BCS-BEC crossover

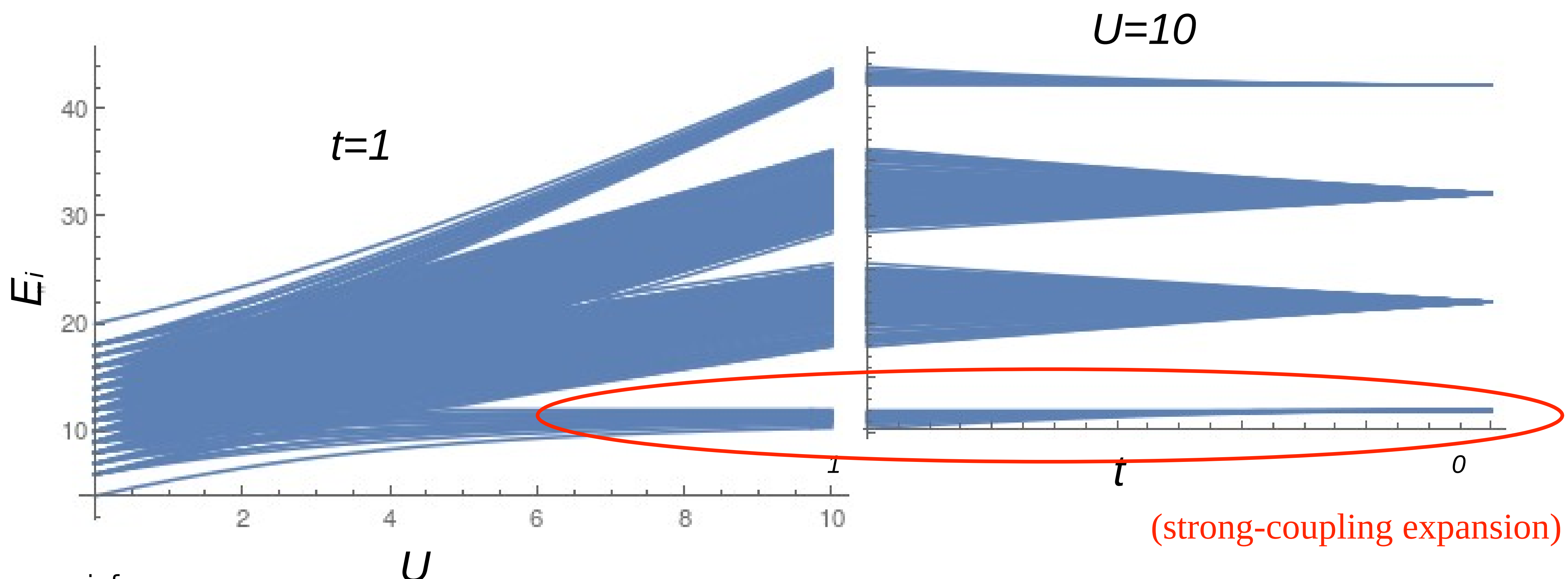
Formation of local moments

$$H = t \sum_{\langle ij \rangle, \sigma} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

$$\boxed{\tilde{H} = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j}$$

$$J = \frac{4t^2}{U}$$

Spectrum of eigenenergies (6-site Hubbard model):
Sector (3,3)

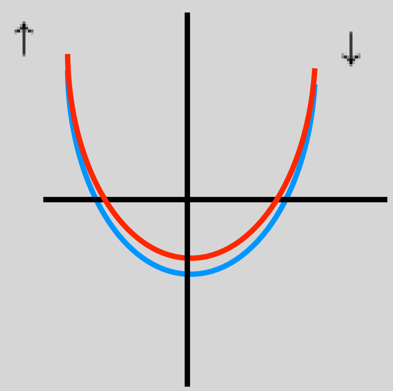


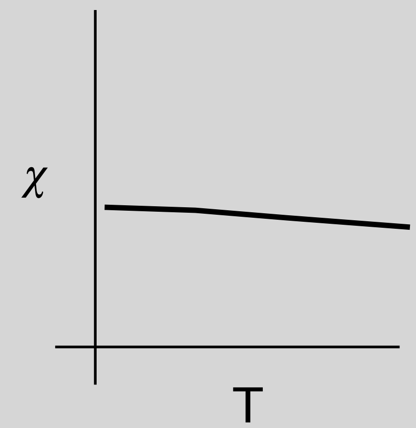
for more info:
ps://tube1.it.tuwien.ac.at/my-account/video-playlists/1faaf358-0774-48c2-aa27-8be8dd3440f3

Beekman, Rademaker and van Wezel, SciPost Phys. Lect. Notes 11 (2019)

ments

Pauli susceptibility:



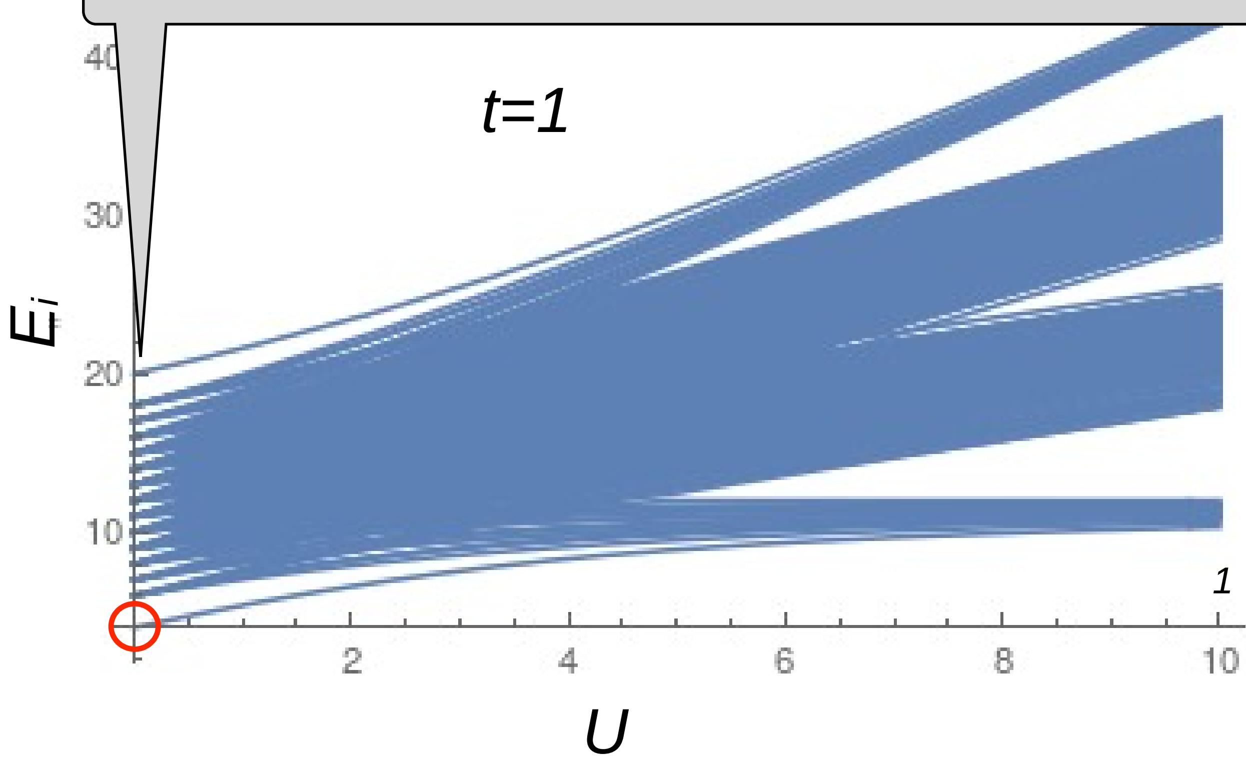


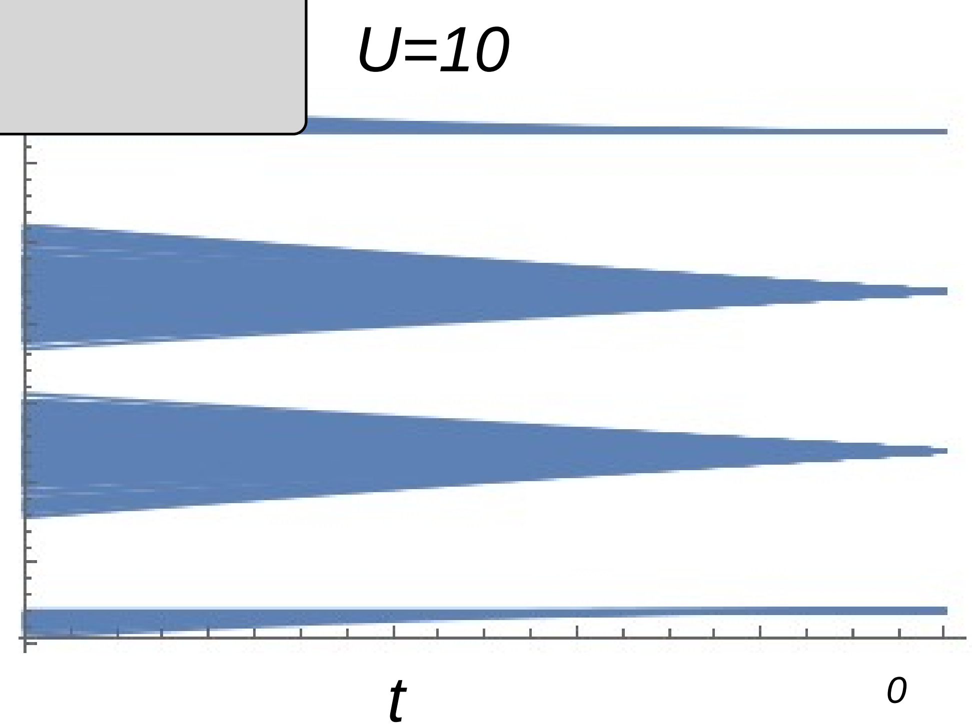
$$\tilde{H} = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

$$J = \frac{4t^2}{U}$$

Spe

Se

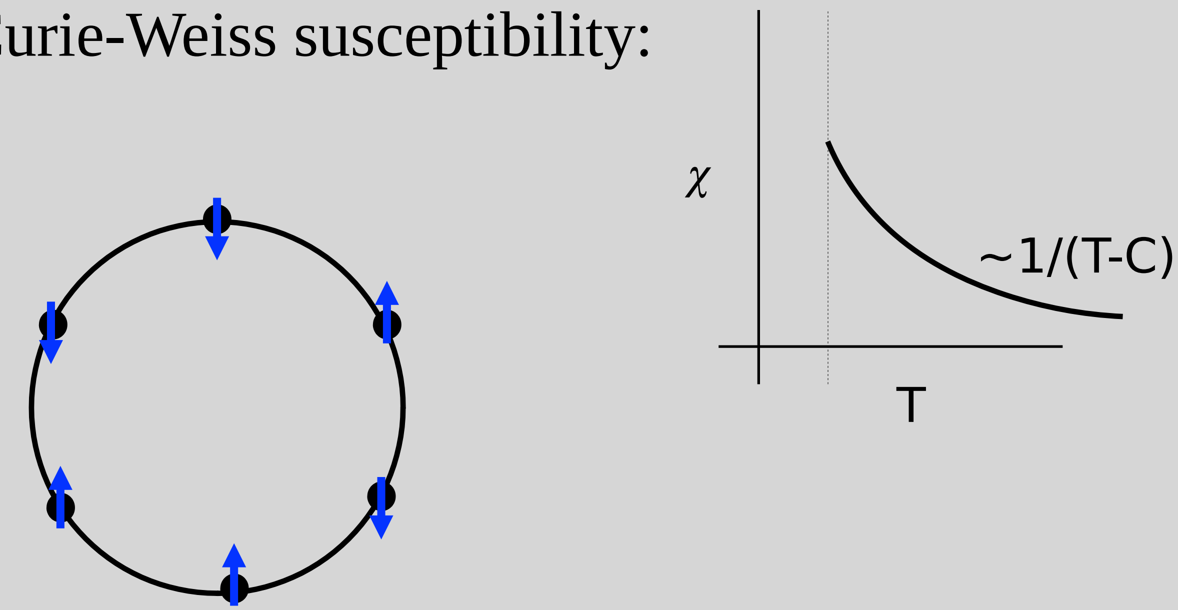




(strong-coupling expansion)

ments

Curie-Weiss susceptibility:

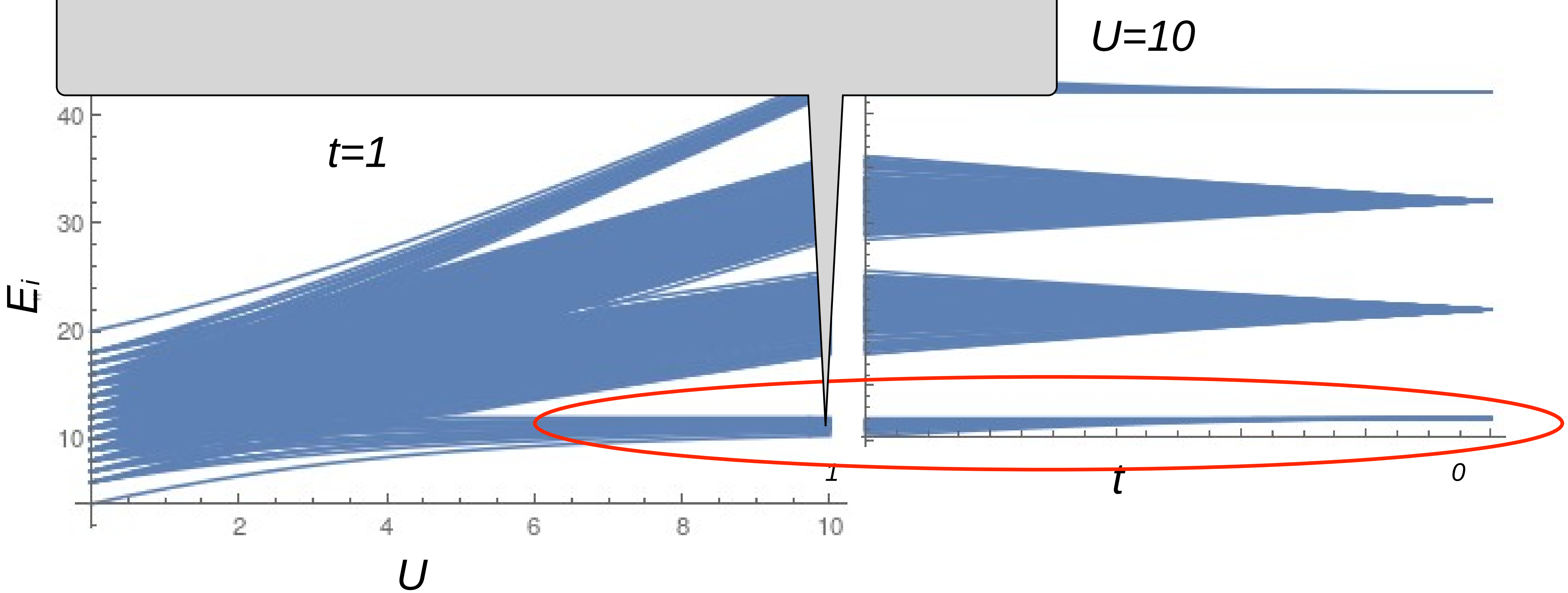


$$\tilde{H} = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

$$J = \frac{4t^2}{U}$$

Spe

Se



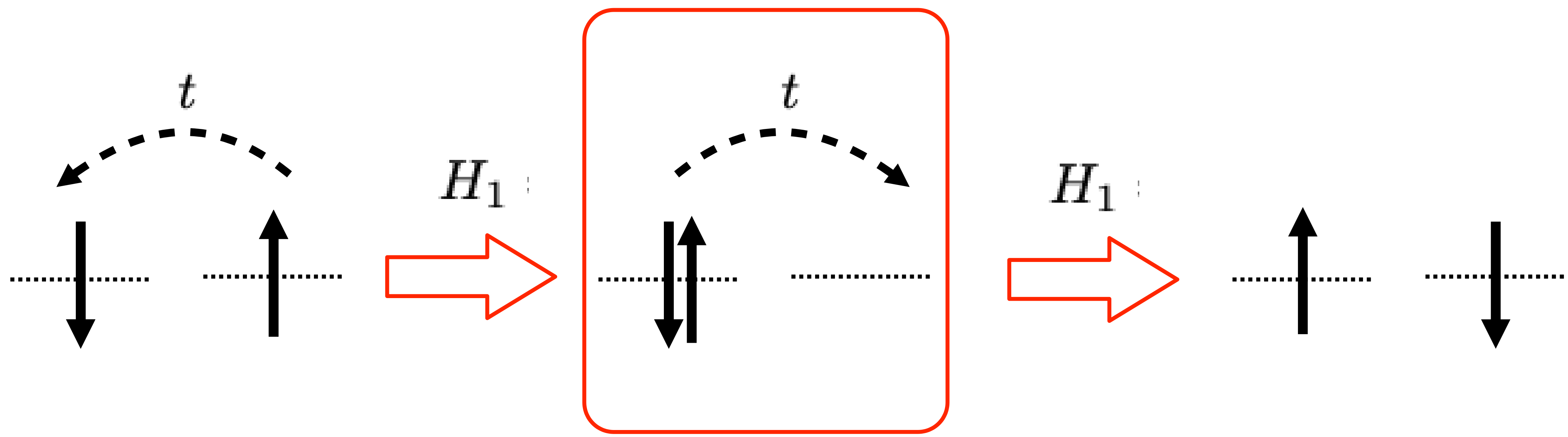
(strong-coupling expansion)

Formation of local moments

$$H = t\sum_{\langle ij\rangle,\sigma} c^\dagger_{i\sigma}c_{j\sigma} + U\sum_i n_{i\uparrow}n_{i\downarrow} \quad \longrightarrow \quad \boxed{\tilde{H} = J\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j}$$

$$J = \frac{4t^2}{U}$$

U is large (U>>t)

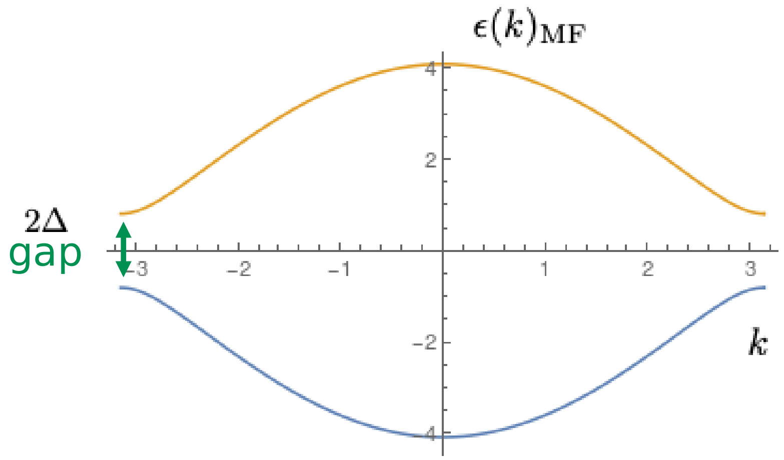


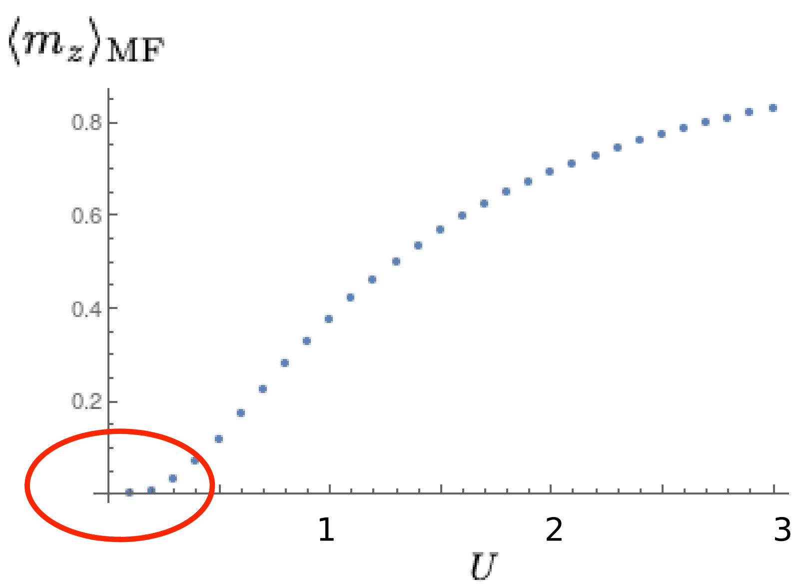
$$\langle\alpha|H_{\text{eff}}|\beta\rangle = \frac{1}{2}\sum_\gamma\left(\frac{\langle\alpha|H_1|\gamma\rangle\langle\gamma|H_1|\beta\rangle}{E_\alpha - E_\gamma} + \frac{\langle\alpha|H_1|\gamma\rangle\langle\gamma|H_1|\beta\rangle}{E_\beta - E_\gamma}\right)$$

BCS-BEC crossover

$$H = -\mu \sum_{i,\sigma} c^\dagger_{i\sigma} c_{i\sigma} + t \sum_{ij,\sigma} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} \qquad M_i = (-1)^i M$$

weak-coupling limit:

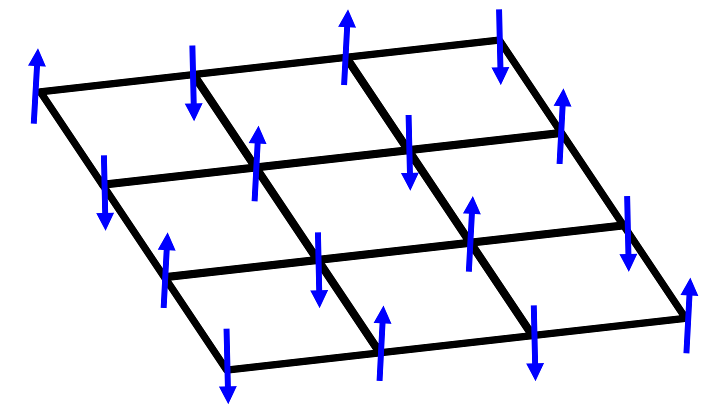


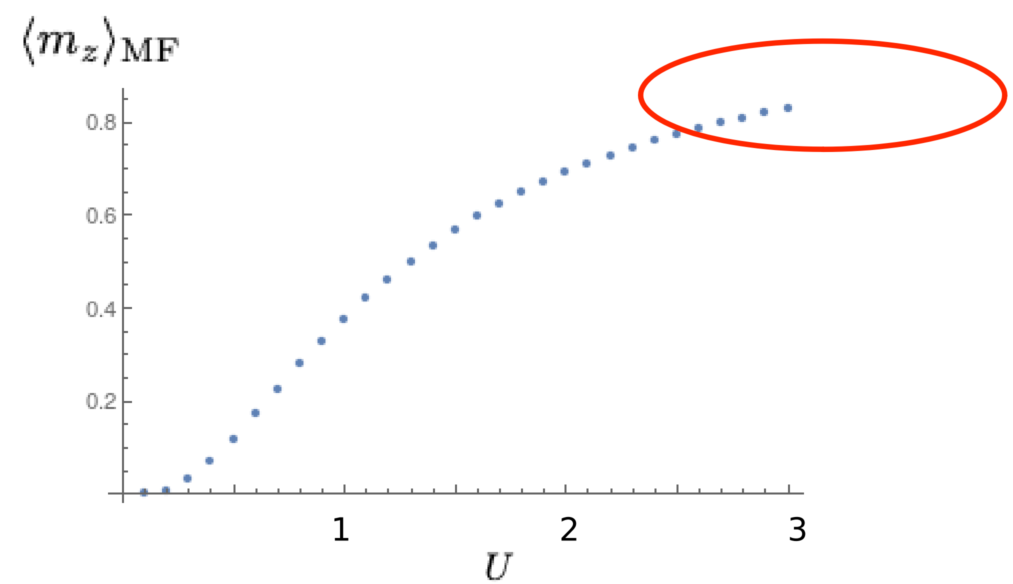


$$\boxed{H_{\mathrm{MF}} = H_0 - U \sum_i M_i m_{iz}}$$

$$\boxed{M_i = \langle m_{iz} \rangle_{\mathrm{MF}}}$$

strong-coupling limit:





$$\tilde{H} = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

BCS-BEC crossover

$$H = -\mu \sum_{i,\sigma} c^\dagger_{i\sigma} c_{i\sigma} + t \sum_{ij,\sigma} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} \qquad M_i = (-1)^i M$$

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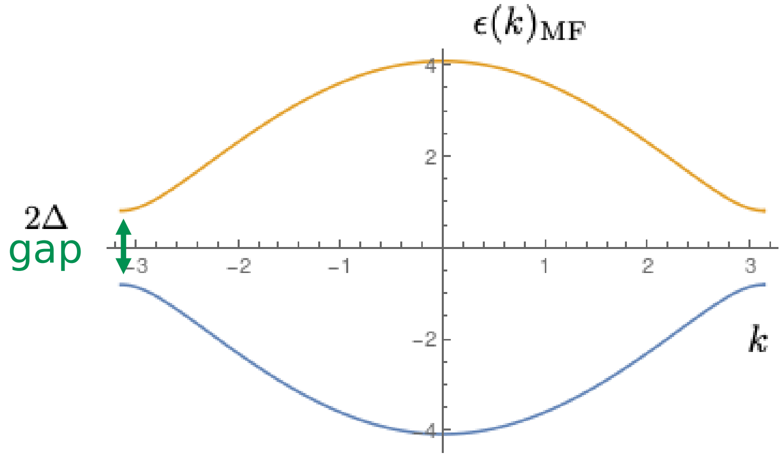

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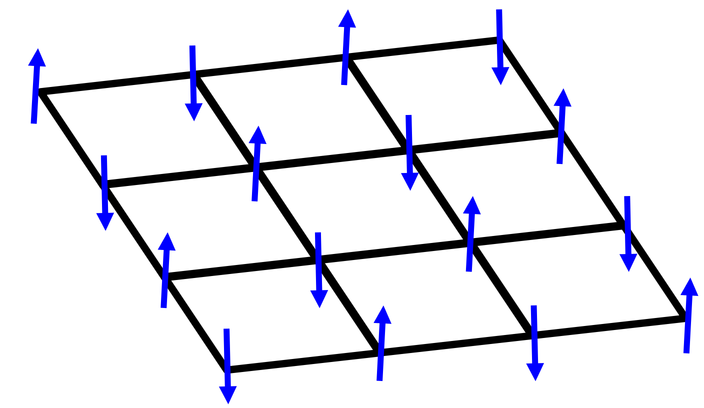

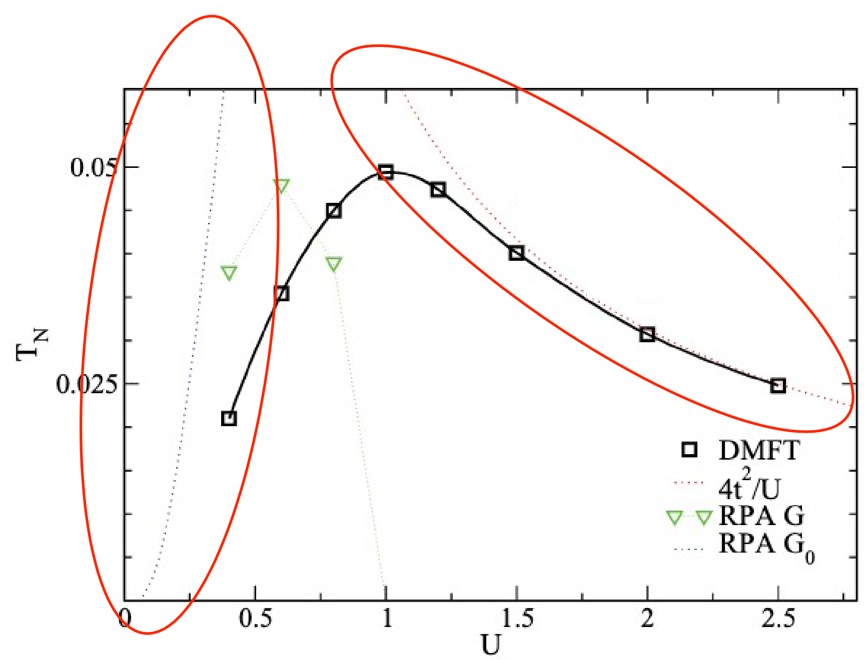

$$T_N \sim N_q(0) U \sim \frac{U}{D}$$

$$T_N \sim \frac{t^2}{U}$$

BCS-BEC crossover

$$H = -\mu \sum_{i,\sigma} c^\dagger_{i\sigma} c_{i\sigma} + t \sum_{ij,\sigma} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} \qquad M_i = (-1)^i M$$

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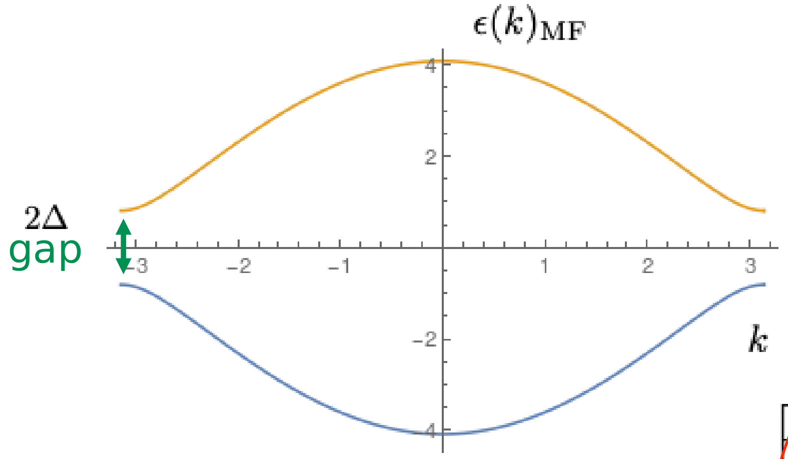
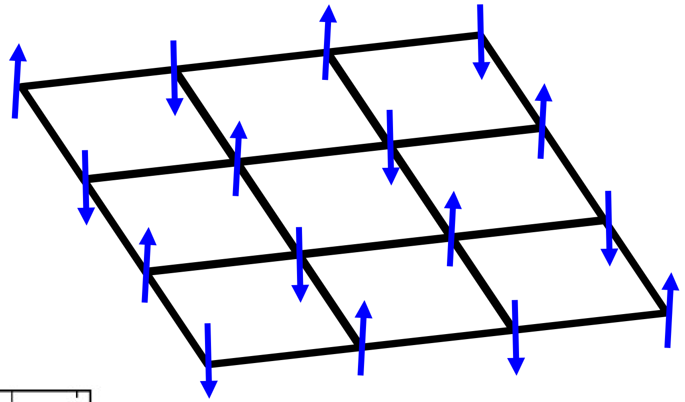
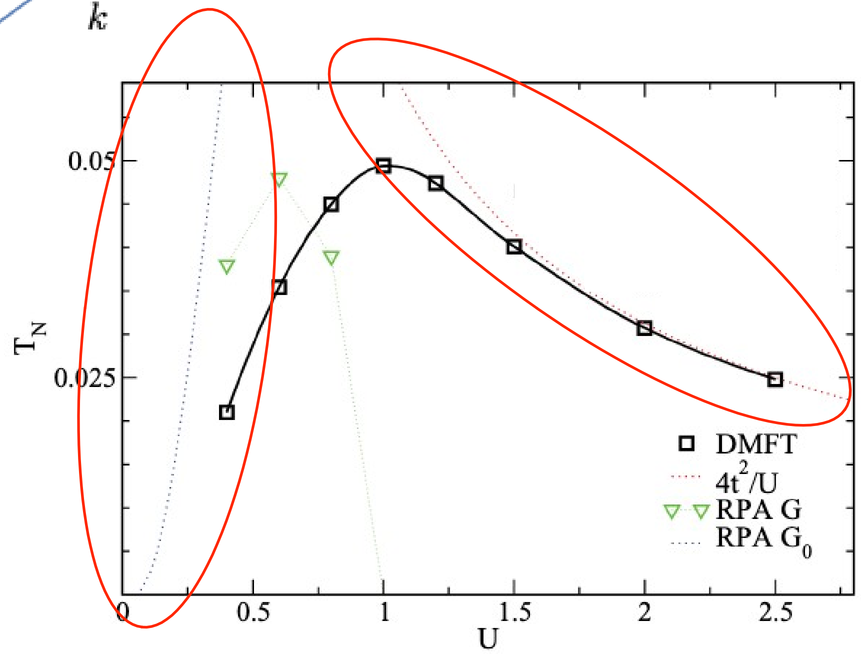


strong-coupling limit:





Bethe-Salpeter equation: $\chi = \chi_0 + \chi_0 \Gamma \chi \qquad \chi = (I - \chi_0 \Gamma)^{-1} \chi_0$

BCS-BEC crossover

$$H = -\mu \sum_{i,\sigma} c^\dagger_{i\sigma} c_{i\sigma} + t \sum_{ij,\sigma} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_{i} c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} \qquad M_i = (-1)^i M$$

weak-

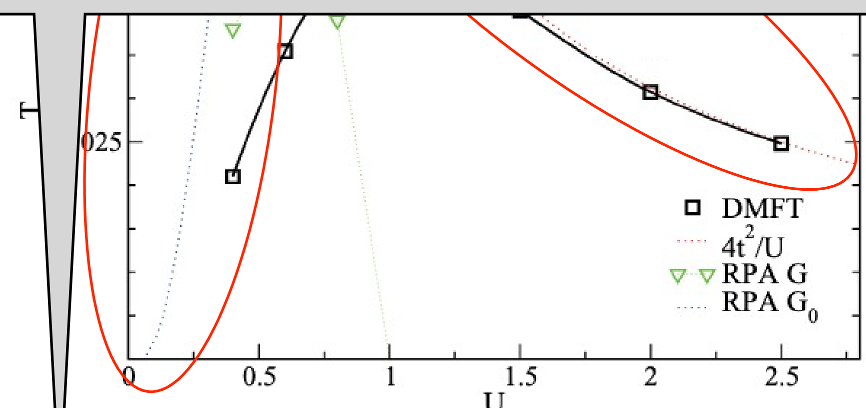
$$\chi(Q; q_1, q_2) = \chi_0(Q; q_1, q_2) + \chi_0(Q; q_1, \bar{k}_1)\Gamma(Q; \bar{k}_1, \bar{k}_2)\chi(Q; \bar{k}_2, q_2)$$

physical susceptibility: $\chi(Q) = T \sum_{q_1, q_2} \chi(Q; q_1, q_2)$

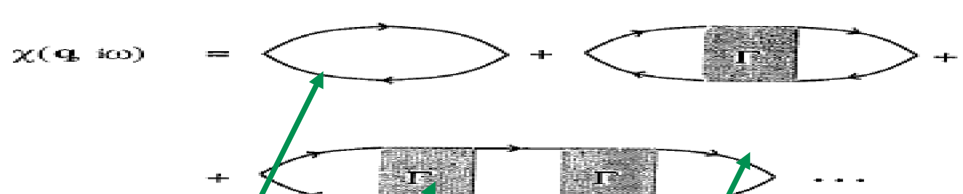
'bubble' $\chi_0(Q; q_1, q_2) = \delta_{q_2, q_1+Q} G(q_1) G(q_1 + Q)$

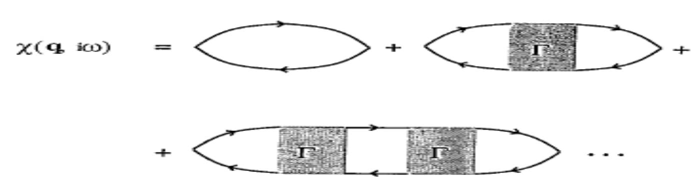
frequency & momentum $q = (\omega, \mathbf{q})$

static response $Q = (0, \mathbf{Q})$



Bethe-Salpeter equation: $\chi = \chi_0 + \chi_0 \Gamma \chi \qquad \chi = (I - \chi_0 \Gamma)^{-1} \chi_0$





electron

hole

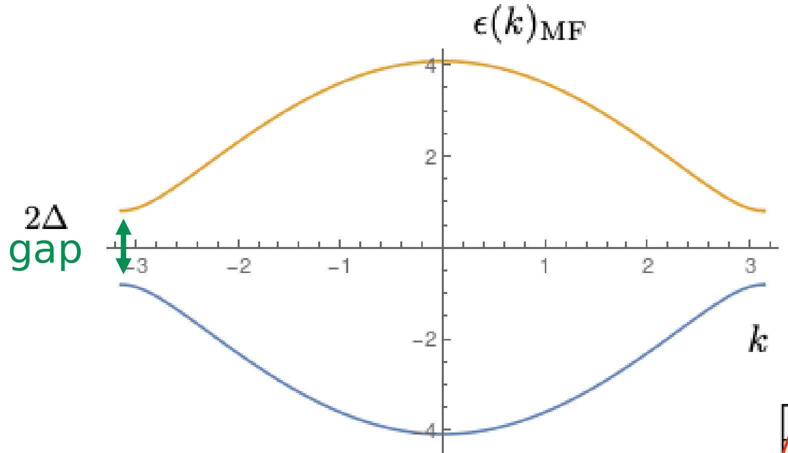
e-h interaction

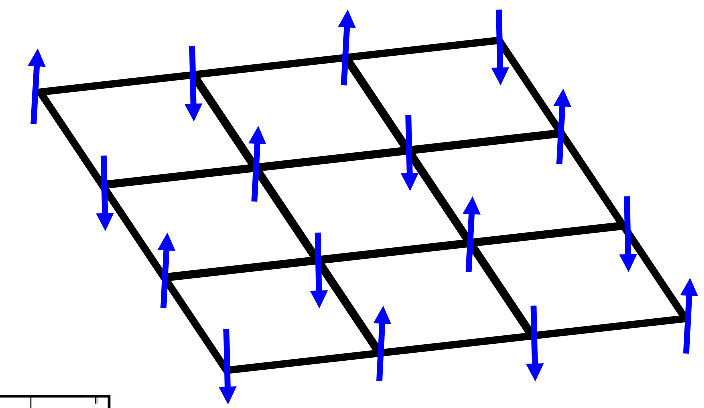
BCS-BEC crossover

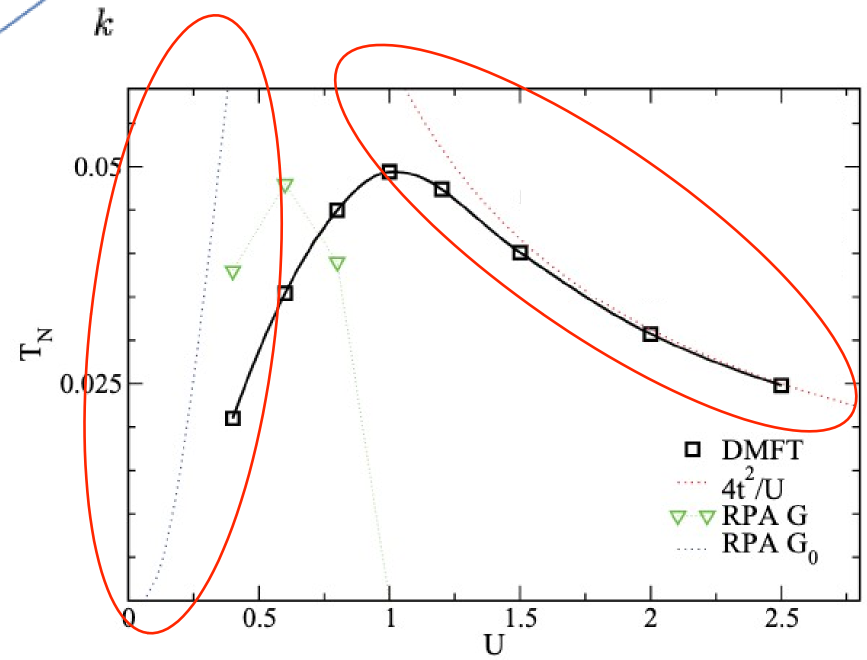
$$H = -\mu \sum_{i,\sigma} c^\dagger_{i\sigma} c_{i\sigma} + t \sum_{ij,\sigma} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_{i} c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} \qquad M_i = (-1)^i M$$

weak-coupling limit:

strong-coupling limit:







Bethe-Salpeter equation: $\chi = \chi_0 + \chi_0 \Gamma \chi \qquad \chi = (I - \chi_0 \Gamma)^{-1} \chi_0$

RPA (random phase approximation)

$\Gamma \approx U$

$\chi_0(T_c) U = I$

Transition is driven by the T-dependence of the 'bubble'

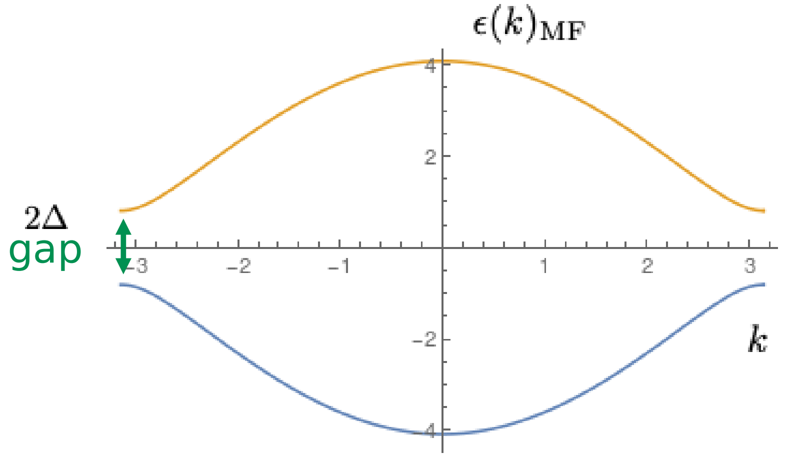
Atomic vertex

Transition is driven by the T-dependence of the vertex

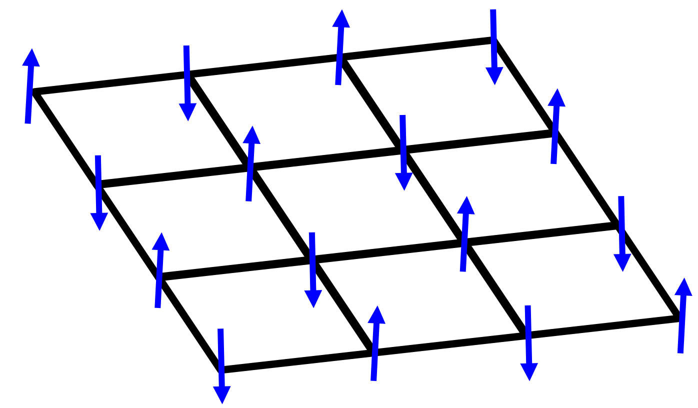
BCS-BEC crossover

$$H = -\mu \sum_{i,\sigma} c^\dagger_{i\sigma} c_{i\sigma} + t \sum_{ij,\sigma} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} \qquad M_i = (-1)^i M$$

weak-coupling limit:



strong-coupling limit:



Normal phase: excitation - almost independent quasiparticles
=> broad peaks, dispersion apparent only close to the transition

excitation - strongly bound pairs of QP
=> exist well above T_c, the formation T and transition T are very different
Curie susceptibility 1/T

Ordered phase: only 1P states close to FL affected
=> large spatial extent of the 'condensed particles'

excitations condense = become coherent
(local moments point in fixed direction)

Superconductivity: Cooper pairs absent in normal phase
CPs have large spatial extent
(many CPs overlap)

Mott antiferromagnetism, exciton condensation

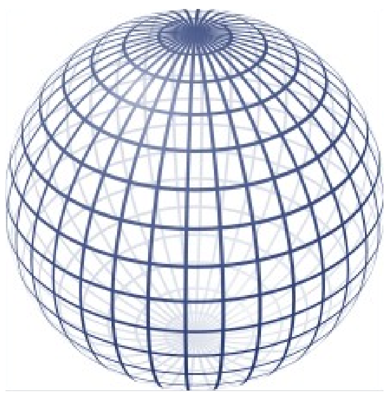
Spontaneous symmetry breaking

3D magnet

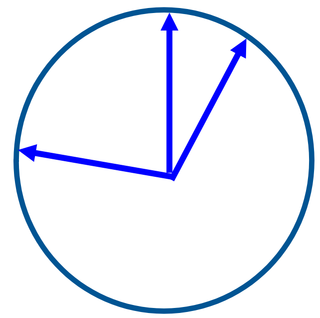
planar magnet

Symmetry group G of the Hamiltonian:

$SO(3) \sim SU(2)$

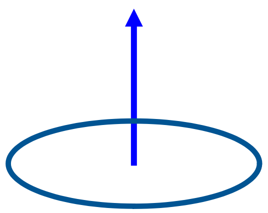


$SO(2)=U(1)$

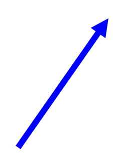


Order parameter and residual symmetry (symmetry of the ground state) H:

$U(1)$



$\{I\}$



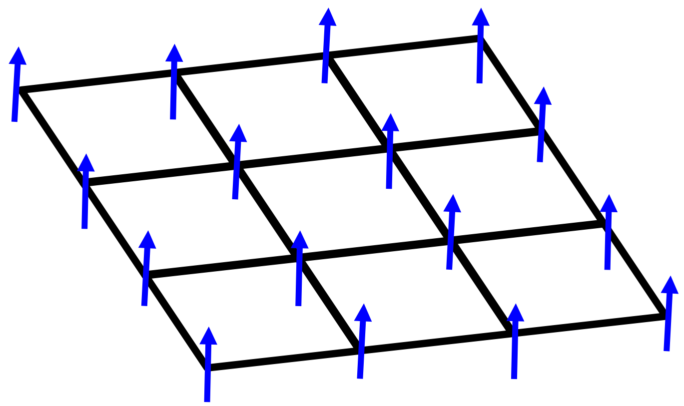
Set of all possible order parameters G/H (coset of G not necessarily a group):

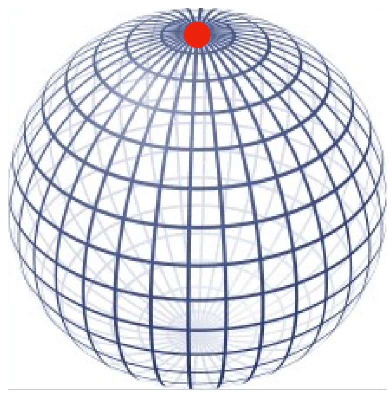
S_2

$U(1)=S_1$

Spontaneous symmetry breaking

S=1

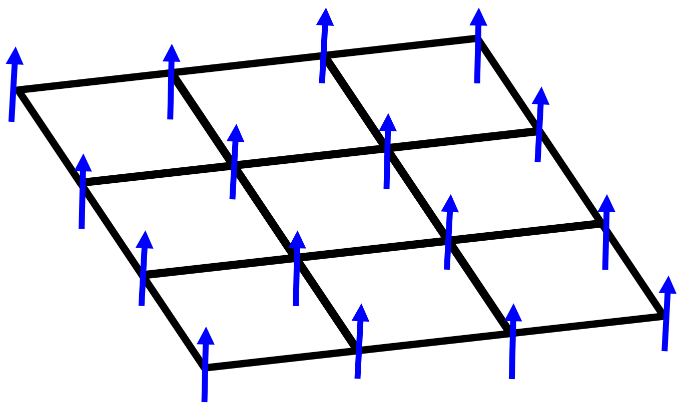


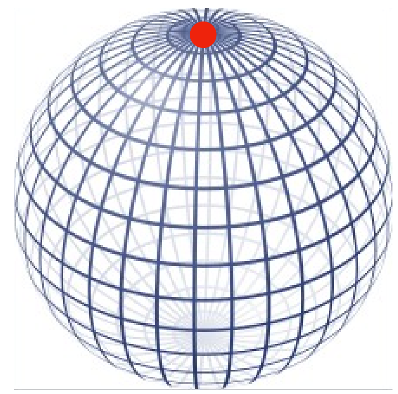


$$H = -J \sum_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$$

Spontaneous symmetry breaking

S=1





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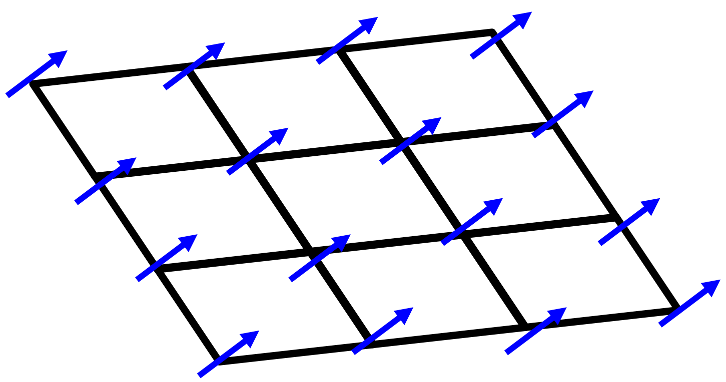
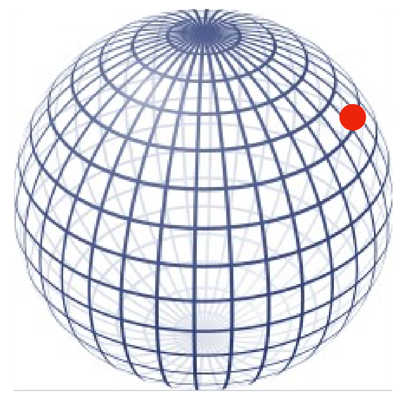


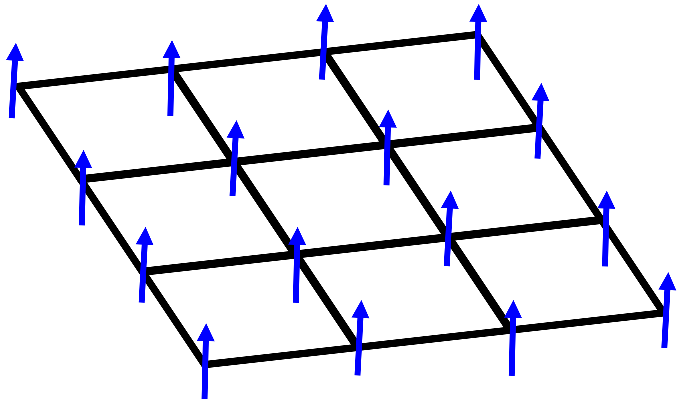


The same order (a different ground state obtained by an operation from *G/H* - orthogonal and degenerate with the above state).

Can we get a different ordered state (assuming uniform order)?

Spontaneous symmetry breaking

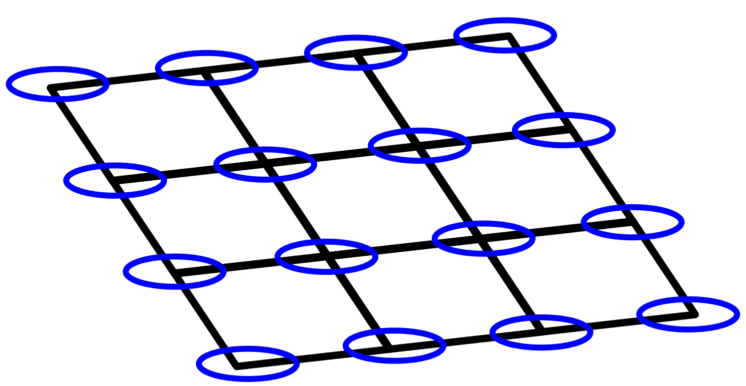
S=1



$$\prod_i |1\rangle_i$$

$$H = -J\sum_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$$

Yes!



$$\prod_i |0\rangle_i$$

$$H = -J\sum_{ij} \left[\mathbf{S}_i \cdot \mathbf{S}_j + 4(\mathbf{S}_i \cdot \mathbf{S}_j)^2 - 2(\mathbf{S}_i \times \mathbf{S}_j)^2\right]$$

There is no *SU(2)* rotation that takes |1> to |0>

Dimensionality

Examples: correlations in everyday life

any D - Coulomb blockade: “in a crowded gym you are stuck
at one station”

Dimensionality

Examples: correlations in everyday life

1D - complete loss of individual identity

Dimensionality

Examples: correlations in everyday life

2D - fading through a crowd “you become heavy”

Density of states

Definition:
$$D(\omega) = \frac{1}{N}\sum_{n,\mathbf{k}} \delta(\omega - \epsilon_{\mathbf{k}n})$$
$$= \frac{\Omega}{(2\pi)^3}\sum_n \int_{\mathrm{BZ}} d^3k \quad \delta(\omega - \epsilon_n(\mathbf{k})) = \frac{\Omega}{(2\pi)^3}\sum_n \int_{\epsilon_n(\mathbf{k})=\omega} \frac{dS}{|\nabla_k \epsilon_n(\mathbf{k})|}$$

Calculation:

- histogram (brute force summation over discrete k-mesh)

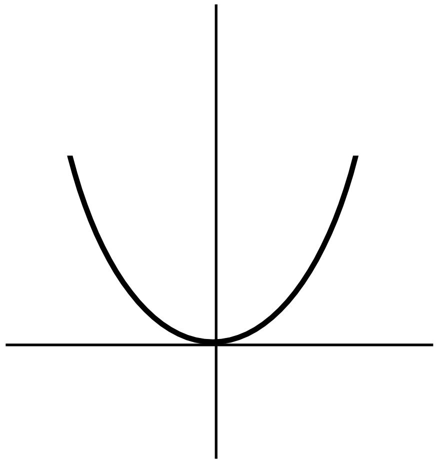
  'trick': use $\delta(\omega-\epsilon) = -\frac{1}{\pi}\,\mathrm{Im}\lim_{\Gamma\to 0}\frac{1}{\omega-\epsilon+i\Gamma}$ and perform the calculation for small finite Γ .

- analytic calculation - band edges (van Hove singularities)

  The states with $\epsilon(\mathbf{k})-\epsilon_0 \sim k^2$ and $\epsilon(\mathbf{k})-\epsilon_0 < \omega$ live in a k-sphere of radius $\omega^{1/2}$

  This gives rise to characteristic behavior close to the band edge:

	$N(\omega)$	$D(\omega)$
1D	$\omega^{1/2}$	$\omega^{-1/2}$
2D	ω	1
3D	$\omega^{3/2}$	$\omega^{1/2}$



Density of states

Definition: $$D(\omega)=\frac{1}{N}\sum_{n,\mathbf{k}}\delta(\omega-\epsilon_{\mathbf{k}n})$$
$$=\frac{\Omega}{(2\pi)^3}\sum_n\int_{\mathrm{BZ}}d^3k\quad\delta(\omega-\epsilon_n(\mathbf{k}))=\frac{\Omega}{(2\pi)^3}\sum_n\int_{\epsilon_n(\mathbf{k})=\omega}\frac{dS}{|\nabla_k\epsilon_n(\mathbf{k})|}$$

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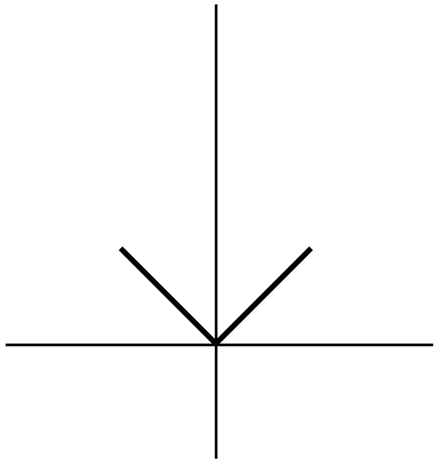
- analytic calculation - band edges (van Hove singularities)

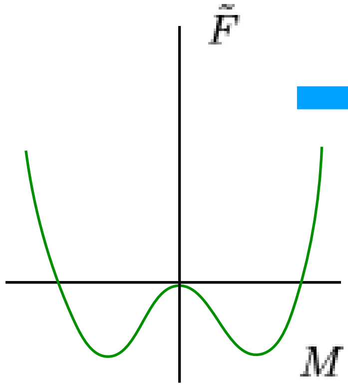
  The states with $\epsilon(\mathbf{k})-\epsilon_0\sim k$ and $\epsilon(\mathbf{k})-\epsilon_0<\omega$ live in a k-sphere of radius ω

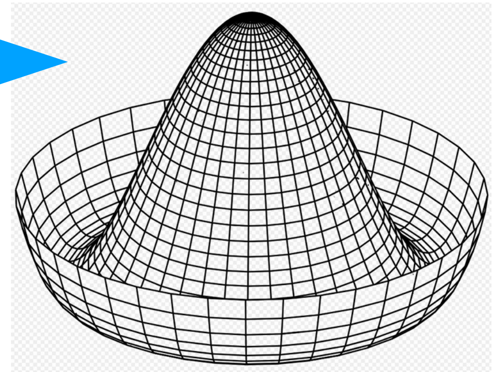
  This gives rise to characteristic behavior close to the band edge:

	$N(\omega)$	$D(\omega)$
1D	ω^1	1
2D	ω^2	ω
3D	ω^3	ω^2



Spontaneous breaking of continuous symmetry

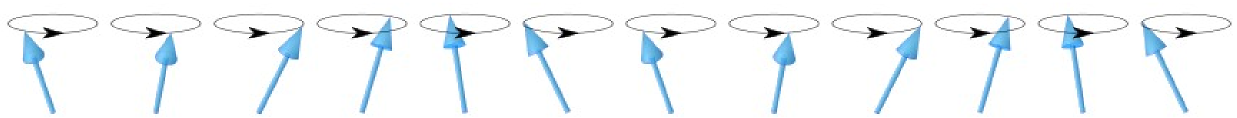


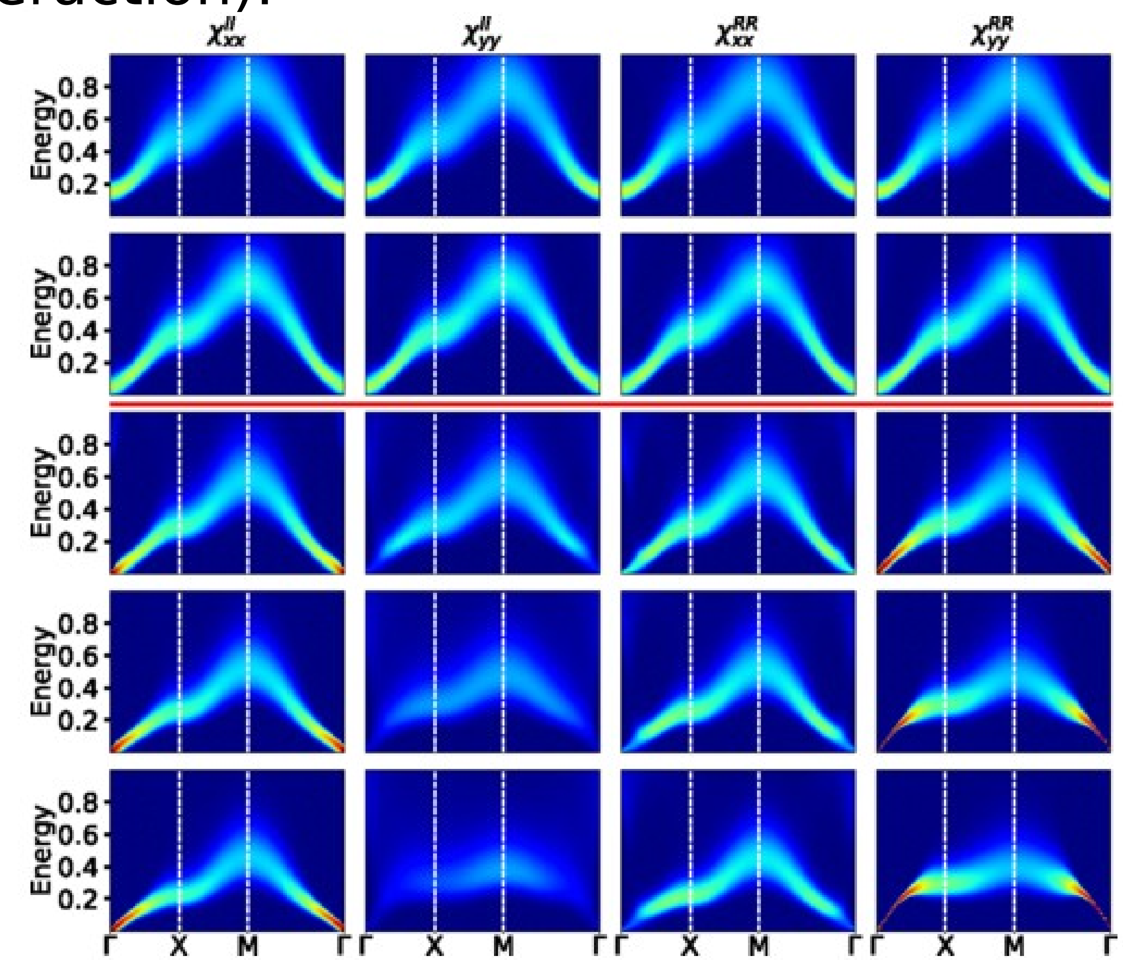


Continuous symmetries: space isotropy (orbital or spin rotations),
space isotropy (translation), gauge symmetries (charge conservation)

Goldstone mode (in systems with short-range interaction):

Long-wave length rotations of the order parameter cost vanishingly low energy.





2 linear modes in 2-orbital Hubbard model (exciton condensate phase)

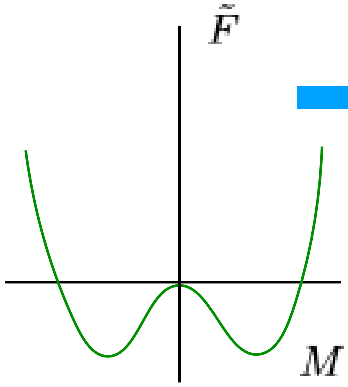
Transition :

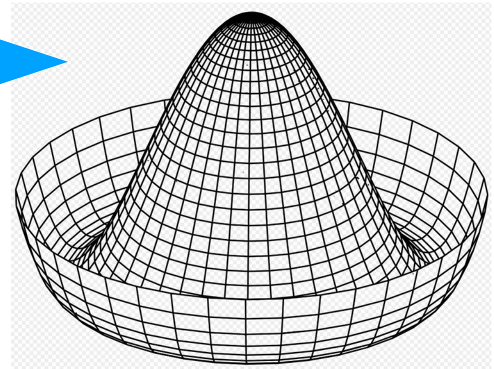
liquid, gas -> solid
paramagnet -> ferromagnet
paramagnet-> antiferromagnet
metal -> superconductor
normal gas -> Bose-Einstein condensate

Goldstone mode

acoustic phonons
(quadratic) magnons
(linear) magnons
massive (due to long-range Coulomb interaction)
'sound' waves

Spontaneous breaking of continuous symmetry





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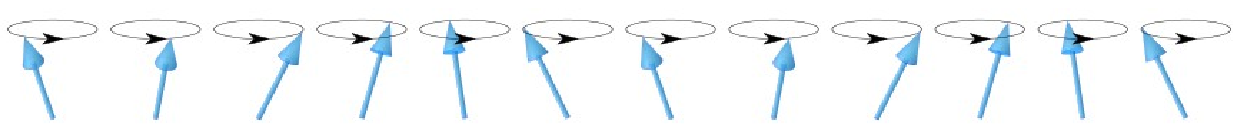
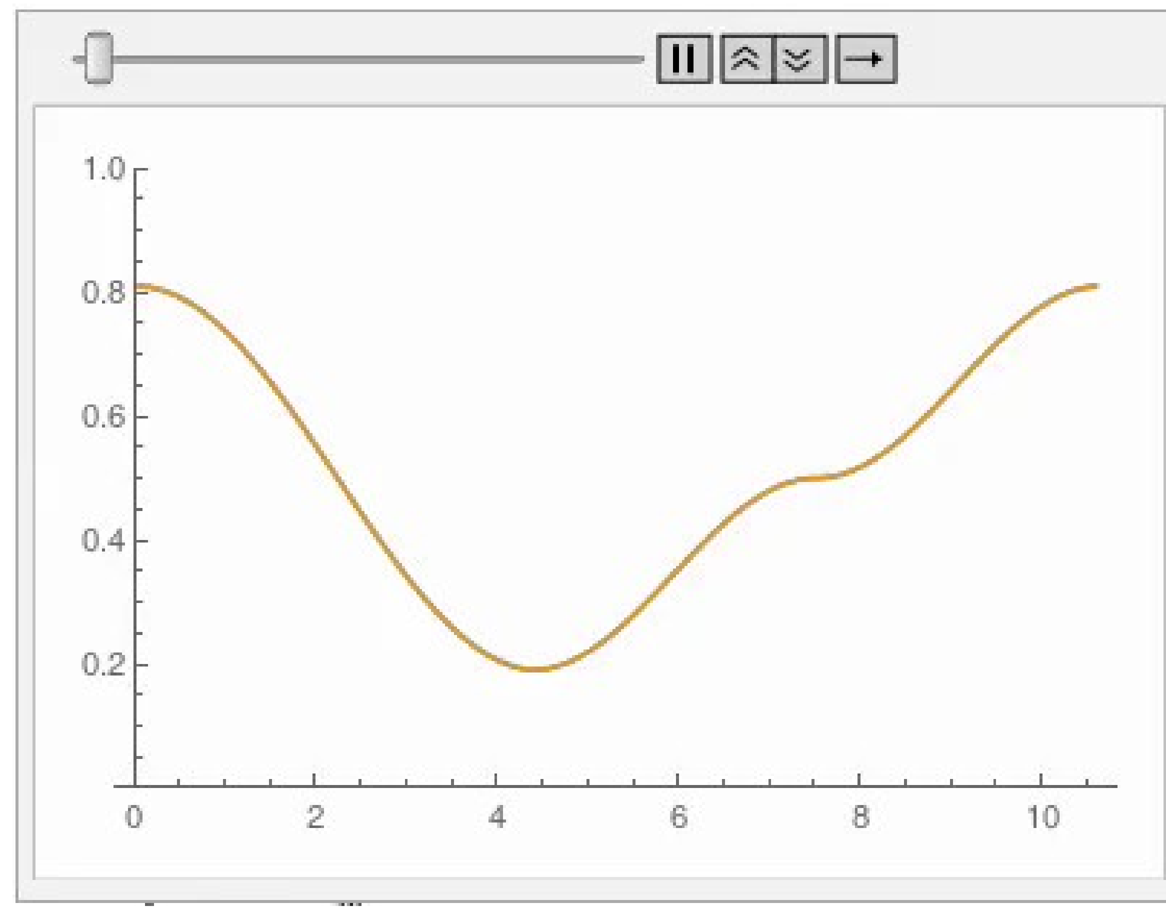
Goldstone mode (in systems with short-range interaction):

Long-wave length rotations of the order parameter cost vanishingly low energy.

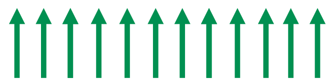


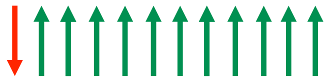
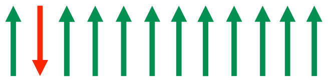
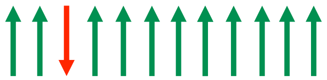
2 linear modes in 2-orbital Hubbard model (exciton condensate phase)

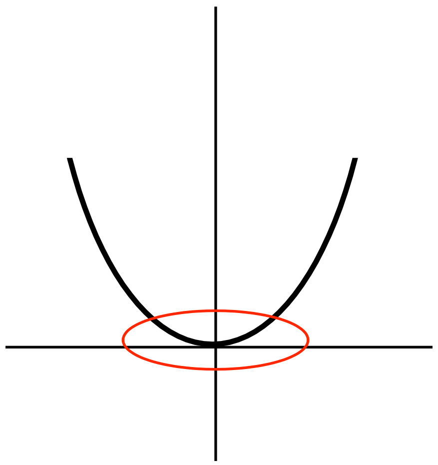


Transition:	**Goldstone mode**
liquid, gas -> solid	acoustic phonons
paramagnet -> ferromagnet	(quadratic) magnons
paramagnet-> antiferromagnet	(linear) magnons
metal -> superconductor	massive (due to long-range Coulomb interaction)
normal gas -> Bose-Einstein condensate	'sound' waves

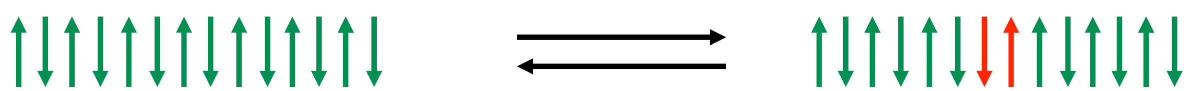
Absence of order in low dimensions

Ferromagnet:  exact groundstate

excitations: $|k\rangle =$  $+e^{ik}$  $+e^{i2k}$  magnon

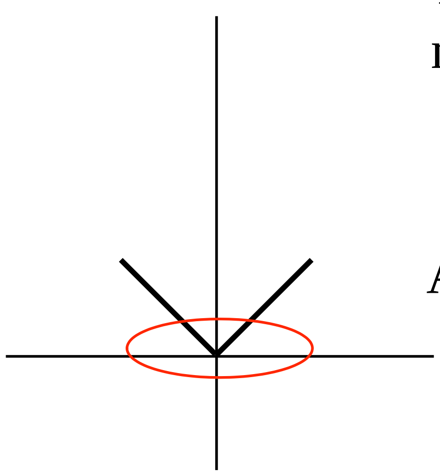


Thermal population of magnons reduced the ordered moment M.
If 'many' magnons are populated M can completely vanish. How many magnons are populated depends on the density of states (dispersion + dimension).

Antiferromagnet: 

quantum fluctuations (T=0)

The Neel state is only approximate ground state. Can the quantum fluctuations destroy it?
The answer is yes in 1D. In D=2 or 3 quantum corrections are finite and only reduce the ordered moment.



AFM magnons destroy the Neel order in 2D for T>0.

Absence of order in low dimensions

Antiferromagnet - quantum correction:

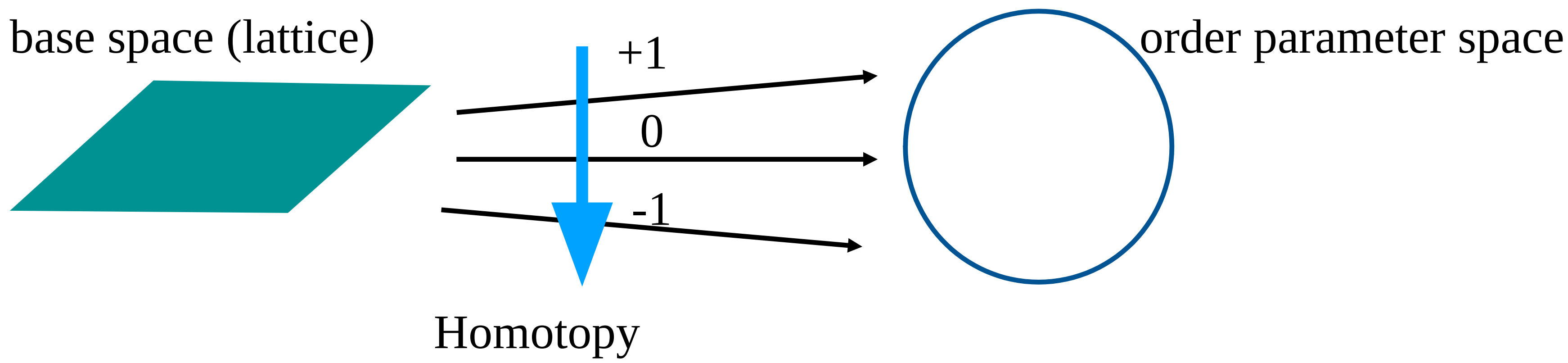
$\langle N^z \rangle / N$	absolute	relative	$S = \frac{1}{2}$		$S = 1$	
$D = 2$	$S - 0.1966$	$19.7/S$ %	0.303	39.3%	0.803	19.7%
$D = 3$	$S - 0.0784$	$7.8/S$ %	0.422	15.6%	0.922	7.8%

Antiferro-, ferro- and ferrimagnet at T=0 and T>0

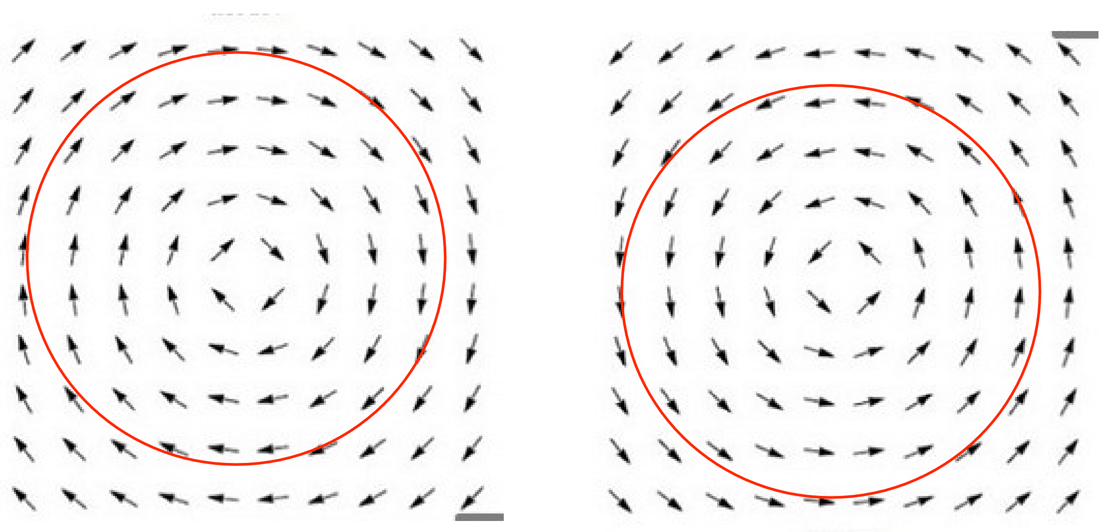
	NG mode dispersion	ground state degeneracy	tower of states	quantum corrections	lower critical dimension	
					$T = 0$	$T > 0$
type-A	$\propto k$	no	yes	yes	1	2
type-B ferro	$\propto k^2$	yes, $\mathcal{O}(N)$	no	no	0	2
type-B ferri	$\propto k^2$, $M + k^2$	yes, $\mathcal{O}(N)$	no	yes	0	2

Beekman, Rademaker and van Wezel, SciPost Phys. Lect. Notes 11 (2019)

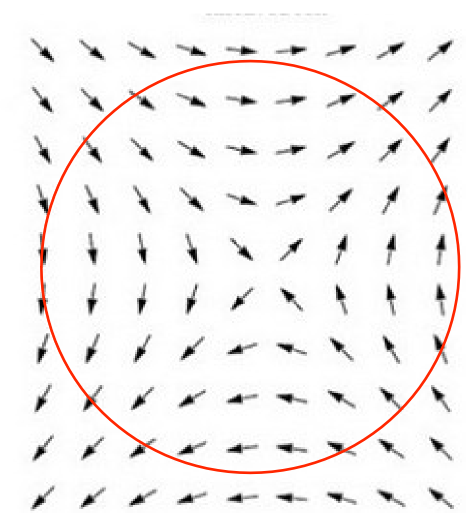
Topological excitations



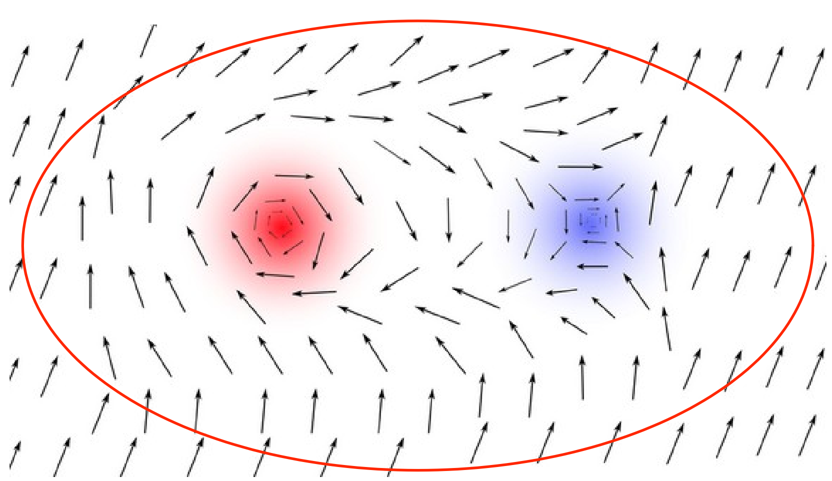
vortex: +1



antivortex: -1



vortex-antivortex pair:

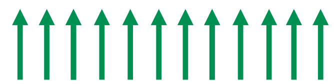
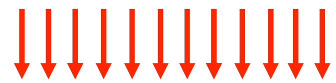


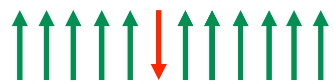
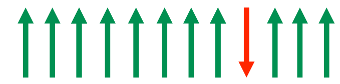
Kosterlitz–Thouless transition

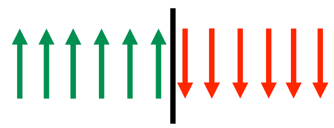
- 2D xy-model
- no local order parameter
- normal phase (exp correlations), ordered phase (power law correlations)
- unbinding of vortex-anti-vortex pairs

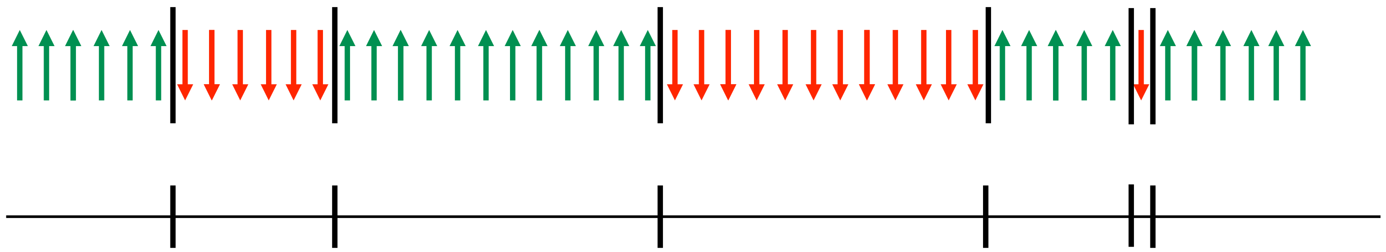
Lack of order in 1D, T>0

$$H = -J \sum_{ij} S_i^z S_j^z, \quad S_z = \pm 1$$ Ising model

T=0, exact ground state:  or 

excitations: (gapped) *E=2J*  

excitations: (gapped) *E=J* 

T>0, typical state (<S>=0 no order): 

What makes 1D different?

The surface does not depend on the volume.

=> growing a domain with 'opposite' order costs finite energy independent of the domain size

=>@T>0 there is always a finite probability do find the domain wall(s)

=> correlation length depends on the concentration of domain walls, finite concentration of domain walls implies a finite correlation length