Differential and Integral Calculus : Recitations

Aakash Jog

2014-15

Contents

1 Instructor Information

Michael Bromberg

E-mail: micbromberg@gmail.com

Part I
Sequences and Series

1 Sequences

Recitation 1 – Exercise 1.
Prove:

$$\lim_{n\to\infty} \frac{2n^2+n+1}{n^2+3} = 2$$

Recitation 1 – Solution 1.
Let

$$\varepsilon > 0$$

$$\begin{aligned}
\left|\frac{2n^2+n+1}{n^2+3} - 2\right| &= \left|\frac{2n^2+n+1-2n^2-6}{n^2+3}\right| \\
&= \left|\frac{n-5}{n^2+3}\right| \\
&\leq \left|\frac{n-5}{n^2}\right| \\
&\leq \frac{1}{n} \\
&< \varepsilon
\end{aligned}$$

Therefore, let $N = \left[\frac{1}{\varepsilon}\right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.

Therefore, $\lim\limits_{n\to\infty} \frac{2n^2+n+1}{n^2+3} = 2$. □

Recitation 1 – Exercise 2.
Prove

$$\lim_{n\to\infty} \frac{n^3+\sin n+n}{2n^4} = 0$$

Recitation 1 – Solution 2.
Let $\varepsilon > 0$

$$\begin{aligned}\left|\frac{n^3+\sin n+n}{2n^4}\right| &\le \left|\frac{n^3+1+n}{2n^4}\right| \\ &\le \left|\frac{3n^3}{2n^4}\right| = \frac{3}{2}\cdot\frac{1}{n} < \varepsilon\end{aligned}$$

Therefore, let $N = \left[\frac{3}{2\varepsilon}\right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.

Therefore, $\lim_{n\to\infty} \frac{n^3+\sin n+n}{2n^4} = 0$ $\square$

Recitation 1 – Exercise 3.
Calculate $\sqrt[3]{n^3+3n} - n$.

Recitation 1 – Solution 3.

$$a^n - b^n = (a-b)\cdot(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Therefore, let

$$a = \sqrt[3]{n^3+3n}$$
$$b = \sqrt[3]{n^3}$$

$$\begin{aligned}a-b &= \frac{a^3-b^3}{a^2+ab+b^2} \\ \therefore \sqrt[3]{n^3+3n}-n &= \frac{n^3+3n-n^3}{(n^3+3n)^{2/3}+(n^3+3n)^{1/3}n+n^2} \\ &= \frac{3}{\left(\frac{n^3+3n}{n^{3/2}}\right)^{2/3}+\left(\frac{n^3+3n}{n^3}\right)^{1/3n}+n}\end{aligned}$$

Therefore, the limit is 0.

Recitation 1 – Exercise 4.
Prove

$$\lim_{n\to\infty}\frac{n!}{n^n} = 0$$

Recitation 1 – Solution 4.

$$0 \le \frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\dots\frac{n}{n} \le \frac{1}{n}$$

Therefore, by the Sandwich Theorem, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Recitation 1 – Exercise 5.

Let $a_1 = 3$, $a_{n+1} = 1 + \sqrt{6 + a_n}$. Prove that a_n converges and find its limit.

Recitation 1 – Solution 5.

If possible, let $\lim_{n\to\infty} a_n = l$.

$$a_{n+1} = 1 + \sqrt{6 + a_n}$$

Taking the limit on both sides,

$$\begin{aligned} l &= 1 + \sqrt{6 + l} \\ \therefore l - 1 &= \sqrt{6 + l} \\ \therefore l &= \frac{3 \pm \sqrt{29}}{2} \end{aligned}$$

As $a_n \ge 0$, $l = \dfrac{3 + \sqrt{29}}{2}$.

$$\begin{aligned} a_2 &= 1 + \sqrt{6 + a_1} \\ &= 1 + \sqrt{6 + 3} \\ &= 4 \\ \therefore a_2 &> a_1 \end{aligned}$$

If possible, let $a_n \ge a_{n-1}$.
Therefore,

$$\begin{aligned} a_{n+1} &= 1 + \sqrt{6 + a_n} \\ &\ge 1 + \sqrt{6 + a_{n+1}} = a_n \end{aligned}$$

Therefore by induction, $\{a_n\}$ is monotonically increasing.

$$\begin{aligned} a_1 &= 3 \\ \therefore a_1 &\le 5 \end{aligned}$$

If possible, let $a_n \leq 5$.
Therefore,

$$a_{n+1} = 1 + \sqrt{6 + a_n} \leq q + \sqrt{11} \leq 5$$

Therefore by induction, $\{a_n\}$ is bounded from above by 5.

1.1 Limit of a Function by Heine

Definition 1.

$$\lim_{x \to x_0} f(x) = l$$

if for every sequence x_n, such that $\lim\limits_{n \to \infty} x_n = x_0$,

$$\lim_{n \to \infty} f(x_n) = l$$

Theorem 1. *If f is continuous at x_0 and $x_n \to x_0$, then*

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right) = f_{x_0}$$

Recitation 2 – Exercise 1.
Calculate $\lim\limits_{n \to \infty} \sqrt[n]{n}$.

Recitation 2 – Solution 1.
Let

$$f(x) = x^{1/x}$$

Therefore,

$$\begin{aligned}\lim_{x \to \infty} x^{1/x} &= \lim_{x \to \infty} e^{\cancelto{0}{\frac{\ln x}{x}}} \\ &= 1\end{aligned}$$

1.2 Sub-sequences

Recitation 2 – Exercise 2.
Find all partial limits and $\overline{\lim}$ and $\underline{\lim}$ of

$$a_n = \left(\cos\frac{\pi n}{4}\right)^n$$

Recitation 2 – Solution 2.
Let $k, z \in \mathbb{Z}$

$$\begin{aligned}
\cos\frac{\pi n}{4} &= \cos\frac{\pi(n+k)}{4} \\
\therefore \frac{\pi n}{4} &= \frac{\pi(n+k)}{4} + 2\pi z \\
\therefore \pi n &= \pi(n+k) + 8\pi z \\
\therefore k &= 8z
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_{8k} &= \left(\cos\frac{\pi\cdot 8k}{4}\right)^{8k} \\
&= (\cos(2\pi k))^{8k} \\
&= 1 \\
a_{8k+1} &= \left(\cos\frac{\pi\cdot(8k+1)}{4}\right)^{8k+1} \\
&= \left(\cos\frac{\pi}{4}\right)^{8k+1} \\
&= \left(\frac{\sqrt{2}}{2}\right)^{8k+1} \\
a_{8k+2} &= \left(\cos\frac{\pi\cdot(8k+2)}{4}\right)^{8k+2} \\
&= \left(\cos\frac{\pi}{2}\right)^{8k+2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{k\to\infty} a_{8k} &= 1 \\
\lim_{k\to\infty} a_{8k+1} &= \lim_{k\to\infty}\left(\frac{\sqrt{2}}{2}\right)^{8k+1} \\
&= 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{k\to\infty} a_{8k+2} &= 0 \\
\lim_{k\to\infty} a_{8k+3} &= 0 \\
\lim_{k\to\infty} a_{8k+4} &= \lim_{k\to\infty} (-1)^{8k+4} \\
&= 1 \\
\lim_{k\to\infty} a_{8k+5} &= 0 \\
\lim_{k\to\infty} a_{8k+6} &= 0 \\
\lim_{k\to\infty} a_{8k+7} &= 0
\end{aligned}$$

Therefore, $\{a_n\}$ has two partial limits, 0 and 1.

$$\begin{aligned}
\overline{\lim} a_n &= 1 \\
\underline{\lim} a_n &= 0
\end{aligned}$$

2 Series

Definition 2 (Convergence of a series)**.** Let $\{a_n\}$ be a sequence. Let S_n be a sequence of partial sums of a_n, s.t.

$$S_n = \sum_{k=1}^{n} a_k$$

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to l if

$$\lim_{n\to\infty} S_n = l$$

that is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n$$

Recitation 2 – Exercise 3.

Does $\displaystyle\sum_{k=0}^{\infty} q^k$ where $-1 < q < 1$ converge?

Recitation 2 – Solution 3.

$$\begin{aligned}\sum_{k=0}^{\infty} q^k &= \lim_{n\to\infty}\sum_{k=0}^{n} q^k \\ &= \lim_{n\to\infty}\frac{1-q^{n+1}}{1-q} \\ &= \frac{1}{1-q}\end{aligned}$$

Therefore, the series converges.

Recitation 2 – Exercise 4.

Does $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ converge?

Recitation 2 – Solution 4.

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k(k+1)} &= \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right) \\ &= \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) \\ &= 1\end{aligned}$$

Recitation 2 – Exercise 5.

Does $\displaystyle\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^k$ converge?

Recitation 2 – Solution 5.

$$\begin{aligned}&\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e \\ \therefore\ &\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k \neq 0\end{aligned}$$

Therefore, the necessary condition is nt satisfied. Hence, the series does not converge.

2.1 Comparison Tests for Positive Series

Theorem 2 (First Comparison Test). *If $a_n \geq 0$, $b_n \geq 0$, and $a_n \leq b_n$, then*

1. *If $\sum b_n$ converges, then $\sum a_n$ converges.*
2. *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Theorem 3 (Second Comparison Test). *If $a_n \geq 0$, $b_n \geq 0$ and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l$$

where $0 < l < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge simultaneously.

Recitation 3 – Exercise 1.
Suppose the sequence a_n satisfies the condition

$$a_{n+1} - a_n > \frac{1}{n}$$

$\forall n \in \mathbb{N}$.
Prove that $\lim\limits_{n\to\infty} a_n = \infty$.

Recitation 3 – Solution 1.

$$\begin{aligned} a_{n+1} &= a_{n+1} - a_n + a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_2 - a_1 + a_1 \\ &= \sum_{k=1}^{n} (a_{k+1} - a_k) + a_1 \\ &\geq \sum_{k=1}^{n} \frac{1}{k} + a_1 \end{aligned}$$

As the harmonic series diverges, $\sum\limits_{k=1}^{n} \dfrac{1}{k} + a_1$ diverges.
Therefore, by the First Comparison Test, $\sum\limits_{k=1}^{\infty} (a_{k+1} - a_k)$ diverges.

Recitation 3 – Exercise 2.
Check the convergence of $\sum\limits_{n=2}^{\infty} \dfrac{n + \sin n}{n^3 + \cos \pi n}$.

Recitation 3 – Solution 2.

The series is non-negative. Therefore, the comparison tests are applicable.

$$\begin{aligned} \frac{n+\sin n}{n^3+\cos \pi n} &\leq \frac{n+1}{n^3-1} \\ \therefore \frac{n+\sin n}{n^3+\cos \pi n} &\leq \frac{2n}{n^3-\frac{n^3}{2}} && \leq \frac{4}{n^2} \end{aligned}$$

Therefore, by the First Comparison Test, as $\frac{4}{n^2}$ converges, $\sum_{n=2}^{\infty} \frac{n+\sin n}{n^3+\cos \pi n}$ also converges.

Recitation 3 – Exercise 3.

Let $a_n \geq 0$ and suppose that $\sum a_n$ converges. Prove that $\sum {a_n}^2$ converges. Is it true without the assumption $a_n \geq 0$?

Recitation 3 – Solution 3.

As $\sum a_n$ converges, $\lim\limits_{n \to \infty} a_n = 0$.
Therefore, $\exists N \in \mathbb{N}$, such that $\forall n > N$, $a_n < 1$.
Therefore, $\forall n > N$, ${a_n}^2 \leq a_n$. Hence, as $\sum\limits_{n=N+1}^{\infty} a_n$ converges, $\sum\limits_{n=N+1}^{\infty} {a_n}^2$ also converges. Hence, $\sum\limits_{n=1}^{\infty} a_n$ also converges.

This is not true without the assumption $a_n \geq 0$, as the argument ${a_n}^2 \leq a_n$ does not hold.

Recitation 3 – Exercise 4.

For which α does $\sum \left(\sqrt{n+1}-\sqrt{n}\right)^{\alpha/2}$ converge?

Recitation 3 – Solution 4.

$$\begin{aligned} \sum \left(\sqrt{n+1}-\sqrt{n}\right)^{\alpha/2} &= \sum \left(\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2} \\ &= \sum \left(\frac{1}{\sqrt{n+1}-\sqrt{n}}\right)^{\alpha/2} \end{aligned}$$

The series is positive. Therefore, the comparison tests are applicable.

Comparing with $\left(\frac{1}{\sqrt{n}}\right)^{\alpha/2}$,

$$\frac{\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2}}{\left(\frac{1}{\sqrt{n}}\right)^{\alpha/2}} = \left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2}$$

$$\therefore \lim_{n\to\infty}\left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2} = \left(\frac{1}{2}\right)^{\alpha/2}$$

$\sum \frac{1}{n^{\alpha/2}}$ converges if and only if $\frac{\alpha}{4} > 1$, i.e. if an inly if $\alpha > 4$.
By the Second Comparison Test, $\sum \frac{1}{n^{\alpha/4}}$ and the series converge or diverge simultaneously.
Therefore, the series converges for $\alpha > 4$.

Recitation 3 – Exercise 5.

Check the convergence of $\sum\limits_{n=1}^{\infty} \sin\frac{1}{n}$.

Recitation 3 – Solution 5.

$\forall n \in \mathbb{N}$, $\sin\frac{1}{n} \geq 0$

$$\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$$

Therefore, by Second Comparison Test, $\sum \frac{1}{n}$ and $\sum \sin\frac{1}{n}$ diverge simultaneously.

2.2 d'Alembert Criteria (Ratio Test)

Definition 3 (Absolute and conditional convergence)**.** The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

Theorem 4. *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 5 (d'Alembert Criteria (Ratio Test)). *1. If*

$$\lim_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right| = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\lim_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right| = L > 1$$

(including $L = \infty$), then $\sum a_n$ converges diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 – Exercise 6.

Check the convergence of $\sum \dfrac{(-1)^n \cdot n^{1000}}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}$.

Recitation 3 – Solution 6.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n \cdot n^{1000}}{1 \cdot \dots \cdot (2n-1)}\right| = \sum_{n=1}^{\infty}\frac{n^{1000}}{1 \cdot \dots \cdot (2n-1)}$$

Therefore, by the d'Alembert Criteria (Ratio Test),

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\frac{(n+1)^{1000}}{1 \cdot \dots \cdot (2n+1)}}{\frac{n^{1000}}{1 \cdot \dots \cdot (2n-1)}} \\ &= \left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1}\end{aligned}$$

$$\begin{aligned}\therefore \lim_{n to \infty}\left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1} &= 0 \\ \therefore \left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1} &< 1\end{aligned}$$

Therefore, by the d'Alembert Criteria (Ratio Test), the series converges absolutely, and hence converges.

2.3 Cauchy Criteria (Cauchy Root Test)

Theorem 6 (Cauchy Criteria (Cauchy Root Test))**.** *1. If*

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

(including $L = \infty$), then $\sum a_n$ diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 – Exercise 7.

Check the convergence of $\sum \left(1 - \frac{2}{n}\right)^{n^2}$.

Recitation 3 – Solution 7.

$$\sqrt[n]{\left(1 - \frac{2}{n}\right)^{n^2}} = \left(1 - \frac{2}{n}\right)^{n}$$

$$\therefore \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^{n} = e^{-2}$$

$$\therefore \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^{n} < 1$$

Therefore, by the Cauchy Criteria (Cauchy Root Test), $\sum \left(1 - \frac{2}{n}\right)^{n^2}$ converges.

2.4 Leibniz's Criteria

Definition 4 (Alternating series)**.** The series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

Theorem 7 (Leibniz's Criteria for Convergence)**.** *If an alternating series $\sum (-1)^{n-1} a_n$ with $a_n > 0$ satisfies*

1. *$a_{n+1} \le a_n$, i.e. $\{a_n\}$ is monotonically decreasing.*

2. $\lim_{n\to\infty} a_n = 0$

then the series $(-1)^{n-1}a_n$ *converges.*

Recitation 3 – Exercise 8.

Prove or disprove: There exists $\{a_n\}$, such that $\sum a_n$ converges and $\sum(1 + a_n)a_n$ diverges.

Recitation 3 – Solution 8.

Let $a_n = \dfrac{(-1)^n}{\sqrt{n}}$.

Therefore, by Leibniz's Criteria for Convergence, $\sum \dfrac{(-1)^n}{\sqrt{n}}$ converges.

$$\begin{aligned}\sum(1+a_n)a_n &= \sum\left(1+\frac{(-1)^n}{\sqrt{n}}\right)\frac{(-1)^n}{\sqrt{n}}\\ &= \sum\left(\frac{(-1)^n}{\sqrt{n}}+\frac{1}{n}\right)\end{aligned}$$

Therefore, as $\sum \dfrac{1}{n}$ diverges, and $\sum \dfrac{(-1)^n}{\sqrt{n}}$ converges, $\sum\left(\dfrac{1}{n}+\dfrac{(-1)^n}{\sqrt{n}}\right)$ diverges.

2.5 Integral Test

Theorem 8 (Integral Test)**.** *If* $f(x) : [1,\infty) \to [0,\infty)$ *is monotonically decreasing. Then,* $\sum\limits_{n=1}^{\infty} f(n)$ *and* $\int\limits_1^{\infty} f(x)\,\mathrm{d}x$ *converge or diverge simultaneously.*

Recitation 3 – Exercise 9.

Check the convergence of $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\ln n}$

Recitation 3 – Solution 9.

Let

$$f(x) = \frac{1}{x\ln x}$$

$f(x)$ is monotonically decreasing. Therefore, the Integral Test is applicable. Therefore,

$$\begin{aligned}\int\limits_2^\infty \frac{1}{x\ln x}\,\mathrm{d}x &= \int\limits_{\ln 2}^\infty \frac{1}{y}\,\mathrm{d}y \\ &= \ln y\big|_{\ln 2}^\infty \\ &= \infty\end{aligned}$$

Therefore, by the integral test, $\sum \dfrac{1}{n\ln n}$ diverges.