

Differential and Integral Calculus : Recitations

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2014-15

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1 Instructor Information

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Part I

Sequences and Series

1 Sequences

Recitation 1 – Exercise 1.

Prove:

$$\lim_{n \rightarrow \infty} \frac{2n^2 + n + 1}{n^2 + 3} = 2$$

Recitation 1 – Solution 1.

Let

$$\varepsilon > 0$$

$$\begin{aligned} \left| \frac{2n^2 + n + 1}{n^2 + 3} - 2 \right| &= \left| \frac{2n^2 + n + 1 - 2n^2 - 6}{n^2 + 3} \right| \\ &= \left| \frac{n - 5}{n^2 + 3} \right| \\ &\leq \left| \frac{n - 5}{n^2} \right| \\ &\leq \frac{1}{n} \\ &< \varepsilon \end{aligned}$$

Therefore, let $N = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$. Hence, for this N , $|a_n - L| < \varepsilon$.

Therefore, $\lim_{n \rightarrow \infty} \frac{2n^2 + n + 1}{n^2 + 3} = 2$. □

Recitation 1 – Exercise 2.

Prove

$$\lim_{n \rightarrow \infty} \frac{n^3 + \sin n + n}{2n^4} = 0$$

Recitation 1 – Solution 2.

Let $\varepsilon > 0$

$$\begin{aligned} \left| \frac{n^3 + \sin n + n}{2n^4} \right| &\leq \left| \frac{n^3 + 1 + n}{2n^4} \right| \\ &\leq \left| \frac{3n^3}{2n^4} \right| = \frac{3}{2} \cdot \frac{1}{n} < \varepsilon \end{aligned}$$

Therefore, let $N = \left\lceil \frac{3}{2\varepsilon} \right\rceil + 1$. Hence, for this N , $|a_n - L| < \varepsilon$.

Therefore, $\lim_{n \rightarrow \infty} \frac{n^3 + \sin n + n}{2n^4} = 0$

□

Recitation 1 – Exercise 3.

Calculate $\sqrt[3]{n^3 + 3n} - n$.

Recitation 1 – Solution 3.

$$a^n - b^n = (a - b) \cdot (a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Therefore, let

$$\begin{aligned} a &= \sqrt[3]{n^3 + 3n} \\ b &= \sqrt[3]{n^3} \end{aligned}$$

$$\begin{aligned} a - b &= \frac{a^3 - b^3}{a^2 + ab + b^2} \\ \therefore \sqrt[3]{n^3 + 3n} - n &= \frac{n^3 + 3n - n^3}{(n^3 + 3n)^{2/3} + (n^3 + 3n)^{1/3}n + n^2} \\ &= \frac{3}{\left(\frac{n^3 + 3n}{n^{3/2}}\right)^{2/3} + \left(\frac{n^3 + 3n}{n^3}\right)^{1/3} + n} \end{aligned}$$

Therefore, the limit is 0.

Recitation 1 – Exercise 4.

Prove

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$$

Recitation 1 – Solution 4.

$$0 \leq \frac{n!}{n^n} = \frac{1}{n} \frac{2}{n} \dots \frac{n}{n} \leq \frac{1}{n}$$

Therefore, by the Sandwich Theorem, $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$.

Recitation 1 – Exercise 5.

Let $a_1 = 3$, $a_{n+1} = 1 + \sqrt{6 + a_n}$. Prove that a_n converges and find its limit.

Recitation 1 – Solution 5.

If possible, let $\lim_{n \rightarrow \infty} a_n = l$.

$$a_{n+1} = 1 + \sqrt{6 + a_n}$$

Taking the limit on both sides,

$$\begin{aligned} l &= 1 + \sqrt{6 + l} \\ \therefore l - 1 &= \sqrt{6 + l} \\ \therefore l &= \frac{3 \pm \sqrt{29}}{2} \end{aligned}$$

$$\text{As } a_n \geq 0, l = \frac{3 + \sqrt{29}}{2}.$$

$$\begin{aligned} a_2 &= 1 + \sqrt{6 + a_1} \\ &= 1 + \sqrt{6 + 3} \\ &= 4 \\ \therefore a_2 &> a_1 \end{aligned}$$

If possible, let $a_n \geq a_{n-1}$.

Therefore,

$$\begin{aligned} a_{n+1} &= 1 + \sqrt{6 + a_n} \\ &\geq 1 + \sqrt{6 + a_{n+1}} = a_n \end{aligned}$$

Therefore by induction, $\{a_n\}$ is monotonically increasing.

$$\begin{aligned} a_1 &= 3 \\ \therefore a_1 &\leq 5 \end{aligned}$$

If possible, let $a_n \leq 5$.

Therefore,

$$a_{n+1} = 1 + \sqrt{6 + a_n} \leq q + \sqrt{11} \leq 5$$

Therefore by induction, $\{a_n\}$ is bounded from above by 5.

1.1 Limit of a Function by Heine

Definition 1.

$$\lim_{x \rightarrow x_0} f(x) = l$$

if for every sequence x_n , such that $\lim_{n \rightarrow \infty} x_n = x_0$,

$$\lim_{n \rightarrow \infty} f(x_n) = l$$

Theorem 1. *If f is continuous at x_0 and $x_n \rightarrow x_0$, then*

$$\lim_{n \rightarrow \infty} f(x_n) = f\left(\lim_{n \rightarrow \infty} x_n\right) = f_{x_0}$$

Recitation 2 – Exercise 1.

Calculate $\lim_{n \rightarrow \infty} \sqrt[n]{n}$.

Recitation 2 – Solution 1.

Let

$$f(x) = x^{1/x}$$

Therefore,

$$\begin{aligned} \lim_{x \rightarrow \infty} x^{1/x} &= \lim_{x \rightarrow \infty} e^{\frac{\ln x}{x}} \\ &= 1 \end{aligned}$$

1.2 Sub-sequences

Recitation 2 – Exercise 2.

Find all partial limits and $\overline{\lim}$ and $\underline{\lim}$ of

$$a_n = \left(\cos \frac{\pi n}{4} \right)^n$$

Recitation 2 – Solution 2.

Let $k, z \in \mathbb{Z}$

$$\begin{aligned} \cos \frac{\pi n}{4} &= \cos \frac{\pi(n+k)}{4} \\ \therefore \frac{\pi n}{4} &= \frac{\pi(n+k)}{4} + 2\pi z \\ \therefore \pi n &= \pi(n+k) + 8\pi z \\ \therefore k &= 8z \end{aligned}$$

Therefore,

$$\begin{aligned} a_{8k} &= \left(\cos \frac{\pi \cdot 8k}{4} \right)^{8k} \\ &= (\cos(2\pi k))^{8k} \\ &= 1 \\ a_{8k+1} &= \left(\cos \frac{\pi \cdot (8k+1)}{4} \right)^{8k+1} \\ &= \left(\cos \frac{\pi}{4} \right)^{8k+1} \\ &= \left(\frac{\sqrt{2}}{2} \right)^{8k+1} \\ a_{8k+2} &= \left(\cos \frac{\pi \cdot (8k+2)}{4} \right)^{8k+2} \\ &= \left(\cos \frac{\pi}{2} \right)^{8k+2} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{k \rightarrow \infty} a_{8k} &= 1 \\ \lim_{k \rightarrow \infty} a_{8k+1} &= \lim_{k \rightarrow \infty} \left(\frac{\sqrt{2}}{2} \right)^{8k+1} \\ &= 0 \end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{k \rightarrow \infty} a_{8k+2} &= 0 \\
\lim_{k \rightarrow \infty} a_{8k+3} &= 0 \\
\lim_{k \rightarrow \infty} a_{8k+4} &= \lim_{k \rightarrow \infty} (-1)^{8k+4} \\
&= 1 \\
\lim_{k \rightarrow \infty} a_{8k+5} &= 0 \\
\lim_{k \rightarrow \infty} a_{8k+6} &= 0 \\
\lim_{k \rightarrow \infty} a_{8k+7} &= 0
\end{aligned}$$

Therefore, $\{a_n\}$ has two partial limits, 0 and 1.

$$\begin{aligned}
\overline{\lim} a_n &= 1 \\
\underline{\lim} a_n &= 0
\end{aligned}$$

2 Series

Definition 2 (Convergence of a series). Let $\{a_n\}$ be a sequence. Let S_n be a sequence of partial sums of a_n , s.t.

$$S_n = \sum_{k=1}^n a_k$$

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to l if

$$\lim_{n \rightarrow \infty} S_n = l$$

that is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = \lim_{n \rightarrow \infty} S_n$$

Recitation 2 – Exercise 3.

Does $\sum_{k=0}^{\infty} q^k$ where $-1 < q < 1$ converge?

Recitation 2 – Solution 3.

$$\begin{aligned}
\sum_{k=0}^{\infty} q^k &= \lim_{n \rightarrow \infty} \sum_{k=0}^n q^k \\
&= \lim_{n \rightarrow \infty} \frac{1 - q^{n+1}}{1 - q} \\
&= \frac{1}{1 - q}
\end{aligned}$$

Therefore, the series converges.

Recitation 2 – Exercise 4.

Does $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converge?

Recitation 2 – Solution 4.

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k(k+1)} &= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) \\
&= 1
\end{aligned}$$

Recitation 2 – Exercise 5.

Does $\sum_{k=1}^{\infty} \left(1 + \frac{1}{k} \right)^k$ converge?

Recitation 2 – Solution 5.

$$\begin{aligned}
\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^k &= e \\
\therefore \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^k &\neq 0
\end{aligned}$$

Therefore, the necessary condition is not satisfied. Hence, the series does not converge.

2.1 Comparison Tests for Positive Series

Theorem 2 (First Comparison Test). *If $a_n \geq 0$, $b_n \geq 0$, and $a_n \leq b_n$, then*

- 1. If $\sum b_n$ converges, then $\sum a_n$ converges.*
- 2. If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Theorem 3 (Second Comparison Test). *If $a_n \geq 0$, $b_n \geq 0$ and*

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l$$

where $0 < l < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge simultaneously.

Recitation 3 – Exercise 1.

Suppose the sequence a_n satisfies the condition

$$a_{n+1} - a_n > \frac{1}{n}$$

$\forall n \in \mathbb{N}$.

Prove that $\lim_{n \rightarrow \infty} a_n = \infty$.

Recitation 3 – Solution 1.

$$\begin{aligned} a_{n+1} &= a_{n+1} - a_n + a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_2 - a_1 + a_1 \\ &= \sum_{k=1}^n (a_{k+1} - a_k) + a_1 \\ &\geq \sum_{k=1}^n \frac{1}{k} + a_1 \end{aligned}$$

As the harmonic series diverges, $\sum_{k=1}^n \frac{1}{k} + a_1$ diverges.

Therefore, by the First Comparison Test, $\sum_{k=1}^{\infty} (a_{k+1} - a_k)$ diverges.

Recitation 3 – Exercise 2.

Check the convergence of $\sum_{n=2}^{\infty} \frac{n + \sin n}{n^3 + \cos \pi n}$.

Recitation 3 – Solution 2.

The series is non-negative. Therefore, the comparison tests are applicable.

$$\begin{aligned} \frac{n + \sin n}{n^3 + \cos \pi n} &\leq \frac{n + 1}{n^3 - 1} \\ \therefore \frac{n + \sin n}{n^3 + \cos \pi n} &\leq \frac{2n}{n^3 - \frac{n^3}{2}} \leq \frac{4}{n^2} \end{aligned}$$

Therefore, by the First Comparison Test, as $\frac{4}{n^2}$ converges, $\sum_{n=2}^{\infty} \frac{n + \sin n}{n^3 + \cos \pi n}$ also converges.

Recitation 3 – Exercise 3.

Let $a_n \geq 0$ and suppose that $\sum a_n$ converges. Prove that $\sum a_n^2$ converges. Is it true without the assumption $a_n \geq 0$?

Recitation 3 – Solution 3.

As $\sum a_n$ converges, $\lim_{n \rightarrow \infty} a_n = 0$.

Therefore, $\exists N \in \mathbb{N}$, such that $\forall n > N$, $a_n < 1$.

Therefore, $\forall n > N$, $a_n^2 \leq a_n$. Hence, as $\sum_{n=N+1}^{\infty} a_n$ converges, $\sum_{n=N+1}^{\infty} a_n^2$ also converges. Hence, $\sum_{n=1}^{\infty} a_n^2$ also converges.

This is not true without the assumption $a_n \geq 0$, as the argument $a_n^2 \leq a_n$ does not hold.

Recitation 3 – Exercise 4.

For which α does $\sum (\sqrt{n+1} - \sqrt{n})^{\alpha/2}$ converge?

Recitation 3 – Solution 4.

$$\begin{aligned} \sum (\sqrt{n+1} - \sqrt{n})^{\alpha/2} &= \sum \left(\frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \right)^{\alpha/2} \\ &= \sum \left(\frac{1}{\sqrt{n+1} + \sqrt{n}} \right)^{\alpha/2} \end{aligned}$$

The series is positive. Therefore, the comparison tests are applicable.

Comparing with $\left(\frac{1}{\sqrt{n}}\right)^{\alpha/2}$,

$$\frac{\left(\frac{1}{\sqrt{n+1} + \sqrt{n}}\right)^{\alpha/2}}{\left(\frac{1}{\sqrt{n}}\right)^{\alpha/2}} = \left(\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right)^{\alpha/2}$$

$$\therefore \lim_{n \rightarrow \infty} \left(\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right)^{\alpha/2} = \left(\frac{1}{2}\right)^{\alpha/2}$$

$\sum \frac{1}{n^{\alpha/2}}$ converges if and only if $\frac{\alpha}{4} > 1$, i.e. if and only if $\alpha > 4$.

By the Second Comparison Test, $\sum \frac{1}{n^{\alpha/4}}$ and the series converge or diverge simultaneously.

Therefore, the series converges for $\alpha > 4$.

Recitation 3 – Exercise 5.

Check the convergence of $\sum_{n=1}^{\infty} \sin \frac{1}{n}$.

Recitation 3 – Solution 5.

$\forall n \in \mathbb{N}, \sin \frac{1}{n} \geq 0$

$$\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$$

Therefore, by Second Comparison Test, $\sum \frac{1}{n}$ and $\sum \sin \frac{1}{n}$ diverge simultaneously.

2.2 d'Alembert Criteria (Ratio Test)

Definition 3 (Absolute and conditional convergence). The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

Theorem 4. *If the series $\sum a_n$ converges absolutely then it converges.*

Theorem 5 (d'Alembert Criteria (Ratio Test)). 1. If

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$$

(including $L = \infty$), then $\sum a_n$ converges diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 – Exercise 6.

Check the convergence of $\sum \frac{(-1)^n \cdot n^{1000}}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}$.

Recitation 3 – Solution 6.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n \cdot n^{1000}}{1 \cdot \dots \cdot (2n-1)} \right| = \sum_{n=1}^{\infty} \frac{n^{1000}}{1 \cdot \dots \cdot (2n-1)}$$

Therefore, by the d'Alembert Criteria (Ratio Test),

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+1)^{1000}}{1 \cdot \dots \cdot (2n+1)} \cdot \frac{1 \cdot \dots \cdot (2n-1)}{n^{1000}} \\ &= \left(\frac{n+1}{n} \right)^{1000} \cdot \frac{1}{2n+1} \\ \therefore \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{1000} \cdot \frac{1}{2n+1} &= 0 \\ \therefore \left(\frac{n+1}{n} \right)^{1000} \cdot \frac{1}{2n+1} &< 1 \end{aligned}$$

Therefore, by the d'Alembert Criteria (Ratio Test), the series converges absolutely, and hence converges.

2.3 Cauchy Criteria (Cauchy Root Test)

Theorem 6 (Cauchy Criteria (Cauchy Root Test)). 1. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

then $\sum a_n$ converges absolutely.

2. If

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

(including $L = \infty$), then $\sum a_n$ diverges.

3. If $L = 1$, the test does not apply.

Recitation 3 – Exercise 7.

Check the convergence of $\sum \left(1 - \frac{2}{n}\right)^{n^2}$.

Recitation 3 – Solution 7.

$$\begin{aligned} \sqrt[n]{\left(1 - \frac{2}{n}\right)^{n^2}} &= \left(1 - \frac{2}{n}\right)^n \\ \therefore \lim_{n \rightarrow \infty} \left(1 - \frac{2}{n}\right)^n &= e^{-2} \\ \therefore \lim_{n \rightarrow \infty} \left(1 - \frac{2}{n}\right)^n &< 1 \end{aligned}$$

Therefore, by the Cauchy Criteria (Cauchy Root Test), $\sum \left(1 - \frac{2}{n}\right)^{n^2}$ converges.

2.4 Leibniz's Criteria

Definition 4 (Alternating series). The series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

Theorem 7 (Leibniz's Criteria for Convergence). If an alternating series $\sum (-1)^{n-1} a_n$ with $a_n > 0$ satisfies

1. $a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.

$$2. \lim_{n \rightarrow \infty} a_n = 0$$

then the series $(-1)^{n-1}a_n$ converges.

Recitation 3 – Exercise 8.

Prove or disprove: There exists $\{a_n\}$, such that $\sum a_n$ converges and $\sum(1 + a_n)a_n$ diverges.

Recitation 3 – Solution 8.

$$\text{Let } a_n = \frac{(-1)^n}{\sqrt{n}}.$$

Therefore, by Leibniz's Criteria for Convergence, $\sum \frac{(-1)^n}{\sqrt{n}}$ converges.

$$\begin{aligned} \sum(1 + a_n)a_n &= \sum \left(1 + \frac{(-1)^n}{\sqrt{n}}\right) \frac{(-1)^n}{\sqrt{n}} \\ &= \sum \left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right) \end{aligned}$$

Therefore, as $\sum \frac{1}{n}$ diverges, and $\sum \frac{(-1)^n}{\sqrt{n}}$ converges, $\sum \left(\frac{1}{n} + \frac{(-1)^n}{\sqrt{n}}\right)$ diverges.

2.5 Integral Test

Theorem 8 (Integral Test). *If $f(x) : [1, \infty) \rightarrow [0, \infty)$ is monotonically decreasing. Then, $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x) dx$ converge or diverge simultaneously.*

Recitation 3 – Exercise 9.

Check the convergence of $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Recitation 3 – Solution 9.

Let

$$f(x) = \frac{1}{x \ln x}$$

$f(x)$ is monotonically decreasing. Therefore, the Integral Test is applicable. Therefore,

$$\begin{aligned}\int_2^{\infty} \frac{1}{x \ln x} dx &= \int_{\ln 2}^{\infty} \frac{1}{y} dy \\ &= \ln y|_{\ln 2}^{\infty} \\ &= \infty\end{aligned}$$

Therefore, by the integral test, $\sum \frac{1}{n \ln n}$ diverges.

Recitation 4 – Exercise 1.

Let $d_n \geq 0$ and suppose

$$\sum_{n=0}^{\infty} d_n = \infty$$

Prove that

$$\sum_{n=0}^{\infty} \frac{d_n}{1 + d_n} = \infty$$

Recitation 4 – Solution 1.

If possible, let d_n be a bounded sequence. Then there exists M , such that $d_n \leq M, \forall n \in \mathbb{N}$.

Therefore,

$$\frac{d_n}{1 + d_n} \geq \frac{d_n}{1 + M}$$

Therefore, by the Second Comparison Test, as $\sum d_n$ diverges, $\sum \frac{d_n}{1 + d_n}$ also diverges.

If d_n is not bounded, then there is a subsequence d_{n_k} which diverges. Therefore,

$$\begin{aligned}\frac{d_{n_k}}{1 + d_{n_k}} &= \frac{1}{\frac{1}{d_{n_k}} + 1} \\ \therefore \lim_{k \rightarrow \infty} \frac{d_{n_k}}{1 + d_{n_k}} &= 1\end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{d_n}{1 + d_n} \neq 0$$

Therefore, the necessary condition for convergence is not fulfilled. Therefore, the series converges.

Recitation 4 – Exercise 2.

Let

$$d_n = \begin{cases} 1 & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$

Does $\sum \frac{d_n}{1 + n \cdot d_n}$ diverge?

Recitation 4 – Solution 2.

$$d_n = \begin{cases} 1 & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$

$$\therefore \frac{d_n}{1 + n \cdot d_n} = \begin{cases} \frac{1}{1 + k^2} & ; \quad n = k^2, k \in \mathbb{N} \\ 0 & ; \quad n \neq k^2, k \in \mathbb{N} \end{cases}$$

As $\frac{1}{1 + k^2} \leq \frac{1}{k^2}$ and as $\frac{1}{k^2}$ converges, $\sum \frac{1}{1 + k^2}$ also converges.

Recitation 4 – Exercise 3.

Let a_n be a sequence such that $|a_{n+1} - a_n| \leq b_{n+1}$ for all $n \in \mathbb{N}$ where $\sum b_k$ converges. Prove that $\{a_n\}$ converges.

Recitation 4 – Solution 3.

Let $\varepsilon > 0$.

$$\begin{aligned} |a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} + \cdots - a_n| \\ &\leq \sum_{k=n+1}^m |a_k - a_{k-1}| \\ &\leq \sum_{k=n+1}^m b_k \end{aligned}$$

Therefore, as $\sum b_n$ converges, the series satisfies the Cauchy Criteria (Cauchy Root Test). Therefore, there exists N , such that $\forall m > n > N$, $\left| \sum_{k=n+1}^m b_k \right| < \varepsilon$. Therefore, for $m > n > N$,

$$|a_m - a_n| \leq \sum_{k=n+1}^m b_k < \varepsilon$$

3 Power Series

Definition 5 (Power series). A power series around x_0 is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

where $\{a_n\}$ is a sequence of real numbers.

Theorem 9 (Abel's Theorem). *For every power series $\sum a_n(x - x_0)^n$, there exists $R \in [0, \infty]$, such that for all x satisfying $|x - x_0| < R$, the series converges and for all x satisfying $|x - x_0| > R$ the series diverges.*

Theorem 10 (Cauchy's Formula for Radius of Convergence).

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$$

Theorem 11 (Hadamard's Formula for Radius of Convergence). *If $\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$ exists, then*

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = R$$

Recitation 4 – Exercise 4.

Find the domain of convergence of $\sum_{n=1}^{\infty} \frac{(2x - 4)^n}{n}$.

Recitation 4 – Solution 4.

$$\sum_{n=1}^{\infty} \frac{(2x-4)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{n}$$

Therefore, by Cauchy's Formula for Radius of Convergence,

$$\begin{aligned} R &= \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}} \\ &= \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{\frac{2^n}{n}}} \\ &= \frac{1}{\lim_{n \rightarrow \infty} \frac{2}{\sqrt[n]{n}}} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, the series converges for

$$|x-2| < \frac{1}{2}$$

and diverges for

$$|x-2| > \frac{1}{2}$$

If $x = \frac{5}{2}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{5}{2} - 2 \right)^n \\ = \sum_{n=1}^{\infty} \frac{1}{n} \end{aligned}$$

Therefore, the series diverges.

If $x = \frac{3}{2}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{3}{2} - 2 \right)^n \\ = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \end{aligned}$$

Therefore, by Leibniz's Criteria for Convergence, the series converges.

Therefore, the domain of convergence is $\left[\frac{3}{2}, \frac{5}{2}\right)$.

Recitation 4 – Exercise 5.

Find the radius of convergence of $\sum_{n=0}^{\infty} n!x^{n!}$.

Recitation 4 – Solution 5.

$$\frac{1}{\sqrt[n]{a_n}} = x + x + 2x^2 + 6x^6 + 24x^{24} + \dots$$

Therefore,

$$a_n = \begin{cases} n & ; \quad n = k^2 \\ 0 & ; \quad n \neq k^2 \end{cases}$$

Therefore,

$$\begin{aligned} R &= \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{a_n}} \\ &= \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k!]{k!}} \\ &= 1 \end{aligned}$$

3.1 Power Series Representation of a Function

Theorem 12. *The power series representation of a function $f(x)$ is equal to its Taylor series if and only if $\lim_{n \rightarrow \infty} R_n(x) = 0$, where $R_n(x)$ is the Lagrange remainder.*

3.2 Differentiation and Integrations of Power Series

Recitation 5 – Exercise 1.

Find the power series representation of $\tan^{-1} x$.

Recitation 5 – Solution 1.

$$\begin{aligned}\frac{d \tan^{-1} x}{dx} &= \frac{1}{1+x^2} \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n}\end{aligned}$$

Integrating term by term,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + c$$

As $\tan^{-1} 0 = 0$, $c = 0$. Therefore,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Recitation 5 – Exercise 2.

Find an explicit formula for $\sum_{n=1}^{\infty} x^n n^2$.

Recitation 5 – Solution 2.

$$\sum_{n=1}^{\infty} x^n n^2 = x \cdot \sum_{n=1}^{\infty} x^{n-1} n^2$$

Let

$$g(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

Integrating term by term,

$$\begin{aligned}\int g(x) dx &= \sum_{n=1}^{\infty} n^2 \frac{x^n}{n} \\ &= \sum_{n=1}^{\infty} n x^n \\ &= x \cdot \sum_{n=1}^{\infty} n x^{n-1}\end{aligned}$$

Let

$$h(x) = \sum_{n=1}^{\infty} nx^{n-1}$$

$$\therefore \int h(x) \, dx = \frac{x}{1-x}$$

Therefore, inside radius of convergence $R = 1$, differentiating $\int h(x) \, dx$,

$$h(x) = \frac{1-x+x}{(1-x)^2}$$

$$= \frac{1}{(1-x)^2}$$

$$\therefore \int g(x) \, dx = xh(x)$$

$$= \frac{x}{(1-x)^2}$$

$$\therefore g(x) = \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4}$$

$$\therefore \sum_{n=1}^{\infty} x^n n^2 = x \cdot \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4}$$

Recitation 5 – Exercise 3.

Find the sum $\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

Recitation 5 – Solution 3.

Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

be a power series with radius R .

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = f\left(\frac{1}{2}\right)$$

Therefore,

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1}$$

$$= \frac{1}{1-x}$$

$$\therefore f(x) = -\ln(1-x) + c$$

As $f(0) = 0$, $c = 0$. Therefore,

$$f(x) = -\ln(1 - x)$$

Therefore,

$$\begin{aligned} f\left(\frac{1}{2}\right) &= -\ln\left(\frac{1}{2}\right) \\ &= \ln 2 \end{aligned}$$

4 Sequences of Functions

Definition 6 (Point-wise convergence and domain of convergence). $\{f_n\}$ is said to converge point-wise in some domain $E \subset D$ if $\forall x \in E$, the sequence $\{f_n(x)\}$ converges. In this case, E is said to be a domain of convergence of $\{f_n\}$.

Recitation 5 – Exercise 4.

Let $f(x) : (0, \infty) \rightarrow \mathbb{R}$ be some function such that $\lim_{x \rightarrow \infty} f(x) = 0$. Let $f_n(x) = f(nx)$. What is the domain of convergence of f_n ? What is the limit function?

Recitation 5 – Solution 4.

Let x be a particular number in $(0, \infty)$.

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} f(nx)$$

Therefore, as $\lim_{x \rightarrow \infty} f(x) = 0$,

$$\lim_{n \rightarrow \infty} f_n(x) = 0$$

Therefore the domain of convergence is $(0, \infty)$ and the limit function is a constant 0.