Differential and Integral Calculus : Recitations

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1 Instructor Information

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Part I
Sequences and Series

1 Sequences

Recitation 1 – Exercise 1.
Prove:

$$\lim_{n\to\infty} \frac{2n^2+n+1}{n^2+3} = 2$$

Recitation 1 – Solution 1.
Let

$$\varepsilon > 0$$

$$\begin{aligned}
\left|\frac{2n^2+n+1}{n^2+3} - 2\right| &= \left|\frac{2n^2+n+1-2n^2-6}{n^2+3}\right| \\
&= \left|\frac{n-5}{n^2+3}\right| \\
&\leq \left|\frac{n-5}{n^2}\right| \\
&\leq \frac{1}{n} \\
&< \varepsilon
\end{aligned}$$

Therefore, let $N = \left[\frac{1}{\varepsilon}\right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.
Therefore, $\lim_{n\to\infty} \frac{2n^2+n+1}{n^2+3} = 2$. □

Recitation 1 – Exercise 2.
Prove

$$\lim_{n\to\infty} \frac{n^3+\sin n+n}{2n^4} = 0$$

Recitation 1 – Solution 2.
Let $\varepsilon > 0$

$$\begin{aligned}\left|\frac{n^3+\sin n+n}{2n^4}\right| &\le \left|\frac{n^3+1+n}{2n^4}\right| \\ &\le \left|\frac{3n^3}{2n^4}\right| = \frac{3}{2}\cdot\frac{1}{n} < \varepsilon\end{aligned}$$

Therefore, let $N = \left[\frac{3}{2\varepsilon}\right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.

Therefore, $\lim\limits_{n\to\infty} \frac{n^3+\sin n+n}{2n^4} = 0$ □

Recitation 1 – Exercise 3.
Calculate $\sqrt[3]{n^3+3n} - n$.

Recitation 1 – Solution 3.

$$a^n - b^n = (a-b)\cdot(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Therefore, let

$$\begin{aligned} a &= \sqrt[3]{n^3+3n} \\ b &= \sqrt[3]{n^3} \end{aligned}$$

$$\begin{aligned} a - b &= \frac{a^3-b^3}{a^2+ab+b^2} \\ \therefore \sqrt[3]{n^3+3n} - n &= \frac{n^3+3n-n^3}{(n^3+3n)^{2/3} + (n^3+3n)^{1/3}n + n^2} \\ &= \frac{3}{\left(\frac{n^3+3n}{n^{3/2}}\right)^{2/3} + \left(\frac{n^3+3n}{n^3}\right)^{1/3n} + n} \end{aligned}$$

Therefore, the limit is 0.

Recitation 1 – Exercise 4.
Prove

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Recitation 1 – Solution 4.

$$0 \le \frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\dots\frac{n}{n} \le \frac{1}{n}$$

Therefore, by the Sandwich Theorem, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Recitation 1 – Exercise 5.

Let $a_1 = 3$, $a_{n+1} = 1 + \sqrt{6 + a_n}$. Prove that a_n converges and find its limit.

Recitation 1 – Solution 5.

If possible, let $\lim_{n\to\infty} a_n = l$.

$$a_{n+1} = 1 + \sqrt{6 + a_n}$$

Taking the limit on both sides,

$$\begin{aligned} l &= 1 + \sqrt{6 + l} \\ \therefore l - 1 &= \sqrt{6 + l} \\ \therefore l &= \frac{3 \pm \sqrt{29}}{2} \end{aligned}$$

As $a_n \ge 0$, $l = \dfrac{3 + \sqrt{29}}{2}$.

$$\begin{aligned} a_2 &= 1 + \sqrt{6 + a_1} \\ &= 1 + \sqrt{6 + 3} \\ &= 4 \\ \therefore a_2 &> a_1 \end{aligned}$$

If possible, let $a_n \ge a_{n-1}$.
Therefore,

$$\begin{aligned} a_{n+1} &= 1 + \sqrt{6 + a_n} \\ &\ge 1 + \sqrt{6 + a_{n+1}} = a_n \end{aligned}$$

Therefore by induction, $\{a_n\}$ is monotonically increasing.

$$\begin{aligned} a_1 &= 3 \\ \therefore a_1 &\le 5 \end{aligned}$$

If possible, let $a_n \leq 5$.
Therefore,

$$a_{n+1} = 1 + \sqrt{6 + a_n} \leq q + \sqrt{11} \leq 5$$

Therefore by induction, $\{a_n\}$ is bounded from above by 5.

1.1 Limit of a Function by Heine

Definition 1.

$$\lim_{x \to x_0} f(x) = l$$

if for every sequence x_n, such that $\lim_{n \to \infty} x_n = x_0$,

$$\lim_{n \to \infty} f(x_n) = l$$

Theorem 1. *If f is continuous at x_0 and $x_n \to x_0$, then*

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right) = f_{x_0}$$

Recitation 2 – Exercise 1.
Calculate $\lim_{n \to \infty} \sqrt[n]{n}$.

Recitation 2 – Solution 1.
Let

$$f(x) = x^{1/x}$$

Therefore,

$$\begin{aligned} \lim_{x \to \infty} x^{1/x} &= \lim_{x \to \infty} e^{\cancelto{0}{\frac{\ln x}{x}}} \\ &= 1 \end{aligned}$$

1.2 Sub-sequences

Recitation 2 – Exercise 2.
Find all partial limits and $\overline{\lim}$ and $\underline{\lim}$ of

$$a_n = \left(\cos\frac{\pi n}{4}\right)^n$$

Recitation 2 – Solution 2.
Let $k, z \in \mathbb{Z}$

$$\begin{aligned}
\cos\frac{\pi n}{4} &= \cos\frac{\pi(n+k)}{4} \\
\therefore \frac{\pi n}{4} &= \frac{\pi(n+k)}{4} + 2\pi z \\
\therefore \pi n &= \pi(n+k) + 8\pi z \\
\therefore k &= 8z
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_{8k} &= \left(\cos\frac{\pi \cdot 8k}{4}\right)^{8k} \\
&= (\cos(2\pi k))^{8k} \\
&= 1 \\
a_{8k+1} &= \left(\cos\frac{\pi \cdot (8k+1)}{4}\right)^{8k+1} \\
&= \left(\cos\frac{\pi}{4}\right)^{8k+1} \\
&= \left(\frac{\sqrt{2}}{2}\right)^{8k+1} \\
a_{8k+2} &= \left(\cos\frac{\pi \cdot (8k+2)}{4}\right)^{8k+2} \\
&= \left(\cos\frac{\pi}{2}\right)^{8k+2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{k\to\infty} a_{8k} &= 1 \\
\lim_{k\to\infty} a_{8k+1} &= \lim_{k\to\infty}\left(\frac{\sqrt{2}}{2}\right)^{8k+1} \\
&= 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{k\to\infty} a_{8k+2} &= 0 \\
\lim_{k\to\infty} a_{8k+3} &= 0 \\
\lim_{k\to\infty} a_{8k+4} &= \lim_{k\to\infty} (-1)^{8k+4} \\
&= 1 \\
\lim_{k\to\infty} a_{8k+5} &= 0 \\
\lim_{k\to\infty} a_{8k+6} &= 0 \\
\lim_{k\to\infty} a_{8k+7} &= 0
\end{aligned}$$

Therefore, $\{a_n\}$ has two partial limits, 0 and 1.

$$\begin{aligned}
\overline{\lim} a_n &= 1 \\
\underline{\lim} a_n &= 0
\end{aligned}$$

2 Series

Definition 2 (Convergence of a series)**.** Let $\{a_n\}$ be a sequence. Let S_n be a sequence of partial sums of a_n, s.t.

$$S_n = \sum_{k=1}^{n} a_k$$

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to l if

$$\lim_{n\to\infty} S_n = l$$

that is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n$$

Recitation 2 – Exercise 3.

Does $\sum\limits_{k=0}^{\infty} q^k$ where $-1 < q < 1$ converge?

Recitation 2 – Solution 3.

$$\begin{aligned}\sum_{k=0}^{\infty} q^k &= \lim_{n\to\infty}\sum_{k=0}^{n} q^k \\ &= \lim_{n\to\infty}\frac{1-q^{n+1}}{1-q} \\ &= \frac{1}{1-q}\end{aligned}$$

Therefore, the series converges.

Recitation 2 – Exercise 4.

Does $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ converge?

Recitation 2 – Solution 4.

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k(k+1)} &= \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right) \\ &= \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) \\ &= 1\end{aligned}$$

Recitation 2 – Exercise 5.

Does $\displaystyle\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^k$ converge?

Recitation 2 – Solution 5.

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e$$

$$\therefore \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k \neq 0$$

Therefore, the necessary condition is nt satisfied. Hence, the series does not converge.

2.1 Comparison Tests for Positive Series

Theorem 2 (First Comparison Test)**.** *If $a_n \geq 0$, $b_n \geq 0$, and $a_n \leq b_n$, then*

1. *If $\sum b_n$ converges, then $\sum a_n$ converges.*
2. *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Theorem 3 (Second Comparison Test)**.** *If $a_n \geq 0$, $b_n \geq 0$ and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l$$

where $0 < l < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge simultaneously.

Recitation 3 – Exercise 1.
Suppose the sequence a_n satisfies the condition

$$a_{n+1} - a_n > \frac{1}{n}$$

$\forall n \in \mathbb{N}$.
Prove that $\lim\limits_{n\to\infty} a_n = \infty$.

Recitation 3 – Solution 1.

$$\begin{aligned} a_{n+1} &= a_{n+1} - a_n + a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_2 - a_1 + a_1 \\ &= \sum_{k=1}^{n} (a_{k+1} - a_k) + a_1 \\ &\geq \sum_{k=1}^{n} \frac{1}{k} + a_1 \end{aligned}$$

As the harmonic series diverges, $\sum\limits_{k=1}^{n} \frac{1}{k} + a_1$ diverges.
Therefore, by the First Comparison Test, $\sum\limits_{k=1}^{\infty} (a_{k+1} - a_k)$ diverges.

Recitation 3 – Exercise 2.
Check the convergence of $\sum\limits_{n=2}^{\infty} \dfrac{n + \sin n}{n^3 + \cos \pi n}$.

Recitation 3 – Solution 2.

The series is non-negative. Therefore, the comparison tests are applicable.

$$
\begin{aligned}
\frac{n+\sin n}{n^3+\cos \pi n} &\leq \frac{n+1}{n^3-1} \\
\therefore \frac{n+\sin n}{n^3+\cos \pi n} &\leq \frac{2n}{n^3-\dfrac{n^3}{2}} && \leq \frac{4}{n^2}
\end{aligned}
$$

Therefore, by the First Comparison Test, as $\frac{4}{n^2}$ converges, $\sum_{n=2}^{\infty} \frac{n+\sin n}{n^3+\cos \pi n}$ also converges.

Recitation 3 – Exercise 3.

Let $a_n \geq 0$ and suppose that $\sum a_n$ converges. Prove that $\sum {a_n}^2$ converges. Is it true without the assumption $a_n \geq 0$?

Recitation 3 – Solution 3.

As $\sum a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$.
Therefore, $\exists N \in \mathbb{N}$, such that $\forall n > N$, $a_n < 1$.
Therefore, $\forall n > N$, ${a_n}^2 \leq a_n$. Hence, as $\sum\limits_{n=N+1}^{\infty} a_n$ converges, $\sum\limits_{n=N+1}^{\infty} {a_n}^2$ also converges. Hence, $\sum\limits_{n=1}^{\infty} a_n$ also converges.

This is not true without the assumption $a_n \geq 0$, as the argument ${a_n}^2 \leq a_n$ does not hold.

Recitation 3 – Exercise 4.

For which α does $\sum \left(\sqrt{n+1}-\sqrt{n}\right)^{\alpha/2}$ converge?

Recitation 3 – Solution 4.

$$
\begin{aligned}
\sum \left(\sqrt{n+1}-\sqrt{n}\right)^{\alpha/2} &= \sum \left(\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2} \\
&= \sum \left(\frac{1}{\sqrt{n+1}-\sqrt{n}}\right)^{\alpha/2}
\end{aligned}
$$

The series is positive. Therefore, the comparison tests are applicable.

Comparing with $\left(\frac{1}{\sqrt{n}}\right)^{\alpha/2}$,

$$\frac{\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2}}{\left(\frac{1}{\sqrt{n}}\right)^{\alpha/2}} = \left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2}$$

$$\therefore \lim_{n\to\infty}\left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2} = \left(\frac{1}{2}\right)^{\alpha/2}$$

$\sum \frac{1}{n^{\alpha/2}}$ converges if and only if $\frac{\alpha}{4} > 1$, i.e. if an inly if $\alpha > 4$.
By the Second Comparison Test, $\sum \frac{1}{n^{\alpha/4}}$ and the series converge or diverge simultaneously.
Therefore, the series converges for $\alpha > 4$.

Recitation 3 – Exercise 5.

Check the convergence of $\sum\limits_{n=1}^{\infty} \sin\frac{1}{n}$.

Recitation 3 – Solution 5.

$\forall n \in \mathbb{N}$, $\sin\frac{1}{n} \geq 0$

$$\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$$

Therefore, by Second Comparison Test, $\sum\frac{1}{n}$ and $\sum\sin\frac{1}{n}$ diverge simultaneously.

2.2 d'Alembert Criteria (Ratio Test)

Definition 3 (Absolute and conditional convergence)**.** The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

Theorem 4. *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 5 (d'Alembert Criteria (Ratio Test))**.** *1. If*

$$\lim_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right| = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\lim_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right| = L > 1$$

(including $L = \infty$), then $\sum a_n$ converges diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 – Exercise 6.

Check the convergence of $\sum \dfrac{(-1)^n \cdot n^{1000}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$.

Recitation 3 – Solution 6.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n \cdot n^{1000}}{1 \cdot \cdots \cdot (2n-1)}\right| = \sum_{n=1}^{\infty}\frac{n^{1000}}{1 \cdot \cdots \cdot (2n-1)}$$

Therefore, by the d'Alembert Criteria (Ratio Test),

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\dfrac{(n+1)^{1000}}{1 \cdot \cdots \cdot (2n+1)}}{\dfrac{n^{1000}}{1 \cdot \cdots \cdot (2n-1)}} \\ &= \left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1}\end{aligned}$$

$$\therefore \lim_{n to \infty}\left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1} = 0$$

$$\therefore \left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1} < 1$$

Therefore, by the d'Alembert Criteria (Ratio Test), the series converges absolutely, and hence converges.

2.3 Cauchy Criteria (Cauchy Root Test)

Theorem 6 (Cauchy Criteria (Cauchy Root Test))**.** *1. If*

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

(including $L = \infty$), then $\sum a_n$ diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 – Exercise 7.

Check the convergence of $\sum \left(1 - \dfrac{2}{n}\right)^{n^2}$.

Recitation 3 – Solution 7.

$$\sqrt[n]{\left(1 - \frac{2}{n}\right)^{n^2}} = \left(1 - \frac{2}{n}\right)^{n}$$

$$\therefore \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^{n} = e^{-2}$$

$$\therefore \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^{n} < 1$$

Therefore, by the Cauchy Criteria (Cauchy Root Test), $\sum \left(1 - \dfrac{2}{n}\right)^{n^2}$ converges.

2.4 Leibniz's Criteria

Definition 4 (Alternating series)**.** The series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

Theorem 7 (Leibniz's Criteria for Convergence)**.** *If an alternating series $\sum (-1)^{n-1} a_n$ with $a_n > 0$ satisfies*

1. $a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.

2. $\lim_{n\to\infty} a_n = 0$

then the series $(-1)^{n-1}a_n$ *converges.*

Recitation 3 – Exercise 8.

Prove or disprove: There exists $\{a_n\}$, such that $\sum a_n$ converges and $\sum(1+a_n)a_n$ diverges.

Recitation 3 – Solution 8.

Let $a_n = \dfrac{(-1)^n}{\sqrt{n}}$.

Therefore, by Leibniz's Criteria for Convergence, $\sum \dfrac{(-1)^n}{\sqrt{n}}$ converges.

$$\begin{aligned}\sum(1+a_n)a_n &= \sum\left(1+\frac{(-1)^n}{\sqrt{n}}\right)\frac{(-1)^n}{\sqrt{n}}\\ &= \sum\left(\frac{(-1)^n}{\sqrt{n}}+\frac{1}{n}\right)\end{aligned}$$

Therefore, as $\sum \dfrac{1}{n}$ diverges, and $\sum \dfrac{(-1)^n}{\sqrt{n}}$ converges, $\sum\left(\dfrac{1}{n}+\dfrac{(-1)^n}{\sqrt{n}}\right)$ diverges.

2.5 Integral Test

Theorem 8 (Integral Test)**.** *If* $f(x) : [1,\infty) \to [0,\infty)$ *is monotonically decreasing. Then,* $\sum\limits_{n=1}^{\infty} f(n)$ *and* $\int\limits_1^{\infty} f(x)\,\mathrm{d}x$ *converge or diverge simultaneously.*

Recitation 3 – Exercise 9.

Check the convergence of $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\ln n}$

Recitation 3 – Solution 9.

Let

$$f(x) = \frac{1}{x\ln x}$$

$f(x)$ is monotonically decreasing. Therefore, the Integral Test is applicable. Therefore,

$$\begin{aligned}\int_2^\infty \frac{1}{x\ln x}\,\mathrm{d}x &= \int_{\ln 2}^\infty \frac{1}{y}\,\mathrm{d}y\\ &= \ln y|_{\ln 2}^\infty\\ &= \infty\end{aligned}$$

Therefore, by the integral test, $\sum \frac{1}{n\ln n}$ diverges.

Recitation 4 – Exercise 1.
Let $d_n \geq 0$ and suppose

$$\sum_{n=0}^\infty d_n = \infty$$

Prove that

$$\sum_{n=0}^\infty \frac{d_n}{1+d_n} = \infty$$

Recitation 4 – Solution 1.
If possible, let d_n be a bounded sequence. Then there exists M, such that $d_n \leq M$, $\forall n \in \mathbb{N}$.
Therefore,

$$\frac{d_n}{1+d_n} \geq \frac{d_n}{1+M}$$

Therefore, by the Second Comparison Test, as $\sum d_n$ diverges, $\sum \frac{d_n}{1+d_n}$ also diverges.

If d_n is not bounded, then there is a subsequence d_{n_k} which diverges. Therefore,

$$\frac{d_{n_k}}{1+d_{n_k}} = \frac{1}{\frac{1}{d_{n_k}}+1}$$

$$\therefore \lim_{k\to\infty} \frac{d_{n_k}}{1+d_{n_k}} = 1$$

Therefore,

$$\lim_{n\to\infty} \frac{d_n}{1+d_n} \neq 0$$

Therefore, the necessary condition for convergence is not fulfilled. Therefore, the series converges.

Recitation 4 – Exercise 2.
Let

$$d_n = \begin{cases} 1 & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$

Does $\sum \dfrac{d_n}{1+n \cdot d_n}$ diverge?

Recitation 4 – Solution 2.

$$d_n = \begin{cases} 1 & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$

$$\therefore \frac{d_n}{1+n \cdot d_n} = \begin{cases} \dfrac{1}{1+k^2} & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$

As $\dfrac{1}{1+k^2} \leq \dfrac{1}{k^2}$ and as $\dfrac{1}{k^2}$ converges, $\sum \dfrac{1}{1+k^2}$ also converges.

Recitation 4 – Exercise 3.
Let a_n be a sequence such that $|a_{n+1} - a_n| \leq b_{n+1}$ for all $n \in \mathbb{N}$ where $\sum b_k$ converges. Prove that $\{a_n\}$ converges.

Recitation 4 – Solution 3.
Let $\varepsilon > 0$.

$$\begin{aligned} |a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} + \cdots - a_n| \\ &\leq \sum_{k=n+1}^{m} |a_k - a_{k-1}| \\ &\leq \sum_{k=n+1}^{m} b_k \end{aligned}$$

Therefore, as $\sum b_n$ converges, the series satisfies the Cauchy Criteria (Cauchy Root Test). Therefore, there exists N, such that $\forall m > n > N$, $\left|\sum\limits_{k=n+1}^{m} b_k\right| < \varepsilon$. Therefore, for $m > n > N$,

$$|a_m - a_n| \leq \sum_{k=n+1}^{m} b_n < \varepsilon$$

3 Power Series

Definition 5 (Power series)**.** A power series around x_0 is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $\{a_n\}$ is a sequence of real numbers.

Theorem 9 (Abel's Theorem)**.** *For every power series $\sum a_n(x - x_0)^n$, there exists $R \in [0, \infty]$, such that for all x satisfying $|x - x_0| < R$, the series converges and for all x satisfying $|x - x_0| > R$ the series diverges.*

Theorem 10 (Cauchy's Formula for Radius of Convergence)**.**

$$R = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$$

Theorem 11 (Hadamard's Formula for Radius of Convergence)**.** *If $\lim\limits_{n\to\infty} \left|\dfrac{a_n}{a_{n+1}}\right|$ exists, then*

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = R$$

Recitation 4 – Exercise 4.

Find the domain of convergence of $\sum\limits_{n=1}^{\infty} \dfrac{(2x-4)^n}{n}$.

Recitation 4 – Solution 4.

$$\sum_{n=1}^{\infty} \frac{(2x-4)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{n}$$

Therefore, by Cauchy's Formula for Radius of Convergence,

$$\begin{aligned} R &= \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}} \\ &= \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{2^n}{n}}} \\ &= \frac{1}{\lim\limits_{n\to\infty} \dfrac{2}{\sqrt[n]{n}}} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, the series converges for

$$|x-2| < \frac{1}{2}$$

and diverges for

$$|x-2| > \frac{1}{2}$$

If $x = \frac{5}{2}$,

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{5}{2} - 2\right)^n \\ &\qquad = \sum_{n=1}^{\infty} \frac{1}{n} \end{aligned}$$

Therefore, the series diverges.
If $x = \frac{3}{2}$,

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{3}{2} - 2\right)^n \\ &\qquad = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \end{aligned}$$

Therefore, by Leibniz's Criteria for Convergence, the series converges.
Therefore, the domain of convergence is $\left[\frac{3}{2}, \frac{5}{2}\right)$.

Recitation 4 – Exercise 5.
Find the radius of convergence of $\sum\limits_{n=0}^{\infty} n!x^{n!}$.

Recitation 4 – Solution 5.

$$\frac{1}{\sqrt[n]{a_n}} = x + x + 2x^2 + 6x^6 + 24x^{24} + \dots$$

Therefore,

$$a_n = \begin{cases} n & ; \quad n = k^2 \\ 0 & ; \quad n \neq k^2 \end{cases}$$

Therefore,

$$\begin{aligned} R &= \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{a_n}} \\ &= \frac{1}{\lim\limits_{k\to\infty} \sqrt[k!]{k!}} \\ &= 1 \end{aligned}$$

3.1 Power Series Representation of a Function

Theorem 12. *The power series representation of a function $f(x)$ is equal to its Taylor series if and only if $\lim\limits_{n\to\infty} R_n(x) = 0$, where $R_n(x)$ is the Lagrange remainder.*

3.2 Differentiation and Integrations of Power Series

Recitation 5 – Exercise 1.
Find the power series representation of $\tan^{-1} x$.

Recitation 5 – Solution 1.

$$\begin{aligned}\frac{\mathrm{d}\tan^{-1} x}{\mathrm{d}x} &= \frac{1}{1+x^2} \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n}\end{aligned}$$

Integrating term by term,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + c$$

As $\tan^{-1} 0 = 0$, $c = 0$. Therefore,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Recitation 5 – Exercise 2.

Find an explicit formula for $\sum\limits_{n=1}^{\infty} x^n n^2$.

Recitation 5 – Solution 2.

$$\sum_{n=1}^{\infty} x^n n^2 = x \cdot \sum_{n=1}^{\infty} x^{n-1} n^2$$

Let

$$g(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

Integrating term by term,

$$\begin{aligned}\int g(x)\,\mathrm{d}x &= \sum_{n=1}^{\infty} n^2 \frac{x^n}{n} \\ &= \sum_{n=1}^{\infty} n x^n \\ &= x \cdot \sum_{n=1}^{\infty} n x^{n-1}\end{aligned}$$

Let

$$
\begin{aligned}
h(x) &= \sum_{n=1}^{\infty} n x^{n-1} \\
\therefore \int h(x)\, \mathrm{d}x &= \frac{x}{1-x}
\end{aligned}
$$

Therefore, inside radius of convergence $R = 1$, differentiating $\int h(x)\,\mathrm{d}x$,

$$
\begin{aligned}
h(x) &= \frac{1-x+x}{(1-x)^2} \\
&= \frac{1}{(1-x)^2} \\
\therefore \int g(x)\, \mathrm{d}x &= x h(x) \\
&= \frac{x}{(1-x)^2} \\
\therefore g(x) &= \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4} \\
\therefore \sum_{n=1}^{\infty} x^n n^2 &= x \cdot \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4}
\end{aligned}
$$

Recitation 5 – Exercise 3.
Find the sum $\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

Recitation 5 – Solution 3.
Let

$$
f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}
$$

be a power series with radius R.
Therefore,

$$
\sum_{n=1}^{\infty} \frac{1}{n2^n} = f\left(\frac{1}{2}\right)
$$

Therefore,

$$
\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} x^{n-1} \\
&= \frac{1}{1-x} \\
\therefore f(x) &= -\ln(1-x) + c
\end{aligned}
$$

As $f(0) = 0$, $c = 0$. Therefore,

$$f(x) = -\ln(1-x)$$

Therefore,

$$\begin{aligned} f\left(\frac{1}{2}\right) &= -\ln\left(\frac{1}{2}\right) \\ &= \ln 2 \end{aligned}$$

4 Sequences of Functions

Definition 6 (Point-wise convergence and domain of convergence)**.** $\{f_n\}$ is said to converge point-wise in some domain $E \subset D$ if $\forall x \in E$, the sequence $\{f_n(x)\}$ converges. In this case, E is said to be a domain of convergence of $\{f_n\}$.

Recitation 5 – Exercise 4.
Let $f(x) : \mathbb{R} \to \mathbb{R}$ be some function such that $\lim\limits_{x\to\infty} f(x) = 0$. Let $f_n(x) = f(nx)$. What is the domain of convergence of f_n? What is the limit function?

Recitation 5 – Solution 4.
Let x be a particular number in $(0, \infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f(nx)$$

Therefore, as $\lim\limits_{x\to\infty} f(x) = 0$,

$$\lim_{n\to\infty} f_n(x) = 0$$

Therefore the domain of convergence is $(0, \infty)$ and the limit function is a constant 0.

Although the all functions in $\{f_n\}$ are continuous, the limit function is not continuous.

Definition 7 (Uniform convergence)**.** A sequence of functions $\{f_n\}$ is said to converge uniformly to f in the domain E, if $\forall \varepsilon$, $\exists N$ such that $\forall n > N$ and $\forall x \in E$, $|f_n(x) - f_(x)| < \varepsilon$. If f_n converges to f uniformly in E, it is denoted as $f_n \overset{E}{\rightrightarrows} f$.

4.1 Supremum and Infimum of Sets

Definition 8 (Supremum)**.** Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the supremum of A if

1. $\forall x \in A$, $x \leq M$, i.e. M is an upper bound of A.

2. $\forall \varepsilon$, $\exists x \in A$, such that $x > M - \varepsilon$.

That is, the supremum of A is the least upper bound of A.
The supremum may or may not be in A.

Definition 9 (Infimum)**.** Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the infimum of A if

1. $\forall x \in A$, $x \geq M$, i.e. M is an upper bound of A.

2. $\forall \varepsilon$, $\exists x \in A$, such that $x < M - \varepsilon$.

That is, the infimum of A is the greatest lower bound of A. The infimum may or may not be in A.

Theorem 13. *Every bounded set A has a supremum and an infimum.*

Theorem 14. $f_n \overset{E}{\rightrightarrows} f$ *if and only if*

$$\lim_{n\to\infty} (\sup\{|f_n(x) - f(x)| : x \in E\}) = 0$$

Recitation 6 – Exercise 1.
Let $f_n(x) = x^n$. Does $\{f_n\}$ converge uniformly?

Recitation 6 – Solution 1.

$$f(x) = \begin{cases} 0 & ; \quad x \in [0,1] \\ 1 & ; \quad x = 1 \end{cases}$$

If the convergence is uniform in $[0, 1]$,

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = 0$$

Let $x = 1 - \frac{1}{n}$.
Therefore, as the supremum is a upper bound,

$$\begin{aligned}
\sup_{x\in[0,1]} |f_n(x) - f(x)| &\geq \left| f_n\left(1 - \frac{1}{n}\right) - f\left(1 - \frac{1}{n}\right) \right| \\
\therefore \sup_{x\in[0,1]} |f_n(x) - f(x)| &\geq \left| \left(1 - \frac{1}{n}\right)^n - 0 \right| \\
\therefore \lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| &\geq \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n \\
\therefore \lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| &\geq \frac{1}{e} \\
\therefore \lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| &\neq 0
\end{aligned}$$

Therefore, the convergence is not uniform.

Recitation 6 – Exercise 2.
Let $f_n(x) = x + \frac{1}{n}$, $x \in \mathbb{R}$. What is its domain of convergence? What is the limit function? Is the convergence uniform?

Recitation 6 – Solution 2.
$\forall x \in \mathbb{R}$,

$$\lim_{n\to\infty} \left(x + \frac{1}{n}\right) = x$$

Therefore $\{f_n\}$ converges pointwise to x, in $\mathbb{R}$.

$$\begin{aligned}
\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| &= \sup_{x\in\mathbb{R}} \left| x + \frac{1}{n} - x \right| \\
&= \frac{1}{n} \\
\therefore \lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| &= 0
\end{aligned}$$

Therefore, the convergence is uniform.

Recitation 6 – Exercise 3.
Let $f_n : [0, \infty) \to \mathbb{R}$.

$$f_n(x) = \begin{cases} 1 & ; \quad n \leq x \leq n+1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Dows f_n converge pointwise in $[0,\infty)$? Dows f_n converge uniformly in $[0,\infty)$?

Recitation 6 – Solution 3.
For every x, the sequence $\{f_n(x)\}$ will be of the form $\{0,\dots,0,1,0,\dots,0\}$ with 1 only when $n \leq x \leq n+1$.
Therefore,

$$\begin{aligned}\lim_{n\to\infty} f_n(x) &= 0\\ &= f(x)\end{aligned}$$

Therefore, f_n converges pointwise in $[0,\infty)$.

$$\begin{aligned}\sup_{x\in[0,\infty)} |f_n(x) - f(x)| &= \max_{x\in[0,\infty)} f_n(x)\\ &= 1\end{aligned}$$

Therefore, as the limit of the supremum is not 0, the convergence is not uniform.

Theorem 15. *If $f_n \overset{D}{\rightrightarrows} f$ and all f_n are continuous is D, then f is also continuous, i.e. uniform convergence preserves continuity.*

Recitation 7 – Exercise 1.
Does x^n converge to

$$f(x) = \begin{cases} 0 & ; \quad x \in [0,1) \\ 1 & ; \quad x = 1 \end{cases}$$

Recitation 7 – Solution 1.
If possible, let x^n converge to $f(x)$.
Therefore, as all $f_n(x)$ are continuous, and as uniform convergence preserves continuity, $f(x)$ also must be continuous.
This contradicts the definition of $f(x)$.
Therefore, the x^n does not converge to $f(x)$.

Recitation 7 – Exercise 2.
Check if $f_n(x) = \frac{x}{1+n^2x^2}$ converges uniformly in $[0,1]$.

Recitation 7 – Solution 2.

$$\begin{aligned}\lim_{n\to\infty} f_n(x) = 0 \\ &= f(x)\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{[0,1]} |f_n(x) - f(x)| &= \sup_{[0,1]} |f_n(x) - 0| \\ &= \sup_{[0,1]} \left|\frac{x}{1+n^2x^2}\right| \\ &= \sup_{[0,1]} \frac{x}{1+n^2x^2}\end{aligned}$$

As the function is continuous and the interval is closed, by the Weierstrass theorem, the function has a maximum. Therefore,

$$\sup_{[0,1]} |f_n(x) - f(x)| = \max_{[0,1]} \frac{x}{1+n^2x^2}$$

Differentiating to find the maximum,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1+n^2x^2}\right) &= \frac{1+n^2x^2-2x^2n^2}{(1+n^2x^2)} \\ &= \frac{1-x^2n^2}{(1+n^2x^2)^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1+n^2x^2}\right) &= 0 \\ \iff \frac{1-x^2n^2}{(1+n^2x^2)^2} &= 0 \\ \iff 1 &= x^2n^2 \\ \iff x &= \frac{1}{n}\end{aligned}$$

Therefore, the values of the function at the critical points and the end points are,

$$
\begin{aligned}
f_n(0) &= 0 \\
f_n(1) &= \frac{1}{1+n^2} \\
f_n\left(\frac{1}{n}\right) &= \frac{\frac{1}{n}}{1+n^2\frac{1}{n^2}} \\
&= \frac{1}{2n}
\end{aligned}
$$

Therefore, the maximum is at $x = \frac{1}{2n}$.
Therefore,

$$
\begin{aligned}
\max_{[0,1]} \frac{x}{1+n^2x^2} &= f_n\left(\frac{1}{n}\right) \\
&= \frac{1}{2n}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{n\to\infty} \sup_{[0,1]} |f_n(x) - f(x)| &= \lim_{n\to\infty} \max_{[0,1]} \frac{x}{1+n^2x^2} \\
&= \lim_{n\to\infty} \frac{1}{2n} \\
&= 0
\end{aligned}
$$

Therefore, the convergence is uniform.

Recitation 7 – Exercise 3.
Check the pointwise and uniform convergence of $f_n(x) = x^n - x^{n+1}$ in $[0,1]$.

Recitation 7 – Solution 3.

$$
\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} x^n - x^{n+1} \\
&= 0 \\
&= f(x)
\end{aligned}
$$

Therefore the function converges pointwise in $[0,1]$.

$$
\sup_{[0,1]} |f_n(x) - f(x)| = \sup_{[0,1]} x^n - x^{n+1}
$$

As the function is continuous and the interval is closed, by the Weierstrass theorem, the function has a maximum. Therefore,

$$\sup_{[0,1]} x^n - x^{n+1} = \max_{[0,1]} x^n - x^{n+1}$$

Differentiating to find the maximum,

$$\frac{\mathrm{d}(x^n - x^{n+1})}{\mathrm{d}x} = nx^{n-1} - (n+1)x^n$$

Therefore,

$$\begin{aligned} \frac{\mathrm{d}(x^n - x^{n+1})}{\mathrm{d}x} &= 0 \\ \iff nx^{n-1} - (n+1)x^n &= 0 \\ \iff n - (n+1)x &= 0 \\ \iff x &= \frac{n}{n+1} \end{aligned}$$

Therefore, the values of the function at the critical points and the end points are

$$\begin{aligned} f_n(0) &= 0 \\ f_n(1) &= 0 \\ f_n\left(\frac{n}{n+1}\right) &= \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \max_{[0,1]} x^n - x^{n+1} &= f_n\left(\frac{n}{n+1}\right) \\ &= \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \sup_{[0,1]} |f_n(x) - f(x)| &= \lim_{n\to\infty} \max_{[0,1]} \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} \\ &= \frac{1}{e} - \frac{1}{e} \\ &= 0 \end{aligned}$$

Therefore, the convergence is uniform.

Theorem 16 (Cauchy's Theorem)**.** $\{f_n\}$ *converges uniformly in* D *if and only if* $\forall \varepsilon \in N$, $\exists N$, *such that* $\forall m, n > N$ *and* $\forall x \in D$,

$$|f_n(x) - f(x)| < \varepsilon$$

Recitation 7 – Exercise 4.
Let $\{f_n\}$ be a sequence of function in D such that $\forall x \in D$, $|f_{n+1}(x) - f_n(x)| \leq a_n$, where $\sum\limits_{n=1}^{\infty} a_n$ converges. Prove that $\{f_n\}$ converges uniformly in D.

Recitation 7 – Solution 4.
As $\sum a_n$ converges, $\exists N$ such that $\forall m > n > N$, $\left|\sum\limits_{k=n}^{m} a_k\right| < \varepsilon$.
Therefore, forall $m > n > N$ and $x \in D$,

$$\begin{aligned}|f_m(x) - f_n(x)| &= |f_m(x) - f_{m-1}(x) + f_{m-1}(x) - \cdots - f_n(x)| \\ &\leq |f_m(x) - f_{m-1}(x)| + |f_{m-1}(x) - f_{m-2}(x) + \cdots + |f_{n+1}(x) - f_n(x)|\end{aligned}$$

$$\therefore |f_m(x) - f_n(x)| \leq \sum_{k=n}^{m-1} |f_{k+1}(x) - f_k(x)|$$

$$\therefore |f_m(x) - f_n(x)| \leq \sum_{k=n}^{m-1} a_k$$

$$\therefore |f_m(x) - f_n(x)| \leq \varepsilon$$

Therefore, $\{f_n\}$ satisfies Cauchy's criterion for uniform convergence.

5 Series of Functions

Definition 10 (Pointwise convergence of series of functions)**.** Let $\{f_n\}$ be a sequence of functions defined in D. Let $S_n(x) = \sum\limits_{k=1}^{n} f_k(x)$.
If $S_n(x)$ converges for every $x \in D$ to a limit S, the series formed by $\{f_n\}$ is said to converge pointwise in D. It is denoted as

$$\sum_{k=1}^{\infty} f_k(x) = \lim_{n \to \infty} S_n(x) = S_x$$

Definition 11 (Uniform convergence of series of functions)**.** The series $\sum\limits_{k=1}^{\infty} f_k(x)$ is said to converge uniformly in D if $S_n \overset{D}{\rightrightarrows} S$.

Theorem 17. *If* $\sum\limits_{k=1}^{\infty} f_k(x)$ *converges uniformly in* D*, then the general term* $f_k(x)$ *must uniformly converge to* 0 *in* D*.*

Recitation 7 – Exercise 5.

Check the uniform convergence of $\sum\limits_{n=1}^{\infty} \frac{x^n}{n^2} - \frac{x^{n+1}}{(n+1)^2}$ in $[-1, 1]$.

Recitation 7 – Solution 5.

$$\begin{aligned} S_n(x) &= \sum_{k=1}^{n} \frac{x^k}{k^2} - \frac{x^{k+1}}{(k+1)^2} \\ &= \frac{x^1}{1^2} - \frac{x^{n+1}}{(n+1)^2} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} S_n(x) &= \lim_{n\to\infty} x = \frac{x^{n+1}}{(n+1)^2} \\ &= x \\ &= S(x) \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{[-1,1]} |S_n(x) - S(x)| &= \sup_{[-1,1]} \left| -\frac{x^{n+1}}{(n+1)^2} \right| \\ &\leq \frac{1}{(n+1)^2} \end{aligned}$$

Therefore,

$$\begin{aligned} &\lim_{n\to\infty} \sup_{[-1,1]} |S_n(x) - S(x)| \leq \lim_{n\to\infty} \frac{1}{(n+1)^2} \\ \therefore\ &\lim_{n\to\infty} \sup_{[-1,1]} |S_n(x) - S(x)| \leq 0 \end{aligned}$$

Therefore the convergence is uniform.

Theorem 18. *If* $\sum\limits_{n=1}^{\infty} f_n(x)$ *converges uniformly in* D *to* $S(x)$ *and the functions* f_n *are continuous in* D*, then the* $S(x)$ *is also continuous in* D*.*

Theorem 19. *A Leibniz series, i.e. a series of the form* $\sum\limits_{n=1}^{\infty} (-1)^n a_n$*, with* a_n *monotonically decreasing and* $\lim\limits_{n\to\infty} a_n = 0$*, converges, and*

$$\sum_{k=n}^{m} (-1)^k a_k \leq a_n$$

Recitation 7 – Exercise 6.

Check for pointwise and uniform convergence of $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{x^2+\sqrt{n}}$ in $\mathbb{R}$.

Recitation 7 – Solution 6.

For $x \in \mathbb{R}$, $\frac{1}{x^2+\sqrt{n}}$ is monotonically decreasing to 0 as $n \to \infty$.

Therefore, for $x \in \mathbb{R}$, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{x^2+\sqrt{n}}$ is a Leibniz series. Hence, it converges pointwise.

$$
\begin{aligned}
\left|\sum_{k=n}^{m} \frac{(-1)^n}{x^2+\sqrt{k}}\right| &\leq \frac{1}{x^2+\sqrt{n}} \\
&\leq \frac{1}{\sqrt{n}}
\end{aligned}
$$

Therefore,

$$
\lim_{n\to\infty} \left|\sum_{k=n}^{m} \frac{(-1)^n}{x^2+\sqrt{k}}\right| = 0
$$

Therefore, $\forall \varepsilon > 0$, there exists N such that $\forall m > n > N$, and $\forall x \in \mathbb{R}$,

$$
\left|\sum_{k=n}^{m} \frac{(-1)^n}{x^2+\sqrt{k}}\right| \leq \frac{1}{\sqrt{n}} < \varepsilon
$$

Therefore, $\left|\sum\limits_{k=n}^{m} \frac{(-1)^n}{x^2+\sqrt{k}}\right|$ satisfies Cauchy's criterion for uniform convergence. Hence it converges uniformly.

Recitation 7 – Exercise 7.

Show that $\sum\limits_{n=1}^{\infty} 3^n \sin\left(\frac{1}{4^n x}\right)$ does not converge uniformly in $(0, \infty)$.

Recitation 7 – Solution 7.

For any $x \in (0, \infty)$, as $\sin\left(\frac{1}{4^n x}\right) \leq \frac{1}{4^n x}$,

$$
\left|3^n \sin\left(\frac{1}{4^n x}\right)\right| \leq 3^n \frac{1}{4^n x}
$$

Therefore, as $\sum \left(\frac{3}{4}\right)^n \cdot \frac{1}{x}$ converges, by the First Comparison Test, $\sum \left|3^n \sin\left(\frac{1}{4^n x}\right)\right|$ also converges.

Therefore, $\sum 3^n \sin(\frac{1}{4^n x})$ converges absolutely. Hence, it converges.

$$\lim_{n\to\infty} 3^n \sin\left(\frac{1}{4^n x}\right) = \lim_{n\to\infty}$$
$$\neq 0$$

Therefore as the general element does not tend to 0, the series does not converge uniformly in $(0, \infty)$.

5.1 Weierstrass M-test

Theorem 20 (Weierstrass M-test)**.** *If $|u_k(x)| \leq c_k$ on D for $k \in \{1, 2, 3, \ldots\}$ and the numerical series $\sum\limits_{k=1}^{\infty} c_k$ converges, then the series of functions $\sum\limits_{k=1}^{\infty} u_k(x)$ converges uniformly on D.*

Recitation 8 – Exercise 1.
Show that $\sum\limits_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ converges uniformly in $[0, 1]$.

Recitation 8 – Solution 1.

$$\left|\sin\frac{x}{n^2}\right| \leq \left|\frac{x}{n^2}\right|$$
$$\therefore \left|\sin\frac{x}{n^2}\right| \leq \frac{1}{n^2}$$

Therefore, as $\sum \frac{1}{n^2}$ converges in $[0, 1]$, by Weierstrass M-test, the series converges uniformly in $[0, 1]$.

Recitation 8 – Exercise 2.
Does $\sum \frac{(-1)^n}{x+n}$ converge on $[0, 1]$?

Recitation 8 – Solution 2.

$$\max_{[0,1]} |f_n| = \max_{[0,1]} \frac{1}{x+n}$$
$$= \frac{1}{n}$$

Therefore, as $\sum \frac{1}{n}$ diverges, the Weierstrass M-test does not apply.
However, $\forall x \in [0, 1]$, $\sum \frac{(-1)^n}{x+n}$ is a Leibniz series. For a Leibniz series, the uniform convergence of the general term to 0 is a necessary and sufficient

condition for the convergence of the series.
Therefore, in $[0,1]$,

$$\lim_{n\to\infty} \frac{1}{x+n} = 0$$

Therefore, as the general term goes to 0, the series converges.

Recitation 8 – Exercise 3.
Does $\sum \frac{n^2x}{1+n^7x^2}$ converge uniformly in $\mathbb{R}$?

Recitation 8 – Solution 3.
As the function is even,

$$\sup_{\mathbb{R}} \left|\frac{n^2x}{1+n^7x^2}\right| = \sup{[0,\infty)} \left|\frac{n^2x}{1+n^7x^2}\right|$$
$$\sup_{[0,\infty)} \frac{n^2x}{1+n^7x^2}$$

Let

$$f_n(x) = \frac{n^2x}{1+n^7x^2}$$

Therefore,

$$f_n{}'(x) = \frac{n^2(1+n^7x^2) - n^9 \cdot 2x^2}{(1+n^7x^2)^2}$$

Therefore, maximizing $f_n(x)$,

$$\begin{aligned} f_n{}'(x) &= 0 \\ \iff n^2(1+n^7x^2 - n^7 \cdot 2x^2) &= 0 \\ \iff 1 - n^7x^2 &= 0 \\ \iff x &= \sqrt{\frac{1}{n^7}} \end{aligned}$$

Therefore, as $f_n{}'(x) \geq 0 \iff x \in \left[0, \sqrt{\frac{1}{n^7}}\right]$ and $f_n{}'(x) \leq 0 \iff x \in \left[\sqrt{\frac{1}{n^7}}, \infty\right)$, $x = \sqrt{\frac{1}{n^7}}$ is a global maximum of f_n in $[0,\infty)$.

Therefore,

$$\begin{aligned}\sup_{\mathbb{R}}\left|\frac{n^2x}{1+n^7x^2}\right| &= \frac{n^2\sqrt{\frac{1}{n^7}}}{1+1}\\ &= \frac{n^2}{2n^{\frac{7}{2}}}\\ &= \frac{1}{2n^{\frac{3}{2}}}\end{aligned}$$

Therefore, as $\sum \frac{1}{2n^{\frac{3}{2}}}$ converges, by the Weierstrass M-test, $\sum \frac{n^2x}{1+n^7x^2}$ converges uniformly in $\mathbb{R}$.

5.2 Application of Uniform Convergence

Theorem 21 (Changing the order of integration and infinite summation)**.** *If the functions $u_k(x)$, $k \in \{1,2,3,\dots\}$ are integrable on $[a,b]$ and the series $\sum\limits_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a,b]$ then*

$$\int\limits_a^b \left(\sum_{k=1}^{\infty} u_k(x)\right) \mathrm{d}x = \sum_{k=1}^{\infty} \int\limits_a^b u_k(x)\,\mathrm{d}x$$

Theorem 22 (Changing the order of integration and limit)**.** *If the functions $f_n(x)$ are integrable on $[a,b]$ and converge uniformly to f on $[a,b]$, then*

$$\lim_{n\to\infty} \int\limits_a^b f_n(x)\,\mathrm{d}x = \int\limits_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \int\limits_a^b f(x)\,\mathrm{d}x$$

Additionally,

$$\int\limits_a^x f_n(t)\,\mathrm{d}t \xrightarrow{[a,b]} \int\limits_a^x f(t)\,\mathrm{d}t$$

Recitation 8 – Exercise 4.

Is $\sum\limits_{n=1}^{\infty} \int\limits_0^2 \frac{(-1)^{n+1}}{x+n} = \int\limits_0^2 \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{x+n}$?

Recitation 8 – Solution 4.

As $\sum \frac{(-1)^{n+1}}{x+n}$ converges uniformly in $[0,2]$, by Theorem 21, the equality holds.

Recitation 8 – Exercise 5.

Is $\int\limits_0^1 \left(\sum\limits_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}\right) \mathrm{d}x = \sum\limits_{n=1}^{\infty} \int\limits_0^1 \left(x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}\right) \mathrm{d}x$?

Recitation 8 – Solution 5.

$$\begin{aligned}
\sum_{n=1}^{N} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} &= -x + \frac{1}{x^{2N+1}} \\
\therefore \lim_{n\to\infty} \sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} &= \sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \\
\lim_{n\to\infty} -x + \frac{1}{x^{2N+1}}
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} = \begin{cases} 1-x & ; \quad x \in (0,1] \\ 0 & ; \quad x = 0 \end{cases}$$

Therefore, as the limit function is not continuous but the function $x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}$ is continuous, the convergence is not uniform. Therefore, Theorem 21 is not applicable.
Therefore, checking directly,

$$\begin{aligned}
\int_0^1 \left(\sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}\right) \mathrm{d}x &= \int_0^1 1 - x \,\mathrm{d}x \\
&= \frac{1}{2}
\end{aligned}$$

$$
\begin{aligned}
\sum_{n=1}^{\infty}\int_0^1\left(x^{\frac{1}{2n+1}}-x^{\frac{1}{2n-1}}\right)\mathrm{d}x &= \sum_{n=1}^{\infty}\left(\frac{x^{\frac{1}{2n+1}+1}}{\frac{1}{2n+1}+1}-\frac{x^{\frac{1}{2n-1}+1}}{\frac{1}{2n-1}+1}\Bigg|_0^1\right)\\
&= \sum_{n=1}^{\infty}\left(\frac{2n+1}{2n+2}-\frac{2n-1}{2n}\right)\\
&= \lim_{N\to\infty}\sum_{n=1}^{N}\frac{2n+1}{2n+2}-\frac{2n-1}{2n}\\
&= \lim_{N\to\infty}\frac{2N+1}{2N+2}-\frac{2\cdot 1-1}{2}\\
&= \lim_{N\to\infty}\frac{2N+1}{2N+2}-\frac{1}{2}\\
&= 1-\frac{1}{2}\\
&= \frac{1}{2}
\end{aligned}
$$

Therefore, although the convergence is not uniform, the equality holds.

Theorem 23 (Changing the order of differentiation and infinite summation)**.** *If the functions $u_k(x)$, $k \in \{1,2,3,\dots\}$ are differentiable on $[a,b]$ and the derivatives are continuous on $[a,b]$, and the series $\sum\limits_{k=1}^{\infty} u_k(x)$ converges pointwise on $[a,b]$ and the series $\sum\limits_{k=1}^{\infty} u_k{}'(x)$ converges uniformly on $[a,b]$, then,*

$$
\left(\sum_{k=1}^{\infty} u_k(x)\right)' = \sum_{k=1}^{\infty} u_k{}'(x)
$$

Recitation 8 – Exercise 6.

If $\sum\limits_{n=1}^{\infty}\left(\tan^{-1}\frac{x}{n^2}\right)' = \left(\sum\limits_{n=1}^{\infty}\tan^{-1}\frac{x}{n^2}\right)$?

Recitation 8 – Solution 6.

$$\begin{aligned}
\left(\tan^{-1}\frac{x}{n^2}\right)' &= \frac{1}{\left(1+\left(\frac{x}{n^2}\right)^2\right)n^2}\\
&= \frac{1}{n^2+\frac{x^2}{n^2}}\\
&= \frac{n^2}{n^4+x^2}\\
\therefore \left(\tan^{-1}\frac{x}{n^2}\right)' &\leq \frac{1}{n^2}
\end{aligned}$$

Therefore, as $\sum \frac{1}{n^2}$ converges, by the Weierstrass M-test, $\sum \left(\tan^{-1}\frac{x}{n^2}\right)'$ converges uniformly.

By Lagrange's Mean Value Theorem, for c between 0 and x,

$$\begin{aligned}
\frac{\tan^{-1}x-\tan^{-1}0}{x-0} &= (\tan^{-1})'(c)\\
\therefore \frac{|\tan^{-1}x|}{|x|} &= \frac{1}{1+c^2}\\
\therefore |\tan^{-1}x| &\leq 1\\
\therefore |\tan^{-1}x| &\leq |x|
\end{aligned}$$

Therefore,

$$\left|\tan^{-1}\frac{x}{n^2}\right| \leq \left|\frac{x}{n^2}\right|$$

Therefore, as $\forall x \in \mathbb{R}\ \sum \frac{x}{n^2}$ converges pointwise, $\sum \tan^{-1}\frac{x}{n^2}$ also converges on $\mathbb{R}$.
Therefore, by Theorem 23, the equality holds.