# Differential and Integral Calculus : Recitations

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## Contents

# 1 Instructor Information

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# Part I
# Sequences and Series

## 1 Sequences

**Recitation 1 – Exercise 1.**
Prove:

$$\lim_{n\to\infty} \frac{2n^2+n+1}{n^2+3} = 2$$

**Recitation 1 – Solution 1.**
Let

$$\varepsilon > 0$$

$$\begin{aligned}
\left|\frac{2n^2+n+1}{n^2+3} - 2\right| &= \left|\frac{2n^2+n+1-2n^2-6}{n^2+3}\right| \\
&= \left|\frac{n-5}{n^2+3}\right| \\
&\leq \left|\frac{n-5}{n^2}\right| \\
&\leq \frac{1}{n} \\
&< \varepsilon
\end{aligned}$$

Therefore, let $N = \left[\frac{1}{\varepsilon}\right] + 1$. Hence, for this $N$, $|a_n - L| < \varepsilon$.
Therefore, $\lim_{n\to\infty} \frac{2n^2+n+1}{n^2+3} = 2$. □

**Recitation 1 – Exercise 2.**
Prove

$$\lim_{n\to\infty} \frac{n^3+\sin n+n}{2n^4} = 0$$

**Recitation 1 – Solution 2.**
Let $\varepsilon > 0$

$$\begin{aligned}\left|\frac{n^3+\sin n+n}{2n^4}\right| &\le \left|\frac{n^3+1+n}{2n^4}\right| \\ &\le \left|\frac{3n^3}{2n^4}\right| = \frac{3}{2}\cdot\frac{1}{n} < \varepsilon\end{aligned}$$

Therefore, let $N = \left[\frac{3}{2\varepsilon}\right] + 1$. Hence, for this $N$, $|a_n - L| < \varepsilon$.

Therefore, $\lim_{n\to\infty} \frac{n^3+\sin n+n}{2n^4} = 0$ □

**Recitation 1 – Exercise 3.**
Calculate $\sqrt[3]{n^3+3n} - n$.

**Recitation 1 – Solution 3.**

$$a^n - b^n = (a-b)\cdot(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Therefore, let

$$\begin{aligned}a &= \sqrt[3]{n^3+3n} \\ b &= \sqrt[3]{n^3}\end{aligned}$$

$$\begin{aligned}a - b &= \frac{a^3-b^3}{a^2+ab+b^2} \\ \therefore\ \sqrt[3]{n^3+3n} - n &= \frac{n^3+3n-n^3}{(n^3+3n)^{2/3} + (n^3+3n)^{1/3}n + n^2} \\ &= \frac{3}{\left(\frac{n^3+3n}{n^{3/2}}\right)^{2/3} + \left(\frac{n^3+3n}{n^3}\right)^{1/3n} + n}\end{aligned}$$

Therefore, the limit is 0.

**Recitation 1 – Exercise 4.**
Prove

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

**Recitation 1 – Solution 4.**

$$0 \le \frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\dots\frac{n}{n} \le \frac{1}{n}$$

Therefore, by the Sandwich Theorem, $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0$.

**Recitation 1 – Exercise 5.**

Let $a_1 = 3$, $a_{n+1} = 1 + \sqrt{6 + a_n}$. Prove that $a_n$ converges and find its limit.

**Recitation 1 – Solution 5.**

If possible, let $\lim\limits_{n\to\infty} a_n = l$.

$$a_{n+1} = 1 + \sqrt{6 + a_n}$$

Taking the limit on both sides,

$$\begin{aligned} l &= 1 + \sqrt{6 + l} \\ \therefore l - 1 &= \sqrt{6 + l} \\ \therefore l &= \frac{3 \pm \sqrt{29}}{2} \end{aligned}$$

As $a_n \ge 0$, $l = \dfrac{3 + \sqrt{29}}{2}$.

$$\begin{aligned} a_2 &= 1 + \sqrt{6 + a_1} \\ &= 1 + \sqrt{6 + 3} \\ &= 4 \\ \therefore a_2 &> a_1 \end{aligned}$$

If possible, let $a_n \ge a_{n-1}$.
Therefore,

$$\begin{aligned} a_{n+1} &= 1 + \sqrt{6 + a_n} \\ &\ge 1 + \sqrt{6 + a_{n+1}} = a_n \end{aligned}$$

Therefore by induction, $\{a_n\}$ is monotonically increasing.

$$\begin{aligned} a_1 &= 3 \\ \therefore a_1 &\le 5 \end{aligned}$$

If possible, let $a_n \leq 5$.
Therefore,

$$a_{n+1} = 1 + \sqrt{6 + a_n} \leq q + \sqrt{11} \leq 5$$

Therefore by induction, $\{a_n\}$ is bounded from above by 5.

## 1.1 Limit of a Function by Heine

**Definition 1.**

$$\lim_{x \to x_0} f(x) = l$$

if for every sequence $x_n$, such that $\lim\limits_{n \to \infty} x_n = x_0$,

$$\lim_{n \to \infty} f(x_n) = l$$

**Theorem 1.** *If $f$ is continuous at $x_0$ and $x_n \to x_0$, then*

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right) = f_{x_0}$$

**Recitation 2 – Exercise 1.**
Calculate $\lim\limits_{n \to \infty} \sqrt[n]{n}$.

**Recitation 2 – Solution 1.**
Let

$$f(x) = x^{1/x}$$

Therefore,

$$\begin{aligned}\lim_{x \to \infty} x^{1/x} &= \lim_{x \to \infty} e^{\cancelto{0}{\frac{\ln x}{x}}} \\ &= 1\end{aligned}$$

## 1.2 Sub-sequences

**Recitation 2 – Exercise 2.**
Find all partial limits and $\overline{\lim}$ and $\underline{\lim}$ of

$$a_n = \left(\cos\frac{\pi n}{4}\right)^n$$

**Recitation 2 – Solution 2.**
Let $k, z \in \mathbb{Z}$

$$\begin{aligned}
\cos\frac{\pi n}{4} &= \cos\frac{\pi(n+k)}{4} \\
\therefore \frac{\pi n}{4} &= \frac{\pi(n+k)}{4} + 2\pi z \\
\therefore \pi n &= \pi(n+k) + 8\pi z \\
\therefore k &= 8z
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_{8k} &= \left(\cos\frac{\pi\cdot 8k}{4}\right)^{8k} \\
&= (\cos(2\pi k))^{8k} \\
&= 1 \\
a_{8k+1} &= \left(\cos\frac{\pi\cdot(8k+1)}{4}\right)^{8k+1} \\
&= \left(\cos\frac{\pi}{4}\right)^{8k+1} \\
&= \left(\frac{\sqrt{2}}{2}\right)^{8k+1} \\
a_{8k+2} &= \left(\cos\frac{\pi\cdot(8k+2)}{4}\right)^{8k+2} \\
&= \left(\cos\frac{\pi}{2}\right)^{8k+2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{k\to\infty} a_{8k} &= 1 \\
\lim_{k\to\infty} a_{8k+1} &= \lim_{k\to\infty}\left(\frac{\sqrt{2}}{2}\right)^{8k+1} \\
&= 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{k\to\infty} a_{8k+2} &= 0 \\
\lim_{k\to\infty} a_{8k+3} &= 0 \\
\lim_{k\to\infty} a_{8k+4} &= \lim_{k\to\infty} (-1)^{8k+4} \\
&= 1 \\
\lim_{k\to\infty} a_{8k+5} &= 0 \\
\lim_{k\to\infty} a_{8k+6} &= 0 \\
\lim_{k\to\infty} a_{8k+7} &= 0
\end{aligned}$$

Therefore, $\{a_n\}$ has two partial limits, 0 and 1.

$$\begin{aligned}
\overline{\lim} a_n &= 1 \\
\underline{\lim} a_n &= 0
\end{aligned}$$

# 2 Series

**Definition 2** (Convergence of a series)**.** Let $\{a_n\}$ be a sequence. Let $S_n$ be a sequence of partial sums of $a_n$, s.t.

$$S_n = \sum_{k=1}^{n} a_k$$

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to $l$ if

$$\lim_{n\to\infty} S_n = l$$

that is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n$$

**Recitation 2 – Exercise 3.**

Does $\displaystyle\sum_{k=0}^{\infty} q^k$ where $-1 < q < 1$ converge?

**Recitation 2 – Solution 3.**

$$\begin{aligned}\sum_{k=0}^{\infty} q^k &= \lim_{n\to\infty} \sum_{k=0}^{n} q^k \\ &= \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q} \\ &= \frac{1}{1-q}\end{aligned}$$

Therefore, the series converges.

**Recitation 2 – Exercise 4.**

Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converge?

**Recitation 2 – Solution 4.**

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k(k+1)} &= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \\ &= 1\end{aligned}$$

**Recitation 2 – Exercise 5.**

Does $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$ converge?

**Recitation 2 – Solution 5.**

$$\begin{aligned}\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k &= e \\ \therefore \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k &\neq 0\end{aligned}$$

Therefore, the necessary condition is nt satisfied. Hence, the series does not converge.

## 2.1 Comparison Tests for Positive Series

**Theorem 2** (First Comparison Test)**.** *If $a_n \geq 0$, $b_n \geq 0$, and $a_n \leq b_n$, then*

1. *If $\sum b_n$ converges, then $\sum a_n$ converges.*
2. *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

**Theorem 3** (Second Comparison Test)**.** *If $a_n \geq 0$, $b_n \geq 0$ and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l$$

*where $0 < l < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge simultaneously.*

**Recitation 3 – Exercise 1.**
Suppose the sequence $a_n$ satisfies the condition

$$a_{n+1} - a_n > \frac{1}{n}$$

$\forall n \in \mathbb{N}$.
Prove that $\lim\limits_{n\to\infty} a_n = \infty$.

**Recitation 3 – Solution 1.**

$$\begin{aligned} a_{n+1} &= a_{n+1} - a_n + a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_2 - a_1 + a_1 \\ &= \sum_{k=1}^{n} (a_{k+1} - a_k) + a_1 \\ &\geq \sum_{k=1}^{n} \frac{1}{k} + a_1 \end{aligned}$$

As the harmonic series diverges, $\sum\limits_{k=1}^{n} \dfrac{1}{k} + a_1$ diverges.
Therefore, by the First Comparison Test, $\sum\limits_{k=1}^{\infty} (a_{k+1} - a_k)$ diverges.

**Recitation 3 – Exercise 2.**
Check the convergence of $\sum\limits_{n=2}^{\infty} \dfrac{n + \sin n}{n^3 + \cos \pi n}$.

**Recitation 3 – Solution 2.**

The series is non-negative. Therefore, the comparison tests are applicable.

$$\begin{aligned}
\frac{n+\sin n}{n^3+\cos \pi n} &\le \frac{n+1}{n^3-1} \\
\therefore \frac{n+\sin n}{n^3+\cos \pi n} &\le \frac{2n}{n^3-\frac{n^3}{2}} && \le \frac{4}{n^2}
\end{aligned}$$

Therefore, by the First Comparison Test, as $\frac{4}{n^2}$ converges, $\sum_{n=2}^{\infty} \frac{n+\sin n}{n^3+\cos \pi n}$ also converges.

**Recitation 3 – Exercise 3.**

Let $a_n \ge 0$ and suppose that $\sum a_n$ converges. Prove that $\sum {a_n}^2$ converges. Is it true without the assumption $a_n \ge 0$?

**Recitation 3 – Solution 3.**

As $\sum a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$.
Therefore, $\exists N \in \mathbb{N}$, such that $\forall n > N$, $a_n < 1$.
Therefore, $\forall n > N$, ${a_n}^2 \le a_n$. Hence, as $\sum\limits_{n=N+1}^{\infty} a_n$ converges, $\sum\limits_{n=N+1}^{\infty} {a_n}^2$ also converges. Hence, $\sum\limits_{n=1}^{\infty} a_n$ also converges.

This is not true without the assumption $a_n \ge 0$, as the argument ${a_n}^2 \le a_n$ does not hold.

**Recitation 3 – Exercise 4.**

For which $\alpha$ does $\sum \left(\sqrt{n+1}-\sqrt{n}\right)^{\alpha/2}$ converge?

**Recitation 3 – Solution 4.**

$$\begin{aligned}
\sum \left(\sqrt{n+1}-\sqrt{n}\right)^{\alpha/2} &= \sum \left(\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2} \\
&= \sum \left(\frac{1}{\sqrt{n+1}-\sqrt{n}}\right)^{\alpha/2}
\end{aligned}$$

The series is positive. Therefore, the comparison tests are applicable.

Comparing with $\left(\frac{1}{\sqrt{n}}\right)^{\alpha/2}$,

$$\frac{\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2}}{\left(\frac{1}{\sqrt{n}}\right)^{\alpha/2}} = \left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2}$$

$$\therefore \lim_{n\to\infty} \left(\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)^{\alpha/2} = \left(\frac{1}{2}\right)^{\alpha/2}$$

$\sum \frac{1}{n^{\alpha/2}}$ converges if and only if $\frac{\alpha}{4} > 1$, i.e. if an inly if $\alpha > 4$.
By the Second Comparison Test, $\sum \frac{1}{n^{\alpha/4}}$ and the series converge or diverge simultaneously.
Therefore, the series converges for $\alpha > 4$.

**Recitation 3 – Exercise 5.**

Check the convergence of $\sum\limits_{n=1}^{\infty} \sin \frac{1}{n}$.

**Recitation 3 – Solution 5.**

$\forall n \in \mathbb{N}$, $\sin \frac{1}{n} \geq 0$

$$\lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$$

Therefore, by Second Comparison Test, $\sum \frac{1}{n}$ and $\sum \sin \frac{1}{n}$ diverge simultaneously.

## 2.2 d'Alembert Criteria (Ratio Test)

**Definition 3** (Absolute and conditional convergence)**.** The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

**Theorem 4.** *If the series $\sum a_n$ converges absolutely then it converges.*

**Theorem 5** (d'Alembert Criteria (Ratio Test)). *1. If*

$$\lim_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right| = L < 1$$

*then $\sum a_n$ converges absolutely.*

*2. If*

$$\lim_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right| = L > 1$$

*(including $L = \infty$), then $\sum a_n$ converges diverges.*

*3. If $L = 1$, the test does not apply.*

**Recitation 3 – Exercise 6.**

Check the convergence of $\sum \dfrac{(-1)^n \cdot n^{1000}}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}$.

**Recitation 3 – Solution 6.**

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n \cdot n^{1000}}{1 \cdot \dots \cdot (2n-1)}\right| = \sum_{n=1}^{\infty}\frac{n^{1000}}{1 \cdot \dots \cdot (2n-1)}$$

Therefore, by the d'Alembert Criteria (Ratio Test),

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\dfrac{(n+1)^{1000}}{1 \cdot \dots \cdot (2n+1)}}{\dfrac{n^{1000}}{1 \cdot \dots \cdot (2n-1)}} \\ &= \left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1}\end{aligned}$$

$$\begin{aligned}\therefore \lim_{nto\infty}\left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1} &= 0 \\ \therefore \left(\frac{n+1}{n}\right)^{1000} \cdot \frac{1}{2n+1} &< 1\end{aligned}$$

Therefore, by the d'Alembert Criteria (Ratio Test), the series converges absolutely, and hence converges.

## 2.3 Cauchy Criteria (Cauchy Root Test)

**Theorem 6** (Cauchy Criteria (Cauchy Root Test))**.** *1. If*

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

*then $\sum a_n$ converges absolutely.*

*2. If*

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

*(including $L = \infty$), then $\sum a_n$ diverges.*

*3. If $L = 1$, the test does not apply.*

**Recitation 3 – Exercise 7.**

Check the convergence of $\sum \left(1 - \dfrac{2}{n}\right)^{n^2}$.

**Recitation 3 – Solution 7.**

$$\begin{aligned}
\sqrt[n]{\left(1 - \frac{2}{n}\right)^{n^2}} &= \left(1 - \frac{2}{n}\right)^{n} \\
\therefore \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^{n} &= e^{-2} \\
\therefore \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^{n} &< 1
\end{aligned}$$

Therefore, by the Cauchy Criteria (Cauchy Root Test), $\sum \left(1 - \dfrac{2}{n}\right)^{n^2}$ converges.

## 2.4 Leibniz's Criteria

**Definition 4** (Alternating series)**.** The series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

**Theorem 7** (Leibniz's Criteria for Convergence)**.** *If an alternating series $\sum (-1)^{n-1} a_n$ with $a_n > 0$ satisfies*

*1. $a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.*

*2.* $\lim_{n\to\infty} a_n = 0$

*then the series* $(-1)^{n-1}a_n$ *converges.*

**Recitation 3 – Exercise 8.**
Prove or disprove: There exists $\{a_n\}$, such that $\sum a_n$ converges and $\sum(1 + a_n)a_n$ diverges.

**Recitation 3 – Solution 8.**
Let $a_n = \frac{(-1)^n}{\sqrt{n}}$.
Therefore, by Leibniz's Criteria for Convergence, $\sum \frac{(-1)^n}{\sqrt{n}}$ converges.

$$\begin{aligned}\sum(1 + a_n)a_n &= \sum \left(1 + \frac{(-1)^n}{\sqrt{n}}\right) \frac{(-1)^n}{\sqrt{n}} \\ &= \sum \left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right)\end{aligned}$$

Therefore, as $\sum \frac{1}{n}$ diverges, and $\sum \frac{(-1)^n}{\sqrt{n}}$ converges, $\sum \left(\frac{1}{n} + \frac{(-1)^n}{\sqrt{n}}\right)$ diverges.

## 2.5 Integral Test

**Theorem 8** (Integral Test)**.** *If* $f(x) : [1, \infty) \to [0, \infty)$ *is monotonically decreasing. Then,* $\sum\limits_{n=1}^{\infty} f(n)$ *and* $\int\limits_{1}^{\infty} f(x)\,\mathrm{d}x$ *converge or diverge simultaneously.*

**Recitation 3 – Exercise 9.**
Check the convergence of $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$

**Recitation 3 – Solution 9.**
Let

$$f(x) = \frac{1}{x \ln x}$$

$f(x)$ is monotonically decreasing. Therefore, the Integral Test is applicable. Therefore,

$$\begin{aligned}\int\limits_2^\infty \frac{1}{x\ln x}\,\mathrm{d}x &= \int\limits_{\ln 2}^\infty \frac{1}{y}\,\mathrm{d}y\\ &= \ln y|_{\ln 2}^\infty\\ &= \infty\end{aligned}$$

Therefore, by the integral test, $\sum \dfrac{1}{n\ln n}$ diverges.

**Recitation 4 – Exercise 1.**
Let $d_n \geq 0$ and suppose

$$\sum_{n=0}^\infty d_n = \infty$$

Prove that

$$\sum_{n=0}^\infty \frac{d_n}{1+d_n} = \infty$$

**Recitation 4 – Solution 1.**
If possible, let $d_n$ be a bounded sequence. Then there exists $M$, such that $d_n \leq M$, $\forall n \in \mathbb{N}$.
Therefore,

$$\frac{d_n}{1+d_n} \geq \frac{d_n}{1+M}$$

Therefore, by the Second Comparison Test, as $\sum d_n$ diverges, $\sum \dfrac{d_n}{1+d_n}$ also diverges.

If $d_n$ is not bounded, then there is a subsequence $d_{n_k}$ which diverges. Therefore,

$$\frac{d_{n_k}}{1+d_{n_k}} = \frac{1}{\frac{1}{d_{n_k}}+1}$$

$$\therefore \lim_{k\to\infty} \frac{d_{n_k}}{1+d_{n_k}} = 1$$

Therefore,

$$\lim_{n\to\infty} \frac{d_n}{1+d_n} \neq 0$$

Therefore, the necessary condition for convergence is not fulfilled. Therefore, the series converges.

**Recitation 4 – Exercise 2.**
Let

$$d_n = \begin{cases} 1 & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$

Does $\sum \dfrac{d_n}{1+n\cdot d_n}$ diverge?

**Recitation 4 – Solution 2.**

$$d_n = \begin{cases} 1 & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$

$$\therefore \frac{d_n}{1+n\cdot d_n} = \begin{cases} \dfrac{1}{1+k^2} & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$

As $\dfrac{1}{1+k^2} \leq \dfrac{1}{k^2}$ and as $\dfrac{1}{k^2}$ converges, $\sum \dfrac{1}{1+k^2}$ also converges.

**Recitation 4 – Exercise 3.**
Let $a_n$ be a sequence such that $|a_{n+1} - a_n| \leq b_{n+1}$ for all $n \in \mathbb{N}$ where $\sum b_k$ converges. Prove that $\{a_n\}$ converges.

**Recitation 4 – Solution 3.**
Let $\varepsilon > 0$.

$$\begin{aligned} |a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} + \cdots - a_n| \\ &\leq \sum_{k=n+1}^{m} |a_k - a_{k-1}| \\ &\leq \sum_{k=n+1}^{m} b_k \end{aligned}$$

Therefore, as $\sum b_n$ converges, the series satisfies the Cauchy Criteria (Cauchy Root Test). Therefore, there exists $N$, such that $\forall m > n > N$, $\left|\sum_{k=n+1}^{m} b_k\right| < \varepsilon$. Therefore, for $m > n > N$,

$$|a_m - a_n| \leq \sum_{k=n+1}^{m} b_n < \varepsilon$$

# 3 Power Series

**Definition 5** (Power series)**.** A power series around $x_0$ is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $\{a_n\}$ is a sequence of real numbers.

**Theorem 9** (Abel's Theorem)**.** *For every power series $\sum a_n (x - x_0)^n$, there exists $R \in [0, \infty]$, such that for all $x$ satisfying $|x - x_0| < R$, the series converges and for all $x$ satisfying $|x - x_0| > R$ the series diverges.*

**Theorem 10** (Cauchy's Formula for Radius of Convergence)**.**

$$R = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$$

**Theorem 11** (Hadamard's Formula for Radius of Convergence)**.** *If $\lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists, then*

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = R$$

**Recitation 4 – Exercise 4.**

Find the domain of convergence of $\sum\limits_{n=1}^{\infty} \frac{(2x-4)^n}{n}$.

**Recitation 4 – Solution 4.**

$$\sum_{n=1}^{\infty} \frac{(2x-4)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n (x-2)^n}{n}$$

Therefore, by Cauchy's Formula for Radius of Convergence,

$$\begin{aligned} R &= \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}} \\ &= \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{2^n}{n}}} \\ &= \frac{1}{\lim\limits_{n\to\infty} \dfrac{2}{\sqrt[n]{n}}} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, the series converges for

$$|x-2| < \frac{1}{2}$$

and diverges for

$$|x-2| > \frac{1}{2}$$

If $x = \frac{5}{2}$,

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{5}{2} - 2\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

Therefore, the series diverges.
If $x = \frac{3}{2}$,

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{3}{2} - 2\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

Therefore, by Leibniz's Criteria for Convergence, the series converges.
Therefore, the domain of convergence is $\left[\frac{3}{2}, \frac{5}{2}\right)$.

**Recitation 4 – Exercise 5.**

Find the radius of convergence of $\sum\limits_{n=0}^{\infty} n! x^{n!}$.

**Recitation 4 – Solution 5.**

$$\frac{1}{\sqrt[n]{a_n}} = x + x + 2x^2 + 6x^6 + 24x^{24} + \dots$$

Therefore,

$$a_n = \begin{cases} n & ; \quad n = k^2 \\ 0 & ; \quad n \neq k^2 \end{cases}$$

Therefore,

$$\begin{aligned} R &= \frac{1}{\lim\limits_{n \to \infty} \sqrt[n]{a_n}} \\ &= \frac{1}{\lim\limits_{k \to \infty} \sqrt[k!]{k!}} \\ &= 1 \end{aligned}$$

## 3.1 Power Series Representation of a Function

**Theorem 12.** *The power series representation of a function $f(x)$ is equal to its Taylor series if and only if $\lim\limits_{n \to \infty} R_n(x) = 0$, where $R_n(x)$ is the Lagrange remainder.*

## 3.2 Differentiation and Integrations of Power Series

**Recitation 5 – Exercise 1.**

Find the power series representation of $\tan^{-1} x$.

**Recitation 5 – Solution 1.**

$$\begin{aligned}\frac{\mathrm{d}\tan^{-1}x}{\mathrm{d}x} &= \frac{1}{1+x^2}\\ &= \sum_{n=0}^{\infty}(-1)^n x^{2n}\end{aligned}$$

Integrating term by term,

$$\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} + c$$

As $\tan^{-1}0 = 0$, $c = 0$. Therefore,

$$\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

**Recitation 5 – Exercise 2.**

Find an explicit formula for $\sum\limits_{n=1}^{\infty} x^n n^2$.

**Recitation 5 – Solution 2.**

$$\sum_{n=1}^{\infty} x^n n^2 = x\cdot\sum_{n=1}^{\infty} x^{n-1}n^2$$

Let

$$g(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

Integrating term by term,

$$\begin{aligned}\int g(x)\,\mathrm{d}x &= \sum_{n=1}^{\infty} n^2\frac{x^n}{n}\\ &= \sum_{n=1}^{\infty} nx^n\\ &= x\cdot\sum_{n=1}^{\infty} nx^{n-1}\end{aligned}$$

Let

$$h(x) = \sum_{n=1}^{\infty} n x^{n-1}$$
$$\therefore \int h(x)\,\mathrm{d}x = \frac{x}{1-x}$$

Therefore, inside radius of convergence $R = 1$, differentiating $\int h(x)\,\mathrm{d}x$,

$$\begin{aligned} h(x) &= \frac{1-x+x}{(1-x)^2} \\ &= \frac{1}{(1-x)^2} \\ \therefore \int g(x)\,\mathrm{d}x &= x h(x) \\ &= \frac{x}{(1-x)^2} \\ \therefore g(x) &= \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4} \\ \therefore \sum_{n=1}^{\infty} x^n n^2 &= x \cdot \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4} \end{aligned}$$

**Recitation 5 – Exercise 3.**
Find the sum $\sum\limits_{n=1}^{\infty} \frac{1}{n2^n}$.

**Recitation 5 – Solution 3.**
Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

be a power series with radius $R$.
Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = f\left(\frac{1}{2}\right)$$

Therefore,

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} x^{n-1} \\ &= \frac{1}{1-x} \\ \therefore f(x) &= -\ln(1-x) + c \end{aligned}$$

As $f(0) = 0$, $c = 0$. Therefore,

$$f(x) = -\ln(1 - x)$$

Therefore,

$$\begin{aligned} f\left(\frac{1}{2}\right) &= -\ln\left(\frac{1}{2}\right) \\ &= \ln 2 \end{aligned}$$

# 4 Sequences of Functions

**Definition 6** (Point-wise convergence and domain of convergence)**.** $\{f_n\}$ is said to converge point-wise in some domain $E \subset D$ if $\forall x \in E$, the sequence $\{f_n(x)\}$ converges. In this case, $E$ is said to be a domain of convergence of $\{f_n\}$.

**Recitation 5 – Exercise 4.**
Let $f(x) : \mathbb{R} \to \mathbb{R}$ be some function such that $\lim\limits_{x\to\infty} f(x) = 0$. Let $f_n(x) = f(nx)$. What is the domain of convergence of $f_n$? What is the limit function?

**Recitation 5 – Solution 4.**
Let $x$ be a particular number in $(0, \infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f(nx)$$

Therefore, as $\lim\limits_{x\to\infty} f(x) = 0$,

$$\lim_{n\to\infty} f_n(x) = 0$$

Therefore the domain of convergence is $(0, \infty)$ and the limit function is a constant 0.

Although the all functions in $\{f_n\}$ are continuous, the limit function is not continuous.

**Definition 7** (Uniform convergence)**.** A sequence of functions $\{f_n\}$ is said to converge uniformly to $f$ in the domain $E$, if $\forall \varepsilon$, $\exists N$ such that $\forall n > N$ and $\forall x \in E$, $|f_n(x) - f_(x)| < \varepsilon$. If $f_n$ converges to $f$ uniformly in $E$, it is denoted as $f_n \overset{E}{\rightrightarrows} f$.

## 4.1 Supremum and Infimum of Sets

**Definition 8** (Supremum)**.** Let $A \subseteq \mathbb{R}$ be a bounded set. $M$ is said to be the supremum of $A$ if

1. $\forall x \in A$, $x \leq M$, i.e. $M$ is an upper bound of $A$.

2. $\forall \varepsilon$, $\exists x \in A$, such that $x > M - \varepsilon$.

That is, the supremum of $A$ is the least upper bound of $A$.
The supremum may or may not be in $A$.

**Definition 9** (Infimum)**.** Let $A \subseteq \mathbb{R}$ be a bounded set. $M$ is said to be the infimum of $A$ if

1. $\forall x \in A$, $x \geq M$, i.e. $M$ is an upper bound of $A$.

2. $\forall \varepsilon$, $\exists x \in A$, such that $x < M - \varepsilon$.

That is, the infimum of $A$ is the greatest lower bound of $A$. The infimum may or may not be in $A$.

**Theorem 13.** *Every bounded set A has a supremum and an infimum.*

**Theorem 14.** $f_n \overset{E}{\rightrightarrows} f$ *if and only if*

$$\lim_{n\to\infty} \left(\sup\{|f_n(x) - f(x)| : x \in E\}\right) = 0$$

**Recitation 6 – Exercise 1.**
Let $f_n(x) = x^n$. Does $\{f_n\}$ converge uniformly?

**Recitation 6 – Solution 1.**

$$f(x) = \begin{cases} 0 & ; \quad x \in [0,1] \\ 1 & ; \quad x = 1 \end{cases}$$

If the convergence is uniform in $[0,1]$,

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = 0$$

Let $x = 1 - \frac{1}{n}$.
Therefore, as the supremum is a upper bound,

$$
\begin{aligned}
\sup_{x\in[0,1]} |f_n(x) - f(x)| &\geq \left| f_n\left(1 - \frac{1}{n}\right) - f\left(1 - \frac{1}{n}\right) \right| \\
\therefore \sup_{x\in[0,1]} |f_n(x) - f(x)| &\geq \left| \left(1 - \frac{1}{n}\right)^n - 0 \right| \\
\therefore \lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| &\geq \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n \\
\therefore \lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| &\geq \frac{1}{e} \\
\therefore \lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| &\neq 0
\end{aligned}
$$

Therefore, the convergence is not uniform.

**Recitation 6 – Exercise 2.**
Let $f_n(x) = x + \frac{1}{n}$, $x \in \mathbb{R}$. What is its domain of convergence? What is the limit function? Is the convergence uniform?

**Recitation 6 – Solution 2.**
$\forall x \in \mathbb{R}$,

$$\lim_{n\to\infty} \left(x + \frac{1}{n}\right) = x$$

Therefore $\{f_n\}$ converges pointwise to $x$, in $\mathbb{R}$.

$$
\begin{aligned}
\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| &= \sup_{x\in\mathbb{R}} \left| x + \frac{1}{n} - x \right| \\
&= \frac{1}{n} \\
\therefore \lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| &= 0
\end{aligned}
$$

Therefore, the convergence is uniform.

**Recitation 6 – Exercise 3.**
Let $f_n : [0, \infty) \to \mathbb{R}$.

$$f_n(x) = \begin{cases} 1 & ; \quad n \leq x \leq n+1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Dows $f_n$ converge pointwise in $[0, \infty)$? Dows $f_n$ converge uniformly in $[0, \infty)$?

**Recitation 6 – Solution 3.**
For every $x$, the sequence $\{f_n(x)\}$ will be of the form $\{0, \dots, 0, 1, 0, \dots, 0\}$ with 1 only when $n \leq x \leq n+1$.
Therefore,

$$\begin{aligned}\lim_{n\to\infty} f_n(x) &= 0\\ &= f(x)\end{aligned}$$

Therefore, $f_n$ converges pointwise in $[0, \infty)$.

$$\begin{aligned}\sup_{x\in[0,\infty)} |f_n(x) - f(x)| &= \max_{x\in[0,\infty)} f_n(x)\\ &= 1\end{aligned}$$

Therefore, as the limit of the supremum is not 0, the convergence is not uniform.

**Theorem 15.** *If $f_n \overset{D}{\rightrightarrows} f$ and all $f_n$ are continuous is $D$, then $f$ is also continuous, i.e. uniform convergence preserves continuity.*

**Recitation 7 – Exercise 1.**
Does $x^n$ converge to

$$f(x) = \begin{cases} 0 & ; \quad x \in [0, 1) \\ 1 & ; \quad x = 1 \end{cases}$$

**Recitation 7 – Solution 1.**
If possible, let $x^n$ converge to $f(x)$.
Therefore, as all $f_n(x)$ are continuous, and as uniform convergence preserves continuity, $f(x)$ also must be continuous.
This contradicts the definition of $f(x)$.
Therefore, the $x^n$ does not converge to $f(x)$.

**Recitation 7 – Exercise 2.**
Check if $f_n(x) = \frac{x}{1+n^2x^2}$ converges uniformly in $[0, 1]$.

**Recitation 7 – Solution 2.**

$$\lim_{n\to\infty} f_n(x) = 0$$
$$= f(x)$$

Therefore,

$$\begin{aligned}
\sup_{[0,1]} |f_n(x) - f(x)| &= \sup_{[0,1]} |f_n(x) - 0| \\
&= \sup_{[0,1]} \left| \frac{x}{1+n^2x^2} \right| \\
&= \sup_{[0,1]} \frac{x}{1+n^2x^2}
\end{aligned}$$

As the function is continuous and the interval is closed, by the Weierstrass theorem, the function has a maximum. Therefore,

$$\sup_{[0,1]} |f_n(x) - f(x)| = \max_{[0,1]} \frac{x}{1+n^2x^2}$$

Differentiating to find the maximum,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1+n^2x^2}\right) &= \frac{1+n^2x^2-2x^2n^2}{(1+n^2x^2)} \\
&= \frac{1-x^2n^2}{(1+n^2x^2)^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1+n^2x^2}\right) = 0 \\
\iff\ &\frac{1-x^2n^2}{(1+n^2x^2)^2} = 0 \\
&\iff 1 = x^2n^2 \\
&\iff x = \frac{1}{n}
\end{aligned}$$

Therefore, the values of the function at the critical points and the end points are,

$$\begin{aligned}
f_n(0) &= 0 \\
f_n(1) &= \frac{1}{1+n^2} \\
f_n\left(\frac{1}{n}\right) &= \frac{\frac{1}{n}}{1+n^2\frac{1}{n^2}} \\
&= \frac{1}{2n}
\end{aligned}$$

Therefore, the maximum is at $x = \frac{1}{2n}$.
Therefore,

$$\begin{aligned}
\max_{[0,1]} \frac{x}{1+n^2x^2} &= f_n\left(\frac{1}{n}\right) \\
&= \frac{1}{2n}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{n\to\infty} \sup_{[0,1]} |f_n(x) - f(x)| &= \lim_{n\to\infty} \max_{[0,1]} \frac{x}{1+n^2x^2} \\
&= \lim_{n\to\infty} \frac{1}{2n} \\
&= 0
\end{aligned}$$

Therefore, the convergence is uniform.

**Recitation 7 – Exercise 3.**
Check the pointwise and uniform convergence of $f_n(x) = x^n - x^{n+1}$ in $[0,1]$.

**Recitation 7 – Solution 3.**

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} x^n - x^{n+1} \\
&= 0 \\
&= f(x)
\end{aligned}$$

Therefore the function converges pointwise in $[0,1]$.

$$\sup_{[0,1]} |f_n(x) - f(x)| = \sup_{[0,1]} x^n - x^{n+1}$$

As the function is continuous and the interval is closed, by the Weierstrass theorem, the function has a maximum. Therefore,

$$\sup_{[0,1]} x^n - x^{n+1} = \max_{[0,1]} x^n - x^{n+1}$$

Differentiating to find the maximum,

$$\frac{\mathrm{d}(x^n - x^{n+1})}{\mathrm{d}x} = nx^{n-1} - (n+1)x^n$$

Therefore,

$$\begin{aligned} \frac{\mathrm{d}(x^n - x^{n+1})}{\mathrm{d}x} &= 0 \\ \iff nx^{n-1} - (n+1)x^n &= 0 \\ \iff n - (n+1)x &= 0 \\ \iff x &= \frac{n}{n+1} \end{aligned}$$

Therefore, the values of the function at the critical points and the end points are

$$\begin{aligned} f_n(0) &= 0 \\ f_n(1) &= 0 \\ f_n\left(\frac{n}{n+1}\right) &= \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \max_{[0,1]} x^n - x^{n+1} &= f_n\left(\frac{n}{n+1}\right) \\ &= \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \sup_{[0,1]} |f_n(x) - f(x)| &= \lim_{n\to\infty} \max_{[0,1]} \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} \\ &= \frac{1}{e} - \frac{1}{e} \\ &= 0 \end{aligned}$$

Therefore, the convergence is uniform.

**Theorem 16** (Cauchy's Theorem)**.** $\{f_n\}$ *converges uniformly in* $D$ *if and only if* $\forall \varepsilon \in N$, $\exists N$, *such that* $\forall m, n > N$ *and* $\forall x \in D$,

$$|f_n(x) - f(x)| < \varepsilon$$

**Recitation 7 – Exercise 4.**
Let $\{f_n\}$ be a sequence of function in $D$ such that $\forall x \in D$, $|f_{n+1}(x) - f_n(x)| \leq a_n$, where $\sum\limits_{n=1}^{\infty} a_n$ converges. Prove that $\{f_n\}$ converges uniformly in $D$.

**Recitation 7 – Solution 4.**
As $\sum a_n$ converges, $\exists N$ such that $\forall m > n > N$, $\left|\sum\limits_{k=n}^{m} a_k\right| < \varepsilon$.
Therefore, forall $m > n > N$ and $x \in D$,

$$\begin{aligned}
|f_m(x) - f_n(x)| &= |f_m(x) - f_{m-1}(x) + f_{m-1}(x) - \cdots - f_n(x)| \\
&\leq |f_m(x) - f_{m-1}(x)| + |f_{m-1}(x) - f_{m-2}(x) + \cdots + |f_{n+1}(x) - f_n(x)| \\
\therefore |f_m(x) - f_n(x)| &\leq \sum_{k=n}^{m-1} |f_{k+1}(x) - f_k(x)| \\
\therefore |f_m(x) - f_n(x)| &\leq \sum_{k=n}^{m-1} a_k \\
\therefore |f_m(x) - f_n(x)| &\leq \varepsilon
\end{aligned}$$

Therefore, $\{f_n\}$ satisfies Cauchy's criterion for uniform convergence.

# 5 Series of Functions

**Definition 10** (Pointwise convergence of series of functions)**.** Let $\{f_n\}$ be a sequence of functions defined in $D$. Let $S_n(x) = \sum\limits_{k=1}^{n} f_k(x)$.
If $S_n(x)$ converges for every $x \in D$ to a limit $S$, the series formed by $\{f_n\}$ is said to converge pointwise in $D$. It is denoted as

$$\sum_{k=1}^{\infty} f_k(x) = \lim_{n \to \infty} S_n(x) = S_x$$

**Definition 11** (Uniform convergence of series of functions)**.** The series $\sum\limits_{k=1}^{\infty} f_k(x)$ is said to converge uniformly in $D$ if $S_n \overset{D}{\rightrightarrows} S$.

**Theorem 17.** *If* $\sum\limits_{k=1}^{\infty} f_k(x)$ *converges uniformly in* $D$*, then the general term* $f_k(x)$ *must uniformly converge to* $0$ *in* $D$*.*

**Recitation 7 – Exercise 5.**

Check the uniform convergence of $\sum\limits_{n=1}^{\infty} \frac{x^n}{n^2} - \frac{x^{n+1}}{(n+1)^2}$ in $[-1, 1]$.

**Recitation 7 – Solution 5.**

$$\begin{aligned} S_n(x) &= \sum_{k=1}^{n} \frac{x^k}{k^2} - \frac{x^{k+1}}{(k+1)^2} \\ &= \frac{x^1}{1^2} - \frac{x^{n+1}}{(n+1)^2} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} x &= \frac{x^{n+1}}{(n+1)^2} \\ &= x \\ &= S(x) \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{[-1,1]} |S_n(x) - S(x)| &= \sup_{[-1,1]} \left| -\frac{x^{n+1}}{(n+1)^2} \right| \\ &\leq \frac{1}{(n+1)^2} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \sup_{[-1,1]} |S_n(x) - S(x)| &\leq \lim_{n\to\infty} \frac{1}{(n+1)^2} \\ \therefore \lim_{n\to\infty} \sup_{[-1,1]} |S_n(x) - S(x)| &\leq 0 \end{aligned}$$

Therefore the convergence is uniform.

**Theorem 18.** *If* $\sum\limits_{n=1}^{\infty} f_n(x)$ *converges uniformly in* $D$ *to* $S(x)$ *and the functions* $f_n$ *are continuous in* $D$*, then the* $S(x)$ *is also continuous in* $D$*.*

**Theorem 19.** *A Leibniz series, i.e. a series of the form* $\sum\limits_{n=1}^{\infty} (-1)^n a_n$*, with* $a_n$ *monotonically decreasing and* $\lim\limits_{n\to\infty} a_n = 0$*, converges, and*

$$\sum_{k=n}^{m} (-1)^k a_k \leq a_n$$

**Recitation 7 – Exercise 6.**

Check for pointwise and uniform convergence of $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{x^2+\sqrt{n}}$ in $\mathbb{R}$.

**Recitation 7 – Solution 6.**

For $x \in \mathbb{R}$, $\frac{1}{x^2+\sqrt{n}}$ is monotonically decreasing to 0 as $n \to \infty$.

Therefore, for $x \in \mathbb{R}$, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{x^2+\sqrt{n}}$ is a Leibniz series. Hence, it converges pointwise.

$$\begin{aligned}\left|\sum_{k=n}^{m} \frac{(-1)^n}{x^2+\sqrt{k}}\right| &\le \frac{1}{x^2+\sqrt{n}} \\ &\le \frac{1}{\sqrt{n}}\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \left|\sum_{k=n}^{m} \frac{(-1)^n}{x^2+\sqrt{k}}\right| = 0$$

Therefore, $\forall \varepsilon > 0$, there exists $N$ such that $\forall m > n > N$, and $\forall x \in \mathbb{R}$,

$$\left|\sum_{k=n}^{m} \frac{(-1)^n}{x^2+\sqrt{k}}\right| \le \frac{1}{\sqrt{n}} < \varepsilon$$

Therefore, $\left|\sum\limits_{k=n}^{m} \frac{(-1)^n}{x^2+\sqrt{k}}\right|$ satisfies Cauchy's criterion for uniform convergence. Hence it converges uniformly.

**Recitation 7 – Exercise 7.**

Show that $\sum\limits_{n=1}^{\infty} 3^n \sin\left(\frac{1}{4^n x}\right)$ does not converge uniformly in $(0, \infty)$.

**Recitation 7 – Solution 7.**

For any $x \in (0, \infty)$, as $\sin\left(\frac{1}{4^n x}\right) \le \frac{1}{4^n x}$,

$$\left|3^n \sin(\frac{1}{4^n x})\right| \le 3^n \frac{1}{4^n x}$$

Therefore, as $\sum \left(\frac{3}{4}\right)^n \cdot \frac{1}{x}$ converges, by the First Comparison Test, $\sum \left|3^n \sin\left(\frac{1}{4^n x}\right)\right|$ also converges.

Therefore, $\sum 3^n \sin(\frac{1}{4^n x})$ converges absolutely. Hence, it converges.

$$\begin{aligned}\lim_{n\to\infty} 3^n \sin\left(\frac{1}{4^n x}\right) &= \lim_{n\to\infty} \\ &\neq 0\end{aligned}$$

Therefore as the general element does not tend to 0, the series does not converge uniformly in $(0, \infty)$.