

Differential and Integral Calculus

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2 Required Reading

Protter and Morrey: *A first Course in Real Analysis*, UTM Series, Springer-Verlag, 1991

3 Additional Reading

Thomas and Finney, *Calculus and Analytic Geometry*, 9th edition, Addison-Wesley, 1996

Part I

Sequences and Series

1 Sequences

Definition 1 (Sequence). A sequence of real numbers is a set of numbers which are written in some order. There are infinitely many terms in a sequence. It is denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Example 1. $1, \frac{1}{2}, \frac{1}{3}, \dots$ is called the harmonic sequence.

$$a_n = \frac{1}{n}$$

Example 2. $1, -\frac{1}{2}, \frac{1}{3}, \dots$ is called the alternating harmonic sequence.

$$a_n = (-1)^{n+1} \frac{1}{n}$$

Example 3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$

$$a_n = \frac{n}{n+1}$$

Example 4. $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \dots$

$$a_n = \frac{n+1}{3^n}$$

Example 5. The Fibonacci sequence is given by

$$f_n = \begin{cases} 1 & ; \quad n = 1, 2 \\ f_{n-1} + f_{n-2} & ; \quad n \geq 3 \end{cases}$$

Example 6. A geometric sequence is given by

$$a_n = a_1 q^{n-1}$$

where q is called the common ratio.

Example 7. A geometric sequence is given by

$$a_n = a_1 + d(n - 1)$$

where d is called the common difference.

Definition 2 (Equal sequences). Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equal if $a_n = b_n, \forall n \in \mathbb{N}$.

Definition 3 (Sequences bounded from above). $\{a_n\}$ is said to be bounded from above if $\exists M \in \mathbb{R}$, s.t. $a_n \leq M, \forall n \in \mathbb{N}$. Each such M is called an upper bound of $\{a_n\}$.

Definition 4 (Sequences bounded from below). $\{a_n\}$ is said to be bounded from below if $\exists m \in \mathbb{R}$, s.t. $a_n \geq m, \forall n \in \mathbb{N}$. Each such m is called a lower bound of $\{a_n\}$.

Definition 5. $\{a_n\}$ is said to be bounded if it is bounded from below and bounded from above.

Example 8. The sequence $a_n = n^2 + 2$ is not bounded from above but is bounded from below, by all $m \leq 3$.

Example 9. $\left\{ \frac{2n-1}{3n} \right\}$ is bounded.

$$m = 0 \leq \frac{2n-1}{3n} \leq \frac{2n}{3n} = \frac{2}{3} = M$$

Definition 6 (Monotonic increasing sequence). A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \leq a_{n+1}, \forall n \geq n_0$.

Definition 7 (Monotonic decreasing sequence). A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \geq a_{n+1}, \forall n \geq n_0$.

Definition 8 (Strongly increasing sequence). A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n < a_{n+1}, \forall n \geq n_0$.

Definition 9 (Strongly decreasing sequence). A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n > a_{n+1}, \forall n \geq n_0$.

Example 10. The sequence $\left\{\frac{n^2}{2^n}\right\}$ is strongly decreasing. However, this is not evident by observing the first few terms. $\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \dots$

$$\begin{aligned}
& a_n > a_{n+1} \\
\iff & \frac{n^2}{2^n} > \frac{(n+1)^2}{2^{n+1}} \\
\iff & 2n^2 > (n+1)^2 \\
\iff & \sqrt{2}n > n+1 \\
\iff & n(\sqrt{2}-1) > 1 \\
\iff & n > \frac{1}{\sqrt{2}-1} \\
\iff & n > 3
\end{aligned}$$

Exercise 1.

Is $a_n = (-1)^n$ monotonic?

Solution 1.

The sequence $-1, 1, -1, 1, \dots$ is not monotonic.

1.1 Limit of a Sequence

Definition 10. Let $\{a_n\}$ be a given sequence. A number L is said to be the limit of the sequence if $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$, s.t. $|a_n - L| < \varepsilon, \forall n \geq n_0$. That is, there are infinitely many terms inside the interval and a finite number of terms outside it.

Example 11. The sequence $\left\{\frac{1}{n}\right\}$ tends to 0, i.e. for any open interval $(-\varepsilon, \varepsilon)$, there are finite number of terms of the sequence outside the interval, and therefore there are infinitely many terms inside the interval.

Exercise 2.

Prove

$$\lim_{n \rightarrow \infty} \frac{n+2}{2n-1} = \frac{1}{2}$$

Solution 2.

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$

Exercise 3.

Prove that 2 is not a limit of $\left\{ \frac{3n+1}{n} \right\}$.

Solution 3.

If possible, let

$$\lim_{n \rightarrow \infty} \frac{3n+1}{n} = 2$$

Then, $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$, s.t. $\left| \frac{3n+1}{n} - 2 \right| < \varepsilon, \forall n \geq n_0$. However,

$$\left| \frac{3n+1}{n} - 2 \right| = 1 + \frac{1}{n} > 1$$

This is a contradiction for $\varepsilon = \frac{1}{2}$. Therefore, 2 is not a limit.

Theorem 1. *If a sequence $\{a_n\}$ has a limit L then the limit is unique.*

Proof. If possible let there exist two limits L_1 and L_2 . Therefore, $\forall \varepsilon > 0$, there exist a finite number of terms in the interval $(L_1 - \varepsilon, L_1 + \varepsilon)$. Therefore, there exist a finite number of terms in the interval $(L_2 - \varepsilon, L_2 + \varepsilon)$. This contradicts the definition of a limit. Therefore, the limit is unique. \square

Theorem 2. *If a sequence $\{a_n\}$ has limit L , then the sequence is bounded.*

Theorem 3. *Let*

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= a \\ \lim_{n \rightarrow \infty} b_n &= b \end{aligned}$$

and let c be a constant. Then,

$$\begin{aligned} \lim c &= c \\ \lim(ca_n) &= c \lim a_n \\ \lim(a_n \pm b_n) &= \lim a_n \pm \lim b_n \\ \lim(a_n b_n) &= \lim a_n \lim b_n \\ \lim\left(\frac{a_n}{b_n}\right) &= \frac{\lim a_n}{\lim b_n} \quad (\text{if } \lim b_n \neq 0) \end{aligned}$$

Theorem 4. Let $\{b_n\}$ be bounded and let $\lim a_n = 0$. Then,

$$\lim(a_nb_n) = 0$$

Theorem 5 (Sandwich Theorem). Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences. If

$$\lim a_n = \lim b_n = L$$

and $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n \leq b_n \leq c_n$. Then,

$$\lim b_n = L$$

Exercise 4.

Calculate $\lim_{n \rightarrow \infty} \sqrt[n]{2^n + 3^n}$

Solution 4.

$$\begin{aligned} \sqrt[n]{3^n} &\leq \sqrt[n]{2^n + 3^n} \leq \sqrt[n]{3^n + 3^n} = \sqrt[n]{2 \cdot 3^n} \\ \therefore 3 &\leq \sqrt[n]{2^n + 3^n} \leq 3\sqrt[n]{2} \end{aligned}$$

Therefore, by the Sandwich Theorem, $\lim_{n \rightarrow \infty} \sqrt[n]{2^n + 3^n} = 3$.

Theorem 6. Any monotonically increasing sequence which is bounded from above converges. Similarly, any monotonically decreasing sequence which is bounded from below converges.

Exercise 5.

Prove that there exists a limit for $a_n = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}_{n \text{ times}}$ and find it.

Solution 5.

$$a_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = a_2$$

If possible, let

$$\begin{aligned} a_{n-1} &< a_n \\ \therefore \sqrt{2 + a_{n-1}} &< \sqrt{2 + a_n} \\ \therefore a_n &< a_{n+1} \end{aligned}$$

Hence, by induction, $\{a_n\}$ is monotonically increasing.

$$a_1 = \sqrt{2} \leq 2$$

If possible, let

$$\begin{aligned} a_n &\leq 2 \therefore \sqrt{2 + a_n} && \leq \sqrt{2 + 2} \\ \therefore a_{n+1} &\leq 2 \end{aligned}$$

Hence, by induction, $\{a_n\}$ is bounded from above by 2. Therefore, by , $\{a_n\}$ converges.

Definition 11 (Limit in a wide sense). The sequence $\{a_n\}$ is said to converge to $+\infty$ if $\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0, a_n > M$.

The sequence $\{a_n\}$ is said to converge to $-\infty$ if $\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0, a_n < M$.

1.2 Sub-sequences

Definition 12 (Sub-sequence). Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_k\}_{k=1}^{\infty}$ be a strongly increasing sequence of natural numbers. Let $\{b_k\}_{k=1}^{\infty}$ be a sequence such that $b_k = a_{n_k}$. Then $\{b_k\}_{k=1}^{\infty}$ is called a sub-sequence of $\{a_n\}_{n=1}^{\infty}$.

Example 12.

$$a_n = \frac{1}{n}$$

If we choose $n_k = k^2$,

$$b_k = a_{n_k} = a_{k^2} = \frac{1}{k^2}$$

Therefore,

$$\{b_k\} = 1, \frac{1}{4}, \frac{1}{9}, \dots$$

Theorem 7. *If the sequence $\{a_n\}$ converges to L in a wide sense, i.e. L can be infinite, then any sub-sequence of $\{a_n\}$ converges to the same limit L .*

Definition 13 (Partial limit). A real number a , which may be infinite, is called a partial limit of the sequence $\{a_n\}$ if there exists a sub-sequence of $\{a_n\}$ which converges to a .

Example 13. Let

$$a_n = (-1)^n$$

Therefore, $\nexists \lim_{n \rightarrow \infty} a_n$. Let

$$b_k = a_{n_k} = a_{2k-1}$$

Therefore,

$$\begin{aligned} \{b_k\} &= -1, -1, -1, \dots \\ \therefore \lim_{k \rightarrow \infty} b_k &= -1 \end{aligned}$$

Therefore, -1 is a partial limit of $\{a_n\}$.

Theorem 8 (Bolzano-Weierstrass Theorem). *For any bounded sequence there exists a subsequence which is convergent, s.t. there exists at least one partial limit.*

Definition 14 (Upper partial limit). The largest partial limit of a sequence is called the upper partial limit. It is denoted by $\overline{\lim} a_n$ or $\limsup a_n$.

Definition 15 (Lower partial limit). The smallest partial limit of a sequence is called the lower partial limit. It is denoted by $\underline{\lim} a_n$ or $\liminf a_n$.

Theorem 9. *If the sequence $\{a_n\}$ is bounded and*

$$\overline{\lim} a_n = \underline{\lim} a_n = a$$

then

$$\exists \lim a_n = a$$

1.3 Cauchy Characterisation of Convergence

Definition 16. A sequence $\{a_n\}$ is called a Cauchy sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall m, n \geq n_0$, $|a_n - a_m| < \varepsilon$.

Theorem 10 (Cauchy Characterisation of Convergence). *A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.*

Proof. Let

$$\lim_{n \rightarrow \infty} a_n = L$$

Then $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$, $|a_n - L| < \frac{\varepsilon}{2}$. Therefore if $n \geq n_0$ and $m \geq n_0$, then

$$\begin{aligned} |a_n - a_m| &= |a_n - L + L - a_m| \\ &\leq |a_n - L| + |L - a_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ \therefore |a_n - a_m| &= \varepsilon \end{aligned}$$

Similarly, the converse can be proved by Theorem 9. \square

Theorem 11 (Another Formulation of the Cauchy Characterisation Theorem). *The sequence $\{a_n\}$ converges if and only if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$.*

Exercise 6.

Prove that the sequence

$$a_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

is convergent.

Solution 6.

$$\begin{aligned} |a_{n+p} - a_n| &= \left| \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(n+p)^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right) \right| \\ &= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(n+p-1)(n+p)} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n} - \frac{1}{\cancel{n+1}} + \frac{1}{\cancel{n+1}} + \cdots + \frac{1}{\cancel{n+p-1}} - \frac{1}{n+p} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n} - \frac{1}{n+p} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n} \end{aligned}$$

Therefore, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$, where $n_0 > \frac{1}{\varepsilon}$. \square

Exercise 7.

Prove that the sequence

$$a_n = \frac{1}{1} + \frac{1}{n} + \cdots + \frac{1}{n}$$

diverges.

Solution 7.

If possible, let the sequence converge. Then, by the Cauchy Characterisation of Convergence, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$. Therefore,

$$\begin{aligned} |a_{n+p} - a_n| &= \left| \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+p} - \left(\frac{1}{n} + \cdots + \frac{1}{n} \right) \right| \\ &= \frac{1}{n+1} + \cdots + \frac{1}{n+p} \\ &\geq p \cdot \frac{1}{n+p} \\ \therefore |a_{n+p} - a_n| &> \frac{p}{n+p} \end{aligned}$$

If $n = p$,

$$\frac{p}{n+p} = \frac{1}{2}$$

This contradicts the result obtained from the Cauchy Characterisation of Convergence, for $\varepsilon = \frac{1}{4}$.

Therefore, the sequence diverges.

2 Series

Definition 17 (Series). Given a sequence $\{a_n\}$, the sum $a_1 + \cdots + a_n + \cdots$ is called an infinite series or series. It is denoted as $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

Definition 18 (Partial sum). The partial sum of the series $\sum a_n$ is defined as

$$S_i = a_1 + \cdots + a_i$$

Definition 19 (Convergent and divergent series). If the sequence $\{S_n\}_{n=1}^{\infty}$ converges, then the series is called convergent. Otherwise, the series is called divergent.

Definition 20 (Sum of a series). If the sequence $\{S_n\}_{n=1}^{\infty}$ converges to $S \neq \pm\infty$, the number S is called the sum of the series.

$$\sum_{n=1}^{\infty} a_n = S$$

Example 14.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

Therefore,

$$S_1 = \frac{1}{2} \tag{1}$$

$$S_2 = \frac{1}{2} + \frac{1}{2^2} \tag{2}$$

$$\vdots S_n = \frac{1}{2} + \cdots + \frac{1}{2^n} \tag{3}$$

$$= \frac{a_1(1 - q^n)}{1 - q} \tag{4}$$

$$= \frac{1/2(1 - 1/2^n)}{1 - 1/2} \tag{5}$$

$$= 1 - \frac{1}{2^n} \tag{6}$$

$$\lim_{n \rightarrow \infty} S_n = 1 \tag{7}$$

Therefore, the series converges.

$$S = \sum_{n=1}^{\infty} = 1$$

Theorem 12. A geometric series $\sum_{n=1}^{\infty} a_1 q^{n-1}$, $a_1 \neq 0$ converges if $|q| < 1$ and then,

$$S = \sum_{n=1}^{\infty} a_1 q^{n-1} = \frac{a_1}{1 - q}$$

Definition 21 (p -series). The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the p -series.

Theorem 13. *The p -series converges for $p > 1$ and diverges for $p \leq 1$.*

Theorem 14. *If $\sum a_n$ converges, then*

$$\lim_{n \rightarrow \infty} a_n = 0$$

Proof.

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ \therefore \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} \\ &= S - S \\ &= 0 \end{aligned}$$

□

Theorem 15. *If $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n \pm b_n)$ and $\sum ca_n$, where c is a constant, also converge. Also,*

$$\begin{aligned} \sum(a_n \pm b_n) &= \sum a_n \pm \sum b_n \\ \sum(ca_n) &= c \sum a_n \end{aligned}$$

2.1 Convergence Criteria

2.1.1 Leibniz's Criteria

Definition 22 (Alternating series). The series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

Theorem 16 (Leibniz's Criteria for Convergence). *If an alternating series $\sum (-1)^{n-1} a_n$ with $a_n > 0$ satisfies*

1. $a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.

2. $\lim_{n \rightarrow \infty} a_n = 0$

then the series $\sum (-1)^{n-1} a_n$ converges.

Proof. Consider the even partial sums of the series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$.

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$$

As $\{a_n\}$ is monotonically increasing, all brackets are non-negative. Therefore,

$$S_{2m+2} \geq S_{2m}$$

Therefore, $\{S_{2m}\}$ is increasing.

Also,

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$$

All brackets and a_{2m} are non-negative. Therefore,

$$S_{2m} \leq a_1$$

Therefore, $\{S_{2m}\}$ is bounded from above by a_1 . Hence,

$$\exists \lim_{m \rightarrow \infty} S_{2m} = S$$

For S_{2m+1} ,

$$\begin{aligned} S_{2m+1} &= S_{2m} + a_{2m+1} \\ \therefore \lim_{m \rightarrow \infty} S_{2m+1} &= \lim_{m \rightarrow \infty} S_{2m} + \lim_{m \rightarrow \infty} a_{2m+1} \\ &= S + 0 \\ &= S \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} S_n = S$$

□

Example 15. The alternating harmonic series $\sum \frac{(-1)^{n-1}}{n}$ converges as $a_n = \frac{1}{n} > 0$, a_n decreases and $\lim a_n = 0$.

2.1.2 Comparison Test

Theorem 17 (Comparison Test for Convergence). Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$.

1. If $a_n \leq b_n$, $\forall n \geq n_0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $a_n \geq b_n$, $\forall n \geq n_0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 18 (Another Formulation of the Comparison Test for Convergence). Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$ and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = a > 0$$

where a is a finite number. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

2.1.3 d'Alembert Criteria (Ratio Test)

Definition 23 (Absolute and conditional convergence). The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

Example 16. The series $\sum \frac{(-1)^{n-1}}{n^2}$ converges absolutely, as $\sum \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum \frac{1}{n^2}$ converges.

Example 17. The series $\sum \frac{(-1)^{n-1}}{n}$ converges conditionally, as it converges, but $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$ diverges.

Theorem 19. *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 20 (d'Alembert Criteria (Ratio Test)). 1. If

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n-1}}{a_n} \right| = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n-1}}{a_n} \right| = L > 1$$

(including $L = \infty$), then $\sum a_n$ converges diverges.

3. If $L = 1$, the test does not apply.

2.1.4 Cauchy Criteria (Cauchy Root Test)

Theorem 21 (Cauchy Criteria (Cauchy Root Test)). 1. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

(including $L = \infty$), then $\sum a_n$ diverges.

3. If $L = 1$, the test does not apply.

2.1.5 Integral Test

Theorem 22 (Integral Test for Series Convergence). *Let $f(x)$ be a continuous, non-negative, monotonic decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x) dx$ converges.*

Exercise 8.

Does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge or diverge?

Solution 8.

Let

$$f(x) = \frac{1}{x^p}$$

with $p > 0$.

Therefore, $f(x)$ is continuous, non-negative and monotonic decreasing on $[1, \infty)$. Therefore, the Integral Test for Series Convergence is applicable.

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^p} dx$$

If $p \neq 1$,

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} &= \lim_{t \rightarrow \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^t \\ &= \lim_{t \rightarrow \infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} \right) \\ &= \frac{1}{p-1} \end{aligned}$$

If $p = 1$,

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} &= \lim_{t \rightarrow \infty} \ln x \Big|_1^t \\ &= \infty \end{aligned}$$

Therefore, the series converges for $p > 1$ and diverges for $p \leq 1$.

Theorem 23. *If the series $\sum a_n$ absolutely converges and the series $\sum b_n$ is obtained from $\sum a_n$ by changing the order of the terms in $\sum a_n$ then $\sum b_n$ also absolutely converges and $\sum b_n = \sum a_n$.*

Theorem 24. *If a series converges then the series with brackets without changing the order of terms also converges. That is, if $\sum a_n$ converges, then any series of the form $(a_1 + a_2) + (a_3 + a_4 + a_5) + a_6 + \dots$ also converges.*

Theorem 25. *If a series with brackets converges and the terms in the brackets have the same sign, then the series without brackets also converges.*

3 Power Series

Definition 24 (Power series). The series $\sum_{n=0}^{\infty} a_n(x - c)^n$ is called a power series.

Theorem 26 (Cauchy-Hadamard Theorem). *For any power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ there exists the limit, which may be infinity,*

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$$

and the series converges for $|x - c| < R$ and diverges for $|x - c| > R$. The end points of the interval, i.e. $x = c - R$ and $x = c + R$ must be separately checked for series convergence.

Definition 25 (Radius of convergence and convergence interval). The number R is called the radius of convergence and the interval $|x - c| < R$ is called the convergence interval of the series. The point c is called the centre of the convergence interval.

Theorem 27. *If $\exists \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$, which may be infinite, then,*

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Theorem 28 (Stirling's Approximation). *For $n \rightarrow \infty$,*

$$n! \approx \left(\frac{n}{e} \right)^n \sqrt{2\pi n}$$

3.1 Differentiation and Integration of Power Series

Theorem 29. If R is a radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ then the function $f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$ is differentiable on $(c - R, c + R)$ and the derivative is

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x - c)^{n-1}$$

Theorem 30. If R is a radius of convergence of the series $\sum_{n=0}^{\infty} a_n(x - c)^n$ then the function $f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$ is integrable in $(c - R, c + R)$ and

$$\int f(x) dx = \sum_{n=0}^{\infty} a_n \frac{(x - c)^{n+1}}{n + 1} + A$$

where $c - R < x < c + R$.

Exercise 9.

Find $\int_0^x e^{-t^2} dt$.

Solution 9.

$\forall s \in \mathbb{R}$,

$$\begin{aligned} e^s &= 1 + \frac{s}{1!} + \frac{s^2}{2!} + \cdots + \frac{s^n}{n!} + \cdots \\ \therefore e^{-t^2} &= 1 - \frac{t^2}{1!} + \frac{t^4}{2!} + \cdots + (-1)^n \frac{t^{2n}}{n!} + \cdots \\ \therefore \int_0^x e^{-t^2} dt &= x - \frac{x^3}{1!3} + \frac{x^5}{2!5} + \cdots + (-1)^n \frac{x^{2n+1}}{n!(2n+1)} + \cdots \end{aligned}$$

Theorem 31. If the series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} B_n x^n$ absolutely converge for $|x| < R$ and $c_n = \sum_{k=0}^n a_k b_{n-k}$, then the series $C(x) = \sum_{n=0}^{\infty} c_n x^n$ also absolutely converges for $|x| < R$ and $C(x) = A(x)B(x)$.

3.2 Taylor Series

Definition 26 (Taylor series). Let $f(x)$ be infinitely differentiable on an open interval about a and let x be an arbitrary point in the interval. Then the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ is called the Taylor series of $f(x)$ at a . If $a = 0$ then it is called the Maclaurin series of $f(x)$ at 0.

Theorem 32. *If there exists a power series which converges to $f(x)$, i.e. if, for $|x-a| < R$,*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

then, for $|x-a| < R$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

that is, $\forall n$,

$$a_n = \frac{f^{(n)}(a)}{n!}$$

Exercise 10.

Show that

$$f(x) = \begin{cases} 0 & ; \quad x = 0 \\ e^{-\frac{1}{x^2}} & ; \quad x \neq 0 \end{cases}$$

is not equal to its Taylor series at $a = 0$.

Solution 10.

If $n = 1$,

$$\begin{aligned} f^{(1)}(0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{e^{-\frac{1}{(\Delta x)^2}}}{\Delta x} \end{aligned}$$

Let $t = \frac{1}{\Delta x}$

$$\begin{aligned}\therefore f'(0) &= \lim_{t \rightarrow \infty} \frac{e^{-t^2}}{\frac{1}{t}} \\ &= \lim_{t \rightarrow \infty} \frac{t}{e^{t^2}} \\ &= \lim_{t \rightarrow \infty} \frac{1}{e^{t^2} 2t} \\ &= 0\end{aligned}$$

Therefore,

$$f'(x) = \begin{cases} 0 & ; \quad x = 0 \\ e^{-\frac{1}{x^2} \cdot 2 \cdot x^{-3}} & ; \quad x \neq 0 \end{cases}$$

Similarly, $\forall n \geq 1, f^{(n)}(0) = 0$

Therefore, the Taylor series is not equal to $f(x)$.

Exercise 11.

Find the Maclaurin series of $f(x) = e^x$ and prove that the series converges to $f(x)$ for any $x \in \mathbb{R}$.

Solution 11.

$\forall n \geq 1, f^{(n)}(x) = e^x$.

Therefore,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c x^{n+1}}{(n+1)!}$$

where c is between 0 and x .

Therefore, as

$$0 \leq |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

by the Sandwich Theorem

$$\lim_{n \rightarrow \infty} |R_n(x)| = 0$$

Therefore,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

4 Series of Real-valued Functions

Definition 27 (Sequence of functions). A sequence $\{f_n\} = f_1(x), f_2(x), \dots$ defined on $D \subseteq \mathbb{R}$ is called a sequence of functions.

Definition 28 (Pointwise convergence and domain of convergence). $\{f_n\}$ converges pointwise in some domain $E \subseteq D$ if for every $x \in E$, the sequence of $\{f_n(x)\}$ converges. In such a case, E is said to be a domain of convergence of $\{f_n\}$.

Exercise 12.

Find the domain of convergence of $f_n(x) = x^n$, defined on some $D \subseteq \mathbb{R}$.

Solution 12.

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & ; \quad -1 < x < 1 \\ 1 & ; \quad x = 1 \\ \text{diverges} & ; \quad x \notin (-1, 1] \end{cases}$$

Therefore, the domain of convergence of $\{f_n\}$ is $(-1, 1]$.

Exercise 13.

Let $f(x) : (0, \infty) \rightarrow \mathbb{R}$ be some function such that $\lim_{x \rightarrow \infty} f(x) = 0$. Let $f_n(x) = f(nx)$. What is the domain of convergence of f_n ? What is the limit function?

Solution 13.

Let x have some fixed value in $(0, \infty)$. Therefore, as $\lim_{x \rightarrow \infty} f(x) = 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(x) &= \lim_{n \rightarrow \infty} f(nx) \\ &= 0 \end{aligned}$$

Therefore, the domain of convergence is $(0, \infty)$ and the limit function is a constant function with value 0.

4.1 Uniform Convergence of Series of Functions

Definition 29 (Pointwise convergence of a sequence of functions). If $\forall x \in D$, $\forall \varepsilon > 0$, $\exists N$ which depends on ε and x , such that $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$, then $\forall x \in D$, $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

Definition 30 (Uniform convergence of a sequence of functions). The sequence $\{f_n(x)\}$ is said to converge uniformly to $f(x)$ in D if $\forall \varepsilon > 0, \exists N = N(\varepsilon)$, such that $\forall n \geq N, \forall x \in D, |f_n(x) - f(x)| < \varepsilon$. It can be denoted as $f_n(x) \xrightarrow{D} f(x)$.

Theorem 33. $f_n(x)$ converges uniformly to $f(x)$ in D if and only if $\lim_{n \rightarrow \infty} \sup_{x \in D} |f_n(x) - f(x)| = 0$.

Exercise 14.

Does $f_n(x) = x^n$ converge in $[0, 1]$?

Solution 14.

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} x^n$$

$$\therefore f(x) = \begin{cases} 0 & ; \quad 0 \leq x < 1 \\ 1 & ; \quad x = 1 \end{cases}$$

Therefore,

If $x = 0$,

$$f_n(0) = 0$$

$$f(0) = 0$$

Therefore, $\forall \varepsilon > 0, N = 1$,

$$|0 - 0| < \varepsilon$$

$$\therefore |f_n(0) - f(0)| < \varepsilon$$

If $x = 1$,

$$f_n(1) = 1$$

$$f(1) = 1$$

Therefore, $\forall \varepsilon > 0, N = 1$,

$$|1 - 1| < \varepsilon$$

$$\therefore |f_n(1) - f(1)| < \varepsilon$$

If $0 < x < 1$,

$$|f_n(x) - f(x)| = |x^n - 0|$$

$$= x^n$$

If possible, let $|f_n(x) - f(x)| = x^n < \varepsilon$.

Therefore,

$$\begin{aligned} x^n &< \varepsilon \\ \therefore \log_x x^n &> \log_x \varepsilon \\ \therefore n &> \log_x \varepsilon \end{aligned}$$

Therefore, for $N = \lfloor \log_x \varepsilon \rfloor + 1$, $|f_n(x) - f(x)| < \varepsilon$.

Therefore, $f_n(x)$ converges pointwise in $[0, 1]$.

If possible let $f_n(x)$ converge uniformly on $[0, 1]$.

Therefore, $\forall \varepsilon > 0$, $\exists N$ dependent on ε , such that $|f_n(x) - f(x)| < \varepsilon$.

Let $\varepsilon = \frac{1}{3}$.

Therefore, $\exists N$ which is dependent on ε , such that $\forall n > N$, $\forall x \in [0, 1]$,

$$|f_n(x) - f(x)| < \frac{1}{3}$$

Let $x = \frac{1}{2}$, $n = N + 1$. Therefore,

$$\begin{aligned} \left| f_n \left(\frac{1}{2} \right) - f \left(\frac{1}{2} \right) \right| &= \left| \frac{1}{2} - 0 \right| \\ &= \frac{1}{2} \\ \therefore \left| f_n \left(\frac{1}{2} \right) - f \left(\frac{1}{2} \right) \right| &> \frac{1}{3} \end{aligned}$$

Therefore, $|f_n(x) - f(x)| > \varepsilon$.

This is a contradiction. Hence, $f_n(x)$ does not converge uniformly.

Definition 31 (Supremum). Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the supremum of A if

1. $\forall x \in A$, $x \leq M$, i.e. M is an upper bound of A .
2. $\forall \varepsilon$, $\exists x \in A$, such that $x > M - \varepsilon$.

That is, the supremum of A is the least upper bound of A .

The supremum may or may not be in A .

Definition 32 (Infimum). Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the infimum of A if

1. $\forall x \in A, x \geq M$, i.e. M is an upper bound of A .

2. $\forall \varepsilon, \exists x \in A$, such that $x < M - \varepsilon$.

That is, the infimum of A is the greatest lower bound of A . The infimum may or may not be in A .

Theorem 34. *Every bounded set A has a supremum and an infimum.*

Theorem 35. $f_n \xrightarrow{E} f$ if and only if

$$\lim_{n \rightarrow \infty} (\sup\{|f_n(x) - f(x)| : x \in E\}) = 0$$

Definition 33 (Remainder of a series of functions). Let $f(x) = \sum_{k=1}^{\infty} u_k(x)$.

Let the partial sums be denoted by $f_n(x) = \sum_{k=1}^n u_k(x)$. Then

$$R_n(x) = f(x) - f_n(x) = \sum_{k=n+1}^{\infty} u_k(x)$$

is called a remainder of the series $f(x) = \sum_{k=1}^{\infty} u_k(x)$.

Definition 34 (Uniform convergence of a series of functions). If $f_n(x)$ converges uniformly to $f(x)$ on D , i.e. if $\lim_{n \rightarrow \infty} R_n(x) = 0$, then the series $\sum_{k=1}^{\infty} u_k(x)$ is said to converge uniformly on D .

Exercise 15.

Show that the series $f(x) = \sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}$ does not converge uniformly on $(-1, 1)$.

Solution 15.

The series converges uniformly if and only if $\lim_{n \rightarrow \infty} R_n(x) = 0$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{(-1,1)} |R_n(x) - 0| &= \lim_{n \rightarrow \infty} \sup_{(-1,1)} \sum_{k=n+1}^{\infty} x^{k-1} \\ &= \lim_{n \rightarrow \infty} \sup_{(-1,1)} \left| \frac{x^n}{1-x} \right| \\ &= \lim_{n \rightarrow \infty} \sup_{(-1,1)} \frac{|x|^n}{1-x} \\ &= \lim_{n \rightarrow \infty} \infty \\ &= \infty \end{aligned}$$

Therefore, the series does not converge uniformly on $(-1, 1)$.

Exercise 16.

Show that the series $f(x) = \sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}$ does not converge uniformly on $(-\frac{1}{2}, \frac{1}{2})$.

Solution 16.

The series converges uniformly if and only if $\lim_{n \rightarrow \infty} R_n(x) = 0$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{(-\frac{1}{2}, \frac{1}{2})} |R_n(x) - 0| &= \lim_{n \rightarrow \infty} \sup_{(-\frac{1}{2}, \frac{1}{2})} \sum_{k=n+1}^{\infty} x^{k-1} \\ &= \lim_{n \rightarrow \infty} \sup_{(-\frac{1}{2}, \frac{1}{2})} \left| \frac{x^n}{1-x} \right| \\ &= \lim_{n \rightarrow \infty} \sup_{(-\frac{1}{2}, \frac{1}{2})} \frac{|x|^n}{1-x} \\ &= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^{n-1} \\ &= 0 \end{aligned}$$

Therefore, the series converges uniformly on $(-\frac{1}{2}, \frac{1}{2})$.

4.2 Weierstrass M-test

Theorem 36 (Weierstrass M-test). *If $|u_k(x)| \leq c_k$ on D for $k \in \{1, 2, 3, \dots\}$ and the numerical series $\sum_{k=1}^{\infty} c_k$ converges, then the series of functions $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on D .*

Exercise 17.

Show that $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx)$ converges uniformly on \mathbb{R} .

Solution 17.

$$|u_k(x)| = \left| \frac{1}{k^2} \sin(kx) \right|$$

$$\therefore |u_k(x)| \leq \frac{1}{k^2}$$

Therefore, let

$$c_k = \frac{1}{k^2}$$

Therefore, as $|u_k(x)| \leq c_k$, and as $\sum_{k=1}^{\infty} c_k$ converges, by the Weierstrass M-test, $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx)$ converges uniformly.

4.3 Application of Uniform Convergence

Theorem 37 (Continuity of a series). *Let functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ be defined on $[a, b]$ and continuous at $x_0 \in [a, b]$. If $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a, b]$ then the function $f(x) = \sum_{k=1}^{\infty} u_k(x)$ is also continuous at x_0 .*

Theorem 38 (Changing the order of integration and infinite summation). *If the functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ are integrable on $[a, b]$ and the series $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a, b]$ then*

$$\int_a^b \left(\sum_{k=1}^{\infty} u_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b u_k(x) dx$$

Exercise 18.

Solve $\int_0^{2\pi} \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx) \right) dx$.

Solution 18.

The series $f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx)$ converges uniformly on $[0, 2\pi]$. Therefore, by the Weierstrass M-test and $u_k(x) = \frac{1}{k^2} \sin(kx)$ are integrable on $[0, 2\pi]$. There-

fore,

$$\begin{aligned}
\int_0^{2\pi} f(x) \, dx &= \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx) \right) \, dx \\
&= \sum_{k=1}^{\infty} \left(\int_0^{2\pi} \frac{1}{k^2} \sin(kx) \, dx \right) \\
&= \sum_{k=1}^{\infty} \left(-\frac{\cos(2\pi k)}{k^3} + \frac{1}{k^3} \right) \\
&= \sum_{k=1}^{\infty} 0 \\
&= 0
\end{aligned}$$

Theorem 39 (Changing the order of differentiation and infinite summation). *If the functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ are differentiable on $[a, b]$ and the derivatives are continuous on $[a, b]$, and the series $\sum_{k=1}^{\infty} u_k(x)$ converges pointwise on $[a, b]$ and the series $\sum_{k=1}^{\infty} u_k'(x)$ converges uniformly on $[a, b]$, then,*

$$\left(\sum_{k=1}^{\infty} u_k(x) \right)' = \sum_{k=1}^{\infty} u_k'(x)$$

Theorem 40 (Changing the order of integration and limit). *If the functions $f_n(x)$ are integrable on $[a, b]$ and converge uniformly to f on $[a, b]$, then*

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) \, dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) \, dx = \int_a^b f(x) \, dx$$

Theorem 41 (Changing the order of differentiation and limit). *If there exists the functions $f_n'(x)$ which are continuous on $[a, b]$, for the functions $f_n(x)$ which $\forall x \in [a, b]$, converge pointwise to $f(x)$ on $[a, b]$, and if $f_n'(x)$ converges uniformly to $g(x)$ on $[a, b]$, then,*

$$f'(x) = \left(\lim_{n \rightarrow \infty} f_n(x) \right)' = \lim_{n \rightarrow \infty} f_n'(x) = g(x)$$

Part II

Functions of Multiple Variables

1 Limits, Continuity, and Differentiability

Definition 35 (Limit of a function of two variables). Let $z = f(x, y)$ be defined on some open neighbourhood about (a, b) , except maybe at the point itself. $L \in \mathbb{R}$ is said to be a limit of $f(x, y)$ at (a, b) , if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$, then,

$$|f(x, y) - L| < \varepsilon$$

Exercise 19.

Does the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2}$ exist?

Solution 19.

Consider the curves $C_1 : y = 0$, and $C_2 : y = x^3$.

Therefore, as $(x, y) \rightarrow (0, 0)$ along these curves, the limit of the function is

$$\begin{aligned} \lim_{(x,y) \xrightarrow{C_1} (0,0)} \frac{3x^2y}{x^2+y^2} &= \lim_{x \rightarrow 0} \frac{3x^2 \cdot 0}{x^2 + y^2} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{(x,y) \xrightarrow{C_2} (0,0)} \frac{3x^2y}{x^2+y^2} &= \lim_{x \rightarrow 0} \frac{3x^2(x^3)}{x^2 + (x^3)^2} \\ &= \lim_{x \rightarrow 0} \frac{3x^5}{x^2 + x^6} \\ &= \lim_{x \rightarrow 0} \frac{3x^3}{x^2 + x^4} \\ &= 0 \end{aligned}$$

If $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2} = 0$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < \sqrt{x^2 + y^2} < \delta$, then,

$$|f(x, y) - L| < \varepsilon$$

Therefore, checking $|f(x, y) - L|$,

$$\begin{aligned} |f(x, y) - L| &= \left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \\ &= \frac{3x^2|y|}{x^2 + y^2} \end{aligned}$$

As $\frac{x^2}{x^2 + y^2} \leq 1$,

$$\begin{aligned} |f(x, y) - L| &\leq 3|y| \\ \therefore |f(x, y) - L| &\leq 3\sqrt{y^2} \\ \therefore |f(x, y) - L| &\leq 3\sqrt{x^2 + y^2} \end{aligned}$$

Therefore, $|f(x, y) - L| < \varepsilon$.

Therefore, for $\delta \leq \frac{\varepsilon}{3}$, the condition is satisfied.

Hence, the limit of the function exists and is 0.

Definition 36 (Iterative limits). The limits $\lim_{x \rightarrow a} \left(\lim_{y \rightarrow b} f(x, y) \right)$ and $\lim_{y \rightarrow b} \left(\lim_{x \rightarrow a} f(x, y) \right)$ are called the iterative limits of $f(x, y)$.

Theorem 42. If $\exists \lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$ and, for some open interval about b , $\forall y \neq b$, $\exists \lim_{x \rightarrow a} f(x, y)$ then

$$\lim_{y \rightarrow b} \left(\lim_{x \rightarrow a} f(x, y) \right) = L$$

If $\exists \lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$ and, for some open interval about a , $\forall x \neq a$, $\exists \lim_{y \rightarrow b} f(x, y)$ then

$$\lim_{x \rightarrow a} \left(\lim_{y \rightarrow b} f(x, y) \right) = L$$

Exercise 20.

Do the iterative limits, as $x \rightarrow 0$, and as $y \rightarrow 0$, of the function

$$f(x, y) = \begin{cases} (x + y) \sin \frac{1}{x+y} & ; \quad x \neq 0, y \neq 0 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

exists? Does the limit of the function at $(0, 0)$ exist?

Solution 20.

$$\begin{aligned}\lim_{x \rightarrow 0} f(x, y) &= \lim_{x \rightarrow 0} (x + y) \sin \frac{1}{x + y} \\ &= \lim_{x \rightarrow 0} y \sin \frac{1}{x + y}\end{aligned}$$

Therefore, as $\sin \frac{1}{x+y}$ oscillates between -1 and 1 , the limits does not exist.

$$\begin{aligned}\lim_{y \rightarrow 0} f(x, y) &= \lim_{y \rightarrow 0} (x + y) \sin \frac{1}{x + y} \\ &= \lim_{y \rightarrow 0} x \sin \frac{1}{x + y}\end{aligned}$$

Therefore, as $\sin \frac{1}{x+y}$ oscillates between -1 and 1 , the limits does not exists. Therefore, the iterative limits do not exist.

$$\begin{aligned}|f(x, y) - 0| &= |x + y| \cdot \left| \sin \frac{1}{xy} \right| \\ \therefore |f(x, y) - 0| &\leq |x| + |y| \\ \therefore |f(x, y) - 0| &\leq \sqrt{2} \sqrt{x^2 + y^2}\end{aligned}$$

Therefore, for $\delta \leq \frac{\varepsilon}{\sqrt{2}}$, the condition is satisfied.

Hence, the limit of the function exists and is 0 .

Therefore, even though the iterative limits do not exist, the limit of the function exists.

Definition 37 (Differential).

$$\begin{aligned}\Delta z &= f(a + \Delta x, b + \Delta y) - f(a, b) \\ dz &= f_x(a, b) dx + f_y(a, b) dy\end{aligned}$$

Definition 38 (Differentiability). The function $x = f(x, y)$ is said to be differentiable at (a, b) if

$$\Delta z = dz + \varepsilon_1(\Delta x, \Delta y) \Delta x + \varepsilon_2(\Delta x, \Delta y) \Delta y$$

where

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \varepsilon_1(\Delta x, \Delta y) = \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \varepsilon_2(\Delta x, \Delta y) = 0$$

Theorem 43. If $f(x, y)$ is differentiable at (a, b) then $f(x, y)$ is continuous at (a, b) .

Theorem 44. If $\exists f_x(a, b)$ and $\exists f_y(a, b)$ on some open neighbourhood of (a, b) and are continuous at (a, b) , then $f(x, y)$ is differentiable at (a, b) .

2 Directional Derivatives and Gradients

Definition 39 (Directional derivative). Let $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$.

Let $\hat{u} = (a, b)$ be a unit vector in the xy -plane.

The directional derivative of $z = f(x, y)$ with respect to the direction $\hat{u} = (a, b)$ at the point (x_0, y_0) is defined as

$$D_{\hat{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

If the limit does not exist, the directional derivative does not exist.

Geometrically the directional derivative of $z = f(x, y)$ is the slope of the tangent of the curve formed due to the intersection of the surface $z = f(x, y)$, and the plane which passes through (x_0, y_0) in the direction of \hat{u} and is perpendicular to the xy -plane.

Definition 40 (Gradient). If the functions $f_x(x, y)$ and $f_y(x, y)$ for $z = f(x, y)$ exist, then the vector function

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y))$$

is called the gradient of $f(x, y)$.

Theorem 45. Let $z = f(x, y)$ be differentiable at (x_0, y_0) . The function $f(x, y)$ has a directional derivative with respect to any direction $\hat{u} = (a, b)$ at (x_0, y_0) and

$$D_{\hat{u}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \nabla f(x_0, y_0) \cdot \hat{u}$$

Exercise 21.

Find the directional derivative of

$$f(x, y) = x^3 + 4xy + y^4$$

with respect to the direction of $\bar{u} = (1, 2)$ at any point (x, y) and at $(0, 1)$.

Solution 21.

$$f(x, y) = x^3 + 4xy + y^4$$

Therefore,

$$\begin{aligned} f_x(x, y) &= 3x^2 + 4y \\ f_y(x, y) &= 4x + 4y^3 \end{aligned}$$

$$\begin{aligned}
\hat{u} &= \frac{\bar{u}}{u} \\
&= \frac{(1, 2)}{\sqrt{5}} \\
&= \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)
\end{aligned}$$

Therefore,

$$D_{\hat{u}}f(x, y) = \frac{1}{\sqrt{5}}(3x^2 + 4y) + \frac{2}{\sqrt{5}}(4x + 4y^2)$$

Therefore,

$$\begin{aligned}
D_{\hat{u}}f(0, 1) &= \frac{4}{\sqrt{5}} + \frac{8}{\sqrt{5}} \\
&= \frac{12}{\sqrt{5}}
\end{aligned}$$

Theorem 46. If $z = f(x, y)$ is differentiable at (x_0, y_0) , then $\exists \hat{u}_0 = (a_0, b_0)$ such that

$$\max_{\hat{u} \in \mathbb{R}} D_{\hat{u}}f(x_0, y_0) = D_{\hat{u}_0}f(x_0, y_0) = |\nabla f(x_0, y_0)|$$

and

$$\hat{u}_0 = \frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$$

Proof.

$$\begin{aligned}
\max_{\hat{u} \in \mathbb{R}} D_{\hat{u}}f(x_0, y_0) &= \max_{\hat{u} \in \mathbb{R}} \nabla f(x_0, y_0) \cdot \hat{u} \\
&= \max_{\hat{u} \in \mathbb{R}} |\nabla f(x_0, y_0)| |\hat{u}| \cos \theta \\
&= |\nabla f(x_0, y_0)| \max_{\hat{u} \in \mathbb{R}} \cos \theta \\
&= |\nabla f(x_0, y_0)|
\end{aligned}$$

□

Theorem 47. If $z = f(x, y)$ is differentiable at (x_0, y_0) , then $\exists \hat{u}_1 = (a_0, b_0)$ such that

$$\min_{\hat{u} \in \mathbb{R}} D_{\hat{u}}f(x_0, y_0) = D_{\hat{u}_1}f(x_0, y_0) = -|\nabla f(x_0, y_0)|$$

and

$$\hat{u}_1 = -\frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$$

Proof.

$$\begin{aligned}
\min_{\hat{u} \in \mathbb{R}} D_{\hat{u}} f(x_0, y_0) &= \min_{\hat{u} \in \mathbb{R}} \nabla f(x_0, y_0) \cdot \hat{u} \\
&= \min_{\hat{u} \in \mathbb{R}} |\nabla f(x_0, y_0)| |\hat{u}| \overset{1}{\cos \theta} \\
&= |\nabla f(x_0, y_0)| \min_{\hat{u} \in \mathbb{R}} \cos \theta \\
&= -|\nabla f(x_0, y_0)|
\end{aligned}$$

□

3 Local Extrema

Theorem 48 (A necessary condition for local extrema existence). *If the function $z = f(x, y)$ has a local extrema at the point (a, b) and $\exists f_x(a, b)$ and $\exists f_y(a, b)$ then $f_x(a, b) = f_y(a, b) = 0$*

Example 18.

$$z = x^2 + y^2$$

Solution 21.

$$f(x, y) \geq f(0, 0)$$

Therefore, $(0, 0)$ is a point of local minimum.

$$\begin{aligned}
f_x &= 2x \\
f_y &= 2y
\end{aligned}$$

Therefore,

$$f_x(0, 0) = f_y(0, 0) = 0$$

Example 19.

$$z = \sqrt{x^2 + y^2}$$

Solution 21.

$$\lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{(\Delta x)^2}}{\Delta x} = \pm 1$$

Therefore, the limit does not exist.

Definition 41 (Critical point). Let the function $z = f(x, y)$ be defined on some open neighbourhood of (a, b) . The point (a, b) is called a critical point of $z = f(x, y)$ if $f_x(a, b) = f_y(a, b) = 0$ or at least one of the partial derivative $f_x(a, b)$ and $f_y(a, b)$ does not exist.

Example 20. Is $(0, 0)$ an local extremum point of

$$z = f(x, y) = y^2 - x^2$$

?

Solution 21.

$$f_x(0, 0) = 0$$

$$f_y(0, 0) = 0$$

Therefore, $(0, 0)$ is a critical point.

If possible let $(0, 0)$ be a local minimum point.

Then, $f(x, y) \geq f(0, 0)$ in some neighbourhood of $(0, 0)$.

Therefore,

$$y^2 - x^2 \geq 0$$

For any point of the form $(x, 0)$, this is a contradiction.

Therefore $(0, 0)$ is not a local minimum point.

Similarly, $(0, 0)$ is not a local maximum point.

Theorem 49 (A sufficient condition for local extrema point). Assume that there exist second order partial derivatives of $z = f(x, y)$, they are continuous on some open neighbourhood of (a, b) and $f_x(a, b) = f_y(a, b) = 0$. Denote

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - \left(f_{xy}(a, b)\right)^2$$

1. If $D(a, b) > 0$ and $f_{xx} < 0$ then (a, b) is a local maximum point.

2. If $D(a, b) > 0$ and $f_{xx} > 0$ then (a, b) is a local minimum point.

3. If $D(a, b) < 0$ then (a, b) is called a saddle point.

Example 21. Find all critical points of

$$z = f(x, y) = x^4 + y^4 - 4xy + 1$$

and classify them.

Solution 21.

$$f_x(x, y) = 4x^3 - 4y$$

$$f_y(x, y) = 4y^3 - 4x$$

For critical points,

$$f_x(x, y) = 0$$

$$f_y(x, y) = 0$$

Solving, $(0, 0)$, $(1, 1)$, $(-1, -1)$ are critical points.

$$f_{xx}(x, y) = 12x^2$$

$$f_{xy}(x, y) = -4$$

$$f_{yy}(x, y) = 12y^2$$

$$\therefore D(x, y) = 144x^2y^2 - 16$$

For $(0, 0)$,

$$D = -16$$

Therefore, $(0, 0)$ is a saddle point.

For $(1, 1)$,

$$D = 144 - 16$$

Therefore, $(1, 1)$ is a local minimum point.

For $(-1, -1)$,

$$D = 144 - 16$$

Therefore, $(-1, -1)$ is a local minimum point.

4 Global Extrema

4.1 Algorithm for Finding Maxima and Minima of a Function

Step 1 Find all critical points of $f(x, y)$ on the domain, excluding the end points.

Step 2 Calculate the values of $f(x, y)$ at the critical points.

Step 3 Calculate the values of $f(x, y)$ at the end points of the domain.

Step 4 Select the maximum and minimum values from Step 2 and Step 3

Example 22. Find the global maxima and minima of

$$z = x^2 - 2xy + 2y$$

in the domain

$$D = \left\{ (x, y) \left| 0 \leq x \leq 3, 0 \leq y \leq -\frac{2}{3}x + 2 \right. \right\}$$

Solution 21.

$$f_x(x, y) = 0$$

$$\therefore 2x - 2y = 0$$

$$f_y(x, y) = 0$$

$$\therefore -2x + 2 = 0$$

Therefore, $(1, 1)$ is a critical point in D .

The boundary of D is $L_1 \cup L_2 \cup L_3$, where

$$L_1 : y = 0, 0 \leq x \leq 3$$

$$L_2 : x = 0, 0 \leq y \leq 2$$

$$L_3 :$$

Therefore,

over L_1 ,

$$f(x, y) = x^2$$

$$\therefore \min_{L_1} f = f(0, 0) = 0$$

$$\therefore \max_{L_1} f = f(3, 0) = 9$$

over L_2 ,

$$\begin{aligned}f(x, y) &= 2y \\ \therefore \min_{L_2} f &= f(0, 0) = 0 \\ \therefore \max_{L_2} f &= f(0, 2) = 4\end{aligned}$$

over L_3 ,

$$\begin{aligned}f(x, y) &= x^2 - 2x \left(-\frac{2}{3}x + 2 \right) + 2 \left(-\frac{2}{3}x + 2 \right) \\ &= \frac{7}{3}x^2 - \frac{16}{3}x + 4 \\ \therefore f' &= \frac{14}{3}x - \frac{16}{3} \\ \therefore f' \left(\frac{8}{7} \right) &= 0 \\ \therefore f \left(\frac{8}{7}, \frac{26}{21} \right) &= 0.952 \\ \therefore \min_{L_3} f &= f \left(\frac{8}{7}, \frac{26}{21} \right) = 0.952 \\ \therefore \max_{L_3} f &= f(3, 0) = 9\end{aligned}$$

Therefore,

$$\begin{aligned}\therefore \min_D f &= f(0, 0) = 0 \\ \therefore \max_D f &= f(3, 0) = 9\end{aligned}$$

5 Taylor's Formula

Theorem 50.

$$\begin{aligned}f(a + h, b + k) &= \sum_{i=0}^n \left(\frac{1}{i!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(a, b) \right) \\ &\quad + \frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(a + ch, b + ck)\end{aligned}$$

where $0 < c < 1$.

6 Vector Functions and Curves in \mathbb{R}^3

Definition 42 (Vector function). A vector function is a function with a domain which consists of a set of real numbers, and with a domain which consists of a set of vectors, i.e. $\bar{r}(t) = (f(t), g(t), h(t)), \forall t \in [a, b]$.

Theorem 51. If $\exists \lim_{t \rightarrow t_0} f(t), \exists \lim_{t \rightarrow t_0} g(t), \exists \lim_{t \rightarrow t_0} h(t)$, then, $\exists \lim_{t \rightarrow t_0} \bar{r}(t) = \left(\lim_{t \rightarrow t_0} f(t), \lim_{t \rightarrow t_0} g(t), \lim_{t \rightarrow t_0} h(t) \right)$.

Definition 43 (Continuous vector function). A vector function $\bar{r}(t)$ is said to be continuous at t_0 if $\lim_{t \rightarrow t_0} \bar{r}(t) = \bar{r}(t_0)$.

Definition 44 (Space curve). Let $f(t), g(t), h(t)$ be continuous functions of $[a, b]$. The set of points (x, y, z) , such that $x = f(t), y = g(t), z = h(t), t \in [a, b]$ is called a space curve.

7 Derivatives of Vector Functions

Definition 45 (Derivative of vector function). The derivative of $\bar{r}(t) = (f(t), g(t), h(t))$, if it exists, is defined as

$$\bar{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\bar{r}(t + \Delta t) - \bar{r}(t)}{\Delta t}$$

Definition 46 (Tangent vector). $\bar{r}'(t_0)$ is called a tangent vector to the curve $C = \bar{r}(t)$ at $P(t_0)$.

Theorem 52. If $\exists f'(t_0), \exists g'(t_0), \exists h'(t_0)$, and $\bar{r}(t) = (f(t), g(t), h(t))$, then,

$$\bar{r}'(t_0) = (f'(t_0), g'(t_0), h'(t_0))$$

Definition 47 (Unit tangent vector). The vector $\hat{T}(t) = \frac{\bar{r}'(t)}{|\bar{r}'(t)|}$ is called the unit tangent vector to $C = r(t)$ at $P(t_0)$.

Definition 48 (Tangent line). A straight line passing through a point $P(t)$ on the curve $C = r(t)$, in the direction $\bar{r}'(t)$, i.e. $\hat{T}(t)$, is called a tangent line to the curve at the point.

Theorem 53. Let $\bar{u}(t)$ and $\bar{v}(t)$ be vector functions, let c be a constant, and let $f(t)$ be a scalar function. Then,

1. $(\bar{u}(t) \pm \bar{v}(t))' = \bar{u}'(t) \pm \bar{v}'(t)$
2. $(c\bar{u}(t))' = c\bar{u}'(t)$

3. $(f(t)\overline{u}(t))' = f'(t)\overline{u}(t) + f(t)\overline{u}'(t)$
4. $(\overline{u}(t) \cdot \overline{v}(t))' = \overline{u}'(t) \cdot \overline{v}(t) + \overline{u}(t) \cdot \overline{v}'(t)$
5. $(\overline{u}(t) \times \overline{v}(t))' = \overline{u}'(t) \times \overline{v}(t) + \overline{u}(t) \times \overline{v}'(t)$
6. $(\overline{u}(f(t)))' = f'(t)\overline{u}'(f(t))$

8 Change of Variables in Double Integrals

Definition 49 (Jacobian). Let

$$T(u, v) = (x, y)$$

be an operator.

The determinant

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

is called the Jacobian of the operator T .

Theorem 54. *Let R and S be domains of the first or second kind.*

Let the operator T from S to R be one-to-one and onto.

Therefore, the inverse operator T^{-1} exists.

Also, let T be a C^1 operator, i.e. $\exists x_u, \exists x_v, \exists y_u, \exists y_v$, which are continuous on S .

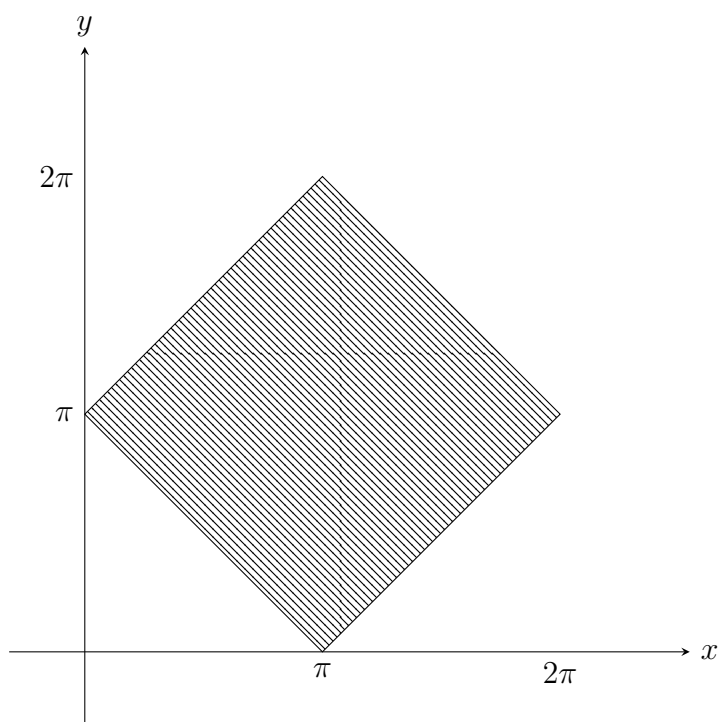
Let $f(x, y)$ be a continuous function on R .

Then,

$$\iint_R f(x, y) \, dx \, dy = \iint_S f(g(u, v), h(u, v)) |J| \, du \, dv$$

Exercise 22.

Calculate $\iint_R (x - y)^2 \sin^2(x + y) \, dx \, dy$, where R is as shown.



Solution 22.

The edges of the domain are

$$x + y = \pi$$

$$x + y = 3\pi$$

$$x - y = \pi$$

$$x - y = -\pi$$

Therefore, let

$$x - y = u$$

$$x + y = v$$

Therefore,

$$x = \frac{u + v}{2}$$

$$y = \frac{v - u}{2}$$

Therefore, the domain R can be written as $S = \{-\pi \leq u \leq \pi, \pi \leq v \leq 3\pi\}$.
Therefore,

$$\begin{aligned} J &= \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} \iint_R f(x, y) \, dx \, dy &= \iint_S f(g(u, v), h(u, v)) |J| \, du \, dv \\ \therefore \iint_R (x - y)^2 \sin^2(x + y) \, dx \, dy &= \int_S u^2 \sin^2 v \left| \frac{1}{2} \right| \, du \, dv \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} u^2 \sin^2 v \, dv \, du \\ &= \frac{1}{2} \int_{-\pi}^{\pi} u^2 \, du \cdot \int_{\pi}^{3\pi} \sin^2 v \, dv \\ &= \frac{1}{2} \left. \frac{u^3}{3} \right|_{-\pi}^{\pi} \cdot \int_{\pi}^{3\pi} \frac{1 - \cos 2v}{2} \, dv \\ &= \frac{1}{2} \frac{2\pi^3}{3} \cdot \frac{1}{2} 2\pi \\ &= \frac{\pi^4}{3} \end{aligned}$$

8.1 Polar Coordinates

Polar coordinates are a special case of change of variables.
The operator for the change of variables is

$$T(r, \theta) = (x, y)$$

where

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

Therefore,

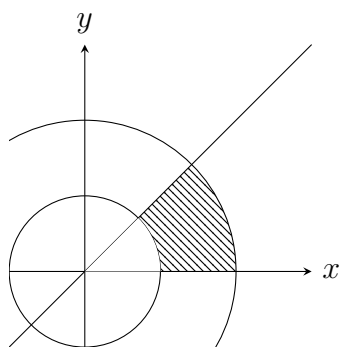
$$\begin{aligned}
 J &= \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} \\
 &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\
 &= r \cos^2 \theta + r \sin^2 \theta \\
 &= r
 \end{aligned}$$

Exercise 23.

Calculate $\iint_R xy \, dx \, dy$, $R = \{(x, y) | 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$.

Solution 23.

The domain R is the region shown.



Therefore, it can be written as $S = \{(r, \theta) | 1 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{4}\}$.
Therefore,

$$\begin{aligned}
 \iint_R xy \, dx \, dy &= \int_0^{\frac{\pi}{4}} \int_1^2 r \cos \theta r \sin \theta \, dr \, d\theta \\
 &= \int_1^2 r^3 \, dr \cdot \int_0^{\frac{\pi}{4}} \cos \theta \sin \theta \, d\theta \\
 &= \frac{15}{4} \cdot \frac{1}{4} \\
 &= \frac{15}{16}
 \end{aligned}$$

Theorem 55. Let D be a domain, written as D_I in polar coordinates, i.e.,

$$D_I = \{(r, \theta) | a \leq r \leq b, g_1(r) \leq \theta \leq g_2(r)\}$$

and let $f(x, y)$ be continuous on D_I .

Then,

$$\iint_{D_I} f(x, y) \, dx \, dy = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r \cos \theta, r \sin \theta) r \, d\theta \, dr$$

Theorem 56. Let D be a domain, written as D_{II} in polar coordinates, i.e.,

$$D_{II} = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

and let $f(x, y)$ be continuous on D_{II} .

Then,

$$\iint_{D_{II}} f(x, y) \, dx \, dy = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$$

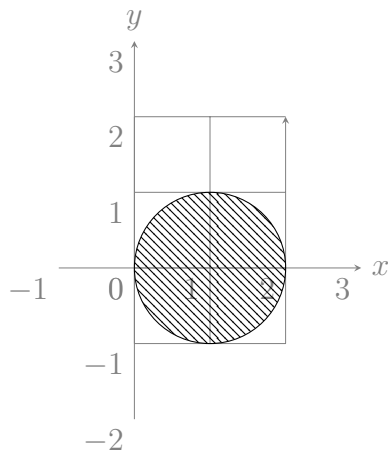
Exercise 24.

Given $D = \{x^2 + y^2 \leq 2x\}$, calculate $\iint_D (x + y) \, dx \, dy$.

Solution 24.

$$\begin{aligned} x^2 + y^2 &= 2x \\ \therefore x^2 - 2x + y^2 &= 0 \\ \therefore (x - 1)^2 + y^2 &= 1 \end{aligned}$$

Therefore, the domain D is as shown.



Therefore, D can be written as $\left\{0 \leq r \leq 2 \cos \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right\}$.
Therefore,

$$\begin{aligned} \iint_D (x+y) \, dx \, dy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2 \cos \theta} (r \cos \theta + r \sin \theta) r \, dr \, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \theta + \sin \theta) \left(\frac{r^3}{3} \right) \bigg|_{z=0}^{z=2 \cos \theta} d\theta \end{aligned}$$

Solving,

$$\iint_D (x+y) \, dx \, dy = \pi$$

9 Change of Variables in Triple Integrals

Definition 50 (Jacobian). Let

$$T(u, v, w) = (x, y, z)$$

be an operator.

The determinant

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

is called the Jacobian of the operator T .

Theorem 57. *Let R and S be domains of the first, second, or third kind.*

Let the operator T from S to R be one-to-one and onto.

Therefore, the inverse operator T^{-1} exists.

Also, let T be a C^1 operator, i.e. $\exists x_u, \exists x_v, \exists x_w, \exists y_u, \exists y_v, \exists y_w, \exists z_u, \exists z_v, \exists z_w$, which are continuous on S .

Let $f(x, y, z)$ be a continuous function on R .

Then,

$$\iiint_R f(x, y, z) \, dx \, dy \, dz = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) |J| \, du \, dv \, dw$$

9.1 Cylindrical Coordinates

Cylindrical coordinates are a special case of change of variables. The operator for the change of variables is

$$T(r, \theta, z) = (x, y, z)$$

where

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

Therefore,

$$\begin{aligned} J &= \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} \\ &= \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= r \cos^2 \theta + r \sin^2 \theta \\ &= r \end{aligned}$$

Exercise 25.

Calculate the iterative integral

$$I = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) \, dz \, dy \, dx$$

Solution 25.

The domain $\{(x, y) \mid -2 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}\}$ is a circle of radius 2.

As $\sqrt{x^2+y^2} \leq z \leq 2$, the domain E , where $-2 \leq x \leq 2$, $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$, $\sqrt{x^2+y^2} \leq z \leq 2$ is a cone, with the circular cross section of radius $\sqrt{x^2+y^2}$.

Therefore,

$$\begin{aligned}
 I &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) \, dz \, dy \, dx \\
 &= \iiint_E (x^2 + y^2) \, dx \, dy \, dz
 \end{aligned}$$

Therefore, let $D_1 = \{(r, \theta, z) | 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi, r \leq z \leq 2\}$.

Therefore,

$$\begin{aligned}
 I &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) \, dz \, dy \, dx \\
 &= \iiint_E (x^2 + y^2) \, dx \, dy \, dz \\
 &= \iiint_{D_1} r^2 \cdot r \, dr \, d\theta \, dz \\
 &= \int_0^{2\pi} \int_0^2 \int_r^2 r^3 \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 r^3 z \Big|_{z=r}^{z=2} \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 (2r^3 - r^4) \, dr \, d\theta \\
 &= \int_0^{2\pi} \left(\frac{r^4}{2} - \frac{r^5}{5} \right) \Big|_0^2 \, d\theta \\
 &= \int_0^{2\pi} \left(8 - \frac{32}{5} \right) \, d\theta \\
 &= \frac{8}{5} \cdot 2\pi \\
 &= \frac{16\pi}{5}
 \end{aligned}$$

9.2 Spherical Coordinates

Spherical coordinates are a special case of change of variables. The operator for the change of variables is

$$T(\rho, \theta, \varphi) = (x, y, z)$$

where

$$x = \rho \cos \theta \sin \varphi$$

$$y = \rho \sin \theta \sin \varphi$$

$$z = \rho \cos \varphi$$

Therefore,

$$\begin{aligned} J &= \begin{vmatrix} x_\rho & x_\theta & x_\varphi \\ y_\rho & y_\theta & y_\varphi \\ z_\rho & z_\theta & z_\varphi \end{vmatrix} \\ &= \begin{vmatrix} \cos \theta \sin \varphi & -r \sin \theta \sin \varphi & r \cos \theta \cos \varphi \\ \sin \theta \sin \varphi & r \cos \theta \sin \varphi & r \sin \theta \cos \varphi \\ \cos \varphi & 0 & -r \sin \varphi \end{vmatrix} \\ &= -\rho^2 \sin \varphi \end{aligned}$$

Exercise 26.

Given the sphere $B : x^2 + y^2 + z^2 \leq 1$, find $I = \iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dx dy dz$.

Solution 26.

$$\begin{aligned}
I &= \iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dx dy dz \\
&= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{\rho^{\frac{3}{2}}} |J| d\rho d\theta d\varphi \\
&= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{\rho^{\frac{3}{2}}} \rho^2 \sin \varphi d\rho d\theta d\varphi \\
&= \int_0^\pi \int_0^{2\pi} \frac{e^{\rho^3}}{3} \sin \varphi \bigg|_{\rho=0}^{\rho=1} d\theta d\varphi \\
&= \frac{e-1}{3} \int_0^\pi \int_0^{2\pi} \sin \varphi d\theta d\varphi \\
&= \frac{e-1}{3} \int_0^\pi \sin \varphi \cdot 2\pi d\varphi \\
&= 2\pi \frac{e-1}{3} (-\cos \theta) \big|_0^\pi \\
&= \frac{4\pi(e-1)}{3}
\end{aligned}$$

Exercise 27.

Calculate the volume of a body which is situated above the cone $z = \sqrt{x^2 + y^2}$ and under the sphere $x^2 + y^2 + z^2 = z$.

Solution 27.

$$\begin{aligned}
x^2 + y^2 + z^2 &= z \\
\therefore x^2 + y^2 + z^2 - z &= 0 \\
\therefore x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 &= \frac{1}{4}
\end{aligned}$$

Therefore, the sphere has centre $(0, 0, \frac{1}{2})$ and radius $\frac{1}{2}$.

Therefore, the cone and the sphere intersect each other at $z = \frac{1}{2}$. The intersection is a circle with radius $\frac{1}{2}$.

Therefore, the body is made of a cone of base radius $\frac{1}{2}$ and height $\frac{1}{2}$, and a hemisphere of radius $\frac{1}{2}$.

In Cartesian coordinates, the sphere is $x^2 + y^2 + z^2 = z$.

Therefore, in spherical coordinates, the sphere is $\rho^2 = \rho \cos \varphi$. Therefore,

$$\begin{aligned}
V &= \iiint dx \, dy \, dz \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos \varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{\rho^3}{3} \sin \varphi \Big|_{\rho=0}^{\rho=\cos \varphi} d\varphi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{1}{3} \cos^3 \varphi \sin \varphi \, d\varphi \, d\theta \\
&= 2\pi \cdot \left(-\frac{\cos^4 \varphi}{4} \right) \Big|_0^{\frac{\pi}{4}} \\
&= 2\pi \left(-\frac{1}{48} + \frac{1}{12} \right) \\
&= 2\pi \left(\frac{3}{48} \right) \\
&= \frac{\pi}{8}
\end{aligned}$$

10 Line Integrals of Scalar Functions

Definition 51 (Line integral of scalar functions). Let C be a curve. Let the curve be divided into n parts, by points P_i .

Let Δs_i be the length of the curve $P_{i-1}P_i$. Let $P_i^* (x_i^*, y_i^*, z_i^*)$ be a point on the curve $P_{i-1}P_i$.

Let

$$\Delta T = \max \{ \Delta s_i \}$$

$$\Delta x_i = x_i - x_{i-1}$$

$$\Delta y_i = y_i - y_{i-1}$$

$$\Delta z_i = z_i - z_{i-1}$$

The line integral of a $f(x, y, z)$ over C is defined as

$$\int_C f(x, y, z) \, ds = \lim_{\Delta T \rightarrow 0} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

The integral

$$\int_C f(x, y, z) \, dx = \lim_{\Delta T \rightarrow 0} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta x_i$$

is called the line integral of $f(x, y, z)$ over C with respect to x .

This integral depends on the direction of C .

The integral

$$\int_C f(x, y, z) \, dy = \lim_{\Delta T \rightarrow 0} \sum_{i=1}^n f(y_i^*, y_i^*, z_i^*) \Delta y_i$$

is called the line integral of $f(x, y, z)$ over C with respect to y .

This integral depends on the direction of C .

The integral

$$\int_C f(x, y, z) \, dz = \lim_{\Delta T \rightarrow 0} \sum_{i=1}^n f(z_i^*, y_i^*, z_i^*) \Delta z_i$$

is called the line integral of $f(x, y, z)$ over C with respect to z .

This integral depends on the direction of C .

Geometrically, the line integral $\int_C f(x, y) \, ds$ is the area under the curve $z = f(x, y)$ above the curve C .

Definition 52 (Smooth curve). A curve C which is parametrically given as $\bar{r}(t) = (x(t), y(t), z(t))$, $t : a \rightarrow b$ is said to be smooth if $\bar{r}(t)$ is a continuous function on $[a, b]$, $\bar{r}'(t) \neq 0$ on (a, b) , and $\bar{r}'(t)$ is continuous on (a, b) .

Theorem 58. If $f(x, y, z)$ is continuous and C is smooth, then

$$\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt$$

Theorem 59. If $f(x, y, z)$ is continuous and C is smooth, then

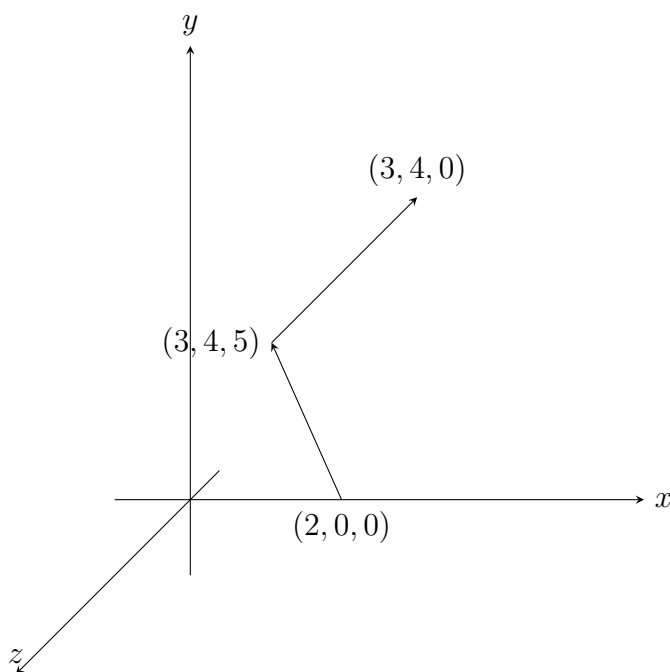
$$\int_C f(x, y, z) \, dx = \int_a^b f(x(t), y(t), z(t)) x'(t) \, dt$$

$$\int_C f(x, y, z) \, dy = \int_a^b f(x(t), y(t), z(t)) y'(t) \, dt$$

$$\int_C f(x, y, z) \, dz = \int_a^b f(x(t), y(t), z(t)) \, z'(t) \, dt$$

Exercise 28.

Calculate $\int_C y \, dx + z \, dy + x \, dz$ for C as shown.



Solution 28.

$$C = C_1 \cup C_2$$

Therefore, for $t : 0 \rightarrow 1$,

$$C_1 : \bar{r}(t) = (2 + 1 \cdot t, 0 + 4 \cdot t, 0 + 5 \cdot t)$$

$$C_2 : \bar{r}(t) = (3 + 0 \cdot t, 4 + 0 \cdot t, 5 - 5 \cdot t)$$

Therefore,

$$\begin{aligned}
\int_C y \, dx + z \, dy + x \, dz &= \int_{C_1} y \, dx + z \, dy + x \, dz + \int_{C_2} y \, dx + z \, dy + x \, dz \\
&= \int_0^1 \left(y_1(t)x_1'(t) + z_1(t)y_1'(t) + x_1(t)z_1'(t) \right) dt \\
&\quad + \int_0^1 \left(y_2(t)x_2'(t) + z_2(t)y_2'(t) + x_2(t)z_2'(t) \right) dt \\
&= \int_0^1 (4t + 5t \cdot 4 + (2+t) \cdot 5) \, dt \\
&\quad + \int_0^1 (4 \cdot 0 + (5-5t) \cdot 0 + 3 \cdot (-5)) \, dt \\
&= \int_0^1 (29t - 5) \, dt \\
&= \left(29\frac{t^2}{2} - 5t \right) \Big|_0^1 \\
&= \frac{19}{2}
\end{aligned}$$