1 The distribution

Here I am much concerned with the observing process ψ_t and CUSUM process $m_t(\lambda)$ defined as such:

$$d\psi_t = \{ \begin{matrix} dB_t & t \leq \theta \\ \mu dt + dB_t & \end{matrix} \}$$

$$\begin{aligned} m_t(\lambda) &= \lambda B_t - \frac{1}{2}\lambda^2 t - \inf_{0\leq s\leq t} B_s - \frac{1}{2}\lambda^2 s \\ &= \lambda[-(-B_t + \frac{1}{2}\lambda t) + \sup_{0\leq s\leq t}(-B_s + \frac{1}{2}\lambda s)] \\ &= \lambda[-(B_t + \frac{1}{2}\lambda t) + \sup_{0\leq s\leq t}(B_s + \frac{1}{2}\lambda s)] \end{aligned}$$

Under the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{1}{2}B_t - \frac{1}{8}t}$, $B_t + \frac{1}{2}\lambda t$ is a standard Brownian Motion under the new measure. Let $x = B_t + \frac{1}{2}\lambda t$ and $y = \sup_{0\leq s\leq t}(B_s + \frac{1}{2}\lambda s))$ and $f(x, y)$ as the joint distribution of (x, y). Since the joint distribution under $\mathbb{Q}$ is $\frac{2(2y-x)}{t\sqrt{2\pi t}} e^{-\frac{1}{2t}(2y-x)^2}$, therefore the joint distribution under $\mathbb{P}$ is:

$$\frac{2(2y-x)}{t\sqrt{2\pi t}} e^{\frac{1}{2}\lambda x - \frac{1}{8}\lambda^2 t - \frac{1}{2t}(2y-x)^2}$$

Therefore we could derive the distribution for $m_t(\lambda)$:

$$\begin{aligned} P(y - x \leq s) &= P(y \leq x + s) \\ &= \int_{-s}^{0} dx \int_{0}^{x+s} \frac{2(2y-x)}{t\sqrt{(2\pi t)}} e^{\frac{1}{2}\lambda x - \frac{1}{8}\lambda^2 t - \frac{1}{2t}(2y-x)^2} dy + \int_{0}^{\infty} dx \int_{x}^{x+s} \frac{2(2y-x)}{t\sqrt{(2\pi t)}} e^{\frac{1}{2}\lambda x - \frac{1}{8}\lambda^2 t - \frac{1}{2t}(2y-x)^2} dy \\ &= \Phi(\frac{s + \frac{\lambda}{2}t}{\sqrt{t}}) - e^{-\lambda s}\Phi(\frac{-s + \frac{\lambda}{2}t}{\sqrt{t}}) \end{aligned}$$

$$\begin{aligned} P(m_t(\lambda) \leq s) &= P(\lambda(y - x) \leq s) \\ &= P(y - x \leq \frac{s}{\lambda}) \\ &= \Phi(\frac{s + \frac{\lambda^2}{2}t}{\sqrt{\lambda^2 t}}) - e^{-s}\Phi(\frac{-s + \frac{\lambda^2 t}{2}}{\sqrt{\lambda^2 t}}) \end{aligned}$$

Therefore we could derive the pdf for $m_t(\lambda) := f(s)$ as:

$$\frac{2}{\sqrt{2\pi\lambda^2 t}} e^{-\frac{s + \frac{\lambda^2 t}{2}}{2\lambda^2 t}} + e^{-s}\Phi(\frac{-s + \frac{\lambda^2 t}{2}}{\sqrt{\lambda^2 t}})$$

Similarly, if the Brownian Motion is drifted from the very begging, that is for any $t > 0$, $d\psi_t = \mu d_t + dB_t$. The distribution for $m_t(\lambda)$ could also be derived likewise in the same way, that is:

$$\frac{2}{\sqrt{2\pi\lambda^2 t}}e^{-\frac{(s+(\frac{\lambda^2}{2}-\mu\lambda)t)^2}{2\lambda^2 t}} + (1-\frac{2\mu}{\lambda})e^{-s+\frac{2\mu}{\lambda}s}\Phi(\frac{-s+(\frac{\lambda^2}{2}-\mu\lambda)t}{\sqrt{\lambda^2 t}})$$

2 P-value and critical value

2.1 P-value

To calculate the p-value of the statistics $m_t(\lambda)$, we shall first calculate the survival function of $\overline{F}_{t,\lambda}(x)$ it:

$$\begin{aligned}\overline{F}_{t,\lambda}(s) &= \int_s^\infty \frac{2}{\sqrt{2\pi\lambda^2 t}}e^{-\frac{s+\frac{\lambda^2 t}{2}}{2\lambda^2 t}} + e^{-s}\Phi(\frac{-s+\frac{\lambda^2 t}{2}}{\sqrt{\lambda^2 t}})ds \\ &= \Phi(\frac{-s-\frac{\lambda^2 t}{2}}{\sqrt{\lambda^2 t}}) + e^{-s}\Phi(\frac{-s+\frac{\lambda^2 t}{2}}{\sqrt{\lambda^2 t}})\end{aligned}$$

Therefore p-value of the statistics could be calculated as such.

2.2 Critical Value

Since p-value could be easily calculated, I would like to find a way to calculate the critical value, that is, given a p-value p, find x such that $P(m_t(\lambda) \geq s) = p$. First let:

$$\tilde{f}(s) = \begin{cases} \frac{1}{2}f(s) & if \quad s \geq 0 \\ \frac{1}{2}f(s) & if \quad s < 0 \end{cases}$$

Therefore $\tilde{f}(s)$ is a symmetrical distribution but judging from the qq-plot, it is long-tailed and has a high kurtosis, and I would like to resort to change of measure to change the kurtosis of the statistic to 1 and approximate the tail probability by a normal distribution.

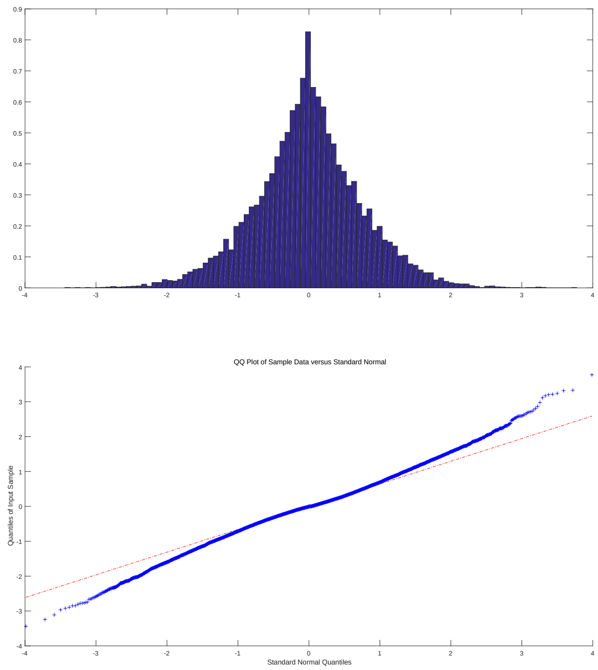

Figure 1: Histogram and QQ-plot for $\tilde{f}(s)$

Suppose the random variable Z has the distribution with the pdf $\tilde{f}(z)$, the Radon-Nikodym derivative of the proposed change is $\frac{d\mathbb{Q}_\theta}{d\mathbb{P}} = e^{\theta z^2 - \psi(\theta)}$, where $\psi(\theta) = log(\int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P})$, and we have several basic results for the mean, variance and kurtosis of the random variable:

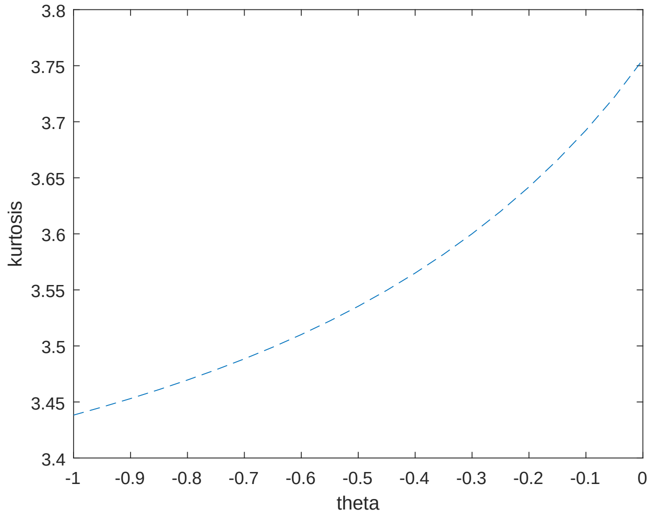

Figure 2: Kurtosis versus θ given $\lambda = t = 1$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_\theta}[Z] &= \int_{\mathbb{R}} z d\mathbb{Q}_\theta \\
&= \int_{\mathbb{R}} z \frac{d\mathbb{Q}_\theta}{d\mathbb{P}} d\mathbb{P} \\
&= \int_{\mathbb{R}} z e^{\theta z^2 - \psi(\theta)} \tilde{f}(z) dz \\
&= 0
\end{aligned}$$

$$\begin{aligned}
Var^{\mathbb{Q}_\theta}[Z] &= \int_{\mathbb{R}} z^2 d\mathbb{Q}_\theta \\
&= \int_{\mathbb{R}} z^2 e^{\theta z^2 - \psi(\theta)} d\mathbb{P}
\end{aligned}$$

$$\begin{aligned}
Kurt^{\mathbb{Q}_\theta}[Z] &= \frac{\mathbb{E}^{\mathbb{Q}_\theta}(Z - \mathbb{E}^{\mathbb{Q}_\theta}[Z])^4}{(Var^{\mathbb{Q}_\theta}[Z])^2} \\
&= \frac{\frac{\int_{\mathbb{R}} z^4 e^{\theta z^2} d\mathbb{P}}{\int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P}}}{(\frac{\int_{\mathbb{R}} z^2 e^{\theta z^2} d\mathbb{P}}{\int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P}})^2} \\
&= \frac{\int_{\mathbb{R}} z^4 e^{\theta z^2} d\mathbb{P} \int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P}}{(\int_{\mathbb{R}} z^2 e^{\theta z^2} d\mathbb{P})^2}
\end{aligned}$$

We also have some results for $\psi(\theta)$:

$$
\begin{aligned}
\dot{\psi}(\theta) &= \frac{\int_{\mathbb{R}} z^2 e^{\theta z^2} d\mathbb{P}}{\int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P}} \\
&= Var^{\mathbb{Q}_\theta}[Z]
\end{aligned}
$$

$$
\ddot{\psi}(\theta) = \frac{\int_{\mathbb{R}} z^4 e^{\theta z^2} d\mathbb{P} \int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P} - (\int_{\mathbb{R}} z^2 e^{\theta z^2} d\mathbb{P})^2}{(\int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P})^2}
$$

$$
\begin{aligned}
\psi^{(3)}(\theta) = &\frac{\int_{\mathbb{R}} z^6 e^{\theta z^2} d\mathbb{P} \int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P} - \int_{\mathbb{R}} z^2 e^{\theta z^2} d\mathbb{P} \int_{\mathbb{R}} z^4 e^{\theta z^2} d\mathbb{P}}{(\int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P})^2} \\
&+ 2\frac{\int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P} (\int_{\mathbb{R}} z^2 e^{\theta z^2} d\mathbb{P})^3 - (\int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P})^2 \int_{\mathbb{R}} z^2 e^{\theta z^2} d\mathbb{P} \int_{\mathbb{R}} z^4 e^{\theta z^2} d\mathbb{P}}{(\int_{\mathbb{R}} e^{\theta z^2} d\mathbb{P})^4}
\end{aligned}
$$

We also have some results for $\psi(\theta)$ at $\theta = 0$:

$$
\begin{aligned}
\psi(0) &= 0 \\
\dot{\psi}(0) &= \mathbb{E}^{\mathbb{P}} Z^2 \\
\ddot{\psi}(0) &= \mathbb{E}^{\mathbb{P}} Z^4 - (\mathbb{E}^{\mathbb{P}} Z^2)^2 \\
\psi^{(3)}(0) &= \mathbb{E}^{\mathbb{P}} Z^6 + 2(\mathbb{E}^{\mathbb{P}} Z^2)^3 - 3(\mathbb{E}^{\mathbb{P}} Z^2)(\mathbb{E}^{\mathbb{P}} Z^4)
\end{aligned}
$$

If we set such $\hat{\theta}$ that the kurtosis for the variable Z under the measure $\mathbb{Q}_{\hat{\theta}}$ would be 3 (the same as the normal distribution), then $\ddot{\psi}(\hat{\theta}) = 2\dot{\psi}(\hat{\theta})^2$, therefore we could approximate the distribution of Z under $\mathbb{Q}_{\hat{\theta}}$ by a normal distribution with mean 0 and variance $\dot{\psi}(\theta)$ Moreover, we could calculate $P(Z \in A)$:

$$
\begin{aligned}
P(Z \in A) &= \int_A d\mathbb{P} \\
&= \int_A \frac{d\mathbb{P}}{d\mathbb{Q}_{\hat{\theta}}} d\mathbb{Q}_{\hat{\theta}} \\
&= \int_A e^{-\theta z^2 + \psi(\theta)} d\mathbb{Q}_{\hat{\theta}} \\
&\sim \int_A e^{-\theta z^2 + \psi(\theta)} \frac{1}{\sqrt{2\pi \dot{\psi}(\theta)}} e^{-\frac{z^2}{2\dot{\psi}(\theta)}} dz
\end{aligned}
$$

Moreover, to solve $\hat{\theta}$ such that $\ddot{\psi}(\hat{\theta}) = 2\dot{\psi}(\hat{\theta})^2$, I approximate $\dot{\psi}(\theta)$ and $\ddot{\psi}(\theta)$ by its first order Taylor Expansion and solve a second-order equation, that is:

$$
\begin{aligned}
\dot{\psi}(\theta) &\approx \dot{\psi}(0) + \ddot{\psi}(0)\theta \\
\ddot{\psi}(\theta) &\approx \ddot{\psi}(0) + \psi^{(3)}(0)\theta
\end{aligned}
$$

From the equation $\ddot{\psi}(\hat{\theta}) = 2\dot{\psi}(\hat{\theta})^2$, we have:

$$
\begin{aligned}
\ddot{\psi}(0) + \psi^{(3)}(0)\theta &= 2(\dot{\psi}(0) + \ddot{\psi}(0)\theta)^2 \\
2\ddot{\psi}(0)\theta^2 + [4\ddot{\psi}(0) - \psi^{(3)}(0)]\theta + [2\dot{\psi}(0)^2 - \ddot{\psi}(0)] &= 0
\end{aligned}
$$

Since the original distribution is long-tailed, here I choose $\hat{\theta}$ to be the negative root of second order equation above, and estimate the critical value x such that $P(m_t(\lambda) > x) = \alpha$, given a certain level of α by:

$$\begin{aligned}
\alpha &= P(m_t(\lambda) > x) \\
&\approx \int_{[x,\infty)} e^{-\hat{\theta}z^2+\psi(\hat{\theta})} \frac{1}{\sqrt{2\pi\dot{\psi}(\hat{\theta})}} e^{-\frac{z^2}{2\dot{\psi}(\hat{\theta})}} dz \\
&\approx \frac{e^{\psi(\hat{\theta})}}{\sqrt{1+2\dot{\psi}(\hat{\theta})\hat{\theta}}} \Phi(\frac{-x}{\sqrt{\frac{\dot{\psi}(\hat{\theta})}{1+2\dot{\psi}(\hat{\theta})\hat{\theta}}}})
\end{aligned}$$