Nonlinear controller analysis

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Kaga 2024

Part 1: Dawdling

Presentation goals

- Don’t get too rusty in writing papers.
- Keep alive a streak of IEEE conference presentations.
- Take advantage of my post-graduate controls courses.
- Contribute proofs for the ArduPilot Controllers

A lot of disappointment

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- This isn’t an IEEE conference.
- I didn’t remember half as much as I thought.
- Didn’t manage half as much as I hoped.

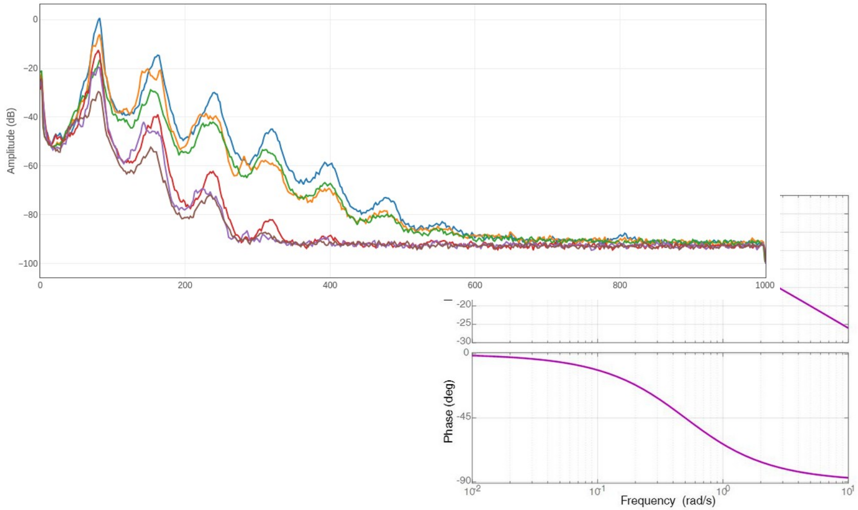
Key takeaways

- Have fun!
- Research is heavy on the body.

Part 2: Meat and potatoes

Nonlinear analysis

We use plenty of FFT and Bode diagrams



Not applicable to nonlinear systems:

- Plant is nonlinear

$$\dot{h} = V sin(\theta)$$

- Controller is nonlinear

$$u = -v^2$$

Nonlinear analysis

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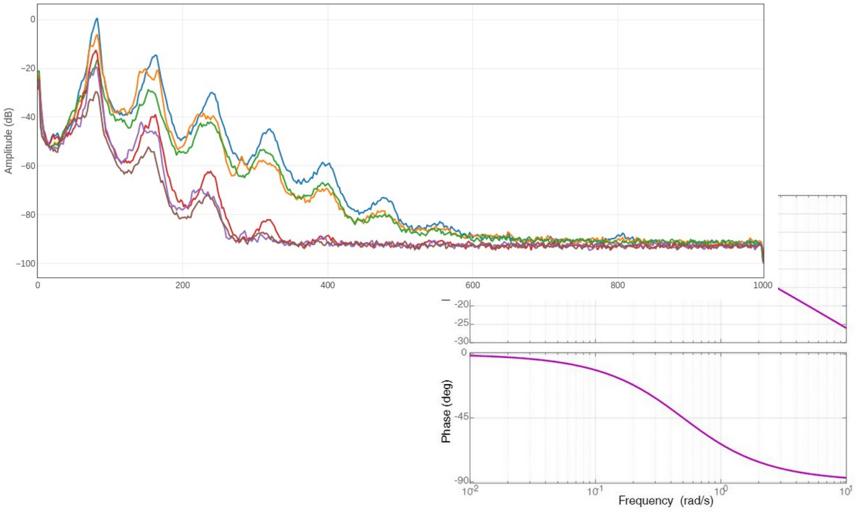

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- Plant is nonlinear

$$\dot{h} = V sin(\theta)$$

- Controller is nonlinear

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TECS!

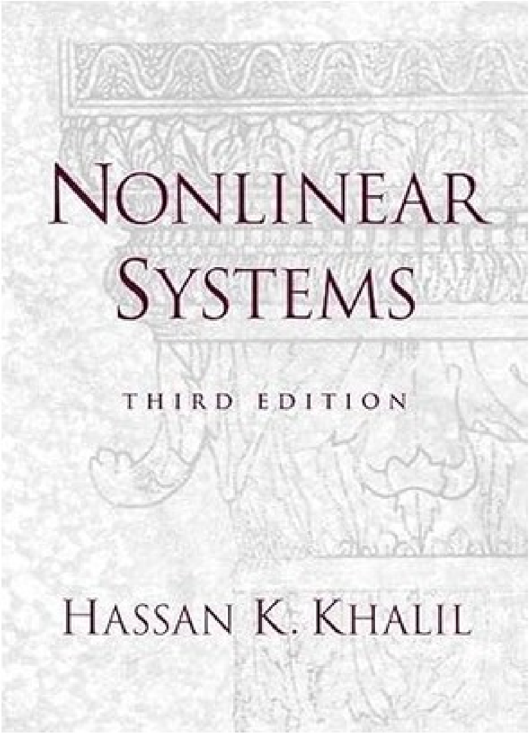
Bearded guy

Aleksandr Lyapunov

- 19th century Russia
- Father of stability theory in dynamic systems

Bibliography

Khalil, H. K. (2002).
Nonlinear Systems (3rd ed.).
Prentice-Hall, Inc.



Literature review

- General nonlinear stability: Literally thousands
- ArduPilot controllers: None that I could find.

  Pointers very welcome!

Goals: revised

1) Study if TECS closes a stable loop (spoiler: it usually does).
2) Uncover any hints on tuning.

Modelling

- Longitudinal airplane dynamics
- Assume perfect pitch tracking
- Simplified thruster model
- Simplified drag
- Limited to constant altitude/velocity tracking

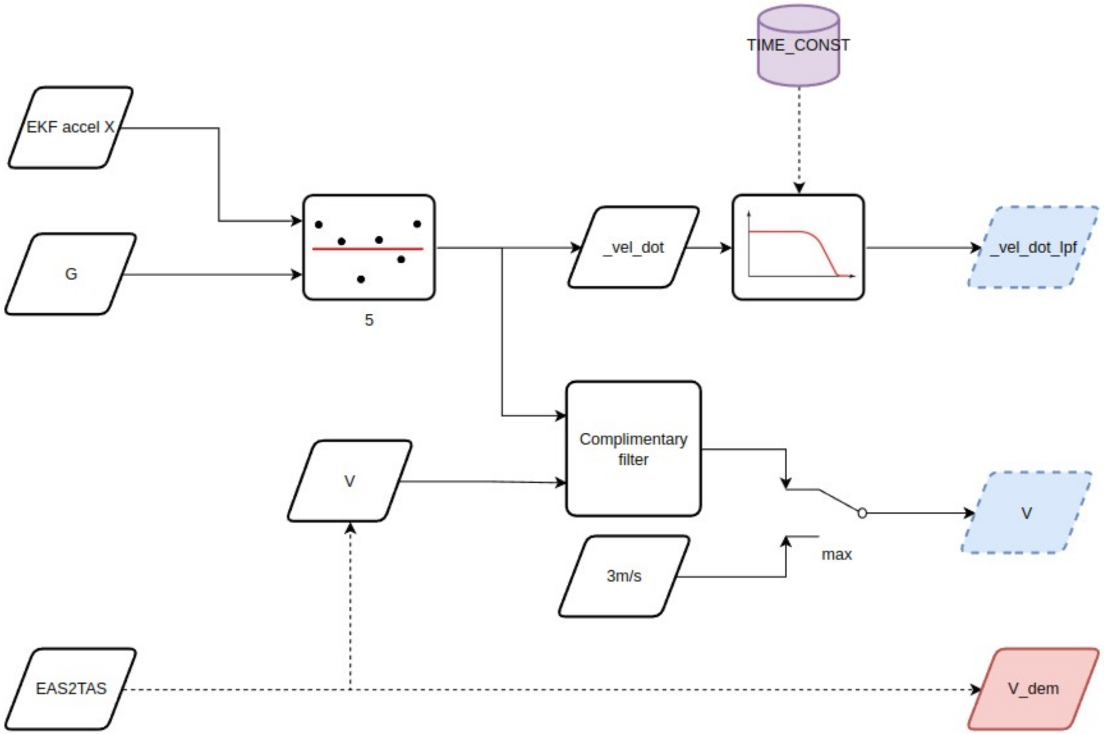
$$D = 0.5\rho V^2 C_D$$

$$\dot{V} = \frac{1}{m}(sin(\theta)g + F_t \delta t - D)$$

$$\dot{h} = V sin(\theta - \theta_0)$$

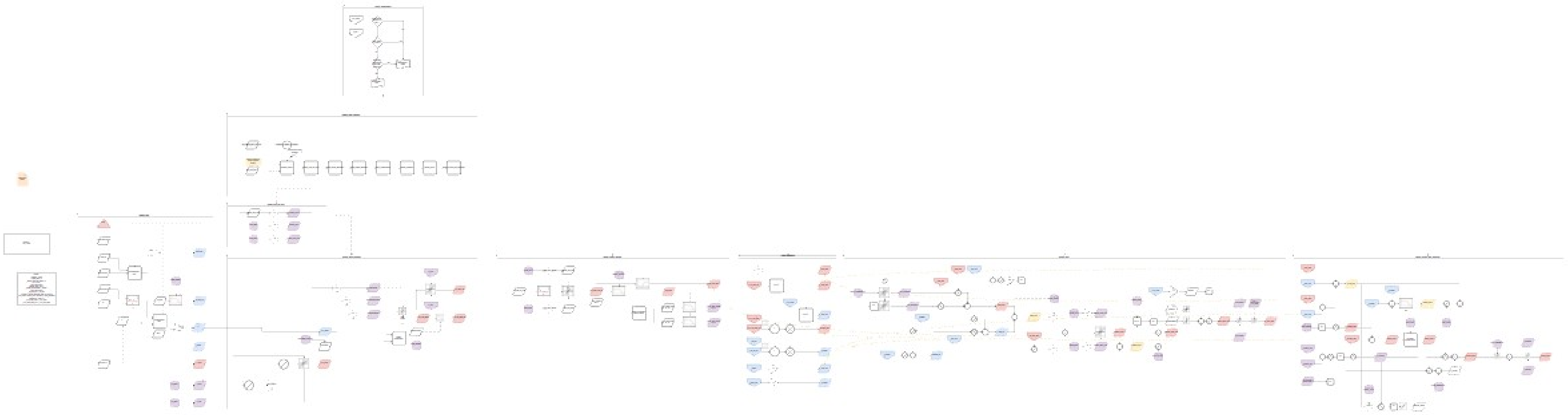
Autonomous system

Condense the controller into a digestible form.



Autonomous system

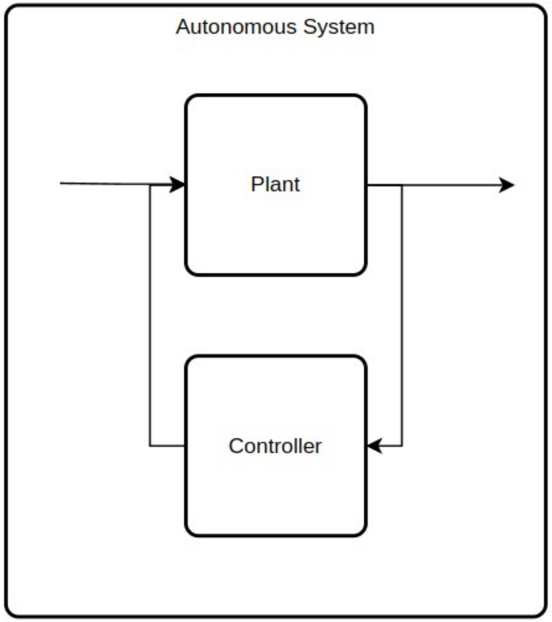
“Digestible”



Autonomous system

Construct the autonomous system:

- Replace the plant inputs with the controller outputs.
- Its state dynamics are a mere function of themselves.



Autonomous system

Construct the autonomous system… easy

$$v(t) = \frac{-0.5C_{drag}S\rho v^2(t) + F_t\left(thr_{trim} + x_{intthr} + \frac{gh_d - gh(t)}{k_{tc}\left(g\dot{h}_{cmax} + g\dot{h}_{smin}\right)} + \frac{gh_d - gh(t) + k_{id}\left(-g\frac{d}{dt}h(t) - v(t)\frac{d}{dt}v(t) + \frac{gh_d - gh(t)}{k_{tc}}\right) + 0.5v_d^2 - 0.5v^2(t)}{k_{tc}\left(g\dot{h}_{cmax} + g\dot{h}_{smin}\right)}\right) + g\sin\left(\frac{gh_{ddot}(2-w) + k_{pd}\left(gh_{ddot}(2-w) - g(2-w)\frac{d}{dt}h(t) + wv(t)\frac{d}{dt}v(t) + \frac{gh_d(2-w) - g(2-w)h(t) - 0.5v_d^2w + 0.5wv^2(t)}{k_{tc}}\right) + x_{intKE} + x_{intsebdot} + \frac{gh_d(2-w) - g(2-w)h(t) - 0.5v_d^2w + 0.5wv^2(t)}{k_{tc}}}{gv(t)}\right)}{m}$$

$$h(t) = -v(t)\sin\left(\theta_0 - \frac{gh_{ddot}(2-w) + k_{pd}\left(gh_{ddot}(2-w) - g(2-w)\frac{d}{dt}h(t) + wv(t)\frac{d}{dt}v(t) + \frac{gh_d(2-w) - g(2-w)h(t) - 0.5v_d^2w + 0.5wv^2(t)}{k_{tc}}\right) + x_{intKE} + x_{intsebdot} + \frac{gh_d(2-w) - g(2-w)h(t) - 0.5v_d^2w + 0.5wv^2(t)}{k_{tc}}}{gv(t)}\right)$$

$$v_d = 0$$

$$h_d = 0$$

$$x_{intsebdot} = k_i\left(-g(2-w)\frac{d}{dt}h(t) + wv(t)\frac{d}{dt}v(t) + \frac{gh_d(2-w) - g(2-w)h(t) - 0.5v_d^2w + 0.5wv^2(t)}{k_{tc}}\right)$$

$$x_{intKE} = \frac{w\left(-0.5v_d^2 + 0.5v^2(t)\right)}{k_{tc}}$$

$$x_{intthr} = \frac{k_i\left(gh_d - gh(t) + 0.5v_d^2 - 0.5v^2(t)\right)}{k_{tc}\left(g\dot{h}_{cmax} + g\dot{h}_{smin}\right)}$$

Equilibrium points

Find the equilibrium points of the system:

- It is basically the height and altitude setpoint.
- The integrators state capture the trim.

$$v(t) = v_d$$

$$h(t) = h_d$$

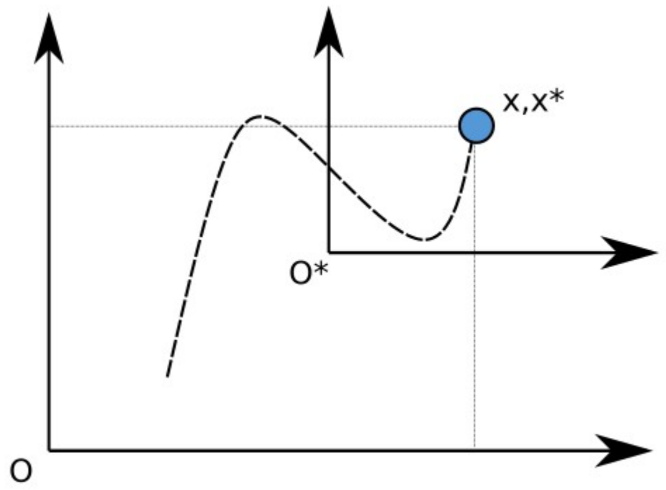
$$x_{intthr} = \frac{0.0025\left(245.0 C_{drag} S v_d^2 - 400.0 F_t thr_{trim} - 3924.0 \sin\left(\theta_0\right)\right)}{F_t}$$

$$x_{ipitch} = 9.81 \theta_0 v_d$$

Translated system

Translate the system so that the new origin is placed onto the trim point.

Most stability analysis techniques assume that the desired stability is for the origin.



Linear analysis

Linear analysis

- Linearization of the autonomous system
- Performed at the trim point.
- Gives an accurate stability assessment.

Linear analysis

$$\begin{bmatrix} -\frac{0.00067\left(-60.0k_{pd}+3.5\cdot10^{2}k_{tc}+1.4\cdot10^{2}\right)}{k_{tc}} & \frac{0.00067\left(3.0k_{tc}(-9.8k_{pd}-9.8)-2.0\cdot10^{2}k_{tc}-1.0\cdot10^{2}k_{td}\right)}{k_{tc}^{2}} & 0.2 & 0.002 \\ \frac{2.0k_{pd}-0.017k_{tc}+2.0}{k_{tc}} & \frac{-1.0k_{pd}-1.0}{k_{tc}} & 0 & 0.1 \\ -\frac{0.68k_{i}}{k_{tc}} & -\frac{0.33k_{i}}{k_{tc}} & 0 & 0 \\ \frac{20.0(k_{i}+1.0)}{k_{tc}} & -\frac{9.8k_{i}}{k_{tc}} & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -0.25 & -0.027 & 0.2 & 0.002 \\ 0.36 & -0.18 & 0 & 0.1 \\ -0.034 & -0.017 & 0 & 0 \\ 4.3 & -0.49 & 0 & 0 \end{bmatrix} \xrightarrow{\text{eigenvalues}}$$

$$-0.016 - 0.25i$$

$$-0.31 + 2.9 \cdot 10^{-65} i$$

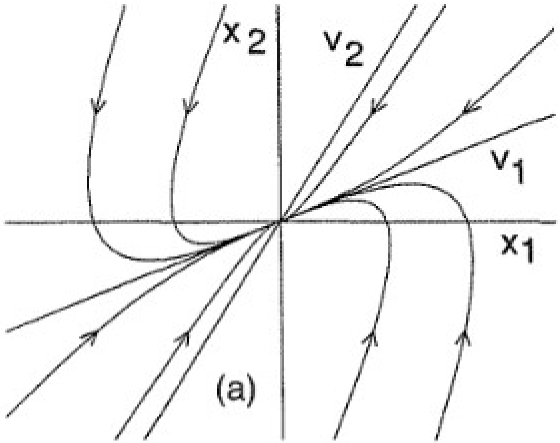
$$-0.097 + 3.6 \cdot 10^{-65} i$$

$$-0.016 + 0.25i$$

Intermission

Qualitative behaviour of linear systems

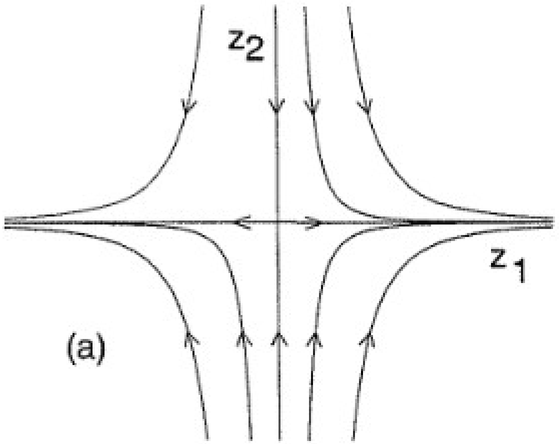
1) Stable node



Intermission

Qualitative behaviour of linear systems

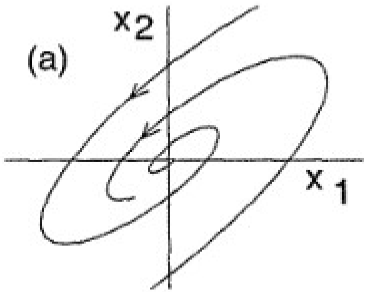
1) Stable node
2) Saddle point



Intermission

Qualitative behaviour of linear systems

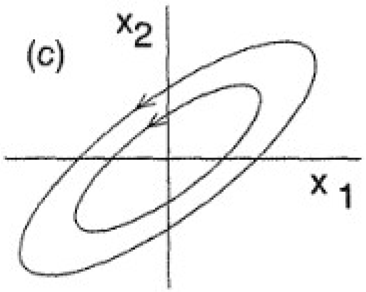
1) Stable node
2) Saddle point
3) Focus point



Intermission

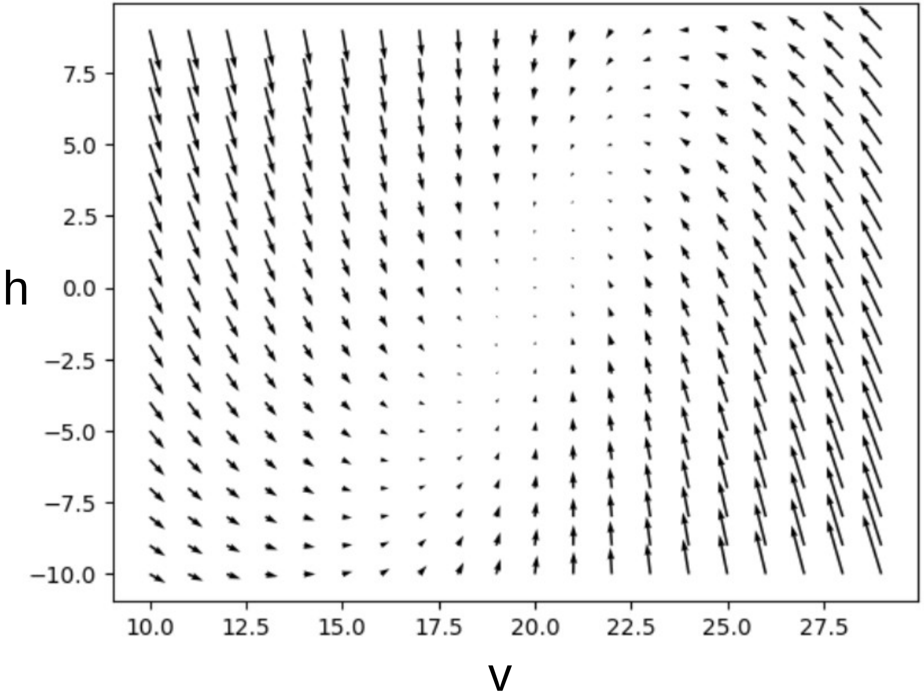
Qualitative behaviour of linear systems

1) Stable node
2) Saddle point
3) Focus point
4) Center
5) … and some more



Phase portrait

- It captures the trajectory of the system via propagation.
- Numerical values needed.
- It is expected to reflect what the linear analysis pointed.



Phase portrait

- It captures the trajectory of the system via propagation.
- Numerical values needed.
- It is expected to reflect what the linear analysis pointed.
- It is not the same as a linear portrait

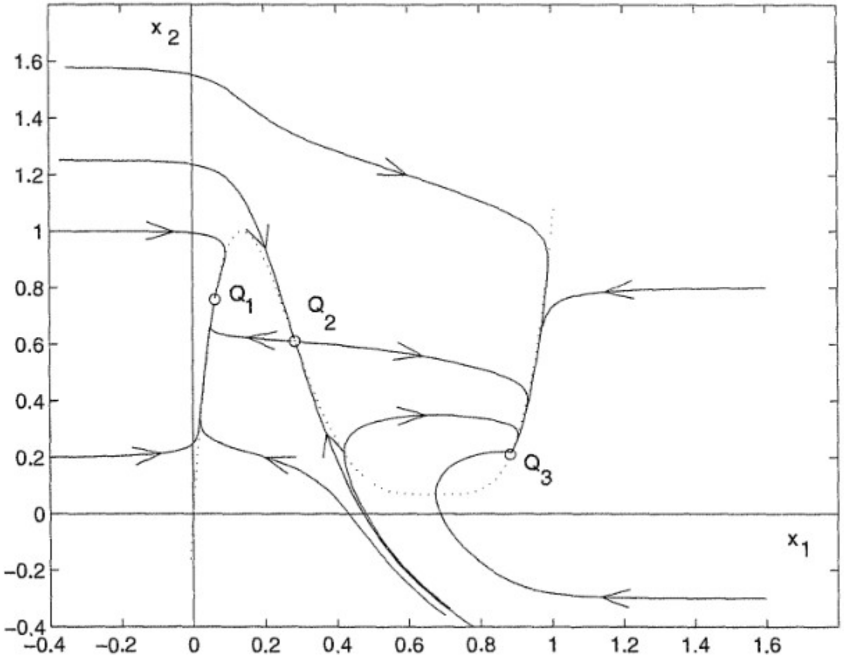


Figure 2.13: Phase portrait of the tunnel-diode circuit of Example 2.1.

Lyapunov's theorem

It is a sufficient proof that the autonomous system is stable.

It is the common source to derive conditions for stability.

Finding a suitable and handy Lyapunov function is the hardest part.

Let: $V(x) \geq 0 \forall x \in D$

If: $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0 \forall x \in D$

then f(x) is stable.

Nonlinear stability

A naive attempt at a Lyapunov function is often doomed:

Let V(x) = $h_{star}^2 + v_{star}^2 + x_{intthr*}^2 + x_{ipitch*}^2$

This yields dV(x)/dt = $2h(t)\frac{d}{dt}h(t) + 2v(t)\frac{d}{dt}v(t) + 2x_{intthr}(t)\frac{d}{dt}x_{intthr}(t) + 2x_{ipitch}(t)\frac{d}{dt}x_{ipitch}(t)$

or equivalently

$$-2h_{star}\left(v_{star}+20\right)\sin\left(0.0174444444444444 - \frac{0.101936799184506\left(\frac{k_{pd}\left(-9.81h_{star}+0.5(v_{star}+20)^2-200.0\right)}{k_{tc}} + x_{ipitch*} + 3.4226 + \frac{-9.81h_{star}+0.5(v_{star}+20)^2-200.0}{k_{tc}}\right)}{v_{star}+20}\right) + \frac{0.0679578661230037k_i x_{intthr*}\left(-9.81h_{star} - 0.5\left(v_{star}+20\right)^2 + 200.0\right)}{k_{tc}} +$$

$$\frac{2v_{star}\left(-\frac{1.66666666666667h_{star}}{k_{tc}} + 5x_{intthr*} - 0.147\left(v_{star}+20\right)^2 + 9.81\sin\left(\frac{0.101936799184506\left(\frac{k_{pd}\left(-9.81h_{star}+0.5(v_{star}+20)^2-200.0\right)}{k_{tc}} + x_{ipitch*} + 3.4226 + \frac{-9.81h_{star}+0.5(v_{star}+20)^2-200.0}{k_{tc}}\right)}{v_{star}+20}\right) + 58.6288786792576 + \frac{0.169894665307509\left(-9.81h_{star} - \frac{9.81h_{star}k_{td}}{k_{tc}} - 0.5(v_{star}+20)^2 + 200.0\right)}{k_{tc}}\right)}{25} +$$

$$2x_{ipitch*}\left(\frac{k_i\left(-9.81h_{star} + 0.5\left(v_{star}+20\right)^2 - 200.0\right)}{k_{tc}} + \frac{0.5\left(v_{star}+20\right)^2 - 200.0}{k_{tc}}\right)$$

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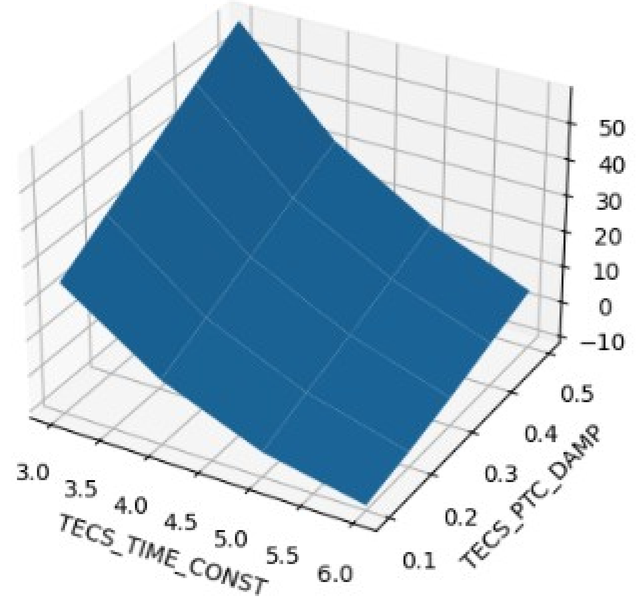
Yeah… no.

Numerical approximations

If you can’t solve them…
calculate them.

```python
def max_v(k_tc_val, k_pd_val):
    X, Y = np.meshgrid(np.arange(10,30,1), np.arange(-10,10,1))
    V = ufuncify((h_star, v_star), V_l_dot_flat.subs({k_tc: k_tc_val, k_pd: k_pd_val}))
    print(V)
    return np.max(np.max(V(X,Y)))

max_v_vec = np.vectorize(max_v)
X, Y = np.meshgrid(np.arange(3,7,1), np.arange(0.1,0.6,0.1))
ax = plt.figure().add_subplot(projection='3d')
ax.plot_surface(X, Y, max_v_vec(X, Y))
ax.set(xlabel='TECS_TIME_CONST', ylabel='TECS_PTC_DAMP', zlabel='Z')
```



Results

1) TECS is proven to be stable with the given assumptions.

That’s a good initial verification for the proposed methodology.

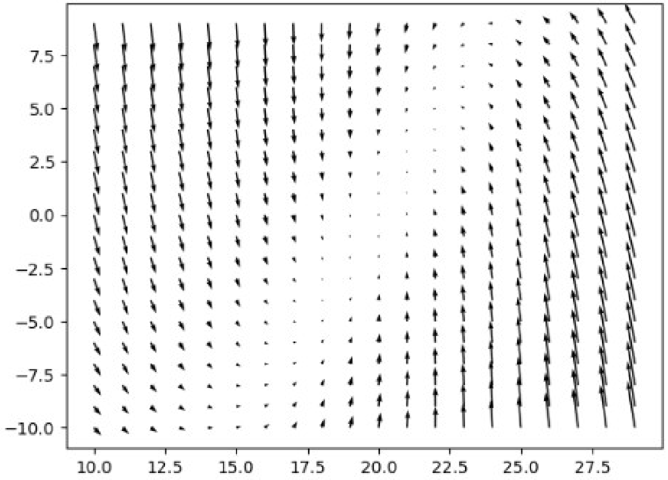
Results

1) TECS is proven to be stable with the given assumptions.

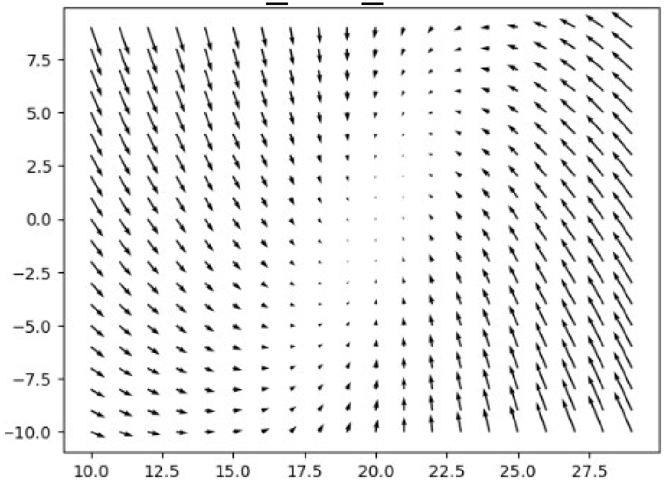
We had to use numerical values to progress on this analysis.

2) Too many variables to keep track of and reconcile.

TECS_TIME_CONST=3
TECS_PTC_DAMP=0.5



TECS_TIME_CONST=6
TECS_PTC_DAMP=0.3



Results

1) TECS is proven to be stable with the given assumptions.
2) Too many variables to keep track of and reconcile.
3) Some parameter identification is necessary.

It probably doesn’t need to be very accurate, so it might be applicable to whole vehicle classes.

Results

1) TECS is proven to be stable with the given assumptions.
2) Too many variables to keep track of and reconcile.
3) Some parameter identification is necessary.
4) Sympy!

All of the mathematics presented were carried out with the computer algebra Python package sympy.

The Jupyter notebook can be found at https://github.com/Georacer/ap_controller_analysis

Future work

Study more!

1) Employ useful theorems that facilitate analysis.
2) Employ passivity theory in systems with independent components.

Thank you!

If you indulged me this far, you have my sincere thanks!

Questions?