# Hamiltonian Monte Carlo on Ising Model

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# Ising Model Depiction

- Let $s = \{s_1, s_2 \cdots s_n\}$, $s_i \in \{0, 1\}^n$ be the binary random vector representing the spin configurations of the ferromagnet
- $s$ is distributed as followed:

$$p(s) \propto \frac{1}{Z} exp(a^\tau s + \frac{1}{2} s^\tau W s)$$

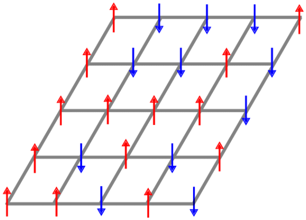

Figure 1: Ising Model on a Lattice

- $W$ is a real-symmetric matrix representing the interactions between different electrons

# Continuous Relaxation on Ising Model

- Conditioning on $s$, the continuous random vector $x \in \mathbb{R}^n$ is normally distributed as:

$$x|s \sim N(A(W+D)s, A^\tau(W+D)A)$$

- $D$ is a diagonal matrix ensuring that $W + D$ is positive-definite
- $A$ is an arbitrary non-singular matrix

# Continuous Relaxation on Ising Model

- Let $d = diag(D)$ and $\alpha_X = A^{-1}x$
- The joint distribution of $(x, s)$ is:
  $p(x,s) \propto exp(-\frac{1}{2}x^\tau (A^{-1})^\tau (W+D)^{-1} A^{-1}x + s^\tau A^{-1}x + (a - \frac{1}{2}d)^\tau s)$
- The marginal distribution for $x$ is:
  $$p(x) \propto exp(-\frac{1}{2}x^\tau (A^{-1})^\tau (W+D)^{-1} A^{-1}x) \prod_{i=1}^{n} (1 + exp(\alpha_{X;i} + a_i - \frac{d_i}{2}))$$
- The conditional distribution of $s$ given $x$ is Bernoulli whose parameter is given by:
  $$p(s_i|x) = \sigma(-\alpha_{X;i} - a_i + \frac{d_i}{2})^{1-s_i} \sigma(\alpha_{X;i} + a_i - \frac{d_i}{2})^{s_i}$$
  where $\sigma(z) = 1/(1 + exp(-z))$

# Continuous Relaxation on Ising Model

- $s_i$ are conditionally independent given $x$
- If $A$ is chosen as $(W+D)^{-1/2}$, $x_i$ are also conditionally independent given $s$
- If $A = I$, $s_i$ is conditionally dependent only on $x_i$

# Hamiltonian Monte Carlo for Discrete Random Vector

- Sampling from $s$ would be easy if we could sample the marginal distribution of $x$ since the conditional distribution of $s$ given $x$ are independent Bernoullis.
- $x$ has a continuous and differentiable probability density function, which guarantees that it could be sampled using Hamiltonian dynamics.
- Let the potential energy function $U(x)$ be:

$$\begin{aligned} U(x) &= -log(p(x)) \\ &= \frac{1}{2}x^{\tau}(A^{-1})^{\tau}(W+D)A^{-1}x - \sum_{i=1}^{n} log(1+exp(\alpha_{X;i}+a_i-\frac{d_i}{2})) \end{aligned}$$

# Hamiltonian Monte Carlo for Discrete Random Vector

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\end{aligned}
$$

- The corresponding kinetic energy function $K(p)$ is:

$$K(p) = \sum_{i=1}^{n} \frac{p_i^2}{2m_i}$$

- The Hamiltonian is therefore:

$$H(x,p) = U(x) + K(p)$$

# Hamiltonian Monte Carlo for Discrete Random Vector

- Let $l_X \in \mathbb{R}^n$ whose $i$th entry is:

$$l_{X;i} = \frac{exp(\alpha_{X;i}+a_i-\frac{d_i}{2})}{1+exp(\alpha_{X;i}+a_i-\frac{d_i}{2})}$$

- The gradient of the potential energy function would therefore be:

$$\frac{\partial U}{\partial x} = (A^{-1})^{\tau}(W+D)^{-1}A^{-1}x - A^{-1}l_X$$

- The Hamoltonian Dynamics for sampling $s$ is

**Algorithm 1:** Hamiltonian MCMC for Ising Model

Initialisation: Choose step-size $\epsilon$, iteration times $N$ and starting point $(x^0, p^0)$;

**for** $0 \leq n \leq N-1$ **do**

$\tilde{p}_i^{n+1} = p_i^n - \frac{\epsilon}{2}\frac{\partial U}{\partial x_i}(x^n)$;

$x_i^{n+1} = x_i^n + \epsilon\frac{\tilde{p}_i^{n+1}}{m_i}$;

$p_i^{n+1} = \tilde{p}_i^{n+1} - \frac{\epsilon}{2}\frac{\partial U}{\partial x_i}(x^{n+1})$;

Accept the new state $(x^{n+1}, p^{n+1})$ with probability:

$$\min[1, exp(-U(x^{n+1}) + U(x^n) - K(p^{n+1}) + K(p^n))]$$

Otherwise set $(x^{n+1}, p^{n+1})$ as $(x^n, p^n)$

Compute $\alpha_X^{n+1} = A^{-1}x^{n+1}$

Sample $p(s_i^{n+1}|x) = \sigma(-\alpha_{X;i}^{n+1} - a_i + \frac{d_i}{2})^{1-s_i^{n+1}}\sigma(\alpha_{X;i}^{n+1} + a_i - \frac{d_i}{2})^{s_i^{n+1}}$

**end**

# References I