# Differential and Integral Calculus

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# Contents

# 1 Lecturer Information

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# 2 Required Reading

Protter and Morrey: *A first Course in Real Analysis*, UTM Series, Springer-Verlag, 1991

# 3 Additional Reading

Thomas and Finney, *Calculus and Analytic Geometry*, 9th edition, Addison-Wesley, 1996

# Part I
# Sequences and Series

## 1 Sequences

**Definition 1** (Sequence)**.** A sequence of real numbers is a set of numbers which are written in some order. There are infinitely many terms in a sequence. It is denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

**Example 1.** $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is called the harmonic sequence.

$$a_n = \frac{1}{n}$$

**Example 2.** $1, -\frac{1}{2}, \frac{1}{3}, \ldots$ is called the alternating harmonic sequence.

$$a_n = (-1)^{n+1}\frac{1}{n}$$

**Example 3.** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$$a_n = \frac{n}{n+1}$$

**Example 4.** $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \ldots$

$$a_n = \frac{n+1}{3^n}$$

**Example 5.** The Fibonacci sequence is given by

$$f_n = \begin{cases} 1 & ; \quad n = 1, 2 \\ f_{n-1} + f_{n-2} & ; \quad n \geq 3 \end{cases}$$

**Example 6.** A geometric sequence is given by

$$a_n = a_1 q^{n-1}$$

where $q$ is called the common ratio.

**Example 7.** A geometric sequence is given by

$$a_n = a_1 + d(n-1)$$

where $d$ is called the common difference.

**Definition 2** (Equal sequences)**.** Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equal if $a_n = b_n$, $\forall n \in \mathbb{N}$.

**Definition 3** (Sequences bounded from above)**.** $\{a_n\}$ is said to be bounded from above if $\exists M \in \mathbb{R}$, s.t. $a_n \leq M$, $\forall n \in \mathbb{N}$. Each such $M$ is called an upper bound of $\{a_n\}$.

**Definition 4** (Sequences bounded from below)**.** $\{a_n\}$ is said to be bounded from below if $\exists m \in \mathbb{R}$, s.t. $a_n \geq M$, $\forall n \in \mathbb{N}$. Each such $M$ is called an lower bound of $\{a_n\}$.

**Definition 5.** $\{a_n\}$ is said to be bounded if it is bounded from below and bounded from above.

**Example 8.** The sequence $a_n = n^2 + 2$ is not bounded from above but is bounded from below, by all $m \leq 3$.

**Example 9.** $\left\{\dfrac{2n-1}{3n}\right\}$ is bounded.

$$m = 0 \leq \frac{2n-1}{3n} \leq \frac{2n}{3n} = \frac{2}{3} = M$$

**Definition 6** (Monotonic increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \leq a_{n+1}$, $\forall n \geq n_0$.

**Definition 7** (Monotonic decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n \geq a_{n+1}$, $\forall n \geq n_0$.

**Definition 8** (Strongly increasing sequence)**.** A sequence $\{a_n\}$ is called monotonic increasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n < a_{n+1}$, $\forall n \geq n_0$.

**Definition 9** (Strongly decreasing sequence)**.** A sequence $\{a_n\}$ is called monotonic decreasing if $\exists n_0 \in \mathbb{N}$, s.t. $a_n > a_{n+1}$, $\forall n \geq n_0$.

**Example 10.** The sequence $\left\{\dfrac{n^2}{2^n}\right\}$ is strongly decreasing. However, this is not evident by observing the first few terms. $\dfrac{1}{2}, 1, \dfrac{9}{8}, 1, \dfrac{25}{32}, \dots$

$$\begin{aligned}
a_n &> a_{n+1} \\
\iff \frac{n^2}{2^n} &> \frac{(n+1)^2}{2^{n+1}} \\
\iff 2n^2 &> (n+1)^2 \\
\iff \sqrt{2}n &> n+1 \\
\iff n(\sqrt{2}-1) &> 1 \\
\iff n &> \frac{1}{\sqrt{2}-1} \\
\iff n &> 3
\end{aligned}$$

**Exercise 1.**
Is $a_n = (-1)^n$ monotonic?

**Solution 1.**
The sequence $-1, 1, -1, 1, \dots$ is not monotonic.

## 1.1 Limit of a Sequence

**Definition 10.** Let $\{a_n\}$ be a given sequence. A number $L$ is said to be the limit of the sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $|a_n - L| < \varepsilon$, $\forall n \geq n_0$. That is, there are infinitely many terms inside the interval and a finite number of terms outside it.

**Example 11.** The sequence $\{\frac{1}{n}\}$ tends to 0, i.e. for any open interval $(-\varepsilon, \varepsilon)$, there are finite number of terms of the sequence outside the interval, and therefore there are infinitely many terms inside the interval.

**Exercise 2.**
Prove

$$\lim_{n\to\infty} \frac{n+2}{2n-1} = \frac{1}{2}$$

**Solution 2.**
$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$

**Exercise 3.**
Prove that 2 is not a limit of $\left\{ \dfrac{3n+1}{n} \right\}$.

**Solution 3.**
If possible, let

$$\lim_{n \to \infty} \frac{3n+1}{n} = 2$$

Then, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\left| \dfrac{3n+1}{n} - 2 \right| < \varepsilon$, $\forall n \geq n_0$. However,

$$\left| \frac{3n+1}{n} - 2 \right| = 1 + \frac{1}{n} > 1$$

This is a contradiction for $\varepsilon = \dfrac{1}{2}$. Therefore, 2 is not a limit.

**Theorem 1.** *If a sequence $\{a_n\}$ has a limit $L$ then the limit is unique.*

*Proof.* If possible let there exist two limits $L_1$ and $L_2$. Therefore, $\forall \varepsilon > 0$, there exist a finite number of terms in the interval $(L_1 - \varepsilon, L_1 + \varepsilon)$. Therefore, there exist a finite number of terms in the interval $(L_2 - \varepsilon, L_2 + \varepsilon)$. This contradicts the definition of a limit. Therefore, the limit is unique. □

**Theorem 2.** *If a sequence $\{a_n\}$ has limit $L$, then the sequence is bounded.*

**Theorem 3.** *Let*

$$\lim_{n \to \infty} a_n = a$$
$$\lim_{n \to \infty} b_n = b$$

*and let $c$ be a constant. Then,*

$$\begin{aligned} \lim c &= c \\ \lim(ca_n) &= c \lim a_n \\ \lim(a_n \pm b_n) &= \lim a_n \pm \lim b_n \\ \lim(a_n b_n) &= \lim a_n \lim b_n \\ \lim\left(\frac{a_n}{b_n}\right) &= \frac{\lim a_n}{\lim b_n} \quad (\text{ if } \lim b \neq 0) \end{aligned}$$

**Theorem 4.** *Let* $\{b_n\}$ *be bounded and let* $\lim a_n = 0$. *Then,*

$$\lim(a_n b_n) = 0$$

**Theorem 5** (Sandwich Theorem)**.** *Let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *be three sequences. If*

$$\lim a_n = \lim b_n = L$$

*and* $\exists n_0 \in \mathbb{N}$, *s.t.* $\forall n \geq n_0$, $a_n \leq b_n \leq c_n$. *Then,*

$$\lim b_n = L$$

**Exercise 4.**
Calculate $\lim\limits_{n\to\infty} \sqrt[n]{2^n + 3^n}$

**Solution 4.**

$$\sqrt[n]{3^n} \leq \sqrt[n]{2^n + 3^n} \leq \sqrt[n]{3^n + 3^n} = \sqrt[3]{2 \cdot 3^n}$$
$$\therefore 3 \leq \sqrt[n]{2^n + 3^n} \leq 3\sqrt[n]{2}$$

Therefore, by the Sandwich Theorem, $\lim\limits_{n\to\infty} \sqrt[n]{2^n + 3^n} = 3$.

**Theorem 6.** *Any monotonically increasing sequence which is bounded from above converges. Similarly, any monotonically decreasing sequence which is bounded from below converges.*

**Exercise 5.**

Prove that there exists a limit for $a_n = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}}_{n \text{ times}}$ and find it.

**Solution 5.**

$$a_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = a_2$$

If possible, let

$$a_{n-1} < a_n$$
$$\therefore \sqrt{2 + a_{n-1}} < \sqrt{2 + a_n}$$
$$\therefore a_n < a_{n+1}$$

Hence, by induction, $\{a_n\}$ is monotonically increasing.

$$a_1 = \sqrt{2} \leq 2$$

If possible, let

$$a_n \leq 2 \therefore \sqrt{2 + a_n} \qquad\qquad \leq \sqrt{2+2}$$
$$\therefore a_{n+1} \leq 2$$

Hence, by induction, $\{a_n\}$ is bounded from above by 2. Therefore, by , $\{a_n\}$ converges.

**Definition 11** (Limit in a wide sense)**.** The sequence $\{a_n\}$ is said to converge to $+\infty$ if $\forall M \in \mathbb{R}$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n > M$.
The sequence $\{a_n\}$ is said to converge to $-\infty$ if $\forall M \in \mathbb{R}$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$, $a_n < M$.

## 1.2 Sub-sequences

**Definition 12** (Sub-sequence)**.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_k\}_{k=1}^{\infty}$ be a strongly increasing sequence of natural numbers. Let $\{b_k\}_{k=1}^{\infty}$ be a sequence such that $b_k = a_{n_k}$. Then $\{b_k\}_{k=1}^{\infty}$ is called a sub-sequence of $\{a_n\}_{n=1}^{\infty}$.

**Example 12.**

$$a_n = \frac{1}{n}$$

If we choose $n_k = k^2$,

$$b_k = a_{n_k} = a_{k^2} = \frac{1}{k^2}$$

Therefore,

$$\{b_k\} = 1, \frac{1}{4}, \frac{1}{9}, \ldots$$

**Theorem 7.** *If the sequence $\{a_n\}$ converges to $L$ in a wide sense, i.e. $L$ can be infinite, then any sub-sequence of $\{a_n\}$ converges to the same limit $L$.*

**Definition 13** (Partial limit)**.** A real number $a$, which may be infinite, is called a partial limit of the sequence $\{a_n\}$ is there exists a sub-sequence of $\{a_n\}$ which converges to $a$.

**Example 13.** Let

$$a_n = (-1)^n$$

Therefore, $\nexists \lim_{n\to\infty} a_n$. Let

$$b_k = a_{n_k} = a_{2n-1}$$

Therefore,

$$\{b_k\} = -1, -1, -1, \dots$$
$$\therefore \lim_{k\to\infty} b_k = 1$$

Therefore, $-1$ is a partial limit of $\{a_n\}$.

**Theorem 8** (Bolzano-Weierstrass Theorem)**.** *For any bounded sequence there exists a subsequence which is convergent, s.t. there exists at least one partial limit.*

**Definition 14** (Upper partial limit)**.** The largest partial limit of a sequence is called the upper partial limit. It is denoted by $\overline{\lim} a_n$ or $\limsup a_n$.

**Definition 15** (Lower partial limit)**.** The smallest partial limit of a sequence is called the upper partial limit. It is denoted by $\underline{\lim} a_n$ or $\liminf a_n$.

**Theorem 9.** *If the sequence $\{a_n\}$ is bounded and*

$$\overline{\lim} a_n = \underline{\lim} a_n = a$$

*then*

$$\exists \lim a_n = a$$

## 1.3 Cauchy Characterisation of Convergence

**Definition 16.** A sequence $\{a_n\}$ is called a Cauchy sequence if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall m, n \geq n_0$, $|a_n - a_m| < \varepsilon$.

**Theorem 10** (Cauchy Characterisation of Convergence)**.** *A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.*

*Proof.* Let

$$\lim_{n\to\infty} a_n = L$$

Then $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$, $|a_n - L| < \dfrac{\varepsilon}{2}$. Therefore if $n \geq n_0$ and $m \geq n_0$, then

$$\begin{aligned}
|a_n - a_m| &= |a_n - L + L - a_m| \\
&\leq |a_n - L| + |L - a_m| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
\therefore |a_n - a_m| &= \varepsilon
\end{aligned}$$

Similarly, the converse can be proved by Theorem 9. □

**Theorem 11** (Another Formulation of the Cauchy Characterisation Theorem)**.** *The sequence* $\{a_n\}$ *converges if and only if* $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, *such that* $\forall n \geq n_0$ *and* $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$*.*

**Exercise 6.**
Prove that the sequence

$$a_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

is convergent.

**Solution 6.**

$$\begin{aligned}
|a_{n+p} - a_n| &= \left| \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(n+p)^2} - \left( \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right) \right| \\
&= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(n+p-1)(n+p)} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} + \cdots + \cancel{\frac{1}{n+p-1}} - \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n} - \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &< \frac{1}{n}
\end{aligned}$$

Therefore, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$, where $n_0 > \frac{1}{\varepsilon}$. $\square$

**Exercise 7.**
Prove that the sequence

$$a_n = \frac{1}{1} + \frac{1}{n} + \cdots + \frac{1}{n}$$

diverges.

**Solution 7.**
If possible, let the sequence converge. Then, by the Cauchy Characterisation of Convergence, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, s.t. $\forall n \geq n_0$ and $\forall p \in \mathbb{N}$, $|a_{n+p} - a_n| < \varepsilon$. Therefore,

$$\begin{aligned}
|a_{n+p} - a_n| &= \left|\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+p} - \left(\frac{1}{n} + \cdots + \frac{1}{n}\right)\right| \\
&= \frac{1}{n+1} + \cdots + \frac{1}{n+p} \\
&\geq p \cdot \frac{1}{n+p} \\
\therefore |a_{n+p} - a_n| &> \frac{p}{n+p}
\end{aligned}$$

If $n = p$,

$$\frac{p}{n+p} = \frac{1}{2}$$

This contradicts the result obtained from the Cauchy Characterisation of Convergence, for $\varepsilon = \frac{1}{4}$.
Therefore, the sequence diverges.

# 2 Series

**Definition 17** (Series)**.** Given a sequence $\{a_n\}$, the sum $a_1 + \cdots + a_n + \ldots$ is called an infinite series or series. It is denoted as $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

**Definition 18** (Partial sum)**.** The partial sum of the series $\sum a_n$ is defined as

$$S_i = a_1 + \cdots + a_i$$

**Definition 19** (Convergent and divergent series)**.** If the sequence $\{S_n\}_{n=1}^{\infty}$ converges, then the series is called convergent. Otherwise, the series is called divergent.

**Definition 20** (Sum of a series)**.** If the sequence $\{S_n\}_{n=1}^{\infty}$ converges to $S \neq \pm\infty$, the number $S$ is called the sum of the series.

$$\sum_{n=1}^{\infty} a_n = S$$

**Example 14.**

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

Therefore,

$$S_1 = \frac{1}{2} \tag{1}$$
$$S_2 = \frac{1}{2} + \frac{1}{2^2} \tag{2}$$
$$\vdots S_n \qquad = \frac{1}{2} + \cdots + \frac{1}{2^n} \tag{3}$$
$$= \frac{a_1\left(1 - q^n\right)}{1 - q} \tag{4}$$
$$= \frac{1/2\left(1 - 1/2^n\right)}{1 - 1/2} \tag{5}$$
$$= 1 - \frac{1}{2^n} \tag{6}$$
$$\lim_{n\to\infty} S_n = 1 \tag{7}$$

Therefore, the series converges.

$$S = \sum_{n=1}^{\infty} = 1$$

**Theorem 12.** *A geometric series* $\sum_{n=1}^{\infty} a_1 q^{n-1}$, $a_1 \neq 0$ *converges if* $|q| < 1$ *and then,*

$$S = \sum_{n=1}^{\infty} a_1 q^{n-1} = \frac{a_1}{1 - q}$$

**Definition 21** ($p$-series)**.** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the $p$-series.

**Theorem 13.** *The p-series converges for $p > 1$ and diverges for $p \leq 1$.*

**Theorem 14.** *If $\sum a_n$ converges, then*

$$\lim_{n\to\infty} a_n = 0$$

*Proof.*

$$\begin{aligned} a_n &= S_n - S_{n-1} \\ \therefore \lim_{n\to\infty} a_n &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \\ &= S - S \\ &= 0 \end{aligned}$$

$\square$

**Theorem 15.** *If $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n \pm b_n)$ and $\sum ca_n$, where $c$ is a constant, also converge. Also,*

$$\begin{aligned} \sum(a_n \pm b_n) &= \sum a_n \pm \sum b_n \\ \sum(ca_n) &= c \sum a_n \end{aligned}$$

## 2.1 Convergence Criteria

### 2.1.1 Leibniz's Criteria

**Definition 22** (Alternating series)**.** The series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n$, where all $a_n > 0$ or all $a_n < 0$ is called an alternating series.

**Theorem 16** (Leibniz's Criteria for Convergence)**.** *If an alternating series $\sum(-1)^{n-1}a_n$ with $a_n > 0$ satisfies*

1. *$a_{n+1} \leq a_n$, i.e. $\{a_n\}$ is monotonically decreasing.*
2. *$\lim\limits_{n\to\infty} a_n = 0$*

*then the series $(-1)^{n-1}a_n$ converges.*

*Proof.* Consider the even partial sums of the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n$.

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$$

As $\{a_n\}$ is monotonically increasing, all brackets are non-negative. Therefore,

$$S_{2m+2} \geq S_{2m}$$

Therefore, $\{S_{2m}\}$ is increasing.
Also,

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$$

All brackets and $a_{2m}$ are non-negative. Therefore,

$$S_{2m} \leq a_1$$

Therefore, $\{S_{2m}\}$ is bounded from above by $a_1$. Hence,

$$\exists \lim_{m\to\infty} S_{2m} = S$$

For $S_{2m+1}$,

$$\begin{aligned} S_{2m+1} &= S_{2m} + a_{2m+1} \\ \therefore \lim_{m\to\infty} S_{2m+1} &= \lim_{m\to\infty} S_{2m} + \lim_{m\to\infty} a_{2m+1} \\ &= S + 0 \\ &= S \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} S_n = S$$

$\square$

**Example 15.** The alternating harmonic series $\sum \frac{(-1)^{n-1}}{n}$ converges as $a_n = \frac{1}{n} > 0$, $a_n$ decreases and $\lim a_n = 0$.

### 2.1.2 Comparison Test

**Theorem 17** (Comparison Test for Convergence)**.** *Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$.*

1. *If $a_n \leq b_n$, $\forall n \geq n_0$ and $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.*
2. *If $a_n \geq b_n$, $\forall n \geq n_0$ and $\sum\limits_{n=1}^{\infty} b_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.*

**Theorem 18** (Another Formulation of the Comparison Test for Convergence)**.** *Assume $\exists n_0 \in \mathbb{N}$, such that $a_n \geq 0$, $b_n \geq 0$, $\forall n \geq n_0$ and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = a > 0$$

*where $a$ is a finite number. Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} b_n$ converges.*

### 2.1.3 d'Alembert Criteria (Ratio Test)

**Definition 23** (Absolute and conditional convergence)**.** The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. The series $\sum a_n$ is said to converge conditionally if it converges but $\sum |a_n|$ diverges.

**Example 16.** The series $\sum \dfrac{(-1)^{n-1}}{n^2}$ converges absolutely, as $\sum \left|\dfrac{(-1)^{n-1}}{n^2}\right| = \sum \dfrac{1}{n^2}$ converges.

**Example 17.** The series $\sum \dfrac{(-1)^{n-1}}{n}$ converges conditionally, as it converges, but $\sum \left|\dfrac{(-1)^{n-1}}{n^2}\right| = \sum \dfrac{1}{n}$ diverges.

**Theorem 19.** *If the series $\sum a_n$ converges absolutely then it converges.*

**Theorem 20** (d'Alembert Criteria (Ratio Test))**.** *1. If*

$$\lim_{n\to\infty} \left|\frac{a_{n-1}}{a_n}\right| = L < 1$$

*then $\sum a_n$ converges absolutely.*

*2. If*

$$\lim_{n\to\infty} \left|\frac{a_{n-1}}{a_n}\right| = L > 1$$

*(including $L = \infty$), then $\sum a_n$ converges diverges.*

*3. If $L = 1$, the test does not apply.*

### 2.1.4 Cauchy Criteria (Cauchy Root Test)

**Theorem 21** (Cauchy Criteria (Cauchy Root Test))**.** *1. If*

$$\overline{\lim} \sqrt[n]{|a_n|} = L < 1$$

*then $\sum a_n$ converges absolutely.*

*2. If*

$$\overline{\lim} \sqrt[n]{|a_n|} = L > 1$$

*(including $L = \infty$), then $\sum a_n$ diverges.*

*3. If $L = 1$, the test does not apply.*