tesseroid_vs_spherical_shell

November 9, 2015

1 A comparison of the effects of tesseroids against a spherical shell

This notebook compares the gravitational fields calculated for a tesseroid model against the analytical solution for a spherical shell.

The tesseroid fields are calculated using a range of distance/size ratios for the recursive division of the tesseroids. The sphell fields are used as a reference, or "ground truth". We use the difference between the tesseroid and reference fields to quantify the error in the tesseroid calculations for each distance/size ratio.

1.1 Provenance information

Load libraries and print the version information.

```
In [1]: %matplotlib inline
        from __future__ import division
        import os
        from timeit import timeit
        import numpy as np
        import pandas as pd
        from matplotlib import pyplot as plt
        from matplotlib import rc_file
        rc_file('matplotlibrc')
        from fatiando.mesher import Tesseroid, TesseroidMesh
        from fatiando.constants import MEAN_EARTH_RADIUS, G, SI2EOTVOS, SI2MGAL
        from fatiando import gridder
        import fatiando
```

```
In [2]: print('Fatiando a Terra version: {}\n'.format(fatiando.__version__))
```

```
Fatiando a Terra version: 0.3
```

```
In [3]: !tessgz --version
```

```
1.2.0
```

1.2 Make a directory for temporary computation files

```
In [4]: filedir = 'tesseroid_vs_spherical_shell_files'
        !mkdir -p $filedir
```

1.3 Gravitational effect of a spherical shell

The analytical solution exists for an observation point located at a geocentric radial coordinate r. We can pretend that the point is at any latitude and longitude because of the symmetry of the shell.

The potential and its first and second derivatives are given by (Grombein et al., 2013):

$$V(r) = \frac{4}{3}\pi G\rho \frac{r_2^3 - r_1^3}{r},$$

$$g_z(r) = \frac{V(r)}{r},$$

$$g_{zz}(r) = 2\frac{V(r)}{r^2},$$

$$g_{xx}(r) = g_{yy}(r) = -\frac{g_{zz}}{2} = -\frac{V(r)}{r^2},,$$

and (because of the symmetry of the shell)

$$g_x(r) = g_y(r) = g_{xy}(r) = g_{xz}(r) = g_{yz}(r) = 0,$$

where ρ is the dentity, r is the radius coordinate of the observation point, r_1 and r_2 are radial coordinates of the bottom and top of the spherical shell, respectively.

The function below calculates the shell fields and returns them in a `pandas.Series`.

```
In [5]: def calc_shell_effect(height, top, bottom, density):
            r = height + MEAN_EARTH_RADIUS
            # top and bottom are heights
            r1 = bottom + MEAN_EARTH_RADIUS
            r2 = top + MEAN_EARTH_RADIUS
            potential = (4/3)*np.pi*G*density*(r2**3 - r1**3)/r
            data = pd.Series({'pot': potential,
                              'gx': 0,
                              'gy': 0,
                              'gz': SI2MGAL*(potential/r),
                              'gxx': SI2EOTVOS*(-potential/r**2),
                              'gxy': 0,
                              'gxz': 0,
                              'gyy': SI2EOTVOS*(-potential/r**2),
                              'gyz': 0,
                              'gzz': SI2EOTVOS*(2*potential/r**2)})
            return data
```

1.4 Tesseroid model of the spherical shell

I'll use the `TesseroidMesh` of Fatiando a Terra to make a model of a spherical shell. The model needs to be written to a file so that I can use the Tesseroids command-line programs on it.

The function below takes the size (in degrees), top, bottom and density, makes the tesseroids and saves them to a file.

```
In [6]: def make_model_file(size, top, bottom, density, verbose=False, return_mesh=False):
            bounds = [0, 360, -90, 90, top, bottom]
            nlon = int(360/size)
            assert nlon - 360/size == 0
            nlat = int(180/size)
            assert nlat - 180/size == 0
            mesh = TesseroidMesh(bounds, (1, nlat, nlon))
            # the 3rd dim is - because of a quirk in TesseroidMesh
            assert mesh.dims == (size, size, -(top - bottom))
            assert mesh.size == nlat*nlon
            fname = os.path.join(filedir, 'model-size{:.1f}.tess'.format(size))
```

```
    with open(fname, 'w') as f:
        for t in mesh:
            f.write('{} {} {} {} {} {} {}\n'.format(
                t.w, t.e, t.s, t.n, t.top, t.bottom, density))
    tmp = !cat $fname | grep -v "#" | grep -e "^$" -v| wc -l
    tess_in_file = int(tmp[0])
    assert mesh.size == tess_in_file
    if verbose:
        print('Model file: {}'.format(fname))
        print('Model size: {}'.format(tess_in_file))
        print('Head:')
        !head $fname
        print('Tail:')
        !tail $fname
    if return_mesh:
        return fname, mesh
    else:
        return fname
```

We'll make a 1 x 1 degree tesseroid model first. Later on, we'll run the computations for other sizes to see if the results are size dependent.

```
In [7]: size = 1
        top, bottom = 1000, 0
        density = 2670
```

```
In [8]: model_file = make_model_file(size, top, bottom, density, verbose=True)
```

```
Model file: tesseroid_vs_spherical_shell_files/model-size1.0.tess
Model size: 64800
Head:
0.0 1.0 -90.0 -89.0 1000.0 0.0 2670
1.0 2.0 -90.0 -89.0 1000.0 0.0 2670
2.0 3.0 -90.0 -89.0 1000.0 0.0 2670
3.0 4.0 -90.0 -89.0 1000.0 0.0 2670
4.0 5.0 -90.0 -89.0 1000.0 0.0 2670
5.0 6.0 -90.0 -89.0 1000.0 0.0 2670
6.0 7.0 -90.0 -89.0 1000.0 0.0 2670
7.0 8.0 -90.0 -89.0 1000.0 0.0 2670
8.0 9.0 -90.0 -89.0 1000.0 0.0 2670
9.0 10.0 -90.0 -89.0 1000.0 0.0 2670
Tail:
350.0 351.0 89.0 90.0 1000.0 0.0 2670
351.0 352.0 89.0 90.0 1000.0 0.0 2670
352.0 353.0 89.0 90.0 1000.0 0.0 2670
353.0 354.0 89.0 90.0 1000.0 0.0 2670
354.0 355.0 89.0 90.0 1000.0 0.0 2670
355.0 356.0 89.0 90.0 1000.0 0.0 2670
356.0 357.0 89.0 90.0 1000.0 0.0 2670
357.0 358.0 89.0 90.0 1000.0 0.0 2670
358.0 359.0 89.0 90.0 1000.0 0.0 2670
359.0 360.0 89.0 90.0 1000.0 0.0 2670
```

1.5 Tesseroid gravitational effect at 2 km height (1 km above the shell)

Now we need to calculate the gravitational effect of this tesseroid model using a range of distance/size ratios. The function below takes a model file name, height of observations, tesseroid size, and a distance to size ratio. It makes a computation grid at the given height. The grid is placed over the North pole from 90° latitude until 90° - the size of the tesseroids. This way, he grid covers all the tesseroids that are around the pole.

```
In [9]: def calc_tess_effect(fname, height, size, ratio, where='pole'):
            if ratio == 0:
                flag = '-a'
            else:
                flag = '-t{:f}'.format(ratio)
            # Make a computation grid above a tesseroid
            if where == 'pole':
                lats, lons = gridder.regular([90 - size, 90, 0, size], (10, 10))
            elif where == 'equator':
                lats, lons = gridder.regular([0, size, 0, size], (10, 10))
            else:
                raise ValueError("Invalid argument where='{}'".format(where))
            tmp = 'effect-{}-ratio{:.1f}-height{:.0f}-{}.txt'.format(os.path.split(fname)[1],
                                                                      ratio, height, where)
            outfile = os.path.join(filedir, tmp)
            grid_text = '\n'.join('{:f} {:f} {:f}'.format(lon, lat, height)
                                  for lon, lat in zip(lons, lats))
            !echo "$grid_text" | \
                tesspot $fname $flag | \
                tessgx $fname $flag | \
                tessgy $fname $flag | \
                tessgz $fname $flag | \
                tessgxx $fname $flag | \
                tessgxy $fname $flag | \
                tessgxz $fname $flag | \
                tessgyy $fname $flag | \
                tessgyz $fname $flag | \
                tessgzz $fname $flag > $outfile
            return outfile
```

Now we can specify a range of distance/size ratios and a computation height...

```
In [10]: ratio = np.arange(0, 10.5, 0.5)
         height = top + 1000
```

... and calculate the effect of the tesseroid model for each ratio.

```
In [11]: tess_out_files = []
         for r in ratio:
             f = calc_tess_effect(model_file, height, size, r)
             print('Done: {}'.format(f))
             tess_out_files.append(f)
```

```
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio0.0-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio0.5-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio1.0-height2000-pole.txt
```

```
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio1.5-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio2.0-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio2.5-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio3.0-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio3.5-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio4.0-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio4.5-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio5.0-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio5.5-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio6.0-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio6.5-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio7.0-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio7.5-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio8.0-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio8.5-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio9.0-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio9.5-height2000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio10.0-height2000-pole.txt
```

1.6 Comparison of spherical shell effect and tesseroid model

Now I can load the calculated fields of the tesseroid model and compare it with the calculated effect for the real spherical shell.

```
In [12]: shell_data = calc_shell_effect(height, top, bottom, density)
         shell_data
```

```
Out[12]: gx         0.000000
         gxx       -0.350758
         gxy        0.000000
         gxz        0.000000
         gy         0.000000
         gyy       -0.350758
         gyz        0.000000
         gz       223.788630
         gzz        0.701517
         pot    14278.021191
         dtype: float64
```

and save the results to a CSV (Comma Sparated Values) file.

```
In [13]: shell_data.to_csv('../data/shell-height-{:.0f}.csv'.format(height))
```

I'll load the tesseroid data into a `pandas.DataFrame` object. This is a like a spreadsheet, but better. The columns are the fields (gx, gy, etc), file name, ratio used, and tesseroid size.

```
In [14]: def load_tess_data(files, ratio, size):
             df = pd.DataFrame(columns=['file', 'ratio', 'size', 'gx', 'gxx', 'gxy', 'gxz', 'gy',
                                        'gyy', 'gyz', 'gz', 'gzz', 'pot'])
             for r, fname in zip(ratio, files):
                 data = np.loadtxt(fname, unpack=True)[3:]
                 tmp = pd.DataFrame({'file': fname,
                                     'ratio': r,
                                     'size': size,
                                     'pot': data[0],
                                     'gx': data[1],
```

```
                                    'gy': data[2],
                                    'gz': data[3],
                                    'gxx': data[4],
                                    'gxy': data[5],
                                    'gxz': data[6],
                                    'gyy': data[7],
                                    'gyz': data[8],
                                    'gzz': data[9]})
            df = df.append(tmp, ignore_index=True)
        df.index.name = 'point'
        return df
```

Lets take a look at the tesseroid data:

```
In [15]: tess_data = load_tess_data(tess_out_files, ratio, size)
         tess_data.head()
```

```
Out[15]:                                                     file          gx  \
         point
         0      tesseroid_vs_spherical_shell_files/effect-mode...   -3.582888
         1      tesseroid_vs_spherical_shell_files/effect-mode...  133.400119
         2      tesseroid_vs_spherical_shell_files/effect-mode... -653.651914
         3      tesseroid_vs_spherical_shell_files/effect-mode... -133.389015
         4      tesseroid_vs_spherical_shell_files/effect-mode...  -45.355184

                       gxx           gxy          gxz            gy        gyy  \
         point
         0       81.667048  2.864375e-14     0.562671  3.860939e-13   0.126986
         1      184.671435 -6.306067e-14   -38.871249 -4.396483e-14 -13.693074
         2     -820.755567 -3.728928e-11  5361.923206  2.995648e-11  37.295314
         3      115.108381  4.096723e-14    24.315601 -3.491651e-14  17.760055
         4       49.847512  7.965850e-15     3.123839 -3.358425e-15   7.142522

                         gyz          gz          gzz           pot  ratio  size
         point
         0     -3.513682e-15  124.244845  -81.794034  14271.388132      0     1
         1      7.327472e-15  138.387916 -170.978361  14278.240812      0     1
         2     -5.456968e-11  910.070144  783.460224  14311.554203      0     1
         3      6.925016e-15  132.259807 -132.868436  14273.562885      0     1
         4      6.210310e-16  120.411687  -56.990034  14263.270443      0     1
```

I'll save this to a CSV file as well for later use. It's easier to load that way than parsing all the output files again.

```
In [16]: tess_data.to_csv(
             '../data/tesseroid-size-{:.0f}-height-{:.0f}-pole.csv'.format(size, height))
```

Now I can use some pandas magic to calculate the difference between the shell effect and the tesseroid effect. This will also be stored in a DataFrame. The values are the absolute value of the differences for each size and ratio.

```
In [17]: def calc_difference(tess_data, shell_data):
             # Subtract the field columns and take the absolute value
             diff = tess_data.drop(['file', 'size', 'ratio'], axis='columns').sub(
                 shell_data, axis='columns').abs()
             # Then, add the size and ratio columns to the difference
```

```python
    diff[['size', 'ratio']] = tess_data[['size', 'ratio']]
    return diff
```

In [18]:
```python
diff = calc_difference(tess_data, shell_data)
diff.head()
```

Out[18]:
```
               gx         gxx           gxy          gxz            gy  \
point
0        3.582888   82.017807  2.864375e-14     0.562671  3.860939e-13
1      133.400119  185.022193  6.306067e-14    38.871249  4.396483e-14
2      653.651914  820.404809  3.728928e-11  5361.923206  2.995648e-11
3      133.389015  115.459139  4.096723e-14    24.315601  3.491651e-14
4       45.355184   50.198270  7.965850e-15     3.123839  3.358425e-15

             gyy           gyz          gz         gzz        pot  size  ratio
point
0       0.477744  3.513682e-15   99.543785   82.495551   6.633058     1      0
1      13.342316  7.327472e-15   85.400714  171.679878   0.219622     1      0
2      37.646072  5.456968e-11  686.281514  782.758707  33.533012     1      0
3      18.110814  6.925016e-15   91.528824  133.569953   4.458306     1      0
4       7.493281  6.210310e-16  103.376943   57.691551  14.750748     1      0
```

I'll save the differences as well to avoid recomputing.

In [19]:
```python
diff.to_csv('../data/difference-size-{:.0f}-height-{:.0f}-pole.csv'.format(size, height))
```

To get the maximum difference per size per ratio, I can use the `DataFrame.groupby` method.

In [20]:
```python
max_diff = diff.groupby(['size', 'ratio']).max()
max_diff
```

Out[20]:
```
                    gx         gxx        gxy          gxz        gy  \
size ratio
1    0.0    769.086112  854.704698  26.822236  5372.013197  0.876313
     0.5     11.215857   85.182630  21.765414    45.271905  1.271564
     1.0      0.463330   19.514930   2.203664    13.430373  0.236333
     1.5      0.070450    1.144707   0.209287     0.557937  0.031353
     2.0      0.022737    0.149761   0.024231     0.120548  0.005499
     2.5      0.008500    0.051286   0.008734     0.026611  0.002261
     3.0      0.003946    0.015951   0.005038     0.008864  0.000848
     3.5      0.001777    0.009913   0.001601     0.005218  0.000305
     4.0      0.001194    0.007222   0.000898     0.002226  0.000269
     4.5      0.000614    0.003432   0.000450     0.000784  0.000065
     5.0      0.000410    0.002540   0.000442     0.001013  0.000058
     5.5      0.000240    0.001280   0.000147     0.000491  0.000030
     6.0      0.000179    0.000925   0.000084     0.000573  0.000020
     6.5      0.000131    0.000641   0.000085     0.000260  0.000015
     7.0      0.000095    0.000510   0.000048     0.000242  0.000014
     7.5      0.000084    0.000569   0.000038     0.000169  0.000010
     8.0      0.000055    0.000345   0.000024     0.000110  0.000009
     8.5      0.000044    0.000249   0.000015     0.000087  0.000005
     9.0      0.000037    0.000216   0.000009     0.000075  0.000004
     9.5      0.000032    0.000178   0.000007     0.000054  0.000004
     10.0     0.000023    0.000117   0.000007     0.000047  0.000002

                   gyy        gyz          gz         gzz        pot
```

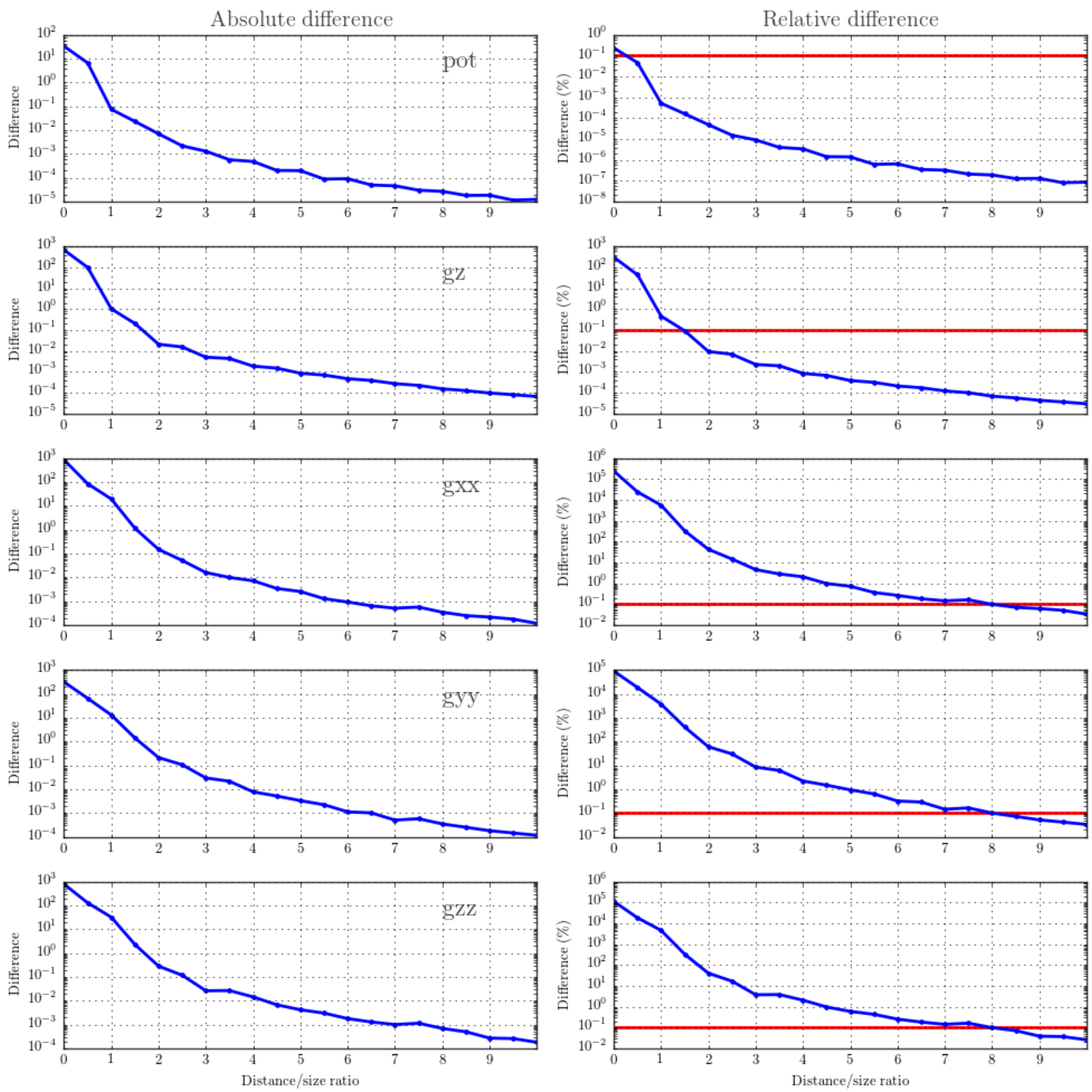
```
size ratio
1    0.0     311.835644  27.800405  686.281514  782.758707  33.533012
     0.5      64.388814  13.539341   99.543785  128.777627   6.633058
     1.0      12.838965   8.384167    1.027770   32.012723   0.074944
     1.5       1.392990   0.662468    0.203132    2.289414   0.023323
     2.0       0.208828   0.123601    0.020666    0.283195   0.007042
     2.5       0.104713   0.027675    0.015305    0.117553   0.002195
     3.0       0.029259   0.008855    0.004941    0.026251   0.001310
     3.5       0.021229   0.004236    0.004276    0.026761   0.000574
     4.0       0.007639   0.002814    0.001837    0.014445   0.000486
     4.5       0.005101   0.001056    0.001475    0.006772   0.000206
     5.0       0.003265   0.001003    0.000840    0.004182   0.000201
     5.5       0.002198   0.000500    0.000687    0.003024   0.000089
     6.0       0.001108   0.000233    0.000459    0.001780   0.000092
     6.5       0.001000   0.000185    0.000379    0.001285   0.000050
     7.0       0.000499   0.000156    0.000266    0.000998   0.000046
     7.5       0.000569   0.000083    0.000219    0.001138   0.000030
     8.0       0.000345   0.000055    0.000150    0.000690   0.000027
     8.5       0.000249   0.000053    0.000123    0.000498   0.000018
     9.0       0.000179   0.000044    0.000095    0.000272   0.000019
     9.5       0.000144   0.000023    0.000079    0.000258   0.000012
     10.0      0.000115   0.000020    0.000065    0.000181   0.000012
```

I'll plot the maximum difference as a function of distance/size ratio. I'll also plot the relative difference (difference divided by the shell value) in percentage.

In [21]:
```python
def plot_difference(diff, shell_data, size=size):
    fig, subplots = plt.subplots(5, 2, figsize=(10, 10))
    for f, axes in zip(['pot', 'gz', 'gxx', 'gyy', 'gzz'], subplots):
        maxd = diff[f].loc[size, :]
        ax1, ax2 = axes
        ax1.text(0.8, 0.8, f, fontsize=16, transform=ax1.transAxes)
        ax1.plot(ratio, maxd, '.-')
        ax1.set_yscale('log')
        ax1.grid(True)
        ax1.set_ylabel('Difference')
        ax2.plot(ratio, 100*maxd/np.abs(shell_data[f]), '.-')
        ax2.set_yscale('log')
        ax2.grid(True)
        ax2.set_ylabel('Difference (\%)')
        ax2.hlines(0.1, ratio.min(), ratio.max(), colors='r')
        ax2.set_xlim(ratio.min(), ratio.max())
        ax1.set_xticks(range(10))
        ax2.set_xticks(range(10))
    ax1.set_xlabel('Distance/size ratio')
    ax2.set_xlabel('Distance/size ratio')
    ax1, ax2 = subplots[0]
    ax1.set_title('Absolute difference')
    ax2.set_title('Relative difference')
    plt.tight_layout()
    return fig
```

In [22]:
```python
plot_difference(max_diff, shell_data)
```

Out[22]:



1.7 Verify the comparison at 260 km height

Run the comparison again to see how big the errors are per ratio and if they are below the threshold when computing at 260 km height (GOCE orbit height).

In [23]:
```
goce_height = 260e3
goce_height_out_files = []
for r in ratio:
    f = calc_tess_effect(model_file, goce_height, size, r, where='pole')
    print('Done: {}'.format(f))
    goce_height_out_files.append(f)
```

```
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio0.0-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio0.5-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio1.0-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio1.5-height260000-pole.txt
```

```
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio2.0-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio2.5-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio3.0-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio3.5-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio4.0-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio4.5-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio5.0-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio5.5-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio6.0-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio6.5-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio7.0-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio7.5-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio8.0-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio8.5-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio9.0-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio9.5-height260000-pole.txt
Done: tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio10.0-height260000-pole.txt
```

Load the data from the output files and save it to a CSV.

```
In [24]: goce_height_data = load_tess_data(goce_height_out_files, ratio, size)
```

```
In [25]: goce_height_data.to_csv(
             '../data/tesseroid-size-{:.0f}-height-{:.0f}-pole.csv'.format(size, goce_height))
```

Calculate the shell fields at the new height as well.

```
In [26]: goce_height_shell_data = calc_shell_effect(goce_height, top, bottom, density)
         goce_height_shell_data
```

```
Out[26]: gx         0.000000
         gxx       -0.311429
         gxy        0.000000
         gxz        0.000000
         gy         0.000000
         gyy       -0.311429
         gyz        0.000000
         gz       206.730999
         gzz        0.622858
         pot    13723.086957
         dtype: float64
```

```
In [27]: goce_height_shell_data.to_csv('../data/shell-height-{:.0f}.csv'.format(goce_height))
```

Calculate and save the differences.

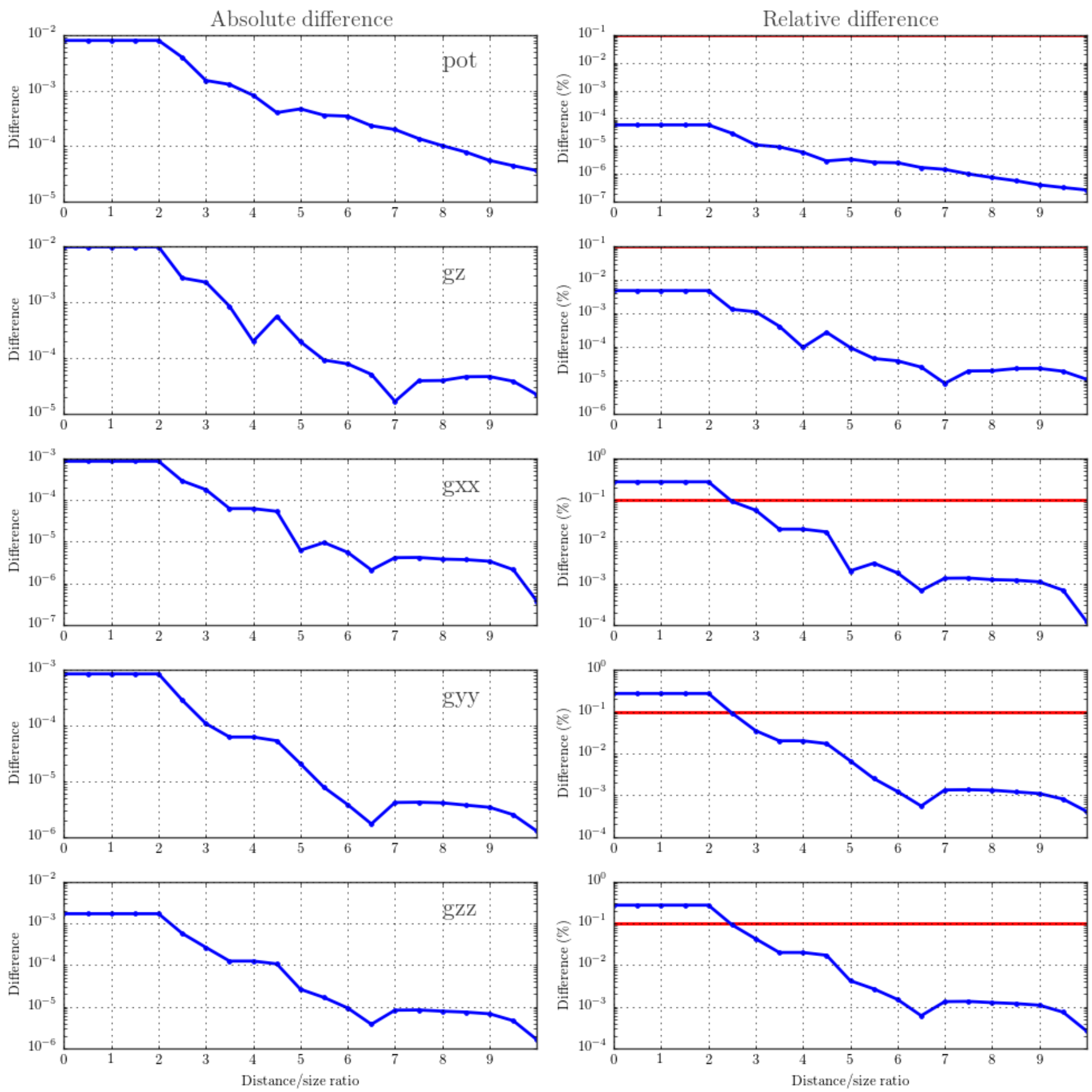
```
In [28]: goce_height_diff = calc_difference(goce_height_data, goce_height_shell_data)
```

```
In [29]: goce_height_diff.to_csv(
             '../data/difference-size-{:.0f}-height-{:.0f}-pole.csv'.format(size, goce_height))
```

Plot the maximum difference per ratio.

```
In [30]: goce_height_max_diff = goce_height_diff.groupby(['size', 'ratio']).max()
         plot_difference(goce_height_max_diff, goce_height_shell_data)
```

Out[30]:



1.8 Very the comparison at the equator

Lets verify that the results are also valid at the equator, where tesseroids have a very different shape.

```
In [31]: equator_out_files = []
         for r in ratio:
             f = calc_tess_effect(model_file, height, size, r, where='equator')
             print('  {}'.format(f))
             equator_out_files.append(f)
```

```
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio0.0-height2000-equator.txt
  tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio0.5-height2000-equator.txt
  tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio1.0-height2000-equator.txt
  tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio1.5-height2000-equator.txt
  tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio2.0-height2000-equator.txt
  tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio2.5-height2000-equator.txt
```

```
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio3.0-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio3.5-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio4.0-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio4.5-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio5.0-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio5.5-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio6.0-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio6.5-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio7.0-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio7.5-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio8.0-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio8.5-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio9.0-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio9.5-height2000-equator.txt
tesseroid_vs_spherical_shell_files/effect-model-size1.0.tess-ratio10.0-height2000-equator.txt
```

In [32]:
```
eq_data = load_tess_data(equator_out_files, ratio, size)
```

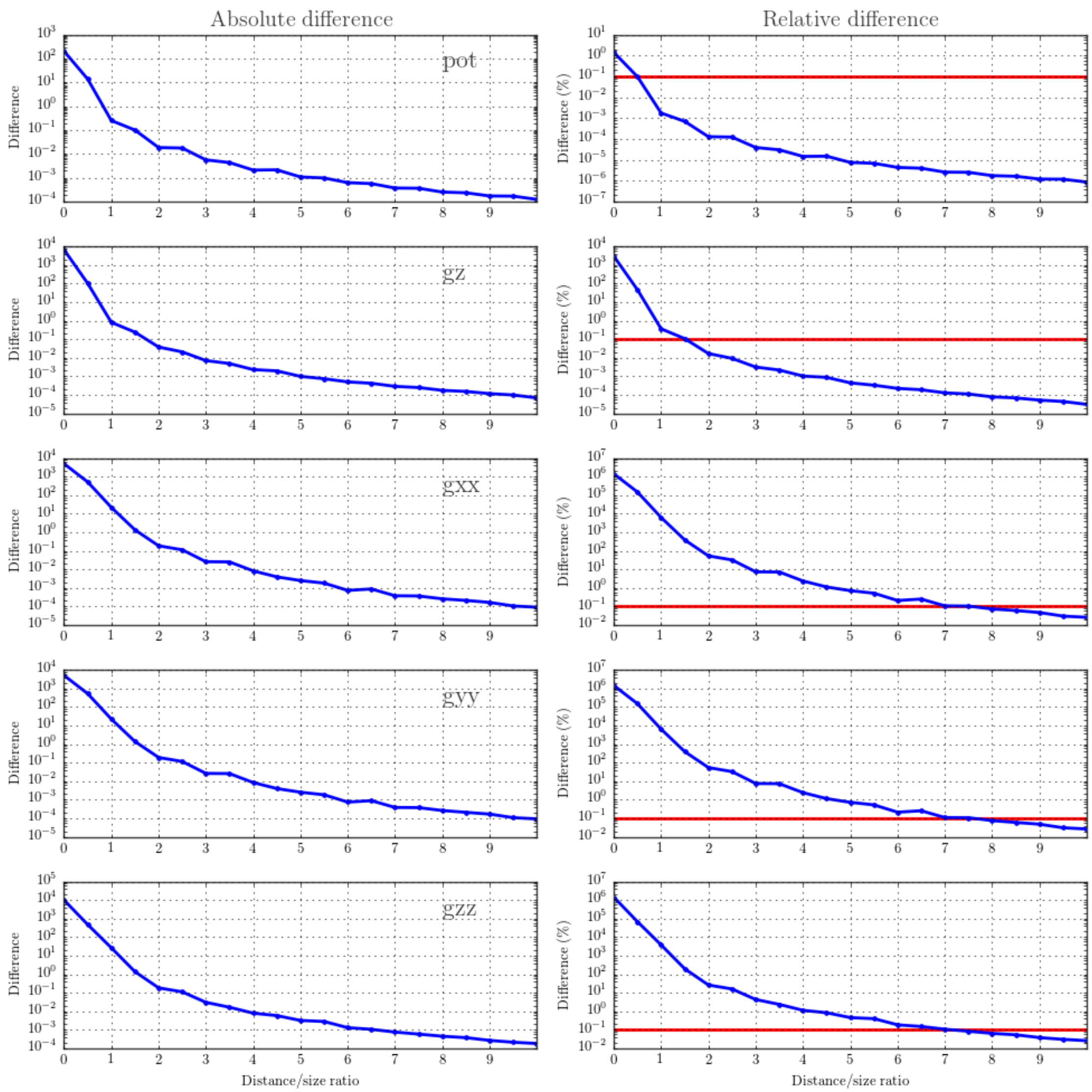
In [33]:
```
eq_data.to_csv(
    '../data/tesseroid-size-{:.0f}-height-{:.0f}-equator.csv'.format(size, height))
```

In [34]:
```
eq_diff = calc_difference(eq_data, shell_data)
```

In [35]:
```
eq_diff.to_csv(
    '../data/difference-size-{:.0f}-height-{:.0f}-equator.csv'.format(size, height))
```

In [36]:
```
eq_max_diff = eq_diff.groupby(['size', 'ratio']).max()
plot_difference(eq_max_diff, shell_data)
```

Out[36]:



1.9 Check how gzz varies with computation height

I want to see how the difference x ratio curve varies with the computation height. We know 2 extremes (2 km and 260 km), but how fast does it decay?

First, we can get the data that we need from the 2 km and 260 km DataFrames.

```
In [37]: tess_gzz_per_height = tess_data['file ratio size gzz'.split()]
         tess_gzz_per_height['height'] = height
         tmp = goce_height_data['file ratio size gzz'.split()]
         tmp['height'] = goce_height
         tess_gzz_per_height = tess_gzz_per_height.append(tmp, ignore_index=True)
```

```
/home/leo/bin/anaconda/lib/python2.7/site-packages/IPython/kernel/__main__.py:2: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead
```

```
See the the caveats in the documentation: http://pandas.pydata.org/pandas-docs/stable/indexing.html#inde
  from IPython.kernel.zmq import kernelapp as app
/home/leo/bin/anaconda/lib/python2.7/site-packages/IPython/kernel/__main__.py:4: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the the caveats in the documentation: http://pandas.pydata.org/pandas-docs/stable/indexing.html#inde
```

Then, set the new heights to compute and make a function to run Tesseroids (only `tessgzz`) at each height (for a grid near the pole).

```
In [38]: heights = [height, 10e3, 50e3, 150e3, goce_height]
```

```
In [39]: def calc_tess_gzz(fname, height, size, ratio):
             if ratio == 0:
                 flag = '-a'
             else:
                 flag = '-t{:f}'.format(ratio)
             # Make a computation grid above a tesseroid
             lats, lons = gridder.regular([90 - size, 90, 0, size], (10, 10))
             tmp = 'gzz-{}-ratio{:.1f}-height{:.0f}.txt'.format(os.path.split(fname)[1],
                                                                  ratio, height)
             outfile = os.path.join(filedir, tmp)
             grid_text = '\n'.join('{:f} {:f} {:f}'.format(lon, lat, height)
                                   for lon, lat in zip(lons, lats))
             !echo "$grid_text" | tessgzz $fname $flag > $outfile
             return outfile
```

We'll need the shell effect at each different height as well. We'll calculate the effects and group them in a `pandas.DataFrame` as well.

```
In [40]: def shell_effect_with_height(height):
             "Helper function to append the height to the pandas.Series with the shell data"
             series = calc_shell_effect(height, top, bottom, density)
             series['height'] = height
             return series

         shell_per_height = pd.DataFrame([shell_effect_with_height(h) for h in heights])
         shell_per_height
```

```
Out[40]:    gx       gxx  gxy  gxz  gy       gyy  gyz          gz       gzz  \
         0   0 -0.350758    0    0   0 -0.350758    0  223.788630  0.701517
         1   0 -0.349442    0    0   0 -0.349442    0  223.228471  0.698884
         2   0 -0.342959    0    0   0 -0.342959    0  220.458972  0.685919
         3   0 -0.327439    0    0   0 -0.327439    0  213.756587  0.654878
         4   0 -0.311429    0    0   0 -0.311429    0  206.730999  0.622858

                     pot  height
         0  14278.021191    2000
         1  14260.140521   10000
         2  14171.404761   50000
         3  13954.322847  150000
         4  13723.086957  260000
```

```
In [41]: shell_per_height.to_csv('../data/shell-per-height.csv')
```

Now we can calculate gzz for each height, load the data into a DataFrame and append it to our dataset.

```
In [42]: for h in heights[1:-1]:
             files = [calc_tess_gzz(model_file, h, size, r) for r in ratio]
             # Load the data files into a DataFrame
             tmp = pd.DataFrame(columns=['file', 'ratio', 'gzz'])
             for r, fname in zip(ratio, files):
                 data = np.loadtxt(fname, usecols=[-1], unpack=True)
                 tmp = tmp.append(pd.DataFrame(dict(file=fname, ratio=r, gzz=data)),
                                  ignore_index=True)
             tmp.index.name = 'point'
             tmp['height'] = h
             tmp['size'] = size
             tess_gzz_per_height = tess_gzz_per_height.append(tmp, ignore_index=True)
             print('Done: height {}'.format(h))
```

```
Done: height 10000.0
Done: height 50000.0
Done: height 150000.0
```

```
In [43]: tess_gzz_per_height.head()
```

```
Out[43]:                                                 file         gzz  height  \
         0  tesseroid_vs_spherical_shell_files/effect-mode...  -81.794034    2000
         1  tesseroid_vs_spherical_shell_files/effect-mode... -170.978361    2000
         2  tesseroid_vs_spherical_shell_files/effect-mode...  783.460224    2000
         3  tesseroid_vs_spherical_shell_files/effect-mode... -132.868436    2000
         4  tesseroid_vs_spherical_shell_files/effect-mode...  -56.990034    2000

            ratio  size
         0      0     1
         1      0     1
         2      0     1
         3      0     1
         4      0     1
```

Again, save this to a CSV for later use.

```
In [44]: tess_gzz_per_height.to_csv('../data/tesseroid-gzz-per-height-size-{:.0f}.csv'.format(size))
```

Calculate the difference for each height. Again, we'll load the differences for 2km and 260km from the previous results first.

```
In [45]: diff_per_height = diff['ratio gzz'.split()]
         diff_per_height['height'] = height
         tmp = goce_height_diff['ratio gzz'.split()]
         tmp['height'] = goce_height
         diff_per_height = diff_per_height.append(tmp, ignore_index=True)
```

```
/home/leo/bin/anaconda/lib/python2.7/site-packages/IPython/kernel/__main__.py:2: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the the caveats in the documentation: http://pandas.pydata.org/pandas-docs/stable/indexing.html#inde
  from IPython.kernel.zmq import kernelapp as app
/home/leo/bin/anaconda/lib/python2.7/site-packages/IPython/kernel/__main__.py:4: SettingWithCopyWarning:
```

```
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the the caveats in the documentation: http://pandas.pydata.org/pandas-docs/stable/indexing.html#inde
```

```
In [46]: for h in heights[1:-1]:
             tmp = tess_gzz_per_height[tess_gzz_per_height['height'] == h]
             tmp_shell = shell_per_height[shell_per_height['height'] == h]['gzz']
             tmp_diff = pd.DataFrame(dict(gzz=(tmp['gzz'] - tmp_shell.values).abs()))
             tmp_diff[['ratio',  'height']] = tmp[['ratio', 'height']]
             diff_per_height = diff_per_height.append(tmp_diff, ignore_index=True)
```

```
In [47]: diff_per_height.head()
```

```
Out[47]:            gzz  height  ratio
         0   82.495551    2000      0
         1  171.679878    2000      0
         2  782.758707    2000      0
         3  133.569953    2000      0
         4   57.691551    2000      0
```

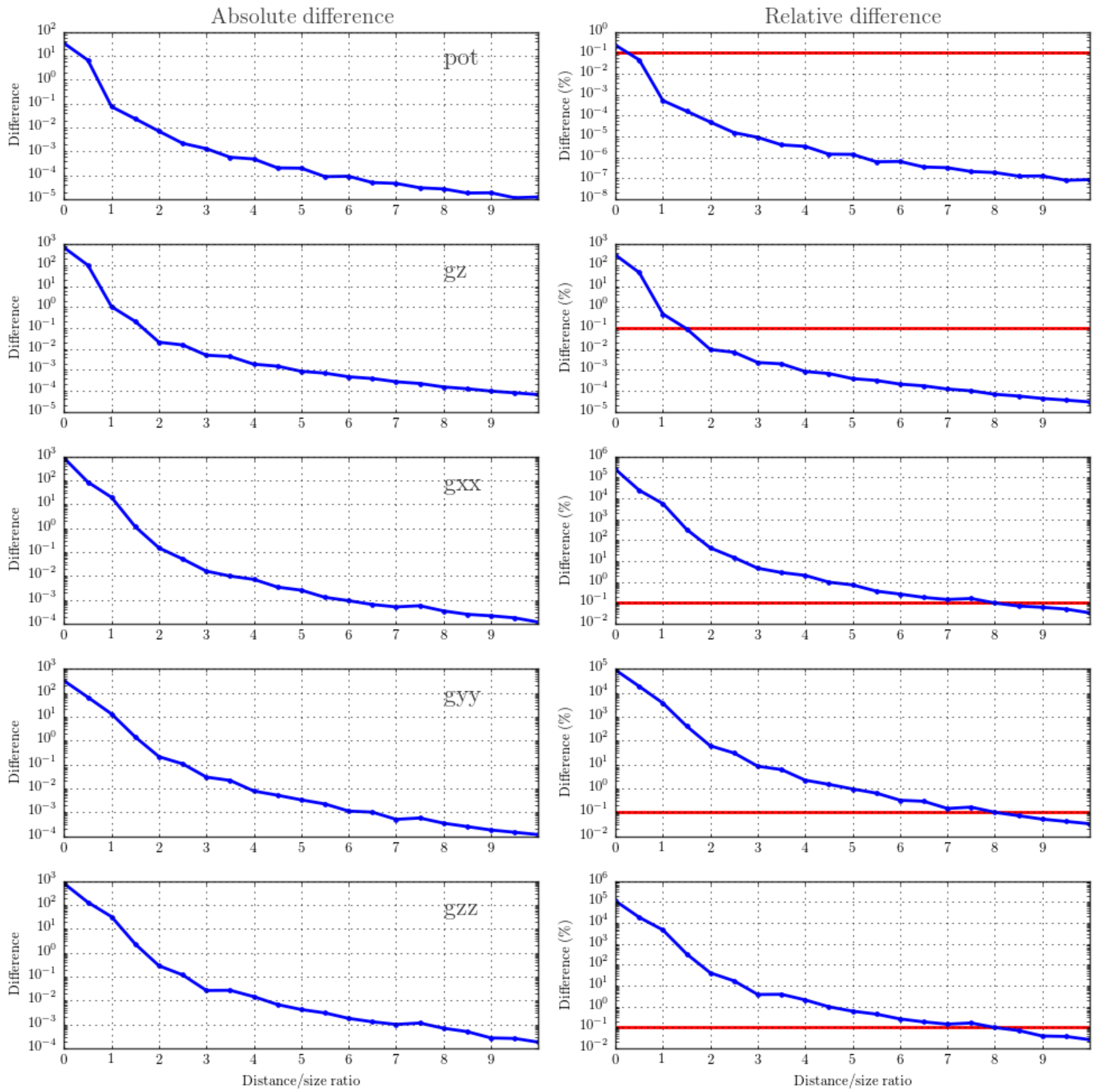
```
In [48]: diff_per_height.to_csv(
             '../data/difference-gzz-per-height-size-{:.0f}-pole.csv'.format(size))
```

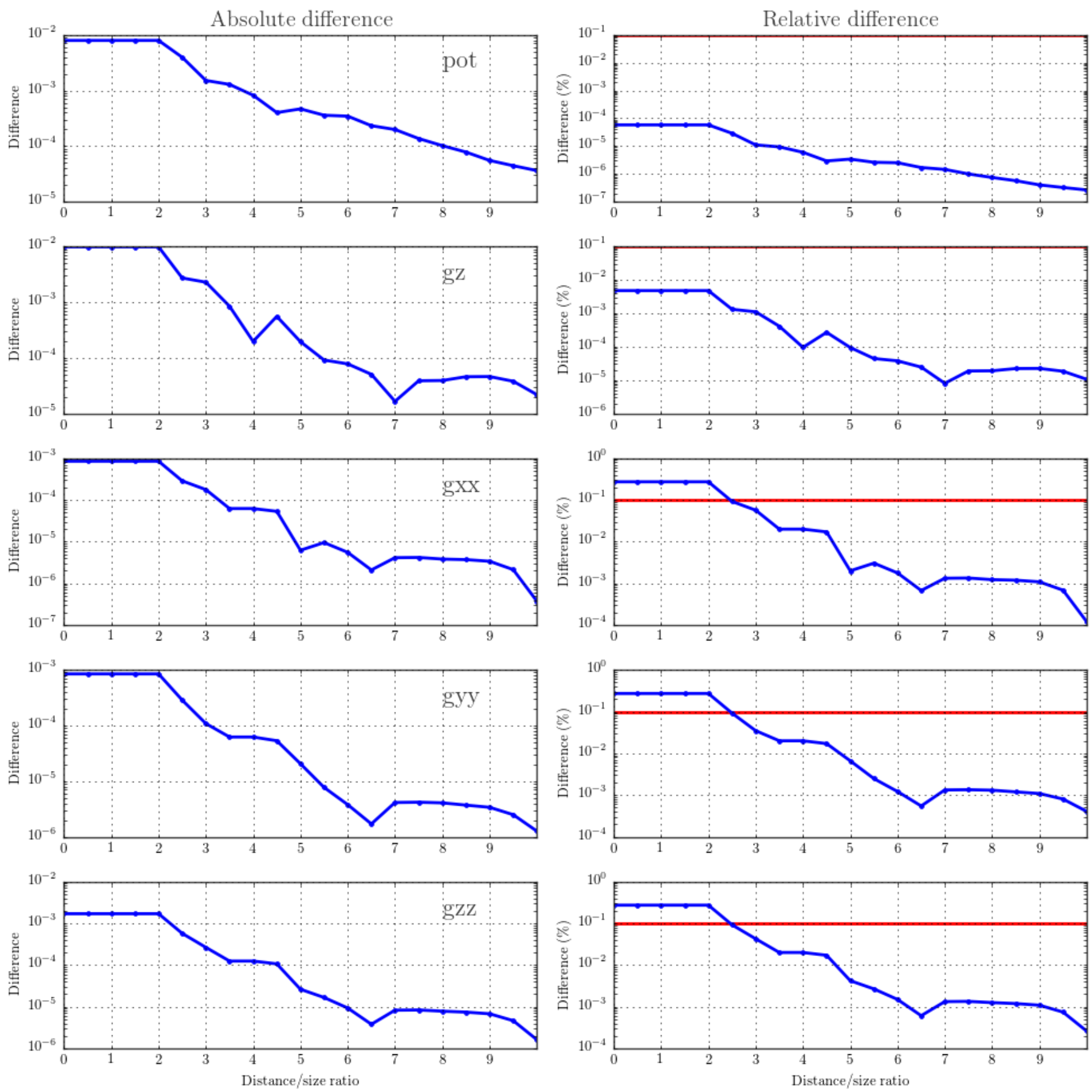
Now I can calculate the maximum difference per ratio and plot a error-ratio curve per computation height.

```
In [49]: max_diff_per_height = diff_per_height.groupby(['height', 'ratio']).max()
```

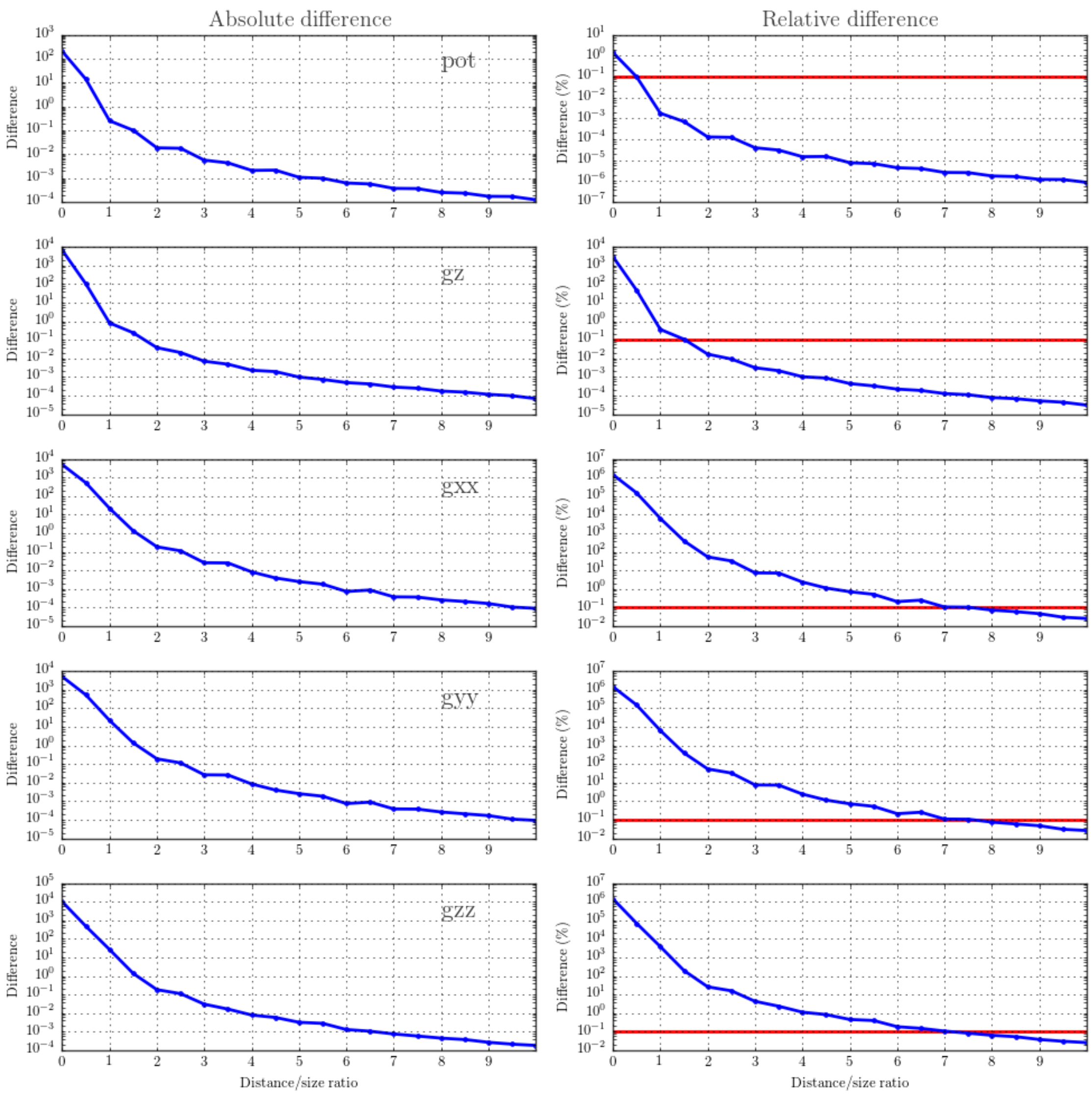
```
In [50]: fig, subplots = plt.subplots(1, 2, figsize=(10, 4))
         ax1, ax2 = subplots
         for h in heights:
             d = max_diff_per_height.loc[h]
             ax1.plot(ratio, d, '.-')
             ax1.set_yscale('log')
             ax1.grid(True)
             ax1.set_ylabel('Difference')
             ref = calc_shell_effect(h, top, bottom, density)['gzz']
             ax2.plot(ratio, 100*d/np.abs(ref), '.-', label='{:.0f} km'.format(h*0.001))
             ax2.set_yscale('log')
             ax2.grid(True)
             ax2.set_ylabel('Difference (\%)')
             ax2.hlines(0.1, ratio.min(), ratio.max(), colors='k')
             ax2.set_xlim(ratio.min(), ratio.max())
         ax2.legend(loc='upper right', numpoints=1)
         ax2.set_xticks(range(11))
         ax1.set_xlabel('Distance/size ratio')
         ax2.set_xlabel('Distance/size ratio')
         ax1.set_title('Absolute difference')
         ax2.set_title('Relative difference')
         plt.tight_layout()
         plt.show()
```

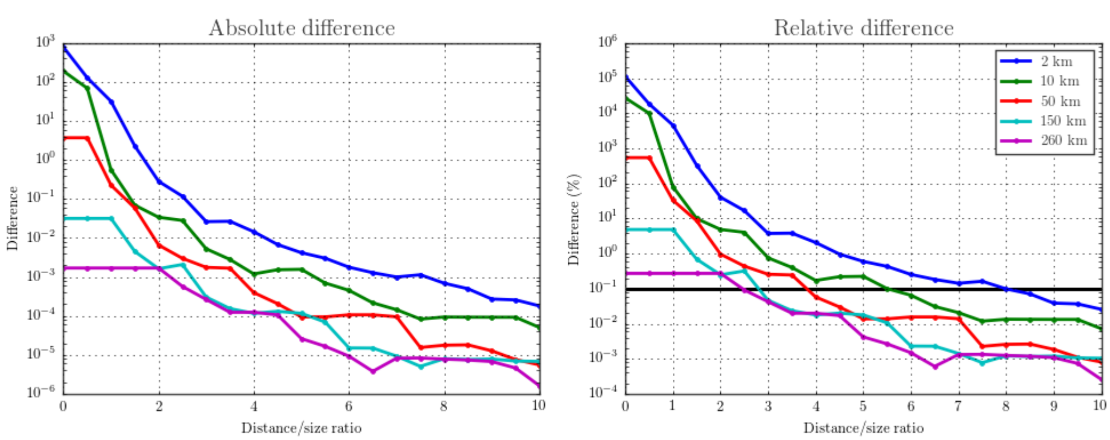
```
/home/leo/bin/anaconda/lib/python2.7/site-packages/pandas/core/index.py:648: FutureWarning: scalar inde
  type(self).__name__),FutureWarning)
```


Absolute difference
Relative difference
pot
gz
gxx
gyy
gzz
Difference
Difference (%)
Distance/size ratio



Absolute difference
Relative difference
pot
gz
gxx
gyy
gzz
Difference
Difference (%)
Distance/size ratio


Absolute difference
Relative difference
pot
gz
gxx
gyy
gzz
Difference
Difference (%)
Distance/size ratio


Absolute difference
Relative difference
2 km
10 km
50 km
150 km
260 km
Difference
Difference (%)
Distance/size ratio

1.10 Check that the results for 2 km height are independent of size

Repeat the computations for a large sized tesseroid and make the difference x ratio curve.

```
In [51]: big_size = 30
```

I'll need a new model file for this.

```
In [52]: big_model_file = make_model_file(big_size, top, bottom, density, verbose=True)
```

```
Model file: tesseroid_vs_spherical_shell_files/model-size30.0.tess
Model size: 72
Head:
0.0 30.0 -90.0 -60.0 1000.0 0.0 2670
30.0 60.0 -90.0 -60.0 1000.0 0.0 2670
60.0 90.0 -90.0 -60.0 1000.0 0.0 2670
90.0 120.0 -90.0 -60.0 1000.0 0.0 2670
120.0 150.0 -90.0 -60.0 1000.0 0.0 2670
150.0 180.0 -90.0 -60.0 1000.0 0.0 2670
180.0 210.0 -90.0 -60.0 1000.0 0.0 2670
210.0 240.0 -90.0 -60.0 1000.0 0.0 2670
240.0 270.0 -90.0 -60.0 1000.0 0.0 2670
270.0 300.0 -90.0 -60.0 1000.0 0.0 2670
Tail:
60.0 90.0 60.0 90.0 1000.0 0.0 2670
90.0 120.0 60.0 90.0 1000.0 0.0 2670
120.0 150.0 60.0 90.0 1000.0 0.0 2670
150.0 180.0 60.0 90.0 1000.0 0.0 2670
180.0 210.0 60.0 90.0 1000.0 0.0 2670
210.0 240.0 60.0 90.0 1000.0 0.0 2670
240.0 270.0 60.0 90.0 1000.0 0.0 2670
270.0 300.0 60.0 90.0 1000.0 0.0 2670
300.0 330.0 60.0 90.0 1000.0 0.0 2670
330.0 360.0 60.0 90.0 1000.0 0.0 2670
```

```
In [53]: big_tess_files = []
         for r in ratio:
             f = calc_tess_effect(big_model_file, height, big_size, r, where='pole')
             print('  {}'.format(f))
             big_tess_files.append(f)
```

```
tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio0.0-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio0.5-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio1.0-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio1.5-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio2.0-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio2.5-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio3.0-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio3.5-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio4.0-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio4.5-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio5.0-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio5.5-height2000-pole.txt
  tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio6.0-height2000-pole.txt
```

```
tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio6.5-height2000-pole.txt
tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio7.0-height2000-pole.txt
tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio7.5-height2000-pole.txt
tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio8.0-height2000-pole.txt
tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio8.5-height2000-pole.txt
tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio9.0-height2000-pole.txt
tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio9.5-height2000-pole.txt
tesseroid_vs_spherical_shell_files/effect-model-size30.0.tess-ratio10.0-height2000-pole.txt
```

Again, load the data to a `DataFrame` and export to CSV.

In [54]:
```
big_data = load_tess_data(big_tess_files, ratio, big_size)
```

In [55]:
```
big_data.to_csv(
        '../data/tesseroid-size-{:.0f}-height-{:.0f}-pole.csv'.format(big_size, height))
```

Do the same for the differences.

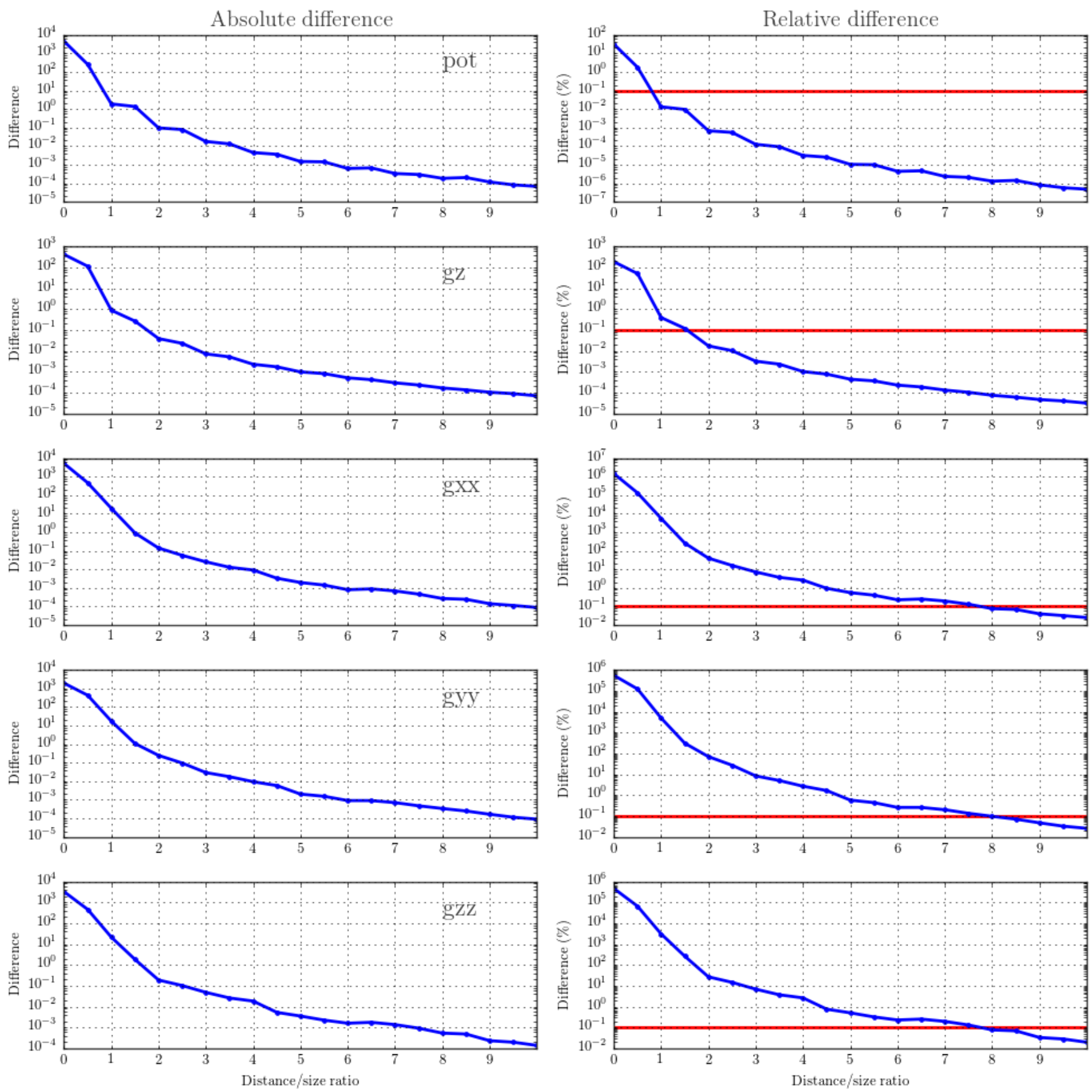
In [56]:
```
big_diff = calc_difference(big_data, shell_data)
```

In [57]:
```
big_diff.to_csv(
        '../data/difference-size-{:.0f}-height-{:.0f}-pole.csv'.format(big_size, height))
```

Plot the error-ratio curves.

In [58]:
```
big_max_diff = big_diff.groupby(['size', 'ratio']).max()
plot_difference(big_max_diff, shell_data, size=big_size)
```

Out[58]:



1.11 Group the above into a single graph

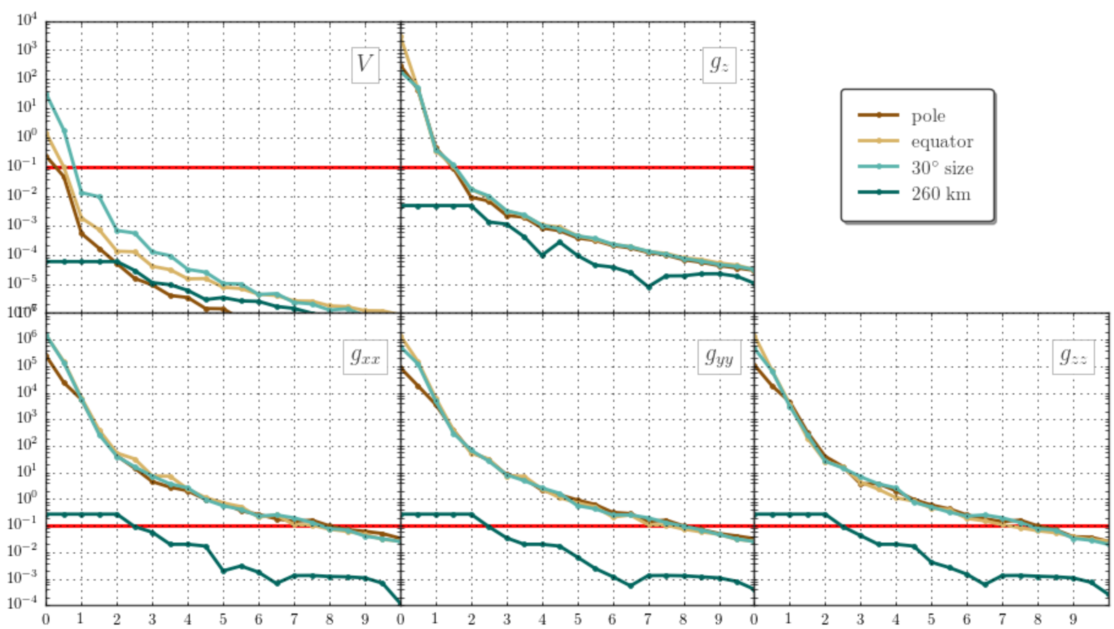
Plot the relative differences for 2 km height at the pole, at the equator, at 260 km height, and for the 30° tesseroid.

```
In [59]: def plot_all_relative(pole, equator, goce, big):
             colors = "#8c510a #d8b365 #5ab4ac #01665e".split()
             fig, subplots = plt.subplots(2, 3, figsize=(9, 5), sharex='col', sharey='row')
             axes = subplots.ravel()[[0, 1, 3, 4, 5]]
             fields = ['pot', 'gz', 'gxx', 'gyy', 'gzz']
             titles = [r'$V$', r'$g_z$', r'$g_{xx}$', r'$g_{yy}$', r'$g_{zz}$']
             for i in xrange(5):
                 ax, f, title = axes[i], fields[i], titles[i]
                 ax.text(0.9, 0.85, title, fontsize=14,
                         horizontalalignment='center', verticalalignment='center',
```

```
                 bbox={'facecolor': 'w',
                       'edgecolor': '#9b9b9b',
                       'linewidth': 0.5, 'pad': 8},
                 transform=ax.transAxes)
         ax.plot(ratio, 100*pole[f]/np.abs(shell_data[f]), '.-',
                 label='pole', color=colors[0])
         ax.plot(ratio, 100*equator[f]/np.abs(shell_data[f]), '.-',
                 label='equator', color=colors[1])
         ax.plot(ratio, 100*big[f]/np.abs(shell_data[f]), '.-',
                 label=r'$30^\circ$ size', color=colors[2])
         ax.plot(ratio, 100*goce[f]/np.abs(goce_height_shell_data[f]), '.-',
                 label='260 km', color=colors[3])
         ax.hlines(0.1, ratio.min(), ratio.max(), colors='r')
         ax.set_xlim(ratio.min(), ratio.max())
         ax.set_yscale('log')
         ax.set_xticks(range(10))
         ax.grid(True)
     subplots[0, 0].legend(borderpad=1, numpoints=1, bbox_to_anchor=(2.7, 0.8),
                           fancybox=True, shadow=True, fontsize=11)
     subplots[0, 2].axison = False
     plt.tight_layout(pad=0, h_pad=0, w_pad=0)
     plt.subplots_adjust(hspace=0, wspace=0)
     return fig
```

```
In [60]: plot_all_relative(max_diff, eq_max_diff, goce_height_max_diff, big_max_diff)
```

Out[60]:



1.12 References

Grombein, T., K. Seitz, and B. Heck (2013), Optimized formulas for the gravitational field of a tesseroid, J Geod, 87(7), 645-660, doi:10.1007/s00190-013-0636-1.