

# Differential and Integral Calculus

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# **2 Required Reading**

Protter and Morrey: *A first Course in Real Analysis*, UTM Series, Springer-Verlag, 1991

# **3 Additional Reading**

Thomas and Finney, *Calculus and Analytic Geometry*, 9th edition, Addison-Wesley, 1996

## Part I

# Sequences and Series

## 1 Sequences

**Definition 1** (Sequence). A sequence of real numbers is a set of numbers which are written in some order. There are infinitely many terms in a sequence. It is denoted by  $\{a_n\}_{n=1}^{\infty}$  or  $\{a_n\}$ .

**Example 1.**  $1, \frac{1}{2}, \frac{1}{3}, \dots$  is called the harmonic sequence.

$$a_n = \frac{1}{n}$$

**Example 2.**  $1, -\frac{1}{2}, \frac{1}{3}, \dots$  is called the alternating harmonic sequence.

$$a_n = (-1)^{n+1} \frac{1}{n}$$

**Example 3.**  $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$

$$a_n = \frac{n}{n+1}$$

**Example 4.**  $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \dots$

$$a_n = \frac{n+1}{3^n}$$

**Example 5.** The Fibonacci sequence is given by

$$f_n = \begin{cases} 1 & ; \quad n = 1, 2 \\ f_{n-1} + f_{n-2} & ; \quad n \geq 3 \end{cases}$$

**Example 6.** A geometric sequence is given by

$$a_n = a_1 q^{n-1}$$

where  $q$  is called the common ratio.

**Example 7.** A geometric sequence is given by

$$a_n = a_1 + d(n - 1)$$

where  $d$  is called the common difference.

**Definition 2** (Equal sequences). Two sequences  $\{a_n\}$  and  $\{b_n\}$  are said to be equal if  $a_n = b_n, \forall n \in \mathbb{N}$ .

**Definition 3** (Sequences bounded from above).  $\{a_n\}$  is said to be bounded from above if  $\exists M \in \mathbb{R}$ , s.t.  $a_n \leq M, \forall n \in \mathbb{N}$ . Each such  $M$  is called an upper bound of  $\{a_n\}$ .

**Definition 4** (Sequences bounded from below).  $\{a_n\}$  is said to be bounded from below if  $\exists m \in \mathbb{R}$ , s.t.  $a_n \geq m, \forall n \in \mathbb{N}$ . Each such  $m$  is called a lower bound of  $\{a_n\}$ .

**Definition 5.**  $\{a_n\}$  is said to be bounded if it is bounded from below and bounded from above.

**Example 8.** The sequence  $a_n = n^2 + 2$  is not bounded from above but is bounded from below, by all  $m \leq 3$ .

**Example 9.**  $\left\{ \frac{2n-1}{3n} \right\}$  is bounded.

$$m = 0 \leq \frac{2n-1}{3n} \leq \frac{2n}{3n} = \frac{2}{3} = M$$

**Definition 6** (Monotonic increasing sequence). A sequence  $\{a_n\}$  is called monotonic increasing if  $\exists n_0 \in \mathbb{N}$ , s.t.  $a_n \leq a_{n+1}, \forall n \geq n_0$ .

**Definition 7** (Monotonic decreasing sequence). A sequence  $\{a_n\}$  is called monotonic decreasing if  $\exists n_0 \in \mathbb{N}$ , s.t.  $a_n \geq a_{n+1}, \forall n \geq n_0$ .

**Definition 8** (Strongly increasing sequence). A sequence  $\{a_n\}$  is called monotonic increasing if  $\exists n_0 \in \mathbb{N}$ , s.t.  $a_n < a_{n+1}, \forall n \geq n_0$ .

**Definition 9** (Strongly decreasing sequence). A sequence  $\{a_n\}$  is called monotonic decreasing if  $\exists n_0 \in \mathbb{N}$ , s.t.  $a_n > a_{n+1}, \forall n \geq n_0$ .

**Example 10.** The sequence  $\left\{\frac{n^2}{2^n}\right\}$  is strongly decreasing. However, this is not evident by observing the first few terms.  $\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \dots$

$$\begin{aligned}
& a_n > a_{n+1} \\
\iff & \frac{n^2}{2^n} > \frac{(n+1)^2}{2^{n+1}} \\
\iff & 2n^2 > (n+1)^2 \\
\iff & \sqrt{2}n > n+1 \\
\iff & n(\sqrt{2}-1) > 1 \\
\iff & n > \frac{1}{\sqrt{2}-1} \\
\iff & n > 3
\end{aligned}$$

**Exercise 1.**

Is  $a_n = (-1)^n$  monotonic?

**Solution 1.**

The sequence  $-1, 1, -1, 1, \dots$  is not monotonic.

## 1.1 Limit of a Sequence

**Definition 10.** Let  $\{a_n\}$  be a given sequence. A number  $L$  is said to be the limit of the sequence if  $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ , s.t.  $|a_n - L| < \varepsilon, \forall n \geq n_0$ . That is, there are infinitely many terms inside the interval and a finite number of terms outside it.

**Example 11.** The sequence  $\left\{\frac{1}{n}\right\}$  tends to 0, i.e. for any open interval  $(-\varepsilon, \varepsilon)$ , there are finite number of terms of the sequence outside the interval, and therefore there are infinitely many terms inside the interval.

**Exercise 2.**

Prove

$$\lim_{n \rightarrow \infty} \frac{n+2}{2n-1} = \frac{1}{2}$$

**Solution 2.**

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$

**Exercise 3.**

Prove that 2 is not a limit of  $\left\{ \frac{3n+1}{n} \right\}$ .

**Solution 3.**

If possible, let

$$\lim_{n \rightarrow \infty} \frac{3n+1}{n} = 2$$

Then,  $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ , s.t.  $\left| \frac{3n+1}{n} - 2 \right| < \varepsilon, \forall n \geq n_0$ . However,

$$\left| \frac{3n+1}{n} - 2 \right| = 1 + \frac{1}{n} > 1$$

This is a contradiction for  $\varepsilon = \frac{1}{2}$ . Therefore, 2 is not a limit.

**Theorem 1.** *If a sequence  $\{a_n\}$  has a limit  $L$  then the limit is unique.*

*Proof.* If possible let there exist two limits  $L_1$  and  $L_2$ . Therefore,  $\forall \varepsilon > 0$ , there exist a finite number of terms in the interval  $(L_1 - \varepsilon, L_1 + \varepsilon)$ . Therefore, there exist a finite number of terms in the interval  $(L_2 - \varepsilon, L_2 + \varepsilon)$ . This contradicts the definition of a limit. Therefore, the limit is unique.  $\square$

**Theorem 2.** *If a sequence  $\{a_n\}$  has limit  $L$ , then the sequence is bounded.*

**Theorem 3.** *Let*

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= a \\ \lim_{n \rightarrow \infty} b_n &= b \end{aligned}$$

*and let  $c$  be a constant. Then,*

$$\begin{aligned} \lim c &= c \\ \lim(ca_n) &= c \lim a_n \\ \lim(a_n \pm b_n) &= \lim a_n \pm \lim b_n \\ \lim(a_n b_n) &= \lim a_n \lim b_n \\ \lim\left(\frac{a_n}{b_n}\right) &= \frac{\lim a_n}{\lim b_n} \quad (\text{if } \lim b_n \neq 0) \end{aligned}$$

**Theorem 4.** Let  $\{b_n\}$  be bounded and let  $\lim a_n = 0$ . Then,

$$\lim(a_nb_n) = 0$$

**Theorem 5** (Sandwich Theorem). Let  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  be three sequences. If

$$\lim a_n = \lim b_n = L$$

and  $\exists n_0 \in \mathbb{N}$ , s.t.  $\forall n \geq n_0$ ,  $a_n \leq b_n \leq c_n$ . Then,

$$\lim b_n = L$$

**Exercise 4.**

Calculate  $\lim_{n \rightarrow \infty} \sqrt[n]{2^n + 3^n}$

**Solution 4.**

$$\begin{aligned} \sqrt[n]{3^n} &\leq \sqrt[n]{2^n + 3^n} \leq \sqrt[n]{3^n + 3^n} = \sqrt[n]{2 \cdot 3^n} \\ \therefore 3 &\leq \sqrt[n]{2^n + 3^n} \leq 3\sqrt[n]{2} \end{aligned}$$

Therefore, by the Sandwich Theorem,  $\lim_{n \rightarrow \infty} \sqrt[n]{2^n + 3^n} = 3$ .

**Theorem 6.** Any monotonically increasing sequence which is bounded from above converges. Similarly, any monotonically decreasing sequence which is bounded from below converges.

**Exercise 5.**

Prove that there exists a limit for  $a_n = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}_{n \text{ times}}$  and find it.

**Solution 5.**

$$a_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = a_2$$

If possible, let

$$\begin{aligned} a_{n-1} &< a_n \\ \therefore \sqrt{2 + a_{n-1}} &< \sqrt{2 + a_n} \\ \therefore a_n &< a_{n+1} \end{aligned}$$



Hence, by induction,  $\{a_n\}$  is monotonically increasing.

$$a_1 = \sqrt{2} \leq 2$$

If possible, let

$$\begin{aligned} a_n &\leq 2 \therefore \sqrt{2 + a_n} && \leq \sqrt{2 + 2} \\ \therefore a_{n+1} &\leq 2 \end{aligned}$$

Hence, by induction,  $\{a_n\}$  is bounded from above by 2. Therefore, by ,  $\{a_n\}$  converges.

**Definition 11** (Limit in a wide sense). The sequence  $\{a_n\}$  is said to converge to  $+\infty$  if  $\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{N}$ , s.t.  $\forall n \geq n_0, a_n > M$ .

The sequence  $\{a_n\}$  is said to converge to  $-\infty$  if  $\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{N}$ , s.t.  $\forall n \geq n_0, a_n < M$ .

## 1.2 Sub-sequences

**Definition 12** (Sub-sequence). Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence. Let  $\{n_k\}_{k=1}^{\infty}$  be a strongly increasing sequence of natural numbers. Let  $\{b_k\}_{k=1}^{\infty}$  be a sequence such that  $b_k = a_{n_k}$ . Then  $\{b_k\}_{k=1}^{\infty}$  is called a sub-sequence of  $\{a_n\}_{n=1}^{\infty}$ .

**Example 12.**

$$a_n = \frac{1}{n}$$

If we choose  $n_k = k^2$ ,

$$b_k = a_{n_k} = a_{k^2} = \frac{1}{k^2}$$

Therefore,

$$\{b_k\} = 1, \frac{1}{4}, \frac{1}{9}, \dots$$

**Theorem 7.** *If the sequence  $\{a_n\}$  converges to  $L$  in a wide sense, i.e.  $L$  can be infinite, then any sub-sequence of  $\{a_n\}$  converges to the same limit  $L$ .*

**Definition 13** (Partial limit). A real number  $a$ , which may be infinite, is called a partial limit of the sequence  $\{a_n\}$  if there exists a sub-sequence of  $\{a_n\}$  which converges to  $a$ .

**Example 13.** Let

$$a_n = (-1)^n$$

Therefore,  $\nexists \lim_{n \rightarrow \infty} a_n$ . Let

$$b_k = a_{n_k} = a_{2k-1}$$

Therefore,

$$\begin{aligned} \{b_k\} &= -1, -1, -1, \dots \\ \therefore \lim_{k \rightarrow \infty} b_k &= -1 \end{aligned}$$

Therefore,  $-1$  is a partial limit of  $\{a_n\}$ .

**Theorem 8** (Bolzano-Weierstrass Theorem). *For any bounded sequence there exists a subsequence which is convergent, s.t. there exists at least one partial limit.*

**Definition 14** (Upper partial limit). The largest partial limit of a sequence is called the upper partial limit. It is denoted by  $\overline{\lim} a_n$  or  $\limsup a_n$ .

**Definition 15** (Lower partial limit). The smallest partial limit of a sequence is called the lower partial limit. It is denoted by  $\underline{\lim} a_n$  or  $\liminf a_n$ .

**Theorem 9.** *If the sequence  $\{a_n\}$  is bounded and*

$$\overline{\lim} a_n = \underline{\lim} a_n = a$$

*then*

$$\exists \lim a_n = a$$

### 1.3 Cauchy Characterisation of Convergence

**Definition 16.** A sequence  $\{a_n\}$  is called a Cauchy sequence if  $\forall \varepsilon > 0$ ,  $\exists n_0 \in \mathbb{N}$ , s.t.  $\forall m, n \geq n_0$ ,  $|a_n - a_m| < \varepsilon$ .

**Theorem 10** (Cauchy Characterisation of Convergence). *A sequence  $\{a_n\}$  converges if and only if it is a Cauchy sequence.*

*Proof.* Let

$$\lim_{n \rightarrow \infty} a_n = L$$

Then  $\forall \varepsilon > 0$ ,  $\exists n_0 \in \mathbb{N}$ , such that  $\forall n \geq n_0$ ,  $|a_n - L| < \frac{\varepsilon}{2}$ . Therefore if  $n \geq n_0$  and  $m \geq n_0$ , then

$$\begin{aligned} |a_n - a_m| &= |a_n - L + L - a_m| \\ &\leq |a_n - L| + |L - a_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ \therefore |a_n - a_m| &= \varepsilon \end{aligned}$$

Similarly, the converse can be proved by Theorem 9.  $\square$

**Theorem 11** (Another Formulation of the Cauchy Characterisation Theorem). *The sequence  $\{a_n\}$  converges if and only if  $\forall \varepsilon > 0$ ,  $\exists n_0 \in \mathbb{N}$ , such that  $\forall n \geq n_0$  and  $\forall p \in \mathbb{N}$ ,  $|a_{n+p} - a_n| < \varepsilon$ .*

**Exercise 6.**

Prove that the sequence

$$a_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

is convergent.

**Solution 6.**

$$\begin{aligned} |a_{n+p} - a_n| &= \left| \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(n+p)^2} - \left( \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right) \right| \\ &= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(n+p-1)(n+p)} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n} - \frac{1}{\cancel{n+1}} + \frac{1}{\cancel{n+1}} + \cdots + \frac{1}{\cancel{n+p-1}} - \frac{1}{n+p} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n} - \frac{1}{n+p} \\ \therefore |a_{n+p} - a_n| &< \frac{1}{n} \end{aligned}$$

Therefore,  $\forall \varepsilon > 0$ ,  $\exists n_0 \in \mathbb{N}$ , s.t.  $\forall n \geq n_0$  and  $\forall p \in \mathbb{N}$ ,  $|a_{n+p} - a_n| < \varepsilon$ , where  $n_0 > \frac{1}{\varepsilon}$ .  $\square$

### Exercise 7.

Prove that the sequence

$$a_n = \frac{1}{1} + \frac{1}{n} + \cdots + \frac{1}{n}$$

diverges.

### Solution 7.

If possible, let the sequence converge. Then, by the Cauchy Characterisation of Convergence,  $\forall \varepsilon > 0$ ,  $\exists n_0 \in \mathbb{N}$ , s.t.  $\forall n \geq n_0$  and  $\forall p \in \mathbb{N}$ ,  $|a_{n+p} - a_n| < \varepsilon$ . Therefore,

$$\begin{aligned} |a_{n+p} - a_n| &= \left| \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+p} - \left( \frac{1}{n} + \cdots + \frac{1}{n} \right) \right| \\ &= \frac{1}{n+1} + \cdots + \frac{1}{n+p} \\ &\geq p \cdot \frac{1}{n+p} \\ \therefore |a_{n+p} - a_n| &> \frac{p}{n+p} \end{aligned}$$

If  $n = p$ ,

$$\frac{p}{n+p} = \frac{1}{2}$$

This contradicts the result obtained from the Cauchy Characterisation of Convergence, for  $\varepsilon = \frac{1}{4}$ .

Therefore, the sequence diverges.

## 2 Series

**Definition 17** (Series). Given a sequence  $\{a_n\}$ , the sum  $a_1 + \cdots + a_n + \cdots$  is called an infinite series or series. It is denoted as  $\sum_{n=1}^{\infty} a_n$  or  $\sum a_n$ .

**Definition 18** (Partial sum). The partial sum of the series  $\sum a_n$  is defined as

$$S_i = a_1 + \cdots + a_i$$

**Definition 19** (Convergent and divergent series). If the sequence  $\{S_n\}_{n=1}^{\infty}$  converges, then the series is called convergent. Otherwise, the series is called divergent.

**Definition 20** (Sum of a series). If the sequence  $\{S_n\}_{n=1}^{\infty}$  converges to  $S \neq \pm\infty$ , the number  $S$  is called the sum of the series.

$$\sum_{n=1}^{\infty} a_n = S$$

**Example 14.**

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

Therefore,

$$S_1 = \frac{1}{2} \tag{1}$$

$$S_2 = \frac{1}{2} + \frac{1}{2^2} \tag{2}$$

$$\vdots S_n = \frac{1}{2} + \cdots + \frac{1}{2^n} \tag{3}$$

$$= \frac{a_1(1 - q^n)}{1 - q} \tag{4}$$

$$= \frac{1/2(1 - 1/2^n)}{1 - 1/2} \tag{5}$$

$$= 1 - \frac{1}{2^n} \tag{6}$$

$$\lim_{n \rightarrow \infty} S_n = 1 \tag{7}$$

Therefore, the series converges.

$$S = \sum_{n=1}^{\infty} = 1$$

**Theorem 12.** A geometric series  $\sum_{n=1}^{\infty} a_1 q^{n-1}$ ,  $a_1 \neq 0$  converges if  $|q| < 1$  and then,

$$S = \sum_{n=1}^{\infty} a_1 q^{n-1} = \frac{a_1}{1 - q}$$

**Definition 21** ( $p$ -series). The series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is called the  $p$ -series.

**Theorem 13.** *The  $p$ -series converges for  $p > 1$  and diverges for  $p \leq 1$ .*

**Theorem 14.** *If  $\sum a_n$  converges, then*

$$\lim_{n \rightarrow \infty} a_n = 0$$

*Proof.*

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ \therefore \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} \\ &= S - S \\ &= 0 \end{aligned}$$

□

**Theorem 15.** *If  $\sum a_n$  and  $\sum b_n$  converge, then  $\sum(a_n \pm b_n)$  and  $\sum ca_n$ , where  $c$  is a constant, also converge. Also,*

$$\begin{aligned} \sum(a_n \pm b_n) &= \sum a_n \pm \sum b_n \\ \sum(ca_n) &= c \sum a_n \end{aligned}$$

## 2.1 Convergence Criteria

### 2.1.1 Leibniz's Criteria

**Definition 22** (Alternating series). The series  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ , where all  $a_n > 0$  or all  $a_n < 0$  is called an alternating series.

**Theorem 16** (Leibniz's Criteria for Convergence). *If an alternating series  $\sum (-1)^{n-1} a_n$  with  $a_n > 0$  satisfies*

1.  $a_{n+1} \leq a_n$ , i.e.  $\{a_n\}$  is monotonically decreasing.

2.  $\lim_{n \rightarrow \infty} a_n = 0$

*then the series  $\sum (-1)^{n-1} a_n$  converges.*

*Proof.* Consider the even partial sums of the series  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ .

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$$

As  $\{a_n\}$  is monotonically increasing, all brackets are non-negative. Therefore,

$$S_{2m+2} \geq S_{2m}$$

Therefore,  $\{S_{2m}\}$  is increasing.

Also,

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$$

All brackets and  $a_{2m}$  are non-negative. Therefore,

$$S_{2m} \leq a_1$$

Therefore,  $\{S_{2m}\}$  is bounded from above by  $a_1$ . Hence,

$$\exists \lim_{m \rightarrow \infty} S_{2m} = S$$

For  $S_{2m+1}$ ,

$$\begin{aligned} S_{2m+1} &= S_{2m} + a_{2m+1} \\ \therefore \lim_{m \rightarrow \infty} S_{2m+1} &= \lim_{m \rightarrow \infty} S_{2m} + \lim_{m \rightarrow \infty} a_{2m+1} \\ &= S + 0 \\ &= S \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} S_n = S$$

□

**Example 15.** The alternating harmonic series  $\sum \frac{(-1)^{n-1}}{n}$  converges as  $a_n = \frac{1}{n} > 0$ ,  $a_n$  decreases and  $\lim a_n = 0$ .

### 2.1.2 Comparison Test

**Theorem 17** (Comparison Test for Convergence). Assume  $\exists n_0 \in \mathbb{N}$ , such that  $a_n \geq 0$ ,  $b_n \geq 0$ ,  $\forall n \geq n_0$ .

1. If  $a_n \leq b_n$ ,  $\forall n \geq n_0$  and  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $a_n \geq b_n$ ,  $\forall n \geq n_0$  and  $\sum_{n=1}^{\infty} b_n$  diverges, then  $\sum_{n=1}^{\infty} a_n$  diverges.

**Theorem 18** (Another Formulation of the Comparison Test for Convergence). Assume  $\exists n_0 \in \mathbb{N}$ , such that  $a_n \geq 0$ ,  $b_n \geq 0$ ,  $\forall n \geq n_0$  and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = a > 0$$

where  $a$  is a finite number. Then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{n=1}^{\infty} b_n$  converges.

### 2.1.3 d'Alembert Criteria (Ratio Test)

**Definition 23** (Absolute and conditional convergence). The series  $\sum a_n$  is said to converge absolutely if  $\sum |a_n|$  converges. The series  $\sum a_n$  is said to converge conditionally if it converges but  $\sum |a_n|$  diverges.

**Example 16.** The series  $\sum \frac{(-1)^{n-1}}{n^2}$  converges absolutely, as  $\sum \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum \frac{1}{n^2}$  converges.

**Example 17.** The series  $\sum \frac{(-1)^{n-1}}{n}$  converges conditionally, as it converges, but  $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$  diverges.

**Theorem 19.** *If the series  $\sum a_n$  converges absolutely then it converges.*

**Theorem 20** (d'Alembert Criteria (Ratio Test)). 1. If

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n-1}}{a_n} \right| = L < 1$$

*then  $\sum a_n$  converges absolutely.*

2. If

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n-1}}{a_n} \right| = L > 1$$

*(including  $L = \infty$ ), then  $\sum a_n$  converges diverges.*

3. If  $L = 1$ , the test does not apply.

### 2.1.4 Cauchy Criteria (Cauchy Root Test)

**Theorem 21** (Cauchy Criteria (Cauchy Root Test)). 1. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

*then  $\sum a_n$  converges absolutely.*

2. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

*(including  $L = \infty$ ), then  $\sum a_n$  diverges.*

3. If  $L = 1$ , the test does not apply.



### 2.1.5 Integral Test

**Theorem 22** (Integral Test for Series Convergence). *Let  $f(x)$  be a continuous, non-negative, monotonic decreasing function on  $[1, \infty)$  and let  $a_n = f(n)$ . Then the series  $\sum_{n=1}^{\infty} a_n$  converges if and only if the improper integral  $\int_1^{\infty} f(x) dx$  converges.*

#### Exercise 8.

Does  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converge or diverge?

#### Solution 8.

Let

$$f(x) = \frac{1}{x^p}$$

with  $p > 0$ .

Therefore,  $f(x)$  is continuous, non-negative and monotonic decreasing on  $[1, \infty)$ . Therefore, the Integral Test for Series Convergence is applicable.

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^p} dx$$

If  $p \neq 1$ ,

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} &= \lim_{t \rightarrow \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^t \\ &= \lim_{t \rightarrow \infty} \left( \frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} \right) \\ &= \frac{1}{p-1} \end{aligned}$$

If  $p = 1$ ,

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} &= \lim_{t \rightarrow \infty} \ln x \Big|_1^t \\ &= \infty \end{aligned}$$

Therefore, the series converges for  $p > 1$  and diverges for  $p \leq 1$ .

**Theorem 23.** *If the series  $\sum a_n$  absolutely converges and the series  $\sum b_n$  is obtained from  $\sum a_n$  by changing the order of the terms in  $\sum a_n$  then  $\sum b_n$  also absolutely converges and  $\sum b_n = \sum a_n$ .*

**Theorem 24.** *If a series converges then the series with brackets without changing the order of terms also converges. That is, if  $\sum a_n$  converges, then any series of the form  $(a_1 + a_2) + (a_3 + a_4 + a_5) + a_6 + \dots$  also converges.*

**Theorem 25.** *If a series with brackets converges and the terms in the brackets have the same sign, then the series without brackets also converges.*

### 3 Power Series

**Definition 24** (Power series). The series  $\sum_{n=0}^{\infty} a_n(x - c)^n$  is called a power series.

**Theorem 26** (Cauchy-Hadamard Theorem). *For any power series  $\sum_{n=0}^{\infty} a_n(x - c)^n$  there exists the limit, which may be infinity,*

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$$

*and the series converges for  $|x - c| < R$  and diverges for  $|x - c| > R$ . The end points of the interval, i.e.  $x = c - R$  and  $x = c + R$  must be separately checked for series convergence.*

**Definition 25** (Radius of convergence and convergence interval). The number  $R$  is called the radius of convergence and the interval  $|x - c| < R$  is called the convergence interval of the series. The point  $c$  is called the centre of the convergence interval.

**Theorem 27.** *If  $\exists \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$ , which may be infinite, then,*

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

**Theorem 28** (Stirling's Approximation). *For  $n \rightarrow \infty$ ,*

$$n! \approx \left( \frac{n}{e} \right)^n \sqrt{2\pi n}$$

### 3.1 Differentiation and Integration of Power Series

**Theorem 29.** If  $R$  is a radius of convergence of the power series  $\sum_{n=0}^{\infty} a_n(x - c)^n$  then the function  $f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$  is differentiable on  $(c - R, c + R)$  and the derivative is

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x - c)^{n-1}$$

**Theorem 30.** If  $R$  is a radius of convergence of the series  $\sum_{n=0}^{\infty} a_n(x - c)^n$  then the function  $f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$  is integrable in  $(c - R, c + R)$  and

$$\int f(x) dx = \sum_{n=0}^{\infty} a_n \frac{(x - c)^{n+1}}{n + 1} + A$$

where  $c - R < x < c + R$ .

#### Exercise 9.

Find  $\int_0^x e^{-t^2} dt$ .

#### Solution 9.

$\forall s \in \mathbb{R}$ ,

$$\begin{aligned} e^s &= 1 + \frac{s}{1!} + \frac{s^2}{2!} + \cdots + \frac{s^n}{n!} + \cdots \\ \therefore e^{-t^2} &= 1 - \frac{t^2}{1!} + \frac{t^4}{2!} + \cdots + (-1)^n \frac{t^{2n}}{n!} + \cdots \\ \therefore \int_0^x e^{-t^2} dt &= x - \frac{x^3}{1!3} + \frac{x^5}{2!5} + \cdots + (-1)^n \frac{x^{2n+1}}{n!(2n+1)} + \cdots \end{aligned}$$

**Theorem 31.** If the series  $A(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $B(x) = \sum_{n=0}^{\infty} B_n x^n$  absolutely converge for  $|x| < R$  and  $c_n = \sum_{k=0}^n a_k b_{n-k}$ , then the series  $C(x) = \sum_{n=0}^{\infty} c_n x^n$  also absolutely converges for  $|x| < R$  and  $C(x) = A(x)B(x)$ .

## 3.2 Taylor Series

**Definition 26** (Taylor series). Let  $f(x)$  be infinitely differentiable on an open interval about  $a$  and let  $x$  be an arbitrary point in the interval. Then the power series  $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$  is called the Taylor series of  $f(x)$  at  $a$ . If  $a = 0$  then it is called the Maclaurin series of  $f(x)$  at 0.

**Theorem 32.** *If there exists a power series which converges to  $f(x)$ , i.e. if, for  $|x-a| < R$ ,*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

*then, for  $|x-a| < R$ ,*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

*that is,  $\forall n$ ,*

$$a_n = \frac{f^{(n)}(a)}{n!}$$

### Exercise 10.

Show that

$$f(x) = \begin{cases} 0 & ; \quad x = 0 \\ e^{-\frac{1}{x^2}} & ; \quad x \neq 0 \end{cases}$$

is not equal to its Taylor series at  $a = 0$ .

### Solution 10.

If  $n = 1$ ,

$$\begin{aligned} f^{(1)}(0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{e^{-\frac{1}{(\Delta x)^2}}}{\Delta x} \end{aligned}$$

Let  $t = \frac{1}{\Delta x}$

$$\begin{aligned}\therefore f'(0) &= \lim_{t \rightarrow \infty} \frac{e^{-t^2}}{\frac{1}{t}} \\ &= \lim_{t \rightarrow \infty} \frac{t}{e^{t^2}} \\ &= \lim_{t \rightarrow \infty} \frac{1}{e^{t^2} 2t} \\ &= 0\end{aligned}$$

Therefore,

$$f'(x) = \begin{cases} 0 & ; \quad x = 0 \\ e^{-\frac{1}{x^2} \cdot 2 \cdot x^{-3}} & ; \quad x \neq 0 \end{cases}$$

Similarly,  $\forall n \geq 1, f^{(n)}(0) = 0$

Therefore, the Taylor series is not equal to  $f(x)$ .

### Exercise 11.

Find the Maclaurin series of  $f(x) = e^x$  and prove that the series converges to  $f(x)$  for any  $x \in \mathbb{R}$ .

### Solution 11.

$\forall n \geq 1, f^{(n)}(x) = e^x$ .

Therefore,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c x^{n+1}}{(n+1)!}$$

where  $c$  is between 0 and  $x$ .

Therefore, as

$$0 \leq |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

by the Sandwich Theorem

$$\lim_{n \rightarrow \infty} |R_n(x)| = 0$$

Therefore,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

## 4 Series of Real-valued Functions

**Definition 27** (Sequence of functions). A sequence  $\{f_n\} = f_1(x), f_2(x), \dots$  defined on  $D \subseteq \mathbb{R}$  is called a sequence of functions.

**Definition 28** (Pointwise convergence and domain of convergence).  $\{f_n\}$  converges pointwise in some domain  $E \subseteq D$  if for every  $x \in E$ , the sequence of  $\{f_n(x)\}$  converges. In such a case,  $E$  is said to be a domain of convergence of  $\{f_n\}$ .

**Exercise 12.**

Find the domain of convergence of  $f_n(x) = x^n$ , defined on some  $D \subseteq \mathbb{R}$ .

**Solution 12.**

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & ; \quad -1 < x < 1 \\ 1 & ; \quad x = 1 \\ \text{diverges} & ; \quad x \notin (-1, 1] \end{cases}$$

Therefore, the domain of convergence of  $\{f_n\}$  is  $(-1, 1]$ .

**Exercise 13.**

Let  $f(x) : (0, \infty) \rightarrow \mathbb{R}$  be some function such that  $\lim_{x \rightarrow \infty} f(x) = 0$ . Let  $f_n(x) = f(nx)$ . What is the domain of convergence of  $f_n$ ? What is the limit function?

**Solution 13.**

Let  $x$  have some fixed value in  $(0, \infty)$ . Therefore, as  $\lim_{x \rightarrow \infty} f(x) = 0$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(x) &= \lim_{n \rightarrow \infty} f(nx) \\ &= 0 \end{aligned}$$

Therefore, the domain of convergence is  $(0, \infty)$  and the limit function is a constant function with value 0.

### 4.1 Uniform Convergence of Series of Functions

**Definition 29** (Pointwise convergence of a sequence of functions). If  $\forall x \in D$ ,  $\forall \varepsilon > 0$ ,  $\exists N$  which depends on  $\varepsilon$  and  $x$ , such that  $\forall n \geq N$ ,  $|f_n(x) - f(x)| < \varepsilon$ , then  $\forall x \in D$ ,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

**Definition 30** (Uniform convergence of a sequence of functions). The sequence  $\{f_n(x)\}$  is said to converge uniformly to  $f(x)$  in  $D$  if  $\forall \varepsilon > 0, \exists N = N(\varepsilon)$ , such that  $\forall n \geq N, \forall x \in D, |f_n(x) - f(x)| < \varepsilon$ . It can be denoted as  $f_n(x) \xrightarrow{D} f(x)$ .

**Theorem 33.**  $f_n(x)$  converges uniformly to  $f(x)$  in  $D$  if and only if  $\lim_{n \rightarrow \infty} \sup_{x \in D} |f_n(x) - f(x)| = 0$ .

**Exercise 14.**

Does  $f_n(x) = x^n$  converge in  $[0, 1]$ ?

**Solution 14.**

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} x^n$$

$$\therefore f(x) = \begin{cases} 0 & ; \quad 0 \leq x < 1 \\ 1 & ; \quad x = 1 \end{cases}$$

Therefore,

If  $x = 0$ ,

$$f_n(0) = 0$$

$$f(0) = 0$$

Therefore,  $\forall \varepsilon > 0, N = 1$ ,

$$|0 - 0| < \varepsilon$$

$$\therefore |f_n(0) - f(0)| < \varepsilon$$

If  $x = 1$ ,

$$f_n(1) = 1$$

$$f(1) = 1$$

Therefore,  $\forall \varepsilon > 0, N = 1$ ,

$$|1 - 1| < \varepsilon$$

$$\therefore |f_n(1) - f(1)| < \varepsilon$$

If  $0 < x < 1$ ,

$$|f_n(x) - f(x)| = |x^n - 0|$$

$$= x^n$$

If possible, let  $|f_n(x) - f(x)| = x^n < \varepsilon$ .

Therefore,

$$\begin{aligned} x^n &< \varepsilon \\ \therefore \log_x x^n &> \log_x \varepsilon \\ \therefore n &> \log_x \varepsilon \end{aligned}$$

Therefore, for  $N = \lfloor \log_x \varepsilon \rfloor + 1$ ,  $|f_n(x) - f(x)| < \varepsilon$ .

Therefore,  $f_n(x)$  converges pointwise in  $[0, 1]$ .

If possible let  $f_n(x)$  converge uniformly on  $[0, 1]$ .

Therefore,  $\forall \varepsilon > 0$ ,  $\exists N$  dependent on  $\varepsilon$ , such that  $|f_n(x) - f(x)| < \varepsilon$ .

Let  $\varepsilon = \frac{1}{3}$ .

Therefore,  $\exists N$  which is dependent on  $\varepsilon$ , such that  $\forall n > N$ ,  $\forall x \in [0, 1]$ ,

$$|f_n(x) - f(x)| < \frac{1}{3}$$

Let  $x = \frac{1}{2}$ ,  $n = N + 1$ . Therefore,

$$\begin{aligned} \left| f_n \left( \frac{1}{2} \right) - f \left( \frac{1}{2} \right) \right| &= \left| \frac{1}{2} - 0 \right| \\ &= \frac{1}{2} \\ \therefore \left| f_n \left( \frac{1}{2} \right) - f \left( \frac{1}{2} \right) \right| &> \frac{1}{3} \end{aligned}$$

Therefore,  $|f_n(x) - f(x)| > \varepsilon$ .

This is a contradiction. Hence,  $f_n(x)$  does not converge uniformly.

**Definition 31** (Supremum). Let  $A \subseteq \mathbb{R}$  be a bounded set.  $M$  is said to be the supremum of  $A$  if

1.  $\forall x \in A$ ,  $x \leq M$ , i.e.  $M$  is an upper bound of  $A$ .
2.  $\forall \varepsilon, \exists x \in A$ , such that  $x > M - \varepsilon$ .

That is, the supremum of  $A$  is the least upper bound of  $A$ .

The supremum may or may not be in  $A$ .

**Definition 32** (Infimum). Let  $A \subseteq \mathbb{R}$  be a bounded set.  $M$  is said to be the infimum of  $A$  if



1.  $\forall x \in A, x \geq M$ , i.e.  $M$  is an upper bound of  $A$ .

2.  $\forall \varepsilon, \exists x \in A$ , such that  $x < M - \varepsilon$ .

That is, the infimum of  $A$  is the greatest lower bound of  $A$ . The infimum may or may not be in  $A$ .

**Theorem 34.** *Every bounded set  $A$  has a supremum and an infimum.*

**Theorem 35.**  $f_n \xrightarrow{E} f$  if and only if

$$\lim_{n \rightarrow \infty} (\sup\{|f_n(x) - f(x)| : x \in E\}) = 0$$

**Definition 33** (Remainder of a series of functions). Let  $f(x) = \sum_{k=1}^{\infty} u_k(x)$ .

Let the partial sums be denoted by  $f_n(x) = \sum_{k=1}^n u_k(x)$ . Then

$$R_n(x) = f(x) - f_n(x) = \sum_{k=n+1}^{\infty} u_k(x)$$

is called a remainder of the series  $f(x) = \sum_{k=1}^{\infty} u_k(x)$ .

**Definition 34** (Uniform convergence of a series of functions). If  $f_n(x)$  converges uniformly to  $f(x)$  on  $D$ , i.e. if  $\lim_{n \rightarrow \infty} R_n(x) = 0$ , then the series  $\sum_{k=1}^{\infty} u_k(x)$  is said to converge uniformly on  $D$ .

**Exercise 15.**

Show that the series  $f(x) = \sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}$  does not converge uniformly on  $(-1, 1)$ .

**Solution 15.**

The series converges uniformly if and only if  $\lim_{n \rightarrow \infty} R_n(x) = 0$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{(-1,1)} |R_n(x) - 0| &= \lim_{n \rightarrow \infty} \sup_{(-1,1)} \sum_{k=n+1}^{\infty} x^{k-1} \\ &= \lim_{n \rightarrow \infty} \sup_{(-1,1)} \left| \frac{x^n}{1-x} \right| \\ &= \lim_{n \rightarrow \infty} \sup_{(-1,1)} \frac{|x|^n}{1-x} \\ &= \lim_{n \rightarrow \infty} \infty \\ &= \infty \end{aligned}$$

Therefore, the series does not converge uniformly on  $(-1, 1)$ .

**Exercise 16.**

Show that the series  $f(x) = \sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}$  does not converge uniformly on  $(-\frac{1}{2}, \frac{1}{2})$ .

**Solution 16.**

The series converges uniformly if and only if  $\lim_{n \rightarrow \infty} R_n(x) = 0$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{(-\frac{1}{2}, \frac{1}{2})} |R_n(x) - 0| &= \lim_{n \rightarrow \infty} \sup_{(-\frac{1}{2}, \frac{1}{2})} \sum_{k=n+1}^{\infty} x^{k-1} \\ &= \lim_{n \rightarrow \infty} \sup_{(-\frac{1}{2}, \frac{1}{2})} \left| \frac{x^n}{1-x} \right| \\ &= \lim_{n \rightarrow \infty} \sup_{(-\frac{1}{2}, \frac{1}{2})} \frac{|x|^n}{1-x} \\ &= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^{n-1} \\ &= 0 \end{aligned}$$

Therefore, the series converges uniformly on  $(-\frac{1}{2}, \frac{1}{2})$ .

## 4.2 Weierstrass M-test

**Theorem 36** (Weierstrass M-test). *If  $|u_k(x)| \leq c_k$  on  $D$  for  $k \in \{1, 2, 3, \dots\}$  and the numerical series  $\sum_{k=1}^{\infty} c_k$  converges, then the series of functions  $\sum_{k=1}^{\infty} u_k(x)$  converges uniformly on  $D$ .*

**Exercise 17.**

Show that  $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx)$  converges uniformly on  $\mathbb{R}$ .

**Solution 17.**

$$|u_k(x)| = \left| \frac{1}{k^2} \sin(kx) \right|$$

$$\therefore |u_k(x)| \leq \frac{1}{k^2}$$

Therefore, let

$$c_k = \frac{1}{k^2}$$

Therefore, as  $|u_k(x)| \leq c_k$ , and as  $\sum_{k=1}^{\infty} c_k$  converges, by the Weierstrass M-test,  $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx)$  converges uniformly.

**4.3 Application of Uniform Convergence**

**Theorem 37** (Continuity of a series). *Let functions  $u_k(x)$ ,  $k \in \{1, 2, 3, \dots\}$  be defined on  $[a, b]$  and continuous at  $x_0 \in [a, b]$ . If  $\sum_{k=1}^{\infty} u_k(x)$  converges uniformly on  $[a, b]$  then the function  $f(x) = \sum_{k=1}^{\infty} u_k(x)$  is also continuous at  $x_0$ .*

**Theorem 38** (Changing the order of integration and infinite summation). *If the functions  $u_k(x)$ ,  $k \in \{1, 2, 3, \dots\}$  are integrable on  $[a, b]$  and the series  $\sum_{k=1}^{\infty} u_k(x)$  converges uniformly on  $[a, b]$  then*

$$\int_a^b \left( \sum_{k=1}^{\infty} u_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b u_k(x) dx$$

**Exercise 18.**

Solve  $\int_0^{2\pi} \left( \sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx) \right) dx$ .

**Solution 18.**

The series  $f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx)$  converges uniformly on  $[0, 2\pi]$ . Therefore, by the Weierstrass M-test and  $u_k(x) = \frac{1}{k^2} \sin(kx)$  are integrable on  $[0, 2\pi]$ . There-

fore,

$$\begin{aligned}
\int_0^{2\pi} f(x) \, dx &= \int_0^{2\pi} \left( \sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx) \right) \, dx \\
&= \sum_{k=1}^{\infty} \left( \int_0^{2\pi} \frac{1}{k^2} \sin(kx) \, dx \right) \\
&= \sum_{k=1}^{\infty} \left( -\frac{\cos(2\pi k)}{k^3} + \frac{1}{k^3} \right) \\
&= \sum_{k=1}^{\infty} 0 \\
&= 0
\end{aligned}$$

**Theorem 39** (Changing the order of differentiation and infinite summation). *If the functions  $u_k(x)$ ,  $k \in \{1, 2, 3, \dots\}$  are differentiable on  $[a, b]$  and the derivatives are continuous on  $[a, b]$ , and the series  $\sum_{k=1}^{\infty} u_k(x)$  converges pointwise on  $[a, b]$  and the series  $\sum_{k=1}^{\infty} u_k'(x)$  converges uniformly on  $[a, b]$ , then,*

$$\left( \sum_{k=1}^{\infty} u_k(x) \right)' = \sum_{k=1}^{\infty} u_k'(x)$$

**Theorem 40** (Changing the order of integration and limit). *If the functions  $f_n(x)$  are integrable on  $[a, b]$  and converge uniformly to  $f$  on  $[a, b]$ , then*

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) \, dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) \, dx = \int_a^b f(x) \, dx$$

**Theorem 41** (Changing the order of differentiation and limit). *If there exists the functions  $f_n'(x)$  which are continuous on  $[a, b]$ , for the functions  $f_n(x)$  which  $\forall x \in [a, b]$ , converge pointwise to  $f(x)$  on  $[a, b]$ , and if  $f_n'(x)$  converges uniformly to  $g(x)$  on  $[a, b]$ , then,*

$$f'(x) = \left( \lim_{n \rightarrow \infty} f_n(x) \right)' = \lim_{n \rightarrow \infty} f_n'(x) = g(x)$$

## Part II

# Functions of Multiple Variables

## 1 Limits, Continuity, and Differentiability

**Definition 35** (Limit of a function of two variables). Let  $z = f(x, y)$  be defined on some open neighbourhood about  $(a, b)$ , except maybe at the point itself.  $L \in \mathbb{R}$  is said to be a limit of  $f(x, y)$  at  $(a, b)$ , if  $\forall \varepsilon > 0, \exists \delta > 0$ , such that  $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$ , then,

$$|f(x, y) - L| < \varepsilon$$

### Exercise 19.

Does the limit  $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2}$  exist?

### Solution 19.

Consider the curves  $C_1 : y = 0$ , and  $C_2 : y = x^3$ .

Therefore, as  $(x, y) \rightarrow (0, 0)$  along these curves, the limit of the function is

$$\begin{aligned} \lim_{(x,y) \xrightarrow{C_1} (0,0)} \frac{3x^2y}{x^2+y^2} &= \lim_{x \rightarrow 0} \frac{3x^2 \cdot 0}{x^2 + y^2} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{(x,y) \xrightarrow{C_2} (0,0)} \frac{3x^2y}{x^2+y^2} &= \lim_{x \rightarrow 0} \frac{3x^2(x^3)}{x^2 + (x^3)^2} \\ &= \lim_{x \rightarrow 0} \frac{3x^5}{x^2 + x^6} \\ &= \lim_{x \rightarrow 0} \frac{3x^3}{x^2 + x^4} \\ &= 0 \end{aligned}$$

If  $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2} = 0, \forall \varepsilon > 0, \exists \delta > 0$  such that  $0 < \sqrt{x^2 + y^2} < \delta$ , then,

$$|f(x, y) - L| < \varepsilon$$

Therefore, checking  $|f(x, y) - L|$ ,

$$\begin{aligned} |f(x, y) - L| &= \left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \\ &= \frac{3x^2|y|}{x^2 + y^2} \end{aligned}$$

As  $\frac{x^2}{x^2 + y^2} \leq 1$ ,

$$\begin{aligned} |f(x, y) - L| &\leq 3|y| \\ \therefore |f(x, y) - L| &\leq 3\sqrt{y^2} \\ \therefore |f(x, y) - L| &\leq 3\sqrt{x^2 + y^2} \end{aligned}$$

Therefore,  $|f(x, y) - L| < \varepsilon$ .

Therefore, for  $\delta \leq \frac{\varepsilon}{3}$ , the condition is satisfied.

Hence, the limit of the function exists and is 0.

**Definition 36** (Iterative limits). The limits  $\lim_{x \rightarrow a} \left( \lim_{y \rightarrow b} f(x, y) \right)$  and  $\lim_{y \rightarrow b} \left( \lim_{x \rightarrow a} f(x, y) \right)$  are called the iterative limits of  $f(x, y)$ .

**Theorem 42.** If  $\exists \lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$  and, for some open interval about  $b$ ,  $\forall y \neq b, \exists \lim_{x \rightarrow a} f(x, y)$  then

$$\lim_{y \rightarrow b} \left( \lim_{x \rightarrow a} f(x, y) \right) = L$$

If  $\exists \lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$  and, for some open interval about  $a$ ,  $\forall x \neq a$ ,  $\exists \lim_{y \rightarrow b} f(x, y)$  then

$$\lim_{x \rightarrow a} \left( \lim_{y \rightarrow b} f(x, y) \right) = L$$

**Exercise 20.**

Do the iterative limits, as  $x \rightarrow 0$ , and as  $y \rightarrow 0$ , of the function

$$f(x, y) = \begin{cases} (x + y) \sin \frac{1}{x+y} & ; \quad x \neq 0, y \neq 0 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

exists? Does the limit of the function at  $(0, 0)$  exist?

**Solution 20.**

$$\begin{aligned}\lim_{x \rightarrow 0} f(x, y) &= \lim_{x \rightarrow 0} (x + y) \sin \frac{1}{x + y} \\ &= \lim_{x \rightarrow 0} y \sin \frac{1}{x + y}\end{aligned}$$

Therefore, as  $\sin \frac{1}{x+y}$  oscillates between  $-1$  and  $1$ , the limits does not exist.

$$\begin{aligned}\lim_{y \rightarrow 0} f(x, y) &= \lim_{y \rightarrow 0} (x + y) \sin \frac{1}{x + y} \\ &= \lim_{y \rightarrow 0} x \sin \frac{1}{x + y}\end{aligned}$$

Therefore, as  $\sin \frac{1}{x+y}$  oscillates between  $-1$  and  $1$ , the limits does not exists.

Therefore, the iterative limits do not exist.

$$\begin{aligned}|f(x, y) - 0| &= |x + y| \cdot \left| \sin \frac{1}{xy} \right| \\ \therefore |f(x, y) - 0| &\leq |x| + |y| \\ \therefore |f(x, y) - 0| &\leq \sqrt{2} \sqrt{x^2 + y^2}\end{aligned}$$

Therefore, for  $\delta \leq \frac{\varepsilon}{\sqrt{2}}$ , the condition is satisfied.

Hence, the limit of the function exists and is  $0$ .

Therefore, even though the iterative limits do not exist, the limit of the function exists.

**Definition 37** (Differential).

$$\begin{aligned}\Delta z &= f(a + \Delta x, b + \Delta y) - f(a, b) \\ dz &= f_x(a, b) dx + f_y(a, b) dy\end{aligned}$$

**Definition 38** (Differentiability). The function  $x = f(x, y)$  is said to be differentiable at  $(a, b)$  if

$$\Delta z = dz + \varepsilon_1(\Delta x, \Delta y) \Delta x + \varepsilon_2(\Delta x, \Delta y) \Delta y$$

where

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \varepsilon_1(\Delta x, \Delta y) = \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \varepsilon_2(\Delta x, \Delta y) = 0$$

**Theorem 43.** If  $f(x, y)$  is differentiable at  $(a, b)$  then  $f(x, y)$  is continuous at  $(a, b)$ .

**Theorem 44.** If  $\exists f_x(a, b)$  and  $\exists f_y(a, b)$  on some open neighbourhood of  $(a, b)$  and are continuous at  $(a, b)$ , then  $f(x, y)$  is differentiable at  $(a, b)$ .

## 2 Directional Derivatives and Gradients

**Definition 39** (Directional derivative). Let  $x_0 \in \mathbb{R}$ ,  $y_0 \in \mathbb{R}$ .

Let  $\hat{u} = (a, b)$  be a unit vector in the  $xy$ -plane.

The directional derivative of  $z = f(x, y)$  with respect to the direction  $\hat{u} = (a, b)$  at the point  $(x_0, y_0)$  is defined as

$$D_{\hat{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

If the limit does not exist, the directional derivative does not exist.

Geometrically the directional derivative of  $z = f(x, y)$  is the slope of the tangent of the curve formed due to the intersection of the surface  $z = f(x, y)$ , and the plane which passes through  $(x_0, y_0)$  in the direction of  $\hat{u}$  and is perpendicular to the  $xy$ -plane.

**Definition 40** (Gradient). If the functions  $f_x(x, y)$  and  $f_y(x, y)$  for  $z = f(x, y)$  exist, then the vector function

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y))$$

is called the gradient of  $f(x, y)$ .

**Theorem 45.** Let  $z = f(x, y)$  be differentiable at  $(x_0, y_0)$ . The function  $f(x, y)$  has a directional derivative with respect to any direction  $\hat{u} = (a, b)$  at  $(x_0, y_0)$  and

$$D_{\hat{u}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \nabla f(x_0, y_0) \cdot \hat{u}$$

**Exercise 21.**

Find the directional derivative of

$$f(x, y) = x^3 + 4xy + y^4$$

with respect to the direction of  $\bar{u} = (1, 2)$  at any point  $(x, y)$  and at  $(0, 1)$ .

**Solution 21.**

$$f(x, y) = x^3 + 4xy + y^4$$

Therefore,

$$\begin{aligned} f_x(x, y) &= 3x^2 + 4y \\ f_y(x, y) &= 4x + 4y^3 \end{aligned}$$



$$\begin{aligned}
\hat{u} &= \frac{\bar{u}}{u} \\
&= \frac{(1, 2)}{\sqrt{5}} \\
&= \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)
\end{aligned}$$

Therefore,

$$D_{\hat{u}}f(x, y) = \frac{1}{\sqrt{5}}(3x^2 + 4y) + \frac{2}{\sqrt{5}}(4x + 4y^2)$$

Therefore,

$$\begin{aligned}
D_{\hat{u}}f(0, 1) &= \frac{4}{\sqrt{5}} + \frac{8}{\sqrt{5}} \\
&= \frac{12}{\sqrt{5}}
\end{aligned}$$

**Theorem 46.** If  $z = f(x, y)$  is differentiable at  $(x_0, y_0)$ , then  $\exists \hat{u}_0 = (a_0, b_0)$  such that

$$\max_{\hat{u} \in \mathbb{R}} D_{\hat{u}}f(x_0, y_0) = D_{\hat{u}_0}f(x_0, y_0) = |\nabla f(x_0, y_0)|$$

and

$$\hat{u}_0 = \frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$$

*Proof.*

$$\begin{aligned}
\max_{\hat{u} \in \mathbb{R}} D_{\hat{u}}f(x_0, y_0) &= \max_{\hat{u} \in \mathbb{R}} \nabla f(x_0, y_0) \cdot \hat{u} \\
&= \max_{\hat{u} \in \mathbb{R}} |\nabla f(x_0, y_0)| |\hat{u}| \cos \theta \\
&= |\nabla f(x_0, y_0)| \max_{\hat{u} \in \mathbb{R}} \cos \theta \\
&= |\nabla f(x_0, y_0)|
\end{aligned}$$

□

**Theorem 47.** If  $z = f(x, y)$  is differentiable at  $(x_0, y_0)$ , then  $\exists \hat{u}_1 = (a_0, b_0)$  such that

$$\min_{\hat{u} \in \mathbb{R}} D_{\hat{u}}f(x_0, y_0) = D_{\hat{u}_1}f(x_0, y_0) = -|\nabla f(x_0, y_0)|$$

and

$$\hat{u}_1 = -\frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$$

*Proof.*

$$\begin{aligned}
\min_{\hat{u} \in \mathbb{R}} D_{\hat{u}} f(x_0, y_0) &= \min_{\hat{u} \in \mathbb{R}} \nabla f(x_0, y_0) \cdot \hat{u} \\
&= \min_{\hat{u} \in \mathbb{R}} |\nabla f(x_0, y_0)| |\hat{u}| \overset{1}{\cos \theta} \\
&= |\nabla f(x_0, y_0)| \min_{\hat{u} \in \mathbb{R}} \cos \theta \\
&= -|\nabla f(x_0, y_0)|
\end{aligned}$$

□

### 3 Local Extrema

**Theorem 48** (A necessary condition for local extrema existence). *If the function  $z = f(x, y)$  has a local extrema at the point  $(a, b)$  and  $\exists f_x(a, b)$  and  $\exists f_y(a, b)$  then  $f_x(a, b) = f_y(a, b) = 0$*

**Example 18.**

$$z = x^2 + y^2$$

**Solution 21.**

$$f(x, y) \geq f(0, 0)$$

Therefore,  $(0, 0)$  is a point of local minimum.

$$\begin{aligned}
f_x &= 2x \\
f_y &= 2y
\end{aligned}$$

Therefore,

$$f_x(0, 0) = f_y(0, 0) = 0$$

**Example 19.**

$$z = \sqrt{x^2 + y^2}$$

**Solution 21.**

$$\lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{(\Delta x)^2}}{\Delta x} = \pm 1$$

Therefore, the limit does not exist.

**Definition 41** (Critical point). Let the function  $z = f(x, y)$  be defined on some open neighbourhood of  $(a, b)$ . The point  $(a, b)$  is called a critical point of  $z = f(x, y)$  if  $f_x(a, b) = f_y(a, b) = 0$  or at least one of the partial derivative  $f_x(a, b)$  and  $f_y(a, b)$  does not exist.

**Example 20.** Is  $(0, 0)$  an local extremum point of

$$z = f(x, y) = y^2 - x^2$$

?

**Solution 21.**

$$f_x(0, 0) = 0$$

$$f_y(0, 0) = 0$$

Therefore,  $(0, 0)$  is a critical point.

If possible let  $(0, 0)$  be a local minimum point.

Then,  $f(x, y) \geq f(0, 0)$  in some neighbourhood of  $(0, 0)$ .

Therefore,

$$y^2 - x^2 \geq 0$$

For any point of the form  $(x, 0)$ , this is a contradiction.

Therefore  $(0, 0)$  is not a local minimum point.

Similarly,  $(0, 0)$  is not a local maximum point.

**Theorem 49** (A sufficient condition for local extrema point). Assume that there exist second order partial derivatives of  $z = f(x, y)$ , they are continuous on some open neighbourhood of  $(a, b)$  and  $f_x(a, b) = f_y(a, b) = 0$ . Denote

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - \left(f_{xy}(a, b)\right)^2$$

1. If  $D(a, b) > 0$  and  $f_{xx} < 0$  then  $(a, b)$  is a local maximum point.

2. If  $D(a, b) > 0$  and  $f_{xx} > 0$  then  $(a, b)$  is a local minimum point.

3. If  $D(a, b) < 0$  then  $(a, b)$  is called a saddle point.

**Example 21.** Find all critical points of

$$z = f(x, y) = x^4 + y^4 - 4xy + 1$$

and classify them.

**Solution 21.**

$$f_x(x, y) = 4x^3 - 4y$$

$$f_y(x, y) = 4y^3 - 4x$$

For critical points,

$$f_x(x, y) = 0$$

$$f_y(x, y) = 0$$

Solving,  $(0, 0)$ ,  $(1, 1)$ ,  $(-1, -1)$  are critical points.

$$f_{xx}(x, y) = 12x^2$$

$$f_{xy}(x, y) = -4$$

$$f_{yy}(x, y) = 12y^2$$

$$\therefore D(x, y) = 144x^2y^2 - 16$$

For  $(0, 0)$ ,

$$D = -16$$

Therefore,  $(0, 0)$  is a saddle point.

For  $(1, 1)$ ,

$$D = 144 - 16$$

Therefore,  $(1, 1)$  is a local minimum point.

For  $(-1, -1)$ ,

$$D = 144 - 16$$

Therefore,  $(-1, -1)$  is a local minimum point.

## 4 Global Extrema

### 4.1 Algorithm for Finding Maxima and Minima of a Function

label = Step 0 Find all critical points of  $f(x, y)$  on the domain, excluding the end points.

lbbel = Step 0 Calculate the values of  $f(x, y)$  at the critical points.

lcbel = Step 0 Calculate the values of  $f(x, y)$  at the end points of the domain.

ldbel = Step 0 Select the maximum and minimum values from b and c

**Example 22.** Find the global maxima and minima of

$$z = x^2 - 2xy + 2y$$

in the domain

$$D = \left\{ (x, y) \left| 0 \leq x \leq 3, 0 \leq y \leq -\frac{2}{3}x + 2 \right. \right\}$$

**Solution 21.**

$$f_x(x, y) = 0$$

$$\therefore 2x - 2y = 0$$

$$f_y(x, y) = 0$$

$$\therefore -2x + 2 = 0$$

Therefore,  $(1, 1)$  is a critical point in  $D$ .

The boundary of  $D$  is  $L_1 \cup L_2 \cup L_3$ , where

$$L_1 : y = 0, 0 \leq x \leq 3$$

$$L_2 : x = 0, 0 \leq y \leq 2$$

$$L_3 :$$

Therefore,

over  $L_1$ ,

$$f(x, y) = x^2$$

$$\therefore \min_{L_1} f = f(0, 0) = 0$$

$$\therefore \max_{L_1} f = f(3, 0) = 9$$

over  $L_2$ ,

$$\begin{aligned}f(x, y) &= 2y \\ \therefore \min_{L_2} f &= f(0, 0) = 0 \\ \therefore \max_{L_2} f &= f(0, 2) = 4\end{aligned}$$

over  $L_3$ ,

$$\begin{aligned}f(x, y) &= x^2 - 2x \left( -\frac{2}{3}x + 2 \right) + 2 \left( -\frac{2}{3}x + 2 \right) \\ &= \frac{7}{3}x^2 - \frac{16}{3}x + 4 \\ \therefore f' &= \frac{14}{3}x - \frac{16}{3} \\ \therefore f' \left( \frac{8}{7} \right) &= 0 \\ \therefore f \left( \frac{8}{7}, \frac{26}{21} \right) &= 0.952 \\ \therefore \min_{L_3} f &= f \left( \frac{8}{7}, \frac{26}{21} \right) = 0.952 \\ \therefore \max_{L_3} f &= f(3, 0) = 9\end{aligned}$$

Therefore,

$$\begin{aligned}\therefore \min_D f &= f(0, 0) = 0 \\ \therefore \max_D f &= f(3, 0) = 9\end{aligned}$$

## 5 Taylor's Formula

**Theorem 50.**

$$\begin{aligned}f(a + h, b + k) &= \sum_{i=0}^n \left( \frac{1}{i!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(a, b) \right) \\ &\quad + \frac{1}{(n+1)!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(a + ch, b + ck)\end{aligned}$$

where  $0 < c < 1$ .