Differential and Integral Calculus

Friday 3rd July, 2015

1 Sequences and Series

1 Sequences

Definition 1 (Sequences bounded from above) $\{a_n\}$ is said to be bounded from above if $\exists M\in\mathbb{R}$, s.t. $a_n\leq M$, $\forall n\in\mathbb{N}$. Each such M is called an upper bound of $\{a_n\}$.

Definition 2 (Sequences bounded from below) $\{a_n\}$ is said to be bounded from below if $\exists m\in\mathbb{R}$, s.t. $a_n\geq M$, $\forall n\in\mathbb{N}$. Each such M is called an lower bound of $\{a_n\}$.

Definition 3 $\{a_n\}$ is said to be bounded if it is bounded from below and bounded from above.

Definition 4 (Monotonic increasing sequence) A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0\in\mathbb{N}$, s.t. $a_n\leq a_{n+1}$, $\forall n\geq n_0$.

Definition 5 (Monotonic decreasing sequence) A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0\in\mathbb{N}$, s.t. $a_n\geq a_{n+1}$, $\forall n\geq n_0$.

Definition 6 (Strongly increasing sequence) A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0\in\mathbb{N}$, s.t. $a_n< a_{n+1}$, $\forall n\geq n_0$.

Definition 7 (Strongly decreasing sequence) A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0\in\mathbb{N}$, s.t. $a_n> a_{n+1}$, $\forall n\geq n_0$.

1.1 Limit of a Sequence

Definition 8 (Limit of a sequence) Let $\{a_n\}$ be a given sequence. A number L is said to be the limit of the sequence if $\forall\varepsilon>0$, $\exists n_0\in\mathbb{N}$, s.t. $|a_n-L|<\varepsilon$, $\forall n\geq n_0$. That is, there are infinitely many terms inside the interval and a finite number of terms outside it.

Exercise 1.

Prove that 2 is not a limit of $\left\{\frac{3n+1}{n}\right\}$.

Solution 1.

If possible, let

$$\lim_{n\to\infty}\frac{3n+1}{n}=2$$

Then, $\forall\varepsilon>0$, $\exists n_0\in\mathbb{N}$, s.t. $\left|\frac{3n+1}{n}-2\right|<\varepsilon$, $\forall n\geq n_0$. However,

$$\left|\frac{3n+1}{n}-2\right|=1+\frac{1}{n}>1$$

This is a contradiction for $\varepsilon=\frac{1}{2}$. Therefore, 2 is not a limit.

Theorem 1 *If a sequence $\{a_n\}$ has a limit L then the limit is unique.*

Theorem 2 *If a sequence $\{a_n\}$ has limit L, then the sequence is bounded.*

Theorem 3 *Let*

$$\lim_{n\to\infty}a_n=a$$
$$\lim_{n\to\infty}b_n=b$$

and let c be a constant. Then,

$$\lim c=c$$
$$\lim(ca_n)=c\lim a_n$$
$$\lim(a_n\pm b_n)=\lim a_n\pm\lim b_n$$
$$\lim(a_nb_n)=\lim a_n\lim b_n$$
$$\lim\left(\frac{a_n}{b_n}\right)=\frac{\lim a_n}{\lim b_n}\quad(\text{ if }\lim b\neq 0)$$

Theorem 4 *Let $\{b_n\}$ be bounded and let $\lim a_n=0$. Then,*
$\lim(a_nb_n)=0$

Theorem 5 (Sandwich Theorem) *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences. If*
$\lim a_n=\lim b_n=L$
and $\exists n_0\in\mathbb{N}$, s.t. $\forall n\geq n_0$, $a_n\leq b_n\leq c_n$. Then,
$\lim b_n=L$

Exercise 2.

Calculate $\lim\limits_{n\to\infty}\sqrt[n]{2^n+3^n}$

Solution 2.

$\sqrt[n]{3^n}\leq\sqrt[n]{2^n+3^n}\leq\sqrt[n]{3^n+3^n}=\sqrt[3]{2\cdot 3^n}$

$\therefore 3\leq\sqrt[n]{2^n+3^n}\leq 3\sqrt[n]{2}$

Therefore, by the Sandwich Theorem, $\lim\limits_{n\to\infty}\sqrt[n]{2^n+3^n}=3$.

Theorem 6 *Any monotonically increasing sequence which is bounded from above converges. Similarly, any monotonically decreasing sequence which is bounded from below converges.*

Exercise 3.

Prove that there exists a limit for $a_n=\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}_{n\text{ times}}$ and find it.

Solution 3.

$a_1=\sqrt{2}<\sqrt{2+\sqrt{2}}=a_2$

If possible, let

$$a_{n-1}<a_n$$
$$\therefore\sqrt{2+a_{n-1}}<\sqrt{2+a_n}$$
$$\therefore a_n<a_{n+1}$$

Hence, by induction, $\{a_n\}$ is monotonically increasing.

$a_1=\sqrt{2}\leq 2$

If possible, let

$$a_n\leq 2$$
$$\therefore\sqrt{2+a_n}\leq\sqrt{2+2}$$
$$\therefore a_{n+1}\leq 2$$

Hence, by induction, $\{a_n\}$ is bounded from above by 2. Therefore, by , $\{a_n\}$ converges.

1.2 Sub-sequences

Definition 9 (Sub-sequence) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_k\}_{k=1}^{\infty}$ be a strongly increasing sequence of natural numbers. Let $\{b_k\}_{k=1}^{\infty}$ be a sequence such that $b_k=a_{n_k}$. Then $\{b_k\}_{k=1}^{\infty}$ is called a sub-sequence of $\{a_n\}_{n=1}^{\infty}$.

Theorem 7 *If the sequence $\{a_n\}$ converges to L in a wide sense, i.e. L can be infinite, then any sub-sequence of $\{a_n\}$ converges to the same limit L.*

Definition 10 (Partial limit) A real number a, which may be infinite, is called a partial limit of the sequence $\{a_n\}$ is there exists a sub-sequence of $\{a_n\}$ which converges to a.

Theorem 8 (Bolzano-Weierstrass Theorem) *For any bounded sequence there exists a subsequence which is convergent, s.t. there exists at least one partial limit.*

Definition 11 (Upper partial limit) The largest partial limit of a sequence is called the upper partial limit. It is denoted by $\overline{\lim}a_n$ or $\limsup a_n$.

Definition 12 (Lower partial limit) The smallest partial limit of a sequence is called the upper partial limit. It is denoted by $\underline{\lim}a_n$ or $\liminf a_n$.

Theorem 9 *If the sequence $\{a_n\}$ is bounded and $\overline{\lim}a_n=\underline{\lim}a_n=a$ then $\exists\lim a_n=a$.*

1.3 Cauchy Characterisation of Convergence

Definition 13 A sequence $\{a_n\}$ is called a Cauchy sequence if $\forall\varepsilon>0$, $\exists n_0\in\mathbb{N}$, s.t. $\forall m,n\geq n_0$, $|a_n-a_m|<\varepsilon$.

Theorem 10 (Cauchy Characterisation of Convergence) *A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.*

Theorem 11 (Another Formulation of the Cauchy Characterisation Theorem) *The sequence $\{a_n\}$ converges if and only if $\forall\varepsilon>0$, $\exists n_0\in\mathbb{N}$, such that $\forall n\geq n_0$ and $\forall p\in\mathbb{N}$, $|a_{n+p}-a_n|<\varepsilon$.*

Exercise 4.
Prove that the sequence $a_n=\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{n^2}$ is convergent.

Solution 4.

$$|a_{n+p}-a_n|=\left|\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{(n+p)^2}-\left(\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{n^2}\right)\right|$$
$$=\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\dots+\frac{1}{(n+p)^2}$$
$$\therefore |a_{n+p}-a_n|<\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\dots+\frac{1}{(n+p-1)(n+p)}$$
$$\therefore |a_{n+p}-a_n|<\frac{1}{n}-\cancel{\frac{1}{n+1}}+\cancel{\frac{1}{n+1}}+\dots+\cancel{\frac{1}{n+p-1}}-\frac{1}{n+p}$$
$$\therefore |a_{n+p}-a_n|<\frac{1}{n}-\frac{1}{n+p}$$
$$\therefore |a_{n+p}-a_n|<\frac{1}{n}$$

Therefore, $\forall\varepsilon>0$, $\exists n_0\in\mathbb{N}$, s.t. $\forall n\geq n_0$ and $\forall p\in\mathbb{N}$, $|a_{n+p}-a_n|<\varepsilon$, where $n_0>\frac{1}{\varepsilon}$. □

Exercise 5.
Prove that the sequence
$$a_n=\frac{1}{1}+\frac{1}{n}+\dots+\frac{1}{n}$$
diverges.

Solution 5.
If possible, let the sequence converge. Then, by the Cauchy Characterisation of Convergence, $\forall\varepsilon>0$, $\exists n_0\in\mathbb{N}$, s.t. $\forall n\geq n_0$ and $\forall p\in\mathbb{N}$, $|a_{n+p}-a_n|<\varepsilon$. Therefore,

$$|a_{n+p}-a_n|=\left|\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{n}+\frac{1}{n+p}-\left(\frac{1}{n}+\dots+\frac{1}{n}\right)\right|$$
$$=\frac{1}{n+1}+\dots+\frac{1}{n+p}$$
$$\geq p\cdot\frac{1}{n+p}$$
$$\therefore |a_{n+p}-a_n|>\frac{p}{n+p}$$

If $n=p$,
$$\frac{p}{n+p}=\frac{1}{2}$$
This contradicts the result obtained from the Cauchy Characterisation of Convergence, for $\varepsilon=\frac{1}{4}$.
Therefore, the sequence diverges.

2 Series

Definition 14 (p-series) The series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is called the p-series.

Theorem 12 *The p-series converges for $p>1$ and diverges for $p\leq 1$.*

Theorem 13 *If $\sum a_n$ converges, then $\lim_{n\to\infty}a_n=0$, but the converse is not true.*

Theorem 14 *If $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n\pm b_n)$ and $\sum ca_n$, where c is a constant, also converge. Also,*

$$\sum(a_n\pm b_n)=\sum a_n\pm\sum b_n$$
$$\sum(ca_n)=c\sum a_n$$

2.1 Convergence Criteria

2.1.1 Leibniz's Criteria

Theorem 15 (Leibniz's Criteria for Convergence) *If an alternating series $\sum(-1)^{n-1}a_n$ with $a_n>0$ satisfies*

(1) $a_{n+1}\leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.
(2) $\lim_{n\to\infty}a_n=0$

then the series $(-1)^{n-1}a_n$ converges.

Example 1. The alternating harmonic series $\sum\frac{(-1)^{n-1}}{n}$ converges as $a_n=\frac{1}{n}>0$, a_n decreases and $\lim a_n=0$.

2.1.2 Comparison Test

Theorem 16 (First Comparison Test for Convergence) *Assume $\exists n_0\in\mathbb{N}$, such that $a_n\geq 0$, $b_n\geq 0$, $\forall n\geq n_0$.*

(1) If $a_n\leq b_n$, $\forall n\geq n_0$ and $\sum_{n=1}^{\infty}b_n$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.
(2) If $a_n\geq b_n$, $\forall n\geq n_0$ and $\sum_{n=1}^{\infty}b_n$ diverges, then $\sum_{n=1}^{\infty}a_n$ diverges.

Theorem 17 (Another Formulation of the Comparison Test for Convergence) *Assume $\exists n_0\in\mathbb{N}$, such that $a_n\geq 0$, $b_n\geq 0$, $\forall n\geq n_0$, then if $\lim_{n\to\infty}\frac{a_n}{b_n}$ exists, then $\sum a_n$ and $\sum b_n$ converge or diverge simultaneously.*

2.1.3 d'Alembert Criteria (Ratio Test)

Definition 15 (Absolute and conditional convergence) The series $\sum a_n$ is said to converge absolutely if $\sum|a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum|a_n|$ diverges.

Theorem 18 *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 19 (d'Alembert Criteria (Ratio Test)) *(1) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=L<1$ then $\sum a_n$ converges absolutely.*
(2) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=L>1$(including $L=\infty$), then $\sum a_n$ converges diverges.
(3) If $L=1$, the test does not apply.

2.1.4 Cauchy Criteria (Cauchy Root Test)

Theorem 20 (Cauchy Criteria (Cauchy Root Test)) *(1) If $\overline{\lim}\sqrt[n]{|a_n|}=L<1$ then $\sum a_n$ converges absolutely.*
(2) If $\overline{\lim}\sqrt[n]{|a_n|}=L>1$ (including $L=\infty$), then $\sum a_n$ diverges.
(3) If $L=1$, the test does not apply.

2.1.5 Integral Test

Theorem 21 (Integral Test for Series Convergence) *Let $f(x)$ be a continuous, non-negative, monotonic decreasing function on $[1,\infty)$ and let $a_n=f(n)$. Then the series $\sum_{n=1}^{\infty}a_n$ converges if and only if the improper integral $\int_1^{\infty}f(x)\mathrm{d}x$ converges.*

Theorem 22 *If the series $\sum a_n$ absolutely converges and the series $\sum b_n$ is obtained from $\sum a_n$ by changing the order of the terms in $\sum a_n$ then $\sum b_n$ also absolutely converges and $\sum b_n=\sum a_n$.*

Theorem 23 *If a series converges then the series with brackets without changing the order of terms also converges. That is, if $\sum a_n$ converges, then any series of the form $(a_1+a_2)+(a_3+a_4+a_5)+a_6+\dots$ also converges.*

Theorem 24 *If a series with brackets converges and the terms in the brackets have the same sign, then the series without brackets also converges.*

3 Power Series

Definition 16 (Power series) The series $\sum_{n=0}^{\infty}a_n(x-c)^n$ is called a power series.

Theorem 25 (Cauchy-Hadamard Theorem) *For any power series $\sum_{n=0}^{\infty}a_n(x-c)^n$ there exists the limit, which may be infinity, $R=\frac{1}{\lim_{n\to\infty}\sqrt[n]{|a_n|}}$ and the series converges for $|x-c|<R$ and diverges for $|x-c|>R$. The end points of the interval, i.e. $x=c-R$ and $x=c+R$ must be separately checked for series convergence.*

Definition 17 (Radius of convergence and convergence interval) The number R is called the radius of convergence and the interval $|x-c|<R$ is called the convergence interval of the series. The point c is called the centre of the convergence interval.

Theorem 26 *If $\exists\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$, which may be infinite, then, $R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$.*

Theorem 27 (Stirling's Approximation) *For $n\to\infty$, $n!\approx\left(\frac{n}{e}\right)^n\sqrt{2\pi n}$.*

Exercise 6.

Find the domain of convergence of $\sum_{n=1}^{\infty}\frac{(2x-4)^n}{n}$.

Solution 6.

$$\sum_{n=1}^{\infty}\frac{(2x-4)^n}{n}=\sum_{n=1}^{\infty}\frac{2^n(x-2)^n}{n}$$

Therefore, by Cauchy's Formula for Radius of Convergence,

$$\begin{aligned}R&=\frac{1}{\overline{\lim}\sqrt[n]{|a_n|}}\\&=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{\frac{2^n}{n}}}\\&=\frac{1}{\lim\limits_{n\to\infty}\frac{2}{\sqrt[n]{n}}}\\&=\frac{1}{2}\end{aligned}$$

Therefore, the series converges for

$$|x-2|<\frac{1}{2}$$

and diverges for

$$|x-2|>\frac{1}{2}$$

If $x=\frac{5}{2}$,

$$\sum_{n=1}^{\infty}\frac{2^n}{n}\left(\frac{5}{2}-2\right)^n$$

$$=\sum_{n=1}^{\infty}\frac{1}{n}$$

Therefore, the series diverges.

If $x=\frac{3}{2}$,

$$\sum_{n=1}^{\infty}\frac{2^n}{n}\left(\frac{3}{2}-2\right)^n$$

$$=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

Therefore, by Leibniz's Criteria for Convergence, the series converges.

Therefore, the domain of convergence is $\left[\frac{3}{2},\frac{5}{2}\right)$.

Exercise 7.

Find the radius of convergence of $\sum\limits_{n=0}^{\infty}n!x^{n!}$.

Solution 7.

$$\frac{1}{\overline{\lim}\sqrt[n]{a_n}}=x+x+2x^2+6x^6+24x^{24}+\dots$$

Therefore,

$$a_n=\begin{cases}n & ; \quad n=k^2\\ 0 & ; \quad n\neq k^2\end{cases}$$

Therefore,

$$\begin{aligned}R&=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{a_n}}\\&=\frac{1}{\lim\limits_{k\to\infty}\sqrt[k!]{k!}}\\&=1\end{aligned}$$

4 Series of Real-valued Functions

Definition 18 (Sequence of functions) A sequence $\{f_n\}=f_1(x),f_2(x),\dots$ defined on $D\subseteq\mathbb{R}$ is called a sequence of functions.

Definition 19 (Pointwise convergence and domain of convergence) $\{f_n\}$ converges pointwise in some domain $E\subseteq D$ if for every $x\in E$, the sequence of $\{f_n(x)\}$ converges. In such a case, E is said to be a domain of convergence of $\{f_n\}$.

Exercise 8.

Find the domain of convergence of $f_n(x)=x^n$, defined on some $D\subseteq\mathbb{R}$.

Solution 8.

$$\lim_{n\to\infty}f_n(x)=\begin{cases}0 & ; \quad -1<x<1\\ 1 & ; \quad x=1\\ \text{diverges} & ; \quad x\notin(-1,1]\end{cases}$$

Therefore, the domain of convergence of $\{f_n\}$ is $(-1,1]$.

Exercise 9.

Let $f(x):(0,\infty)\to\mathbb{R}$ be some function such that $\lim\limits_{x\to\infty}f(x)=0$. Let $f_n(x)=f(nx)$. What is the domain of convergence of f_n? What is the limit function?

Solution 9.

Let x have some fixed value in $(0,\infty)$. Therefore, as $\lim\limits_{x\to\infty}f(x)=0$,

$$\begin{aligned}\lim_{n\to\infty}f_n(x)&=\lim_{n\to\infty}f(nx)\\&=0\end{aligned}$$

Therefore, the domain of convergence is $(0,\infty)$ and the limit function is a constant function with value 0.

4.1 Uniform Convergence of Series of Functions

Definition 20 (Pointwise convergence of a sequence of functions) If $\forall x\in D$, $\forall\varepsilon>0$, $\exists N$ which depends on ε and x, such that $\forall n\geq N$, $|f_n(x)-f(x)|<\varepsilon$, then $\forall x\in D$, $\lim\limits_{n\to\infty}=f(x)$.

Definition 21 (Uniform convergence of a sequence of functions) The sequence $\{f_n(x)\}$ is said to converge uniformly to $f(x)$ in D if $\forall\varepsilon>0$, $\exists N=N(\varepsilon)$, such that $\forall n\geq N$, $\forall x\in D$, $|f_n(x)-f(x)|<\varepsilon$. It can be denoted as $f_n(x)\overset{D}{\rightrightarrows}f(x)$.

Theorem 28 *$f_n(x)$ converges uniformly to $f(x)$ in D if and only if* $\lim\limits_{n\to\infty}\sup\limits_{x\in D}|f_n(x)-f(x)|=0$.

Exercise 10.

Does $f_n(x)=x^n$ converge in $[0,1]$?

Solution 10.

$$\begin{aligned}\lim_{n\to\infty}f_n(x)&=\lim_{n\to\infty}x^n\\ \therefore f(x)&=\begin{cases}0 & ; \quad 0\leq x<1\\ 1 & ; \quad x=1\end{cases}\end{aligned}$$

Therefore,
If $x=0$,

$$f_n(0)=0$$
$$f(0)=0$$

Therefore, $\forall\varepsilon>0$, $N=1$,

$$|0-0|<\varepsilon$$
$$\therefore|f_n(0)-f(0)|<\varepsilon$$

If $x=1$,

$$f_n(1)=1$$
$$f(1)=1$$

Therefore, $\forall\varepsilon>0$, $N=1$,

$$|1-1|<\varepsilon$$
$$\therefore|f_n(1)-f(1)|<\varepsilon$$

If $0<x<1$,

$$\begin{aligned}|f_n(x)-f(x)|&=|x^n-0|\\&=x^n\end{aligned}$$

If possible, let $|f_n(x)-f(x)|=x^n<\varepsilon$.
Therefore,

$$\begin{aligned}x^n&<\varepsilon\\ \therefore\log_x x^n&>\log_x\varepsilon\\ \therefore n&>\log_x\varepsilon\end{aligned}$$

Therefore, for $N=\lfloor\log_x\varepsilon\rfloor+1$, $|f_n(x)-f(x)|<\varepsilon$.

Therefore, $f_n(x)$ converges pointwise in $[0,1]$.

If possible let $f_n(x)$ converge uniformly on $[0,1]$.
Therefore, $\forall\varepsilon>0$, $\exists N$ dependent on ε, such that $|f_n(x)-f(x)|<\varepsilon$.
Let $\varepsilon=\frac{1}{3}$.
Therefore, $\exists N$ which is dependent on ε, such that $\forall n>N$, $\forall x\in[0,1]$,

$$|f_n(x)-f(x)|<\frac{1}{3}$$

Let $x=\frac{1}{2}$, $n=N+1$. Therefore,

$$\left|f_n\left(\frac{1}{2}\right)-f\left(\frac{1}{2}\right)\right|=\left|\frac{1}{2}-0\right|$$

$$=\frac{1}{2}$$

$$\therefore \left|f_n\left(\frac{1}{2}\right)-f\left(\frac{1}{2}\right)\right|>\frac{1}{3}$$

Therefore, $|f_n(x)-f(x)|>\varepsilon$.
This is a contradiction. Hence, $f_n(x)$ is does not converge uniformly.

Definition 22 (Supremum) Let $A\subseteq\mathbb{R}$ be a bounded set. M is said to be the supremum of A if

(1) $\forall x\in A$, $x\le M$, i.e. M is an upper bound of A.
(2) $\forall\varepsilon$, $\exists x\in A$, such that $x>M-\varepsilon$.

That is, the supremum of A is the least upper bound of A.
The supremum may or may not be in A.

Definition 23 (Infimum) Let $A\subseteq\mathbb{R}$ be a bounded set. M is said to be the infimum of A if

(1) $\forall x\in A$, $x\ge M$, i.e. M is an upper bound of A.
(2) $\forall\varepsilon$, $\exists x\in A$, such that $x<M-\varepsilon$.

That is, the infimum of A is the greatest lower bound of A. The infimum may or may not be in A.

Theorem 29 *Every bounded set A has a supremum and an infimum.*

Theorem 30 $f_n \overset{E}{\rightrightarrows} f$ *if and only if*

$$\lim_{n\to\infty}\left(\sup\{|f_n(x)-f(x)|:x\in E\}\right)=0$$

Definition 24 (Remainder of a series of functions) Let $f(x)=\sum_{k=1}^{\infty}u_k(x)$.

Let the partial sums be denoted by $f_n(x)=\sum_{k=1}^{n}u_k(x)$. Then

$$R_n(x)=f(x)-f_n(x)=\sum_{k=n+1}^{\infty}u_k(x)$$

is called a remainder of the series $f(x)=\sum_{k=1}^{\infty}u_k(x)$.

Definition 25 (Uniform convergence of a series of functions) If $f_n(x)$ converges uniformly to $f(x)$ on D, i.e. if $\lim_{n\to\infty}R_n(x)=0$, then the series $\sum_{k=1}^{\infty}u_k(x)$ is said to converge uniformly on D..

Exercise 11.

Show that the series $f(x)=\sum_{k=1}^{\infty}x^{k-1}=\frac{1}{1-k}$ does not converge uniformly on $(-1,1)$.

Solution 11.
The series converges uniformly if and only if $\lim_{n\to\infty}R_n(x)=0$.

$$\begin{aligned}\lim_{n\to\infty}\sup_{(-1,1)}|R_n(x)-0| &= \lim_{n\to\infty}\sup_{(-1,1)}\sum_{k=n+1}^{\infty}x^{k-1}\\ &= \lim_{n\to\infty}\sup_{(-1,1)}\left|\frac{x^n}{1-x}\right|\\ &= \lim_{n\to\infty}\sup_{(-1,1)}\frac{|x|^n}{1-x}\\ &= \lim_{n\to\infty}\infty\\ &=\infty\end{aligned}$$

Therefore, the series does not converge uniformly on $(-1,1)$.

Exercise 12.

Show that the series $f(x)=\sum_{k=1}^{\infty}x^{k-1}=\frac{1}{1-k}$ does not converge uniformly on $\left(-\frac{1}{2},\frac{1}{2}\right)$.

Solution 12.
The series converges uniformly if and only if $\lim_{n\to\infty}R_n(x)=0$.

$$\begin{aligned}\lim_{n\to\infty}\sup_{\left(-\frac{1}{2},\frac{1}{2}\right)}|R_n(x)-0| &= \lim_{n\to\infty}\sup_{\left(-\frac{1}{2},\frac{1}{2}\right)}\sum_{k=n+1}^{\infty}x^{k-1}\\ &= \lim_{n\to\infty}\sup_{\left(-\frac{1}{2},\frac{1}{2}\right)}\left|\frac{x^n}{1-x}\right|\\ &= \lim_{n\to\infty}\sup_{\left(-\frac{1}{2},\frac{1}{2}\right)}\frac{|x|^n}{1-x}\\ &= \lim_{n\to\infty}\frac{\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\\ &= \lim_{n\to\infty}\left(\frac{1}{2}\right)^{n-1}\\ &=0\end{aligned}$$

Therefore, the series converges uniformly on $\left(-\frac{1}{2},\frac{1}{2}\right)$.

4.2 Weierstrass M-test

Theorem 31 (Weierstrass M-test) *If $|u_k(x)|\le c_k$ on D for $k\in\{1,2,3,\ldots\}$ and the numerical series $\sum_{k=1}^{\infty}c_k$ converges, then the series of functions $\sum_{k=1}^{\infty}u_k(x)$ converges uniformly on D.*

Exercise 13.

Show that $\sum_{k=1}^{\infty}\frac{1}{k^2}\sin(kx)$ converges uniformly on $\mathbb{R}$.

Solution 13.

$$|u_k(x)|=\left|\frac{1}{k^2}\sin(kx)\right|$$

$$\therefore|u_k(x)|\le\frac{1}{k^2}$$

Therefore, let

$$c_k=\frac{1}{k^2}$$

Therefore, as $|u_k(x)|\le c_k$, and as $\sum_{k=1}^{\infty}c_k$ converges, by the Weierstrass M-test, $\sum_{k=1}^{\infty}\frac{1}{k^2}\sin(kx)$ converges uniformly.

4.3 Application of Uniform Convergence

Theorem 32 (Continuity of a series) *Let functions $u_k(x)$, $k\in\{1,2,3,\ldots\}$ be defined on $[a,b]$ and continuous at $x_0\in[a,b]$. If $\sum_{k=1}^{\infty}u_k(x)$ converges uniformly on $[a,b]$ then the function $f(x)=\sum_{k=1}^{\infty}$ is also continuous at x_0.*

Theorem 33 (Changing the order of integration and infinite summation) *If the functions $u_k(x)$, $k\in\{1,2,3,\ldots\}$ are integrable on $[a,b]$ and the series $\sum_{k=1}^{\infty}u_k(x)$ converges uniformly on $[a,b]$ then*

$$\int_a^b\left(\sum_{k=1}^{\infty}u_k(x)\right)\mathrm{d}x=\sum_{k=1}^{\infty}\int_a^b u_k(x)\mathrm{d}x$$

Exercise 14.

Solve $\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\sin(kx)\right)$.

Solution 14.

The series $f(x)=\sum_{k=1}^{\infty}\frac{1}{k^2}\sin(kx)$ converges uniformly on $[0,2\pi]$. Therefore, by the Weierstrass M-test and $u_k(x)=\frac{1}{k^2}(kx)$ are integrable on $[0,2\pi]$. Therefore,

$$\begin{aligned}\int_0^{2\pi}f(x)\mathrm{d}x &= \int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\sin(kx)\right)\mathrm{d}x\\ &=\sum_{k=1}^{\infty}\left(\int_0^{2\pi}\frac{1}{k^2}\sin(kx)\mathrm{d}x\right)\\ &=\sum_{k=1}^{\infty}\left(-\frac{\cos(2\pi k)}{k^3}+\frac{1}{k^3}\right)\end{aligned}$$

$$=\sum_{k=1}^{\infty}0$$
$$=0$$

Theorem 34 (Changing the order of differentiation and infinite summation) *If the functions $u_k(x)$, $k\in\{1,2,3,...\}$ are differentiable on $[a,b]$ and the derivatives are continuous on $[a,b]$, and the series $\sum\limits_{k=1}^{\infty}u_k(x)$ converges pointwise on $[a,b]$ and the series $\sum\limits_{k=1}^{\infty}{u_k}'(x)$ converges uniformly on $[a,b]$, then,*

$$\left(\sum_{k=1}^{\infty}u_k(x)\right)'=\sum_{k=1}^{\infty}{u_k}'(x)$$

Theorem 35 (Changing the order of integration and limit) *If the functions $f_n(x)$ are integrable on $[a,b]$ and converge uniformly to f on $[a,b]$, then*

$$\lim_{n\to\infty}\int_a^b f_n(x)\mathrm{d}x=\int_a^b \lim_{n\to\infty}f_n(x)\mathrm{d}x=\int_a^b f(x)\mathrm{d}x$$

Theorem 36 (Changing the order of differentiation and limit) *If there exists the functions ${f_n}'(x)$ which are continuous on $[a,b]$, for the functions $f_n(x)$ which $\forall x\in[a,b]$, converge pointwise to $f(x)$ on $[a,b]$, and if ${f_n}'(x)$ converges uniformly to $g(x)$ on $[a,b]$, then,*

$$f'(x)=\left(\lim_{n\to\infty}f_n(x)\right)'=\lim_{n\to\infty}{f_n}'(x)=g(x)$$

2 Functions of Multiple Variables

1 Limits, Continuity, and Differentiability

Definition 26 (Limit of a function of two variables) Let $z=f(x,y)$ be defined on some open neighbourhood about (a,b), except maybe at the point itself. $L\in\mathbb{R}$ is said to be a limit of $f(x,y)$ at (a,b), if $\forall\varepsilon>0$, $\exists d>0$, such that $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$, then,

$|f(x,y)-L|<\varepsilon$

Exercise 15.

Does the limit $\lim\limits_{(x,y)\to(0,0)}\frac{3x^2y}{x^2+y^2}$ exist?

Solution 15.

Consider the curves $C_1:y=0$, and $C_2:y=x^3$.
Therefore, as $(x,y)\to(0,0)$ along these curves, the limit of the function is

$$\lim_{(x,y)\xrightarrow{C_1}(0,0)}\frac{3x^2y}{x^2+y^2}=\lim_{x\to0}\frac{3x^2\cdot0}{x^2+y^2}$$
$$=0$$

$$\lim_{(x,y)\xrightarrow{C_2}(0,0)}\frac{3x^2y}{x^2+y^2}=\lim_{x\to0}\frac{3x^2(x^3)}{x^2+(x^3)^2}$$
$$=\lim_{x\to0}\frac{3x^5}{x^2+x^6}$$
$$=\lim_{x\to0}\frac{3x^3}{x^2+x^4}$$
$$=0$$

If $\lim\limits_{(x,y)\to(0,0)}\frac{3x^2y}{x^2+y^2}=0$, $\forall\varepsilon>0$, $\exists\delta>0$ such that $0<\sqrt{x^2+y^2}<\delta$, then,
$|f(x,y)-L|<\varepsilon$
Therefore, checking $|f(x,y)-L|$,

$$\left|f(x,y)-L\right|=\left|\frac{3x^2y}{x^2+y^2}-0\right|$$
$$=\frac{3x^2|y|}{x^2+y^2}$$

As $\frac{x^2}{x^2+y^2}\leq1$,

$$\left|f(x,y)-L\right|\leq3|y|$$
$$\therefore\left|f(x,y)-L\right|\leq3\sqrt{y^2}$$
$$\therefore\left|f(x,y)-L\right|\leq3\sqrt{x^2+y^2}$$

Therefore, $|f(x,y)-L|<\varepsilon$.
Therefore, for $\delta\leq\frac{\varepsilon}{3}$, the condition is satisfied.
Hence, the limit of the function exists and is 0.

Definition 27 (Iterative limits) The limits $\lim\limits_{x\to a}\left(\lim\limits_{y\to b}f(x,y)\right)$ and $\lim\limits_{y\to b}\left(\lim\limits_{x\to a}f(x,y)\right)$ are called the iterative limits of $f(x,y)$.

Theorem 37 *If $\exists\lim\limits_{(x,y)\to(a,b)}f(x,y)=L$ and, for some open interval about b, $\forall y\neq b$, $\exists\lim\limits_{x\to a}f(x,y)$ then*

$$\lim_{y\to b}\left(\lim_{x\to a}f(x,y)\right)=L$$

If $\exists\lim\limits_{(x,y)\to(a,b)}f(x,y)=L$ and, for some open interval about a, $\forall x\neq a$, $\exists\lim\limits_{y\to b}f(x,y)$ then

$$\lim_{x\to a}\left(\lim_{y\to b}f(x,y)\right)=L$$

Definition 28 (Differential)
$\Delta z=f(a+\Delta x,b+\Delta y)-f(a,b)$
$\mathrm{d}z=f_x(a,b)\mathrm{d}x+f_y(a,b)\mathrm{d}y$

Definition 29 (Differentiability) The function $x=f(x,y)$ is said to be differentiable at (a,b) if
$\Delta z=\mathrm{d}z+\varepsilon_1(\Delta x,\Delta y)\Delta x+\varepsilon_2(\Delta x,\Delta y)\Delta y$
where
$\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_1(\Delta x,\Delta y)=\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_2(\Delta x,\Delta y)=0$

Theorem 38 *If $f(x,y)$ is differentiable at (a,b) then $f(x,y)$ is continuous at (a,b).*

Theorem 39 *If $\exists f_x(a,b)$ and $\exists f_y(a,b)$ on some open neighbourhood of (a,b) and are continuous at (a,b), then $f(x,y)$ is differentiable at (a,b).*

2 Directional Derivatives and Gradients

Definition 30 (Directional derivative) Let $x_0\in\mathbb{R}$, $y_0\in\mathbb{R}$.
Let $\hat{u}=(a,b)$ be a unit vector in the xy-plane.
The directional derivative of $z=f(x,y)$ with respect to the direction $\hat{u}=(a,b)$ at the point $(x_0,y_0$ is defined as

$$D_{\hat{u}}f(x_0,y_0)=\lim_{h\to0}\frac{f(x_0+ah,y_0+bh)-f(x_0,y_0)}{h}$$

If the limit does not exist, the directional derivative does not exist.

Geometrically the directional derivative of $z=f(x,y)$ is the slope of the tangent of the curve formed due to the intersection of the curve $z=f(x,y)$, and the plane which passes through (x_0,y_0) in the direction of $\hat{u}$ and is perpendicular to the xy-plane.

Definition 31 (Gradient) If the functions $f_x(x,y)$ and $f_y(x,y)$ for $z=f(x,y)$ exist, then the vector function

$$\nabla f(x,y)=\big(f_x(x,y),f_y(x,y)\big)$$

is called the gradient of $f(x,y)$.

Theorem 40 *Let $z=f(x,y)$ be differential at (x_0,y_0). The function $f(x,y)$ has a directional derivative with respect to any direction $\hat{u}=(a,b)$ at (x_0,y_0) and*
$D_{\hat{u}}f(x_0,y_0)=f_x(x_0,y_0)a+f_y(x_0,y_0)b=\nabla f(x_0,y_0)\cdot\hat{u}$

Exercise 16.

Find the directional derivative of
$f(x,y)=x^3+4xy+y^4$
with respect to the direction of $\overline{u}=(1,2)$ at any point (x,y) and at $(0,1)$.

Solution 16.

$f(x,y)=x^3+4xy+y^4$
Therefore,
$f_x(x,y)=3x^2+4y$
$f_y(x,y)=4x+4y^3$

$$\hat{u}=\frac{\overline{u}}{u}$$
$$=\frac{(1,2)}{\sqrt{5}}$$
$$=\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)$$

Therefore,

$$D_{\hat{u}}f(x,y)=\frac{1}{\sqrt{5}}(3x^2+4y)+\frac{2}{\sqrt{5}}(4x+4y^2)$$

Therefore,

$$D_{\hat{u}}f(0,1)=\frac{4}{\sqrt{5}}+\frac{8}{\sqrt{5}}$$
$$=\frac{12}{\sqrt{5}}$$

Theorem 41 *If $z=f(x,y)$ is differentiable at (x_0,y_0), then $\exists\hat{u_0}=(a_0,b_0)$ such that*

$$\max_{\hat{u}\in\mathbb{R}} D_{\hat{u}}f(x_0,y_0)=D_{\hat{u_0}}f(x_0,y_0)=\left|\nabla f(x_0,y_0)\right|$$

and

$$\hat{u_0}=\frac{\nabla f(x_0,y_0)}{\left|\nabla f(x_0,y_0)\right|}$$

Theorem 42 *If $z=f(x,y)$ is differentiable at (x_0,y_0), then $\exists\hat{u_1}=(a_0,b_0)$ such that*

$$\min_{\hat{u}\in\mathbb{R}} D_{\hat{u}}f(x_0,y_0)=D_{\hat{u_1}}f(x_0,y_0)=-\left|\nabla f(x_0,y_0)\right|$$

and

$$\hat{u_1}=-\frac{\nabla f(x_0,y_0)}{\left|\nabla f(x_0,y_0)\right|}$$

3 Local Extrema

Theorem 43 (A necessary condition for local extrema existence) *If the function $z=f(x,y)$ has a local extrema at the point (a,b) and $\exists f_x(a,b)$ and $\exists f_y(a,b)$ then $f_x(a,b)=f_y(a,b)=0$*

Definition 32 (Critical point) Let the function $z=f(x,y)$ be defined on some open neighbourhood of (a,b). The point (a,b) is called a critical point of $z=f(x,y)$ if $f_x(a,b)=f_y(a,b)=0$ or at least one of the partial derivative $f_x(a,b)$ and $f_y(a,b)$ does not exist.

Theorem 44 (A sufficient condition for local extrema point) *Assume that there exist second order partial derivates of $z=f(x,y)$, they are continuous on some open neighbourhood of (a,b) and $f_x(a,b)=f_y(a,b)=0$. Denote*

$$D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left(f_{xy}(a,b)\right)^2$$

(1) If $D(a,b)>0$ and $f_{xx}<0$ then (a,b) is a local maximum point.
(2) If $D(a,b)>0$ and $f_{xx}>0$ then (a,b) is a local minimum point.
(3) If $D(a,b)<0$ then (a,b) is called a saddle point.

4 Global Extrema

4.1 Algorithm for Finding Maxima and Minima of a Function

Step 1 Find all critical points of $f(x,y)$ on the domain, excluding the end points.
Step 2 Calculate the values of $f(x,y)$ at the critical points.
Step 3 Calculate the values of $f(x,y)$ at the end points of the domain.
Step 4 Select the maximum and minimum values from Step 2 and Step 3

5 Taylor's Formula

Theorem 45

$$f(a+h,b+k)=\sum_{i=0}^{n}\left(\frac{1}{i!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^i f(a,b)\right)$$
$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(a+ch,b+ck)$$

where $0<c<1$.

6 Vector Functions and Curves in $\mathbb{R}^3$

Definition 33 (Vector function) A vector function is a function with a domain which consists of a set of real numbers, and with a domain which consists of a set of vectors, i.e. $\overline{r}(t)=\left(f(t),g(t),h(t)\right)$, $\forall t\in[a,b]$.

Theorem 46 *If $\exists\lim_{t\to t_0}f(t)$, $\exists\lim_{t\to t_0}g(t)$, $\exists\lim_{t\to t_0}h(t)$, then, $\exists\lim_{t\to t_0}=\left(\lim_{t\to t_0}f(t),\lim_{t\to t_0}g(t),\lim_{t\to t_0}h(t)\right)$.*

Definition 34 (Continuous vector function) A vector function $\overline{r}(t)$ is said to be continuous at t_0 if $\lim_{t\to t_0}\overline{r}(t)=\overline{r}(t_0)$.

Definition 35 (Space curve) Let $f(t)$, $g(t)$, $h(t)$ be continuous functions of $[a,b]$. The set of points (x,y,z), such that $x=f(t)$, $y=g(t)$, $z=h(t)$, $t\in[a,b]$ is called a space curve.

7 Derivatives of Vector Functions

Definition 36 (Derivative of vector function) The derivative of $\overline{r}(t)=\left(f(t),g(t),h(t)\right)$, if it exists, is defined as

$$\overline{r}'(t)=\lim_{\Delta t\to 0}\frac{\overline{r}(t+\Delta t)-\overline{r}(t)}{\Delta t}$$

Definition 37 (Tangent vector) $\overline{r}'(t_0)$ is called a tangent vector to the curve $C=\overline{r}(t)$ at $P(t_0)$.

Theorem 47 *If $\exists f'(t_0)$, $\exists g'(t_0)$, $\exists h'(t_0)$, and $\overline{r}(t)=\left(f(t),g(t),h(t)\right)$, then,*

$$\overline{r}'(t_0)=\left(f'(t_0),g'(t_0),h'(t_0)\right)$$

Definition 38 (Unit tangent vector) The vector $\hat{T}(t)=\frac{\overline{r}'(t)}{|\overline{r}'(t)|}$ is called the unit tangent vector to $C=r(t)$ at $P(t_0)$.

Definition 39 (Tangent line) A straight line passing through a point $P(t)$ on the curve $C=r(t)$, in the direction $\overline{r}'(t)$, i.e. $\hat{T}(t)$, is called a tangent line to the curve at the point.

Theorem 48 *Let $\overline{u}(t)$ and $\overline{v}(t)$ be vector functions, let c be a constant, and let $f(t)$ be a scalar function. Then,*

(1) $\left(\overline{u}(t)\pm\overline{v}(t)\right)'=\overline{u}'(t)\pm\overline{v}'(t)$
(2) $\left(c\overline{u}(t)\right)'=c\overline{u}'(t)$
(3) $\left(f(t)\overline{u}(t)\right)'=f'(t)\overline{u}(t)+f(t)\overline{u}'(t)$
(4) $\left(\overline{u}(t)\cdot\overline{v}(t)\right)'=\overline{u}'(t)\cdot\overline{v}(t)+\overline{u}(t)\cdot\overline{v}'(t)$
(5) $\left(\overline{u}(t)\times\overline{v}(t)\right)'=\overline{u}'(t)\times\overline{v}(t)+\overline{u}(t)\times\overline{v}'(t)$
(6) $\left(\overline{u}\left(f(t)\right)\right)'=f'(t)\overline{u}'\left(f(t)\right)$

8 Change of Variables in Double Integrals

Definition 40 (Jacobian) Let

$$T(u,v)=(x,y)$$

be an operator.
The determinant

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}x_u & x_v\\ y_u & y_v\end{vmatrix}$$

is called the Jacobian of the operator T.

Theorem 49 *Let R and S be domains of the first or second kind.*
Let the operator T from S to R be one-to-one and onto.
Therefore, the inverse operator T^{-1} exists.
Also, let T be a C^1 operator, i.e. $\exists x_u$, $\exists x_v$, $\exists y_u$, $\exists y_v$, which are continuous on S.
Let $f(x,y)$ be a continuous function on R.
Then,

$$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y=\iint_S f\left(g(u,v),h(u,v)\right)|J|\mathrm{d}u\mathrm{d}v$$

Exercise 17.

Calculate $\iint_R (x-y)^2\sin^2(x+y)\mathrm{d}x\mathrm{d}y$, where R the area bounded by the square with vertices at $(\pi,0)$, $(2\pi,\pi)$, $(\pi,2\pi)$, and $(0,\pi)$.

Solution 17.

The edges of the domain are

$$x+y=\pi$$
$$x+y=3\pi$$
$$x-y=\pi$$
$$x-y=-\pi$$

Therefore, let

$$x-y=u$$
$$x+y=v$$

Therefore,

$$x=\frac{u+v}{2}$$
$$y=\frac{v-u}{2}$$

Therefore, the domain R can be written as $S=\{-\pi\le u\le\pi,\pi\le v\le 3\pi\}$.
Therefore,

$$J=\begin{pmatrix}x_u & x_v\\ y_u & y_v\end{pmatrix}$$
$$=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}$$
$$=\frac{1}{2}$$

Therefore,

$$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y=\iint_S f\big(g(u,v),h(u,v)\big)|J|\mathrm{d}u\mathrm{d}v$$

$$\therefore \iint_R (x-y)^2\sin^2(x+y)\mathrm{d}x\mathrm{d}y=\int_S u^2\sin^2 v\left|\frac{1}{2}\right|\mathrm{d}u\mathrm{d}v$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}\int_{\pi}^{3\pi} u^2\sin^2 v\mathrm{d}v\mathrm{d}u$$

$$=\frac{1}{2}\int_{-\pi}^{\pi} u^2\mathrm{d}u\cdot\int_{\pi}^{3\pi}\sin^2 v\mathrm{d}v$$

$$=\frac{1}{2}\left.\frac{u^3}{3}\right|_{-\pi}^{\pi}\cdot\int_{\pi}^{3\pi}\frac{1-\cos 2v}{2}\mathrm{d}v$$

$$=\frac{1}{2}\frac{2\pi^3}{3}\cdot\frac{1}{2}2\pi$$

$$=\frac{\pi^4}{3}$$

8.1 Polar Coordinates

Polar coordinates are a special case of change of variables.
The operator for the change of variables is
$T(r,\theta)=(x,y)$
where
$x=r\cos\theta$
$y=r\sin\theta$
Therefore,

$$J=\begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$=\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta$$

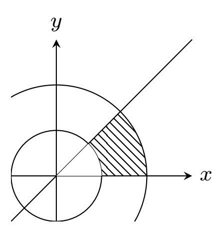
$$=r$$

Exercise 18.

Calculate $\iint_R xy\mathrm{d}x\mathrm{d}y$, $R=\left\{(x,y)|1\le x^2+y^2\le 4, 0\le y\le x\right\}$.

Solution 18.

The domain R is the region shown.



Therefore, it can be written as $S=\left\{(r,\theta)|1\le r\le 2, 0\le\theta\le\frac{\pi}{4}\right\}$.
Therefore,

$$\iint_R xy\mathrm{d}x\mathrm{d}y=\int_0^{\frac{\pi}{4}}\int_1^2 r\cos\theta r\sin\theta r\mathrm{d}r\mathrm{d}\theta$$

$$=\int_1^2 r^3\mathrm{d}r\cdot\int_0^{\frac{\pi}{4}}\cos\theta\sin\theta\mathrm{d}\theta$$

$$=\frac{15}{4}\cdot\frac{1}{4}$$

$$=\frac{15}{16}$$

Theorem 50 *Let D be a domain, written as D_{I} in polar coordinates, i.e.,*
$D_{\mathrm{I}}=\left\{(r,\theta)|a\le r\le b, g_1(r)\le\theta\le g_2(r)\right\}$
and let $f(x,y)$ be continuous on D_{I}.
Then,

$$\iint_{D_{\mathrm{I}}} f(x,y)\mathrm{d}x\mathrm{d}y=\int_a^b\int_{g_1(r)}^{g_2(r)} f(r\cos\theta,r\sin\theta)r\mathrm{d}\theta\mathrm{d}r$$

Theorem 51 *Let D be a domain, written as D_{II} in polar coordinates, i.e.,*
$D_{\mathrm{I}}=\left\{(r,\theta)|\alpha\le\theta\le\beta, h_1(\theta)\le r\le h_2(\theta)\right\}$
and let $f(x,y)$ be continuous on D_{II}.
Then,

$$\iint_{D_{\mathrm{II}}} f(x,y)\mathrm{d}x\mathrm{d}y=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,r\sin\theta)r\mathrm{d}r\mathrm{d}\theta$$

9 Change of Variables in Triple Integrals

Definition 41 (Jacobian) Let
$T(u,v,w)=(x,y,z)$
be an operator.
The determinant

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

is called the Jacobian of the operator T.

Theorem 52 *Let R and S be domains of the first, second, or third kind.*
Let the operator T from S to R be one-to-one and onto.
Therefore, the inverse operator T^{-1} exists.
Also, let T be a C^1 operator, i.e. $\exists x_u$, $\exists x_v$, $\exists x_w$, $\exists y_u$, $\exists y_v$, $\exists y_w$, $\exists z_u$, $\exists z_v$, $\exists z_w$, which are continuous on S.
Let $f(x,y,z)$ be a continuous function on R.
Then,

$$\iiint_R f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint_S f\big(x(u,v,w),y(u,v,w),z(u,v,w)\big)|J|\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

9.1 Cylindrical Coordinates

Cylindrical coordinates are a special case of change of variables.
The operator for the change of variables is
$T(r,\theta,z)=(x,y,z)$
where
$x=r\cos\theta$
$y=r\sin\theta$
$z=z$
Therefore,

$$J=\begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix}$$

$$=\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta$$

$$=r$$

Exercise 19.

Calculate the iterative integral

$$I=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}\left(x^2+y^2\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

Solution 19.

The domain $\left\{(x,y)|-2\le x\le 2, -\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\right\}$ is a circle of radius 2.
As $\sqrt{x^2+y^2}\le z\le 2$, the domain E, where $-2\le x\le 2$, $-\sqrt{4-x^2}\le y\le\sqrt{4-x^2}$, $\sqrt{x^2+y^2}\le z\le 2$ is a cone, with the circular cross section of radius x^2+y^2.
Therefore,

$$I=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}\left(x^2+y^2\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

$$=\iiint_E\left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Therefore, let $D_{\mathrm{I}}=\left\{(r,\theta,z)|0\le r\le 2,0\le\theta\le 2\pi,r\le z\le 2\right\}$.
Therefore,

$$I=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}\left(x^2+y^2\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$
$$=\iiint_E\left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\iiint_{D_{\mathrm{I}}}r^2\cdot r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$$
$$=\int_0^{2\pi}\int_0^2\int_r^2 r^3\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$$
$$=\int_0^{2\pi}\int_0^2 r^3z\Big|_{z=r}^{z=2}\mathrm{d}r\mathrm{d}\theta$$
$$=\int_0^{2\pi}\int_0^2\left(2r^3-r^4\right)\mathrm{d}r\mathrm{d}\theta$$
$$=\frac{16\pi}{5}$$

9.2 Spherical Coordinates

Spherical coordinates are a special case of change of variables.
The operator for the change of variables is
$T(\rho,\theta,\varphi)=(x,y,z)$
where
$x=\rho\cos\theta\sin\varphi$
$y=\rho\sin\theta\sin\varphi$
$z=\rho\cos\varphi$
Therefore,

$$J=\begin{vmatrix}x_\rho & x_\theta & x_\varphi\\ y_\rho & y_\theta & y_\varphi\\ z_\rho & z_\theta & z_\varphi\end{vmatrix}$$
$$=\begin{vmatrix}\cos\theta\sin\theta & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi\\ \sin\theta\sin\theta & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi\\ \cos\varphi & 0 & -r\sin\varphi\end{vmatrix}$$
$$=-\rho^2\sin\varphi$$

Exercise 20.

Given the sphere $B:x^2+y^2+z^2\le 1$, find $I=\iiint\limits_B e^{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\mathrm{d}x\mathrm{d}y\mathrm{d}z$.

Solution 20.

$$I=\iiint_B e^{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{\rho^{\frac{3}{2}}}|J|\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi$$
$$=\int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{\rho^{\frac{3}{2}}}\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi$$
$$=\int_0^{\pi}\int_0^{2\pi}\frac{e^{\rho^3}}{3}\sin\varphi\Bigg|_{\rho=0}^{\rho=1}\mathrm{d}\theta\mathrm{d}\varphi$$
$$=\frac{e-1}{3}\int_0^{\pi}\int_0^{2\pi}\sin\varphi\mathrm{d}\theta\mathrm{d}\varphi$$
$$=\frac{e-1}{3}\int_0^{\pi}\sin\varphi\cdot 2\pi\mathrm{d}\varphi$$
$$=2\pi\frac{e-1}{3}(-\cos\theta)\Big|_0^{\pi}$$
$$=\frac{4\pi(e-1)}{3}$$

Exercise 21.
Calculate the volume of a body which is situated above the cone $z=\sqrt{x^2+y^2}$ and under the sphere $x^2+y^2+z^2=z$.

Solution 21.

$$x^2+y^2+z^2=z$$
$$\therefore x^2+y^2+z^2-z=0$$
$$\therefore x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}$$

Therefore, the sphere has centre $\left(0,0,\frac{1}{2}\right)$ and radius $\frac{1}{2}$.
Therefore, the cone and the sphere intersect each other at $z=\frac{1}{2}$. The intersection is a circle with radius $\frac{1}{2}$.
Therefore, the body is made of a cone of base radius $\frac{1}{2}$ and height $\frac{1}{2}$, and a hemisphere of radius $\frac{1}{2}$.
In Cartesian coordinates, the sphere is $x^2+y^2+z^2=z$.
Therefore, in spherical coordinates, the sphere is $\rho^2=\rho\cos\varphi$. Therefore,

$$V=\iiint\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi}\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$$
$$=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\frac{\rho^3}{3}\sin\varphi\Bigg|_{\rho=0}^{\rho=\cos\varphi}\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$$
$$=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\frac{1}{3}\cos^3\varphi\sin\varphi\mathrm{d}\varphi\mathrm{d}\theta$$
$$=2\pi\cdot\left(-\frac{\cos^4\pi}{12}\right)\Bigg|_0^{\frac{\pi}{4}}$$
$$=2\pi\left(-\frac{1}{48}+\frac{1}{12}\right)$$
$$=2\pi\left(\frac{3}{48}\right)$$
$$=\frac{\pi}{8}$$

10 Line Integrals of Scalar Functions

Definition 42 (Smooth curve) A curve C which is parametrically given as $\overline{r}(t)=\left(x(t),y(t),z(t)\right)$, $t:a\to b$ is said to be smooth if $\overline{r}(t)$ is a continuous function on $[a,b]$, $\overline{r}'(t)\neq 0$ on (a,b), and $\overline{r}'(t)$is continuous on (a,b).

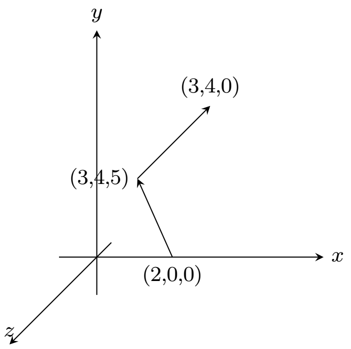
Theorem 53 *If $f(x,y,z)$ is continuous and C is smooth, then*

$$\int_C f(x,y,z)\mathrm{d}s=\int_a^b f\left(x(t),y(t),z(t)\right)\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2+\left(z'(t)\right)^2}\mathrm{d}t$$

Theorem 54 *If $f(x,y,z)$ is continuous and C is smooth, then*

$$\int_C f(x,y,z)\mathrm{d}x=\int_a^b f\left(x(t),y(t),z(t)\right)x'(t)\mathrm{d}t$$
$$\int_C f(x,y,z)\mathrm{d}y=\int_a^b f\left(x(t),y(t),z(t)\right)y'(t)\mathrm{d}t$$
$$\int_C f(x,y,z)\mathrm{d}z=\int_a^b f\left(x(t),y(t),z(t)\right)z'(t)\mathrm{d}t$$

Exercise 22.
Calculate $\int\limits_C y\mathrm{d}x+z\mathrm{d}y+x\mathrm{d}z$ for C as shown.



Solution 22.

$C = C_1 \cup C_2$

Therefore, for $t: 0 \to 1$,

$C_1: \overline{r}(t) = (2 + 1\cdot t, 0 + 4\cdot t, 0 + 5\cdot t)$

$C_2: \overline{r}(t) = (3 + 0\cdot t, 4 + 0\cdot t, 5 - 5\cdot t)$

Therefore,

$$\int_C y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z = \int_{C_1} y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z + \int_{C_2} y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z$$

$$= \int_0^1 \left(y_1(t){x_1}'(t) + z_1(t){y_1}'(t) + x_1(t){z_1}'(t)\right)\mathrm{d}t$$

$$+ \int_0^1 \left(y_2(t){x_2}'(t) + z_2(t){y_2}'(t) + x_2(t){z_2}'(t)\right)\mathrm{d}t$$

$$= \int_0^1 \left(4t + 5t\cdot 4 + (2+t)\cdot 5\right)\mathrm{d}t$$

$$+ \int_0^1 \left(4\cdot 0 + (5-5t)\cdot 0 + 3\cdot(-5)\right)\mathrm{d}t$$

$$= \int_0^1 (29t - 5)\mathrm{d}t$$

$$= \left(29\frac{t^2}{2} - 5t\right)\Bigg|_0^1$$

$$= \frac{19}{2}$$

11 Line Integrals of Vector Functions

Theorem 55 *If $C: \overline{r}(t) = \left(x(t), y(t), z(t)\right)$, $t: a \to b$, then*

$$W = \int_C \overline{F}\cdot\hat{T}\mathrm{d}s$$

$$= \int_a^b \left(\overline{F}\left(\overline{r}(t)\right)\right)\cdot\overline{r}'(t)\mathrm{d}t$$

$$= \int_C \overline{F}\cdot\mathrm{d}\overline{r}$$

$$= \int_a^b \left(P\left(\overline{r}(t)\right)x'(t) + Q\left(\overline{r}(t)\right)y'(t) + R\left(\overline{r}(t)\right)z'(t)\right)\mathrm{d}t$$

$$= \int_C P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$$

Theorem 56 (Fundamental Theorem of Line Integrals) *Let C be a smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ given parametrically by $\overline{r}(t)$, $t: a \to b$. Let f be a continuous function of (x,y) or (x,y,z), on C, and ∇f be a continuous vector function in a connected domain D which contains C. Then*

$$W = \int_C \nabla f\cdot\hat{T}\mathrm{d}s$$

$$= f\left(\overline{r}(b)\right) - f\left(\overline{r}(a)\right)$$

$$= f(B) - f(A)$$

Definition 43 (Simple curve) A curve C is called a simple curve if it does not intersect itself.

Definition 44 (Connected domain) A domain $D \subset \mathbb{R}^2$ is called connected if for any two points from D, the is a path C which connects the points and remains in D.

Definition 45 (Simple connected domain) A connected domain $D \subset \mathbb{R}^2$ is called simple connected if any simple closed curve from D contains inside itself only points in D.

Definition 46 (Curve with positive orientation) A simple closed curve C is called a curve with a positive orientation, or with anti-clockwise orientation if the domain D bounded by C always remains on the left when we circulate over C by $\overline{r}(t), t: a \to b$.

12 Surface Integrals of Scalar Functions

Definition 47 (Parametric representation of surfaces) Let the surface S be given by

$\overline{r}(u,v) = \left(f(u,v), g(u,v), h(u,v)\right)$

The equations

$x = f(u,v)$

$y = g(u,v)$

$z = h(u,v)$

are called the parametric equations of S

Definition 48 If a smooth surface S is given by $\overline{r}(u,v) = \left(x(u,v), y(u,v), z(u,v)\right)$, $u, v \in D$ and $\overline{r}(u,v)$ is one-to-one, then the surface area of S is

$$A = \iint_D |\overline{r}_u \times \overline{r}_v|\mathrm{d}u\mathrm{d}v$$

where

$\overline{r}_u = (x_u, y_u, z_u)$

$\overline{r}_v = (x_v, y_v, z_v)$

Theorem 57 *If S is smooth and given by $z = g(x,y)$, $(x,y) \in D$, then*

$$\iint_S f(x,y,z)\mathrm{d}S = \iint_D f\left(x, y, g(x,y)\right)\sqrt{1 + (g_x)^2 + (g_y)^2}\mathrm{d}x\mathrm{d}y$$

Theorem 58 *If S is smooth and given parametrically by $\overline{r}(u,v) = \left(x(u,v), y(u,v), z(u,v)\right)$, $(u,v) \in D$, then*

$$\iint_S f(x,y,z)\mathrm{d}S = \iint_D f\left(\overline{r}(u,v)\right)|\overline{r}_u \times \overline{r}_v|\mathrm{d}u\mathrm{d}v$$

Exercise 23.

Find $\iint_S x^2\mathrm{d}S$ where $S: x^2 + y^2 + z^2 = 1$.

Solution 23.

In spherical coordinates with $\rho = 1$,

$x = \cos\theta\sin\varphi$

$y = \sin\theta\sin\varphi$

$z = \cos\varphi$

Therefore,

$\overline{r}(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

Therefore,

$\overline{r}_\theta = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$

$\overline{r}_\varphi = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$

Therefore,

$$\overline{r}_\theta \times \overline{r}_\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \end{vmatrix}$$

$$= \hat{i}\left(-\sin^2\varphi\cos\theta\right)$$

$$-\hat{j}\left(\sin^2\varphi\sin\theta\right)$$

$$+\hat{k}\left(-\sin\varphi\cos\varphi\sin^2\theta - \sin\varphi\cos\varphi\cos^2\theta\right)$$

Therefore,

$\left|\overline{r}_\theta \times \overline{r}_\varphi\right| = \sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \sin^2\varphi\cos^2\varphi}$

$$=\sqrt{\sin^4\varphi+\sin^2\varphi\cos^2\varphi}$$
$$=\sqrt{\sin^2\varphi}$$
$$=\sin\varphi$$

Therefore,

$$\iint\limits_S x^2 \mathrm{d}S=\iint\limits_D (\cos\theta\sin\varphi)^2\sin\varphi\mathrm{d}\theta\mathrm{d}\varphi$$
$$=\int\limits_0^{2\pi}\int\limits_0^{\pi}\sin^3\varphi\cos^2\theta\mathrm{d}\varphi\mathrm{d}\theta$$
$$=\int\limits_0^{\pi}\sin^3\varphi\mathrm{d}\varphi\int\limits_0^{2\pi}\cos^2\theta\mathrm{d}\theta$$
$$=\int\limits_0^{\pi}\left(1-\cos^2\varphi\right)\sin\varphi\mathrm{d}\varphi\int\limits_0^{2\pi}\frac{1+\cos2\theta}{2}\mathrm{d}\theta$$
$$=\int\limits_0^{\pi}\left(\sin\varphi-\cos^2\varphi\sin\varphi\right)\mathrm{d}\varphi\left(\frac{\theta}{2}+\frac{\sin2\theta}{4}\bigg|_0^{\pi}\right)$$
$$=\frac{4\pi}{3}$$

13 Surface Integrals of Vector Functions

Definition 49 (Oriented surface) If a normal vector $\overline{n}(x,y,z)$ to the surface S is continuously changing on S then S is said to be an oriented surface.

Theorem 59 *If a surface is given by $F(x,y,z)=k$, then ∇F is a normal vector to the surface at a point on it.*

Definition 50 (Surface with positive orientation) A surface S is said to have positive orientation if $\hat{n}$ is positive.
A closed surface S is said to have positive orientation if $\hat{n}$ is directed outwards.

Definition 51 (Surface Integral of Vector Functions) If
$\overline{F}(x,y,z)=\big(P(x,y,z),Q(x,y,z),R(x,y,z)\big)$
is a continuous vector function on S with orientation $\hat{n}$, then the surface integral of $\overline{F}$ over $\overline{S}$ is

$$\iint\limits_S \overline{F}\cdot\mathrm{d}\overline{S}=\iint\limits_S \overline{F}\cdot\hat{n}\mathrm{d}S$$

This integral is also called the flux of $\overline{F}$ through $\overline{S}$ in direction $\hat{n}$.

Theorem 60 *Let*
$\overline{F}(x,y,z)=\big(P(x,y,z),Q(x,y,z),R(x,y,z)\big)$

If $S:z=g(x,y)$, $(x,y)\in D$, then,

$$\iint\limits_S \overline{F}\cdot\mathrm{d}\overline{S}=\iint\limits_S \overline{F}\cdot\hat{n}\mathrm{d}S$$
$$=\iint\limits_D \left(-Pg_x-Qg_y+R\right)\mathrm{d}x\mathrm{d}y$$

for S with positive orientation, and

$$\iint\limits_S \overline{F}\cdot\mathrm{d}\overline{S}=\iint\limits_S \overline{F}\cdot\hat{n}\mathrm{d}S$$
$$=-\iint\limits_D \left(-Pg_x-Qg_y+R\right)\mathrm{d}x\mathrm{d}y$$

for S with negative orientation.
If S is given parametrically as
$\overline{r}(u,v)=\big(x(u,v),y(u,v),z(u,v)\big)$
for $(u,v)\in D$, then

$$\iint\limits_S \overline{F}\cdot\mathrm{d}\overline{S}=\iint\limits_S \overline{F}\cdot\hat{n}\mathrm{d}S$$
$$=\iint\limits_D \overline{F}\cdot(\overline{r}_u\times\overline{r}_v)\mathrm{d}u\mathrm{d}v$$

If S is closed and given parametrically, it can be solved as above.
If S is closed and not given parametrically, it can be divided into surfaces of the first kind, and each of the integrals over the smaller surfaces can be solved as above.

Exercise 24.
Given
$\overline{F}=(x,y,z)$
Calculate $\iint\limits_S \overline{F}\cdot\hat{n}\mathrm{d}S$, where $S:x^2+y^2+z^2=1$.

Solution 24.
The surface S is given by
$x^2+y^2+z^2=1$

$$\therefore z=\pm\sqrt{1-x^2-y^2}$$

Therefore, let
$S_1=-\sqrt{1-x^2-y^2}$
$S_2=\sqrt{1-x^2-y^2}$
Therefore,

$$\iint\limits_S \overline{F}\cdot\hat{n}\mathrm{d}S=\iint\limits_{S_1} \overline{F}\cdot\hat{n}\mathrm{d}S+\iint\limits_{S_2} \overline{F}\cdot\hat{n}\mathrm{d}S$$
$$=-\iint\limits_D \left(-P(g_1)_x-Q(g_1)_y+R\right)\mathrm{d}x\mathrm{d}y$$
$$+\iint\limits_D \left(-P(g_1)_x-Q(g_1)_y+R\right)\mathrm{d}x\mathrm{d}y$$
$$=2\iint\limits_D \left(\frac{x^2}{\sqrt{1-x^2-y^2}}+\frac{y^2}{\sqrt{1-x^2-y^2}}+\sqrt{1-x^2-y^2}\right)\mathrm{d}A$$
$$=2\iint\limits_D \frac{1}{\sqrt{1-x^2-y^2}}\mathrm{d}x\mathrm{d}y$$
$$=2\int\limits_0^1\int\limits_0^{2\pi}\frac{1}{1-r^2}r\mathrm{d}\theta\mathrm{d}r$$
$$=4\pi$$

Exercise 25.
Given
$\overline{F}=(x,y,z)$
Calculate $\iint\limits_S \overline{F}\cdot\hat{n}\,\mathrm{d}S$, where $S: x^2+y^2+z^2=1$, using parametric representation.

Solution 25.
S is given parametrically by
$\overline{r}(\theta,\varphi)=\big(x(\theta,\varphi),y(\theta,\varphi),z(\theta,\varphi)\big)$
where
$x(\theta,\varphi)=\cos\theta\sin\varphi$
$y(\theta,\varphi)=\sin\theta\sin\varphi$
$z(\theta,\varphi)=\cos\varphi$
with $D:\{0\le\theta\le2\pi, 0\le\varphi\le\pi\}$.
Therefore,
$\overline{r}_\theta\times\overline{r}_\varphi=\left(-\cos\theta\sin^2\varphi,-\sin\theta\sin^2\varphi,-\sin\varphi\cos\varphi\right)$
If $\theta=\frac{\pi}{2}$, $\varphi=\frac{\pi}{2}$,
$\overline{r}_\theta\times\overline{r}_\varphi=(0,-1,0)$
However, the positive normal to S at that point is positively directed.
Therefore,

$$\iint\limits_S \overline{F}\cdot\hat{n}\mathrm{d}S=-\iint\limits_D \overline{F}\cdot\left(\overline{r}_\theta\times\overline{r}_\varphi\right)\mathrm{d}\theta\mathrm{d}\varphi$$
$$=-\iint\limits_D \left(-\cos^2\theta\sin^3\varphi-\sin^2\theta\sin^3\varphi-\cos^2\varphi\sin\varphi\right)\mathrm{d}\theta\mathrm{d}\varphi$$
$$=\iint\limits_D \left(\sin^3\varphi+\cos^2\varphi\sin\varphi\right)\mathrm{d}\theta\mathrm{d}\varphi$$
$$=\iint\limits_D \sin\varphi\mathrm{d}\theta\mathrm{d}\varphi$$

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^{\pi} \sin\varphi \mathrm{d}\varphi \mathrm{d}\theta \\
&= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \sin\varphi \mathrm{d}\varphi \\
&= 2\pi\left(-\cos\varphi\right)\Big|_0^{\pi} \\
&= 4\pi
\end{aligned}
$$

14 Green's Theorem

Definition 52 (Curl/Rotor) If
$\overline{F}(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$
then

$$
\mathrm{curl}\overline{R} = \nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}
$$

Definition 53 (Divergence) If
$\overline{F}(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$
then

$$
\mathrm{div}\overline{R} = \nabla\cdot\overline{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
$$

Theorem 61 *If a vector field $\overline{F}(x,y,z)$ is defined on $\mathbb{R}^3$, if there exist continuous first order partial derivatives of P, Q, R, and if $\mathrm{curl}\overline{F} = 0$, then $\overline{F}$ is a conservative vector field.*
In this case, $\exists f(x,y,z)$, such that $\overline{F} = \nabla f$.

Theorem 62 (Green's Theorem) *Let C be a piecewise smooth, simple, and closed curve in $\mathbb{R}^2$ with positive orientation. Let D be a domain bounded by C. If there exist continuous first order partial derivatives of $P(x,y)$ and $Q(x,y)$ in an open domain which contains D, then*

$$
\begin{aligned}
W &= \int_C \overline{F}\cdot\hat{T}\mathrm{d}s = \int_C P\mathrm{d}x + Q\mathrm{d}y \\
&= \iint_D (Q_x - P_y)\mathrm{d}A = \iint_D \mathrm{curl}\overline{F}\cdot\hat{k}\mathrm{d}A = \iint_D \mathrm{div}\overline{F}\mathrm{d}A
\end{aligned}
$$

15 Stoke's Theorem

Definition 54 (Curve with positive orientation) Let S be an oriented surface with normal $\hat{n}$ and let C be a curve bounding S. C is called a curve with positive orientation with respect to S if, as we walk on C in this direction and with our head in the direction of $\hat{n}$, the surface S is always on our left.

Theorem 63 (Stoke's Theorem) *Let S be a piecewise smooth surface with normal $\hat{n}$ and let S be bounded by a curve C which is piecewise smooth, simple, closed and with positive orientation with respect to S. Let $\overline{F}(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$ be a vector field such that there exist continuous first order partial derivatives of P, Q, R in an open domain of $\mathbb{R}^3$ which contains S. Then*

$$
\int_C \overline{F}\cdot\hat{T}\mathrm{d}s = \iint_S \mathrm{curl}\overline{F}\cdot\hat{n}\mathrm{d}S
$$

Stoke's Theorem is a generalization of Green's Theorem.

16 Gauss' Theorem

Theorem 64 *Let E be a body bounded by a surface S, with a positive orientation of S. Let*
$\overline{F} = (P,Q,R)$
be a vector field such that there exist continuous first order partial derivatives of P, Q, and Q, in some open domain which contains E. Then,

$$
\iint_S \overline{F}\cdot\hat{n}\mathrm{d}S = \iiint_E \mathrm{div}\overline{F}\mathrm{d}V
$$

Exercise 26.
Find $\iint_S \overline{F}\cdot\hat{n}\mathrm{d}S$ where

$$
\overline{F} = \left(xy, y^2 + e^{xz^2}, \sin xy\right)
$$

and S is a lateral surface of a body E which is bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z =$, $y = 0$, and $y + z = 2$.

Solution 26.

$$
\begin{aligned}
\iint_S \overline{F}\cdot\hat{n}\mathrm{d}S &= \iiint_E \mathrm{div}\overline{F}\mathrm{d}V \\
&= \iiint_E (y + 2y + 0)\mathrm{d}V \\
&= 3\iiint_{E_{\mathrm{II}}} y\mathrm{d}V \\
&= 3\iint_D \left(\int_0^{2-z} y\mathrm{d}y\right)\mathrm{d}A \\
&= 3\iint_D \frac{y^2}{2}\Bigg|_{y=0}^{y=2-z}\mathrm{d}A \\
&= \frac{3}{2}\iint_D (2-z)^2\mathrm{d}A \\
&= \frac{3}{2}\int_{-1}^{1}\int_0^{1-x^2} (2-z)^2\mathrm{d}z\mathrm{d}x \\
&= \frac{184}{35}
\end{aligned}
$$

Exercise 27.
Verify Stoke's Theorem when $\overline{F} = \left(-y^2, x, z^2\right)$ and C is the intersecton like between the plane $y + z = 2$ and the culinder $x^2 + y^2 = 1$. The direction of C is clockwise, when seen from above.

Solution 27.
Let S be the circular surface enclosed by C.
As C is clockwise, when seen from above, $\hat{n}$ is negative.
Let
$x = \cos t$
$y = \sin t$
Therefore, as $y + z = 2$,
$z = 2 - \sin t$
where, $t: 2\pi \to 0$.
t goes from 2π to 0 and not from 0 to 2π, as C is directed clockwise, when seen from above.
Therefore, the LHS is,

$$
\begin{aligned}
\int_C \overline{F}\cdot\hat{T}\mathrm{d}S &= \int_{2\pi}^{0} \Big(Px'(t) + Qy'(t) + Rz'(t)\Big)\mathrm{d}t \\
&= \int_{2\pi}^{0} \Big(-\sin^2 t\cdot -\sin t + \cos t\cdot\cos t + (2-\sin t)^2\cdot -\cos t\Big)\mathrm{d}t \\
&= \int_{2\pi}^{0} \left(\left(1-\cos^2 t\right)\sin t + \frac{1+\cos 2t}{2} - (2-\sin t)^2\cos t\right)\mathrm{d}t \\
&= -\cos t + \frac{\cos^3 t}{3} + \frac{t}{2} + \frac{\sin 2t}{4} + \frac{(2-\sin t)^3}{3}\Bigg|_{2\pi}^{0} \\
&= -\pi
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{curl}\overline{F} &= \nabla\times\overline{F} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} \\
&= (0-0)\hat{i} - (0-0)\hat{j} + (1+2y)\hat{k} \\
&= (1+2y)\hat{k} \\
&= \tilde{P}\hat{i} + \tilde{Q}\hat{j} + \tilde{R}\hat{k}
\end{aligned}
$$

As C is clockwise, when seen from above, $\hat{n}$ is negative. Therefore, the RHS is,

$$\iint_S \mathrm{curl}\overline{F}\cdot\hat{n}\mathrm{d}S = -\iint_D \left(-\tilde{P}g_x - \tilde{Q}q_y + \tilde{R}\right)\mathrm{d}A$$

$$= -\iint_D \tilde{R}\mathrm{d}A$$

$$= -\iint_D (1+2y)\mathrm{d}A$$

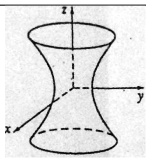
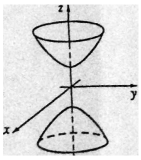
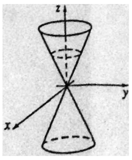
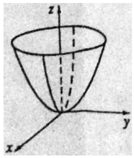
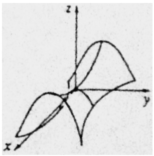
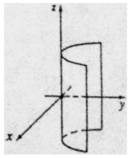
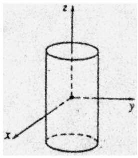
$$= -\int_0^1\int_0^{2\pi}(1+2r\sin\theta)r\mathrm{d}\theta\mathrm{d}r$$

$$= -\int_0^1\int_0^{2\pi} r\mathrm{d}\theta\mathrm{d}r - \int_0^{2\pi} 2r^2\sin\theta\mathrm{d}\theta\mathrm{d}r$$

$$= -\int_0^1 r\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta$$

$$= -\pi$$

□

Surface	Equation	Trace			Graph
		$z=0$	$y=0$	$x=0$	
Hyperboloid of One Sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	ellipse	hyperbola	hyperbola	
Hyperboloid of Two Sheets	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	none	hyperbola	hyperbola	
Elliptic Cone	$z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	(0,0,0)	2 intersecting lines	2 intersecting lines	
Elliptic Paraboloid	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	(0,0,0)	upwards parabola	upwards parabola	
Hyperboloid Paraboloid	$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$	2 intersecting lines	upwards parabola	downwards parabola	
Parabolic Cylinder	$x^2=4ay$	parabola	z-axis	z-axis	
Elliptic Cylinder	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	ellipse	2 parallel lines	2 parallel lines	
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	ellipse	ellipse	ellipse