2+y^2 &= x \\
\therefore \left(x-\frac{1}{2}\right)^2 + y^2 &= \frac{1}{4}
\end{aligned}$$

Therefore, the cylinder is centred at $\left(\frac{1}{2}, 0\right)$, and has radius $\frac{1}{2}$.

Therefore, the volume of the body is

$$
\begin{aligned}
&\iiint\limits_{E_{\mathrm{I}}} \mathrm{d}V \\
&\qquad = \iint\limits_{D} \int\limits_{0}^{\sqrt{x^2+y^2}} \mathrm{d}z \, \mathrm{d}A \\
&\qquad = \iint\limits_{D} \sqrt{x^2+y^2} \, \mathrm{d}A
\end{aligned}
$$

Let

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}
$$

Therefore, the D is given by $-\frac{\pi}{2} \leq \theta\frac{\pi}{2}$ and $0 \leq r \leq \cos\theta$.

Therefore,

$$
\begin{aligned}
\iint\limits_{E_{\mathrm{I}}} \mathrm{d}V &= \iint\limits_{D} \sqrt{x^2+y^2}\,\mathrm{d}A \\
&= \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{0}^{\cos\theta} r \cdot r \,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left.\frac{r^3}{3}\right|_{0}^{\cos\theta} \mathrm{d}\theta \\
&= \frac{1}{3}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta\,\mathrm{d}\theta \\
&= \frac{2}{3}\int\limits_{0}^{\frac{\pi}{2}} \cos^3\theta\,\mathrm{d}\theta \\
&= \frac{2}{3}\int\limits_{0}^{\frac{\pi}{2}} \cos^2\theta\cos\theta\,\mathrm{d}\theta \\
&= \frac{2}{3}\int\limits_{0}^{\frac{\pi}{2}} \left(1-\sin^2\theta\right)\cos\theta\,\mathrm{d}\theta \\
&= \frac{2}{3}\int\limits_{0}^{\frac{\pi}{2}} \left(\cos\theta-\sin^2\theta\cos\theta\right)\mathrm{d}\theta \\
&= \frac{4}{9}
\end{aligned}
$$