DIFFERENTIAL AND INTEGRAL CALCULUS
ASSIGNMENT 2

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Exercise 1.
Find the following limits (using the sandwich theorem)

1. $\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2}$
2. $\frac{\sin n}{n + \cos n}$
3. $\sqrt[n]{3n - \sqrt{n}}$

Solution 1.

1. $$\begin{aligned} 0 &\leq \frac{1}{n^2} + \dots + \frac{1}{(2n)^2} \leq n \cdot \frac{1}{n^2} \\ \therefore \lim_{n\to\infty} 0 &\leq \lim_{n\to\infty} \frac{1}{n^2} + \dots + \frac{1}{(2n)^2} \leq \lim_{n\to\infty} \frac{1}{n} \\ \therefore 0 &\leq \lim_{n\to\infty} \frac{1}{n^2} + \dots + \frac{1}{(2n)^2} \leq 0 \end{aligned}$$
   Therefore, by the Sandwich Theorem,
   $$\lim_{n\to\infty} \frac{1}{n^2} + \dots + \frac{1}{(2n)^2} = 0$$

2. $$\begin{aligned} \frac{-1}{n+1} &\leq \frac{\sin n}{n + \cos n} \leq \frac{1}{n-1} \\ \therefore \lim_{n\to\infty} \frac{-1}{n+1} &\leq \lim_{n\to\infty} \frac{\sin n}{n + \cos n} \leq \lim_{n\to\infty} \frac{1}{n-1} \\ \therefore 0 &\leq \lim_{n\to\infty} \frac{\sin n}{n + \cos n} \leq 0 \end{aligned}$$
   Therefore, by the Sandwich Theorem,
   $$\lim_{n\to\infty} \frac{\sin n}{n + \cos n} = 0$$

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3.

$$\sqrt[n]{3n-n} \leq \sqrt[n]{3n-\sqrt{n}} \leq \sqrt[n]{3n}$$
$$\therefore \lim_{n\to\infty} \sqrt[n]{3n-n} \leq \lim_{n\to\infty} \sqrt[n]{3n-\sqrt{n}} \leq \lim_{n\to\infty} \sqrt[n]{3n}$$
$$\therefore \lim_{n\to\infty} \sqrt[n]{2n} \leq \lim_{n\to\infty} \sqrt[n]{3n-\sqrt{n}} \leq \lim_{n\to\infty} \sqrt[n]{3n}$$
$$\therefore \lim_{n\to\infty} 2^{1/n}\sqrt[n]{n} \leq \lim_{n\to\infty} \sqrt[n]{3n-\sqrt{n}} \leq \lim_{n\to\infty} 3^{1/n}\sqrt[n]{n}$$
$$\therefore 1 \leq \lim_{n\to\infty} \sqrt[n]{3n-\sqrt{n}} \leq 1$$

Therefore, by the Sandwich Theorem,

$$\lim_{n\to\infty} \sqrt[n]{3n-\sqrt{n}} = 1$$

Exercise 2.
Let $a, b > 0$. Find the limit $\lim_{n\to\infty} \sqrt[n]{a^n+b^n}$.

Solution 2.
If $a > b$,

$$\sqrt[n]{a^n} \leq \sqrt[n]{a^n+b^n} \leq \sqrt[n]{2a^n}$$
$$\therefore \lim_{n\to\infty} \sqrt[n]{a^n} \leq \lim_{n\to\infty} \sqrt[n]{a^n+b^n} \leq \lim_{n\to\infty} \sqrt[n]{2a^n}$$
$$\therefore a \leq \lim_{n\to\infty} \sqrt[n]{a^n+b^n} \leq a$$

Therefore, by the Sandwich Theorem,

$$\lim_{n\to\infty} \sqrt[n]{a^n+b^n} = a$$

Similarly, if $b > a$,

$$\lim_{n\to\infty} \sqrt[n]{a^n+b^n} = b$$

Exercise 3.
Check whether the following sequence are bounded from above or from below (or both): (keep in mind that a convergent sequence is always bounded).

1. $a_n = \dfrac{n^2+1}{n+2}$
2. $a_n = \dfrac{n^5}{2^n}$
3. $a_n = \sqrt{n^2-n} - \sqrt{n}$
4. $a_n = \tan\left(\dfrac{\pi}{2} - \dfrac{1}{n}\right)$

Solution 3.

1.

$$\begin{aligned} a_n &= \frac{n^2+1}{n+2} \\ &= (n-2) + \frac{5}{n+2} \end{aligned}$$

Therefore, for $n \geq 1$, $\{a_n\}$ is monotonically increasing.
Therefore, the smallest term is

$$\begin{aligned} a_1 &= \frac{1+1}{1+2} \\ &= \frac{2}{3} \end{aligned}$$

Therefore, the series is bounded from below by $\frac{2}{3}$.
As the sequence is monotonically increasing, it is not bounded from above.

2.

$$a_n = \frac{n^5}{2^n}$$

Let

$$f(x) = \frac{x^5}{2^x}$$

Differentiating and maximizing,

$$f(x)_{\max} = \frac{5}{\ln 2}$$

Therefore, as $f(x)$ is bounded from above by $\frac{5}{\ln 2}$, $\{a_n\}$ is also bounded from above be $\frac{5}{\ln 2}$.

$$\begin{aligned} &\lim_{x\to\infty} f(x) = 0 \\ &\therefore \lim_{n\to\infty} a_n = 0 \end{aligned}$$

Therefore, $\{a_n\}$ is bounded from below by 0.

3.

$$a_n = \sqrt{n^2 - n} - \sqrt{n}$$

Let

$$f(x) = \sqrt{x^2 - x} - \sqrt{x}$$

$$\therefore f'(x) = \frac{1}{2}\left(\frac{2x-1}{\sqrt{x(x-1)}} - \frac{1}{\sqrt{x}}\right)$$

Therefore, minimizing,

$$f(x)_{\min} = -1$$

Therefore, $f(x)$ has minimum value -1, but no maximum value.
Therefore, $\{a_n\}$ is bounded from below by -1.

4.

$$a_n = \tan\left(\frac{\pi}{2} - \frac{1}{n}\right)$$

Let

$$f(x) = \tan\left(\frac{\pi}{2} - \frac{1}{x}\right)$$

$$= \cot\frac{1}{n}$$

Therefore, as $\frac{1}{n}$ is monotonically decreasing $\cot\frac{1}{n}$ is monotonically increasing.
Therefore, the sequence is not bounded from above. The minimum value of the sequence is

$$a_1 = \cot 1$$

Therefore, the sequence is bounded from below by $\cot 1$.

Exercise 4.

Check whether the following sequences are eventually monotone (i.e whether there exists $N \in \mathbb{N}$ such that a_n is monotone for all $n > N$.

1. $a_n = \sqrt{n} - \frac{1}{n}$

2. $a_n = \sin(\pi n)$

Solution 4.

1.
$$a_n = \sqrt{n} - \frac{1}{n}$$
Let
$$f(x) = \sqrt{x} - \frac{1}{x}$$
$$\therefore f'(x) = \frac{1}{x^2} + \frac{1}{2\sqrt{x}}$$
Therefore, $f(x)$ is monotonically increasing on $(0, \infty)$.
Therefore, $\{a_n\}$ is monotonically increasing for all $n > 1$.

2.
$$\begin{aligned} a_n &= \sin(\pi n) \\ \therefore \{a_n\} &= \sin(\pi), \sin(2\pi), \sin(3\pi), \ldots \\ &= 0, 0, 0, \ldots \end{aligned}$$
Therefore, for all $n \geq 1$, the sequence is monotonically increasing.

Exercise 5.
Prove that the following sequences converge and find their limits

1. $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$

2. $a_1 = 2$, $a_{n+1} = \sqrt{2a_n - 1}$

3. $a_1 = 2$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right)$

Solution 5.

1.

$$a_2 = \sqrt{2+\sqrt{2}} \qquad \leq \sqrt{2}$$
$$\therefore a_2 \leq a_1$$

If possible, let $a_{n-1} \leq a_n$.
Therefore,

$$\begin{aligned} a_n &= \sqrt{2+a_{n-1}} \\ &\leq \sqrt{2+a_n} \\ \therefore a_n &\leq a_{n+1} \end{aligned}$$

Therefore, by induction, the sequence is monotonically increasing.

If possible, let

$$\lim_{n\to\infty} a_n = l \geq a_1$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \sqrt{2+a_{n-1}} \\ \therefore l &= \sqrt{2+l} \\ \therefore l &= 2 \end{aligned}$$

$$\begin{aligned} a_1 &= \sqrt{2} \\ \therefore a_1 &\leq l \end{aligned}$$

If possible, let $a_n \leq l$.
Therefore,

$$\begin{aligned} a_{n+1} &= \sqrt{2+a_n} \\ &\leq \sqrt{2+l} \\ \therefore a_{n+1} &\leq l \end{aligned}$$

Therefore, by induction, the sequence is bounded from above.

Therefore, the sequence is monotonically increasing and bounded from above by $l = 2$.
Therefore, it converges to $l = 2$.

2.

$$
\begin{aligned}
a_2 &= \sqrt{2a_1 - 1} \\
&= \sqrt{3} \\
\therefore a_2 &\leq a_1
\end{aligned}
$$

If possible, let $a_{n-1} \leq a_n$.
Therefore,

$$
\begin{aligned}
a_n &= \sqrt{2a_{n-1} - 1} \\
&\geq \sqrt{2a_n - 1} \\
\therefore a_n &\geq a_{n+1}
\end{aligned}
$$

Therefore, by induction, the sequence is monotonically decreasing.

If possible, let

$$\lim_{n\to\infty} a_n = l$$

Therefore,

$$
\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \sqrt{2a_{n-1} - 1} \\
\therefore l &= \sqrt{2l - 1} \\
\therefore l &= 1
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= 2 \\
&\geq 1 \\
\therefore a_1 &\geq l
\end{aligned}
$$

If possible, let $a_n \geq l$.
Therefore,

$$
\begin{aligned}
a_{n+1} &= \sqrt{2a_n - 1} \\
&\geq \sqrt{2l - 1} \\
&\geq \sqrt{2 - 1} \\
&\geq 1 \\
\therefore a_{n+1} &\geq l
\end{aligned}
$$

Therefore, as the sequence is monotonically decreasing and is bounded from below by $l = 1$, it converges to $l = 1$.

3.

$$\begin{aligned}
a_2 &= \frac{1}{2}\left(a_1 + \frac{1}{a_1}\right) \\
&= \frac{1}{2}\left(2 + \frac{1}{2}\right) \\
&= \frac{5}{4} \\
\therefore a_2 &\leq a_1
\end{aligned}$$

If possible let $a_{n-1} \geq a_n$.
Therefore,

$$\begin{aligned}
a_n &= \frac{1}{2}\left(a_{n-1} + \frac{1}{a_{n-1}}\right) \\
&\geq \frac{1}{2}\left(a_n + \frac{1}{a_n}\right) \\
\therefore a_n &\geq a_{n+1}
\end{aligned}$$

Therefore, by induction, the sequence is monotonically decreasing.

If possible, let

$$\lim_{n\to\infty} a_n = l$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{1}{a_n}\right) \\
\therefore l &= \frac{1}{2}\left(l + \frac{1}{l}\right) \\
\therefore l &= 1
\end{aligned}$$

$$\begin{aligned}
a_1 &= 2 \\
&\geq 1 \\
\therefore a_1 &\geq l
\end{aligned}$$

If possible, let $a_n \geq l$.
Therefore,

$$\begin{aligned}
a_{n+1} &= \frac{1}{2}\left(a_n + \frac{1}{a_n}\right) \\
&\geq \frac{1}{l + \frac{1}{l}} \\
\therefore a_{n+1} &\geq l
\end{aligned}$$

Therefore, by induction, the sequence is bounded from below.
Therefore, as the sequence is monotonically decreasing and is bounded from below by $l = 1$, it converges to $l = 1$.

Exercise 6.
Prove or disprove: If a_n and b_n are bounded sequences then $a_n b_n$ is bounded.

Solution 6.
Let
$$a \leq a_n \leq A$$
and
$$b \leq b_n \leq B$$
Therefore,
$$a_n b \leq a_n b_n \leq a_n B$$
and
$$b_n a \leq a_n b_n \leq b_n A$$
Therefore,
$$\min\{a_n b, b_n a\} \leq a_n b_n \leq \max\{a_n B, b_n A\}$$
Therefore, $a_n b_n$ is bounded.