Differential and Integral Calculus : Recitations

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1 Instructor Information

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Part I
Sequences and Series

1 Sequences

Recitation 1 − Exercise 1.
Prove:
$$\lim_{n\to\infty} \frac{2n^2 + n + 1}{n^2 + 3} = 2$$

Recitation 1 − Solution 1.
Let
$$\varepsilon > 0$$

$$
\begin{aligned}
\left| \frac{2n^2 + n + 1}{n^2 + 3} - 2 \right| &= \left| \frac{2n^2 + n + 1 - 2n^2 - 6}{n^2 + 3} \right| \\
&= \left| \frac{n - 5}{n^2 + 3} \right| \\
&\leq \left| \frac{n - 5}{n^2} \right| \\
&\leq \frac{1}{n} \\
&< \varepsilon
\end{aligned}
$$

Therefore, let $N = \left[\dfrac{1}{\varepsilon} \right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.

Therefore, $\displaystyle\lim_{n\to\infty} \frac{2n^2 + n + 1}{n^2 + 3} = 2$. $\qquad\qquad\square$

Recitation 1 − Exercise 2.
Prove
$$\lim_{n\to\infty} \frac{n^3 + \sin n + n}{2n^4} = 0$$

Recitation 1 – Solution 2.

Let $\varepsilon > 0$

$$\left|\frac{n^3 + \sin n + n}{2n^4}\right| \leq \left|\frac{n^3 + 1 + n}{2n^4}\right|$$

$$\leq \left|\frac{3n^3}{2n^4}\right| = \frac{3}{2} \cdot \frac{1}{n} < \varepsilon$$

Therefore, let $N = \left[\dfrac{3}{2\varepsilon}\right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.

Therefore, $\displaystyle\lim_{n\to\infty} \frac{n^3 + \sin n + n}{2n^4} = 0$ $\qquad\qquad$ □

Recitation 1 – Exercise 3.

Calculate $\sqrt[3]{n^3 + 3n} - n$.

Recitation 1 – Solution 3.

$$a^n - b^n = (a - b) \cdot (a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Therefore, let

$$a = \sqrt[3]{n^3 + 3n}$$
$$b = \sqrt[3]{n^3}$$

$$a - b = \frac{a^3 - b^3}{a^2 + ab + b^2}$$

$$\therefore \sqrt[3]{n^3 + 3n} - n = \frac{n^3 + 3n - n^3}{(n^3 + 3n)^{2/3} + (n^3 + 3n)^{1/3}n + n^2}$$

$$= \frac{3}{\left(\dfrac{n^3 + 3n}{n^{3/2}}\right)^{2/3} + \left(\dfrac{n^3 + 3n}{n^3}\right)^{1/3n} + n}$$

Therefore, the limit is 0.

Recitation 1 – Exercise 4.

Prove

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

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Recitation 1 − Solution 4.

$$0 \leq \frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\ldots\frac{n}{n} \leq \frac{1}{n}$$

Therefore, by the Sandwich Theorem, $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.

Recitation 1 − Exercise 5.

Let $a_1 = 3$, $a_{n+1} = 1 + \sqrt{6 + a_n}$. Prove that a_n converges and find its limit.

Recitation 1 − Solution 5.

If possible, let $\lim\limits_{n\to\infty} a_n = l$.

$$a_{n+1} = 1 + \sqrt{6 + a_n}$$

Taking the limit on both sides,

$$l = 1 + \sqrt{6 + l}$$
$$\therefore l - 1 = \sqrt{6 + l}$$
$$\therefore l = \frac{3 \pm \sqrt{29}}{2}$$

As $a_n \geq 0$, $l = \dfrac{3 + \sqrt{29}}{2}$.

$$a_2 = 1 + \sqrt{6 + a_1}$$
$$= 1 + \sqrt{6 + 3}$$
$$= 4$$
$$\therefore a_2 > a_1$$

If possible, let $a_n \geq a_{n-1}$.
Therefore,

$$a_{n+1} = 1 + \sqrt{6 + a_n}$$
$$\geq 1 + \sqrt{6 + a_{n+1}} = a_n$$

Therefore by induction, $\{a_n\}$ is monotonically increasing.

$$a_1 = 3$$
$$\therefore a_1 \leq 5$$

If possible, let $a_n \leq 5$.
Therefore,

$$a_{n+1} = 1 + \sqrt{6 + a_n} \leq q + \sqrt{11} \leq 5$$

Therefore by induction, $\{a_n\}$ is bounded from above by 5.

1.1 Limit of a Function by Heine

Definition 1.

$$\lim_{x \to x_0} f(x) = l$$

if for every sequence x_n, such that $\lim_{n \to \infty} x_n = x_0$,

$$\lim_{n \to \infty} f(x_n) = l$$

Theorem 1. *If f is continuous at x_0 and $x_n \to x_0$, then*

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right) = f_{x_0}$$

Recitation 2 – Exercise 1.
Calculate $\lim_{n \to \infty} \sqrt[n]{n}$.

Recitation 2 – Solution 1.
Let

$$f(x) = x^{1/x}$$

Therefore,

$$\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} e^{\frac{\ln x}{x}}^{\,0}$$
$$= 1$$

1.2 Sub-sequences

Recitation 2 – Exercise 2.

Find all partial limits and $\overline{\lim}$ and $\underline{\lim}$ of

$$a_n = \left(\cos \frac{\pi n}{4} \right)^n$$

Recitation 2 – Solution 2.

Let $k, z \in \mathbb{Z}$

$$\cos \frac{\pi n}{4} = \cos \frac{\pi(n+k)}{4}$$
$$\therefore \frac{\pi n}{4} = \frac{\pi(n+k)}{4} + 2\pi z$$
$$\therefore \pi n = \pi(n+k) + 8\pi z$$
$$\therefore k = 8z$$

Therefore,

$$a_{8k} = \left(\cos \frac{\pi \cdot 8k}{4} \right)^{8k}$$
$$= (\cos(2\pi k))^{8k}$$
$$= 1$$

$$a_{8k+1} = \left(\cos \frac{\pi \cdot (8k+1)}{4} \right)^{8k+1}$$
$$= \left(\cos \frac{\pi}{4} \right)^{8k+1}$$
$$= \left(\frac{\sqrt{2}}{2} \right)^{8k+1}$$

$$a_{8k+2} = \left(\cos \frac{\pi \cdot (8k+2)}{4} \right)^{8k+2}$$
$$= \left(\cos \frac{\pi}{2} \right)^{8k+2}$$

Therefore,

$$\lim_{k \to \infty} a_{8k} = 1$$
$$\lim_{k \to \infty} a_{8k+1} = \lim_{k \to \infty} \left(\frac{\sqrt{2}}{2} \right)^{8k+1}$$
$$= 0$$

Similarly,

$$\lim_{k \to \infty} a_{8k+2} = 0$$
$$\lim_{k \to \infty} a_{8k+3} = 0$$
$$\lim_{k \to \infty} a_{8k+4} = \lim_{k \to \infty} (-1)^{8k+4}$$
$$= 1$$
$$\lim_{k \to \infty} a_{8k+5} = 0$$
$$\lim_{k \to \infty} a_{8k+6} = 0$$
$$\lim_{k \to \infty} a_{8k+7} = 0$$

Therefore, $\{a_n\}$ has two partial limits, 0 and 1.

$$\overline{\lim} a_n = 1$$
$$\underline{\lim} a_n = 0$$

2 Series

Definition 2 (Convergence of a series). Let $\{a_n\}$ be a sequence. Let S_n be a sequence of partial sums of a_n, s.t.

$$S_n = \sum_{k=1}^{n} a_k$$

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to l if

$$\lim_{n \to \infty} S_n = l$$

that is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \lim_{n \to \infty} S_n$$

Recitation 2 − Exercise 3.

Does $\sum_{k=0}^{\infty} q^k$ where $-1 < q < 1$ converge?

Recitation 2 – Solution 3.

$$\sum_{k=0}^{\infty} q^k = \lim_{n \to \infty} \sum_{k=0}^{n} q^k$$
$$= \lim_{n \to \infty} \frac{1 - q^{n+1}}{1 - q}$$
$$= \frac{1}{1 - q}$$

Therefore, the series converges.

Recitation 2 – Exercise 4.

Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converge?

Recitation 2 – Solution 4.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$
$$= \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$
$$= \lim_{n \to \infty} \left(1 - \frac{1}{n+1} \right)$$
$$= 1$$

Recitation 2 – Exercise 5.

Does $\displaystyle\sum_{k=1}^{\infty} \left(1 + \frac{1}{k} \right)^k$ converge?

Recitation 2 – Solution 5.

$$\lim_{k \to \infty} \left(1 + \frac{1}{k} \right)^k = e$$
$$\therefore \lim_{k \to \infty} \left(1 + \frac{1}{k} \right)^k \neq 0$$

Therefore, the necessary condition is nt satisfied. Hence, the series does not converge.

2.1 Comparison Tests for Positive Series

Theorem 2 (First Comparison Test). *If $a_n \geq 0$, $b_n \geq 0$, and $a_n \leq b_n$, then*

1. *If $\sum b_n$ converges, then $\sum a_n$ converges.*

2. *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Theorem 3 (Second Comparison Test). *If $a_n \geq 0$, $b_n \geq 0$ and*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = l$$

where $0 < l < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge simultaneously.

Recitation 3 – Exercise 1.
Suppose the sequence a_n satisfies the condition

$$a_{n+1} - a_n > \frac{1}{n}$$

$\forall n \in \mathbb{N}$.
Prove that $\lim\limits_{n \to \infty} a_n = \infty$.

Recitation 3 – Solution 1.

$$
\begin{aligned}
a_{n+1} &= a_{n+1} - a_n + a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_2 - a_1 + a_1 \\
&= \sum_{k=1}^{n} \left(a_{k+1} - a_k \right) + a_1 \\
&\geq \sum_{k=1}^{n} \frac{1}{k} + a_1
\end{aligned}
$$

As the harmonic series diverges, $\sum\limits_{k=1}^{n} \frac{1}{k} + a_1$ diverges.

Therefore, by the First Comparison Test, $\sum\limits_{k=1}^{\infty} \left(a_{k+1} - a_k \right)$ diverges.

Recitation 3 – Exercise 2.
Check the convergence of $\sum\limits_{n=2}^{\infty} \frac{n + \sin n}{n^3 + \cos \pi n}$.

Recitation 3 − Solution 2.

The series is non-negative. Therefore, the comparison tests are applicable.

$$\frac{n + \sin n}{n^3 + \cos \pi n} \leq \frac{n + 1}{n^3 - 1}$$

$$\therefore \frac{n + \sin n}{n^3 + \cos \pi n} \leq \frac{2n}{n^3 - \dfrac{n^3}{2}} \qquad\qquad \leq \frac{4}{n^2}$$

Therefore, by the First Comparison Test, as $\dfrac{4}{n^2}$ converges, $\sum_{n=2}^{\infty} \dfrac{n + \sin n}{n^3 + \cos \pi n}$ also converges.

Recitation 3 − Exercise 3.

Let $a_n \geq 0$ and suppose that $\sum a_n$ converges. Prove that $\sum a_n{}^2$ converges. Is it true without the assumption $a_n \geq 0$?

Recitation 3 − Solution 3.

As $\sum a_n$ converges, $\lim\limits_{n \to \infty} a_n = 0$.
Therefore, $\exists N \in \mathbb{N}$, such that $\forall n > N$, $a_n < 1$.
Therefore, $\forall n > N$, $a_n{}^2 \leq a_n$. Hence, as $\sum\limits_{n=N+1}^{\infty} a_n$ converges, $\sum\limits_{n=N+1}^{\infty} a_n{}^2$ also converges. Hence, $\sum\limits_{n=1}^{\infty} a_n$ also converges.

This is not true without the assumption $a_n \geq 0$, as the argument $a_n{}^2 \leq a_n$ does not hold.

Recitation 3 − Exercise 4.

For which α does $\sum \left(\sqrt{n+1} - \sqrt{n} \right)^{\alpha/2}$ converge?

Recitation 3 − Solution 4.

$$\sum \left(\sqrt{n+1} - \sqrt{n} \right)^{\alpha/2} = \sum \left(\frac{n + 1 - n}{\sqrt{n+1} + \sqrt{n}} \right)^{\alpha/2}$$

$$= \sum \left(\frac{1}{\sqrt{n+1} - \sqrt{n}} \right)^{\alpha/2}$$

The series is positive. Therefore, the comparison tests are applicable.

Comparing with $\left(\dfrac{1}{\sqrt{n}}\right)^{\alpha/2}$,

$$\frac{\left(\dfrac{1}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2}}{\left(\dfrac{1}{\sqrt{n}}\right)^{\alpha/2}} = \left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2}$$

$$\therefore \lim_{n\to\infty}\left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2} = \left(\frac{1}{2}\right)^{\alpha/2}$$

$\sum \dfrac{1}{n^{\alpha/2}}$ converges if and only if $\dfrac{\alpha}{4} > 1$, i.e. if an inly if $\alpha > 4$.

By the Second Comparison Test, $\sum \dfrac{1}{n^{\alpha/4}}$ and the series converge or diverge simultaneously.

Therefore, the series converges for $\alpha > 4$.

Recitation 3 – Exercise 5.

Check the convergence of $\sum\limits_{n=1}^{\infty} \sin \dfrac{1}{n}$.

Recitation 3 – Solution 5.

$\forall n \in \mathbb{N}, \sin \dfrac{1}{n} \geq 0$

$$\lim_{n\to\infty} \frac{\sin \dfrac{1}{n}}{\dfrac{1}{n}} = 1$$

Therefore, by Second Comparison Test, $\sum \dfrac{1}{n}$ and $\sum \sin \dfrac{1}{n}$ diverge simultaneously.

2.2 d'Alembert Criteria (Ratio Test)

Definition 3 (Absolute and conditional convergence)**.** The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

Theorem 4. *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 5 (d'Alembert Criteria (Ratio Test)). *1. If*

$$\lim_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right| = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\lim_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right| = L > 1$$

(including $L = \infty$), then $\sum a_n$ converges diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 − Exercise 6.

Check the convergence of $\sum \dfrac{(-1)^n \cdot n^{1000}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$.

Recitation 3 − Solution 6.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n \cdot n^{1000}}{1 \cdots (2n-1)}\right| = \sum_{n=1}^{\infty}\frac{n^{1000}}{1 \cdots (2n-1)}$$

Therefore, by the d'Alembert Criteria (Ratio Test),

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^{1000}}{1 \cdots (2n+1)}}{\dfrac{n^{1000}}{1 \cdots (2n-1)}}$$

$$= \left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1}$$

$$\therefore \lim_{nto\infty} \left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1} = 0$$

$$\therefore \left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1} < 1$$

Therefore, by the d'Alembert Criteria (Ratio Test), the series converges absolutely, and hence converges.

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2.3   Cauchy Criteria (Cauchy Root Test)

Theorem 6 (Cauchy Criteria (Cauchy Root Test)).   *1. If*

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

(including $L = \infty$), then $\sum a_n$ diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 − Exercise 7.

Check the convergence of $\sum \left(1 - \dfrac{2}{n}\right)^{n^2}$.

Recitation 3 − Solution 7.

$$\sqrt[n]{\left(1 - \frac{2}{n}\right)^{n^2}} = \left(1 - \frac{2}{n}\right)^{n}$$

$$\therefore \lim_{n \to \infty} \left(1 - \frac{2}{n}\right)^{n} = e^{-2}$$

$$\therefore \lim_{n \to \infty} \left(1 - \frac{2}{n}\right)^{n} < 1$$

Therefore, by the Cauchy Criteria (Cauchy Root Test), $\sum \left(1 - \dfrac{2}{n}\right)^{n^2}$ converges.

2.4   Leibniz's Criteria

Definition 4 (Alternating series). The series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

Theorem 7 (Leibniz's Criteria for Convergence). *If an alternating series $\sum (-1)^{n-1} a_n$ with $a_n > 0$ satisfies*

1. $a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.

2. $\lim_{n\to\infty} a_n = 0$

then the series $(-1)^{n-1}a_n$ converges.

Recitation 3 – Exercise 8.

Prove or disprove: There exists $\{a_n\}$, such that $\sum a_n$ converges and $\sum(1 + a_n)a_n$ diverges.

Recitation 3 – Solution 8.

Let $a_n = \dfrac{(-1)^n}{\sqrt{n}}$.

Therefore, by Leibniz's Criteria for Convergence, $\sum \dfrac{(-1)^n}{\sqrt{n}}$ converges.

$$\sum(1 + a_n)a_n = \sum \left(1 + \frac{(-1)^n}{\sqrt{n}}\right)\frac{(-1)^n}{\sqrt{n}}$$
$$= \sum \left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right)$$

Therefore, as $\sum \dfrac{1}{n}$ diverges, and $\sum \dfrac{(-1)^n}{\sqrt{n}}$ converges, $\sum \left(\dfrac{1}{n} + \dfrac{(-1)^n}{\sqrt{n}}\right)$ diverges.

2.5 Integral Test

Theorem 8 (Integral Test). *If $f(x) : [1, \infty) \to [0, \infty)$ is monotonically decreasing. Then, $\sum\limits_{n=1}^{\infty} f(n)$ and $\int\limits_{1}^{\infty} f(x)\,dx$ converge or diverge simultaneously.*

Recitation 3 – Exercise 9.

Check the convergence of $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$

Recitation 3 – Solution 9.

Let

$$f(x) = \frac{1}{x \ln x}$$

$f(x)$ is monotonically decreasing. Therefore, the Integral Test is applicable. Therefore,

$$\int\limits_2^\infty \frac{1}{x \ln x}\, \mathrm{d}x = \int\limits_{\ln 2}^\infty \frac{1}{y}\, \mathrm{d}y$$
$$= \ln y\big|_{\ln 2}^\infty$$
$$= \infty$$

Therefore, by the integral test, $\sum \dfrac{1}{n \ln n}$ diverges.

Recitation 4 − Exercise 1.

Let $d_n \geq 0$ and suppose

$$\sum_{n=0}^\infty d_n = \infty$$

Prove that

$$\sum_{n=0}^\infty \frac{d_n}{1 + d_n} = \infty$$

Recitation 4 − Solution 1.

If possible, let d_n be a bounded sequence. Then there exists M, such that $d_n \leq M$, $\forall n \in \mathbb{N}$.
Therefore,

$$\frac{d_n}{1 + d_n} \geq \frac{d_n}{1 + M}$$

Therefore, by the Second Comparison Test, as $\sum d_n$ diverges, $\sum \dfrac{d_n}{1 + d_n}$ also diverges.

If d_n is not bounded, then there is a subsequence d_{n_k} which diverges. Therefore,

$$\frac{d_{n_k}}{1 + d_{n_k}} = \frac{1}{\dfrac{1}{d_{n_k}} + 1}$$

$$\therefore \lim_{k \to \infty} \frac{d_{n_k}}{1 + d_{n_k}} = 1$$

Therefore,

$$\lim_{n \to \infty} \frac{d_n}{1 + d_n} \neq 0$$

Therefore, the necessary condition for convergence is not fulfilled. Therefore, the series converges.

Recitation 4 − Exercise 2.
Let

$$d_n = \begin{cases} 1 & ; & n = k^2, k \in \mathbb{N} \\ 0 & ; & n \neq k^2, k \in \mathbb{N} \end{cases}$$

Does $\sum \dfrac{d_n}{1 + n \cdot d_n}$ diverge?

Recitation 4 − Solution 2.

$$d_n = \begin{cases} 1 & ; & n = k^2, k \in \mathbb{N} \\ 0 & ; & n \neq k^2, k \in \mathbb{N} \end{cases}$$

$$\therefore \frac{d_n}{1 + n \cdot d_n} = \begin{cases} \dfrac{1}{1 + k^2} & ; & n = k^2, k \in \mathbb{N} \\ 0 & ; & n \neq k^2, k \in \mathbb{N} \end{cases}$$

As $\dfrac{1}{1 + k^2} \leq \dfrac{1}{k^2}$ and as $\dfrac{1}{k^2}$ converges, $\sum \dfrac{1}{1 + k^2}$ also converges.

Recitation 4 − Exercise 3.
Let a_n be a sequence such that $|a_{n+1} - a_n| \leq b_{n+1}$ for all $n \in \mathbb{N}$ where $\sum b_k$ converges. Prove that $\{a_n\}$ converges.

Recitation 4 − Solution 3.
Let $\varepsilon > 0$.

$$\begin{aligned} |a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} + \cdots - a_n| \\ &\leq \sum_{k=n+1}^{m} |a_k - a_{k-1}| \\ &\leq \sum_{k=n+1}^{m} b_k \end{aligned}$$

Therefore, as $\sum b_n$ converges, the series satisfies the Cauchy Criteria (Cauchy Root Test). Therefore, there exists N, such that $\forall m > n > N, \left| \sum_{k=n+1}^{m} b_k \right| < \varepsilon.$ Therefore, for $m > n > N$,

$$|a_m - a_n| \leq \sum_{k=n+1}^{m} b_n < \varepsilon$$

3   Power Series

Definition 5 (Power series). A power series around x_0 is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $\{a_n\}$ is a sequence of real numbers.

Theorem 9 (Abel's Theorem). *For every power series $\sum a_n(x - x_0)^n$, there exists $R \in [0, \infty]$, such that for all x satisfying $|x - x_0| < R$, the series converges and for all x satisfying $|x - x_0| > R$ the series diverges.*

Theorem 10 (Cauchy's Formula for Radius of Convergence).

$$R = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$$

Theorem 11 (Hadamard's Formula for Radius of Convergence). *If $\lim\limits_{n \to \infty} \left| \dfrac{a_n}{a_{n+1}} \right|$ exists, then*

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = R$$

Recitation 4 – Exercise 4.

Find the domain of convergence of $\sum\limits_{n=1}^{\infty} \dfrac{(2x - 4)^n}{n}$.

Recitation 4 – Solution 4.

$$\sum_{n=1}^{\infty} \frac{(2x-4)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{n}$$

Therefore, by Cauchy's Formula for Radius of Convergence,

$$R = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$$

$$= \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{2^n}{n}}}$$

$$= \frac{1}{\lim\limits_{n\to\infty} \dfrac{2}{\sqrt[n]{n}}}$$

$$= \frac{1}{2}$$

Therefore, the series converges for

$$|x-2| < \frac{1}{2}$$

and diverges for

$$|x-2| > \frac{1}{2}$$

If $x = \dfrac{5}{2}$,

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{5}{2} - 2 \right)^n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

Therefore, the series diverges.

If $x = \dfrac{3}{2}$,

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{3}{2} - 2 \right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

Therefore, by Leibniz's Criteria for Convergence, the series converges.

Therefore, the domain of convergence is $\left[\dfrac{3}{2}, \dfrac{5}{2}\right)$.

Recitation 4 – Exercise 5.

Find the radius of convergence of $\sum\limits_{n=0}^{\infty} n!x^{n!}$.

Recitation 4 – Solution 5.

$$\frac{1}{\sqrt[n]{a_n}} = x + x + 2x^2 + 6x^6 + 24x^{24} + \dots$$

Therefore,

$$a_n = \begin{cases} n & ; & n = k^2 \\ 0 & ; & n \neq k^2 \end{cases}$$

Therefore,

$$
\begin{aligned}
R &= \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{a_n}} \\
&= \frac{1}{\lim\limits_{k\to\infty} \sqrt[k!]{k!}} \\
&= 1
\end{aligned}
$$

3.1 Power Series Representation of a Function

Theorem 12. *The power series representation of a function $f(x)$ is equal to its Taylor series if and only if $\lim\limits_{n\to\infty} R_n(x) = 0$, where $R_n(x)$ is the Lagrange remainder.*

3.2 Differentiation and Integrations of Power Series

Recitation 5 – Exercise 1.

Find the power series representation of $\tan^{-1} x$.

Recitation 5 – Solution 1.

$$\frac{\mathrm{d}\tan^{-1}x}{\mathrm{d}x} = \frac{1}{1+x^2}$$
$$= \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

Integrating term by term,

$$\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + c$$

As $\tan^{-1}0 = 0$, $c = 0$. Therefore,

$$\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

Recitation 5 – Exercise 2.

Find an explicit formula for $\sum\limits_{n=1}^{\infty} x^n n^2$.

Recitation 5 – Solution 2.

$$\sum_{n=1}^{\infty} x^n n^2 = x \cdot \sum_{n=1}^{\infty} x^{n-1} n^2$$

Let

$$g(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

Integrating term by term,

$$\int g(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} n^2 \frac{x^n}{n}$$
$$= \sum_{n=1}^{\infty} n x^n$$
$$= x \cdot \sum_{n=1}^{\infty} n x^{n-1}$$

Let

$$h(x) = \sum_{n=1}^{\infty} nx^{n-1}$$

$$\therefore \int h(x)\,\mathrm{d}x = \frac{x}{1-x}$$

Therefore, inside radius of convergence $R = 1$, differentiating $\int h(x)\,\mathrm{d}x$,

$$h(x) = \frac{1-x+x}{(1-x)^2}$$

$$= \frac{1}{(1-x)^2}$$

$$\therefore \int g(x)\,\mathrm{d}x = xh(x)$$

$$= \frac{x}{(1-x)^2}$$

$$\therefore g(x) = \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4}$$

$$\therefore \sum_{n=1}^{\infty} x^n n^2 = x \cdot \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4}$$

Recitation 5 − Exercise 3.

Find the sum $\sum\limits_{n=1}^{\infty} \dfrac{1}{n2^n}$.

Recitation 5 − Solution 3.

Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

be a power series with radius R.
Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = f\left(\frac{1}{2}\right)$$

Therefore,

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1}$$

$$= \frac{1}{1-x}$$

$$\therefore f(x) = -\ln(1-x) + c$$

As $f(0) = 0$, $c = 0$. Therefore,

$$f(x) = -\ln(1 - x)$$

Therefore,

$$f\left(\frac{1}{2}\right) = -\ln\left(\frac{1}{2}\right)$$
$$= \ln 2$$

4   Sequences of Functions

Definition 6 (Point-wise convergence and domain of convergence). $\{f_n\}$ is said to converge point-wise in some domain $E \subset D$ if $\forall x \in E$, the sequence $\{f_n(x)\}$ converges. In this case, E is said to be a domain of convergence of $\{f_n\}$.

Recitation 5 − Exercise 4.
Let $f(x) : \mathbb{R} \to \mathbb{R}$ be some function such that $\lim_{x \to \infty} f(x) = 0$. Let $f_n(x) = f(nx)$. What is the domain of convergence of f_n? What is the limit function?

Recitation 5 − Solution 4.
Let x be a particular number in $(0, \infty)$.

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} f(nx)$$

Therefore, as $\lim_{x \to \infty} f(x) = 0$,

$$\lim_{n \to \infty} f_n(x) = 0$$

Therefore the domain of convergence is $(0, \infty)$ and the limit function is a constant 0.

Although the all functions in $\{f_n\}$ are continuous, the limit function is not continuous.

Definition 7 (Uniform convergence). A sequence of functions $\{f_n\}$ is said to converge uniformly to f in the domain E, if $\forall \varepsilon$, $\exists N$ such that $\forall n > N$ and $\forall x \in E$, $|f_n(x) - f(x)| < \varepsilon$. If f_n converges to f uniformly in E, it is denoted as $f_n \overset{E}{\rightrightarrows} f$.

4.1 Supremum and Infimum of Sets

Definition 8 (Supremum). Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the supremum of A if

1. $\forall x \in A$, $x \leq M$, i.e. M is an upper bound of A.

2. $\forall \varepsilon$, $\exists x \in A$, such that $x > M - \varepsilon$.

That is, the supremum of A is the least upper bound of A.
The supremum may or may not be in A.

Definition 9 (Infimum). Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the infimum of A if

1. $\forall x \in A$, $x \geq M$, i.e. M is an upper bound of A.

2. $\forall \varepsilon$, $\exists x \in A$, such that $x < M - \varepsilon$.

That is, the infimum of A is the greatest lower bound of A. The infimum may or may not be in A.

Theorem 13. *Every bounded set A has a supremum and an infimum.*

Theorem 14. $f_n \overset{E}{\Rightarrow} f$ *if and only if*

$$\lim_{n \to \infty} \left(\sup\{|f_n(x) - f(x)| : x \in E\} \right) = 0$$

Recitation 6 – Exercise 1.
Let $f_n(x) = x^n$. Does $\{f_n\}$ converge uniformly?

Recitation 6 – Solution 1.

$$f(x) = \begin{cases} 0 & ; \quad x \in [0, 1] \\ 1 & ; \quad x = 1 \end{cases}$$

If the convergence is uniform in $[0, 1]$,

$$\lim_{n \to \infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| = 0$$

Let $x = 1 - \frac{1}{n}$.

Therefore, as the supremum is a upper bound,

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| \geq \left| f_n\left(1 - \frac{1}{n}\right) - f\left(1 - \frac{1}{n}\right) \right|$$

$$\therefore \sup_{x \in [0,1]} |f_n(x) - f(x)| \geq \left| \left(1 - \frac{1}{n}\right)^n - 0 \right|$$

$$\therefore \lim_{n \to \infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| \geq \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n$$

$$\therefore \lim_{n \to \infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| \geq \frac{1}{e}$$

$$\therefore \lim_{n \to \infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| \neq 0$$

Therefore, the convergence is not uniform.

Recitation 6 − Exercise 2.

Let $f_n(x) = x + \frac{1}{n}$, $x \in \mathbb{R}$. What is its domain of convergence? What is the limit function? Is the convergence uniform?

Recitation 6 − Solution 2.

$\forall x \in \mathbb{R}$,

$$\lim_{n \to \infty} \left(x + \frac{1}{n}\right) = x$$

Therefore $\{f_n\}$ converges pointwise to x, in $\mathbb{R}$.

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} \left| x + \frac{1}{n} - x \right|$$

$$= \frac{1}{n}$$

$$\therefore \lim_{n \to \infty} \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = 0$$

Therefore, the convergence is uniform.

Recitation 6 − Exercise 3.

Let $f_n : [0, \infty) \to \mathbb{R}$.

$$f_n(x) = \begin{cases} 1 & ; \quad n \leq x \leq n + 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Dows f_n converge pointwise in $[0, \infty)$? Dows f_n converge uniformly in $[0, \infty)$?

Recitation 6 – Solution 3.

For every x, the sequence $\{f_n(x)\}$ will be of the form $\{0, \ldots, 0, 1, 0, \ldots, 0\}$ with 1 only when $n \le x \le n + 1$.

Therefore,

$$\lim_{n \to \infty} f_n(x) = 0$$
$$= f(x)$$

Therefore, f_n converges pointwise in $[0, \infty)$.

$$\sup_{x \in [0, \infty)} |f_n(x) - f(x)| = \max_{x \in [0, \infty)} f_n(x)$$
$$= 1$$

Therefore, as the limit of the supremum is not 0, the convergence is not uniform.

Theorem 15. *If $f_n \overset{D}{\rightrightarrows} f$ and all f_n are continuous is D, then f is also continuous, i.e. uniform convergence preserves continuity.*

Recitation 7 – Exercise 1.

Does x^n converge to

$$f(x) = \begin{cases} 0 & ; \quad x \in [0, 1) \\ 1 & ; \quad x = 1 \end{cases}$$

Recitation 7 – Solution 1.

If possible, let x^n converge to $f(x)$.

Therefore, as all $f_n(x)$ are continuous, and as uniform convergence preserves continuity, $f(x)$ also must be continuous.

This contradicts the definition of $f(x)$.

Therefore, the x^n does not converge to $f(x)$.

Recitation 7 – Exercise 2.

Check if $f_n(x) = \frac{x}{1+n^2x^2}$ converges uniformly in $[0, 1]$.

Recitation 7 – Solution 2.

$$\lim_{n \to \infty} f_n(x) = 0$$
$$= f(x)$$

Therefore,

$$\sup_{[0,1]} |f_n(x) - f(x)| = \sup_{[0,1]} |f_n(x) - 0|$$

$$= \sup_{[0,1]} \left| \frac{x}{1 + n^2 x^2} \right|$$

$$= \sup_{[0,1]} \frac{x}{1 + n^2 x^2}$$

As the function is continuous and the interval is closed, by the Weierstrass theorem, the function has a maximum. Therefore,

$$\sup_{[0,1]} |f_n(x) - f(x)| = \max_{[0,1]} \frac{x}{1 + n^2 x^2}$$

Differentiating to find the maximum,

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{x}{1 + n^2 x^2} \right) = \frac{1 + n^2 x^2 - 2x^2 n^2}{(1 + n^2 x^2)}$$

$$= \frac{1 - x^2 n^2}{(1 + n^2 x^2)^2}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{x}{1 + n^2 x^2} \right) = 0$$

$$\Longleftrightarrow \frac{1 - x^2 n^2}{(1 + n^2 x^2)^2} = 0$$

$$\Longleftrightarrow 1 = x^2 n^2$$

$$\Longleftrightarrow x = \frac{1}{n}$$

Therefore, the values of the function at the critical points and the end points

are,

$$f_n(0) = 0$$
$$f_n(1) = \frac{1}{1 + n^2}$$
$$f_n\left(\frac{1}{n}\right) = \frac{\frac{1}{n}}{1 + n^2 \frac{1}{n^2}}$$
$$= \frac{1}{2n}$$

Therefore, the maximum is at $x = \frac{1}{2n}$.
Therefore,

$$\max_{[0,1]} \frac{x}{1 + n^2 x^2} = f_n\left(\frac{1}{n}\right)$$
$$= \frac{1}{2n}$$

Therefore

$$\lim_{n\to\infty} \sup_{[0,1]} |f_n(x) - f(x)| = \lim_{n\to\infty} \max_{[0,1]} \frac{x}{1 + n^2 x^2}$$
$$= \lim_{n\to\infty} \frac{1}{2n}$$
$$= 0$$

Therefore, the convergence is uniform.

Recitation 7 – Exercise 3.
Check the pointwise and uniform convergence of $f_n(x) = x^n - x^{n+1}$ in $[0, 1]$.

Recitation 7 – Solution 3.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n - x^{n+1}$$
$$= 0$$
$$= f(x)$$

Therefore the function converges pointwise in $[0, 1]$.

$$\sup_{[0,1]} |f_n(x) - f(x)| = \sup_{[0,1]} x^n - x^{n+1}$$

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As the function is continuous and the interval is closed, by the Weierstrass theorem, the function has a maximum. Therefore,

$$\sup_{[0,1]} x^n - x^{n+1} = \max_{[0,1]} x^n - x^{n+1}$$

Differentiating to find the maximum,

$$\frac{\mathrm{d}(x^n - x^{n+1})}{\mathrm{d}x} = nx^{n-1} - (n+1)x^n$$

Therefore,

$$\frac{\mathrm{d}(x^n - x^{n+1})}{\mathrm{d}x} = 0$$
$$\iff nx^{n-1} - (n+1)x^n = 0$$
$$\iff n - (n+1)x = 0$$
$$\iff x = \frac{n}{n+1}$$

Therefore, the values of the function at the critical points and the end points are

$$f_n(0) = 0$$
$$f_n(1) = 0$$
$$f_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1}$$

Therefore,

$$\max_{[0,1]} x^n - x^{n+1} = f_n\left(\frac{n}{n+1}\right)$$
$$= \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1}$$

Therefore,

$$\lim_{n\to\infty} \sup_{[0,1]} |f_n(x) - f(x)| = \lim_{n\to\infty} \max_{[0,1]} \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1}$$
$$= \frac{1}{e} - \frac{1}{e}$$
$$= 0$$

Therefore, the convergence is uniform.

Theorem 16 (Cauchy's Theorem). $\{f_n\}$ *converges uniformly in D if and only if $\forall \varepsilon \in N$, $\exists N$, such that $\forall m, n > N$ and $\forall x \in D$,*

$$|f_n(x) - f(x)| < \varepsilon$$

Recitation 7 − Exercise 4.

Let $\{f_n\}$ be a sequence of function in D such that $\forall x \in D$, $|f_{n+1}(x) - f_n(x)| \leq a_n$, where $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\{f_n\}$ converges uniformly in D.

Recitation 7 − Solution 4.

As $\sum a_n$ converges, $\exists N$ such that $\forall m > n > N$, $\left| \sum_{k=n}^{m} a_k \right| < \varepsilon$.

Therefore, forall $m > n > N$ and $x \in D$,

$$
\begin{aligned}
|f_m(x) - f_n(x)| &= |f_m(x) - f_{m-1}(x) + f_{m-1}(x) - \cdots - f_n(x)| \\
&\leq |f_m(x) - f_{m-1}(x)| + |f_{m-1}(x) - f_{m-2}(x) + \cdots + |f_{n+1}(x) - f_n(x)| \\
\therefore |f_m(x) - f_n(x)| &\leq \sum_{k=n}^{m-1} |f_{k+1}(x) - f_k(x)| \\
\therefore |f_m(x) - f_n(x)| &\leq \sum_{k=n}^{m-1} a_k \\
\therefore |f_m(x) - f_n(x)| &\leq \varepsilon
\end{aligned}
$$

Therefore, $\{f_n\}$ satisfies Cauchy's criterion for uniform convergence.

5 Series of Functions

Definition 10 (Pointwise convergence of series of functions). Let $\{f_n\}$ be a sequence of functions defined in D. Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$.

If $S_n(x)$ converges for every $x \in D$ to a limit S, the series formed by $\{f_n\}$ is said to converge pointwise in D. It is denoted as

$$\sum_{k=1}^{\infty} f_k(x) = \lim_{n \to \infty} S_n(x) = S_x$$

Definition 11 (Uniform convergence of series of functions). The series $\sum_{k=1}^{\infty} f_k(x)$ is said to converge uniformly in D if $S_n \overset{D}{\rightrightarrows} S$.

Theorem 17. *If $\sum\limits_{k=1}^{\infty} f_k(x)$ converges uniformly in D, then the general term $f_k(x)$ must uniformly converge to 0 in D.*

Recitation 7 – Exercise 5.

Check the uniform convergence of $\sum\limits_{n=1}^{\infty} \frac{x^n}{n^2} - \frac{x^{n+1}}{(n+1)^2}$ in $[-1, 1]$.

Recitation 7 – Solution 5.

$$S_n(x) = \sum_{k=1}^{n} \frac{x^k}{k^2} - \frac{x^{k+1}}{(k+1)^2}$$
$$= \frac{x^1}{1^2} - \frac{x^{n+1}}{(n+1)^2}$$

Therefore,

$$\lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} x = \frac{x^{n+1}}{(n+1)^2}$$
$$= x$$
$$= S(x)$$

Therefore,

$$\sup_{[-1,1]} |S_n(x) - S(x)| = \sup_{[-1,1]} \left| -\frac{x^{n+1}}{(n+1)^2} \right|$$
$$\leq \frac{1}{(n+1)^2}$$

Therefore,

$$\lim_{n \to \infty} \sup_{[-1,1]} |S_n(x) - S(x)| \leq \lim_{n \to \infty} \frac{1}{(n+1)^2}$$
$$\therefore \lim_{n \to \infty} \sup_{[-1,1]} |S_n(x) - S(x)| \leq 0$$

Therefore the convergence is uniform.

Theorem 18. *If $\sum\limits_{n=1}^{\infty} f_n(x)$ converges uniformly in D to $S(x)$ and the functions f_n are continuous in D, then the $S(x)$ is also continuous in D.*

Theorem 19. *A Leibniz series, i.e. a series of the form $\sum\limits_{n=1}^{\infty} (-1)^n a_n$, with a_n monotonically decreasing and $\lim\limits_{n \to \infty} a_n = 0$, converges, and*

$$\sum_{k=n}^{m} (-1)^k a_k \leq a_n$$

Recitation 7 – Exercise 6.

Check for pointwise and uniform convergence of $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{x^2+\sqrt{n}}$ in $\mathbb{R}$.

Recitation 7 – Solution 6.

For $x \in \mathbb{R}$, $\frac{1}{x^2+\sqrt{n}}$ is monotonically decreasing to 0 as $n \to \infty$.

Therefore, for $x \in \mathbb{R}$, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{x^2+\sqrt{n}}$ is a Leibniz series. Hence, it converges pointwise.

$$\left| \sum_{k=n}^{m} \frac{(-1)^n}{x^2 + \sqrt{k}} \right| \leq \frac{1}{x^2 + \sqrt{n}}$$

$$\leq \frac{1}{\sqrt{n}}$$

Therefore,

$$\lim_{n \to \infty} \left| \sum_{k=n}^{m} \frac{(-1)^n}{x^2 + \sqrt{k}} \right| = 0$$

Therefore, $\forall \varepsilon > 0$, there exists N such that $\forall m > n > N$, and $\forall x \in \mathbb{R}$,

$$\left| \sum_{k=n}^{m} \frac{(-1)^n}{x^2 + \sqrt{k}} \right| \leq \frac{1}{\sqrt{n}} < \varepsilon$$

Therefore, $\left| \sum\limits_{k=n}^{m} \frac{(-1)^n}{x^2+\sqrt{k}} \right|$ satisfies Cauchy's criterion for uniform convergence. Hence it converges uniformly.

Recitation 7 – Exercise 7.

Show that $\sum\limits_{n=1}^{\infty} 3^n \sin\left(\frac{1}{4^n x}\right)$ does not converge uniformly in $(0, \infty)$.

Recitation 7 – Solution 7.

For any $x \in (0, \infty)$, as $\sin\left(\frac{1}{4^n x}\right) \leq \frac{1}{4^n x}$,

$$\left| 3^n \sin(\frac{1}{4^n x}) \right| \leq 3^n \frac{1}{4^n x}$$

Therefore, as $\sum \left(\frac{3}{4}\right)^n \cdot \frac{1}{x}$ converges, by the First Comparison Test, $\sum \left| 3^n \sin\left(\frac{1}{4^n x}\right) \right|$ also converges.

Therefore, $\sum 3^n \sin(\frac{1}{4^n x})$ converges absolutely. Hence, it converges.

$$\lim_{n\to\infty} 3^n \sin\left(\frac{1}{4^n x}\right) = \lim_{n\to\infty}$$
$$\neq 0$$

Therefore as the general element does not tend to 0, the series does not converge uniformly in $(0, \infty)$.

5.1 Weierstrass M-test

Theorem 20 (Weierstrass M-test). *If $|u_k(x)| \le c_k$ on D for $k \in \{1, 2, 3, \dots\}$ and the numerical series $\sum_{k=1}^{\infty} c_k$ converges, then the series of functions $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on D.*

Recitation 8 − Exercise 1.
Show that $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ converges uniformly in $[0, 1]$.

Recitation 8 − Solution 1.

$$\left|\sin \frac{x}{n^2}\right| \le \left|\frac{x}{n^2}\right|$$
$$\therefore \left|\sin \frac{x}{n^2}\right| \le \frac{1}{n^2}$$

Therefore, as $\sum \frac{1}{n^2}$ converges in $[0, 1]$, by Weierstrass M-test, the series converges uniformly in $[0, 1]$.

Recitation 8 − Exercise 2.
Does $\sum \frac{(-1)^n}{x+n}$ converge on $[0, 1]$?

Recitation 8 − Solution 2.

$$\max_{[0,1]} |f_n| = \max_{[0,1]} \frac{1}{x+n}$$
$$= \frac{1}{n}$$

Therefore, as $\sum \frac{1}{n}$ diverges, the Weierstrass M-test does not apply.
However, $\forall x \in [0, 1]$, $\sum \frac{(-1)^n}{x+n}$ is a Leibniz series. For a Leibniz series, the uniform convergence of the general term to 0 is a necessary and sufficient

condition for the convergence of the series.

Therefore, in $[0, 1]$,

$$\lim_{n \to \infty} \frac{1}{x + n} = 0$$

Therefore, as the general term goes to 0, the series converges.

Recitation 8 – Exercise 3.

Does $\sum \frac{n^2 x}{1 + n^7 x^2}$ converge uniformly in $\mathbb{R}$?

Recitation 8 – Solution 3.

As the function is even,

$$\sup_{\mathbb{R}} \left| \frac{n^2 x}{1 + n^7 x^2} \right| = \sup [0, \infty) \left| \frac{n^2 x}{1 + n^7 x^2} \right|$$

$$\sup_{[0,\infty)} \frac{n^2 x}{1 + n^7 x^2}$$

Let

$$f_n(x) = \frac{n^2 x}{1 + n^7 x^2}$$

Therefore,

$$f_n{}'(x) = \frac{n^2(1 + n^7 x^2) - n^9 \cdot 2x^2}{(1 + n^7 x^2)^2}$$

Therefore, maximizing $f_n(x)$,

$$
\begin{aligned}
f_n{}'(x) &= 0 \\
\iff n^2(1 + n^7 x^2 - n^7 \cdot 2x^2) &= 0 \\
\iff 1 - n^7 x^2 &= 0 \\
\iff x &= \sqrt{\frac{1}{n^7}}
\end{aligned}
$$

Therefore, as $f_n{}'(x) \geq 0 \iff x \in \left[0, \sqrt{\frac{1}{n^7}}\right]$ and $f_n{}'(x) \leq 0 \iff x \in \left[\sqrt{\frac{1}{n^7}}, \infty\right)$, $x = \sqrt{\frac{1}{n^7}}$ is a global maximum of f_n in $[0, \infty)$.

Therefore,

$$\sup_{\mathbb{R}} \left| \frac{n^2 x}{1 + n^7 x^2} \right| = \frac{n^2 \sqrt{\frac{1}{n^7}}}{1 + 1}$$

$$= \frac{n^2}{2n^{\frac{7}{2}}}$$

$$= \frac{1}{2n^{\frac{3}{2}}}$$

Therefore, as $\sum \frac{1}{2n^{\frac{3}{2}}}$ converges, by the Weierstrass M-test, $\sum \frac{n^2 x}{1+n^7 x^2}$ converges uniformly in $\mathbb{R}$.

5.2 Application of Uniform Convergence

Theorem 21 (Changing the order of integration and infinite summation). *If the functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ are integrable on $[a, b]$ and the series $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a, b]$ then*

$$\int_a^b \left(\sum_{k=1}^{\infty} u_k(x) \right) \mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b u_k(x)\, \mathrm{d}x$$

Theorem 22 (Changing the order of integration and limit). *If the functions $f_n(x)$ are integrable on $[a, b]$ and converge uniformly to f on $[a, b]$, then*

$$\lim_{n \to \infty} \int_a^b f_n(x)\, \mathrm{d}x = \int_a^b \lim_{n \to \infty} f_n(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x$$

Additionally,

$$\int_a^x f_n(t)\, \mathrm{d}t \overset{[a,b]}{\Longrightarrow} \int_a^x f(t)\, \mathrm{d}t$$

Recitation 8 – Exercise 4.
Is $\sum_{n=1}^{\infty} \int_0^2 \frac{(-1)^{n+1}}{x+n} = \int_0^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{x+n}$?

Recitation 8 − Solution 4.

As $\sum \frac{(-1)^{n+1}}{x+n}$ converges uniformly in $[0, 2]$, by Theorem 21, the equality holds.

Recitation 8 − Exercise 5.

Is $\int\limits_0^1 \left(\sum\limits_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \right) \mathrm{d}x = \sum\limits_{n=1}^{\infty} \int\limits_0^1 \left(x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \right) \mathrm{d}x$?

Recitation 8 − Solution 5.

$$\sum_{n=1}^{N} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} = -x + \frac{1}{x^{2N+1}}$$

$$\therefore \lim_{n \to \infty} \sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} = \sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}$$

$$\lim_{n \to \infty} -x + \frac{1}{x^{2N+1}}$$

Therefore,

$$\sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} = \begin{cases} 1 - x & ; & x \in (0, 1] \\ 0 & ; & x = 0 \end{cases}$$

Therefore, as the limit function is not continuous but the function $x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}$ is continuous, the convergence is not uniform. Therefore, Theorem 21 is not applicable.
Therefore, checking directly,

$$\int\limits_0^1 \left(\sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \right) \mathrm{d}x = \int\limits_0^1 1 - x \, \mathrm{d}x$$

$$= \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \int_0^1 \left(x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \right) \mathrm{d}x = \sum_{n=1}^{\infty} \left(\frac{x^{\frac{1}{2n+1}+1}}{\frac{1}{2n+1}+1} - \frac{x^{\frac{1}{2n-1}+1}}{\frac{1}{2n-1}+1} \Bigg|_0^1 \right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{2n+1}{2n+2} - \frac{2n-1}{2n} \right)$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \frac{2n+1}{2n+2} - \frac{2n-1}{2n}$$

$$= \lim_{N \to \infty} \frac{2N+1}{2N+2} - \frac{2 \cdot 1 - 1}{2}$$

$$= \lim_{N \to \infty} \frac{2N+1}{2N+2} - \frac{1}{2}$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

Therefore, although the convergence is not uniform, the equality holds.

Theorem 23 (Changing the order of differentiation and infinite summation). *If the functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ are differentiable on $[a, b]$ and the derivatives are continuous on $[a, b]$, and the series $\sum_{k=1}^{\infty} u_k(x)$ converges pointwise on $[a, b]$ and the series $\sum_{k=1}^{\infty} u_k'(x)$ converges uniformly on $[a, b]$, then,*

$$\left(\sum_{k=1}^{\infty} u_k(x) \right)' = \sum_{k=1}^{\infty} u_k'(x)$$

Recitation 8 − Exercise 6.

If $\sum_{n=1}^{\infty} \left(\tan^{-1} \frac{x}{n^2} \right)' = \left(\sum_{n=1}^{\infty} \tan^{-1} \frac{x}{n^2} \right)$?

Recitation 8 – Solution 6.

$$\left(\tan^{-1}\frac{x}{n^2}\right)' = \frac{1}{\left(1+\left(\frac{x}{n^2}\right)^2\right)n^2}$$

$$= \frac{1}{n^2 + \frac{x^2}{n^2}}$$

$$= \frac{n^2}{n^4 + x^2}$$

$$\therefore \left(\tan^{-1}\frac{x}{n^2}\right)' \leq \frac{1}{n^2}$$

Therefore, as $\sum \frac{1}{n^2}$ converges, by the Weierstrass M-test, $\sum \left(\tan^{-1}\frac{x}{n^2}\right)'$ converges uniformly.

By Lagrange's Mean Value Theorem, for c between 0 and x,

$$\frac{\tan^{-1}x - \tan^{-1}0}{x - 0} = (\tan^{-1})'(c)$$

$$\therefore \frac{|\tan^{-1}x|}{|x|} = \frac{1}{1+c^2}$$

$$\therefore |\tan^{-1}x| \leq 1$$

$$\therefore |\tan^{-1}x| \leq |x|$$

Therefore,

$$\left|\tan^{-1}\frac{x}{n^2}\right| \leq \left|\frac{x}{n^2}\right|$$

Therefore, as $\forall x \in \mathbb{R}$ $\sum \frac{x}{n^2}$ converges pointwise, $\sum \tan^{-1}\frac{x}{n^2}$ also converges on $\mathbb{R}$.

Therefore, by Theorem 23, the equality holds.

Part II
Functions of Multiple Variables

1 Change of Variables in Double Integrals

Definition 12 (Jacobian)**.** Let

$$T(u, v) = (x, y)$$

be an operator.
The determinant

$$J = D_T = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

is called the Jacobian of the operator T.

Theorem 24. *Let $T : \Delta \to D$ be a one-to-one, onto, continuously differentiable transformation, with $\det |D_T| \neq 0$, such that $(x, y) = T(u, v)$.* [1] *be non-zero. Let f be integrable on D.*
Then,

$$\iint_D f \, \mathrm{d}A = \iint_\Delta f \circ T \cdot |DT| \, \mathrm{d}A$$

Recitation 9 − Exercise 1.
Calculate the area of the region bounded by

$$y = 0$$
$$y = 2$$
$$y = 2x$$
$$y = 2x + 3$$

---

[1]Continuously differentiable means that all partial derivatives x_u, x_v, y_u, y_v, exist and are continuous.

Recitation 9 – Solution 1.
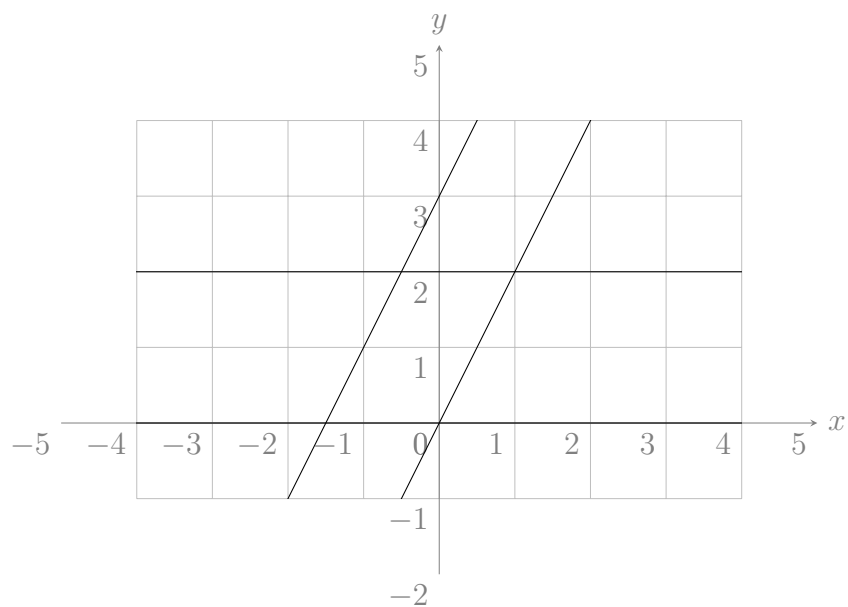
The change of variables depends on the boundaries of the region. Therefore, let

$$u = y$$
$$v = 2x - y$$

Therefore,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Therefore,

$y = 0$	$\rightarrow$	$u = 0$
$y = 2$	$\rightarrow$	$u = 2$
$y = 2x$	$\rightarrow$	$v = 0$
$y = 2x + 3$	$\rightarrow$	$v = 3$

The Jacobian can be calculated by finding T^{-1} and then finding its determinant, or using the formula

$$|D_T| = \frac{1}{|D_T{}^{-1}|}$$
$$\therefore |D_T| = \frac{1}{|-2|}$$
$$= \frac{1}{2}$$

Therefore,

$$\iint_D \mathrm{d}A = \iint_\Delta \frac{1}{2}\,\mathrm{d}A$$
$$= \int_0^2 \int_0^3 \frac{1}{2}\,\mathrm{d}v\,\mathrm{d}u$$
$$= 3$$

Recitation 9 − Exercise 2.

Calculate $\iint_D \frac{x+3y}{x^4} e^{\frac{y}{x^3}}$, where D is bounded by

$$y = x^3$$
$$y = 4x^3$$
$$x + y = 1$$
$$x + y = \frac{1}{2}$$

Recitation 9 − Solution 2.
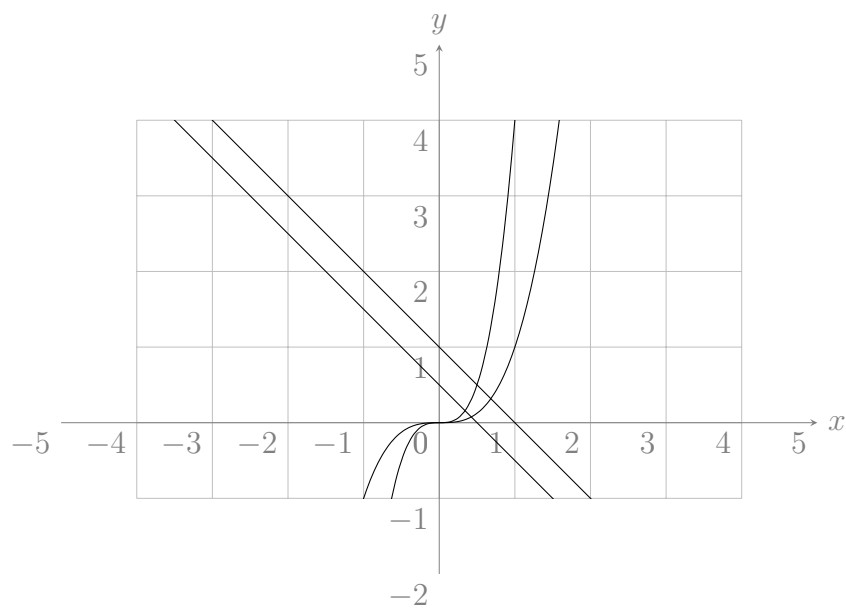
Let

$$u = \frac{y}{x^3}$$
$$v = x + y$$

Therefore, the domain D is transformed to $\left\{1 \le u \le 4, \frac{1}{2} \le v \le 1\right\}$.
Therefore,

$$J^{-1} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$
$$= \begin{vmatrix} -\frac{3y}{x^4} & \frac{1}{x^3} \\ 1 & 1 \end{vmatrix}$$
$$= \frac{3y + x}{x^4}$$

Therefore,

$$J = \frac{1}{J^{-1}}$$
$$= \frac{x^4}{3y + x}$$

Therefore,

$$\iint_D \frac{x+3y}{x^4} e^{\frac{y}{x^3}} \, \mathrm{d}A = \iint_\Delta \frac{x+3y}{x^4} e^u J \, \mathrm{d}u \, \mathrm{d}v$$

$$= \iint_\Delta \frac{x+3y}{x^4} \frac{x^4}{x+3y} e^u \, \mathrm{d}u \, \mathrm{d}v$$

$$= \int_1^4 \int_{\frac{1}{2}}^1 e^u \, \mathrm{d}v \, \mathrm{d}u$$

$$= \int_1^4 ve^u \big|_{\frac{1}{2}}^1$$

$$= \int_1^4 \frac{1}{2} e^u \, \mathrm{d}y$$

$$= \frac{1}{2} e^u \bigg|_1^4$$

$$= \frac{1}{2} e^4 - \frac{1}{2} e^1$$

$$= \frac{e^4 - e}{2}$$

1.1 Polar Coordinates

For polar coordinates,

$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$J = r$$

Recitation 9 – Exercise 3.

Find $\iint_D \frac{x}{y} \, \mathrm{d}A$ where D is $\left\{ (x,y) : 9 \le x^2 + y^2 \le 16, \frac{\sqrt{3}}{3} x \le y \le \sqrt{3} x \right\}$.

Recitation 9 – Solution 3.
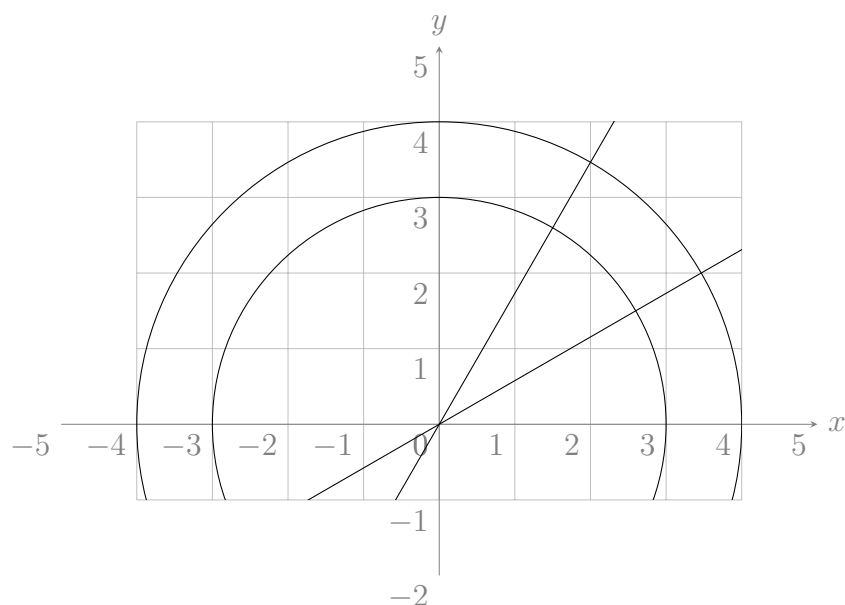
$$x^2 + y^2 = r^2$$
$$\frac{y}{x} = \tan \theta$$

Therefore, the domain D is transformed to the domain $\left\{3 \le r \le 4,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\right\}$.
Therefore,

$$\iint_D \frac{x}{y}\, \mathrm{d}A = \iint_\Delta \frac{1}{\tan\theta} J\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \iint_\Delta r \cot\theta\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin\theta}\, \mathrm{d}t \cdot \int_3^4 r\, \mathrm{d}r$$

$$= \left(\ln|\sin\theta| \big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \right) \left(\frac{r^2}{2} \Big|_3^4 \right)$$

$$= \frac{7}{2} \ln(\sqrt{3})$$

$$= \frac{7}{4} \ln 3$$

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1.2   Generalized Polar Coordinates

In generalized polar coordinates or elliptical coordinates, the transformation is

$$x = ar\cos\theta$$
$$y = br\sin\theta$$

Therefore,

$$J = abr$$

Recitation 9 – Exercise 4.

Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > 0$, $b > 0$.

Recitation 9 – Solution 4.

$$
\begin{aligned}
\iint\limits_{D} \mathrm{d}A &= \iint\limits_{\Delta} J\,\mathrm{d}A \\
&= \iint\limits_{\Delta} abr\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{0}^{2\pi}\int_{0}^{1} abr\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{2}ab \cdot 2\pi \\
&= \pi ab
\end{aligned}
$$

2   Change of Variables in Triple Integrals

Theorem 25. *Let $T : \Delta \to D$ be a one-to-one, onto, continuously differentiable transformation, with $\det|D_T| \neq 0$, such that $(x,y,z) = T(u,v,w)$.* [2] *be non-zero. Let f be integrable on D.*
Then,

$$\iint\limits_{D} f\,\mathrm{d}A = \iint\limits_{\Delta} f \circ T \cdot |DT|\,\mathrm{d}A$$

---

[2]Continuously differentiable means that all partial derivatives x_u, x_v, x_w, y_u, y_v, y_w, z_u, z_v, z_w, exist and are continuous.

2.1 Cylindrical Coordinates

For polar coordinates,

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = r$$
$$J = r$$

Recitation 10 − Exercise 1.

Calculate $\iiint\limits_{V} \frac{z}{\sqrt{z^2+y^2}}\, dV$, where $V = \left\{ z \le \sqrt{x^2+y^2}+1,\, x^2+y^2 \le 1,\, z \ge 0 \right\}$

Recitation 10 − Solution 1.

$z \le \sqrt{x^2+y^2}+1$ is the volume under a cone with its apex at $(0,0,1)$.
Therefore, as $x^2+y^2 \le 1$ and $z \ge 0$, V is the volume under the cone, from $z = 0$ to $z = 2$.
Therefore, in cylindrical coordinates,

$$
\begin{array}{ccc}
z = \sqrt{x^2+y^2}+1 & \rightarrow & z = r+1 \\
x^2+y^2 = 1 & \rightarrow & r^2 = 1 \\
z = 0 & \rightarrow & z = 0
\end{array}
$$

Therefore,

$$
\begin{aligned}
\iiint\limits_{V} \frac{z}{\sqrt{x^2+y^2}}\, dV &= \iiint\limits_{\Delta} \frac{z}{r} r\, dV \\
&= \int\limits_{0}^{2\pi}\int\limits_{0}^{1}\int\limits_{0}^{r+1} z\, dz\, dr\, d\theta \\
&= 2\pi \int\limits_{0}^{1} \frac{(r+1)^2}{2}\, dr \\
&= 2\pi \left(\frac{(r+1)^3}{6} \right)\Bigg|_{0}^{1} \\
&= 2\pi \left(\frac{8}{6} - \frac{1}{6} \right) \\
&= \frac{14\pi}{6}
\end{aligned}
$$

2.2 Spherical Coordinates

For spherical coordinates,

$$x = r\cos\theta\sin\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\varphi$$
$$J = r^2\sin t$$

where ρ is the distance from $(0,0,0)$ to (x,y,z); θ is the angle between the line joining $(0,0,0)$ and (x,y,z), and the x axis; and φ is the angle between the line joining $(0,0,0)$ and (x,y,z), and the z axis.

Recitation 10 − Exercise 2.

Calculate the volume of the body

$$V = \left\{(x,y,z) : x^2 + y^2 + z^2 \leq 1, z \geq \sqrt{x^2+y^2}\right\}$$

Recitation 10 − Solution 2.

$x^2 + y^2 + z^2 \leq 1$ is the volume inside a sphere of radius 1 centred at $(0,0,0)$, and $z \geq \sqrt{x^2+y^2}$ is the area above a right angled cone.
Therefore, in spherical coordinates,

$$x^2 + y^2 + z^2 = 1 \quad \rightarrow \quad r^2 = 1$$
$$\therefore x^2 + y^2 + z^2 = 1 \quad \rightarrow \quad r = 1$$
$$\sqrt{x^2+y^2} = z \quad \rightarrow \quad r\cos\varphi = \sqrt{(r\cos\theta\sin\varphi)^2 + (r\sin\theta\sin\varphi)^2}$$
$$\sqrt{x^2+y^2} = z \quad \rightarrow \quad r\cos\varphi = r\sin\varphi$$

Therefore,

$$\iiint_V \mathrm{d}V = \iiint_\Delta r^2\sin\varphi\,\mathrm{d}V$$
$$= \int_0^{\frac{\pi}{4}}\int_0^1\int_0^{2\pi} r^2\sin\varphi\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\varphi$$
$$= \frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

Recitation 10 − Exercise 3.

Calculate $\iiint\limits_{V} \frac{1}{\sqrt{x^2+y^2+z^2}} \, dV$ where

$$V = \left\{ (x, y, z) : x^2 + y^2 + z^2 \leq 2z \right\}$$

Recitation 10 − Solution 3.

$$x^2 + y^2 + z^2 \leq 2z$$
$$\therefore x^2 + y^2 + (z-1)^2 \leq 1$$

Therefore, the region is a sphere of radius 1 centred at $(0, 0, 1)$.
Therefore, let

$$x = r \cos \theta \sin \varphi$$
$$y = r \sin \theta \sin \varphi$$
$$z = r \cos \varphi + 1$$

Therefore,

$$J = r^2 \sin \varphi$$

Therefore,

$$\iiint\limits_{V} \frac{1}{\sqrt{x^2 + y^2 + z^2}} \, dV = \iiint\limits_{\Delta} \frac{r^2 \sin \varphi}{\sqrt{x^2 + y^2 + z^2}} \, dV$$

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{(r \cos \theta \sin \varphi)^2 (r \sin \theta \sin \varphi)^2 + (r \cos \varphi + 1)^2}$$
$$= r^2 + 2r \cos \varphi + 1$$

Therefore,

$$\iiint\limits_{V} \frac{1}{\sqrt{x^2 + y^2 + z^2}} \, dV = \int_0^{2\pi} \int_0^1 \int_0^{\pi} \frac{r^2 \sin \varphi}{\sqrt{r^2 + 2r \cos \varphi + 1}} \, d\varphi \, dr \, d\theta$$

Let

$$t = r^2 + 2r \cos \varphi + 1$$
$$\therefore dt = -2r \sin \varphi \, d\varphi$$

Therefore,

$$\int_0^\pi \frac{r^2 \sin \varphi}{\sqrt{r^2 + 2r \cos \varphi + 1}} \, \mathrm{d}\varphi = -\frac{r}{2} \int_{r^2+2r+1}^{r^2-2r+1} \frac{1}{\sqrt{t}} \, \mathrm{d}t$$

$$= -r \cdot \sqrt{t} \Big|_{r^2+2r+1}^{r^2-2r+1}$$

$$= -r \left(\sqrt{r^2 + 2r + 1} - \sqrt{r^2 - 2r + 1} \right)$$

$$= -r \left(\sqrt{(r+1)^2} - \sqrt{(r-1)^2} \right)$$

As $r \le 1$, $r - 1 \le 0$. Therefore, $\sqrt{(r-1)^2} = (r-1)$. Therefore,

$$\int_0^\pi \frac{r^2 \sin \varphi}{\sqrt{r^2 + 2r \cos \varphi + 1}} \, \mathrm{d}\varphi = r \left((r+1) - (1-r) \right)$$

$$= 2r^2$$

Therefore,

$$\iiint_V \frac{1}{\sqrt{x^2 + y^2 + z^2}} \, \mathrm{d}V = \int_0^{2\pi} \int_0^1 2r^2 \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= 2\pi \int_0^1 2r^2 \, \mathrm{d}r$$

$$= 2\pi \cdot \frac{2}{3}$$

$$= \frac{4\pi}{3}$$

Recitation 10 − Exercise 4.
Calculate $\iiint_V x^5 yz - y^5 xz \, \mathrm{d}V$, where V is bounded by $y = 0$, $z = 0$, $z = 1$, and satisfies $x^2 - y^2 \ge 1$ and $0 \le x^2 + y^2 \le 4$.

Recitation 10 − Solution 4.
Let

$$u = z$$
$$v = x^2 - y^2$$
$$w = x^2 + y^2$$

Therefore,

$$J^{-1} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2x & -2y & 0 \\ 2x & 2y & 0 \end{vmatrix}$$

$$= 8xy$$

$$\therefore J = \frac{1}{8xy}$$

Therefore,

$$
\begin{array}{ccc}
z = 0 & \rightarrow & u = 0 \\
z = 1 & \rightarrow & u = 1 \\
y = 0 & \rightarrow & v = w \\
x^2 - y^2 \geq 1 & \rightarrow & v \geq 1 \\
x^2 + y^2 = 0 & \rightarrow & w = 0 \\
x^2 + y^2 = 4 & \rightarrow & w = 4
\end{array}
$$

Therefore,

$$\iiint_V x^5 yz - y^5 xz \, \mathrm{d}V = \iiint_\Delta \left(x^5 yz - y^5 xz \right) \frac{1}{8xy} \, \mathrm{d}V$$

$$= \int_0^1 \int_1^4 \int_1^w \frac{x^5 yz - y^5 xz}{8xy} \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}u$$

$$= \frac{1}{8} \int_0^1 \int_1^4 \int_1^w x^4 z - y^4 z \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}u$$

$$= \frac{1}{8} \int_0^1 \int_1^4 \int_1^w z(x^4 - y^4) \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}u$$

$$= \frac{1}{8} \int_0^1 \int_1^4 \int_1^w z(x^2 + y^2)(x^2 - y^2) \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}u$$

$$= \frac{1}{8} \int_0^1 \int_1^4 \int_1^w uvw \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}u$$

$$= \frac{1}{8} \int_0^1 \int_1^4 \left. \frac{uv^2 w}{2} \right|_{v=1}^{v=w} \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}u$$

$$= \frac{1}{8} \int_0^1 \int_1^4 \frac{uw^3}{3} - \frac{uw}{2} \, \mathrm{d}w \, \mathrm{d}u$$

$$= \frac{1}{8} \int_0^1 \left. \frac{uw^4}{4} - \frac{uw^2}{2} \right|_{w=1}^{w=4} \mathrm{d}u$$

$$= \frac{1}{8} \int_0^1 \frac{255u}{4} - \frac{15u}{2} \, \mathrm{d}u$$

$$= \frac{1}{8} \int_0^1 \frac{225u}{4} \, \mathrm{d}u$$

$$= \frac{1}{8} \left. \left(\frac{225u^2}{8} \right) \right|_0^1$$

$$= \frac{1}{8} \cdot \frac{225}{8}$$

$$= \frac{225}{64}$$