

Differential and Integral Calculus : Recitations

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1 Instructor Information

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Part I

Sequences and Series

1 Sequences

Recitation 1 – Exercise 1.

Prove:

$$\lim_{n \rightarrow \infty} \frac{2n^2 + n + 1}{n^2 + 3} = 2$$

Recitation 1 – Solution 1.

Let

$$\varepsilon > 0$$

$$\begin{aligned} \left| \frac{2n^2 + n + 1}{n^2 + 3} - 2 \right| &= \left| \frac{2n^2 + n + 1 - 2n^2 - 6}{n^2 + 3} \right| \\ &= \left| \frac{n - 5}{n^2 + 3} \right| \\ &\leq \left| \frac{n - 5}{n^2} \right| \\ &\leq \frac{1}{n} \\ &< \varepsilon \end{aligned}$$

Therefore, let $N = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$. Hence, for this N , $|a_n - L| < \varepsilon$.

Therefore, $\lim_{n \rightarrow \infty} \frac{2n^2 + n + 1}{n^2 + 3} = 2$. □

Recitation 1 – Exercise 2.

Prove

$$\lim_{n \rightarrow \infty} \frac{n^3 + \sin n + n}{2n^4} = 0$$

Recitation 1 – Solution 2.

Let $\varepsilon > 0$

$$\begin{aligned} \left| \frac{n^3 + \sin n + n}{2n^4} \right| &\leq \left| \frac{n^3 + 1 + n}{2n^4} \right| \\ &\leq \left| \frac{3n^3}{2n^4} \right| = \frac{3}{2} \cdot \frac{1}{n} < \varepsilon \end{aligned}$$

Therefore, let $N = \left\lceil \frac{3}{2\varepsilon} \right\rceil + 1$. Hence, for this N , $|a_n - L| < \varepsilon$.

Therefore, $\lim_{n \rightarrow \infty} \frac{n^3 + \sin n + n}{2n^4} = 0$

□

Recitation 1 – Exercise 3.

Calculate $\sqrt[3]{n^3 + 3n} - n$.

Recitation 1 – Solution 3.

$$a^n - b^n = (a - b) \cdot (a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Therefore, let

$$\begin{aligned} a &= \sqrt[3]{n^3 + 3n} \\ b &= \sqrt[3]{n^3} \end{aligned}$$

$$\begin{aligned} a - b &= \frac{a^3 - b^3}{a^2 + ab + b^2} \\ \therefore \sqrt[3]{n^3 + 3n} - n &= \frac{n^3 + 3n - n^3}{(n^3 + 3n)^{2/3} + (n^3 + 3n)^{1/3}n + n^2} \\ &= \frac{3}{\left(\frac{n^3 + 3n}{n^{3/2}}\right)^{2/3} + \left(\frac{n^3 + 3n}{n^3}\right)^{1/3} + n} \end{aligned}$$

Therefore, the limit is 0.

Recitation 1 – Exercise 4.

Prove

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$$

Recitation 1 – Solution 4.

$$0 \leq \frac{n!}{n^n} = \frac{1}{n} \frac{2}{n} \dots \frac{n}{n} \leq \frac{1}{n}$$

Therefore, by the Sandwich Theorem, $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$.

Recitation 1 – Exercise 5.

Let $a_1 = 3$, $a_{n+1} = 1 + \sqrt{6 + a_n}$. Prove that a_n converges and find its limit.

Recitation 1 – Solution 5.

If possible, let $\lim_{n \rightarrow \infty} a_n = l$.

$$a_{n+1} = 1 + \sqrt{6 + a_n}$$

Taking the limit on both sides,

$$\begin{aligned} l &= 1 + \sqrt{6 + l} \\ \therefore l - 1 &= \sqrt{6 + l} \\ \therefore l &= \frac{3 \pm \sqrt{29}}{2} \end{aligned}$$

$$\text{As } a_n \geq 0, l = \frac{3 + \sqrt{29}}{2}.$$

$$\begin{aligned} a_2 &= 1 + \sqrt{6 + a_1} \\ &= 1 + \sqrt{6 + 3} \\ &= 4 \\ \therefore a_2 &> a_1 \end{aligned}$$

If possible, let $a_n \geq a_{n-1}$.

Therefore,

$$\begin{aligned} a_{n+1} &= 1 + \sqrt{6 + a_n} \\ &\geq 1 + \sqrt{6 + a_{n+1}} = a_n \end{aligned}$$

Therefore by induction, $\{a_n\}$ is monotonically increasing.

$$\begin{aligned} a_1 &= 3 \\ \therefore a_1 &\leq 5 \end{aligned}$$

If possible, let $a_n \leq 5$.

Therefore,

$$a_{n+1} = 1 + \sqrt{6 + a_n} \leq q + \sqrt{11} \leq 5$$

Therefore by induction, $\{a_n\}$ is bounded from above by 5.

1.1 Limit of a Function by Heine

Definition 1.

$$\lim_{x \rightarrow x_0} f(x) = l$$

if for every sequence x_n , such that $\lim_{n \rightarrow \infty} x_n = x_0$,

$$\lim_{n \rightarrow \infty} f(x_n) = l$$

Theorem 1. *If f is continuous at x_0 and $x_n \rightarrow x_0$, then*

$$\lim_{n \rightarrow \infty} f(x_n) = f\left(\lim_{n \rightarrow \infty} x_n\right) = f_{x_0}$$

Recitation 2 – Exercise 1.

Calculate $\lim_{n \rightarrow \infty} \sqrt[n]{n}$.

Recitation 2 – Solution 1.

Let

$$f(x) = x^{1/x}$$

Therefore,

$$\begin{aligned} \lim_{x \rightarrow \infty} x^{1/x} &= \lim_{x \rightarrow \infty} e^{\frac{\ln x}{x}} \\ &= 1 \end{aligned}$$

1.2 Sub-sequences

Recitation 2 – Exercise 2.

Find all partial limits and $\overline{\lim}$ and $\underline{\lim}$ of

$$a_n = \left(\cos \frac{\pi n}{4} \right)^n$$

Recitation 2 – Solution 2.

Let $k, z \in \mathbb{Z}$

$$\begin{aligned} \cos \frac{\pi n}{4} &= \cos \frac{\pi(n+k)}{4} \\ \therefore \frac{\pi n}{4} &= \frac{\pi(n+k)}{4} + 2\pi z \\ \therefore \pi n &= \pi(n+k) + 8\pi z \\ \therefore k &= 8z \end{aligned}$$

Therefore,

$$\begin{aligned} a_{8k} &= \left(\cos \frac{\pi \cdot 8k}{4} \right)^{8k} \\ &= (\cos(2\pi k))^{8k} \\ &= 1 \\ a_{8k+1} &= \left(\cos \frac{\pi \cdot (8k+1)}{4} \right)^{8k+1} \\ &= \left(\cos \frac{\pi}{4} \right)^{8k+1} \\ &= \left(\frac{\sqrt{2}}{2} \right)^{8k+1} \\ a_{8k+2} &= \left(\cos \frac{\pi \cdot (8k+2)}{4} \right)^{8k+2} \\ &= \left(\cos \frac{\pi}{2} \right)^{8k+2} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{k \rightarrow \infty} a_{8k} &= 1 \\ \lim_{k \rightarrow \infty} a_{8k+1} &= \lim_{k \rightarrow \infty} \left(\frac{\sqrt{2}}{2} \right)^{8k+1} \\ &= 0 \end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{k \rightarrow \infty} a_{8k+2} &= 0 \\
\lim_{k \rightarrow \infty} a_{8k+3} &= 0 \\
\lim_{k \rightarrow \infty} a_{8k+4} &= \lim_{k \rightarrow \infty} (-1)^{8k+4} \\
&= 1 \\
\lim_{k \rightarrow \infty} a_{8k+5} &= 0 \\
\lim_{k \rightarrow \infty} a_{8k+6} &= 0 \\
\lim_{k \rightarrow \infty} a_{8k+7} &= 0
\end{aligned}$$

Therefore, $\{a_n\}$ has two partial limits, 0 and 1.

$$\begin{aligned}
\overline{\lim} a_n &= 1 \\
\underline{\lim} a_n &= 0
\end{aligned}$$

2 Series

Definition 2 (Convergence of a series). Let $\{a_n\}$ be a sequence. Let S_n be a sequence of partial sums of a_n , s.t.

$$S_n = \sum_{k=1}^n a_k$$

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to l if

$$\lim_{n \rightarrow \infty} S_n = l$$

that is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = \lim_{n \rightarrow \infty} S_n$$

Recitation 2 – Exercise 3.

Does $\sum_{k=0}^{\infty} q^k$ where $-1 < q < 1$ converge?

Recitation 2 – Solution 3.

$$\begin{aligned}\sum_{k=0}^{\infty} q^k &= \lim_{n \rightarrow \infty} \sum_{k=0}^n q^k \\ &= \lim_{n \rightarrow \infty} \frac{1 - q^{n+1}}{1 - q} \\ &= \frac{1}{1 - q}\end{aligned}$$

Therefore, the series converges.

Recitation 2 – Exercise 4.

Does $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converge?

Recitation 2 – Solution 4.

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k(k+1)} &= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) \\ &= 1\end{aligned}$$

Recitation 2 – Exercise 5.

Does $\sum_{k=1}^{\infty} \left(1 + \frac{1}{k} \right)^k$ converge?

Recitation 2 – Solution 5.

$$\begin{aligned}\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^k &= e \\ \therefore \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^k &\neq 0\end{aligned}$$

Therefore, the necessary condition is not satisfied. Hence, the series does not converge.

2.1 Comparison Tests for Positive Series

Theorem 2 (First Comparison Test). *If $a_n \geq 0$, $b_n \geq 0$, and $a_n \leq b_n$, then*

1. *If $\sum b_n$ converges, then $\sum a_n$ converges.*
2. *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Theorem 3 (Second Comparison Test). *If $a_n \geq 0$, $b_n \geq 0$ and*

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l$$

where $0 < l < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge simultaneously.

Recitation 3 – Exercise 1.

Suppose the sequence a_n satisfies the condition

$$a_{n+1} - a_n > \frac{1}{n}$$

$\forall n \in \mathbb{N}$.

Prove that $\lim_{n \rightarrow \infty} a_n = \infty$.

Recitation 3 – Solution 1.

$$\begin{aligned} a_{n+1} &= a_{n+1} - a_n + a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_2 - a_1 + a_1 \\ &= \sum_{k=1}^n (a_{k+1} - a_k) + a_1 \\ &\geq \sum_{k=1}^n \frac{1}{k} + a_1 \end{aligned}$$

As the harmonic series diverges, $\sum_{k=1}^n \frac{1}{k} + a_1$ diverges.

Therefore, by the First Comparison Test, $\sum_{k=1}^{\infty} (a_{k+1} - a_k)$ diverges.

Recitation 3 – Exercise 2.

Check the convergence of $\sum_{n=2}^{\infty} \frac{n + \sin n}{n^3 + \cos \pi n}$.

Recitation 3 – Solution 2.

The series is non-negative. Therefore, the comparison tests are applicable.

$$\begin{aligned} \frac{n + \sin n}{n^3 + \cos \pi n} &\leq \frac{n + 1}{n^3 - 1} \\ \therefore \frac{n + \sin n}{n^3 + \cos \pi n} &\leq \frac{2n}{n^3 - \frac{n^3}{2}} \leq \frac{4}{n^2} \end{aligned}$$

Therefore, by the First Comparison Test, as $\frac{4}{n^2}$ converges, $\sum_{n=2}^{\infty} \frac{n + \sin n}{n^3 + \cos \pi n}$ also converges.

Recitation 3 – Exercise 3.

Let $a_n \geq 0$ and suppose that $\sum a_n$ converges. Prove that $\sum a_n^2$ converges. Is it true without the assumption $a_n \geq 0$?

Recitation 3 – Solution 3.

As $\sum a_n$ converges, $\lim_{n \rightarrow \infty} a_n = 0$.

Therefore, $\exists N \in \mathbb{N}$, such that $\forall n > N$, $a_n < 1$.

Therefore, $\forall n > N$, $a_n^2 \leq a_n$. Hence, as $\sum_{n=N+1}^{\infty} a_n$ converges, $\sum_{n=N+1}^{\infty} a_n^2$ also converges. Hence, $\sum_{n=1}^{\infty} a_n$ also converges.

This is not true without the assumption $a_n \geq 0$, as the argument $a_n^2 \leq a_n$ does not hold.

Recitation 3 – Exercise 4.

For which α does $\sum (\sqrt{n+1} - \sqrt{n})^{\alpha/2}$ converge?

Recitation 3 – Solution 4.

$$\begin{aligned} \sum (\sqrt{n+1} - \sqrt{n})^{\alpha/2} &= \sum \left(\frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \right)^{\alpha/2} \\ &= \sum \left(\frac{1}{\sqrt{n+1} + \sqrt{n}} \right)^{\alpha/2} \end{aligned}$$

The series is positive. Therefore, the comparison tests are applicable.

Comparing with $\left(\frac{1}{\sqrt{n}}\right)^{\alpha/2}$,

$$\frac{\left(\frac{1}{\sqrt{n+1} + \sqrt{n}}\right)^{\alpha/2}}{\left(\frac{1}{\sqrt{n}}\right)^{\alpha/2}} = \left(\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right)^{\alpha/2}$$

$$\therefore \lim_{n \rightarrow \infty} \left(\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right)^{\alpha/2} = \left(\frac{1}{2}\right)^{\alpha/2}$$

$\sum \frac{1}{n^{\alpha/2}}$ converges if and only if $\frac{\alpha}{4} > 1$, i.e. if and only if $\alpha > 4$.

By the Second Comparison Test, $\sum \frac{1}{n^{\alpha/4}}$ and the series converge or diverge simultaneously.

Therefore, the series converges for $\alpha > 4$.

Recitation 3 – Exercise 5.

Check the convergence of $\sum_{n=1}^{\infty} \sin \frac{1}{n}$.

Recitation 3 – Solution 5.

$\forall n \in \mathbb{N}, \sin \frac{1}{n} \geq 0$

$$\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$$

Therefore, by Second Comparison Test, $\sum \frac{1}{n}$ and $\sum \sin \frac{1}{n}$ diverge simultaneously.

2.2 d'Alembert Criteria (Ratio Test)

Definition 3 (Absolute and conditional convergence). The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

Theorem 4. *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 5 (d'Alembert Criteria (Ratio Test)). 1. If

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$$

(including $L = \infty$), then $\sum a_n$ converges diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 – Exercise 6.

Check the convergence of $\sum \frac{(-1)^n \cdot n^{1000}}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}$.

Recitation 3 – Solution 6.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n \cdot n^{1000}}{1 \cdot \dots \cdot (2n-1)} \right| = \sum_{n=1}^{\infty} \frac{n^{1000}}{1 \cdot \dots \cdot (2n-1)}$$

Therefore, by the d'Alembert Criteria (Ratio Test),

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+1)^{1000}}{1 \cdot \dots \cdot (2n+1)} \cdot \frac{1 \cdot \dots \cdot (2n-1)}{n^{1000}} \\ &= \left(\frac{n+1}{n} \right)^{1000} \cdot \frac{1}{2n+1} \\ \therefore \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{1000} \cdot \frac{1}{2n+1} &= 0 \\ \therefore \left(\frac{n+1}{n} \right)^{1000} \cdot \frac{1}{2n+1} &< 1 \end{aligned}$$

Therefore, by the d'Alembert Criteria (Ratio Test), the series converges absolutely, and hence converges.

2.3 Cauchy Criteria (Cauchy Root Test)

Theorem 6 (Cauchy Criteria (Cauchy Root Test)). 1. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

(including $L = \infty$), then $\sum a_n$ diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 – Exercise 7.

Check the convergence of $\sum \left(1 - \frac{2}{n}\right)^{n^2}$.

Recitation 3 – Solution 7.

$$\begin{aligned}\sqrt[n]{\left(1 - \frac{2}{n}\right)^{n^2}} &= \left(1 - \frac{2}{n}\right)^n \\ \therefore \lim_{n \rightarrow \infty} \left(1 - \frac{2}{n}\right)^n &= e^{-2} \\ \therefore \lim_{n \rightarrow \infty} \left(1 - \frac{2}{n}\right)^n &< 1\end{aligned}$$

Therefore, by the Cauchy Criteria (Cauchy Root Test), $\sum \left(1 - \frac{2}{n}\right)^{n^2}$ converges.

2.4 Leibniz's Criteria

Definition 4 (Alternating series). The series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

Theorem 7 (Leibniz's Criteria for Convergence). If an alternating series $\sum (-1)^{n-1} a_n$ with $a_n > 0$ satisfies

1. $a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.

$$2. \lim_{n \rightarrow \infty} a_n = 0$$

then the series $(-1)^{n-1}a_n$ converges.

Recitation 3 – Exercise 8.

Prove or disprove: There exists $\{a_n\}$, such that $\sum a_n$ converges and $\sum(1 + a_n)a_n$ diverges.

Recitation 3 – Solution 8.

$$\text{Let } a_n = \frac{(-1)^n}{\sqrt{n}}.$$

Therefore, by Leibniz's Criteria for Convergence, $\sum \frac{(-1)^n}{\sqrt{n}}$ converges.

$$\begin{aligned} \sum(1 + a_n)a_n &= \sum \left(1 + \frac{(-1)^n}{\sqrt{n}}\right) \frac{(-1)^n}{\sqrt{n}} \\ &= \sum \left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right) \end{aligned}$$

Therefore, as $\sum \frac{1}{n}$ diverges, and $\sum \frac{(-1)^n}{\sqrt{n}}$ converges, $\sum \left(\frac{1}{n} + \frac{(-1)^n}{\sqrt{n}}\right)$ diverges.

2.5 Integral Test

Theorem 8 (Integral Test). *If $f(x) : [1, \infty) \rightarrow [0, \infty)$ is monotonically decreasing. Then, $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x) dx$ converge or diverge simultaneously.*

Recitation 3 – Exercise 9.

Check the convergence of $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Recitation 3 – Solution 9.

Let

$$f(x) = \frac{1}{x \ln x}$$

$f(x)$ is monotonically decreasing. Therefore, the Integral Test is applicable. Therefore,

$$\begin{aligned}\int_2^{\infty} \frac{1}{x \ln x} dx &= \int_{\ln 2}^{\infty} \frac{1}{y} dy \\ &= \ln y|_{\ln 2}^{\infty} \\ &= \infty\end{aligned}$$

Therefore, by the integral test, $\sum \frac{1}{n \ln n}$ diverges.

Recitation 4 – Exercise 1.

Let $d_n \geq 0$ and suppose

$$\sum_{n=0}^{\infty} d_n = \infty$$

Prove that

$$\sum_{n=0}^{\infty} \frac{d_n}{1 + d_n} = \infty$$

Recitation 4 – Solution 1.

If possible, let d_n be a bounded sequence. Then there exists M , such that $d_n \leq M$, $\forall n \in \mathbb{N}$.

Therefore,

$$\frac{d_n}{1 + d_n} \geq \frac{d_n}{1 + M}$$

Therefore, by the Second Comparison Test, as $\sum d_n$ diverges, $\sum \frac{d_n}{1 + d_n}$ also diverges.

If d_n is not bounded, then there is a subsequence d_{n_k} which diverges. Therefore,

$$\begin{aligned}\frac{d_{n_k}}{1 + d_{n_k}} &= \frac{1}{\frac{1}{d_{n_k}} + 1} \\ \therefore \lim_{k \rightarrow \infty} \frac{d_{n_k}}{1 + d_{n_k}} &= 1\end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{d_n}{1 + d_n} \neq 0$$

Therefore, the necessary condition for convergence is not fulfilled. Therefore, the series converges.

Recitation 4 – Exercise 2.

Let

$$d_n = \begin{cases} 1 & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$

Does $\sum \frac{d_n}{1 + n \cdot d_n}$ diverge?

Recitation 4 – Solution 2.

$$d_n = \begin{cases} 1 & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$
$$\therefore \frac{d_n}{1 + n \cdot d_n} = \begin{cases} \frac{1}{1 + k^2} & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$

As $\frac{1}{1 + k^2} \leq \frac{1}{k^2}$ and as $\frac{1}{k^2}$ converges, $\sum \frac{1}{1 + k^2}$ also converges.

Recitation 4 – Exercise 3.

Let a_n be a sequence such that $|a_{n+1} - a_n| \leq b_{n+1}$ for all $n \in \mathbb{N}$ where $\sum b_k$ converges. Prove that $\{a_n\}$ converges.

Recitation 4 – Solution 3.

Let $\varepsilon > 0$.

$$\begin{aligned} |a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} + \cdots - a_n| \\ &\leq \sum_{k=n+1}^m |a_k - a_{k-1}| \\ &\leq \sum_{k=n+1}^m b_k \end{aligned}$$

Therefore, as $\sum b_n$ converges, the series satisfies the Cauchy Criteria (Cauchy Root Test). Therefore, there exists N , such that $\forall m > n > N$, $\left| \sum_{k=n+1}^m b_k \right| < \varepsilon$. Therefore, for $m > n > N$,

$$|a_m - a_n| \leq \sum_{k=n+1}^m b_k < \varepsilon$$

3 Power Series

Definition 5 (Power series). A power series around x_0 is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

where $\{a_n\}$ is a sequence of real numbers.

Theorem 9 (Abel's Theorem). *For every power series $\sum a_n(x - x_0)^n$, there exists $R \in [0, \infty]$, such that for all x satisfying $|x - x_0| < R$, the series converges and for all x satisfying $|x - x_0| > R$ the series diverges.*

Theorem 10 (Cauchy's Formula for Radius of Convergence).

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$$

Theorem 11 (Hadamard's Formula for Radius of Convergence). *If $\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$ exists, then*

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = R$$

Recitation 4 – Exercise 4.

Find the domain of convergence of $\sum_{n=1}^{\infty} \frac{(2x - 4)^n}{n}$.

Recitation 4 – Solution 4.

$$\sum_{n=1}^{\infty} \frac{(2x-4)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{n}$$

Therefore, by Cauchy's Formula for Radius of Convergence,

$$\begin{aligned} R &= \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}} \\ &= \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{\frac{2^n}{n}}} \\ &= \frac{1}{\lim_{n \rightarrow \infty} \frac{2}{\sqrt[n]{n}}} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, the series converges for

$$|x-2| < \frac{1}{2}$$

and diverges for

$$|x-2| > \frac{1}{2}$$

If $x = \frac{5}{2}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{5}{2} - 2\right)^n \\ = \sum_{n=1}^{\infty} \frac{1}{n} \end{aligned}$$

Therefore, the series diverges.

If $x = \frac{3}{2}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{3}{2} - 2\right)^n \\ = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \end{aligned}$$

Therefore, by Leibniz's Criteria for Convergence, the series converges.

Therefore, the domain of convergence is $\left[\frac{3}{2}, \frac{5}{2}\right)$.

Recitation 4 – Exercise 5.

Find the radius of convergence of $\sum_{n=0}^{\infty} n!x^{n!}$.

Recitation 4 – Solution 5.

$$\frac{1}{\sqrt[n]{a_n}} = x + x + 2x^2 + 6x^6 + 24x^{24} + \dots$$

Therefore,

$$a_n = \begin{cases} n & ; \quad n = k^2 \\ 0 & ; \quad n \neq k^2 \end{cases}$$

Therefore,

$$\begin{aligned} R &= \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{a_n}} \\ &= \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k!]{k!}} \\ &= 1 \end{aligned}$$

3.1 Power Series Representation of a Function

Theorem 12. *The power series representation of a function $f(x)$ is equal to its Taylor series if and only if $\lim_{n \rightarrow \infty} R_n(x) = 0$, where $R_n(x)$ is the Lagrange remainder.*

3.2 Differentiation and Integrations of Power Series

Recitation 5 – Exercise 1.

Find the power series representation of $\tan^{-1} x$.

Recitation 5 – Solution 1.

$$\begin{aligned}\frac{d \tan^{-1} x}{dx} &= \frac{1}{1+x^2} \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n}\end{aligned}$$

Integrating term by term,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + c$$

As $\tan^{-1} 0 = 0$, $c = 0$. Therefore,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Recitation 5 – Exercise 2.

Find an explicit formula for $\sum_{n=1}^{\infty} x^n n^2$.

Recitation 5 – Solution 2.

$$\sum_{n=1}^{\infty} x^n n^2 = x \cdot \sum_{n=1}^{\infty} x^{n-1} n^2$$

Let

$$g(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

Integrating term by term,

$$\begin{aligned}\int g(x) dx &= \sum_{n=1}^{\infty} n^2 \frac{x^n}{n} \\ &= \sum_{n=1}^{\infty} n x^n \\ &= x \cdot \sum_{n=1}^{\infty} n x^{n-1}\end{aligned}$$

Let

$$h(x) = \sum_{n=1}^{\infty} nx^{n-1}$$

$$\therefore \int h(x) \, dx = \frac{x}{1-x}$$

Therefore, inside radius of convergence $R = 1$, differentiating $\int h(x) \, dx$,

$$h(x) = \frac{1-x+x}{(1-x)^2}$$

$$= \frac{1}{(1-x)^2}$$

$$\therefore \int g(x) \, dx = xh(x)$$

$$= \frac{x}{(1-x)^2}$$

$$\therefore g(x) = \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4}$$

$$\therefore \sum_{n=1}^{\infty} x^n n^2 = x \cdot \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4}$$

Recitation 5 – Exercise 3.

Find the sum $\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

Recitation 5 – Solution 3.

Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

be a power series with radius R .

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = f\left(\frac{1}{2}\right)$$

Therefore,

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1}$$

$$= \frac{1}{1-x}$$

$$\therefore f(x) = -\ln(1-x) + c$$

As $f(0) = 0$, $c = 0$. Therefore,

$$f(x) = -\ln(1 - x)$$

Therefore,

$$\begin{aligned} f\left(\frac{1}{2}\right) &= -\ln\left(\frac{1}{2}\right) \\ &= \ln 2 \end{aligned}$$

4 Sequences of Functions

Definition 6 (Point-wise convergence and domain of convergence). $\{f_n\}$ is said to converge point-wise in some domain $E \subset D$ if $\forall x \in E$, the sequence $\{f_n(x)\}$ converges. In this case, E is said to be a domain of convergence of $\{f_n\}$.

Recitation 5 – Exercise 4.

Let $f(x) : \mathbb{R} \rightarrow \mathbb{R}$ be some function such that $\lim_{x \rightarrow \infty} f(x) = 0$. Let $f_n(x) = f(nx)$. What is the domain of convergence of f_n ? What is the limit function?

Recitation 5 – Solution 4.

Let x be a particular number in $(0, \infty)$.

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} f(nx)$$

Therefore, as $\lim_{x \rightarrow \infty} f(x) = 0$,

$$\lim_{n \rightarrow \infty} f_n(x) = 0$$

Therefore the domain of convergence is $(0, \infty)$ and the limit function is a constant 0.

Although the all functions in $\{f_n\}$ are continuous, the limit function is not continuous.

Definition 7 (Uniform convergence). A sequence of functions $\{f_n\}$ is said to converge uniformly to f in the domain E , if $\forall \varepsilon$, $\exists N$ such that $\forall n > N$ and $\forall x \in E$, $|f_n(x) - f(x)| < \varepsilon$. If f_n converges to f uniformly in E , it is denoted as $f_n \xrightarrow{E} f$.

4.1 Supremum and Infimum of Sets

Definition 8 (Supremum). Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the supremum of A if

1. $\forall x \in A, x \leq M$, i.e. M is an upper bound of A .
2. $\forall \varepsilon, \exists x \in A$, such that $x > M - \varepsilon$.

That is, the supremum of A is the least upper bound of A .
The supremum may or may not be in A .

Definition 9 (Infimum). Let $A \subseteq \mathbb{R}$ be a bounded set. M is said to be the infimum of A if

1. $\forall x \in A, x \geq M$, i.e. M is an upper bound of A .
2. $\forall \varepsilon, \exists x \in A$, such that $x < M + \varepsilon$.

That is, the infimum of A is the greatest lower bound of A . The infimum may or may not be in A .

Theorem 13. *Every bounded set A has a supremum and an infimum.*

Theorem 14. $f_n \xrightarrow{E} f$ if and only if

$$\lim_{n \rightarrow \infty} (\sup\{|f_n(x) - f(x)| : x \in E\}) = 0$$

Recitation 6 – Exercise 1.

Let $f_n(x) = x^n$. Does $\{f_n\}$ converge uniformly?

Recitation 6 – Solution 1.

$$f(x) = \begin{cases} 0 & ; \quad x \in [0, 1] \\ 1 & ; \quad x = 1 \end{cases}$$

If the convergence is uniform in $[0, 1]$,

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, 1]} |f_n(x) - f(x)| = 0$$

Let $x = 1 - \frac{1}{n}$.

Therefore, as the supremum is an upper bound,

$$\begin{aligned}\sup_{x \in [0,1]} |f_n(x) - f(x)| &\geq \left| f_n\left(1 - \frac{1}{n}\right) - f\left(1 - \frac{1}{n}\right) \right| \\ \therefore \sup_{x \in [0,1]} |f_n(x) - f(x)| &\geq \left| \left(1 - \frac{1}{n}\right)^n - 0 \right| \\ \therefore \lim_{n \rightarrow \infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| &\geq \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n \\ \therefore \lim_{n \rightarrow \infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| &\geq \frac{1}{e} \\ \therefore \lim_{n \rightarrow \infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| &\neq 0\end{aligned}$$

Therefore, the convergence is not uniform.

Recitation 6 – Exercise 2.

Let $f_n(x) = x + \frac{1}{n}$, $x \in \mathbb{R}$. What is its domain of convergence? What is the limit function? Is the convergence uniform?

Recitation 6 – Solution 2.

$\forall x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \left(x + \frac{1}{n}\right) = x$$

Therefore $\{f_n\}$ converges pointwise to x , in \mathbb{R} .

$$\begin{aligned}\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| &= \sup_{x \in \mathbb{R}} \left|x + \frac{1}{n} - x\right| \\ &= \frac{1}{n} \\ \therefore \lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| &= 0\end{aligned}$$

Therefore, the convergence is uniform.

Recitation 6 – Exercise 3.

Let $f_n : [0, \infty) \rightarrow \mathbb{R}$.

$$f_n(x) = \begin{cases} 1 & ; \quad n \leq x \leq n+1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Does f_n converge pointwise in $[0, \infty)$? Does f_n converge uniformly in $[0, \infty)$?

Recitation 6 – Solution 3.

For every x , the sequence $\{f_n(x)\}$ will be of the form $\{0, \dots, 0, 1, 0, \dots, 0\}$ with 1 only when $n \leq x \leq n+1$.

Therefore,

$$\begin{aligned}\lim_{n \rightarrow \infty} f_n(x) &= 0 \\ &= f(x)\end{aligned}$$

Therefore, f_n converges pointwise in $[0, \infty)$.

$$\begin{aligned}\sup_{x \in [0, \infty)} |f_n(x) - f(x)| &= \max_{x \in [0, \infty)} f_n(x) \\ &= 1\end{aligned}$$

Therefore, as the limit of the supremum is not 0, the convergence is not uniform.

Theorem 15. If $f_n \xrightarrow{D} f$ and all f_n are continuous on D , then f is also continuous, i.e. uniform convergence preserves continuity.

Recitation 7 – Exercise 1.

Does x^n converge to

$$f(x) = \begin{cases} 0 & ; \quad x \in [0, 1) \\ 1 & ; \quad x = 1 \end{cases}$$

Recitation 7 – Solution 1.

If possible, let x^n converge to $f(x)$.

Therefore, as all $f_n(x)$ are continuous, and as uniform convergence preserves continuity, $f(x)$ also must be continuous.

This contradicts the definition of $f(x)$.

Therefore, the x^n does not converge to $f(x)$.

Recitation 7 – Exercise 2.

Check if $f_n(x) = \frac{x}{1+n^2x^2}$ converges uniformly in $[0, 1]$.

Recitation 7 – Solution 2.

$$\begin{aligned}\lim_{n \rightarrow \infty} f_n(x) &= 0 \\ &= f(x)\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{[0,1]} |f_n(x) - f(x)| &= \sup_{[0,1]} |f_n(x) - 0| \\ &= \sup_{[0,1]} \left| \frac{x}{1 + n^2 x^2} \right| \\ &= \sup_{[0,1]} \frac{x}{1 + n^2 x^2}\end{aligned}$$

As the function is continuous and the interval is closed, by the Weierstrass theorem, the function has a maximum. Therefore,

$$\sup_{[0,1]} |f_n(x) - f(x)| = \max_{[0,1]} \frac{x}{1 + n^2 x^2}$$

Differentiating to find the maximum,

$$\begin{aligned}\frac{d}{dx} \left(\frac{x}{1 + n^2 x^2} \right) &= \frac{1 + n^2 x^2 - 2x^2 n^2}{(1 + n^2 x^2)^2} \\ &= \frac{1 - x^2 n^2}{(1 + n^2 x^2)^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{dx} \left(\frac{x}{1 + n^2 x^2} \right) &= 0 \\ \iff \frac{1 - x^2 n^2}{(1 + n^2 x^2)^2} &= 0 \\ \iff 1 &= x^2 n^2 \\ \iff x &= \frac{1}{n}\end{aligned}$$

Therefore, the values of the function at the critical points and the end points

are,

$$\begin{aligned}f_n(0) &= 0 \\f_n(1) &= \frac{1}{1+n^2} \\f_n\left(\frac{1}{n}\right) &= \frac{\frac{1}{n}}{1+n^2\frac{1}{n^2}} \\&= \frac{1}{2n}\end{aligned}$$

Therefore, the maximum is at $x = \frac{1}{2n}$.
Therefore,

$$\begin{aligned}\max_{[0,1]} \frac{x}{1+n^2x^2} &= f_n\left(\frac{1}{n}\right) \\&= \frac{1}{2n}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{n \rightarrow \infty} \sup_{[0,1]} |f_n(x) - f(x)| &= \lim_{n \rightarrow \infty} \max_{[0,1]} \frac{x}{1+n^2x^2} \\&= \lim_{n \rightarrow \infty} \frac{1}{2n} \\&= 0\end{aligned}$$

Therefore, the convergence is uniform.

Recitation 7 – Exercise 3.

Check the pointwise and uniform convergence of $f_n(x) = x^n - x^{n+1}$ in $[0, 1]$.

Recitation 7 – Solution 3.

$$\begin{aligned}\lim_{n \rightarrow \infty} f_n(x) &= \lim_{n \rightarrow \infty} x^n - x^{n+1} \\&= 0 \\&= f(x)\end{aligned}$$

Therefore the function converges pointwise in $[0, 1]$.

$$\sup_{[0,1]} |f_n(x) - f(x)| = \sup_{[0,1]} x^n - x^{n+1}$$

As the function is continuous and the interval is closed, by the Weierstrass theorem, the function has a maximum. Therefore,

$$\sup_{[0,1]} x^n - x^{n+1} = \max_{[0,1]} x^n - x^{n+1}$$

Differentiating to find the maximum,

$$\frac{d(x^n - x^{n+1})}{dx} = nx^{n-1} - (n+1)x^n$$

Therefore,

$$\begin{aligned} \frac{d(x^n - x^{n+1})}{dx} &= 0 \\ \iff nx^{n-1} - (n+1)x^n &= 0 \\ \iff n - (n+1)x &= 0 \\ \iff x &= \frac{n}{n+1} \end{aligned}$$

Therefore, the values of the function at the critical points and the end points are

$$\begin{aligned} f_n(0) &= 0 \\ f_n(1) &= 0 \\ f_n\left(\frac{n}{n+1}\right) &= \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \max_{[0,1]} x^n - x^{n+1} &= f_n\left(\frac{n}{n+1}\right) \\ &= \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{[0,1]} |f_n(x) - f(x)| &= \lim_{n \rightarrow \infty} \max_{[0,1]} \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} \\ &= \frac{1}{e} - \frac{1}{e} \\ &= 0 \end{aligned}$$

Therefore, the convergence is uniform.

Theorem 16 (Cauchy's Theorem). $\{f_n\}$ converges uniformly in D if and only if $\forall \varepsilon \in \mathbb{R}, \exists N$, such that $\forall m, n > N$ and $\forall x \in D$,

$$|f_n(x) - f(x)| < \varepsilon$$

Recitation 7 – Exercise 4.

Let $\{f_n\}$ be a sequence of function in D such that $\forall x \in D, |f_{n+1}(x) - f_n(x)| \leq a_n$, where $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\{f_n\}$ converges uniformly in D .

Recitation 7 – Solution 4.

As $\sum a_n$ converges, $\exists N$ such that $\forall m > n > N, \left| \sum_{k=n}^m a_k \right| < \varepsilon$.

Therefore, for all $m > n > N$ and $x \in D$,

$$\begin{aligned} |f_m(x) - f_n(x)| &= |f_m(x) - f_{m-1}(x) + f_{m-1}(x) - \cdots - f_n(x)| \\ &\leq |f_m(x) - f_{m-1}(x)| + |f_{m-1}(x) - f_{m-2}(x) + \cdots + f_{n+1}(x) - f_n(x)| \\ \therefore |f_m(x) - f_n(x)| &\leq \sum_{k=n}^{m-1} |f_{k+1}(x) - f_k(x)| \\ \therefore |f_m(x) - f_n(x)| &\leq \sum_{k=n}^{m-1} a_k \\ \therefore |f_m(x) - f_n(x)| &\leq \varepsilon \end{aligned}$$

Therefore, $\{f_n\}$ satisfies Cauchy's criterion for uniform convergence.

5 Series of Functions

Definition 10 (Pointwise convergence of series of functions). Let $\{f_n\}$ be a sequence of functions defined in D . Let $S_n(x) = \sum_{k=1}^n f_k(x)$.

If $S_n(x)$ converges for every $x \in D$ to a limit S , the series formed by $\{f_n\}$ is said to converge pointwise in D . It is denoted as

$$\sum_{k=1}^{\infty} f_k(x) = \lim_{n \rightarrow \infty} S_n(x) = S_x$$

Definition 11 (Uniform convergence of series of functions). The series $\sum_{k=1}^{\infty} f_k(x)$

is said to converge uniformly in D if $S_n \xrightarrow{D} S$.

Theorem 17. If $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly in D , then the general term $f_k(x)$ must uniformly converge to 0 in D .

Recitation 7 – Exercise 5.

Check the uniform convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n^2} - \frac{x^{n+1}}{(n+1)^2}$ in $[-1, 1]$.

Recitation 7 – Solution 5.

$$\begin{aligned} S_n(x) &= \sum_{k=1}^n \frac{x^k}{k^2} - \frac{x^{k+1}}{(k+1)^2} \\ &= \frac{x^1}{1^2} - \frac{x^{n+1}}{(n+1)^2} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n(x) &= \lim_{n \rightarrow \infty} x = \frac{x^{n+1}}{(n+1)^2} \\ &= x \\ &= S(x) \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{[-1,1]} |S_n(x) - S(x)| &= \sup_{[-1,1]} \left| -\frac{x^{n+1}}{(n+1)^2} \right| \\ &\leq \frac{1}{(n+1)^2} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{[-1,1]} |S_n(x) - S(x)| &\leq \lim_{n \rightarrow \infty} \frac{1}{(n+1)^2} \\ \therefore \lim_{n \rightarrow \infty} \sup_{[-1,1]} |S_n(x) - S(x)| &\leq 0 \end{aligned}$$

Therefore the convergence is uniform.

Theorem 18. If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly in D to $S(x)$ and the functions f_n are continuous in D , then the $S(x)$ is also continuous in D .

Theorem 19. A Leibniz series, i.e. a series of the form $\sum_{n=1}^{\infty} (-1)^n a_n$, with a_n monotonically decreasing and $\lim_{n \rightarrow \infty} a_n = 0$, converges, and

$$\sum_{k=n}^m (-1)^k a_k \leq a_n$$

Recitation 7 – Exercise 6.

Check for pointwise and uniform convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 + \sqrt{n}}$ in \mathbb{R} .

Recitation 7 – Solution 6.

For $x \in \mathbb{R}$, $\frac{1}{x^2 + \sqrt{n}}$ is monotonically decreasing to 0 as $n \rightarrow \infty$.

Therefore, for $x \in \mathbb{R}$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 + \sqrt{n}}$ is a Leibniz series. Hence, it converges pointwise.

$$\begin{aligned} \left| \sum_{k=n}^m \frac{(-1)^k}{x^2 + \sqrt{k}} \right| &\leq \frac{1}{x^2 + \sqrt{n}} \\ &\leq \frac{1}{\sqrt{n}} \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \left| \sum_{k=n}^m \frac{(-1)^k}{x^2 + \sqrt{k}} \right| = 0$$

Therefore, $\forall \varepsilon > 0$, there exists N such that $\forall m > n > N$, and $\forall x \in \mathbb{R}$,

$$\left| \sum_{k=n}^m \frac{(-1)^k}{x^2 + \sqrt{k}} \right| \leq \frac{1}{\sqrt{n}} < \varepsilon$$

Therefore, $\left| \sum_{k=n}^m \frac{(-1)^k}{x^2 + \sqrt{k}} \right|$ satisfies Cauchy's criterion for uniform convergence. Hence it converges uniformly.

Recitation 7 – Exercise 7.

Show that $\sum_{n=1}^{\infty} 3^n \sin\left(\frac{1}{4^n x}\right)$ does not converge uniformly in $(0, \infty)$.

Recitation 7 – Solution 7.

For any $x \in (0, \infty)$, as $\sin\left(\frac{1}{4^n x}\right) \leq \frac{1}{4^n x}$,

$$\left| 3^n \sin\left(\frac{1}{4^n x}\right) \right| \leq 3^n \frac{1}{4^n x}$$

Therefore, as $\sum \left(\frac{3}{4}\right)^n \cdot \frac{1}{x}$ converges, by the First Comparison Test, $\sum \left| 3^n \sin\left(\frac{1}{4^n x}\right) \right|$ also converges.

Therefore, $\sum 3^n \sin(\frac{1}{4^n x})$ converges absolutely. Hence, it converges.

$$\lim_{n \rightarrow \infty} 3^n \sin\left(\frac{1}{4^n x}\right) = \lim_{n \rightarrow \infty} \neq 0$$

Therefore as the general element does not tend to 0, the series does not converge uniformly in $(0, \infty)$.

5.1 Weierstrass M-test

Theorem 20 (Weierstrass M-test). *If $|u_k(x)| \leq c_k$ on D for $k \in \{1, 2, 3, \dots\}$ and the numerical series $\sum_{k=1}^{\infty} c_k$ converges, then the series of functions $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on D .*

Recitation 8 – Exercise 1.

Show that $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ converges uniformly in $[0, 1]$.

Recitation 8 – Solution 1.

$$\begin{aligned} \left| \sin \frac{x}{n^2} \right| &\leq \left| \frac{x}{n^2} \right| \\ \therefore \left| \sin \frac{x}{n^2} \right| &\leq \frac{1}{n^2} \end{aligned}$$

Therefore, as $\sum \frac{1}{n^2}$ converges in $[0, 1]$, by Weierstrass M-test, the series converges uniformly in $[0, 1]$.

Recitation 8 – Exercise 2.

Does $\sum \frac{(-1)^n}{x+n}$ converge on $[0, 1]$?

Recitation 8 – Solution 2.

$$\begin{aligned} \max_{[0,1]} |f_n| &= \max_{[0,1]} \frac{1}{x+n} \\ &= \frac{1}{n} \end{aligned}$$

Therefore, as $\sum \frac{1}{n}$ diverges, the Weierstrass M-test does not apply.

However, $\forall x \in [0, 1]$, $\sum \frac{(-1)^n}{x+n}$ is a Leibniz series. For a Leibniz series, the uniform convergence of the general term to 0 is a necessary and sufficient

condition for the convergence of the series.

Therefore, in $[0, 1]$,

$$\lim_{n \rightarrow \infty} \frac{1}{x + n} = 0$$

Therefore, as the general term goes to 0, the series converges.

Recitation 8 – Exercise 3.

Does $\sum \frac{n^2 x}{1 + n^7 x^2}$ converge uniformly in \mathbb{R} ?

Recitation 8 – Solution 3.

As the function is even,

$$\begin{aligned} \sup_{\mathbb{R}} \left| \frac{n^2 x}{1 + n^7 x^2} \right| &= \sup_{[0, \infty)} \left| \frac{n^2 x}{1 + n^7 x^2} \right| \\ &= \sup_{[0, \infty)} \frac{n^2 x}{1 + n^7 x^2} \end{aligned}$$

Let

$$f_n(x) = \frac{n^2 x}{1 + n^7 x^2}$$

Therefore,

$$f_n'(x) = \frac{n^2(1 + n^7 x^2) - n^9 \cdot 2x^2}{(1 + n^7 x^2)^2}$$

Therefore, maximizing $f_n(x)$,

$$\begin{aligned} f_n'(x) &= 0 \\ \iff n^2(1 + n^7 x^2 - n^7 \cdot 2x^2) &= 0 \\ \iff 1 - n^7 x^2 &= 0 \\ \iff x &= \sqrt{\frac{1}{n^7}} \end{aligned}$$

Therefore, as $f_n'(x) \geq 0 \iff x \in \left[0, \sqrt{\frac{1}{n^7}}\right]$ and $f_n'(x) \leq 0 \iff x \in \left[\sqrt{\frac{1}{n^7}}, \infty\right)$, $x = \sqrt{\frac{1}{n^7}}$ is a global maximum of f_n in $[0, \infty)$.

Therefore,

$$\begin{aligned}\sup_{\mathbb{R}} \left| \frac{n^2 x}{1 + n^7 x^2} \right| &= \frac{n^2 \sqrt{\frac{1}{n^7}}}{1 + 1} \\ &= \frac{n^2}{2n^{\frac{7}{2}}} \\ &= \frac{1}{2n^{\frac{3}{2}}}\end{aligned}$$

Therefore, as $\sum \frac{1}{2n^{\frac{3}{2}}}$ converges, by the Weierstrass M-test, $\sum \frac{n^2 x}{1 + n^7 x^2}$ converges uniformly in \mathbb{R} .

5.2 Application of Uniform Convergence

Theorem 21 (Changing the order of integration and infinite summation). *If the functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ are integrable on $[a, b]$ and the series $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on $[a, b]$ then*

$$\int_a^b \left(\sum_{k=1}^{\infty} u_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b u_k(x) dx$$

Theorem 22 (Changing the order of integration and limit). *If the functions $f_n(x)$ are integrable on $[a, b]$ and converge uniformly to f on $[a, b]$, then*

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx = \int_a^b f(x) dx$$

Additionally,

$$\int_a^x f_n(t) dt \xrightarrow{[a,b]} \int_a^x f(t) dt$$

Recitation 8 – Exercise 4.

Is $\sum_{n=1}^{\infty} \int_0^2 \frac{(-1)^{n+1}}{x+n} = \int_0^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{x+n}$?

Recitation 8 – Solution 4.

As $\sum \frac{(-1)^{n+1}}{x+n}$ converges uniformly in $[0, 2]$, by Theorem 21, the equality holds.

Recitation 8 – Exercise 5.

$$\text{Is } \int_0^1 \left(\sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \right) dx = \sum_{n=1}^{\infty} \int_0^1 \left(x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \right) dx?$$

Recitation 8 – Solution 5.

$$\begin{aligned} \sum_{n=1}^N x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} &= -x + \frac{1}{x^{2N+1}} \\ \therefore \lim_{n \rightarrow \infty} \sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} &= \sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \\ &= \lim_{n \rightarrow \infty} -x + \frac{1}{x^{2N+1}} \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} = \begin{cases} 1 - x & ; \quad x \in (0, 1] \\ 0 & ; \quad x = 0 \end{cases}$$

Therefore, as the limit function is not continuous but the function $x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}$ is continuous, the convergence is not uniform. Therefore, Theorem 21 is not applicable.

Therefore, checking directly,

$$\begin{aligned} \int_0^1 \left(\sum_{n=1}^{\infty} x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \right) dx &= \int_0^1 1 - x \, dx \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_0^1 \left(x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}} \right) dx &= \sum_{n=1}^{\infty} \left(\frac{x^{\frac{1}{2n+1}+1}}{\frac{1}{2n+1}+1} - \frac{x^{\frac{1}{2n-1}+1}}{\frac{1}{2n-1}+1} \Big|_0^1 \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{2n+1}{2n+2} - \frac{2n-1}{2n} \right) \\
&= \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{2n+1}{2n+2} - \frac{2n-1}{2n} \\
&= \lim_{N \rightarrow \infty} \frac{2N+1}{2N+2} - \frac{2 \cdot 1 - 1}{2} \\
&= \lim_{N \rightarrow \infty} \frac{2N+1}{2N+2} - \frac{1}{2} \\
&= 1 - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Therefore, although the convergence is not uniform, the equality holds.

Theorem 23 (Changing the order of differentiation and infinite summation). *If the functions $u_k(x)$, $k \in \{1, 2, 3, \dots\}$ are differentiable on $[a, b]$ and the derivatives are continuous on $[a, b]$, and the series $\sum_{k=1}^{\infty} u_k(x)$ converges pointwise on $[a, b]$ and the series $\sum_{k=1}^{\infty} u_k'(x)$ converges uniformly on $[a, b]$, then,*

$$\left(\sum_{k=1}^{\infty} u_k(x) \right)' = \sum_{k=1}^{\infty} u_k'(x)$$

Recitation 8 – Exercise 6.

If $\sum_{n=1}^{\infty} \left(\tan^{-1} \frac{x}{n^2} \right)' = \left(\sum_{n=1}^{\infty} \tan^{-1} \frac{x}{n^2} \right)'$?

Recitation 8 – Solution 6.

$$\begin{aligned}\left(\tan^{-1} \frac{x}{n^2}\right)' &= \frac{1}{\left(1 + \left(\frac{x}{n^2}\right)^2\right) n^2} \\ &= \frac{1}{n^2 + \frac{x^2}{n^2}} \\ &= \frac{n^2}{n^4 + x^2} \\ \therefore \left(\tan^{-1} \frac{x}{n^2}\right)' &\leq \frac{1}{n^2}\end{aligned}$$

Therefore, as $\sum \frac{1}{n^2}$ converges, by the Weierstrass M-test, $\sum \left(\tan^{-1} \frac{x}{n^2}\right)'$ converges uniformly.

By Lagrange's Mean Value Theorem, for c between 0 and x ,

$$\begin{aligned}\frac{\tan^{-1} x - \tan^{-1} 0}{x - 0} &= (\tan^{-1})'(c) \\ \therefore \frac{|\tan^{-1} x|}{|x|} &= \frac{1}{1 + c^2} \\ \therefore |\tan^{-1} x| &\leq 1 \\ \therefore |\tan^{-1} x| &\leq |x|\end{aligned}$$

Therefore,

$$\left|\tan^{-1} \frac{x}{n^2}\right| \leq \left|\frac{x}{n^2}\right|$$

Therefore, as $\forall x \in \mathbb{R} \sum \frac{x}{n^2}$ converges pointwise, $\sum \tan^{-1} \frac{x}{n^2}$ also converges on \mathbb{R} .

Therefore, by Theorem 23, the equality holds.

Part II

Functions of Multiple Variables

1 Change of Variables in Double Integrals

Definition 12 (Jacobian). Let

$$T(u, v) = (x, y)$$

be an operator.

The determinant

$$J = D_T = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

is called the Jacobian of the operator T .

Theorem 24. Let $T : \Delta \rightarrow D$ be a one-to-one, onto, continuously differentiable transformation, with $\det |D_T| \neq 0$, such that $(x, y) = T(u, v)$.¹ be non-zero. Let f be integrable on D .

Then,

$$\iint_D f \, dA = \iint_{\Delta} f \circ T \cdot |DT| \, dA$$

Recitation 9 – Exercise 1.

Calculate the area of the region bounded by

$$y = 0$$

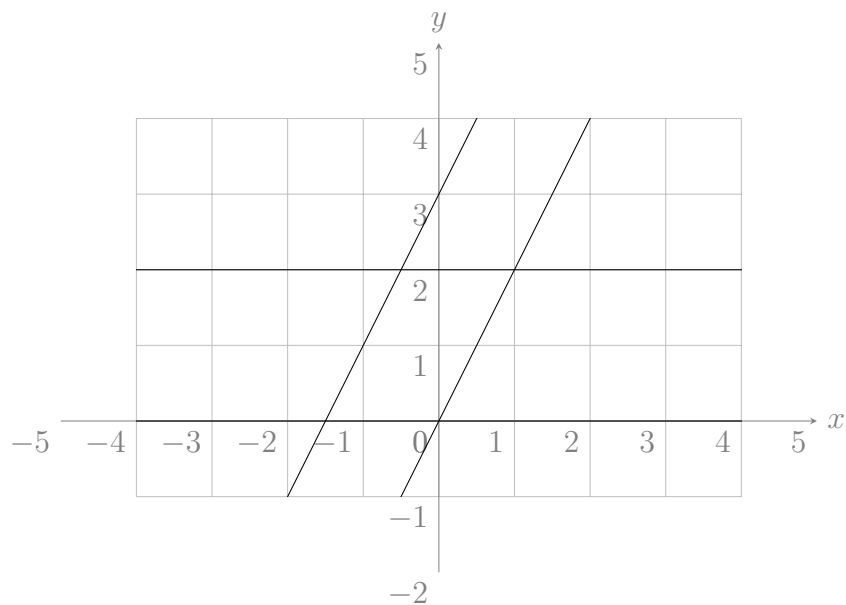
$$y = 2$$

$$y = 2x$$

$$y = 2x + 3$$

¹Continuously differentiable means that all partial derivatives x_u, x_v, y_u, y_v , exist and are continuous.

Recitation 9 – Solution 1.



The change of variables depends on the boundaries of the region.
Therefore, let

$$\begin{aligned} u &= y \\ v &= 2x - y \end{aligned}$$

Therefore,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Therefore,

$$\begin{aligned} y = 0 & \rightarrow u = 0 \\ y = 2 & \rightarrow u = 2 \\ y = 2x & \rightarrow v = 0 \\ y = 2x + 3 & \rightarrow v = 3 \end{aligned}$$

The Jacobian can be calculated by finding T^{-1} and then finding its determinant, or using the formula

$$\begin{aligned} |D_T| &= \frac{1}{|D_{T^{-1}}|} \\ \therefore |D_T| &= \frac{1}{|-2|} \\ &= \frac{1}{2} \end{aligned}$$

Therefore,

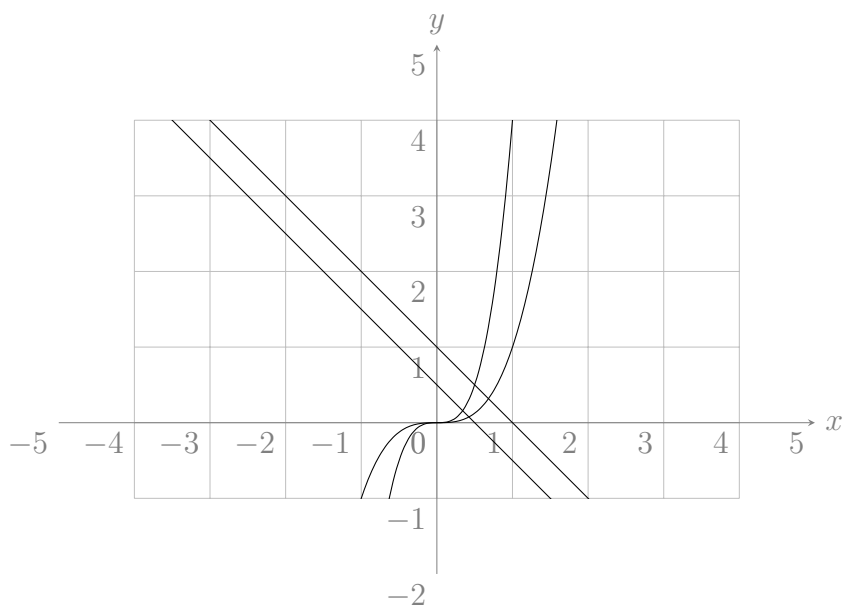
$$\begin{aligned} \iint_D dA &= \iint_{\Delta} \frac{1}{2} dA \\ &= \int_0^2 \int_0^3 \frac{1}{2} dv du \\ &= 3 \end{aligned}$$

Recitation 9 – Exercise 2.

Calculate $\iint_D \frac{x+3y}{x^4} e^{\frac{y}{x^3}}$, where D is bounded by

$$\begin{aligned} y &= x^3 \\ y &= 4x^3 \\ x + y &= 1 \\ x + y &= \frac{1}{2} \end{aligned}$$

Recitation 9 – Solution 2.



Let

$$u = \frac{y}{x^3}$$

$$v = x + y$$

Therefore, the domain D is transformed to $\{1 \leq u \leq 4, \frac{1}{2} \leq v \leq 1\}$.
Therefore,

$$J^{-1} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{3y}{x^4} & \frac{1}{x^3} \\ 1 & 1 \end{vmatrix}$$

$$= \frac{3y + x}{x^4}$$

Therefore,

$$J = \frac{1}{J^{-1}}$$

$$= \frac{x^4}{3y + x}$$

Therefore,

$$\begin{aligned}
 \iint_D \frac{x+3y}{x^4} e^{\frac{y}{x^3}} \, dA &= \iint_{\Delta} \frac{x+3y}{x^4} e^u J \, du \, dv \\
 &= \iint_{\Delta} \frac{x+3y}{x^4} \frac{x^4}{x+3y} e^u \, du \, dv \\
 &= \int_1^4 \int_{\frac{1}{2}}^1 e^u \, dv \, du \\
 &= \int_1^4 v e^u \Big|_{\frac{1}{2}}^1 \\
 &= \int_1^4 \frac{1}{2} e^u \, dy \\
 &= \frac{1}{2} e^u \Big|_1^4 \\
 &= \frac{1}{2} e^4 - \frac{1}{2} e^1 \\
 &= \frac{e^4 - e}{2}
 \end{aligned}$$

1.1 Polar Coordinates

For polar coordinates,

$$x = r \cos \theta$$

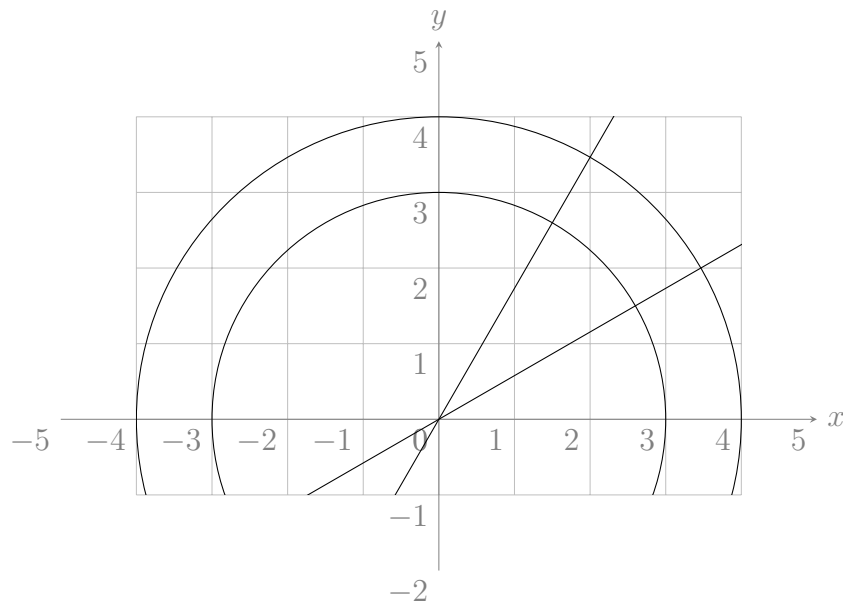
$$y = r \sin \theta$$

$$J = r$$

Recitation 9 – Exercise 3.

Find $\iint_D \frac{x}{y} \, dA$ where D is $\left\{ (x, y) : 9 \leq x^2 + y^2 \leq 16, \frac{\sqrt{3}}{3}x \leq y \leq \sqrt{3}x \right\}$.

Recitation 9 – Solution 3.



$$x^2 + y^2 = r^2$$

$$\frac{y}{x} = \tan \theta$$

Therefore, the domain D is transformed to the domain $\left\{3 \leq r \leq 4, \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}\right\}$.
Therefore,

$$\begin{aligned} \iint_D \frac{x}{y} dA &= \iint_{\Delta} \frac{1}{\tan \theta} J dr d\theta \\ &= \iint_{\Delta} r \cot \theta dr d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos \theta}{\sin \theta} dt \cdot \int_3^4 r dr \\ &= \left(\ln |\sin \theta| \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \right) \left(\frac{r^2}{2} \Big|_3^4 \right) \\ &= \frac{7}{2} \ln(\sqrt{3}) \\ &= \frac{7}{4} \ln 3 \end{aligned}$$

1.2 Generalized Polar Coordinates

In generalized polar coordinates or elliptical coordinates, the transformation is

$$\begin{aligned}x &= ar \cos \theta \\y &= br \sin \theta\end{aligned}$$

Therefore,

$$J = abr$$

Recitation 9 – Exercise 4.

Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > 0$, $b > 0$.

Recitation 9 – Solution 4.

$$\begin{aligned}\iint_D dA &= \iint_{\Delta} J dA \\&= \iint_{\Delta} abr dr d\theta \\&= \int_0^{2\pi} \int_0^1 abr dr d\theta \\&= \frac{1}{2}ab \cdot 2\pi \\&= \pi ab\end{aligned}$$

2 Change of Variables in Triple Integrals

Theorem 25. Let $T : \Delta \rightarrow D$ be a one-to-one, onto, continuously differentiable transformation, with $\det |D_T| \neq 0$, such that $(x, y, z) = T(u, v, w)$.² be non-zero. Let f be integrable on D .

Then,

$$\iiint_D f dA = \iiint_{\Delta} f \circ T \cdot |DT| dA$$

²Continuously differentiable means that all partial derivatives $x_u, x_v, x_w, y_u, y_v, y_w, z_u, z_v, z_w$, exist and are continuous.

2.1 Cylindrical Coordinates

For polar coordinates,

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = r$$

$$J = r$$

Recitation 10 – Exercise 1.

Calculate $\iiint_V \frac{z}{\sqrt{z^2+y^2}} dV$, where $V = \{z \leq \sqrt{x^2+y^2} + 1, x^2+y^2 \leq 1, z \geq 0\}$

Recitation 10 – Solution 1.

$z \leq \sqrt{x^2+y^2} + 1$ is the volume under a cone with its apex at $(0,0,1)$.

Therefore, as $x^2+y^2 \leq 1$ and $z \geq 0$, V is the volume under the cone, from $z=0$ to $z=2$.

Therefore, in cylindrical coordinates,

$$\begin{array}{lll} z = \sqrt{x^2+y^2} + 1 & \rightarrow & z = r + 1 \\ x^2 + y^2 = 1 & \rightarrow & r^2 = 1 \\ z = 0 & \rightarrow & z = 0 \end{array}$$

Therefore,

$$\begin{aligned} \iiint_V \frac{z}{\sqrt{x^2+y^2}} dV &= \iiint_{\Delta} \frac{z}{r} r dV \\ &= \int_0^{2\pi} \int_0^1 \int_0^{r+1} z dz dr d\theta \\ &= 2\pi \int_0^1 \frac{(r+1)^2}{2} dr \\ &= 2\pi \left(\frac{(r+1)^3}{6} \right) \Big|_0^1 \\ &= 2\pi \left(\frac{8}{6} - \frac{1}{6} \right) \\ &= \frac{14\pi}{6} \end{aligned}$$

2.2 Spherical Coordinates

For spherical coordinates,

$$x = r \cos \theta \sin \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \varphi$$

$$J = r^2 \sin \varphi$$

where ρ is the distance from $(0, 0, 0)$ to (x, y, z) ; θ is the angle between the line joining $(0, 0, 0)$ and (x, y, z) , and the x axis; and φ is the angle between the line joining $(0, 0, 0)$ and (x, y, z) , and the z axis.

Recitation 10 – Exercise 2.

Calculate the volume of the body

$$V = \left\{ (x, y, z) : x^2 + y^2 + z^2 \leq 1, z \geq \sqrt{x^2 + y^2} \right\}$$

Recitation 10 – Solution 2.

$x^2 + y^2 + z^2 \leq 1$ is the volume inside a sphere of radius 1 centred at $(0, 0, 0)$, and $z \geq \sqrt{x^2 + y^2}$ is the area above a right angled cone.

Therefore, in spherical coordinates,

$$\begin{aligned} x^2 + y^2 + z^2 = 1 &\rightarrow r^2 = 1 \\ \therefore x^2 + y^2 + z^2 = 1 &\rightarrow r = 1 \\ \sqrt{x^2 + y^2} = z &\rightarrow r \cos \varphi = \sqrt{(r \cos \theta \sin \varphi)^2 + (r \sin \theta \sin \varphi)^2} \\ \sqrt{x^2 + y^2} = z &\rightarrow r \cos \varphi = r \sin \varphi \end{aligned}$$

Therefore,

$$\begin{aligned} \iiint_V dV &= \iiint_{\Delta} r^2 \sin \varphi \, dV \\ &= \int_0^{\frac{\pi}{4}} \int_0^1 \int_0^{2\pi} r^2 \sin \varphi \, d\theta \, dr \, d\varphi \\ &= \frac{2\pi}{3} \left(1 - \frac{1}{\sqrt{2}} \right) \end{aligned}$$

Recitation 10 – Exercise 3.

Calculate $\iiint_V \frac{1}{\sqrt{x^2+y^2+z^2}} dV$ where

$$V = \{(x, y, z) : x^2 + y^2 + z^2 \leq 2z\}$$

Recitation 10 – Solution 3.

$$\begin{aligned} x^2 + y^2 + z^2 &\leq 2z \\ \therefore x^2 + y^2 + (z-1)^2 &\leq 1 \end{aligned}$$

Therefore, the region is a sphere of radius 1 centred at $(0, 0, 1)$.
Therefore, let

$$\begin{aligned} x &= r \cos \theta \sin \varphi \\ y &= r \sin \theta \sin \varphi \\ z &= r \cos \varphi + 1 \end{aligned}$$

Therefore,

$$J = r^2 \sin \varphi$$

Therefore,

$$\begin{aligned} \iiint_V \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV &= \iiint_{\Delta} \frac{r^2 \sin \varphi}{\sqrt{x^2 + y^2 + z^2}} dV \\ \sqrt{x^2 + y^2 + z^2} &= \sqrt{(r \cos \theta \sin \varphi)^2 + (r \sin \theta \sin \varphi)^2 + (r \cos \varphi + 1)^2} \\ &= r^2 + 2r \cos \varphi + 1 \end{aligned}$$

Therefore,

$$\iiint_V \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV = \int_0^{2\pi} \int_0^1 \int_0^\pi \frac{r^2 \sin \varphi}{\sqrt{r^2 + 2r \cos \varphi + 1}} d\varphi dr d\theta$$

Let

$$\begin{aligned} t &= r^2 + 2r \cos \varphi + 1 \\ \therefore dt &= -2r \sin \varphi d\varphi \end{aligned}$$

Therefore,

$$\begin{aligned}
 \int_0^\pi \frac{r^2 \sin \varphi}{\sqrt{r^2 + 2r \cos \varphi + 1}} d\varphi &= -\frac{r}{2} \int_{r^2+2r+1}^{r^2-2r+1} \frac{1}{\sqrt{t}} dt \\
 &= -r \cdot \sqrt{t} \Big|_{r^2+2r+1}^{r^2-2r+1} \\
 &= -r \left(\sqrt{r^2 + 2r + 1} - \sqrt{r^2 - 2r + 1} \right) \\
 &= -r \left(\sqrt{(r+1)^2} - \sqrt{(r-1)^2} \right)
 \end{aligned}$$

As $r \leq 1$, $r - 1 \leq 0$. Therefore, $\sqrt{(r-1)^2} = (r-1)$. Therefore,

$$\begin{aligned}
 \int_0^\pi \frac{r^2 \sin \varphi}{\sqrt{r^2 + 2r \cos \varphi + 1}} d\varphi &= r((r+1) - (1-r)) \\
 &= 2r^2
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \iiint_V \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV &= \int_0^{2\pi} \int_0^1 2r^2 dr d\theta \\
 &= 2\pi \int_0^1 2r^2 dr \\
 &= 2\pi \cdot \frac{2}{3} \\
 &= \frac{4\pi}{3}
 \end{aligned}$$

Recitation 10 – Exercise 4.

Calculate $\iiint_V x^5 y z - y^5 x z dV$, where V is bounded by $y = 0$, $z = 0$, $z = 1$, and satisfies $x^2 - y^2 \geq 1$ and $0 \leq x^2 + y^2 \leq 4$.

Recitation 10 – Solution 4.

Let

$$\begin{aligned}
 u &= z \\
 v &= x^2 - y^2 \\
 w &= x^2 + y^2
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 J^{-1} &= \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \\
 &= \begin{vmatrix} 0 & 0 & 1 \\ 2x & -2y & 0 \\ 2x & 2y & 0 \end{vmatrix} \\
 &= 8xy \\
 \therefore J &= \frac{1}{8xy}
 \end{aligned}$$

Therefore,

$$\begin{array}{lll}
 z = 0 & \rightarrow & u = 0 \\
 z = 1 & \rightarrow & u = 1 \\
 y = 0 & \rightarrow & v = w \\
 x^2 - y^2 \geq 1 & \rightarrow & v \geq 1 \\
 x^2 + y^2 = 0 & \rightarrow & w = 0 \\
 x^2 + y^2 = 4 & \rightarrow & w = 4
 \end{array}$$

Therefore,

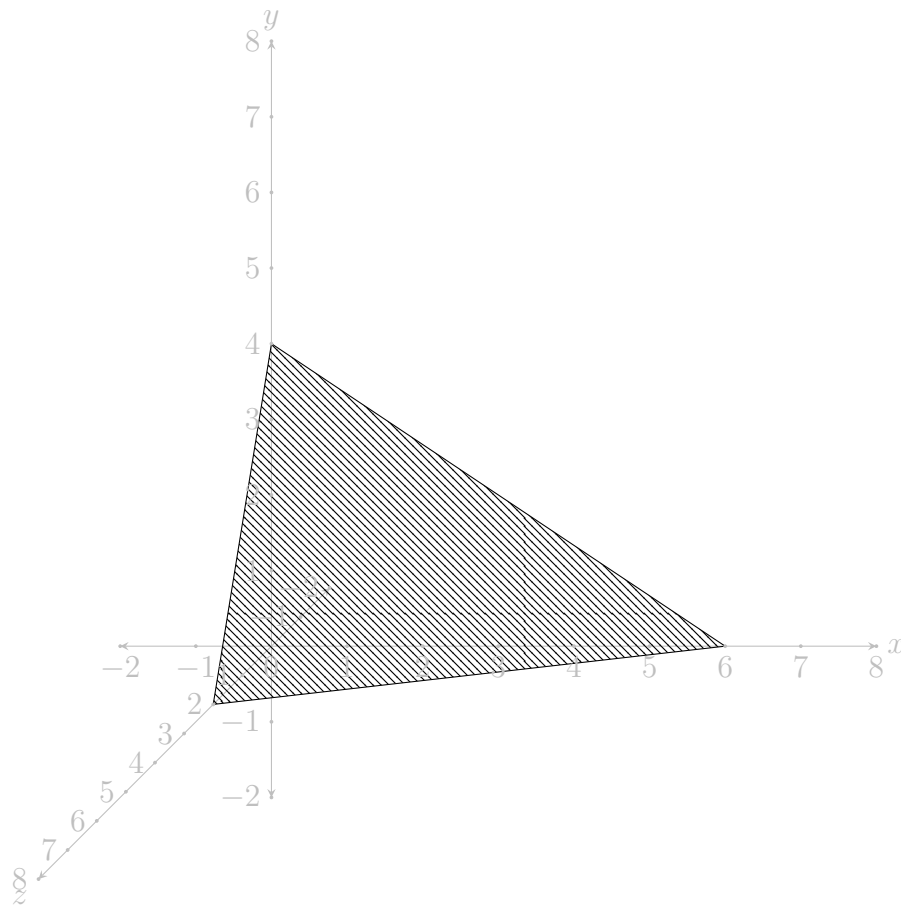
$$\begin{aligned}
\iiint_V x^5 y z - y^5 x z \, dV &= \iiint_{\Delta} (x^5 y z - y^5 x z) \frac{1}{8xy} \, dV \\
&= \int_0^1 \int_1^4 \int_1^w \frac{x^5 y z - y^5 x z}{8xy} \, dv \, dw \, du \\
&= \frac{1}{8} \int_0^1 \int_1^4 \int_1^w x^4 z - y^4 z \, dv \, dw \, du \\
&= \frac{1}{8} \int_0^1 \int_1^4 \int_1^w z(x^4 - y^4) \, dv \, dw \, du \\
&= \frac{1}{8} \int_0^1 \int_1^4 \int_1^w z(x^2 + y^2)(x^2 - y^2) \, dv \, dw \, du \\
&= \frac{1}{8} \int_0^1 \int_1^4 \int_1^w uvw \, dv \, dw \, du \\
&= \frac{1}{8} \int_0^1 \int_1^4 \frac{uv^2 w}{2} \Big|_{v=1}^{v=w} \, dv \, dw \, du \\
&= \frac{1}{8} \int_0^1 \int_1^4 \frac{uw^3}{3} - \frac{uw}{2} \, dw \, du \\
&= \frac{1}{8} \int_0^1 \frac{uw^4}{4} - \frac{uw^2}{2} \Big|_{w=1}^{w=4} \, dw \, du \\
&= \frac{1}{8} \int_0^1 \frac{255u}{4} - \frac{15u}{2} \, du \\
&= \frac{1}{8} \int_0^1 \frac{225u}{4} \, du \\
&= \frac{1}{8} \left(\frac{225u^2}{8} \right) \Big|_0^1 \\
&= \frac{1}{8} \cdot \frac{225}{8} \\
&= \frac{225}{64}
\end{aligned}$$

3 Surface Integrals of Scalar Functions

Recitation 11 – Exercise 1.

Calculate the area of the plane $z = 2 - \frac{1}{2}y - \frac{1}{3}x$, bounded by $x = 0$, $y = 0$, $z = 0$.

Recitation 11 – Solution 1.



As the surface is the graph of a function of x and y , it can be parametrized as,

$$\begin{aligned}\bar{r}(x, y) &= (x, y, f(x, y)) \\ &= \left(x, y, 2 - \frac{1}{3}x - \frac{1}{2}y\right)\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{r}_u &= \left(1, 0, -\frac{1}{3}\right) \\ \bar{r}_v &= \left(0, 1, -\frac{1}{2}\right)\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{r}_u \times \bar{r}_v &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} \\ &= \frac{1}{3}\hat{i} + \frac{1}{2}\hat{j} + \hat{k}\end{aligned}$$

Therefore,

$$\begin{aligned}|\bar{r}_u \times \bar{r}_v| &= \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2} \\ &= \sqrt{\frac{1}{9} + \frac{1}{4} + 1} \\ &= \sqrt{\frac{49}{36}} \\ &= \frac{7}{6}\end{aligned}$$

Therefore,

$$\begin{aligned}
A &= \iint_S dS \\
&= \iint_D |\bar{r}_u \times \bar{r}_v| dx dy \\
&= \iint_D \frac{7}{6} dx dy \\
&= \frac{7}{6} \iint_D dx dy \\
&= \frac{7}{6} \int_0^6 \int_0^{4-\frac{2}{3}x} dy dx \\
&= \frac{7}{6} \int_0^6 \left(4 - \frac{2}{3}x\right) dx \\
&= \frac{7}{6} \left(4 \cdot 6 - \frac{2}{3} \cdot \frac{36}{2}\right) \\
&= \frac{7}{6} (24 - 12) \\
&= \frac{7}{6} \cdot 12 = 14
\end{aligned}$$

Recitation 11 – Exercise 2.

Calculate the area of the surface of the part of the cone $z = \sqrt{x^2 + y^2}$ which satisfies $x^2 + y^2 \leq 2x$.

Recitation 11 – Solution 2.

$$\begin{aligned}
x^2 + y^2 &\leq 2x \\
\therefore (x-1)^2 + y^2 &\leq 1
\end{aligned}$$

Therefore, $x^2 + y^2 \leq 2x$ is the region inside a cone of radius 1 centred at $(1, 0, 0)$.

As the surface is the graph of a function of x and y , it can be parametrized as,

$$\begin{aligned}
\bar{r}(x, y) &= (x, y, f(x, y)) \\
&= \left(x, y, \sqrt{x^2 + y^2}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{r}_u &= \left(1, 0, \frac{x}{\sqrt{x^2 + y^2}}\right) \\ \bar{r}_v &= \left(0, 1, \frac{y}{\sqrt{x^2 + y^2}}\right)\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{r}_u \times \bar{r}_v &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{x}{\sqrt{x^2 + y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2 + y^2}} \end{vmatrix} \\ &= \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2 \hat{i} + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 \hat{j} + \hat{k}\end{aligned}$$

Therefore,

$$\begin{aligned}|\bar{r}_u \times \bar{r}_v| &= \sqrt{\left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2 + 1} \\ &= \sqrt{2}\end{aligned}$$

Therefore,

$$\begin{aligned}A &= \iint_S dS \\ &= \iint_D \sqrt{2} \, dx \, dy\end{aligned}$$

Let

$$\begin{aligned}x &= r \cos \theta + 1 \\ y &= r \sin \theta\end{aligned}$$

Therefore,

$$\begin{aligned}A &= \iint_D \sqrt{2} \, dx \, dy \\ &= \sqrt{2}\pi\end{aligned}$$

Recitation 11 – Exercise 3.

Calculate the area of the surface of the part of the cone $z = \sqrt{x^2 + y^2}$ which satisfies $x^2 + y^2 \leq 2x$.

Recitation 11 – Solution 3.

Let

$$\begin{aligned}x &= r \cos \theta \\y &= r \sin \theta \\\therefore z &= r\end{aligned}$$

Therefore,

$$\bar{s}(r, \theta) = (r \cos \theta, r \sin \theta, r)$$

Therefore,

$$\begin{aligned}\bar{s}_r &= (\cos \theta, \sin \theta, 1) \\\bar{s}_\theta &= (-r \sin \theta, r \cos \theta, 0)\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{s}_r \times \bar{s}_\theta &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & 1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\&= r \cos \theta \hat{i} + r \sin \theta \hat{j} + (-r \sin^2 \theta - r \cos^2 \theta) \hat{k} \\&= r \cos \theta \hat{i} + r \sin \theta \hat{j} - (r) \hat{k}\end{aligned}$$

Therefore,

$$\begin{aligned}|\bar{s}_r \times \bar{s}_\theta| &= \sqrt{(r \cos \theta)^2 + (r \sin \theta)^2 + r^2} \\&= r\sqrt{2}\end{aligned}$$

Therefore,

$$\begin{aligned}
A &= \iint_S dS \\
&= \iint_D r\sqrt{2} \, dr \, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \sqrt{2}r \, dr \, d\theta \\
&= \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2\theta \, d\theta \\
&= \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta + 1 \, d\theta \\
&= \pi\sqrt{2}
\end{aligned}$$

Recitation 11 – Exercise 4.

Calculate $\iint_S x^2 \, dS$, where $S : \{(x, y, z) : x^2 + y^2 = 1, 0 \leq z \leq 1\}$.

Recitation 11 – Solution 4.

The surface is a cylinder with radius \sqrt{a} .

Therefore, in cylindrical coordinates with $r = \sqrt{a}$,

$$\begin{aligned}
x &= \sqrt{a} \cos \theta \\
y &= \sqrt{a} \sin \theta \\
z &= z
\end{aligned}$$

Therefore,

$$\vec{s}(\theta, z) = (\sqrt{a} \cos \theta, \sqrt{a} \sin \theta, z)$$

Therefore,

$$\begin{aligned}
\vec{s}_\theta &= (-\sqrt{a} \sin \theta, \sqrt{a} \cos \theta, 0) \\
\vec{s}_z &= (0, 0, 1)
\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{s}_\theta \times \bar{s}_z &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sqrt{a} \sin \theta & \sqrt{a} \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \sqrt{a} \cos \theta \hat{i} + \sqrt{a} \sin \theta \hat{j}\end{aligned}$$

Therefore,

$$\begin{aligned}|\bar{s}_\theta \times \bar{s}_z| &= \sqrt{(\sqrt{a} \cos \theta)^2 + (\sqrt{a} \sin \theta)^2} \\ &= \sqrt{a}\end{aligned}$$

Therefore,

$$\begin{aligned}\iint_S x^2 \, dS &= \iint_D \sqrt{a} \, dx \, dy \\ &= \int_0^{2\pi} \int_0^1 a \cos^2 \theta \cdot \sqrt{a} \, dz \\ &= a\sqrt{a} \int_0^{2\pi} \cos^2 \theta \, d\theta \\ &= a\sqrt{a} \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} \, d\theta \\ &= a\sqrt{a}\pi\end{aligned}$$

Recitation 11 – Exercise 5.

Calculate $\iint_S z^2 \, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Recitation 11 – Solution 5.

In spherical coordinates with $\rho = 1$,

$$\begin{aligned}x &= \cos \theta \sin \varphi \\ y &= \sin \theta \sin \varphi \\ z &= \cos \varphi\end{aligned}$$

Therefore,

$$\bar{r} = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$$

Therefore,

$$\begin{aligned}\bar{r}_\theta &= (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0) \\ \bar{r}_\varphi &= (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi)\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{r}_\theta \times \bar{r}_\varphi &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin \theta \sin \varphi & \cos \theta \sin \varphi & 0 \\ \cos \theta \cos \varphi & \sin \theta \cos \varphi & -\sin \varphi \end{vmatrix} \\ &= -\cos \theta \sin^2 \varphi \hat{i} - \sin \theta \sin^2 \varphi \hat{j} - (\sin^2 \theta \cos \varphi \sin \varphi - \cos^2 \theta \sin \varphi \cos \varphi) \hat{k} \\ &= -\cos \theta \sin^2 \varphi \hat{i} - \sin \theta \sin^2 \varphi \hat{j} - \cos \varphi \sin \varphi \hat{k}\end{aligned}$$

Therefore,

$$\begin{aligned}|\bar{r}_\theta \times \bar{r}_\varphi| &= \sqrt{\cos^2 \theta \sin^4 \varphi + \sin^2 \theta \sin^4 \varphi + \cos^2 \varphi \sin^2 \varphi} \\ &= \sqrt{\sin^4 \varphi + \cos^2 \varphi \sin^2 \varphi} \\ &= \sqrt{\sin^2 \varphi} \\ &= \sin \varphi\end{aligned}$$

Therefore,

$$\begin{aligned}\iint_S z^2 \, dS &= \iint_D \sin \varphi \, d\theta \, d\varphi \\ &= \int_0^\pi \int_0^{2\pi} \cos^2 \varphi \sin \varphi \, d\theta \, d\varphi \\ &= 2\pi \int_0^\pi \cos^2 \varphi \sin \varphi \, d\varphi \\ &= 2\pi \left(-\frac{\cos^3 \varphi}{3} \Big|_0^\pi \right) \\ &= 2\pi \left(\frac{1}{3} + \frac{1}{3} \right) \\ &= \frac{4\pi}{3}\end{aligned}$$

4 Surface Integrals of Vector Functions

Recitation 12 – Exercise 1.

Find the outward unit normal vector for the surface $z = x^2 + y^2$.

Recitation 12 – Solution 1.

If $z = g(x, y)$, $\overline{N} = (g_x, g_y, -1)$.

As the z coordinate is negative, \overline{N} points outwards.

Therefore,

$$\hat{n} = \pm \frac{(2x, 2y, -1)}{\sqrt{4x^2 + 4y^2 + 1}}$$

Recitation 12 – Exercise 2.

Find the unit normal vectors which point outwards for $S = T_1 \cup T_2 \cup T_3$, where

$$T_1 = \{(x, y, z) | x^2 + y^2 = 4, 1 \leq z \leq 4\}$$

$$T_2 = \{(x, y, z) | x^2 + y^2 \leq 4, z = 1\}$$

$$T_3 = \{(x, y, z) | x^2 + y^2 \leq 4, z = 4\}$$

Recitation 12 – Solution 2.

In cylindrical coordinates with $r = 2$,

$$x = 2 \cos \theta$$

$$y = 2 \sin \theta$$

$$z = z$$

Therefore, for T_1 , $0 \leq \theta \leq 2\pi$, $1 \leq z \leq 4$.

Therefore,

$$T_1(\theta, z) = (2 \cos \theta, 2 \sin \theta, z)$$

Therefore,

$$(T_1)_\theta = (-2 \sin \theta, 2 \cos \theta, 0)$$

$$(T_2)_z = (0, 0, 1)$$

Therefore,

$$\begin{aligned}(T_1)_\theta \times (T_1)_z &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 \sin \theta & 2 \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= (2 \cos \theta, 2 \sin \theta, 0)\end{aligned}$$

For $\theta = 0$, $(T_1)_\theta \times (T_1)_z = (2, 0, 0)$.

Therefore, it is directed outwards, with respect to the cylinder.

Therefore, the unit normal is

$$\hat{n}_1 = \frac{(2 \cos \theta, 2 \sin \theta, 0)}{2}$$

T_2 and T_3 are parts of the planes $z = 1$ and $z = 4$ respectively.

Therefore, the unit normals directed outwards with respect to the cylinder are

$$\hat{n}_2 = (0, 0, -1)$$

$$\hat{n}_3 = (0, 0, 1)$$

Recitation 12 – Exercise 3.

Calculate $\iint_S \bar{F} \cdot \hat{n} \, dS$ where $\bar{F} = (-x^2, -y, z^2)$. S is the cone $z = \sqrt{x^2 + y^2}$ bounded by $z = 1$ and $z = 2$, with orientation outwards with respect to the cone.

Recitation 12 – Solution 3.

The surface S can be parametrized as

$$\bar{r} = \left(x, y, \sqrt{x^2 + y^2} \right)$$

with $(x, y) \in D$, where $D : \{1 \leq x^2 + y^2 \leq 4\}$.

Therefore, D is the region between circles of radii 1 and 2, centred at $(0, 0, 0)$.

Therefore, the outward normals are given by

$$\begin{aligned}\hat{n} &= (g_x, g_y, -1) \\ &= \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1 \right)\end{aligned}$$

Therefore,

$$\begin{aligned}
\iint_S \vec{F} \cdot \hat{n} \, dS &= \iint_D (-x, -y, z^2) \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1 \right) dS \\
&= \iint_D (-x, -y, x^2 + y^2) \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1 \right) dx \, dy \\
&= \iint_D -\frac{x^2}{\sqrt{x^2 + y^2}} - \frac{y^2}{\sqrt{x^2 + y^2}} - (x^2 + y^2) dx \, dy \\
&= \int_0^{2\pi} \int_1^2 \left(-\frac{r^2}{r} - r^2 \right) r \, dr \, d\theta \\
&= 2\pi \int_1^2 -r^2 - r^3 \, dr \\
&= -\frac{73\pi}{6}
\end{aligned}$$

Alternatively, by S can be parametrized as

$$\vec{\Gamma} = (r \cos \theta, r \sin \theta, r)$$

with $(r, \theta) \in D$, where $D : \{1 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$.

Therefore, D is the region between circles of radii 1 and 2, centred at $(0, 0, 0)$.

Therefore,

$$\begin{aligned}
\vec{\Gamma}_r \times \vec{\Gamma}_\theta &= (-r \cos \theta, -r \sin \theta, r \cos^2 \theta + r \sin^2 \theta) \\
&= (-r \cos \theta, -r \sin \theta, r)
\end{aligned}$$

As this vector is directed inwards with respect to the cone,

$$\begin{aligned}
\hat{n} &= -\vec{\Gamma}_r \times \vec{\Gamma}_\theta \\
&= (r \cos \theta, r \sin \theta, -r)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\iint_S \vec{F} \cdot \hat{n} \, dS &= \int_0^{2\pi} \int_1^2 (-r \cos \theta, -r \sin \theta, r^2) \cdot (r \cos \theta, r \sin \theta, -r) \, dr \, d\theta \\
&= 2\pi \int_1^2 -r^2 - r^2 \, dr \\
&= -\frac{73\pi}{6}
\end{aligned}$$

5 Stoke's Theorem

Recitation 12 – Exercise 4.

Let C be the circle $x^2 + y^2 = 4$, and let $\overline{F} = (2z, 3x, 5y)$. Calculate $\int_C \overline{F} \, dr$, where the direction of C is positive.

Recitation 12 – Solution 4.

$$\begin{aligned}\operatorname{curl} \overline{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} \\ &= (5, 2, 3)\end{aligned}$$

Let S be the paraboloid $z = 4 - x^2 - y^2$, $z \geq 0$.
Parametrizing \overline{F} ,

$$\begin{aligned}\overline{\Gamma} &= (x, y, 4 - x^2 - y^2) \\ \therefore \overline{\Gamma}_x \times \overline{\Gamma}_y &= (-2x, -2y, -1)\end{aligned}$$

As this vector is directed inwards,

$$\begin{aligned}\hat{n} &= -\overline{\Gamma}_x \times \overline{\Gamma}_y \\ &= (2x, 2y, 1)\end{aligned}$$

Therefore, by Stoke's Theorem,

$$\begin{aligned}\int_C \overline{F} \, dr &= \int_S \operatorname{curl} \overline{F} \cdot \hat{n} \, dS \\ &= \iint_D (5, 2, 3) \cdot (2x, 2y, 1) \, dA \\ &= \iint_D (10x + 4y + 3) \, dx \, dy \\ &= \int_0^{2\pi} \int_0^2 (10r \cos \theta + 4r \sin \theta + 3) \, dr \, d\theta \\ &= 12\pi\end{aligned}$$

Alternatively, let S be the circular region of radius 2. Parametrizing \overline{F} ,

$$\begin{aligned}\overline{\Gamma} &= (x, y, 0) \\ \therefore \overline{\Gamma}_x \times \overline{\Gamma}_y &= (0, 0, 1)\end{aligned}$$

As this vector is directed outwards,

$$\hat{n} = (0, 0, 1)$$

Therefore, by Stoke's Theorem,

$$\begin{aligned}\int_C \overline{F} \, dr &= \int_S \operatorname{curl} \overline{F} \cdot \hat{n} \, dS \\ &= \iint_D (5, 2, 3) \cdot (0, 0, 1) \, dA \\ &= 12\pi\end{aligned}$$