Differential and Integral Calculus

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Contents

1 Lecturer Information

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2 Required Reading

Protter and Morrey: *A first Course in Real Analysis*, UTM Series, Springer-Verlag, 1991

3 Additional Reading

Thomas and Finney, *Calculus and Analytic Geometry*, 9th edition, Addison-Wesley, 1996

Part I
Sequences and Series

1 Sequences

Definition 1 (Sequence)**.** A sequence of real numbers is a set of numbers which are written in some order. There are infinitely many terms in a sequence. It is denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Example 1. $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is called the harmonic sequence.

$$a_n = \frac{1}{n}$$

Example 2. $1, -\frac{1}{2}, \frac{1}{3}, \ldots$ is called the alternating harmonic sequence.

$$a_n = (-1)^{n+1}\frac{1}{n}$$

Example 3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$$a_n = \frac{n}{n+1}$$

Example 4. $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \ldots$

$$a_n = \frac{n+1}{3^n}$$

Example 5. The Fibonacci sequence is given by

$$f_n = \begin{cases} 1 & ; \quad n = 1, 2 \\ f_{n-1} + f_{n-2} & ; \quad n \geq 3 \end{cases}$$

Example 6. A geometric sequence is given by

$$a_n = a_1 q^{n-1}$$

where q is called the common ratio.

Example 7. A geometric sequence is given by

$$a_n = a_1 + d(n-1)$$

where d is called the common difference.

Definition 2 (Equal sequences)**.** Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equal if $a_n = b_n$, $\forall n \in \mathbb{N}$.

Definition 3 (Sequences bounded from above)**.** $\{a_n\}$ is said to be bounded from above if $\exists M \in \mathbb{R}$, s.t. $a_n \leq M$, $\forall n \in \mathbb{N}$. Each such M is called an upper bound of $\{a_n\}$.

Definition 4 (Sequences bounded from below)**.** $\{a_n\}$ is said to be bounded from below if $\exists m \in \mathbb{R}$, s.t. $a_n \geq M$, $\forall n \in \mathbb{N}$. Each such M is called an lower bound of $\{a_n\}$.

Definition 5. $\{a_n\}$ is said to be bounded if it is bounded from below and bounded from above.

Example 8. The sequence $a_n = n^2 + 2$ is not bounded from above but is bounded from below, by all $m \leq 3$.

Example 9. $\left\{\dfrac{2n-1}{3n}\right\}$ is bounded.

$$m = 0 \leq \frac{2n-1}{3n} \leq \frac{2n}{3n} = \frac{2}{3} = M$$

Definition 6 (Monotonic increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \leq a_{n+1}$, $\forall n \geq n_0$.

Definition 7 (Monotonic decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \geq a_{n+1}$, $\forall n \geq n_0$.

Definition 8 (Strongly increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n < a_{n+1}$, $\forall n \geq n_0$.

Definition 9 (Strongly decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n > a_{n+1}$, $\forall n \geq n_0$.

Example 10. The sequence $\left\{\dfrac{n^2}{2^n}\right\}$ is strongly decreasing. However, this is not evident by observing the first few terms. $\dfrac{1}{2}, 1, \dfrac{9}{8}, 1, \dfrac{25}{32}, \ldots$

$$\begin{aligned}
a_n &> a_{n+1} \\
\iff \frac{n^2}{2^n} &> \frac{(n+1)^2}{2^{n+1}} \\
\iff 2n^2 &> (n+1)^2 \\
\iff \sqrt{2}n &> n+1 \\
\iff n(\sqrt{2}-1) &> 1 \\
\iff n &> \frac{1}{\sqrt{2}-1} \\
\iff n &> 3
\end{aligned}$$

Exercise 1.
Is $a_n = (-1)^n$ monotonic?

Solution 1.
The sequence $-1, 1, -1, 1, \ldots$ is not monotonic.

1.1 Limit of a Sequence

Definition 10. Let $\{a_n\}$ be a given sequence. A number L is said to be the limit of the sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $|a_n - L| < \varepsilon$, $\forall n \geq n_0$. That is, there are infinitely many terms inside the interval and a finite number of terms outside it.

Example 11. The sequence $\{\frac{1}{n}\}$ tends to 0, i.e. for any open interval $(-\varepsilon, \varepsilon)$, there are finite number of terms of the sequence outside the interval, and therefore there are infinitely many terms inside the interval.

Exercise 2.
Prove

$$\lim_{n\to\infty} \frac{n+2}{2n-1} = \frac{1}{2}$$

Solution 2.
$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$

Exercise 3.
Prove that 2 is not a limit of $\left\{\dfrac{3n+1}{n}\right\}$.

Solution 3.
If possible, let

$$\lim_{n\to\infty} \frac{3n+1}{n} = 2$$

Then, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\left|\dfrac{3n+1}{n} - 2\right| < \varepsilon$, $\forall n \geq n_0$. However,

$$\left|\frac{3n+1}{n} - 2\right| = 1 + \frac{1}{n} > 1$$

This is a contradiction for $\varepsilon = \dfrac{1}{2}$. Therefore, 2 is not a limit.

Theorem 1. *If a sequence $\{a_n\}$ has a limit L then the limit is unique.*

Proof. If possible let there exist two limits L_1 and L_2. Therefore, $\forall \varepsilon > 0$, there exist a finite number of terms in the interval $(L_1 - \varepsilon, L_1 + \varepsilon)$. Therefore, there exist a finite number of terms in the interval $(L_2 - \varepsilon, L_2 + \varepsilon)$. This contradicts the definition of a limit. Therefore, the limit is unique. $\square$

Theorem 2. *If a sequence $\{a_n\}$ has limit L, then the sequence is bounded.*

Theorem 3. *Let*

$$\lim_{n\to\infty} a_n = a$$
$$\lim_{n\to\infty} b_n = b$$

and let c be a constant. Then,

$$\begin{aligned}
\lim c &= c \\
\lim(ca_n) &= c \lim a_n \\
\lim(a_n \pm b_n) &= \lim a_n \pm \lim b_n \\
\lim(a_n b_n) &= \lim a_n \lim b_n \\
\lim\left(\frac{a_n}{b_n}\right) &= \frac{\lim a_n}{\lim b_n} \quad (\text{ if } \lim b \neq 0)
\end{aligned}$$

Theorem 4. *Let $\{b_n\}$ be bounded and let $\lim a_n = 0$. Then,*

$$\lim(a_n b_n) = 0$$

Theorem 5 (Sandwich Theorem)**.** *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences. If*

$$\lim a_n = \lim b_n = L$$

and $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n \leq b_n \leq c_n$. Then,

$$\lim b_n = L$$

Exercise 4.
Calculate $\lim\limits_{n\to\infty} \sqrt[n]{2^n + 3^n}$

Solution 4.

$$\sqrt[n]{3^n} \leq \sqrt[n]{2^n + 3^n} \leq \sqrt[n]{3^n + 3^n} = \sqrt[3]{2 \cdot 3^n}$$
$$\therefore 3 \leq \sqrt[n]{2^n + 3^n} \leq 3\sqrt[n]{2}$$

Therefore, by the Sandwich Theorem, $\lim\limits_{n\to\infty} \sqrt[n]{2^n + 3^n} = 3$.

Theorem 6. *Any monotonically increasing sequence which is bounded from above converges. Similarly, any monotonically decreasing sequence which is bounded from below converges.*

Exercise 5.

Prove that there exists a limit for $a_n = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}_{n \text{ times}}$ and find it.

Solution 5.

$$a_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = a_2$$

If possible, let

$$\begin{aligned} a_{n-1} &< a_n \\ \therefore \sqrt{2 + a_{n-1}} &< \sqrt{2 + a_n} \\ \therefore a_n &< a_{n+1} \end{aligned}$$

Hence, by induction, $\{a_n\}$ is monotonically increasing.

$$a_1 = \sqrt{2} \leq 2$$

If possible, let

$$a_n \leq 2 \therefore \sqrt{2+a_n} \qquad \leq \sqrt{2+2}$$
$$\therefore a_{n+1} \leq 2$$

Hence, by induction, $\{a_n\}$ is bounded from above by 2. Therefore, by , $\{a_n\}$ converges.

Definition 11 (Limit in a wide sense)**.** The sequence $\{a_n\}$ is said to converge to $+\infty$ if $\forall M \in \mathbb{R}$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n > M$.
The sequence $\{a_n\}$ is said to converge to $-\infty$ if $\forall M \in \mathbb{R}$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n < M$.

1.2 Sub-sequences

Definition 12 (Sub-sequence)**.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_k\}_{k=1}^{\infty}$ be a strongly increasing sequence of natural numbers. Let $\{b_k\}_{k=1}^{\infty}$ be a sequence such that $b_k = a_{n_k}$. Then $\{b_k\}_{k=1}^{\infty}$ is called a sub-sequence of $\{a_n\}_{n=1}^{\infty}$.

Example 12.

$$a_n = \frac{1}{n}$$

If we choose $n_k = k^2$,

$$b_k = a_{n_k} = a_{k^2} = \frac{1}{k^2}$$

Therefore,

$$\{b_k\} = 1, \frac{1}{4}, \frac{1}{9}, \ldots$$

Theorem 7. *If the sequence $\{a_n\}$ converges to L in a wide sense, i.e. L can be infinite, then any sub-sequence of $\{a_n\}$ converges to the same limit L.*

Definition 13 (Partial limit)**.** A real number a, which may be infinite, is called a partial limit of the sequence $\{a_n\}$ is there exists a sub-sequence of $\{a_n\}$ which converges to a.

Example 13. Let

$$a_n = (-1)^n$$

Therefore, $\nexists \lim_{n \to \infty} a_n$. Let

$$b_k = a_{n_k} = a_{2n-1}$$

Therefore,

$$\{b_k\} = -1, -1, -1, \ldots$$
$$\therefore \lim_{k \to \infty} b_k = 1$$

Therefore, -1 is a partial limit of $\{a_n\}$.

Theorem 8 (Bolzano-Weierstrass Theorem)**.** *For any bounded sequence there exists a subsequence which is convergent, s.t. there exists at least one partial limit.*

Definition 14 (Upper partial limit)**.** The largest partial limit of a sequence is called the upper partial limit. It is denoted by $\overline{\lim} a_n$ or $\limsup a_n$.

Definition 15 (Lower partial limit)**.** The smallest partial limit of a sequence is called the upper partial limit. It is denoted by $\underline{\lim} a_n$ or $\liminf a_n$.

Theorem 9. *If the sequence $\{a_n\}$ is bounded and*

$$\overline{\lim} a_n = \underline{\lim} a_n = a$$

then

$$\exists \lim a_n = a$$

1.3 Cauchy Characterisation of Convergence

Definition 16. A sequence $\{a_n\}$ is called a Cauchy sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall m, n \geq n_0$, $|a_n - a_m| < \varepsilon$.

Theorem 10 (Cauchy Characterisation of Convergence)**.** *A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.*

Proof. Let

$$\lim_{n\to\infty} a_n = L$$

Then $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$, $|a_n - L| < \dfrac{\varepsilon}{2}$. Therefore if $n \geq n_0$ and $m \geq n_0$, then

$$\begin{aligned}
|a_n - a_m| &= |a_n - L + L - a_m| \\
&\leq |a_n - L| + |L - a_m| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
\therefore |a_n - a_m| &= \varepsilon
\end{aligned}$$

Similarly, the converse can be proved by Theorem 9. $\square$

Theorem 11 (Another Formulation of the Cauchy Characterisation Theorem)**.** *The sequence* $\{a_n\}$ *converges if and only if* $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, *such that* $\forall n \geq n_0$ *and* $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$*.*

Exercise 6.
Prove that the sequence

$$a_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

is convergent.

Solution 6.

$$\begin{aligned}
|a_{n+p} - a_n| &= \left| \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(n+p)^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right) \right| \\
&= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(n+p-1)(n+p)} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} + \cdots + \cancel{\frac{1}{n+p-1}} - \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n} - \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n}
\end{aligned}$$

Therefore, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$, where $n_0 > \frac{1}{\varepsilon}$. $\square$

Exercise 7.
Prove that the sequence

$$a_n = \frac{1}{1} + \frac{1}{n} + \cdots + \frac{1}{n}$$

diverges.

Solution 7.
If possible, let the sequence converge. Then, by the Cauchy Characterisation of Convergence, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$.
Therefore,

$$\begin{aligned}
|a_{n+p} - a_n| &= \left| \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+p} - \left(\frac{1}{n} + \cdots + \frac{1}{n} \right) \right| \\
&= \frac{1}{n+1} + \cdots + \frac{1}{n+p} \\
&\geq p \cdot \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &> \frac{p}{n+p}
\end{aligned}$$

If $n = p$,

$$\frac{p}{n+p} = \frac{1}{2}$$

This contradicts the result obtained from the Cauchy Characterisation of Convergence, for $\varepsilon = \frac{1}{4}$.
Therefore, the sequence diverges.

2 Series

Definition 17 (Series)**.** Given a sequence $\{a_n\}$, the sum $a_1 + \cdots + a_n + \ldots$ is called an infinite series or series. It is denoted as $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

Definition 18 (Partial sum)**.** The partial sum of the series $\sum a_n$ is defined as

$$S_i = a_1 + \cdots + a_i$$

Definition 19 (Convergent and divergent series)**.** If the sequence $\{S_n\}_{n=1}^{\infty}$ converges, then the series is called convergent. Otherwise, the series is called divergent.

Definition 20 (Sum of a series)**.** If the sequence $\{S_n\}_{n=1}^{\infty}$ converges to $S \neq \pm\infty$, the number S is called the sum of the series.

$$\sum_{n=1}^{\infty} a_n = S$$

Example 14.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

Therefore,

$$S_1 = \frac{1}{2} \tag{1}$$
$$S_2 = \frac{1}{2} + \frac{1}{2^2} \tag{2}$$
$$\vdots S_n = \frac{1}{2} + \cdots + \frac{1}{2^n} \tag{3}$$
$$= \frac{a_1\left(1 - q^n\right)}{1 - q} \tag{4}$$
$$= \frac{{}^1\!/_2\left(1 - {}^1\!/_{2^n}\right)}{1 - {}^1\!/_2} \tag{5}$$
$$= 1 - \frac{1}{2^n} \tag{6}$$
$$\lim_{n\to\infty} S_n = 1 \tag{7}$$

Therefore, the series converges.

$$S = \sum_{n=1}^{\infty} = 1$$

Theorem 12. *A geometric series* $\sum_{n=1}^{\infty} a_1 q^{n-1}$, $a_1 \neq 0$ *converges if* $|q| < 1$ *and then,*

$$S = \sum_{n=1}^{\infty} a_1 q^{n-1} = \frac{a_1}{1-q}$$

Definition 21 (p-series)**.** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the p-series.

Theorem 13. *The p-series converges for $p > 1$ and diverges for $p \leq 1$.*

Theorem 14. *If $\sum a_n$ converges, then*

$$\lim_{n\to\infty} a_n = 0$$

Proof.

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ \therefore \lim_{n\to\infty} a_n &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \\ &= S - S \\ &= 0 \end{aligned}$$

□

Theorem 15. *If $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n \pm b_n)$ and $\sum ca_n$, where c is a constant, also converge. Also,*

$$\begin{aligned} \sum(a_n \pm b_n) &= \sum a_n \pm \sum b_n \\ \sum(ca_n) &= c \sum a_n \end{aligned}$$

2.1 Convergence Criteria

2.1.1 Leibniz's Criteria

Definition 22 (Alternating series)**.** The series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

Theorem 16 (Leibniz's Criteria for Convergence)**.** *If an alternating series $\sum(-1)^{n-1}a_n$ with $a_n > 0$ satisfies*

1. *$a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.*
2. *$\lim\limits_{n\to\infty} a_n = 0$*

then the series $(-1)^{n-1}a_n$ converges.

Proof. Consider the even partial sums of the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n$.

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$$

As $\{a_n\}$ is monotonically increasing, all brackets are non-negative. Therefore,

$$S_{2m+2} \geq S_{2m}$$

Therefore, $\{S_{2m}\}$ is increasing.
Also,

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$$

All brackets and a_{2m} are non-negative. Therefore,

$$S_{2m} \leq a_1$$

Therefore, $\{S_{2m}\}$ is bounded from above by a_1. Hence,

$$\exists \lim_{m\to\infty} S_{2m} = S$$

For S_{2m+1},

$$\begin{aligned} S_{2m+1} &= S_{2m} + a_{2m+1} \\ \therefore \lim_{m\to\infty} S_{2m+1} &= \lim_{m\to\infty} S_{2m} + \lim_{m\to\infty} a_{2m+1} \\ &= S + 0 \\ &= S \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} S_n = S$$

□

Example 15. The alternating harmonic series $\sum \dfrac{(-1)^{n-1}}{n}$ converges as $a_n = \dfrac{1}{n} > 0$, a_n decreases and $\lim a_n = 0$.

2.1.2 Comparison Test

Theorem 17 (Comparison Test for Convergence)**.** *Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$.*

1. *If $a_n \leq b_n$, $\forall n \geq n_0$ and $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.*

2. *If $a_n \geq b_n$, $\forall n \geq n_0$ and $\sum\limits_{n=1}^{\infty} b_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.*

Theorem 18 (Another Formulation of the Comparison Test for Convergence)**.** *Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$ and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = a > 0$$

where a is a finite number. Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} b_n$ converges.

2.1.3 d'Alembert Criteria (Ratio Test)

Definition 23 (Absolute and conditional convergence)**.** The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

Example 16. The series $\sum \frac{(-1)^{n-1}}{n^2}$ converges absolutely, as $\sum \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum \frac{1}{n^2}$ converges.

Example 17. The series $\sum \frac{(-1)^{n-1}}{n}$ converges conditionally, as it converges, but $\sum \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum \frac{1}{n}$ diverges.

Theorem 19. *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 20 (d'Alembert Criteria (Ratio Test))**.** *1. If*

$$\lim_{n\to\infty} \left|\frac{a_{n-1}}{a_n}\right| = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\lim_{n\to\infty} \left|\frac{a_{n-1}}{a_n}\right| = L > 1$$

(including $L = \infty$), then $\sum a_n$ converges diverges.

3. If $L = 1$, the test does not apply.

2.1.4 Cauchy Criteria (Cauchy Root Test)

Theorem 21 (Cauchy Criteria (Cauchy Root Test))**.** *1. If*

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

(including $L = \infty$), then $\sum a_n$ diverges.

3. If $L = 1$, the test does not apply.

2.1.5 Integral Test

Theorem 22 (Integral Test for Series Convergence)**.** *Let $f(x)$ be a continuous, non-negative, monotonic decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int\limits_1^{\infty} f(x)\,\mathrm{d}x$ converges.*

Exercise 8.
Does $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ converge or diverge?

Solution 8.
Let

$$f(x) = \frac{1}{x^p}$$

with $p > 0$.
Therefore, $f(x)$ is continuous, non-negative and monotonic decreasing on $[1,\infty)$. Therefore, the Integral Test for Series Convergence is applicable.

$$\int_1^{\infty} \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^p}\,\mathrm{d}x$$

If $p \neq 1$,

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^p} &= \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^t \\ &= \lim_{t\to\infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right) \\ &= \frac{1}{p-1}\end{aligned}$$

If $p = 1$,

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^p} &= \lim_{t\to\infty} \ln x\big|_1^t \\ &= \infty\end{aligned}$$

Therefore, the series converges for $p > 1$ and diverges for $p \leq 1$.

Theorem 23. *If the series $\sum a_n$ absolutely converges and the series $\sum b_n$ is obtained from $\sum a_n$ by changing the order of the terms in $\sum a_n$ then $\sum b_n$ also absolutely converges and $\sum b_n = \sum a_n$.*

Theorem 24. *If a series converges then the series with brackets without changing the order of terms also converges. That is, if $\sum a_n$ converges, then any series of the form $(a_1 + a_2) + (a_3 + a_4 + a_5) + a_6 + \dots$ also converges.*

Theorem 25. *If a series with brackets converges and the terms in the brackets have the same sign, then the series without brackets also converges.*

3 Power Series

Definition 24 (Power series)**.** The series $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ is called a power series.

Theorem 26 (Cauchy-Hadamard Theorem)**.** *For any power series $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ there exists the limit, which may be infinity,*

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}}$$

and the series converges for $|x-c| < R$ and diverges for $|x-c| > R$. The end points of the interval, i.e. $x = c - R$ and $x = c + R$ must be separately checked for series convergence.

Definition 25 (Radius of convergence and convergence interval)**.** The number R is called the radius of convergence and the interval $|x-c| < R$ is called the convergence interval of the series. The point c is called the centre of the convergence interval.

Theorem 27. *If $\exists \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, which may be infinite, then,*

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

Theorem 28 (Stirling's Approximation)**.** *For $n \to \infty$,*

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

3.1 Differentiation and Integration of Power Series

Theorem 29. *If R is a radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ then the function $f(x) = \sum\limits_{n=0}^{\infty} a_n(x-c)^n$ is differentiable on $(c-R, c+R)$ and the derivative is*

$$f'(x) = \sum_{n=0}^{\infty} na_n(x-c)^{n-1}$$

Theorem 30. *If R is a radius of convergence of the series $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ then the function $f(x) = \sum\limits_{n=0}^{\infty} a_n(x-c)^n$ is integrable in $(c-R, c+R)$ and*

$$\int f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} + A$$

where $c-R < x < c+R$.

Exercise 9.
Find $\int\limits_0^x e^{-t^2}\,\mathrm{d}t$.

Solution 9.
$\forall s \in \mathbb{R}$,

$$\begin{aligned}
e^s &= 1 + \frac{s}{1!} + \frac{s^2}{2!} + \cdots + \frac{s^n}{n!} + \dots \\
\therefore e^{-t^2} &== 1 - \frac{t^2}{1!} + \frac{t^4}{2!} + \cdots + (-1)^n \frac{t^{2n}}{n!} + \dots \\
\therefore \int_0^x e^{-t^2}\,\mathrm{d}t &= x - \frac{x^3}{1!3} + \frac{x^5}{2!5} + \cdots + (-1)^n \frac{x^{2n-1}}{n!(2n+1} + \dots
\end{aligned}$$

Theorem 31. *If the series $A(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum\limits_{n=0}^{\infty} B_n x^n$ absolutely converge for $|x| < R$ and $c_n = \sum\limits_{k=0}^{n} a_k b_{n-k}$, then the series $C(x) = \sum\limits_{n=0}^{\infty} c_n x^n$ also absolutely converges for $|x| < R$ and $C(x) = A(x)B(x)$.*

3.2 Taylor Series

Definition 26 (Taylor series)**.** Let $f(x)$ be infinitely differentiable on an open interval about a and let x be an arbitrary point in the interval. Then the power series $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the Taylor series of $f(x)$ at a. If $a = 0$ then it is called the Maclaurin series of $f(x)$ at 0.

Theorem 32. *If there exists a power series which converges to $f(x)$, i.e. if, for $|x-a| < R$,*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

then, for $|x-a| < R$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

that is, $\forall n$,

$$a_n = \frac{f^{(n)}(a)}{n!}$$

Exercise 10.
Show that

$$f(x) = \begin{cases} 0 & ; \quad x = 0 \\ e^{-\frac{1}{x^2}} & ; \quad x \neq 0 \end{cases}$$

is not equal to it's Taylor series at $a = 0$.

Solution 10.
If $n = 1$,

$$\begin{aligned} f^{(n)}(0) &= \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{e^{-\frac{1}{(\Delta x)^2}}}{\Delta x} \end{aligned}$$

Let $t = \dfrac{1}{\Delta x}$

$$\begin{aligned} \therefore f'(0) &= \lim_{t \to \infty} \frac{e^{-t^2}}{\frac{1}{t}} \\ &= \lim_{t\ to \infty} \frac{t}{e^{t^2}} \\ &= \lim_{t \to \infty} \frac{1}{e^{t^2} 2t} \\ &= 0 \end{aligned}$$

Therefore,

$$f'(x) = \begin{cases} 0 & ; \quad x = 0 \\ e^{-\frac{1}{x^2}} \cdot 2 \cdot x^{-3} & ; \quad x \neq 0 \end{cases}$$

Similarly, $\forall n \geq 1$, $f^{(n)}(0) = 0$
Therefore, the Taylor series is not equal to $f(x)$.

Exercise 11.
Find the Maclaurin series of $f(x) = e^x$ and prove that the series converges to $f(x)$ for any $x \in \mathbb{R}$.

Solution 11.
$\forall n \geq 1$, $f^{(n)}(x) = e^x$.
Therefore,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c x^{n+1}}{(n+1)!}$$

where c is between 0 and x.
Therefore, as

$$0 \leq |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

by the Sandwich Theorem

$$\lim_{n \to \infty} |R_n(x)| = 0$$

Therefore,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \ldots$$

4 Series of Real-valued Functions

Definition 27 (Sequence of functions)**.** A sequence $\{f_n\} = f_1(x), f_2(x), \ldots$ defined on $D \subseteq \mathbb{R}$ is called a sequence of functions.

Definition 28 (Pointwise convergence and domain of convergence)**.** $\{f_n\}$ converges pointwise in some domain $E \subseteq D$ if for every $x \in E$, the sequence of $\{f_n(x)\}$ converges. In such a case, E is said to be a domain of convergence of $\{f_n\}$.

Exercise 12.
Find the domain of convergence of $f_n(x) = x^n$, defined on some $D \subseteq \mathbb{R}$.

Solution 12.

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & ; \quad -1 < x < 1 \\ 1 & ; \quad x = 1 \\ \text{diverges} & ; \quad x \notin (-1, 1] \end{cases}$$

Therefore, the domain of convergence of $\{f_n\}$ is $(-1, 1]$.

Exercise 13.
Let $f(x) : (0, \infty) \to \mathbb{R}$ be some function such that $\lim\limits_{x\to\infty} f(x) = 0$. Let $f_n(x) = f(nx)$. What is the domain of convergence of f_n? What is the limit function?

Solution 13.
Let x have some fixed value in $(0, \infty)$. Therefore, as $\lim\limits_{x\to\infty} f(x) = 0$,

$$\begin{aligned} \lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} f(nx) \\ &= 0 \end{aligned}$$

Therefore, the domain of convergence is $(0, \infty)$ and the limit function is a constant function with value 0.

4.1 Uniform Convergence of Series of Functions

Definition 29 (Pointwise convergence of a sequence of functions)**.** If $\forall x \in D$, $\forall \varepsilon > 0$, $\exists N$ which depends on ε and x, such that $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$, then $\forall x \in D$, $\lim\limits_{n\to\infty} = f(x)$.

Definition 30 (Uniform convergence of a sequence of functions)**.** The sequence $\{f_n(x)\}$ is said to converge uniformly to $f(x)$ in D if $\forall \varepsilon > 0$, $\exists N = N(\varepsilon)$, such that $\forall n \geq N$, $\forall x \in D$, $|f_n(x) - f(x)| < \varepsilon$. It can be denoted as $f_n(x) \overset{D}{\rightrightarrows} f(x)$.

Theorem 33. *$f_n(x)$ converges uniformly to $f(x)$ in D if and only if $\lim\limits_{n\to\infty} \sup\limits_{x\in D} |f_n(x) - f(x)| = 0$.*

Exercise 14.
Does $f_n(x) = x^n$ converge in $[0, 1]$?

Solution 14.

$$\begin{aligned} \lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} x^n \\ \therefore f(x) &= \begin{cases} 0 & ; \quad 0 \leq x < 1 \\ 1 & ; \quad x = 1 \end{cases} \end{aligned}$$

Therefore,
If $x = 0$,

$$\begin{aligned} f_n(0) &= 0 \\ f(0) &= 0 \end{aligned}$$

Therefore, $\forall \varepsilon > 0$, $N = 1$,

$$\begin{aligned} |0 - 0| &< \varepsilon \\ \therefore |f_n(0) - f(0)| &< \varepsilon \end{aligned}$$

If $x = 1$,

$$\begin{aligned} f_n(1) &= 1 \\ f(1) &= 1 \end{aligned}$$

Therefore, $\forall \varepsilon > 0$, $N = 1$,

$$\begin{aligned} |1 - 1| &< \varepsilon \\ \therefore |f_n(1) - f(1)| &< \varepsilon \end{aligned}$$

If $0 < x < 1$,

$$\begin{aligned} |f_n(x) - f(x)| &= |x^n - 0| \\ &= x^n \end{aligned}$$

If possible, let $|f_n(x) - f(x)| = x^n < \varepsilon$.
Therefore,

$$\begin{aligned} x^n &< \varepsilon \\ \therefore \log_x x^n &> \log_x \varepsilon \\ \therefore n &> \log_x \varepsilon \end{aligned}$$

Therefore, for $N = \lfloor \log_x \varepsilon \rfloor + 1$, $|f_n(x) - f(x)| < \varepsilon$.

Therefore, $f_n(x)$ converges pointwise in $[0, 1]$.

If possible let $f_n(x)$ converge uniformly on $[0, 1]$.
Therefore, $\forall \varepsilon > 0$, $\exists N$ dependent on ε, such that $|f_n(x) - f(x)| < \varepsilon$.
Let $\varepsilon = \frac{1}{3}$.
Therefore, $\exists N$ which is dependent on ε, such that $\forall n > N$, $\forall x \in [0, 1]$,

$$|f_n(x) - f(x)| < \frac{1}{3}$$

Let $x = \frac{1}{2}$, $n = N + 1$. Therefore,

$$\begin{aligned} \left| f_n\left(\frac{1}{2}\right) - f\left(\frac{1}{2}\right) \right| &= \left| \frac{1}{2} - 0 \right| \\ &= \frac{1}{2} \\ \therefore \left| f_n\left(\frac{1}{2}\right) - f\left(\frac{1}{2}\right) \right| &> \frac{1}{3} \end{aligned}$$

Therefore, $|f_n(x) - f(x)| > \varepsilon$.
This is a contradiction. Hence, $f_n(x)$ is does not converge uniformly.

Definition 31 (Supremum)**.** Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the supremum of A if

1. $\forall x \in A$, $x \leq M$, i.e. M is an upper bound of A.

2. $\forall \varepsilon$, $\exists x \in A$, such that $x > M - \varepsilon$.

That is, the supremum of A is the least upper bound of A.
The supremum may or may not be in A.

Definition 32 (Infimum)**.** Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the infimum of A if

1. $\forall x \in A$, $x \geq M$, i.e. M is an upper bound of A.

2. $\forall \varepsilon$, $\exists x \in A$, such that $x < M - \varepsilon$.

That is, the infimum of A is the greatest lower bound of A. The infimum may or may not be in A.

Theorem 34. *Every bounded set A has a supremum and an infimum.*

Theorem 35. $f_n \overset{E}{\rightrightarrows} f$ *if and only if*

$$\lim_{n\to\infty} (\sup\{|f_n(x) - f(x)| : x \in E\}) = 0$$

Definition 33 (Remainder of a series of functions)**.** Let $f(x) = \sum\limits_{k=1}^{\infty} u_k(x)$. Let the partial sums be denoted by $f_n(x) = \sum\limits_{k=1}^{n} u_k(x)$. Then

$$R_n(x) = f(x) - f_n(x) = \sum_{k=n+1}^{\infty} u_k(x)$$

is called a remainder of the series $f(x) = \sum\limits_{k=1}^{\infty} u_k(x)$.

Definition 34 (Uniform convergence of a series of functions)**.** If $f_n(x)$ converges uniformly to $f(x)$ on D, i.e. if $\lim\limits_{n\to\infty} R_n(x) = 0$, then the series $\sum\limits_{k=1}^{\infty} u_k(x)$ is said to converge uniformly on D..

Exercise 15.

Show that the series $f(x) = \sum\limits_{k=1}^{\infty} x^{k-1} = \frac{1}{1-k}$ does not converge uniformly on $(-1, 1)$.

Solution 15.

The series converges uniformly if and only if $\lim\limits_{n\to\infty} R_n(x) = 0$.

$$\begin{aligned}
\lim_{n\to\infty} \sup_{(-1,1)} |R_n(x) - 0| &= \lim_{n\to\infty} \sup_{(-1,1)} \sum_{k=n+1}^{\infty} x^{k-1} \\
&= \lim_{n\to\infty} \sup_{(-1,1)} \left| \frac{x^n}{1-x} \right| \\
&= \lim_{n\to\infty} \sup_{(-1,1)} \frac{|x|^n}{1-x} \\
&= \lim_{n\to\infty} \infty \\
&= \infty
\end{aligned}$$

Therefore, the series does not converge uniformly on $(-1, 1)$.

Exercise 16.

Show that the series $f(x) = \sum\limits_{k=1}^{\infty} x^{k-1} = \frac{1}{1-k}$ does not converge uniformly on $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Solution 16.

The series converges uniformly if and only if $\lim\limits_{n\to\infty} R_n(x) = 0$.

$$\begin{aligned}
\lim_{n\to\infty} \sup_{\left(-\frac{1}{2},\frac{1}{2}\right)} |R_n(x) - 0| &= \lim_{n\to\infty} \sup_{\left(-\frac{1}{2},\frac{1}{2}\right)} \sum_{k=n+1}^{\infty} x^{k-1} \\
&= \lim_{n\to\infty} \sup_{\left(-\frac{1}{2},\frac{1}{2}\right)} \left|\frac{x^n}{1-x}\right| \\
&= \lim_{n\to\infty} \sup_{\left(-\frac{1}{2},\frac{1}{2}\right)} \frac{|x|^n}{1-x} \\
&= \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} \\
&= \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n-1} \\
&= 0
\end{aligned}$$

Therefore, the series converges uniformly on $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

4.2 Weierstrass M-test

Theorem 36 (Weierstrass M-test)**.** *If $|u_k(x)| \le c_k$ on D for $k \in \{1, 2, 3, \dots\}$ and the numerical series $\sum\limits_{k=1}^{\infty} c_k$ converges, then the series of functions $\sum\limits_{k=1}^{\infty} u_k(x)$ converges uniformly on D.*

Exercise 17.

Show that $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} \sin(kx)$ converges uniformly on $\mathbb{R}$.

Solution 17.

$$|u_k(x)| = \left|\frac{1}{k^2}\sin(kx)\right|$$
$$\therefore |u_k(x)| \leq \frac{1}{k^2}$$

Therefore, let

$$c_k = \frac{1}{k^2}$$

Therefore, as $|u_k(x)| \leq c_k$, and as $\sum\limits_{k=1}^{\infty} c_k$ converges, by the Weierstrass M-test, $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}\sin(kx)$ converges uniformly.

4.3 Application of Uniform Convergence

Theorem 37 (Continuity of a series)**.** *Let functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ be defined on $[a, b]$ and continuous at $x_0 \in [a, b]$. If $\sum\limits_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a, b]$ then the function $f(x) = \sum\limits_{k=1}^{\infty}$ is also continuous at x_0.*

Theorem 38 (Changing the order of integration and infinite summation)**.** *If the functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ are integrable on $[a, b]$ and the series $\sum\limits_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a, b]$ then*

$$\int_a^b \left(\sum_{k=1}^{\infty} u_k(x)\right) \mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b u_k(x)\,\mathrm{d}x$$

Exercise 18.
Solve $\int\limits_0^{2\pi} \left(\sum\limits_{k=1}^{\infty} \frac{1}{k^2}\sin(kx)\right)$.

Solution 18.
The series $f(x) = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}\sin(kx)$ converges uniformly on $[0, 2\pi]$. Therefore, by the Weierstrass M-test and $u_k(x) = \frac{1}{k^2}(kx)$ are integrable on $[0, 2\pi]$. There-

fore,

$$\begin{aligned}
\int_0^{2\pi} f(x)\,\mathrm{d}x &= \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx) \right) \mathrm{d}x \\
&= \sum_{k=1}^{\infty} \left(\int_0^{2\pi} \frac{1}{k^2} \sin(kx)\,\mathrm{d}x \right) \\
&= \sum_{k=1}^{\infty} \left(-\frac{\cos(2\pi k)}{k^3} + \frac{1}{k^3} \right) \\
&= \sum_{k=1}^{\infty} 0 \\
&= 0
\end{aligned}$$

Theorem 39 (Changing the order of differentiation and infinite summation)**.** *If the functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ are differentiable on $[a, b]$ and the derivatives are continuous on $[a, b]$, and the series $\sum_{k=1}^{\infty} u_k(x)$ converges pointwise on $[a, b]$ and the series $\sum_{k=1}^{\infty} {u_k}'(x)$ converges uniformly on $[a, b]$, then,*

$$\left(\sum_{k=1}^{\infty} u_k(x) \right)' = \sum_{k=1}^{\infty} {u_k}'(x)$$

Theorem 40 (Changing the order of integration and limit)**.** *If the functions $f_n(x)$ are integrable on $[a, b]$ and converge uniformly to f on $[a, b]$, then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Theorem 41 (Changing the order of differentiation and limit)**.** *If there exists the functions ${f_n}'(x)$ which are continuous on $[a, b]$, for the functions $f_n(x)$ which $\forall x \in [a, b]$, converge pointwise to $f(x)$ on $[a, b]$, and if ${f_n}'(x)$ converges uniformly to $g(x)$ on $[a, b]$, then,*

$$f'(x) = \left(\lim_{n\to\infty} f_n(x) \right)' = \lim_{n\to\infty} {f_n}'(x) = g(x)$$

Part II
Functions of Multiple Variables

1 Limits, Continuity, and Differentiability

Definition 35 (Limit of a function of two variables)**.** Let $z = f(x, y)$ be defined on some open neighbourhood about (a, b), except maybe at the point itself. $L \in \mathbb{R}$ is said to be a limit of $f(x, y)$ at (a, b), if $\forall \varepsilon > 0$, $\exists d > 0$, such that $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then,

$$|f(x, y) - L| < \varepsilon$$

Exercise 19.

Does the limit $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ exist?

Solution 19.

Consider the curves $C_1 : y = 0$, and $C_2 : y = x^3$.
Therefore, as $(x, y) \to (0, 0)$ along these curves, the limit of the function is

$$\begin{aligned} \lim_{(x,y)\xrightarrow{C_1}(0,0)} \frac{3x^2y}{x^2+y^2} &= \lim_{x\to 0} \frac{3x^2 \cdot 0}{x^2+y^2} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{(x,y)\xrightarrow{C_2}(0,0)} \frac{3x^2y}{x^2+y^2} &= \lim_{x\to 0} \frac{3x^2(x^3)}{x^2+(x^3)^2} \\ &= \lim_{x\to 0} \frac{3x^5}{x^2+x^6} \\ &= \lim_{x\to 0} \frac{3x^3}{x^2+x^4} \\ &= 0 \end{aligned}$$

If $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < \sqrt{x^2+y^2} < \delta$, then,

$$|f(x, y) - L| < \varepsilon$$

Therefore, checking $|f(x,y) - L|$,

$$\begin{aligned} |f(x,y) - L| &= \left| \frac{3x^2y}{x^2+y^2} - 0 \right| \\ &= \frac{3x^2|y|}{x^2+y^2} \end{aligned}$$

As $\frac{x^2}{x^2+y^2} \leq 1$,

$$\begin{aligned} |f(x,y) - L| &\leq 3|y| \\ \therefore |f(x,y) - L| &\leq 3\sqrt{y^2} \\ \therefore |f(x,y) - L| &\leq 3\sqrt{x^2+y^2} \end{aligned}$$

Therefore, $|f(x,y) - L| < \varepsilon$.
Therefore, for $\delta \leq \frac{\varepsilon}{3}$, the condition is satisfied.
Hence, the limit of the function exists and is 0.

Definition 36 (Iterative limits)**.** The limits $\lim\limits_{x \to a} \left(\lim\limits_{y \to b} f(x,y) \right)$ and $\lim\limits_{y \to b} \left(\lim\limits_{x \to a} f(x,y) \right)$ are called the iterative limits of $f(x,y)$.

Theorem 42. *If* $\exists \lim\limits_{(x,y) \to (a,b)} f(x,y) = L$ *and, for some open interval about* b, $\forall y \neq b$, $\exists \lim\limits_{x \to a} f(x,y)$ *then*

$$\lim_{y \to b} \left(\lim_{x \to a} f(x,y) \right) = L$$

If $\exists \lim\limits_{(x,y) \to (a,b)} f(x,y) = L$ *and, for some open interval about* a, $\forall x \neq a$, $\exists \lim\limits_{y \to b} f(x,y)$ *then*

$$\lim_{x \to a} \left(\lim_{y \to b} f(x,y) \right) = L$$

Exercise 20.
Do the iterative limits, as $x \to 0$, and as $y \to 0$, of the function

$$f(x,y) = \begin{cases} (x+y)\sin\frac{1}{x+y} & ; \quad x \neq 0, y \neq 0 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

exists? Does the limit of the function at $(0,0)$ exist?

Solution 20.

$$\begin{aligned}\lim_{x\to 0} f(x,y) &= \lim_{x\to 0}(x+y)\sin\frac{1}{x+y}\\ &= \lim_{x\to 0} y\sin\frac{1}{x+y}\end{aligned}$$

Therefore, as $\sin\frac{1}{x+y}$ oscillates between -1 and 1, the limits does not exist.

$$\begin{aligned}\lim_{y\to 0} f(x,y) &= \lim_{y\to 0}(x+y)\sin\frac{1}{x+y}\\ &= \lim_{y\to 0} x\sin\frac{1}{x+y}\end{aligned}$$

Therefore, as $\sin\frac{1}{x+y}$ oscillates between -1 and 1, the limits does not exists.
Therefore, the iterative limits do not exist.

$$\begin{aligned}|f(x,y)-0| &= |x+y|\cdot\left|\sin\frac{1}{xy}\right|\\ \therefore |f(x,y)-0| &\leq |x|+|y|\\ \therefore |f(x,y)-0| &\leq \sqrt{2}\sqrt{x^2+y^2}\end{aligned}$$

Therefore, for $\delta \leq \frac{\varepsilon}{\sqrt{2}}$, the condition is satisfied.
Hence, the limit of the function exists and is 0.
Therefore, even though the iterative limits do not exist, the limit of the function exists.

Definition 37 (Differential)**.**

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

$$\mathrm{d}z = f_x(a,b)\,\mathrm{d}x + f_y(a,b)\,\mathrm{d}y$$

Definition 38 (Differentiability)**.** The function $x = f(x,y)$ is said to be differentiable at (a,b) if

$$\Delta z = \mathrm{d}z + \varepsilon_1(\Delta x, \Delta y)\Delta x + \varepsilon_2(\Delta x, \Delta y)\Delta y$$

where

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1(\Delta x,\Delta y) = \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2(\Delta x,\Delta y) = 0$$

Theorem 43. *If $f(x,y)$ is differentiable at (a,b) then $f(x,y)$ is continuous at (a,b).*

Theorem 44. *If $\exists f_x(a,b)$ and $\exists f_y(a,b)$ on some open neighbourhood of (a,b) and are continuous at (a,b), then $f(x,y)$ is differentiable at (a,b).*

2 Directional Derivatives and Gradients

Definition 39 (Directional derivative)**.** Let $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$.
Let $\hat{u} = (a, b)$ be a unit vector in the xy-plane.
The directional derivative of $z = f(x, y)$ with respect to the direction $\hat{u} = (a, b)$ at the point $(x_0, y_0$ is defined as

$$D_{\hat{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

If the limit does not exist, the directional derivative does not exist.

Geometrically the directional derivative of $z = f(x, y)$ is the slope of the tangent of the curve formed due to the intersection of the curve $z = f(x, y)$, and the plane which passes through (x_0, y_0) in the direction of $\hat{u}$ and is perpendicular to the xy-plane.

Definition 40 (Gradient)**.** If the functions $f_x(x, y)$ and $f_y(x, y)$ for $z = f(x, y)$ exist, then the vector function

$$\nabla f(x, y) = \big(f_x(x, y), f_y(x, y)\big)$$

is called the gradient of $f(x, y)$.

Theorem 45. *Let $z = f(x, y)$ be differential at (x_0, y_0). The function $f(x, y)$ has a directional derivative with respect to any direction $\hat{u} = (a, b)$ at (x_0, y_0) and*

$$D_{\hat{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \nabla f(x_0, y_0) \cdot \hat{u}$$

Exercise 21.
Find the directional derivative of

$$f(x, y) = x^3 + 4xy + y^4$$

with respect to the direction of $\overline{u} = (1, 2)$ at any point (x, y) and at $(0, 1)$.

Solution 21.

$$f(x, y) = x^3 + 4xy + y^4$$

Therefore,

$$f_x(x, y) = 3x^2 + 4y$$
$$f_y(x, y) = 4x + 4y^3$$

$$
\begin{aligned}
\hat{u} &= \frac{\overline{u}}{u} \\
&= \frac{(1,2)}{\sqrt{5}} \\
&= \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)
\end{aligned}
$$

Therefore,

$$
D_{\hat{u}} f(x,y) = \frac{1}{\sqrt{5}}(3x^2 + 4y) + \frac{2}{\sqrt{5}}(4x + 4y^2)
$$

Therefore,

$$
\begin{aligned}
D_{\hat{u}} f(0,1) &= \frac{4}{\sqrt{5}} + \frac{8}{\sqrt{5}} \\
&= \frac{12}{\sqrt{5}}
\end{aligned}
$$

Theorem 46. *If $z = f(x,y)$ is differentiable at (x_0, y_0), then $\exists \hat{u_0} = (a_0, b_0)$ such that*

$$
\max_{\hat{u} \in \mathbb{R}} D_{\hat{u}} f(x_0, y_0) = D_{\hat{u_0}} f(x_0, y_0) = |\nabla f(x_0, y_0)|
$$

and

$$
\hat{u_0} = \frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}
$$

Proof.

$$
\begin{aligned}
\max_{\hat{u} \in \mathbb{R}} D_{\hat{u}} f(x_0, y_0) &= \max_{\hat{u} \in \mathbb{R}} \nabla f(x_0, y_0) \cdot \hat{u} \\
&= \max_{\hat{u} \in \mathbb{R}} |\nabla f(x_0, y_0)| \, |\cancelto{1}{\hat{u}}| \cos\theta \\
&= |\nabla f(x_0, y_0)| \max_{\hat{u} \in \mathbb{R}} \cos\theta \\
&= |\nabla f(x_0, y_0)|
\end{aligned}
$$

□

Theorem 47. *If $z = f(x,y)$ is differentiable at (x_0, y_0), then $\exists \hat{u_1} = (a_0, b_0)$ such that*

$$
\min_{\hat{u} \in \mathbb{R}} D_{\hat{u}} f(x_0, y_0) = D_{\hat{u_1}} f(x_0, y_0) = -|\nabla f(x_0, y_0)|
$$

and

$$
\hat{u_1} = -\frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}
$$

Proof.

$$
\begin{aligned}
\min_{\hat{u}\in\mathbb{R}} D_{\hat{u}} f(x_0, y_0) &= \min_{\hat{u}\in\mathbb{R}} \nabla f(x_0, y_0) \cdot \hat{u} \\
&= \min_{\hat{u}\in\mathbb{R}} |\nabla f(x_0, y_0)| \, \overset{1}{\cancel{|\hat{u}|}} \cos\theta \\
&= |\nabla f(x_0, y_0)| \min_{\hat{u}\in\mathbb{R}} \cos\theta \\
&= -\,|\nabla f(x_0, y_0)|
\end{aligned}
$$

□

3 Local Extrema

Theorem 48 (A necessary condition for local extrema existence)**.** *If the function $z = f(x, y)$ has a local extrema at the point (a, b) and $\exists f_x(a, b)$ and $\exists f_y(a, b)$ then $f_x(a, b) = f_y(a, b) = 0$*

Example 18.

$$z = x^2 + y^2$$

Solution 21.

$$f(x, y) \geq f(0, 0)$$

Therefore, $(0, 0)$ is a point of local minimum.

$$
\begin{aligned}
f_x &= 2x \\
f_y &= 2y
\end{aligned}
$$

Therefore,

$$f_x(0, 0) = f_y(0, 0) = 0$$

Example 19.

$$z = \sqrt{x^2 + y^2}$$

Solution 21.

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{(\Delta x)^2}}{\Delta x}$$
$$= \pm 1$$

Therefore, the limit does not exist.

Definition 41 (Critical point)**.** Let the function $z = f(x, y)$ be defined on some open neighbourhood of (a, b). The point (a, b) is called a critical point of $z = f(x, y)$ if $f_x(a, b) = f_y(a, b) = 0$ or at least one of the partial derivative $f_x(a, b)$ and $f_y(a, b)$ does not exist.

Example 20. Is $(0, 0)$ an local extremum point of

$$z = f(x, y) = y^2 - z^2$$

?

Solution 21.

$$f_x(0,0) = 0$$
$$f_y(0,0) = 0$$

Therefore, $(0, 0)$ is a critical point.
If possible let $(0, 0)$ be a local minimum point.
Then, $f(x, y) \geq f(0, 0)$ in some neighbourhood of $(0, 0)$.
Therefore,

$$y^2 - x^2 \geq 0$$

For any point of the form $(x, 0)$, this is a contradiction.
Therefore $(0, 0)$ is not a local minimum point.
Similarly, $(0, 0)$ is not a local maximum point.

Theorem 49 (A sufficient condition for local extrema point)**.** *Assume that there exist second order partial derivates of $z = f(x, y)$, they are continuous on some open neighbourhood of (a, b) and $f_x(a, b) = f_y(a, b) = 0$. Denote*

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - \Big(f_{xy}(a, b)\Big)^2$$

1. *If $D(a, b) > 0$ and $f_{xx} < 0$ then (a, b) is a local maximum point.*

2. *If $D(a,b) > 0$ and $f_{xx} > 0$ then (a,b) is a local minimum point.*

3. *If $D(a,b) < 0$ then (a,b) is called a saddle point.*

Example 21. Find all critical points of

$$z = f(x,y) = x^4 + y^4 - 4xy + 1$$

and classify them.

Solution 21.

$$\begin{aligned} f_x(x,y) &= 4x^3 - 4y \\ f_y(x,y) &= 4y^3 - 4x \end{aligned}$$

For critical points,

$$\begin{aligned} f_x(x,y) &= 0 \\ f_y(x,y) &= 0 \end{aligned}$$

Solving, $(0,0)$, $(1,1)$, $(-1,-1)$ are critical points.

$$\begin{aligned} f_{xx}(x,y) &= 12x^2 \\ f_{xy}(x,y) &= -4 \\ f_{yy}(x,y) &= 12y^2 \\ \therefore D(x,y) &= 144x^2y^2 - 16 \end{aligned}$$

For $(0,0)$,

$$D = -16$$

Therefore, $(0,0)$ is a saddle point.
For $(1,1)$,

$$D = 144 - 16$$

Therefore, $(1,1)$ is a local minimum point.
For $(-1,-1)$,

$$D = 144 - 16$$

Therefore, $(-1,-1)$ is a local minimum point.

4 Global Extrema

4.1 Algorithm for Finding Maxima and Minima of a Function

Step 1 Find all critical points of $f(x, y)$ on the domain, excluding the end points.

Step 2 Calculate the values of $f(x, y)$ at the critical points.

Step 3 Calculate the values of $f(x, y)$ at the end points of the domain.

Step 4 Select the maximum and minimum values from Step 2 and Step 3

Example 22. Find the global maxima and minima of

$$z = x^2 - 2xy + 2y$$

in the domain

$$D = \left\{ (x, y) \middle| 0 \leq x \leq 3, 0 \leq y \leq -\frac{2}{3}x + 2 \right\}$$

Solution 21.

$$\begin{aligned} f_x(x, y) &= 0 \\ \therefore 2x - 2y &= 0 \\ f_y(x, y) &= 0 \\ \therefore -2x + 2 &= 0 \end{aligned}$$

Therefore, $(1, 1)$ is a critical point in D.
The boundary of D is $L_1 \cup L_2 \cup L_3$, where

$$\begin{aligned} &L_1 : y = 0, 0 \leq x \leq 3 \\ &L_2 : x = 0, 0 \leq y \leq 2 \\ &L_3 : \end{aligned}$$

Therefore,
over L_1,

$$\begin{aligned} f(x, y) &= x^2 \\ \therefore \min_{L_1} f &= f(0, 0) = 0 \\ \therefore \max_{L_1} f &= f(3, 0) = 9 \end{aligned}$$

over L_2,

$$
\begin{aligned}
f(x,y) &= 2y \\
\therefore \min_{L_2} f &= f(0,0) = 0 \\
\therefore \max_{L_2} f &= f(0,2) = 4
\end{aligned}
$$

over L_3,

$$
\begin{aligned}
f(x,y) &= x^2 - 2x\left(-\frac{2}{3}x + 2\right) + 2\left(-\frac{2}{3}x + 2\right) \\
&= \frac{7}{3}x^2 - \frac{16}{3}x + 4 \\
\therefore f' &= \frac{14}{3}x - \frac{16}{3} \\
\therefore f'\left(\frac{8}{7}\right) &= 0 \\
\therefore f\left(\frac{8}{7}, \frac{26}{21}\right) &= 0.952 \\
\therefore \min_{L_3} f &= f\left(\frac{8}{7}, \frac{26}{21}\right) = 0.952 \\
\therefore \max_{L_3} f &= f(3,0) = 9
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\therefore \min_{D} f &= f(0,0) = 0 \\
\therefore \max_{D} f &= f(3,0) = 9
\end{aligned}
$$

5 Taylor's Formula

Theorem 50.

$$
\begin{aligned}
f(a+h, b+k) = &\sum_{i=0}^{n}\left(\frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(a,b)\right) \\
&+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+ch, b+ck)
\end{aligned}
$$

where $0 < c < 1$.