1 Dual Formulation of Elastic Net with Intercept

First, to write the optimization problem of elastic net in a matrix form, and denote $D = [0, I]$, the optimization problem becomes:

$$\hat{\beta} = \arg\min_{\beta} \frac{1}{2}||y - X\beta||_2^2 + \lambda_1||D\beta||_1 + \lambda_2||D\beta||_2^2$$

The augmented Lagrangian for the problem is:

$$L = \frac{1}{2}||y - z||_2^2 + \lambda_1||\omega||_1 + \lambda_2||\omega||_2^2 + u^\tau(z - X\beta) + v^\tau(\omega - D\beta)$$

By taking the derivatives with respect to z and β, we could obtain:

$$0 = \frac{\partial L}{\partial z} = z - y + u$$
$$0 = \frac{\partial L}{\partial \beta} = -X^\tau u - D^\tau v$$

Since the first column of X is filled with ones and the first column of D is filled with zeros, the first row of $-X^\tau u - D^\tau v = 0$ gives $\mathbf{1}^\tau u = 0$ and due to $D = [0, I]$, the rest rows give that $-X_j^\tau u = v_j$. Moreover, since the rest dimensions of ω is penalized element-wisely in the augmented Lagrangian, we can minimize over ω by minimizing over each $\omega_i, \quad i \geq 2$, that is, we have to minimize $\lambda_1|\omega_i| + \lambda_2\omega_i^2 - u^\tau X_i\omega_i$ for each dimension of ω, where X_i denotes the ith column of X, therefore:

$$\min_{\omega_i} \lambda_1|\omega_i| + \lambda_2\omega_i^2 - u^\tau X_i\omega_i$$
$$= \begin{cases} 0 & if \quad |u^\tau X_i| \leq \lambda_1 \\ -\dfrac{(\lambda_1 - |u^\tau X_i|)^2}{4\lambda_2} & if \quad |u^\tau X_i| > \lambda_1 \end{cases}$$

By taking all the above back to the Lagrangian, we obtain the dual problem as:

$$d^* = \min_u \frac{1}{2}||y - u||_2^2 + \sum_{j:|X_j^\tau u| > \lambda_1} \frac{(\lambda_1 - |u^\tau X_i|)^2}{4\lambda_2}$$
$$subject \quad to \quad \mathbf{1}^\tau u = 0$$

First denote $f(u) = \sum_{j:|X_j^\tau u| > \lambda_1} \frac{(\lambda_1 - |u^\tau X_i|)^2}{4\lambda_2}$, clearly $f(u)$ is of quadratic form, $f(u) = \frac{1}{2}u^\tau Au + a^\tau u + b$, where b is a constant and does not matter in the optimization of the dual problem, A and a are:

$$A = \frac{1}{2\lambda_2} X_E X_E^\tau$$
$$E := \{i : |X_i^\tau u| > \lambda\}$$

$$a = \frac{\lambda_1}{2\lambda_2}\Big(\sum_{i:X_i^\tau < -\lambda_1} X_i - \sum_{i:X_i^\tau > \lambda_1} X_i\Big)$$

The dual problem could also be written in a proximal form:

$$\hat{u} = \mathbf{prox}_{\tilde{f}}(y)$$
$$\tilde{f} = \mathbf{I}(\mathbf{1}^\tau u = 0) f(u) + \mathbf{I}(\mathbf{1}^\tau u \neq 0)\infty$$

After transforming $f(u)$ into a quadratic form, we could write the Lagrangian for the dual problem back again:

$$L = \frac{1}{2}||y - u||_2^2 + \frac{1}{2}u^\tau A u + a^\tau u + b + \lambda \mathbf{1}^\tau u$$

By taking the derivative with respect to u, we could obtain:

$$\frac{\partial L}{\partial u} = u - y + Au + a + \lambda \mathbf{1} = 0$$

By shifting the terms, u could be written as a formula of y and λ: $u = (I + A)^{-1}(y - a - \lambda \mathbf{1})$, by taking the derivative with respect to y at both sides, we could obtain the Jacobian matrix J of the proximal operator $\mathbf{prox}(\tilde{f})$ at y as:

$$J = (I + A)^{-1} - (I + A)^{-1}\mathbf{1}\nabla(\hat{\lambda})^\tau$$

where $\nabla(\hat{\lambda})^\tau$ denotes the gradient of λ as a function of y.

By taking $u = (I+A)^{-1}(y-a-\lambda\mathbf{1})$ back to the Lagrangian, the dual problem will become a second-order equation of λ:

$$d^* = \max_\lambda \frac{1}{2}||y - (I + A)^{-1}(y - a - \lambda\mathbf{1})||_2^2 + \frac{1}{2}(y - a - \lambda\mathbf{1})^\tau (I + A)^{-1} A (I + A)^{-1}(y - a - \lambda\mathbf{1}) + a^\tau (I + A)^{-1}(y - a - \lambda\mathbf{1}) + \lambda\mathbf{1}^\tau (I + A)^{-1}(y - a - \lambda\mathbf{1})$$

More specifically, the second-order term is:

$$\frac{1}{2}\mathbf{1}^\tau (I + A)^{-2}\mathbf{1} + \frac{1}{2}\mathbf{1}^\tau (I + A)^{-1} A (I + A)^{-1}\mathbf{1} - \mathbf{1}^\tau (I + A)^{-1}\mathbf{1}$$

and the first-order term is:

$$2\mathbf{1}^\tau (I + A)^{-1}(y - a) - \mathbf{1}^\tau (I + A)^{-2}(y - a) - \mathbf{1}^\tau (I + A)^{-1} A (I + A)^{-1}(y - a)$$

Thus by solving the second-order equation, we could obtain

$$\hat{\lambda} = \frac{2\mathbf{1}^\tau (I+A)^{-1}(y-a) - \mathbf{1}^\tau (I+A)^{-2}(y-a) - \mathbf{1}^\tau (I+A)^{-1}A(I+A)^{-1}(y-a)}{\mathbf{1}^\tau (I+A)^{-2}\mathbf{1} + \mathbf{1}^\tau (I+A)^{-1}A(I+A)^{-1}\mathbf{1} - 2\mathbf{1}^\tau (I+A)^{-1}\mathbf{1}}$$

and the gradient

$$\nabla(\hat{\lambda}) = \frac{2(I+A)^{-1}\mathbf{1} - (I+A)^{-2}\mathbf{1} - (I+A)^{-1}A(I+A)^{-1}\mathbf{1}}{\mathbf{1}^\tau (I+A)^{-2}\mathbf{1} + \mathbf{1}^\tau (I+A)^{-1}A(I+A)^{-1}\mathbf{1} - 2\mathbf{1}^\tau (I+A)^{-1}\mathbf{1}}$$

By taking the gradient back to $J = (I + A)^{-1} - (I + A)^{-1}\mathbf{1}\nabla(\hat{\lambda})^\tau = (I + A)^{-1} - \frac{(I+A)^{-1}\mathbf{1}\mathbf{1}^\tau (I+A)^{-1}}{\mathbf{1}^\tau (I+A)^{-1}\mathbf{1}}$, we could obtain the Jacobian.

2 Proof of the Equivalence between the primal and the dual solutions

First recall the alo formula from the dual approach $y^{/i} = y_i - \frac{u_i}{J_{ii}} = \frac{J_{ii}-1}{J_{ii}} y_i + \frac{1}{J_{ii}} x_i \hat{\beta}$ and the primal formula from the primal approach $y^{/i} = x_i\hat{\beta} + \frac{H_{ii}}{1-H_{ii}}(x_i\hat{\beta} - y_i) = -\frac{H_{ii}}{1-H_{ii}} y_i + \frac{1}{1-H_{ii}} x_i\hat{\beta}$. In the following section, we're going to show that $H + J = I$, thus giving $H_{ii} + J_{ii} = 1$, and that the solutions given by both the primal and the dual approach are equivalent.

First, using matrix inverse lemma, we could calculate the inverse of $(I + A) = (I + \frac{1}{2\lambda_2} X_E X_E^\tau)$ as:

$$\begin{aligned}
(I + A)^{-1} &= (I + \frac{1}{2\lambda_2} X_E X_E^\tau)^{-1} \\
&= I - X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau
\end{aligned}$$

therefore the matrix J is:

$$\begin{aligned}
J &= (I + A)^{-1} - \frac{(I + A)^{-1}\mathbf{1}\mathbf{1}^\tau (I + A)^{-1}}{\mathbf{1}^\tau (I + A)^{-1}\mathbf{1}} \\
&= I - X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau - \frac{(\mathbf{1} - X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau \mathbf{1})(\mathbf{1}^\tau - \mathbf{1}^\tau X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau)}{\mathbf{1}^\tau (I - X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau)\mathbf{1}}
\end{aligned}$$

Now recall that $H = [1, X_E]([1, X_E]^\tau [1, X_E] + diag(0, 2\lambda_2, \cdots, 2\lambda_2))[1, X_E]^\tau$, by adopting block inverse, we could derive H as:

$$\begin{aligned}
H &= [1, X_E] \begin{pmatrix} n & \mathbf{1}^\tau X_E \\ X_E^\tau \mathbf{1} & X_E^\tau X_E + 2\lambda_2 I \end{pmatrix} [1, X_E]^\tau \\
&= [1, X_e] \\
&\begin{pmatrix} \frac{1}{\mathbf{1}^\tau (I - X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau)\mathbf{1}} & \frac{-\mathbf{1}^\tau X_E(2\lambda_2 I + X_E^\tau X_E)^{-1}}{\mathbf{1}^\tau (I - X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau)\mathbf{1}} \\ \frac{-(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau \mathbf{1}}{\mathbf{1}^\tau (I - X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau)\mathbf{1}} & (2\lambda_2 I + X_E^\tau X_E)^{-1} + \frac{(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau \mathbf{1}\mathbf{1}^\tau X_E (2\lambda_2 I + X_E^\tau X_E)^{-1}}{\mathbf{1}^\tau (I - X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau)\mathbf{1}} \end{pmatrix} \\
&[1, X_E]^\tau \\
&= X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau + \frac{(\mathbf{1} - X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau \mathbf{1})(\mathbf{1}^\tau - \mathbf{1}^\tau X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau)}{\mathbf{1}^\tau (I - X_E(2\lambda_2 I + X_E^\tau X_E)^{-1} X_E^\tau)\mathbf{1}} \\
&= I - J
\end{aligned}$$

Now that we have showed that $H + J = I$, we could also conclude that $J_{ii} + H_{ii} = 1$ and that the alo solutions given by both the primal and dual approaches are equivalent.