DIFFERENTIAL AND INTEGRAL CALCULUS
ASSIGNMENT 3

AAKASH JOG
ID : 989323563

Exercise 1.
Prove or disprove:

(1) There exist two sequence $\{a_n\}$, $\{b_n\}$ such that $b_n \to -\infty$ and $a_n + b_n \to \infty$.
(2) If a_n and b_n are divergent sequences, then $a_n b_n$ is divergent.
(3) If $\{a_n\}$ has a subsequence that tends to infinity and $\{b_n\}$ has a subsequence that tends to infinity, then $a_n + b_n$ is divergent.
(4) If a_n is a convergent sequence, then $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$.

Solution 1.

(1) Let

$$\begin{aligned} a_n &= 2n \\ b_n &= -n \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} b_n &= -\infty \\ \lim_{n\to\infty} a_n + b_n &= \lim_{n\to\infty} n \\ &= \infty \end{aligned}$$

$\square$

(2) Let

$$\begin{aligned} a_n &= (-1)^n \\ b_n &= (-1)^n \end{aligned}$$

Therefore

$$\begin{aligned} a_n b_n &= (-1)^{2n} \\ &= 1 \end{aligned}$$

Therefore $a_n b_n$ converges.
Therefore the statement is false.

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(3) Let $k \in \mathbb{N}$.
Let

$$a_n = \begin{cases} n & ; \quad n = 2k \\ -n & ; \quad n \neq 2k \end{cases}$$

$$b_n = \begin{cases} -n & ; \quad n = 2k \\ n & ; \quad n \neq 2k \end{cases}$$

Therefore,

$$\begin{aligned} a_n + b_n &= \begin{cases} n + (-n) & ; \quad n = 2k \\ -n + n & ; \quad n \neq 2k \end{cases} \\ &= 0 \end{aligned}$$

Therefore $a_n + b_n$ converges.
Therefore the statement is false.

(4) Let

$$\lim_{n\to\infty} a_n = l$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} - a_n &= \lim_{n\to\infty} a_{n+1} - \lim_{n\to\infty} a_n \\ &= l - l \\ &= 0 \end{aligned}$$

□

Exercise 2.
Let $\{a_n\}$ be a sequence. Prove that if the subsequences a_{2k} and a_{2k+1} converge to the same limit L then $\lim_{n\to\infty} a_n = L$.

Solution 2.

$$\begin{aligned} \{a_n\} &= a_1, a_2, \ldots, a_{2k}, a_{2k+1}, \ldots \\ \therefore, \{a_n\} &= \{a_k, a_{2k+1}\} \end{aligned}$$

Therefore, as $n \to \infty$, if the even terms of $\{a_n\}$ and the odd terms of $\{a_n\}$ all tend to L, then $\{a_n\}$ itself tends to L. □

Exercise 3.
Show that if $\{a_n\}$ is a sequence that is unbounded from above (i.e $\forall M > 0$ there exists $n \in \mathbb{N}$ such that $a_n > M$) then there exists a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = \infty$.

Solution 3.
If possible, $\nexists\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = \infty$.
Therefore, there must be a maximum term in $\{a_n\}$, say a_p.
Therefore, $\{a_n\}$ is bounded from above by any number greater than or equal to a_p.
However, this contradicts the assumption that $\{a_n\}$ is unbounded from above. Therefore, there must exist a subsequence $\{a_{n_k}\}$ which tends to infinity. $\square$

Exercise 4.
Let $\{a_n\}$ be a sequence. Show that if $a_{n+1}a_n \leq 0$, $\forall n \in \mathbb{N}$ and the limit $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} a_n = 0$.

Solution 4.

$$a_{n+1}a_n \leq 0$$

Therefore, either a_{n+1} and a_n must have opposite parity, or at least one must be zero.

If $\forall n \in \mathbb{N}$, a_{n+1} and a_n have opposite parity, then $\{a_n\}$ diverges. This contradicts the existence of $\lim_{n\to\infty} a_n$. Therefore at least one of them must be zero.
Therefore $\lim_{n\to\infty} a_n = 0$.

Exercise 5.
Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ that satisfies $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$, but the limit $\lim_{n\to\infty} a_n$ does not exist (in the strict sense).

Solution 5.
Let

$$\begin{aligned}
a_n &= \ln n \\
\therefore \lim_{n\to\infty} a_{n+1} - a_n &= \lim_{n\to\infty} \ln(n+1) - \ln n \\
&= \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) \\
&= \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \ln n \\
&= \infty
\end{aligned}$$

Therefore, $\lim_{n\to\infty} a_{n+1} - a_n = 0$, but $\lim_{n\to\infty} a_n$ does not exist.

Exercise 6.
Find the following limits:
(1) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ for $x\in\mathbb{R}$.
(2) $\lim_{n\to\infty} n\tan^{-1}\left(\frac{1}{n}\right)$

Solution 6.
(1)

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n}{x}\cdot x}\\ &= \lim_{n\to\infty}\left(\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right)^x\\ &= e^x\end{aligned}$$

(2)

$$\begin{aligned}\tan^{-1}x &= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots\\ \therefore \tan^{-1}\left(\frac{1}{n}\right) &= \frac{1}{n}-\frac{1}{3n^3}+\frac{1}{5n^5}-\frac{1}{7n^7}+\dots\\ \therefore n\tan^{-1}\left(\frac{1}{n}\right) &= n\left(\frac{1}{n}-\frac{1}{3n^3}+\frac{1}{5n^5}-\frac{1}{7n^7}+\dots\right)\\ &= 1-\frac{1}{3n^2}+\frac{1}{5n^4}-\frac{1}{7n^6}+\dots\\ &= \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)n^{2i}}\end{aligned}$$