# DIFFERENTIAL AND INTEGRAL CALCULUS
# ASSIGNMENT 7

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**Exercise 1.**
Check pointwise and uniform convergence of the following series of functions

(1) $\sum\limits_{n=0}^{\infty} \left(x^{n+1} - x^n\right)$ in $[0,1]$.
(2) $\sum\limits_{n=0}^{\infty} x^n$ in $[0,1]$.
(3) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{x^2+n}$ in $\mathbb{R}$.
(4) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{x^2+n^3}$ in $\mathbb{R}$.
(5) $\sum\limits_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2+x^2}\right)$ in $\mathbb{R}$.
(6) $\sum\limits_{n=1}^{\infty} \frac{1}{3^n \sqrt[3]{1+n^2x^2}}$ in $\mathbb{R}$.
(7) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n x^2}{(1+x^2)^n}$ in $\mathbb{R}$.

**Solution 1.**
(1)

$$\begin{aligned}
S_k &= \sum_{n=0}^{k} x^{n+1} - x^n \\
&= x^{k+1} - x^0 \\
&= x^{k+1} - 1
\end{aligned}$$

Therefore

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} x^{k+1} - 1$$

If $0 \le x < 1$,

$$\begin{aligned}
\lim_{k\to\infty} S_k &= \lim_{k\to\infty} x^{k+1} - 1 \\
&= 0 - 1 \\
&= -1
\end{aligned}$$

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If $x = 1$,

$$\begin{aligned}\lim_{k\to\infty} S_k &= \lim_{k\to\infty} 1^{k+1} - 1 \\ &= 0\end{aligned}$$

Therefore,

$$S(x) = \begin{cases} -1 & ; \quad 0 \leq x < 1 \\ 0 & ; \quad x = 1 \end{cases}$$

Therefore, $S_n(x)$ converges pointwise to $S(x)$.
As $S(x)$ is not continuous in $[0, 1]$ but all $x^{n+1} - x^n$ are, the convergence cannot be uniform.

(2)

$$S_k = \sum_{n=0}^{k} x^n$$

Therefore,

$$\begin{aligned}\lim_{k\to\infty} S_k &= \lim_{k\to\infty} \sum_{n=0}^{k} x^n \\ &= \frac{x^{k+1} - 1}{x - 1}\end{aligned}$$

If $0 \leq x < 1$,

$$\begin{aligned}\lim_{k\to\infty} S_k &= \lim_{k\to\infty} \frac{x^{k+1} - 1}{x - 1} \\ &= \lim_{k\to\infty} \frac{-1}{x - 1} \\ &= 1\end{aligned}$$

If $x = 1$,

$$\begin{aligned}\lim_{k\to\infty} S_k &= \lim_{k\to\infty} \sum_{n=0}^{k} 1^n \\ &= \lim_{k\to\infty} k + 1 \\ &= \infty\end{aligned}$$

Therefore,

$$S(x) = \begin{cases} -\frac{1}{x-1} & ; \quad 0 \leq x < 1 \\ \infty & ; \quad x = 1 \end{cases}$$

Therefore, $S_n(x)$ does not converge pointwise to $S(x)$ as $S(x)$ is not defined at $x = 1$.
Hence, there is no uniform convergence.

(3)

$$\lim_{n\to\infty} \frac{1}{x^2 + n} = 0$$

Therefore, as $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{x^2+n}$ is a Leibniz series, and as $\lim\limits_{n\to\infty} \frac{1}{x^2+n} = 0$, the series converges pointwise.

$$\left|\frac{(-1)^n}{x^2+n}\right| \leq \frac{1}{n}$$

Therefore, by the Weierstrass M-test, as $\sum \frac{1}{n}$ converges, the series converges uniformly on $\mathbb{R}$.

(4)

$$\lim_{n\to\infty} \frac{1}{x^2+n^3} = 0$$

Therefore, as $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{x^2+n^3}$ is a Leibniz series, and as $\lim\limits_{n\to\infty} \frac{1}{x^2+n^3} = 0$, the series converges pointwise.

$$\left|\frac{(-1)^n}{x^2+n^3}\right| \leq \frac{1}{n^3}$$

Therefore, by the Weierstrass M-test, as $\sum \frac{1}{n^3}$ converges, the series converges uniformly on $\mathbb{R}$.

(5)

$$\left|\ln\left(1+\frac{1}{n^2+x^2}\right)\right| \leq \frac{1}{n^2+x^2}$$

$$\therefore \ln\left(1+\frac{1}{n^2+x^2}\right) \leq \frac{1}{n^2}$$

Therefore, by the Weierstrass M-test, as $\sum \frac{1}{n^2}$ converges, the series converges uniformly on $\mathbb{R}$.
Hence, the series also converges pointwise on $\mathbb{R}$.

(6)

$$\left|\frac{1}{3^n\sqrt[3]{1+n^2x^2}}\right| \leq \frac{1}{3^n}$$

Therefore, by the Weierstrass M-test, as $\sum \frac{1}{3^n}$ converges, the series converges uniformly on $\mathbb{R}$
Hence, the series also converges pointwise on $\mathbb{R}$.

(7)

$$\lim_{n\to\infty} \frac{x^2}{(1+x^2)^n} = 0$$

Therefore, as $\sum\limits_{n=1}^{\infty} \frac{(-1)^n x^2}{(1+x^2)^n}$ is a Leibniz series, and as $\lim\limits_{n\to\infty} \frac{1}{x^2+n} = 0$, the series converges pointwise.

$$\begin{aligned}\sup_{\mathbb{R}} \left|f_n(x) - f(x)\right| &= \sup_{\mathbb{R}} \left|\frac{x^2}{(1+x^2)^n} - 0\right| \\ &= \sup_{\mathbb{R}} \frac{x^2}{(1+x^2)^n}\end{aligned}$$

Therefore, differentiating, the critical points are

$$x = 0$$
$$x = \pm \frac{1}{\sqrt{n^2+1}}$$

Therefore, the maximum value of the function is at $x = \pm \frac{1}{\sqrt{n^2+1}}$.
Therefore,

$$\begin{aligned}\lim_{n\to\infty} \sup_{\mathbb{R}} |f_n(x) - f(x)| &= \lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\left(1+\frac{1}{n^2+1}\right)^2} \\ &= 0\end{aligned}$$

Therefore, the convergence is uniform.

**Exercise 2.**
Let $\{f_n(x)\}$ be a sequence of functions defined in the domain $I$.

(1) Prove that if the series $\sum\limits_{n=1}^{\infty} |f_n(x)|$ converges uniformly on $I$ then $\sum\limits_{n=0}^{\infty} f_n(x)$ converges uniformly on $I$.
(2) Show that the converse is not true, i.e. uniform convergence of $\sum\limits_{n=0}^{\infty} f_n(x)$ does not imply uniform convergence of $\sum\limits_{n=0}^{\infty} |f_n(x)|$.

**Solution 2.**

(1) As $\sum |f_n(x)|$ converges uniformly,

$$\lim_{k\to\infty} \sum_{n=1}^{k} |f_n(x)| = 0$$

Therefore, as $|f_n(x)| = \pm f_n(x)$,

$$\lim_{k\to\infty} \sum_{n=1}^{k} f_n(x) = \pm \lim_{k\to\infty} \sum_{n=1}^{k} |f_n(x)|$$
$$\therefore \lim_{k\to\infty} \sum_{n=1}^{k} f_n(x) = \pm 0$$
$$\therefore \lim_{k\to\infty} \sum_{n=1}^{k} f_n(x) = 0$$

Therefore, $\sum f_n(x)$ converges uniformly on $I$. □

(2) Let

$$f_n(x) = \frac{(-1)^n}{n}$$
$$\therefore f_n(x) = \frac{1}{n}$$

Therefore, $\sum \frac{(-1)^n}{n}$ converges, but $\sum \frac{1}{n}$ diverges.
Hence, uniform convergence of $\sum\limits_{n=0}^{\infty} f_n(x)$ does not imply convergence of $\sum\limits_{n=0}^{\infty} |f_n(x)|$. □

**Exercise 3.**
Let $f(x) = \sum\limits_{n=1}^{\infty} (-1)^n \frac{\cos\left(\frac{x}{n}\right)}{n^2+1}$. Show that $f(x)$ is continuous on $\mathbb{R}$. Is it possible to differentiate $f(x)$ term by term?

**Solution 3.**

$$\lim_{n\to\infty} \frac{\cos\left(\frac{x}{n}\right)}{n^2+1} = 0$$

Therefore, as $\sum\limits_{n=1}^{\infty} (-1)^n \frac{\cos\left(\frac{x}{n}\right)}{n^2+1}$ is a Leibniz series, and as $\lim\limits_{n\to\infty} \frac{\cos\left(\frac{x}{n}\right)}{n^2+1} = 0$, the series converges pointwise.

$$\begin{aligned} \left|\frac{\cos\left(\frac{x}{n}\right)}{n^2+1}\right| &\leq \frac{1}{n^2+1} \\ \therefore \left|\frac{\cos\left(\frac{x}{n}\right)}{n^2+1}\right| &\leq \frac{1}{n^2} \end{aligned}$$

Therefore, by the Weierstrass M-test, as $\sum \frac{1}{n^2}$ converges, the series converges uniformly. Therefore, the limit function $f(x)$ is continuous.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos\left(\frac{x}{n}\right)}{n^2+1}\right) &= \frac{-\frac{1}{n}\sin\left(\frac{x}{n}\right)}{n^2+1} \\ &= -\frac{\sin\left(\frac{x}{n}\right)}{n^3+n} \end{aligned}$$

As the derivative exists and is continuous on $\mathbb{R}$, it is possible to differentiate $f(x)$ term by term.

**Exercise 4.**
Define $f(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{2+n}$. Find the domain of convergence of this series. In what domain can we use term by term differentiation to show that $\left(x^2 f(x)\right)' = \frac{x}{1-x}$?

**Solution 4.**

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{2+(n+1)}{2+n}\right| \\ &= \lim_{n\to\infty}\left|\frac{n+3}{n+2}\right| \\ &= 1 \end{aligned}$$

If $x = -1$,

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2+n}$$

Therefore, as the series is a Leibniz series, and as $\lim_{n\to\infty} \frac{1}{2+n} = 0$, the series converges pointwise. If $x = 1$,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{1^n}{2+n} \\ &= \sum_{n=0}^{\infty} \frac{1}{2+n} \end{aligned}$$

Therefore, the series diverges.
Therefore, the domain of convergence is $[-1, 1)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2+n}\right) = \frac{2x}{2+n}$$

As the derivative is continuous on $[-1, 1)$ and the series converges in $[-1, 1)$, we can use term by term differentiation in $[-1, 1)$.