Differential and Integral Calculus

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Contents

1 Lecturer Information

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2 Required Reading

Protter and Morrey: *A first Course in Real Analysis*, UTM Series, Springer-Verlag, 1991

3 Additional Reading

Thomas and Finney, *Calculus and Analytic Geometry*, 9th edition, Addison-Wesley, 1996

Part I
Sequences and Series

1 Sequences

Definition 1 (Sequence)**.** A sequence of real numbers is a set of numbers which are written in some order. There are infinitely many terms in a sequence. It is denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Example 1. $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is called the harmonic sequence.

$$a_n = \frac{1}{n}$$

Example 2. $1, -\frac{1}{2}, \frac{1}{3}, \ldots$ is called the alternating harmonic sequence.

$$a_n = (-1)^{n+1}\frac{1}{n}$$

Example 3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$$a_n = \frac{n}{n+1}$$

Example 4. $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \ldots$

$$a_n = \frac{n+1}{3^n}$$

Example 5. The Fibonacci sequence is given by

$$f_n = \begin{cases} 1 & ; \quad n = 1, 2 \\ f_{n-1} + f_{n-2} & ; \quad n \geq 3 \end{cases}$$

Example 6. A geometric sequence is given by

$$a_n = a_1 q^{n-1}$$

where q is called the common ratio.

Example 7. A geometric sequence is given by

$$a_n = a_1 + d(n-1)$$

where d is called the common difference.

Definition 2 (Equal sequences)**.** Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equal if $a_n = b_n$, $\forall n \in \mathbb{N}$.

Definition 3 (Sequences bounded from above)**.** $\{a_n\}$ is said to be bounded from above if $\exists M \in \mathbb{R}$, s.t. $a_n \leq M$, $\forall n \in \mathbb{N}$. Each such M is called an upper bound of $\{a_n\}$.

Definition 4 (Sequences bounded from below)**.** $\{a_n\}$ is said to be bounded from below if $\exists m \in \mathbb{R}$, s.t. $a_n \geq M$, $\forall n \in \mathbb{N}$. Each such M is called an lower bound of $\{a_n\}$.

Definition 5. $\{a_n\}$ is said to be bounded if it is bounded from below and bounded from above.

Example 8. The sequence $a_n = n^2 + 2$ is not bounded from above but is bounded from below, by all $m \leq 3$.

Example 9. $\left\{\dfrac{2n-1}{3n}\right\}$ is bounded.

$$m = 0 \leq \frac{2n-1}{3n} \leq \frac{2n}{3n} = \frac{2}{3} = M$$

Definition 6 (Monotonic increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \leq a_{n+1}$, $\forall n \geq n_0$.

Definition 7 (Monotonic decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \geq a_{n+1}$, $\forall n \geq n_0$.

Definition 8 (Strongly increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n < a_{n+1}$, $\forall n \geq n_0$.

Definition 9 (Strongly decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n > a_{n+1}$, $\forall n \geq n_0$.

Example 10. The sequence $\left\{\frac{n^2}{2^n}\right\}$ is strongly decreasing. However, this is not evident by observing the first few terms. $\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \dots$

$$\begin{aligned} a_n &> a_{n+1} \\ \iff \frac{n^2}{2^n} &> \frac{(n+1)^2}{2^{n+1}} \\ \iff 2n^2 &> (n+1)^2 \\ \iff \sqrt{2}n &> n+1 \\ \iff n(\sqrt{2}-1) &> 1 \\ \iff n &> \frac{1}{\sqrt{2}-1} \\ \iff n &> 3 \end{aligned}$$

Exercise 1.
Is $a_n = (-1)^n$ monotonic?

Solution 1.
The sequence $-1, 1, -1, 1, \dots$ is not monotonic.

1.1 Limit of a Sequence

Definition 10. Let $\{a_n\}$ be a given sequence. A number L is said to be the limit of the sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $|a_n - L| < \varepsilon$, $\forall n \geq n_0$. That is, there are infinitely many terms inside the interval and a finite number of terms outside it.

Example 11. The sequence $\{\frac{1}{n}\}$ tends to 0, i.e. for any open interval $(-\varepsilon, \varepsilon)$, there are finite number of terms of the sequence outside the interval, and therefore there are infinitely many terms inside the interval.

Exercise 2.
Prove

$$\lim_{n\to\infty} \frac{n+2}{2n-1} = \frac{1}{2}$$

Solution 2.
$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$

Exercise 3.
Prove that 2 is not a limit of $\left\{\dfrac{3n+1}{n}\right\}$.

Solution 3.
If possible, let

$$\lim_{n\to\infty} \frac{3n+1}{n} = 2$$

Then, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\left|\dfrac{3n+1}{n} - 2\right| < \varepsilon$, $\forall n \geq n_0$. However,

$$\left|\frac{3n+1}{n} - 2\right| = 1 + \frac{1}{n} > 1$$

This is a contradiction for $\varepsilon = \dfrac{1}{2}$. Therefore, 2 is not a limit.

Theorem 1. *If a sequence $\{a_n\}$ has a limit L then the limit is unique.*

Proof. If possible let there exist two limits L_1 and L_2. Therefore, $\forall \varepsilon > 0$, there exist a finite number of terms in the interval $(L_1 - \varepsilon, L_1 + \varepsilon)$. Therefore, there exist a finite number of terms in the interval $(L_2 - \varepsilon, L_2 + \varepsilon)$. This contradicts the definition of a limit. Therefore, the limit is unique. □

Theorem 2. *If a sequence $\{a_n\}$ has limit L, then the sequence is bounded.*

Theorem 3. *Let*

$$\lim_{n\to\infty} a_n = a$$
$$\lim_{n\to\infty} b_n = b$$

and let c be a constant. Then,

$$\begin{aligned}
\lim c &= c \\
\lim(ca_n) &= c \lim a_n \\
\lim(a_n \pm b_n) &= \lim a_n \pm \lim b_n \\
\lim(a_n b_n) &= \lim a_n \lim b_n \\
\lim\left(\frac{a_n}{b_n}\right) &= \frac{\lim a_n}{\lim b_n} \quad (\text{ if } \lim b \neq 0)
\end{aligned}$$

Theorem 4. *Let $\{b_n\}$ be bounded and let $\lim a_n = 0$. Then,*

$$\lim(a_n b_n) = 0$$

Theorem 5 (Sandwich Theorem)**.** *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences. If*

$$\lim a_n = \lim b_n = L$$

and $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n \leq b_n \leq c_n$. Then,

$$\lim b_n = L$$

Exercise 4.
Calculate $\lim\limits_{n\to\infty} \sqrt[n]{2^n + 3^n}$

Solution 4.

$$\begin{aligned}
&\sqrt[n]{3^n} \leq \sqrt[n]{2^n + 3^n} \leq \sqrt[n]{3^n + 3^n} = \sqrt[3]{2 \cdot 3^n} \\
&\therefore 3 \leq \sqrt[n]{2^n + 3^n} \leq 3\sqrt[n]{2}
\end{aligned}$$

Therefore, by the Sandwich Theorem, $\lim\limits_{n\to\infty} \sqrt[n]{2^n + 3^n} = 3$.

Theorem 6. *Any monotonically increasing sequence which is bounded from above converges. Similarly, any monotonically decreasing sequence which is bounded from below converges.*

Exercise 5.

Prove that there exists a limit for $a_n = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}_{n \text{ times}}$ and find it.

Solution 5.

$$a_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = a_2$$

If possible, let

$$\begin{aligned}
a_{n-1} &< a_n \\
\therefore \sqrt{2 + a_{n-1}} &< \sqrt{2 + a_n} \\
\therefore a_n &< a_{n+1}
\end{aligned}$$

Hence, by induction, $\{a_n\}$ is monotonically increasing.

$$a_1 = \sqrt{2} \leq 2$$

If possible, let

$$a_n \leq 2 \therefore \sqrt{2 + a_n} \qquad \leq \sqrt{2+2}$$
$$\therefore a_{n+1} \leq 2$$

Hence, by induction, $\{a_n\}$ is bounded from above by 2. Therefore, by , $\{a_n\}$ converges.

Definition 11 (Limit in a wide sense)**.** The sequence $\{a_n\}$ is said to converge to $+\infty$ if $\forall M \in \mathbb{R}$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n > M$.
The sequence $\{a_n\}$ is said to converge to $-\infty$ if $\forall M \in \mathbb{R}$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n < M$.

1.2 Sub-sequences

Definition 12 (Sub-sequence)**.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_k\}_{k=1}^{\infty}$ be a strongly increasing sequence of natural numbers. Let $\{b_k\}_{k=1}^{\infty}$ be a sequence such that $b_k = a_{n_k}$. Then $\{b_k\}_{k=1}^{\infty}$ is called a sub-sequence of $\{a_n\}_{n=1}^{\infty}$.

Example 12.

$$a_n = \frac{1}{n}$$

If we choose $n_k = k^2$,

$$b_k = a_{n_k} = a_{k^2} = \frac{1}{k^2}$$

Therefore,

$$\{b_k\} = 1, \frac{1}{4}, \frac{1}{9}, \dots$$

Theorem 7. *If the sequence $\{a_n\}$ converges to L in a wide sense, i.e. L can be infinite, then any sub-sequence of $\{a_n\}$ converges to the same limit L.*

Definition 13 (Partial limit)**.** A real number a, which may be infinite, is called a partial limit of the sequence $\{a_n\}$ is there exists a sub-sequence of $\{a_n\}$ which converges to a.

Example 13. Let

$$a_n = (-1)^n$$

Therefore, $\nexists \lim_{n\to\infty} a_n$. Let

$$b_k = a_{n_k} = a_{2n-1}$$

Therefore,

$$\{b_k\} = -1, -1, -1, \dots$$
$$\therefore \lim_{k\to\infty} b_k = 1$$

Therefore, -1 is a partial limit of $\{a_n\}$.

Theorem 8 (Bolzano-Weierstrass Theorem)**.** *For any bounded sequence there exists a subsequence which is convergent, s.t. there exists at least one partial limit.*

Definition 14 (Upper partial limit)**.** The largest partial limit of a sequence is called the upper partial limit. It is denoted by $\overline{\lim} a_n$ or $\limsup a_n$.

Definition 15 (Lower partial limit)**.** The smallest partial limit of a sequence is called the upper partial limit. It is denoted by $\underline{\lim} a_n$ or $\liminf a_n$.

Theorem 9. *If the sequence $\{a_n\}$ is bounded and*

$$\overline{\lim} a_n = \underline{\lim} a_n = a$$

then

$$\exists \lim a_n = a$$

1.3 Cauchy Characterisation of Convergence

Definition 16. A sequence $\{a_n\}$ is called a Cauchy sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall m, n \geq n_0$, $|a_n - a_m| < \varepsilon$.

Theorem 10 (Cauchy Characterisation of Convergence)**.** *A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.*

Proof. Let

$$\lim_{n\to\infty} a_n = L$$

Then $\forall\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$, $|a_n - L| < \dfrac{\varepsilon}{2}$. Therefore if $n \geq n_0$ and $m \geq n_0$, then

$$\begin{aligned}
|a_n - a_m| &= |a_n - L + L - a_m| \\
&\leq |a_n - L| + |L - a_m| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
\therefore |a_n - a_m| &= \varepsilon
\end{aligned}$$

Similarly, the converse can be proved by Theorem 9. □

Theorem 11 (Another Formulation of the Cauchy Characterisation Theorem)**.** *The sequence $\{a_n\}$ converges if and only if $\forall\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$.*

Exercise 6.
Prove that the sequence

$$a_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

is convergent.

Solution 6.

$$\begin{aligned}
|a_{n+p} - a_n| &= \left|\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(n+p)^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right)\right| \\
&= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(n+p-1)(n+p)} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} + \cdots + \cancel{\frac{1}{n+p-1}} - \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n} - \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n}
\end{aligned}$$

Therefore, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$, where $n_0 > \frac{1}{\varepsilon}$. $\square$

Exercise 7.
Prove that the sequence

$$a_n = \frac{1}{1} + \frac{1}{n} + \cdots + \frac{1}{n}$$

diverges.

Solution 7.
If possible, let the sequence converge. Then, by the Cauchy Characterisation of Convergence, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$. Therefore,

$$\begin{aligned}
|a_{n+p} - a_n| &= \left| \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+p} - \left(\frac{1}{n} + \cdots + \frac{1}{n} \right) \right| \\
&= \frac{1}{n+1} + \cdots + \frac{1}{n+p} \\
&\geq p \cdot \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &> \frac{p}{n+p}
\end{aligned}$$

If $n = p$,

$$\frac{p}{n+p} = \frac{1}{2}$$

This contradicts the result obtained from the Cauchy Characterisation of Convergence, for $\varepsilon = \frac{1}{4}$.
Therefore, the sequence diverges.

2 Series

Definition 17 (Series)**.** Given a sequence $\{a_n\}$, the sum $a_1 + \cdots + a_n + \ldots$ is called an infinite series or series. It is denoted as $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

Definition 18 (Partial sum)**.** The partial sum of the series $\sum a_n$ is defined as

$$S_i = a_1 + \cdots + a_i$$

Definition 19 (Convergent and divergent series)**.** If the sequence $\{S_n\}_{n=1}^{\infty}$ converges, then the series is called convergent. Otherwise, the series is called divergent.

Definition 20 (Sum of a series)**.** If the sequence $\{S_n\}_{n=1}^{\infty}$ converges to $S \neq \pm\infty$, the number S is called the sum of the series.

$$\sum_{n=1}^{\infty} a_n = S$$

Example 14.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

Therefore,

$$S_1 = \frac{1}{2} \tag{1}$$
$$S_2 = \frac{1}{2} + \frac{1}{2^2} \tag{2}$$
$$\vdots S_n = \frac{1}{2} + \cdots + \frac{1}{2^n} \tag{3}$$
$$= \frac{a_1\left(1-q^n\right)}{1-q} \tag{4}$$
$$= \frac{1/2\left(1-1/2^n\right)}{1-1/2} \tag{5}$$
$$= 1 - \frac{1}{2^n} \tag{6}$$
$$\lim_{n\to\infty} S_n = 1 \tag{7}$$

Therefore, the series converges.

$$S = \sum_{n=1}^{\infty} = 1$$

Theorem 12. *A geometric series* $\sum_{n=1}^{\infty} a_1 q^{n-1}$, $a_1 \neq 0$ *converges if* $|q| < 1$ *and then,*

$$S = \sum_{n=1}^{\infty} a_1 q^{n-1} = \frac{a_1}{1-q}$$

Definition 21 (p-series)**.** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the p-series.

Theorem 13. *The p-series converges for $p > 1$ and diverges for $p \leq 1$.*

Theorem 14. *If $\sum a_n$ converges, then*

$$\lim_{n\to\infty} a_n = 0$$

Proof.

$$\begin{aligned}
a_n &= S_n - S_{n-1} \\
\therefore \lim_{n\to\infty} a_n &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \\
&= S - S \\
&= 0
\end{aligned}$$

□

Theorem 15. *If $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n \pm b_n)$ and $\sum ca_n$, where c is a constant, also converge. Also,*

$$\begin{aligned}
\sum(a_n \pm b_n) &= \sum a_n \pm \sum b_n \\
\sum(ca_n) &= c \sum a_n
\end{aligned}$$

2.1 Convergence Criteria

2.1.1 Leibniz's Criteria

Definition 22 (Alternating series)**.** The series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

Theorem 16 (Leibniz's Criteria for Convergence)**.** *If an alternating series $\sum(-1)^{n-1}a_n$ with $a_n > 0$ satisfies*

1. *$a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.*
2. *$\lim\limits_{n\to\infty} a_n = 0$*

then the series $(-1)^{n-1}a_n$ converges.

Proof. Consider the even partial sums of the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n$.

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$$

As $\{a_n\}$ is monotonically increasing, all brackets are non-negative. Therefore,

$$S_{2m+2} \geq S_{2m}$$

Therefore, $\{S_{2m}\}$ is increasing.
Also,

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$$

All brackets and a_{2m} are non-negative. Therefore,

$$S_{2m} \leq a_1$$

Therefore, $\{S_{2m}\}$ is bounded from above by a_1. Hence,

$$\exists \lim_{m\to\infty} S_{2m} = S$$

For S_{2m+1},

$$\begin{aligned} S_{2m+1} &= S_{2m} + a_{2m+1} \\ \therefore \lim_{m\to\infty} S_{2m+1} &= \lim_{m\to\infty} S_{2m} + \lim_{m\to\infty} a_{2m+1} \\ &= S + 0 \\ &= S \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} S_n = S$$

$\square$

Example 15. The alternating harmonic series $\sum \dfrac{(-1)^{n-1}}{n}$ converges as $a_n = \dfrac{1}{n} > 0$, a_n decreases and $\lim a_n = 0$.

2.1.2 Comparison Test

Theorem 17 (Comparison Test for Convergence)**.** *Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$.*

1. *If $a_n \leq b_n$, $\forall n \geq n_0$ and $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.*
2. *If $a_n \geq b_n$, $\forall n \geq n_0$ and $\sum\limits_{n=1}^{\infty} b_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.*

Theorem 18 (Another Formulation of the Comparison Test for Convergence)**.** *Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$ and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = a > 0$$

where a is a finite number. Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} b_n$ converges.

2.1.3 d'Alembert Criteria (Ratio Test)

Definition 23 (Absolute and conditional convergence)**.** The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

Example 16. The series $\sum \dfrac{(-1)^{n-1}}{n^2}$ converges absolutely, as $\sum \left|\dfrac{(-1)^{n-1}}{n^2}\right| = \sum \dfrac{1}{n^2}$ converges.

Example 17. The series $\sum \dfrac{(-1)^{n-1}}{n}$ converges conditionally, as it converges, but $\sum \left|\dfrac{(-1)^{n-1}}{n^2}\right| = \sum \dfrac{1}{n}$ diverges.

Theorem 19. *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 20 (d'Alembert Criteria (Ratio Test))**.** *1. If*

$$\lim_{n\to\infty} \left|\frac{a_{n-1}}{a_n}\right| = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\lim_{n\to\infty} \left|\frac{a_{n-1}}{a_n}\right| = L > 1$$

(including $L = \infty$), then $\sum a_n$ converges diverges.

3. If $L = 1$, the test does not apply.

2.1.4 Cauchy Criteria (Cauchy Root Test)

Theorem 21 (Cauchy Criteria (Cauchy Root Test))**.** *1. If*

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

(including $L = \infty$), then $\sum a_n$ diverges.

3. If $L = 1$, the test does not apply.

2.1.5 Integral Test

Theorem 22 (Integral Test for Series Convergence)**.** *Let $f(x)$ be a continuous, non-negative, monotonic decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int\limits_{1}^{\infty} f(x)\,\mathrm{d}x$ converges.*

Exercise 8.
Does $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ converge or diverge?

Solution 8.
Let

$$f(x) = \frac{1}{x^p}$$

with $p > 0$.
Therefore, $f(x)$ is continuous, non-negative and monotonic decreasing on $[1, \infty)$. Therefore, the Integral Test for Series Convergence is applicable.

$$\int_1^{\infty} \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^p}\,\mathrm{d}x$$

If $p \neq 1$,

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p} &= \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^t \\
&= \lim_{t\to\infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right) \\
&= \frac{1}{p-1}
\end{aligned}$$

If $p = 1$,

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p} &= \lim_{t\to\infty} \ln x|_1^t \\
&= \infty
\end{aligned}$$

Therefore, the series converges for $p > 1$ and diverges for $p \leq 1$.

Theorem 23. *If the series $\sum a_n$ absolutely converges and the series $\sum b_n$ is obtained from $\sum a_n$ by changing the order of the terms in $\sum a_n$ then $\sum b_n$ also absolutely converges and $\sum b_n = \sum a_n$.*

Theorem 24. *If a series converges then the series with brackets without changing the order of terms also converges. That is, if $\sum a_n$ converges, then any series of the form $(a_1 + a_2) + (a_3 + a_4 + a_5) + a_6 + \ldots$ also converges.*

Theorem 25. *If a series with brackets converges and the terms in the brackets have the same sign, then the series without brackets also converges.*

3 Power Series

Definition 24 (Power series)**.** The series $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ is called a power series.

Theorem 26 (Cauchy-Hadamard Theorem)**.** *For any power series $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ there exists the limit, which may be infinity,*

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}}$$

and the series converges for $|x-c| < R$ and diverges for $|x-c| > R$. The end points of the interval, i.e. $x = c - R$ and $x = c + R$ must be separately checked for series convergence.

Definition 25 (Radius of convergence and convergence interval)**.** The number R is called the radius of convergence and the interval $|x-c| < R$ is called the convergence interval of the series. The point c is called the centre of the convergence interval.

Theorem 27. *If $\exists \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, which may be infinite, then,*

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

Theorem 28 (Stirling's Approximation)**.** *For $n \to \infty$,*

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

3.1 Differentiation and Integration of Power Series

Theorem 29. *If R is a radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ then the function $f(x) = \sum\limits_{n=0}^{\infty} a_n(x-c)^n$ is differentiable on $(c-R, c+R)$ and the derivative is*

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$

Theorem 30. *If R is a radius of convergence of the series $\sum\limits_{n=0}^{\infty} a_n(x-c)^n$ then the function $f(x) = \sum\limits_{n=0}^{\infty} a_n(x-c)^n$ is integrable in $(c-R, c+R)$ and*

$$\int f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} + A$$

where $c-R < x < c+R$.

Exercise 9.
Find $\int\limits_0^x e^{-t^2}\,\mathrm{d}t$.

Solution 9.
$\forall s \in \mathbb{R}$,

$$\begin{aligned} e^s &= 1 + \frac{s}{1!} + \frac{s^2}{2!} + \cdots + \frac{s^n}{n!} + \ldots \\ \therefore e^{-t^2} &== 1 - \frac{t^2}{1!} + \frac{t^4}{2!} + \cdots + (-1)^n \frac{t^{2n}}{n!} + \ldots \\ \therefore \int_0^x e^{-t^2}\,\mathrm{d}t &= x - \frac{x^3}{1!3} + \frac{x^5}{2!5} + \cdots + (-1)^n \frac{x^{2n-1}}{n!(2n+1} + \ldots \end{aligned}$$

Theorem 31. *If the series $A(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum\limits_{n=0}^{\infty} B_n x^n$ absolutely converge for $|x| < R$ and $c_n = \sum\limits_{k=0}^{n} a_k b_{n-k}$, then the series $C(x) = \sum\limits_{n=0}^{\infty} c_n x^n$ also absolutely converges for $|x| < R$ and $C(x) = A(x)B(x)$.*

3.2 Taylor Series

Definition 26 (Taylor series)**.** Let $f(x)$ be infinitely differentiable on an open interval about a and let x be an arbitrary point in the interval. Then the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the Taylor series of $f(x)$ at a. If $a = 0$ then it is called the Maclaurin series of $f(x)$ at 0.

Theorem 32. *If there exists a power series which converges to $f(x)$, i.e. if, for $|x-a| < R$,*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

then, for $|x-a| < R$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

that is, $\forall n$,

$$a_n = \frac{f^{(n)}(a)}{n!}$$

Exercise 10.
Show that

$$f(x) = \begin{cases} 0 & ; \quad x = 0 \\ e^{-\frac{1}{x^2}} & ; \quad x \neq 0 \end{cases}$$

is not equal to it's Taylor series at $a = 0$.

Solution 10.
If $n = 1$,

$$\begin{aligned} f^{(n)}(0) &= \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{e^{-\frac{1}{(\Delta x)^2}}}{\Delta x} \end{aligned}$$

Let $t = \dfrac{1}{\Delta x}$

$$\begin{aligned}
\therefore f'(0) &= \lim_{t \to \infty} \frac{e^{-t^2}}{\frac{1}{t}} \\
&= \lim_{t\ to\infty} \frac{t}{e^{t^2}} \\
&= \lim_{t \to \infty} \frac{1}{e^{t^2} 2t} \\
&= 0
\end{aligned}$$

Therefore,

$$f'(x) = \begin{cases} 0 & ; \quad x = 0 \\ e^{-\frac{1}{x^2}} \cdot 2 \cdot x^{-3} & ; \quad x \neq 0 \end{cases}$$

Similarly, $\forall n \geq 1,\ f^{(n)}(0) = 0$
Therefore, the Taylor series is not equal to $f(x)$.

Exercise 11.
Find the Maclaurin series of $f(x) = e^x$ and prove that the series converges to $f(x)$ for any $x \in \mathbb{R}$.

Solution 11.
$\forall n \geq 1,\ f^{(n)}(x) = e^x$.
Therefore,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c x^{n+1}}{(n+1)!}$$

where c is between 0 and x.
Therefore, as

$$0 \leq |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

by the Sandwich Theorem

$$\lim_{n \to \infty} |R_n(x)| = 0$$

Therefore,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \dots$$

4 Series of Real-valued Functions

Definition 27 (Sequence of functions)**.** A sequence $\{f_n\} = f_1(x), f_2(x), \ldots$ defined on $D \subseteq \mathbb{R}$ is called a sequence of functions.

Definition 28 (Pointwise convergence and domain of convergence)**.** $\{f_n\}$ converges pointwise in some domain $E \subseteq D$ if for every $x \in E$, the sequence of $\{f_n(x)\}$ converges. In such a case, E is said to be a domain of convergence of $\{f_n\}$.

Exercise 12.
Find the domain of convergence of $f_n(x) = x^n$, defined on some $D \subseteq \mathbb{R}$.

Solution 12.

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & ; \quad -1 < x < 1 \\ 1 & ; \quad x = 1 \\ \text{diverges} & ; \quad x \notin (-1, 1] \end{cases}$$

Therefore, the domain of convergence of $\{f_n\}$ is $(-1, 1]$.

Exercise 13.
Let $f(x) : (0, \infty) \to \mathbb{R}$ be some function such that $\lim\limits_{x\to\infty} f(x) = 0$. Let $f_n(x) = f(nx)$. What is the domain of convergence of f_n? What is the limit function?

Solution 13.
Let x have some fixed value in $(0, \infty)$. Therefore, as $\lim\limits_{x\to\infty} f(x) = 0$,

$$\begin{aligned} \lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} f(nx) \\ &= 0 \end{aligned}$$

Therefore, the domain of convergence is $(0, \infty)$ and the limit function is a constant function with value 0.

4.1 Uniform Convergence of Series of Functions

Definition 29 (Pointwise convergence of a sequence of functions)**.** If $\forall x \in D$, $\forall \varepsilon > 0$, $\exists N$ which depends on ε and x, such that $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$, then $\forall x \in D$, $\lim\limits_{n\to\infty} = f(x)$.

Definition 30 (Uniform convergence of a sequence of functions)**.** The sequence $\{f_n(x)\}$ is said to converge uniformly to $f(x)$ in D if $\forall \varepsilon > 0$, $\exists N = N(\varepsilon)$, such that $\forall n \geq N$, $\forall x \in D$, $|f_n(x) - f(x)| < \varepsilon$. It can be denoted as $f_n(x) \overset{D}{\rightrightarrows} f(x)$.

Theorem 33. *$f_n(x)$ converges uniformly to $f(x)$ in D if and only if $\lim\limits_{n\to\infty} \sup\limits_{x\in D} |f_n(x) - f(x)| = 0$.*

Exercise 14.
Does $f_n(x) = x^n$ converge in $[0,1]$?

Solution 14.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n$$
$$\therefore f(x) = \begin{cases} 0 & ; \quad 0 \leq x < 1 \\ 1 & ; \quad x = 1 \end{cases}$$

Therefore,
If $x = 0$,

$$f_n(0) = 0$$
$$f(0) = 0$$

Therefore, $\forall \varepsilon > 0$, $N = 1$,

$$|0 - 0| < \varepsilon$$
$$\therefore |f_n(0) - f(0)| < \varepsilon$$

If $x = 1$,

$$f_n(1) = 1$$
$$f(1) = 1$$

Therefore, $\forall \varepsilon > 0$, $N = 1$,

$$|1 - 1| < \varepsilon$$
$$\therefore |f_n(1) - f(1)| < \varepsilon$$

If $0 < x < 1$,

$$\begin{aligned} |f_n(x) - f(x)| &= |x^n - 0| \\ &= x^n \end{aligned}$$

If possible, let $|f_n(x) - f(x)| = x^n < \varepsilon$.
Therefore,

$$\begin{aligned} x^n &< \varepsilon \\ \therefore \log_x x^n &> \log_x \varepsilon \\ \therefore n &> \log_x \varepsilon \end{aligned}$$

Therefore, for $N = \lfloor \log_x \varepsilon \rfloor + 1$, $|f_n(x) - f(x)| < \varepsilon$.

Therefore, $f_n(x)$ converges pointwise in $[0, 1]$.

If possible let $f_n(x)$ converge uniformly on $[0, 1]$.
Therefore, $\forall \varepsilon > 0$, $\exists N$ dependent on ε, such that $|f_n(x) - f(x)| < \varepsilon$.
Let $\varepsilon = \frac{1}{3}$.
Therefore, $\exists N$ which is dependent on ε, such that $\forall n > N$, $\forall x \in [0, 1]$,

$$|f_n(x) - f(x)| < \frac{1}{3}$$

Let $x = \frac{1}{2}$, $n = N + 1$. Therefore,

$$\begin{aligned} \left| f_n\left(\frac{1}{2}\right) - f\left(\frac{1}{2}\right) \right| &= \left| \frac{1}{2} - 0 \right| \\ &= \frac{1}{2} \\ \therefore \left| f_n\left(\frac{1}{2}\right) - f\left(\frac{1}{2}\right) \right| &> \frac{1}{3} \end{aligned}$$

Therefore, $|f_n(x) - f(x)| > \varepsilon$.
This is a contradiction. Hence, $f_n(x)$ is does not converge uniformly.

Definition 31 (Supremum)**.** Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the supremum of A if

1. $\forall x \in A$, $x \leq M$, i.e. M is an upper bound of A.
2. $\forall \varepsilon$, $\exists x \in A$, such that $x > M - \varepsilon$.

That is, the supremum of A is the least upper bound of A.
The supremum may or may not be in A.

Definition 32 (Infimum)**.** Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the infimum of A if

1. $\forall x \in A$, $x \geq M$, i.e. M is an upper bound of A.
2. $\forall \varepsilon$, $\exists x \in A$, such that $x < M - \varepsilon$.

That is, the infimum of A is the greatest lower bound of A. The infimum may or may not be in A.

Theorem 34. *Every bounded set A has a supremum and an infimum.*

Theorem 35. $f_n \overset{E}{\rightrightarrows} f$ *if and only if*

$$\lim_{n\to\infty} (\sup\{|f_n(x) - f(x)| : x \in E\}) = 0$$

Definition 33 (Remainder of a series of functions)**.** Let $f(x) = \sum_{k=1}^{\infty} u_k(x)$. Let the partial sums be denoted by $f_n(x) = \sum_{k=1}^{n} u_k(x)$. Then

$$R_n(x) = f(x) - f_n(x) = \sum_{k=n+1}^{\infty} u_k(x)$$

is called a remainder of the series $f(x) = \sum_{k=1}^{\infty} u_k(x)$.

Definition 34 (Uniform convergence of a series of functions)**.** If $f_n(x)$ converges uniformly to $f(x)$ on D, i.e. if $\lim_{n\to\infty} R_n(x) = 0$, then the series $\sum_{k=1}^{\infty} u_k(x)$ is said to converge uniformly on D..

Exercise 15.

Show that the series $f(x) = \sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-k}$ does not converge uniformly on $(-1, 1)$.

Solution 15.

The series converges uniformly if and only if $\lim_{n\to\infty} R_n(x) = 0$.

$$\begin{aligned}
\lim_{n\to\infty} \sup_{(-1,1)} |R_n(x) - 0| &= \lim_{n\to\infty} \sup_{(-1,1)} \sum_{k=n+1}^{\infty} x^{k-1} \\
&= \lim_{n\to\infty} \sup_{(-1,1)} \left| \frac{x^n}{1-x} \right| \\
&= \lim_{n\to\infty} \sup_{(-1,1)} \frac{|x|^n}{1-x} \\
&= \lim_{n\to\infty} \infty \\
&= \infty
\end{aligned}$$

Therefore, the series does not converge uniformly on $(-1, 1)$.

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Exercise 16.

Show that the series $f(x) = \sum\limits_{k=1}^{\infty} x^{k-1} = \frac{1}{1-k}$ does not converge uniformly on $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Solution 16.

The series converges uniformly if and only if $\lim\limits_{n\to\infty} R_n(x) = 0$.

$$\begin{aligned}
\lim_{n\to\infty} \sup_{\left(-\frac{1}{2},\frac{1}{2}\right)} |R_n(x) - 0| &= \lim_{n\to\infty} \sup_{\left(-\frac{1}{2},\frac{1}{2}\right)} \sum_{k=n+1}^{\infty} x^{k-1} \\
&= \lim_{n\to\infty} \sup_{\left(-\frac{1}{2},\frac{1}{2}\right)} \left| \frac{x^n}{1-x} \right| \\
&= \lim_{n\to\infty} \sup_{\left(-\frac{1}{2},\frac{1}{2}\right)} \frac{|x|^n}{1-x} \\
&= \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} \\
&= \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n-1} \\
&= 0
\end{aligned}$$

Therefore, the series converges uniformly on $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

4.2 Weierstrass M-test

Theorem 36 (Weierstrass M-test)**.** *If $|u_k(x)| \le c_k$ on D for $k \in \{1, 2, 3, \dots\}$ and the numerical series $\sum\limits_{k=1}^{\infty} c_k$ converges, then the series of functions $\sum\limits_{k=1}^{\infty} u_k(x)$ converges uniformly on D.*

Exercise 17.

Show that $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} \sin(kx)$ converges uniformly on $\mathbb{R}$.

Solution 17.

$$|u_k(x)| = \left|\frac{1}{k^2}\sin(kx)\right|$$
$$\therefore |u_k(x)| \leq \frac{1}{k^2}$$

Therefore, let

$$c_k = \frac{1}{k^2}$$

Therefore, as $|u_k(x)| \leq c_k$, and as $\sum\limits_{k=1}^{\infty} c_k$ converges, by the Weierstrass M-test, $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}\sin(kx)$ converges uniformly.

4.3 Application of Uniform Convergence

Theorem 37 (Continuity of a series)**.** *Let functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ be defined on $[a, b]$ and continuous at $x_0 \in [a, b]$. If $\sum\limits_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a, b]$ then the function $f(x) = \sum\limits_{k=1}^{\infty}$ is also continuous at x_0.*

Theorem 38 (Changing the order of integration and infinite summation)**.** *If the functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ are integrable on $[a, b]$ and the series $\sum\limits_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a, b]$ then*

$$\int]limits_a^b \left(\sum_{k=1}^{\infty} u_k(x)\right) \mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b u_k(x)\, \mathrm{d}x$$

Exercise 18.
Solve $\int\limits_0^{2\pi} \left(\sum\limits_{k=1}^{\infty} \frac{1}{k^2}\sin(kx)\right)$.

¡++¿

Solution 18.
The series $f(x) = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}\sin(kx)$ converges uniformly on $[0, 2\pi]$. Therefore, by the Weierstrass M-test and $u_k(x) = \frac{1}{k^2}(kx)$ are integrable on $[0, 2\pi]$. There-

fore,

$$\begin{aligned}\int\limits_0^{2\pi} f(x)\,\mathrm{d}x &= \int\limits_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\sin(kx)\right)\mathrm{d}x\\ &= \sum_{k=1}^{\infty}\left(\int\limits_0^{2\pi}\frac{1}{k^2}\sin(kx)\,\mathrm{d}x\right)\\ &= \sum_{k=1}^{\infty}\left(-\frac{\cos(2\pi k)}{k^3}+\frac{1}{k^3}\right)\\ &= \sum_{k=1}^{\infty} 0\\ &= 0\end{aligned}$$