DIFFERENTIAL AND INTEGRAL CALCULUS
ASSIGNMENT 4

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Exercise 1.

Let $\sum\limits_{n=0}^{\infty}$ be a non-negative series (i.e. $a_n \leq 0$). Prove that the series either converges to a finite limit or else, diverges to ∞.

Solution 1.

As the series is non-negative, the sum must always be non-negative. Therefore, as $n \to \infty$, the sum will go on increasing, and will always remain non-negative. Therefore, the series will converge in a wide sense. Therefore, it will either converge to a finite limit, or an infinite one, i.e. either it will converge to a finite limit or diverge to ∞.

Exercise 2.

Check whether the following series converge:

(a) $\sum\limits_{n=1}^{\infty} \frac{\sqrt{7n}}{n^2+3n+5}$

(b) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n(n+3)}}$

(c) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n^3+4n^2+8}}$

(d) $\sum\limits_{n=1}^{\infty} \frac{1}{n2^n}$

(e) $\sum\limits_{n=1}^{\infty} \frac{1}{2^n} \cdot \sin \frac{10\pi}{n^2}$

(f) $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

(g) $\sum\limits_{n=1}^{\infty} \frac{n^n}{2^n n!}$

(h) $\sum\limits_{n=1}^{\infty} \frac{n^n}{3^n n!}$

(i) $\sum\limits_{n=1}^{\infty} \frac{1}{(\ln n)^n}$

(j) $\sum\limits_{n=1}^{\infty} \left(\frac{2n+1}{3n-1}\right)^n$

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(k) $\sum_{n=1}^{\infty} \frac{n!}{a^n}$, $a > 0$

Solution 2.

(a)

$$a_n = \frac{\sqrt{7n}}{n^2 + 3n + 5}$$

Therefore, let

$$b_n = \frac{1}{n^{\frac{3}{2}}}$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{7n}}{n^2+3n+5}}{\frac{1}{n^{\frac{3}{2}}}}$$
$$= \sqrt{7}$$

As $\sum b_n$ is a p-series, and as $\frac{3}{2} > 1$, $\sum b_n$ converges.
Therefore, by the second comparison test, as $\sum b_n$ converges, $\sum a_n$ also converges.

(b)

$$a_n = \frac{1}{\sqrt{n(n+3)}}$$

Therefore, Let

$$b_n = \frac{1}{n}$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n(n+3)}}{\frac{1}{n}}$$
$$= 1$$

Therefore, by the second comparison test, as $\sum b_n$ diverges, $\sum b_n$ also diverges.

(c)

$$a_n = \frac{1}{\sqrt{n^3 + 4n^2 + 8}}$$

Therefore, let

$$b_n = \frac{1}{n^{\frac{3}{2}}}$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3+4n^2+8}}}{\frac{1}{n^{\frac{3}{2}}}}$$
$$= 1$$

Therefore, by the second comparison test, as $\sum b_n$ diverges, $\sum b_n$ also diverges.

(d)

$$a_n = \frac{1}{n2^n}$$

Therefore,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n2^n}{(n+1)2^{n+1}}\right|$$
$$= \frac{1}{2}$$
$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

Therefore, by the d'Alembert Criteria, $\sum a_n$ converges.

(e)

$$a_n = \frac{1}{2^n} \cdot \sin \frac{10\pi}{n^2}$$

Therefore,

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{2^n} \cdot \sin \frac{10\pi}{n^2}\right|} = \lim_{n\to\infty} \frac{1}{2} \sqrt[n]{\sin \frac{10\pi}{n^2}}$$
$$\therefore \lim_{n\to\infty} \sqrt[n]{\frac{1}{2^n} \cdot \sin \frac{10\pi}{n^2}} < 1$$

Therefore, by the Cauchy Root Test, $\sum a_n$ converges.

(f)

$$a_n = \frac{(n!)^2}{(2n)!}$$

Therefore,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\left((n+1)!\right)^2 (2n)!}{2(n+1)!(n!)^2}\right|$$
$$= \lim_{n\to\infty} \left|\frac{(n+1)^2}{(2n+1)(2n+2)}\right|$$
$$= \frac{1}{4}$$

Therefore, by the d'Alembert Criteria, $\sum a_n$ converges.

(g)

$$a_n = \frac{n^n}{2^n n!}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{2^n n!}} \\ &= \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{2^n \left(\frac{n}{e}\right)^n \sqrt{2\pi n}}} \\ &= \lim_{n\to\infty} \frac{e}{2} \frac{1}{\sqrt[2n]{2\pi n}} \\ &= \frac{e}{2}\end{aligned}$$

Therefore, by the Cauchy Root Test, $\sum a_n$ diverges.

(h)

$$a_n = \frac{n^n}{3^n n!}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{3^n n!}} \\ &= \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{3^n \left(\frac{n}{e}\right)^n \sqrt{2\pi n}}} \\ &= \lim_{n\to\infty} \frac{e}{3} \frac{1}{\sqrt[2n]{2\pi n}} \\ &= \frac{e}{3}\end{aligned}$$

Therefore, by the Cauchy Root Test, $\sum a_n$ diverges.

(i)

$$a_n = \frac{1}{(\ln n)^n}$$

Therefore, a_1 is infinite.

Therefore, as the series is non-negative, and as $\lim\limits_{n\to\infty} a_n = 0$, $\sum a_1$ cannot converge to any finite value.

Therefore, $\sum a_n$ diverges.

(j)

$$a_n = \left(\frac{2n+1}{3n-1}\right)^n$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+1}{3n-1}\right)^n} \\ &= \lim_{n\to\infty} \frac{2n+1}{3n-1} \\ &= \frac{2}{3}\end{aligned}$$

Therefore, by the Cauchy Root Test, $\sum a_n$ converges.

(k)

$$a_n = \frac{n!}{a^n}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[n]{\frac{n!}{a^n}} \\ &= \lim_{n\to\infty} \sqrt[2]{\frac{\left(\frac{n}{2}\right)^n \sqrt{2\pi n}}{a^n}} \\ &= \lim_{n\to\infty} \frac{n}{ae} \sqrt[n]{\sqrt{2\pi n}} \\ &= \infty\end{aligned}$$

Therefore, by the Cauchy Root Test, $\sum a_n$ diverges.

Exercise 3.
Check whether the following series converge, converge absolutely or diverge

(a) $\lim\limits_{n\to\infty} (-1)^{\frac{n^2+n}{2}} \cdot \frac{1}{n^{2-\frac{1}{n}}}$
(b) $\lim\limits_{n\to\infty} (-1)^n \cdot \left(\frac{n-1}{n}\right)^{n^2}$
(c) $\lim\limits_{n\to\infty} (-1)^n \cdot \left(\frac{n-1}{n}\right)^n$
(d) $\lim\limits_{n\to\infty} (-1)^n \cdot \left(\frac{2n+100}{3n+1}\right)^n$
(e) $\lim\limits_{n\to\infty} (-1)^n \cdot \frac{1}{n+2}$
(f) $\frac{\sin n\alpha}{n^4}$

Solution 3.
(a)

$$a_n = (-1)^{\frac{n^2+n}{2}} \cdot \frac{1}{n^{2-\frac{1}{n}}}$$

Therefore,

$$\begin{aligned}|a_n| &= \frac{1}{n^{2-\frac{1}{n}}}\\ \therefore \lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \frac{1}{n^{2-\frac{1}{n}}}\\ &= 0\end{aligned}$$

Therefore, by Leibnitz Rule, as a_n is monotonically increasing, and as $\lim_{n\to\infty} |a_n| = 0$, $\sum a_n$ converges absolutely.

(b)

$$a_n = (-1)^n \cdot \left(\frac{n-1}{n}\right)^{n^2}$$

Therefore,

$$\begin{aligned}|a_n| &= \left|\left(\frac{n-1}{n}\right)^{n^2}\right|\\ \therefore \lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\left(\frac{n-1}{n}\right)^{n^2}\right|\\ &= 0\end{aligned}$$

Therefore, by Leibnitz Rule, as a_n is monotonically increasing, and as $\lim_{n\to\infty} |a_n| = 0$, $\sum a_n$ converges absolutely.

(c)

$$a_n = (-1)^n \cdot \left(\frac{n-1}{n}\right)^{n}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\left(\frac{n-1}{n}\right)^{n}\right|\\ &= e\end{aligned}$$

Therefore, $\sum a_n$ diverges.

(d)

$$a_n = (-1)^n \cdot \left(\frac{2n+100}{3n+1}\right)^{n}$$

Therefore

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \frac{2n+100}{3n+1}\\ &= \frac{2}{3}\end{aligned}$$

Therefore, by the Cauchy Root Test, $\sum a_n$ converges absolutely.

(e)

$$a_n = (-1)^n \cdot \frac{1}{n+2}$$

Therefore,

$$|a_n| = \frac{1}{n+2}$$

$$\therefore \lim_{n\to\infty} |a_n| = 0$$

Therefore, as $\lim_{n\to\infty} |a_n| = 0$, and as a_n is monotonically decreasing, $\sum a_n$ converges absolutely.

(f)

$$a_n = \frac{\sin n\alpha}{n^4}$$

Therefore, let

$$b_n = \frac{1}{n^3}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\sin n\alpha}{n} \\ &= 1\end{aligned}$$

Therefore, by the second comparison test, as $\sum b_n$ converges, $\sum a_n$ also converges.

Exercise 4.

Prove or disprove the following claims

(a) There exists a non-negative sequence $\{a_n\}$ such that $\sum\limits_{n=1}^{\infty} a_n$ converges and $\sum\limits_{n=1}^{\infty} {a_n}^2$ diverges.

(b) There exists a sequence $\{a_n\}$ such that the series $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely and the series $\sum\limits_{n=1}^{\infty} {a_n}^2$ diverges.

(c) There exists a sequence $\{a_n\}$ such that $\sum\limits_{n=1}^{\infty} a_n$ converges and $\sum\limits_{n=1}^{\infty} {a_n}^2$ diverges.

(d) Let $\{a_n\}$, $\{b_n\}$ be two sequences such that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$. If $\sum\limits_{n=1}^{\infty} a_n$ converges then $\sum\limits_{n=1}^{\infty} b_n$ converges.

Solution 4.

(a) Let $\{a_n\}$ be a sequence bounded between 0 and 1.
Therefore, $\forall n$, ${a_n}^2 < a_n$.
Therefore, by the first comparison test, as $\sum a_n$ converges, $\sum {a_n}^n$ must also converge.
Therefore the statement is false.

(b) Let $\{a_n\}$ be a sequence bounded between -1 and 1.
Therefore, $\{|a_n|\}$ is bounded between -1 and 1.
Therefore, as proved above, $\sum {a_n}^2$ also must converge.
Therefore, $\sum {a_n}^2$ cannot diverge.
Therefore the statement is false.

(c) Let

$$a_n = (-1)^n \frac{1}{\sqrt{n}}$$

$$\therefore {a_n}^2 = \frac{1}{n}$$

Therefore, $\sum a_n$ converges, but $\sum {a_n}^2$ diverges.
Therefore, such a sequence exists. Hence, the statement is true. □

(d)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

Therefore, by the second comparison test, as $\sum a_n$ converges, $\sum b_n$ also converges. □

Exercise 5.

Let $\{a_n\}$ be a non-negative sequence such that $\sum\limits_{n=1}^{\infty} a_n$ converges. Prove that $\sum\limits_{n=1}^{\infty} a_n a_{n+1}$ also converges.

Solution 5.

As $\sum a_n$ converges, $\{a_n\}$ is monotonically decreasing. Therefore, $\exists a_n$, such that

$$a_{n+1} < 1$$

$$\therefore a_n a_{n+1} < a_n$$

$$\therefore \sum a_n a_{n+1} < \sum a_n$$

Therefore, by the first comparison test, as $\sum a_n$ converges, $\sum a_n a_{n+1}$ also converges.