Differential and Integral Calculus : Recitations

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1 Instructor Information

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Part I
Sequences and Series

1 Sequences

Exercise 1.
Prove:

$$\lim_{n\to\infty} \frac{2n^2+n+1}{n^2+3} = 2$$

Solution 1.
Let

$$\varepsilon > 0$$

$$\begin{aligned}
\left|\frac{2n^2+n+1}{n^2+3} - 2\right| &= \left|\frac{2n^2+n+1-2n^2-6}{n^2+3}\right| \\
&= \left|\frac{n-5}{n^2+3}\right| \\
&\leq \left|\frac{n-5}{n^2}\right| \\
&\leq \frac{1}{n} \\
&< \varepsilon
\end{aligned}$$

Therefore, let $N = \left[\frac{1}{\varepsilon}\right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.
Therefore, $\lim_{n\to\infty} \frac{2n^2+n+1}{n^2+3} = 2$. □

Exercise 2.
Prove

$$\lim_{n\to\infty} \frac{n^3+\sin n+n}{2n^4} = 0$$

Solution 2.
Let $\varepsilon > 0$

$$\begin{aligned}\left|\frac{n^3 + \sin n + n}{2n^4}\right| &\leq \left|\frac{n^3 + 1 + n}{2n^4}\right| \\ &\leq \left|\frac{3n^3}{2n^4}\right| = \frac{3}{2} \cdot \frac{1}{n} < \varepsilon\end{aligned}$$

Therefore, let $N = \left[\frac{3}{2\varepsilon}\right] + 1$. Hence, for this N, $|a_n - L| < \varepsilon$.

Therefore, $\lim_{n\to\infty} \frac{n^3 + \sin n + n}{2n^4} = 0$ □

Exercise 3.
Calculate $\sqrt[3]{n^3 + 3n} - n$.

Solution 3.

$$a^n - b^n = (a - b) \cdot (a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Therefore, let

$$\begin{aligned}a &= \sqrt[3]{n^3 + 3n} \\ b &= \sqrt[3]{n^3}\end{aligned}$$

$$\begin{aligned}a - b &= \frac{a^3 - b^3}{a^2 + ab + b^2} \\ \therefore \sqrt[3]{n^3 + 3n} - n &= \frac{n^3 + 3n - n^3}{(n^3 + 3n)^{2/3} + (n^3 + 3n)^{1/3}n + n^2} \\ &= \frac{3}{\left(\frac{n^3 + 3n}{n^{3/2}}\right)^{2/3} + \left(\frac{n^3 + 3n}{n^3}\right)^{1/3n} + n}\end{aligned}$$

Therefore, the limit is 0.

Exercise 4.
Prove

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Solution 4.

$$0 \leq \frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\dots\frac{n}{n} \leq \frac{1}{n}$$

Therefore, by the Sandwich Theorem, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Exercise 5.

Let $a_1 = 3$, $a_{n+1} = 1 + \sqrt{6 + a_n}$. Prove that a_n converges and find its limit.

Solution 5.

If possible, let $\lim_{n\to\infty} a_n = l$.

$$a_{n+1} = 1 + \sqrt{6 + a_n}$$

Taking the limit on both sides,

$$\begin{aligned} l &= 1 + \sqrt{6 + l} \\ \therefore l - 1 &= \sqrt{6 + l} \\ \therefore l &= \frac{3 \pm \sqrt{29}}{2} \end{aligned}$$

As $a_n \geq 0$, $l = \frac{3 + \sqrt{29}}{2}$.

$$\begin{aligned} a_2 &= 1 + \sqrt{6 + a_1} \\ &= 1 + \sqrt{6 + 3} \\ &= 4 \\ \therefore a_2 &> a_1 \end{aligned}$$

If possible, let $a_n \geq a_{n-1}$.
Therefore,

$$\begin{aligned} a_{n+1} &= 1 + \sqrt{6 + a_n} \\ &\geq 1 + \sqrt{6 + a_{n+1}} = a_n \end{aligned}$$

Therefore by induction, $\{a_n\}$ is monotonically increasing.

$$\begin{aligned} a_1 &= 3 \\ \therefore a_1 &\leq 5 \end{aligned}$$

If possible, let $a_n \leq 5$.
Therefore,

$$a_{n+1} = 1 + \sqrt{6 + a_n} \leq q + \sqrt{11} \leq 5$$

Therefore by induction, $\{a_n\}$ is bounded from above by 5.

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