Differential and Integral Calculus

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1 Lecturer Information

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2 Required Reading

Protter and Morrey: *A first Course in Real Analysis*, UTM Series, Springer-Verlag, 1991

3 Additional Reading

Thomas and Finney, *Calculus and Analytic Geometry*, 9th edition, Addison-Wesley, 1996

Part I
Sequences and Series

1 Sequences

Definition 1 (Sequence)**.** A sequence of real numbers is a set of numbers which are written in some order. There are infinitely many terms in a sequence. It is denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Example 1. $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is called the harmonic sequence.

$$a_n = \frac{1}{n}$$

Example 2. $1, -\frac{1}{2}, \frac{1}{3}, \ldots$ is called the alternating harmonic sequence.

$$a_n = (-1)^{n+1}\frac{1}{n}$$

Example 3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$$a_n = \frac{n}{n+1}$$

Example 4. $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \ldots$

$$a_n = \frac{n+1}{3^n}$$

Example 5. The Fibonacci sequence is given by

$$f_n = \begin{cases} 1 & ; \quad n = 1, 2 \\ f_{n-1} + f_{n-2} & ; \quad n \geq 3 \end{cases}$$

Example 6. A geometric sequence is given by

$$a_n = a_1 q^{n-1}$$

where q is called the common ratio.

Example 7. A geometric sequence is given by

$$a_n = a_1 + d(n-1)$$

where d is called the common difference.

Definition 2 (Equal sequences)**.** Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equal if $a_n = b_n$, $\forall n \in \mathbb{N}$.

Definition 3 (Sequences bounded from above)**.** $\{a_n\}$ is said to be bounded from above if $\exists M \in \mathbb{R}$, s.t. $a_n \leq M$, $\forall n \in \mathbb{N}$. Each such M is called an upper bound of $\{a_n\}$.

Definition 4 (Sequences bounded from below)**.** $\{a_n\}$ is said to be bounded from below if $\exists m \in \mathbb{R}$, s.t. $a_n \geq M$, $\forall n \in \mathbb{N}$. Each such M is called an lower bound of $\{a_n\}$.

Definition 5. $\{a_n\}$ is said to be bounded if it is bounded from below and bounded from above.

Example 8. The sequence $a_n = n^2 + 2$ is not bounded from above but is bounded from below, by all $m \leq 3$.

Example 9. $\left\{\dfrac{2n-1}{3n}\right\}$ is bounded.

$$m = 0 \leq \frac{2n-1}{3n} \leq \frac{2n}{3n} = \frac{2}{3} = M$$

Definition 6 (Monotonic increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \leq a_{n+1}$, $\forall n \geq n_0$.

Definition 7 (Monotonic decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \geq a_{n+1}$, $\forall n \geq n_0$.

Definition 8 (Strongly increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n < a_{n+1}$, $\forall n \geq n_0$.

Definition 9 (Strongly decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n > a_{n+1}$, $\forall n \geq n_0$.

Example 10. The sequence $\left\{\frac{n^2}{2^n}\right\}$ is strongly decreasing. However, this is not evident by observing the first few terms. $\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \ldots$

$$\begin{aligned} a_n &> a_{n+1} \\ \iff \frac{n^2}{2^n} &> \frac{(n+1)^2}{2^{n+1}} \\ \iff 2n^2 &> (n+1)^2 \\ \iff \sqrt{2}n &> n+1 \\ \iff n(\sqrt{2}-1) &> 1 \\ \iff n &> \frac{1}{\sqrt{2}-1} \\ \iff n &> 3 \end{aligned}$$

Exercise 1.
Is $a_n = (-1)^n$ monotonic?

Solution 1.
The sequence $-1, 1, -1, 1, \ldots$ is not monotonic.

1.1 Limit of a Sequence

Definition 10. Let $\{a_n\}$ be a given sequence. A number L is said to be the limit of the sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $|a_n - L| < \varepsilon$, $\forall n \geq n_0$. That is, there are infinitely many terms inside the interval and a finite number of terms outside it.

Example 11. The sequence $\{\frac{1}{n}\}$ tends to 0, i.e. for any open interval $(-\varepsilon, \varepsilon)$, there are finite number of terms of the sequence outside the interval, and therefore there are infinitely many terms inside the interval.

Exercise 2.
Prove

$$\lim_{n\to\infty} \frac{n+2}{2n-1} = \frac{1}{2}$$

Solution 2.
$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$

Exercise 3.
Prove that 2 is not a limit of $\left\{ \dfrac{3n+1}{n} \right\}$.

Solution 3.
If possible, let

$$\lim_{n \to \infty} \frac{3n+1}{n} = 2$$

Then, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\left| \dfrac{3n+1}{n} - 2 \right| < \varepsilon$, $\forall n \geq n_0$. However,

$$\left| \frac{3n+1}{n} - 2 \right| = 1 + \frac{1}{n} > 1$$

This is a contradiction for $\varepsilon = \dfrac{1}{2}$. Therefore, 2 is not a limit.

Theorem 1. *If a sequence $\{a_n\}$ has a limit L then the limit is unique.*

Proof. If possible let there exist two limits L_1 and L_2. Therefore, $\forall \varepsilon > 0$, there exist a finite number of terms in the interval $(L_1 - \varepsilon, L_1 + \varepsilon)$. Therefore, there exist a finite number of terms in the interval $(L_2 - \varepsilon, L_2 + \varepsilon)$. This contradicts the definition of a limit. Therefore, the limit is unique. □

Theorem 2. *If a sequence $\{a_n\}$ has limit L, then the sequence is bounded.*

Theorem 3. *Let*

$$\lim_{n \to \infty} a_n = a$$
$$\lim_{n \to \infty} b_n = b$$

and let c be a constant. Then,

$$\begin{aligned}
\lim c &= c \\
\lim(ca_n) &= c \lim a_n \\
\lim(a_n \pm b_n) &= \lim a_n \pm \lim b_n \\
\lim(a_n b_n) &= \lim a_n \lim b_n \\
\lim\left(\frac{a_n}{b_n}\right) &= \frac{\lim a_n}{\lim b_n} \quad (\text{ if } \lim b \neq 0)
\end{aligned}$$

Theorem 4. *Let $\{b_n\}$ be bounded and let $\lim a_n = 0$. Then,*

$$\lim(a_n b_n) = 0$$

Theorem 5 (Sandwich Theorem)**.** *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences. If*

$$\lim a_n = \lim b_n = L$$

and $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n \leq b_n \leq c_n$. Then,

$$\lim b_n = L$$

Exercise 4.
Calculate $\lim\limits_{n\to\infty} \sqrt[n]{2^n + 3^n}$

Solution 4.

$$\begin{aligned} &\sqrt[n]{3^n} \leq \sqrt[n]{2^n + 3^n} \leq \sqrt[n]{3^n + 3^n} = \sqrt[3]{2 \cdot 3^n} \\ &\therefore 3 \leq \sqrt[n]{2^n + 3^n} \leq 3\sqrt[n]{2} \end{aligned}$$

Therefore, by the Sandwich Theorem, $\lim\limits_{n\to\infty} \sqrt[n]{2^n + 3^n} = 3$.

Theorem 6. *Any monotonically increasing sequence which is bounded from above converges. Similarly, any monotonically decreasing sequence which is bounded from below converges.*

Exercise 5.

Prove that there exists a limit for $a_n = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}_{n \text{ times}}$ and find it.

Solution 5.

$$a_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = a_2$$

If possible, let

$$\begin{aligned} a_{n-1} &< a_n \\ \therefore \sqrt{2 + a_{n-1}} &< \sqrt{2 + a_n} \\ \therefore a_n &< a_{n+1} \end{aligned}$$

Hence, by induction, $\{a_n\}$ is monotonically increasing.

$$a_1 = \sqrt{2} \leq 2$$

If possible, let

$$a_n \leq 2 \therefore \sqrt{2 + a_n} \qquad \leq \sqrt{2+2}$$
$$\therefore a_{n+1} \leq 2$$

Hence, by induction, $\{a_n\}$ is bounded from above by 2. Therefore, by , $\{a_n\}$ converges.

Definition 11 (Limit in a wide sense)**.** The sequence $\{a_n\}$ is said to converge to $+\infty$ if $\forall M \in \mathbb{R}$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n > M$.
The sequence $\{a_n\}$ is said to converge to $-\infty$ if $\forall M \in \mathbb{R}$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n < M$.

1.2 Sub-sequences

Definition 12 (Sub-sequence)**.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_k\}_{k=1}^{\infty}$ be a strongly increasing sequence of natural numbers. Let $\{b_k\}_{k=1}^{\infty}$ be a sequence such that $b_k = a_{n_k}$. Then $\{b_k\}_{k=1}^{\infty}$ is called a sub-sequence of $\{a_n\}_{n=1}^{\infty}$.

Example 12.

$$a_n = \frac{1}{n}$$

If we choose $n_k = k^2$,

$$b_k = a_{n_k} = a_{k^2} = \frac{1}{k^2}$$

Therefore,

$$\{b_k\} = 1, \frac{1}{4}, \frac{1}{9}, \dots$$

Theorem 7. *If the sequence $\{a_n\}$ converges to L in a wide sense, i.e. L can be infinite, then any sub-sequence of $\{a_n\}$ converges to the same limit L.*

Definition 13 (Partial limit)**.** A real number a, which may be infinite, is called a partial limit of the sequence $\{a_n\}$ is there exists a sub-sequence of $\{a_n\}$ which converges to a.

Example 13. Let

$$a_n = (-1)^n$$

Therefore, $\nexists \lim_{n\to\infty} a_n$. Let

$$b_k = a_{n_k} = a_{2n-1}$$

Therefore,

$$\{b_k\} = -1, -1, -1, \ldots$$
$$\therefore \lim_{k\to\infty} b_k = 1$$

Therefore, -1 is a partial limit of $\{a_n\}$.

Theorem 8 (Bolzano-Weierstrass Theorem)**.** *For any bounded sequence there exists a subsequence which is convergent, s.t. there exists at least one partial limit.*

Definition 14 (Upper partial limit)**.** The largest partial limit of a sequence is called the upper partial limit. It is denoted by $\overline{\lim} a_n$ or $\limsup a_n$.

Definition 15 (Lower partial limit)**.** The smallest partial limit of a sequence is called the upper partial limit. It is denoted by $\underline{\lim} a_n$ or $\liminf a_n$.

Theorem 9. *If the sequence $\{a_n\}$ is bounded and*

$$\overline{\lim} a_n = \underline{\lim} a_n = a$$

then

$$\exists \lim a_n = a$$

1.3 Cauchy Characterisation of Convergence

Definition 16. A sequence $\{a_n\}$ is called a Cauchy sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall m, n \geq n_0$, $|a_n - a_m| < \varepsilon$.

Theorem 10 (Cauchy Characterisation of Convergence)**.** *A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.*

Proof. Let

$$\lim_{n\to\infty} a_n = L$$

Then $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$, $|a_n - L| < \dfrac{\varepsilon}{2}$. Therefore if $n \geq n_0$ and $m \geq n_0$, then

$$\begin{aligned}
|a_n - a_m| &= |a_n - L + L - a_m| \\
&\leq |a_n - L| + |L - a_m| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
\therefore |a_n - a_m| &= \varepsilon
\end{aligned}$$

Similarly, the converse can be proved by Theorem 9. □

Theorem 11 (Another Formulation of the Cauchy Characterisation Theorem)**.** *The sequence* $\{a_n\}$ *converges if and only if* $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, *such that* $\forall n \geq n_0$ *and* $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$.

Exercise 6.
Prove that the sequence

$$a_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

is convergent.

Solution 6.

$$\begin{aligned}
|a_{n+p} - a_n| &= \left| \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(n+p)^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right) \right| \\
&= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(n+p-1)(n+p)} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} + \cdots + \cancel{\frac{1}{n+p-1}} - \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n} - \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n}
\end{aligned}$$

Therefore, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$, where $n_0 > \dfrac{1}{\varepsilon}$. $\square$

Exercise 7.
Prove that the sequence

$$a_n = \frac{1}{1} + \frac{1}{n} + \cdots + \frac{1}{n}$$

diverges.

Solution 7.
If possible, let the sequence converge. Then, by the Cauchy Characterisation of Convergence, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$. Therefore,

$$\begin{aligned}
|a_{n+p} - a_n| &= \left| \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+p} - \left(\frac{1}{n} + \cdots + \frac{1}{n} \right) \right| \\
&= \frac{1}{n+1} + \cdots + \frac{1}{n+p} \\
&\geq p \cdot \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &> \frac{p}{n+p}
\end{aligned}$$

If $n = p$,

$$\frac{p}{n+p} = \frac{1}{2}$$

This contradicts the result obtained from the Cauchy Characterisation of Convergence, for $\varepsilon = \dfrac{1}{4}$.
Therefore, the sequence diverges.