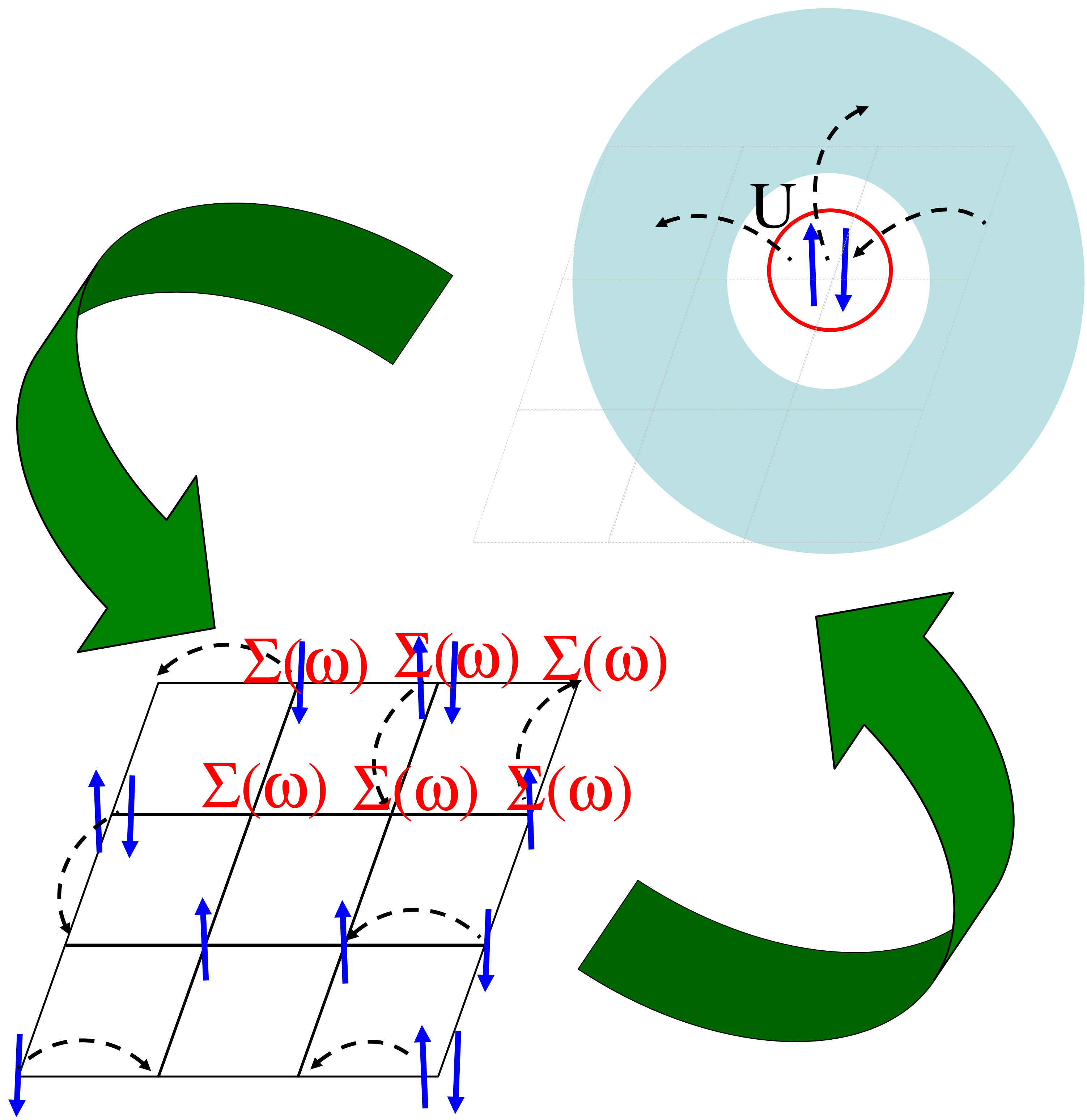
Embedding

Quantum impurity problem



Large dimension limit - classical Heisenberg model

$$\boxed{Z = \sum_{\{s_i\}} \exp(-S)}$$

$$S = \beta \left(\sum_{i,j} J_{ij} s_i s_j + h \sum_i s_i \right)$$

Cavity construction: $$S = S_0 + \Delta S + S_{(0)} \qquad \Delta S = \sum_i J_{0i} s_0 s_i$$

Expansion in 'hybridization':

$$Z = Z_{(0)} \sum_{s_0} \exp(-S_0) \left(1 + \sum_i J_{i0} \langle s_i \rangle_{(0)} s_0 + \tfrac{1}{2} \sum_{i,j} J_{i0} J_{j0} \langle s_i s_j \rangle_{(0)} s_0^2 + ... \right)$$

Cumulant expansion: $$\boxed{Z = Z_{(0)} \sum_{s_0} \exp(-S_{\text{eff}})}$$

$$S_{\text{eff}} = h s_0 + \sum_i J_{i0} \langle s_i \rangle_{(0)} s_0 + \tfrac{1}{2} \sum_{i,j} J_{i0} J_{j0} \left(\langle s_i s_j \rangle_{(0)} - \langle s_i \rangle_{(0)} \langle s_j \rangle_{(0)} \right) s_0^2 + ...$$

Scaling: $$\boxed{J = \frac{J^*}{d}}$$

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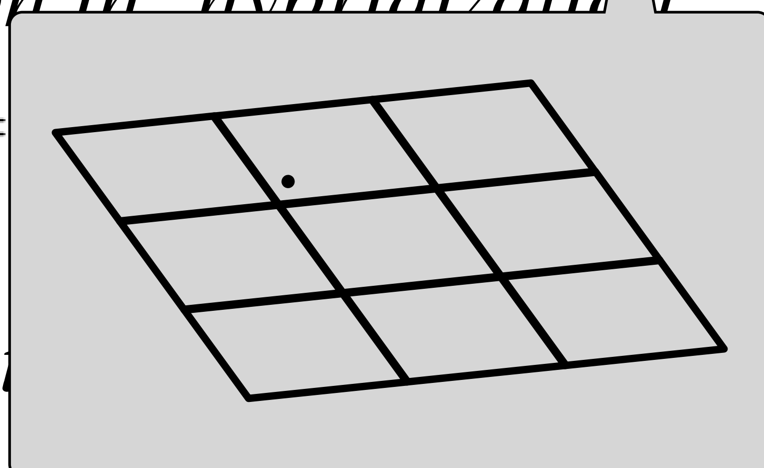

Expansion in ‘hybridization’:

$$Z = \ldots \sum_i J_{i0} \langle s_i \rangle_{(0)} s_0 + \tfrac{1}{2} \sum_{i,j} J_{i0} J_{j0} \langle s_i s_j \rangle_{(0)} s_0^2 + \ldots \Big)$$

Cumulant

$$\ldots Z_{(0)} \sum_{s_0} \exp(-S_{\mathrm{eff}})$$

$$S_{\mathrm{eff}} = h s_0 + \sum_i J_{i0} \langle s_i \rangle_{(0)} s_0 + \tfrac{1}{2} \sum_{i,j} J_{i0} J_{j0} \Big(\langle s_i s_j \rangle_{(0)} - \langle s_i \rangle_{(0)} \langle s_j \rangle_{(0)} \Big) s_0^2 + \ldots$$

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Expansion in ‘hybridization’:

$$Z = Z_{(0)} \sum \quad \langle s_i \rangle_{(0)} s_0 + \tfrac{1}{2} \sum_{i,j} J_{i0} J_{j0} \langle s_i s_j \rangle_{(0)} s_0^2 + \ldots \Big)$$

Cumulant expa

$$\sum_{s_0} \exp(-S_{\text{eff}})$$

$$S_{\text{eff}} = h s_0 + \sum_i J_{i0} \langle s_i \rangle_{(0)} s_0 + \tfrac{1}{2} \sum_{i,j} J_{i0} J_{j0} \Big(\langle s_i s_j \rangle_{(0)} - \langle s_i \rangle_{(0)} \langle s_j \rangle_{(0)} \Big) s_0^2 + \ldots$$

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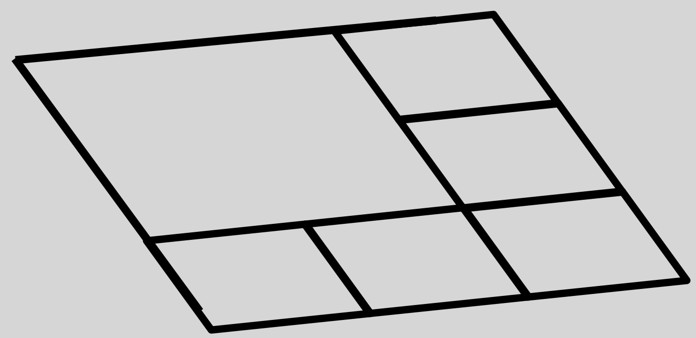

Cumulant expansion:

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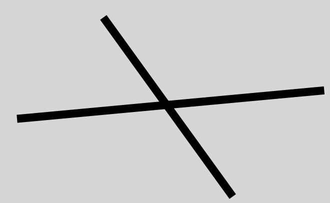

Cumulant expansion ... $(-S_{\text{eff}})$

$$S_{\text{eff}} = h s_0 + \sum_i J_{i0} \langle s_i \rangle_{(0)} s_0 + \tfrac{1}{2} \sum_{i,j} J_{i0} J_{j0} \Big(\langle s_i s_j \rangle_{(0)} - \langle s_i \rangle_{(0)} \langle s_j \rangle_{(0)} \Big) s_0^2 + \ldots$$

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Scaling:

$$\boxed{J = \frac{J^*}{d}}$$

Hubbard model in d= ∞

How to construct non-trivial d= ∞ limit?

$\langle E_{kin} \rangle \approx \langle E_{int} \rangle$

How to scale hopping?

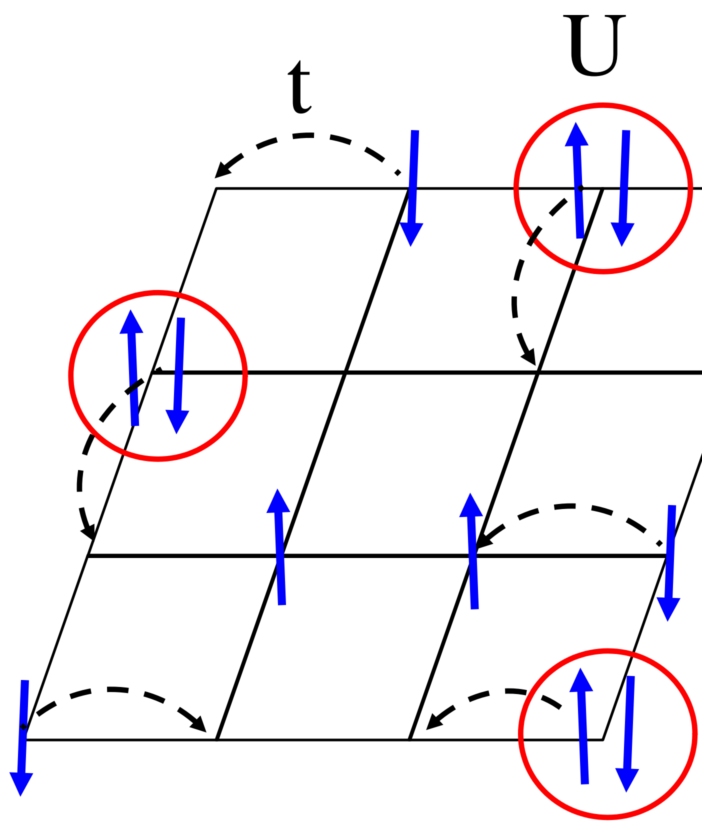
$$H = t \sum_{i,\sigma} \sum_{p=1}^{d} c^\dagger_{i\pm p,\sigma} c_{i,\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

$$\langle c^\dagger_{i+p,\sigma} c_{i,\sigma} \rangle_0 \sim \frac{1}{\sqrt{d}}$$

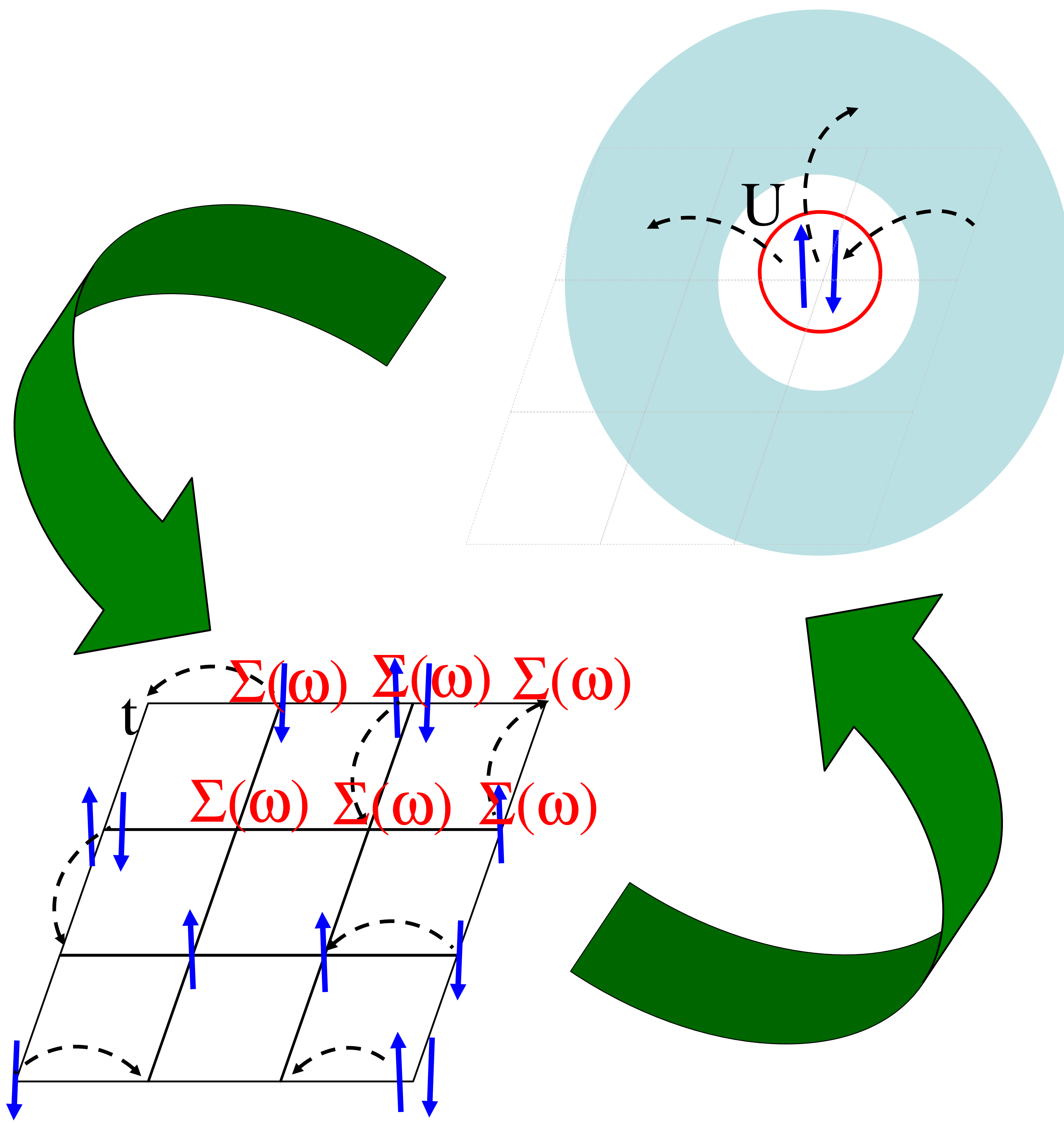
$$\boxed{t = \frac{t^*}{\sqrt{d}}}$$

Metzner and Vollhardt,
*PRL **62**, 324 (1998)*

Dynamical mean-field theory (DMFT)



A. Georges et al. RMP ***68****, 13 (1996)*)



Physics Today (March 2004) Kotliar, Vollhardt

DMFT

$$H=\sum_{i,j,\sigma} t_{ij}c^{\dagger}_{i\sigma}c_{j\sigma}+U\sum_{i} n_{i\uparrow}n_{i\downarrow}$$

$$G_{ii}(\tau)=-\langle Tc_i(\tau)c_i^{\dagger}(0)\rangle$$

$$H_{\mathrm{loc}} = \sum_{k,\alpha=\uparrow,\downarrow}\epsilon_k c^{\dagger}_{k\alpha}c_{k\alpha}+\epsilon_d(n_\uparrow+n_\downarrow)+Un_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow}(V_k c^{\dagger}_{k\alpha}d_\alpha+V_k^* d^{\dagger}_\alpha c_{k\alpha})$$

Anderson impurity model

$$G_{ii}(\omega)=\sum_k \frac{1}{\omega+\mu-\varepsilon_k-\Sigma(\omega)}$$

$$G_{ii}(\omega)=\omega+\mu-\epsilon-\Delta(\omega)-\Sigma(\omega)$$

Weiss molecular field

$$H=\sum_{i,j} J_{ij}S_iS_j+h\sum_i S_i$$

$$s_i=\langle S_i\rangle$$

$$H_{\mathrm{loc}}=\tilde{h}S$$

$$\tilde{h}=\sum_i J_{0i}s_i+h$$

Local moments in metals

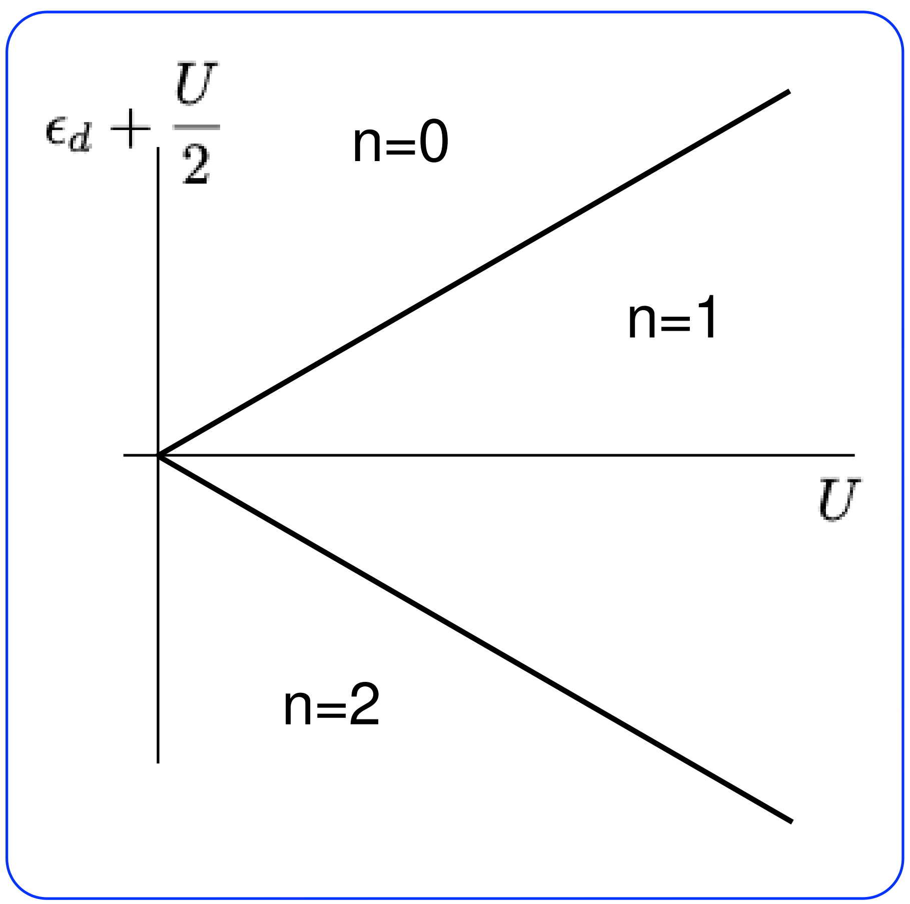
What is the fate of a local moment (=interacting atom) submerged into a Fermi sea?

Anderson impurity model

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d (n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V_k^* d^\dagger_\alpha c_{k\alpha})$$

Impurity states:

$\|0\rangle$	0
$\|\uparrow\rangle, \|\downarrow\rangle$	ϵ_d
$\|\uparrow\downarrow\rangle$	$2\epsilon_d + U$



Mean-field theory

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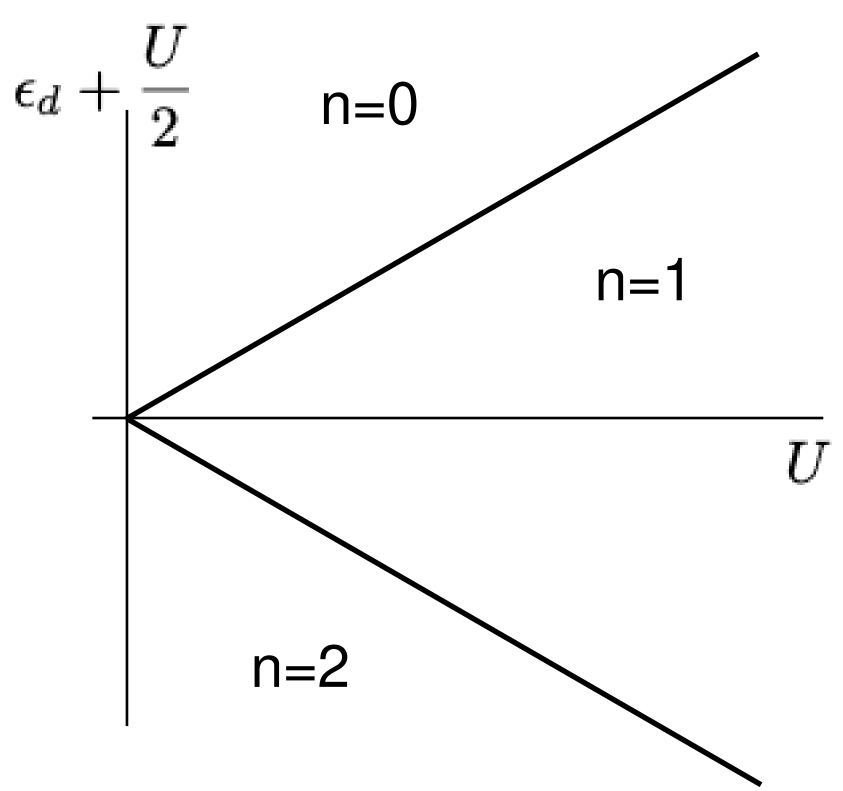
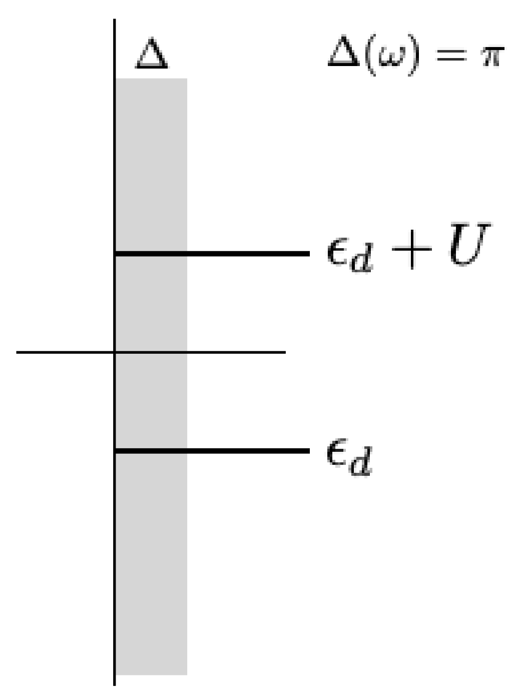


$$H_{\mathrm{MF}} = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \sum_\sigma E_\alpha d^\dagger_\alpha d_\alpha + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V^*_k d^\dagger_\alpha c_{k\alpha})$$

$$E_\alpha = \epsilon_d + U\langle n_{-\alpha}\rangle$$

Anderson's solution (special case)



$$\Delta(\omega) = \pi \sum_k V_k^2 \delta(\omega - \epsilon_k)$$

$$\langle n_\alpha\rangle = \frac{1}{\pi}\,\mathrm{cotg}^{-1}\left(\frac{\epsilon_d + U\langle n_{-\alpha}\rangle}{\Delta}\right)$$

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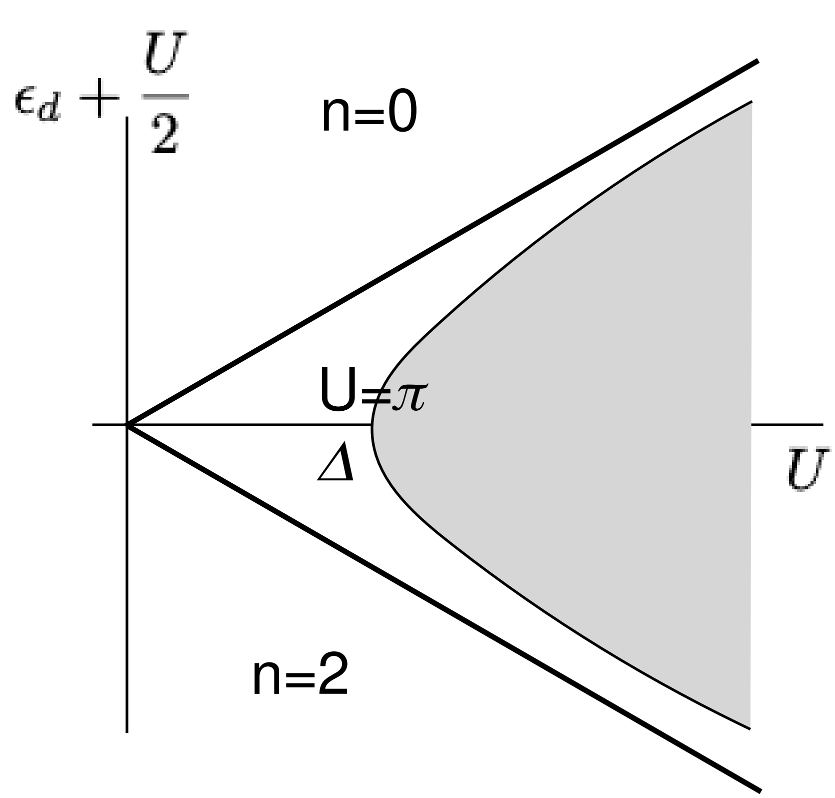

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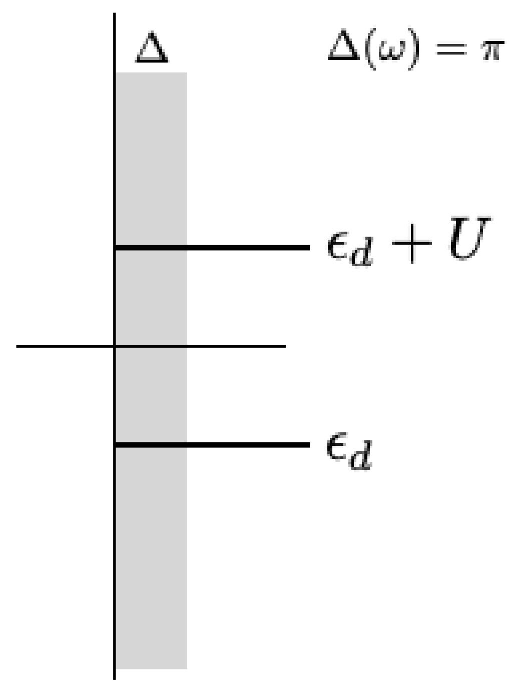

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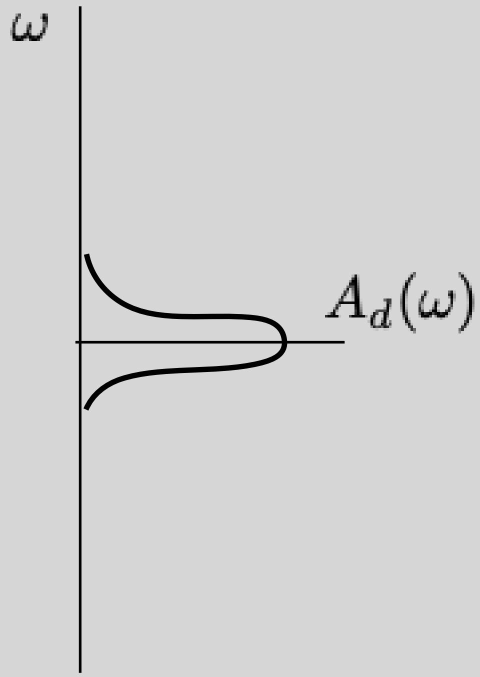
...bmerged into a Fermi sea?

Impurity spectral function:



$$\ldots + \sum_{\sigma} E_\alpha d^\dagger_\alpha d_\alpha + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V_k^* d^\dagger_\alpha c_{k\alpha})$$

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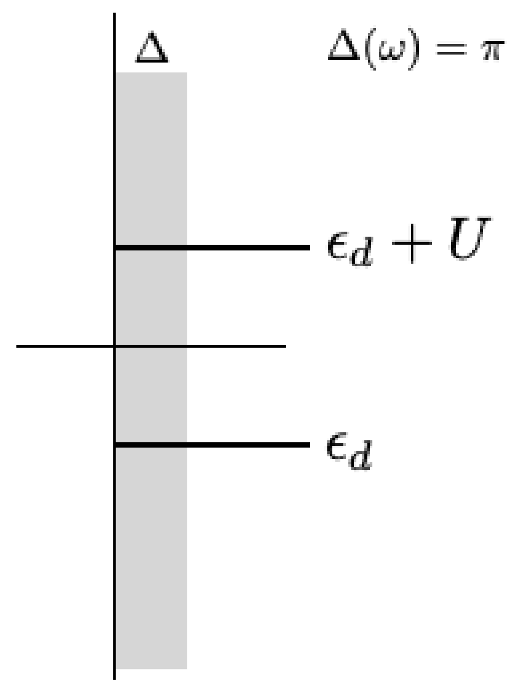

$$\Delta(\omega) = \pi \sum_k V_k^2 \delta(\omega - \epsilon_k)$$

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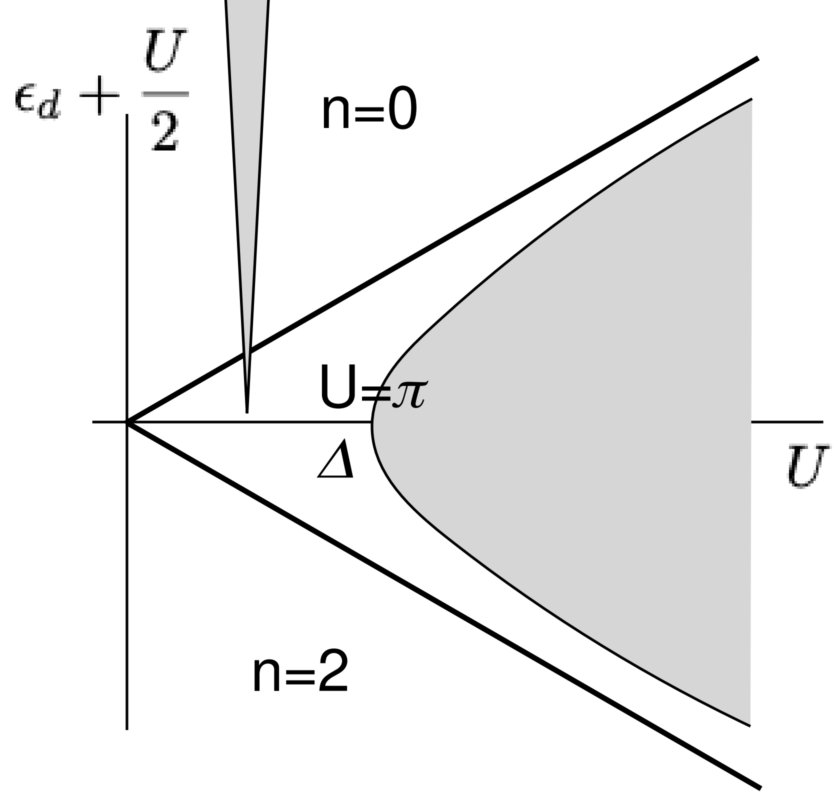
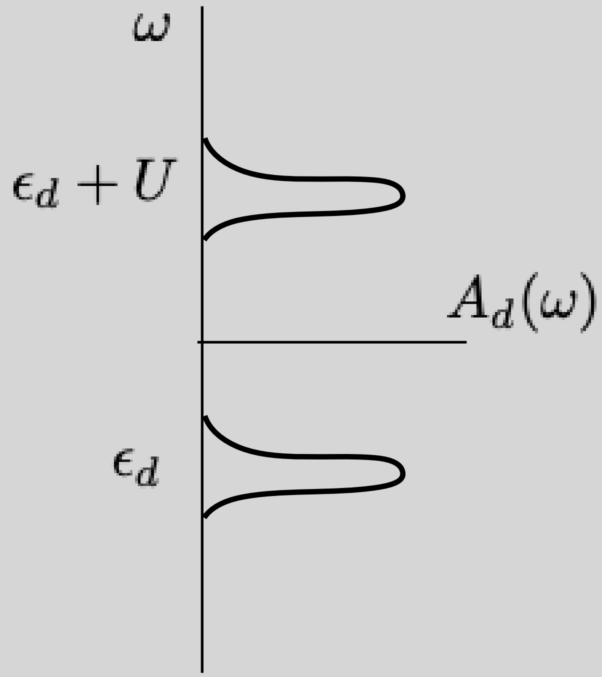


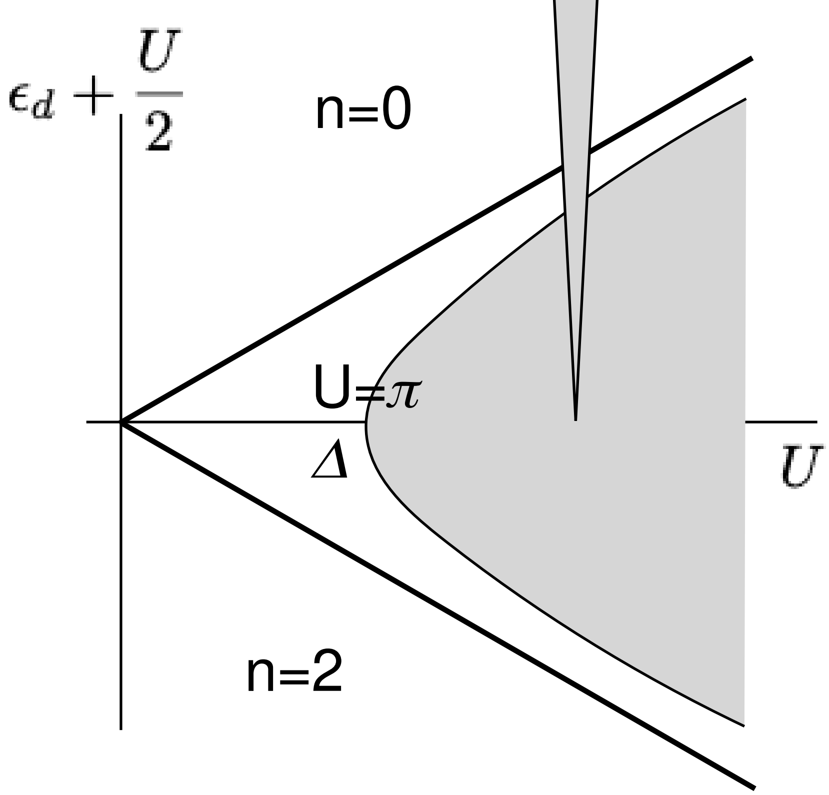
…ubmerged into a Fermi sea?

Impurity spectral function:



$$\ldots + \sum_{\sigma} E_\alpha d^\dagger_\alpha d_\alpha + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V_k^* d^\dagger_\alpha c_{k\alpha})$$

$$E_\alpha = \epsilon_d + U\langle n_{-\alpha}\rangle$$



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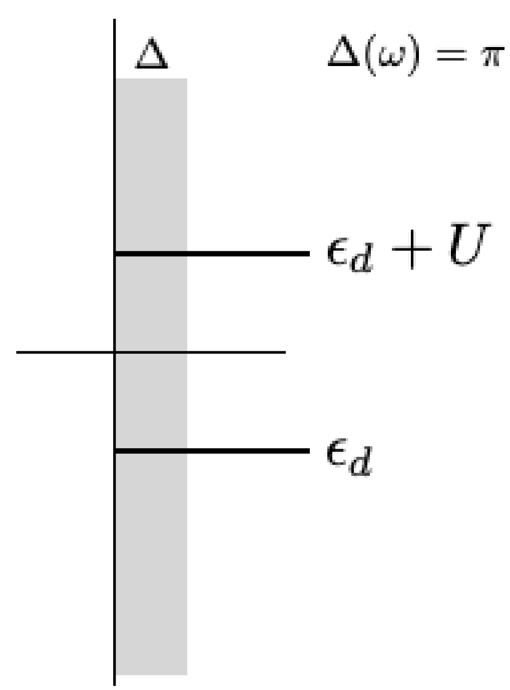

$$\Delta(\omega) = \pi \sum_k V_k^2 \delta(\omega - \epsilon_k)$$

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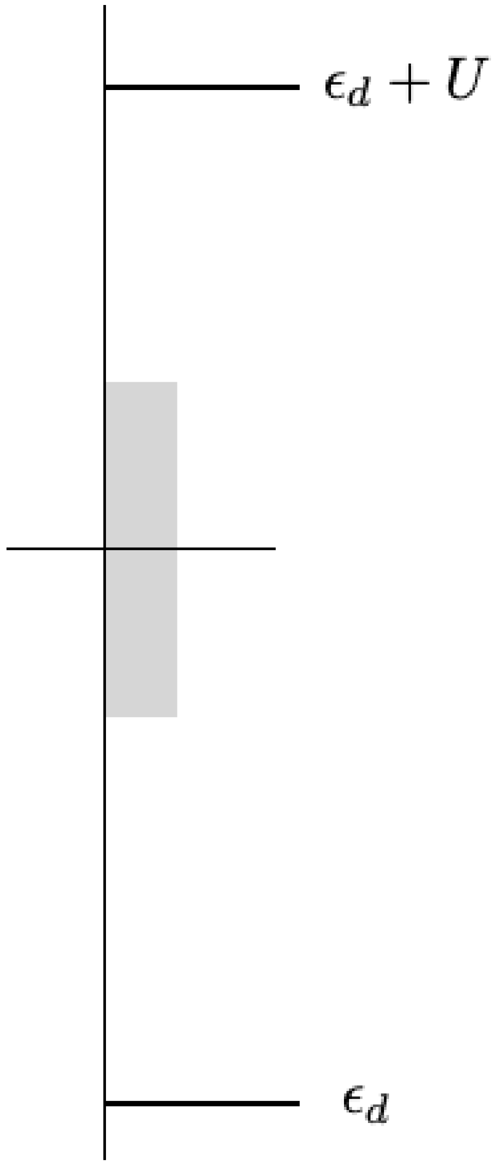
Anderson to Kondo model

Let us look at the hard case of large U



Let us eliminate the $\cancel{|0\rangle \quad 0}$ state of the impurity

$$
\begin{array}{ll}
\cancel{|0\rangle} & \cancel{0} \\
|\uparrow\rangle, |\downarrow\rangle & \epsilon_d \\
\cancel{|\uparrow\downarrow\rangle} & \cancel{2\epsilon_d + U}
\end{array}
$$

$$\langle\alpha|H_{\text{eff}}|\beta\rangle = \frac{1}{2}\sum_{\gamma}\left(\frac{\langle\alpha|H_1|\gamma\rangle\langle\gamma|H_1|\beta\rangle}{E_\alpha - E_\gamma} + \frac{\langle\alpha|H_1|\gamma\rangle\langle\gamma|H_1|\beta\rangle}{E_\beta - E_\gamma}\right)$$

$$H = \underbrace{\sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d(n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow}_{H_0} + \underbrace{\sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V^*_k d^\dagger_\alpha c_{k\alpha})}_{H_1}$$

$$E_\alpha \approx E_\beta \approx 0 \qquad E_\gamma \approx -\epsilon_d, \epsilon_d + U \qquad \sum_\gamma |\gamma\rangle\langle\gamma| = \begin{cases} n_\uparrow n_\downarrow & (a) \\ (1-n_\uparrow)(1-n_\downarrow) & (b) \end{cases}$$

$$H_{\text{eff}} = \sum_{k,\alpha} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \sum_{k,k',\delta,\gamma} V_k V^*_{k'} \left(\frac{1}{\epsilon_d + U} - \frac{1}{\epsilon_d}\right) \boldsymbol{S}\cdot\boldsymbol{\sigma}_{\gamma\delta} c^\dagger_{k\gamma} c_{k'\delta} + \frac{1}{2}\sum_{k,k',\alpha} V_k V^*_{k'} \left(\frac{1}{\epsilon_d + U} + \frac{1}{\epsilon_d}\right) c^\dagger_{k\alpha} c_{k'\alpha}$$

Kondo model

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + J \sum_{\substack{\alpha,\beta=\uparrow,\downarrow \\ k,k'}} \boldsymbol{S}\cdot\boldsymbol{\sigma}_{\alpha\beta} c^\dagger_{k\alpha} c_{k'\beta}$$

Kondo model

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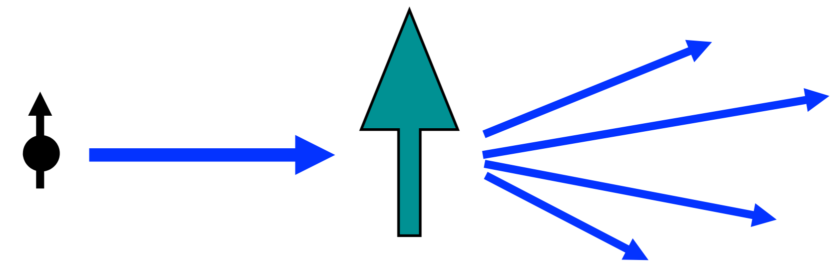
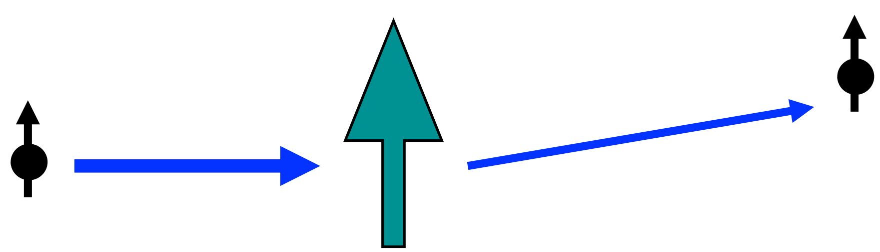


states: $|\uparrow||n_{1\uparrow}n_{2\uparrow}n_{3\uparrow}\ldots||n_{1\downarrow}n_{2\downarrow}n_{3\downarrow}\ldots\rangle \quad n=0,1$

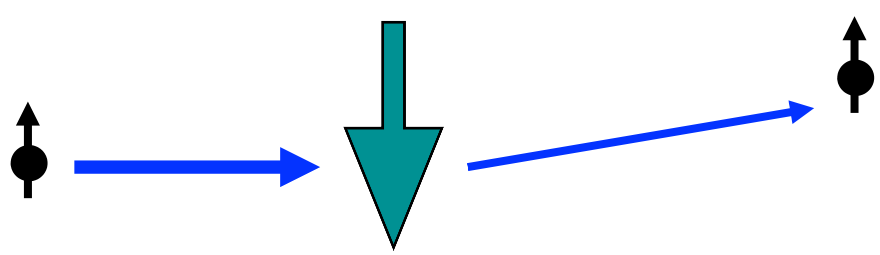
$$\vdots$$

$$|\downarrow||n_{1\uparrow}n_{2\uparrow}n_{3\uparrow}\ldots||n_{1\downarrow}n_{2\downarrow}n_{3\downarrow}\ldots\rangle$$

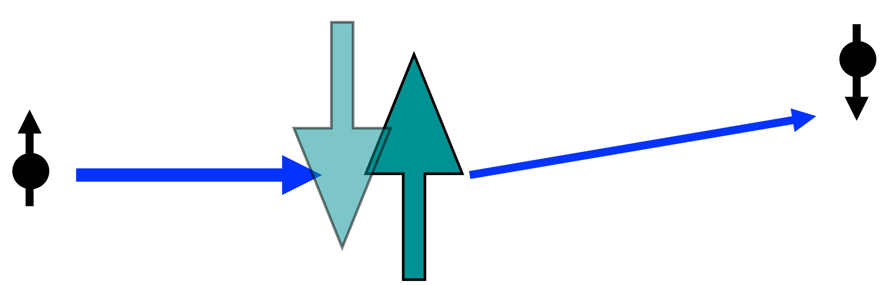
processes:



J/2

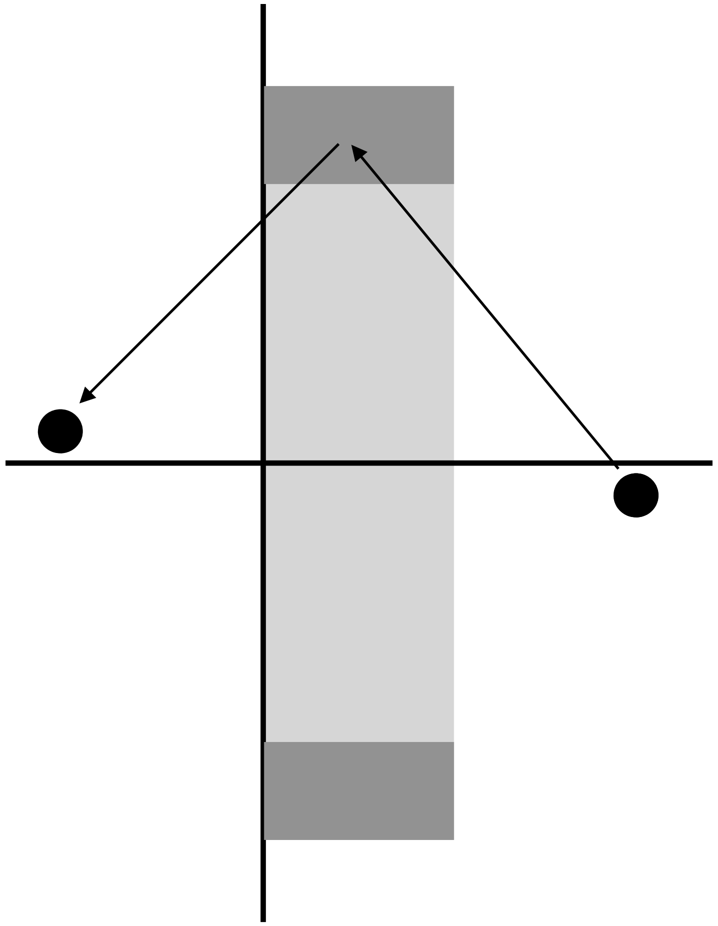


-J/2



J

Poor man's scaling (renormalization)



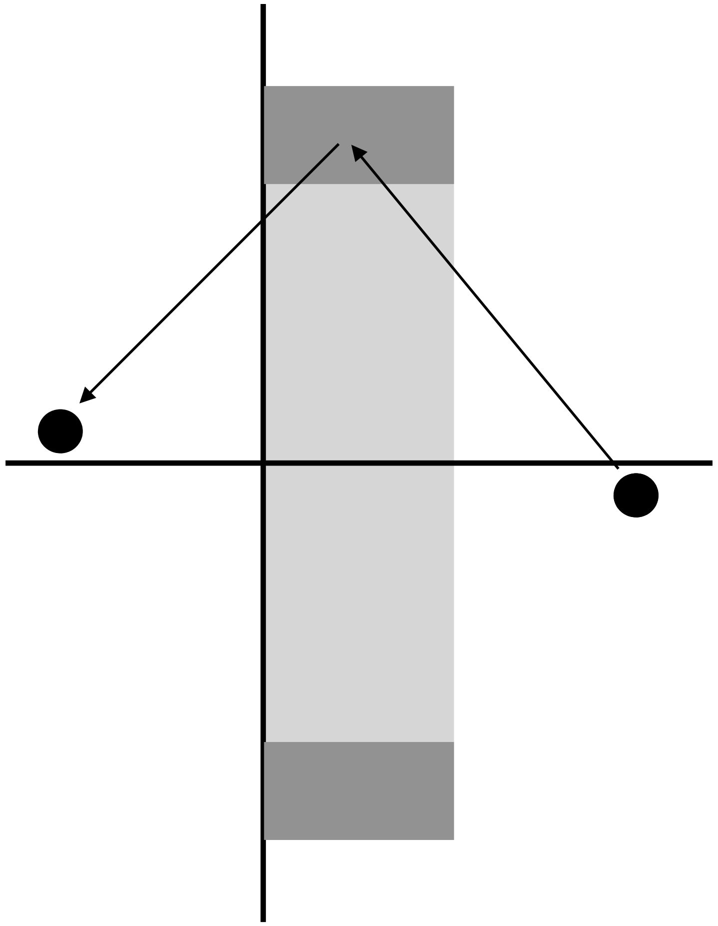
$$H(D) = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + J(D) \sum_{\substack{\alpha,\beta=\uparrow,\downarrow \\ |\epsilon_k|,|\epsilon_{k'}|<D}} \boldsymbol{S}\cdot\boldsymbol{\sigma}_{\alpha\beta} c^\dagger_{k\alpha} c_{k'\beta}$$

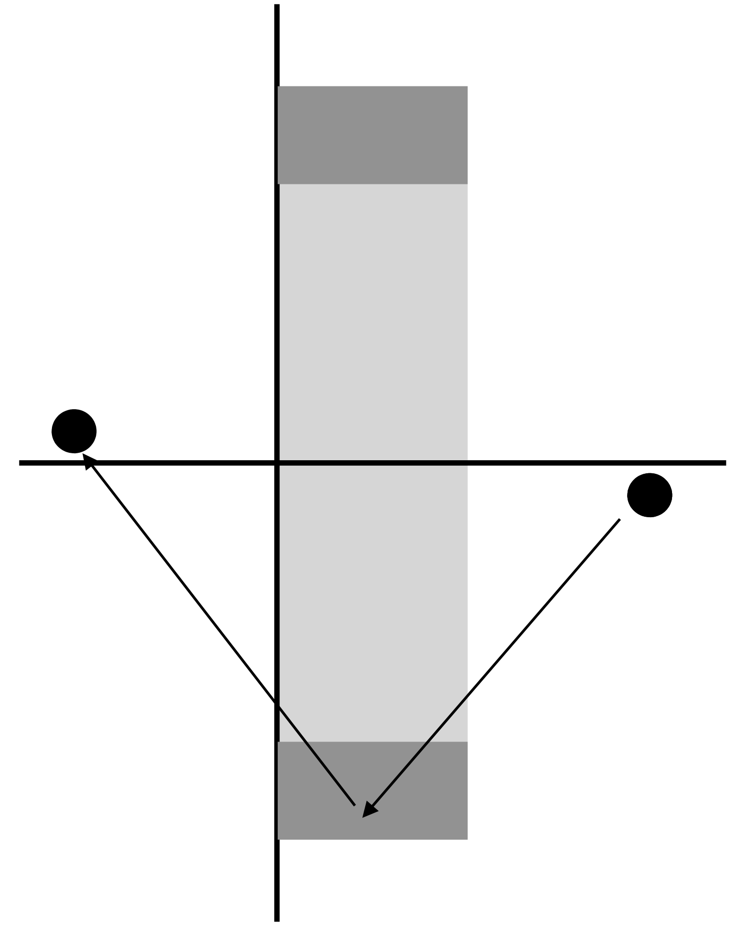
$$H(D') = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + J(D) \sum_{\substack{\alpha,\beta=\uparrow,\downarrow \\ |\epsilon_k|,|\epsilon_{k'}|<D'}} \boldsymbol{S}\cdot\boldsymbol{\sigma}_{\alpha\beta} c^\dagger_{k\alpha} c_{k'\beta} + \Delta H$$

$$\Delta H = \frac{1}{2}\sum_\lambda |m\rangle \frac{\langle m|H_I|\lambda\rangle\langle\lambda|H_I|n\rangle}{E_m - E_\lambda}\langle n| + \frac{1}{2}\sum_\lambda |m\rangle \frac{\langle m|H_I|\lambda\rangle\langle\lambda|H_I|n\rangle}{E_n - E_\lambda}\langle n|$$

$$\begin{aligned}
\Delta H &\approx -J^2\rho\frac{|\delta D|}{D}\boldsymbol{S}\cdot\boldsymbol{\sigma}_{\alpha\lambda} c^\dagger_{k\alpha} c_{K\lambda}(c^\dagger_{K\lambda} c_{K\lambda})\boldsymbol{S}\cdot\boldsymbol{\sigma}_{\lambda\beta} c^\dagger_{K\lambda} c_{k'\beta} \\
&= -J^2\rho\frac{|\delta D|}{D} c^\dagger_{k\alpha} c_{k'\beta}\sigma^{\bar a}_{\alpha\bar\lambda}\sigma^{\bar b}_{\bar\lambda\beta} S^{\bar a} S^{\bar b} \\
&= -J^2\rho\frac{|\delta D|}{D}\left(c^\dagger_{k\alpha} c_{k'\beta}\sigma^{\bar c}_{\alpha\beta}\frac{1}{2}S^{\bar c}(i\varepsilon_{\bar a\bar b\bar c})^2 + \delta_{\bar a\bar b}\frac{1}{4}c^\dagger_{k\alpha} c_{k'\alpha}\right) \\
&= J^2\rho\frac{|\delta D|}{D} c^\dagger_{k\alpha} c_{k'\beta}\boldsymbol{\sigma}_{\alpha\beta}\cdot\boldsymbol{S} - \frac{3}{4}J^2\rho\frac{|\delta D|}{D} c^\dagger_{k\alpha} c_{k'\alpha}
\end{aligned}$$

Poor man's scaling (renormalization)





$$
\begin{aligned}
\Delta H &\approx -J^2\rho\frac{|\delta D|}{D}\boldsymbol{S}\cdot\boldsymbol{\sigma}_{\alpha\lambda}c^\dagger_{k\alpha}c_{K\lambda}(c^\dagger_{K\lambda}c_{K\lambda})\boldsymbol{S}\cdot\boldsymbol{\sigma}_{\lambda\beta}c^\dagger_{K\lambda}c_{k'\beta}\\
&= -J^2\rho\frac{|\delta D|}{D}c^\dagger_{k\alpha}c_{k'\beta}\sigma^{\bar a}_{\alpha\bar\lambda}\sigma^{\bar b}_{\bar\lambda\beta}S^{\bar a}S^{\bar b}\\
&= -J^2\rho\frac{|\delta D|}{D}\left(c^\dagger_{k\alpha}c_{k'\beta}\sigma^{\bar c}_{\alpha\beta}\frac{1}{2}S^{\bar c}(i\varepsilon_{\bar a\bar b\bar c})^2+\delta_{\bar a\bar b}\frac{1}{4}c^\dagger_{k\alpha}c_{k'\alpha}\right)\\
&= J^2\rho\frac{|\delta D|}{D}c^\dagger_{k\alpha}c_{k'\beta}\boldsymbol{\sigma}_{\alpha\beta}\cdot\boldsymbol{S}-\frac{3}{4}J^2\rho\frac{|\delta D|}{D}c^\dagger_{k\alpha}c_{k'\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\Delta H &\approx -J^2\rho\frac{|\delta D|}{D}\boldsymbol{S}\cdot\boldsymbol{\sigma}_{\lambda\beta}c^\dagger_{K\lambda}c_{k'\beta}(c_{K\lambda}c^\dagger_{K\lambda})\boldsymbol{S}\cdot\boldsymbol{\sigma}_{\alpha\lambda}c^\dagger_{k\alpha}c_{K\lambda}\\
&= -J^2\rho\frac{|\delta D|}{D}c_{k'\beta}c^\dagger_{k\alpha}\sigma^{\bar a}_{\bar\lambda\beta}\sigma^{\bar b}_{\alpha\bar\lambda}S^{\bar a}S^{\bar b}\\
&= J^2\rho\frac{|\delta D|}{D}c^\dagger_{k\alpha}c_{k'\beta}\sigma^{\bar b}_{\alpha\bar\lambda}\sigma^{\bar a}_{\bar\lambda\beta}(\frac{1}{2}\delta_{\bar a\bar b}-S^{\bar b}S^{\bar a})\\
&= J^2\rho\frac{|\delta D|}{D}c^\dagger_{k\alpha}c_{k'\beta}\boldsymbol{\sigma}_{\alpha\beta}\cdot\boldsymbol{S}+\frac{3}{4}J^2\rho\frac{|\delta D|}{D}c^\dagger_{k\alpha}c_{k'\alpha}
\end{aligned}
$$

Poor man's scaling (renormalization)

$$J(D-|dD|)=J(D)+2J^2\rho\frac{|dD|}{D}$$

$$J(D')=\frac{J}{1-2J\rho\ln\frac{D}{D'}}$$

$$J(D+dD)=J(D)-2J^2\rho\frac{dD}{D}$$

$$\int_J^{J'}\frac{dJ}{J^2}=-2\rho\int_D^{D'}\frac{dD}{D}$$

$$\frac{1}{J'}-\frac{1}{J}=-2\rho\ln\frac{D}{D'}$$

Kondo temperature

The perturbative procedure breaks down at the energy scale: $T_K\approx D\exp\left(-\frac{1}{2\rho J}\right)$

$$T_K=D\exp\left[-\Phi(2\rho J)\right],\quad \Phi(y)=\frac{1}{y}-\frac{1}{2}\ln(y)+1.58y+O(y^2)$$

Accurate numerical calculations can be performed at any temperature

- analytic (IPT, NCA) - diagrammatic expansions
- diagonalization (ED, NRG, DMRG) - Hamiltonian based
- QMC (Hirsch-Fye, CT-QMC) - action based

- Numerical renormalization group (K. G. Wilson - Nobel prize)
- The basic idea is similar to poor man's scaling -> create a sequence of Hamiltonians that describe progressively lower energy scale
- renormalization is one of the key concepts in physics, which allows to describe phenomena where perturbative methods fail (e.g. critical points - phase transitions)

Universality

Impurity observables = 'system with impurity' - 'system without impurity'

Magnetic susceptibility:

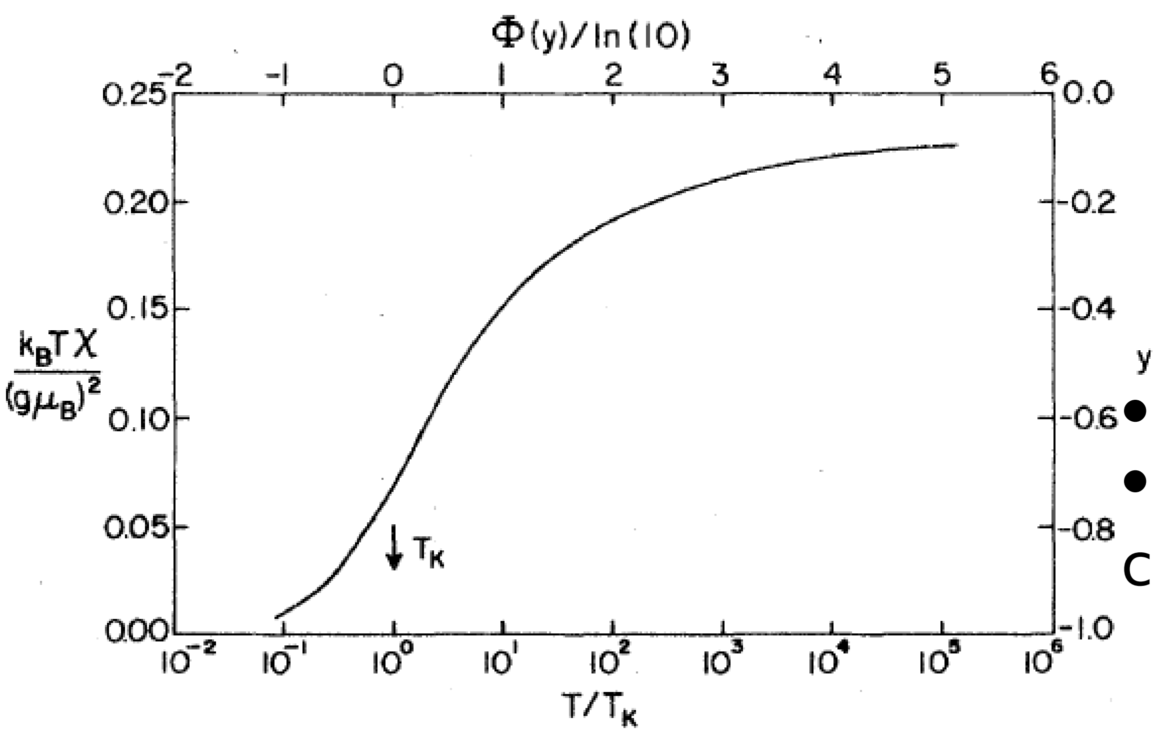
$$T\chi(T) = F\left(\frac{T}{T_K}\right) + O\left(\frac{T}{D}\right)$$

$$\Phi(4T\chi(T) - 1) = \ln\left(\frac{T}{T_K}\right)$$

$$T_K = D\exp\left[-\Phi(2\rho J)\right], \quad \Phi(y) = \frac{1}{y} - \frac{1}{2}\ln(y) + 1.58y + O(y^2)$$

$$T\chi(T) = F\left(\frac{T}{T_K}\right) \text{ for } T \ll D$$

Universal function



Krishnamurthy,Wilkins and Wilson, PRB 1980

Universality: $\rho, J, D \to T_K$

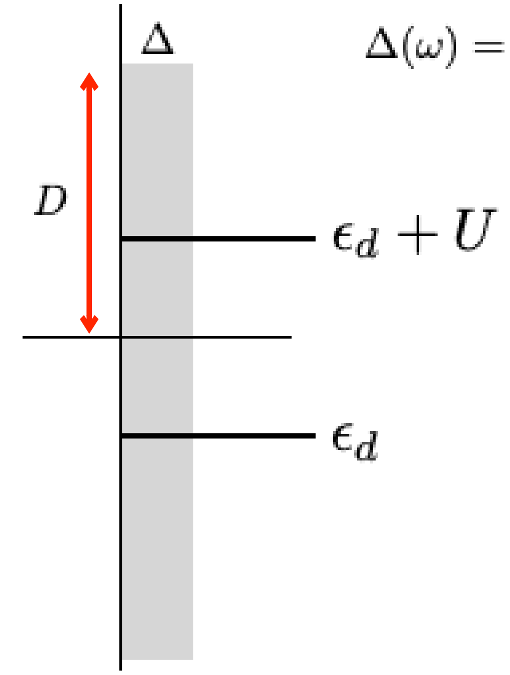
$\rho, D, U, V \to T_K$

$\epsilon_k, U, V_k \to T_K$

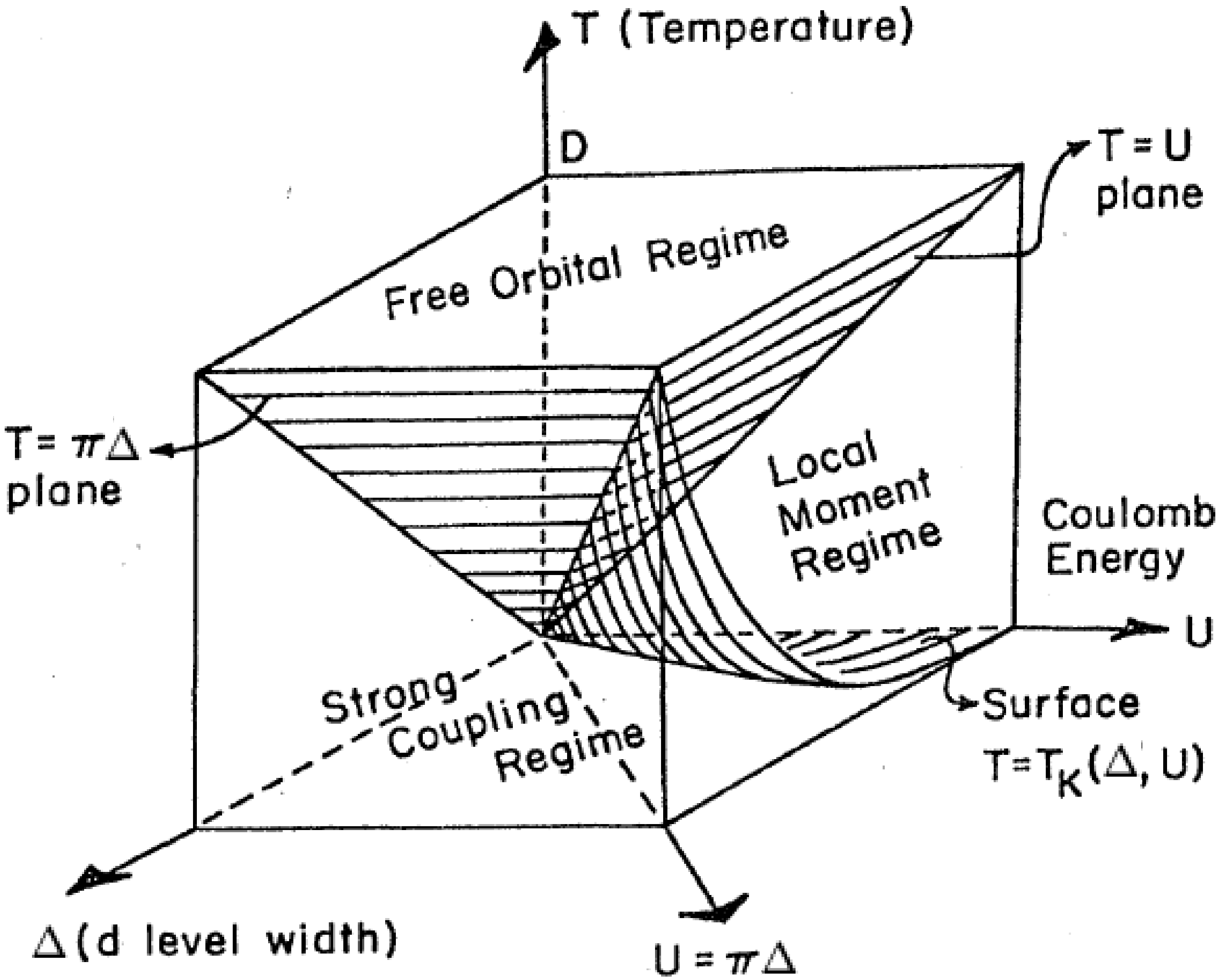
- Many different systems have the same behavior
- Several (infinite number of) microscopic parameters combine in a single number T_K

Anderson/Kondo model 'phase diagram'

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d (n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V^*_k d^\dagger_\alpha c_{k\alpha})$$



$$\Delta(\omega) = \pi \sum_k V_k^2 \delta(\omega - \epsilon_k)$$



Krishnamurthy,Wilkins and Wilson, PRB
1980

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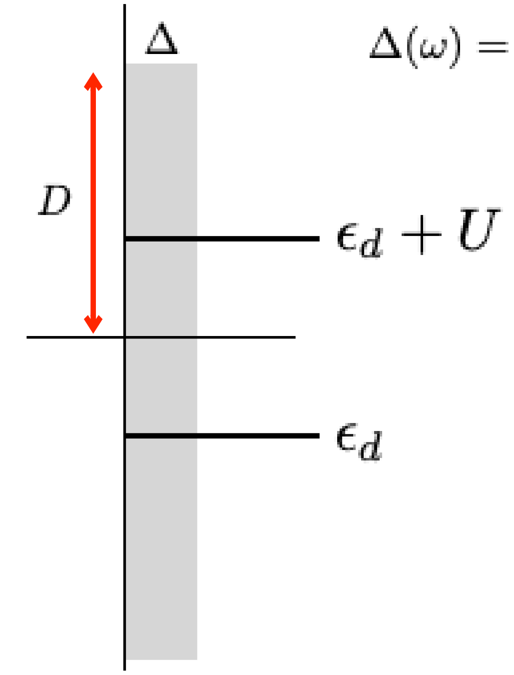
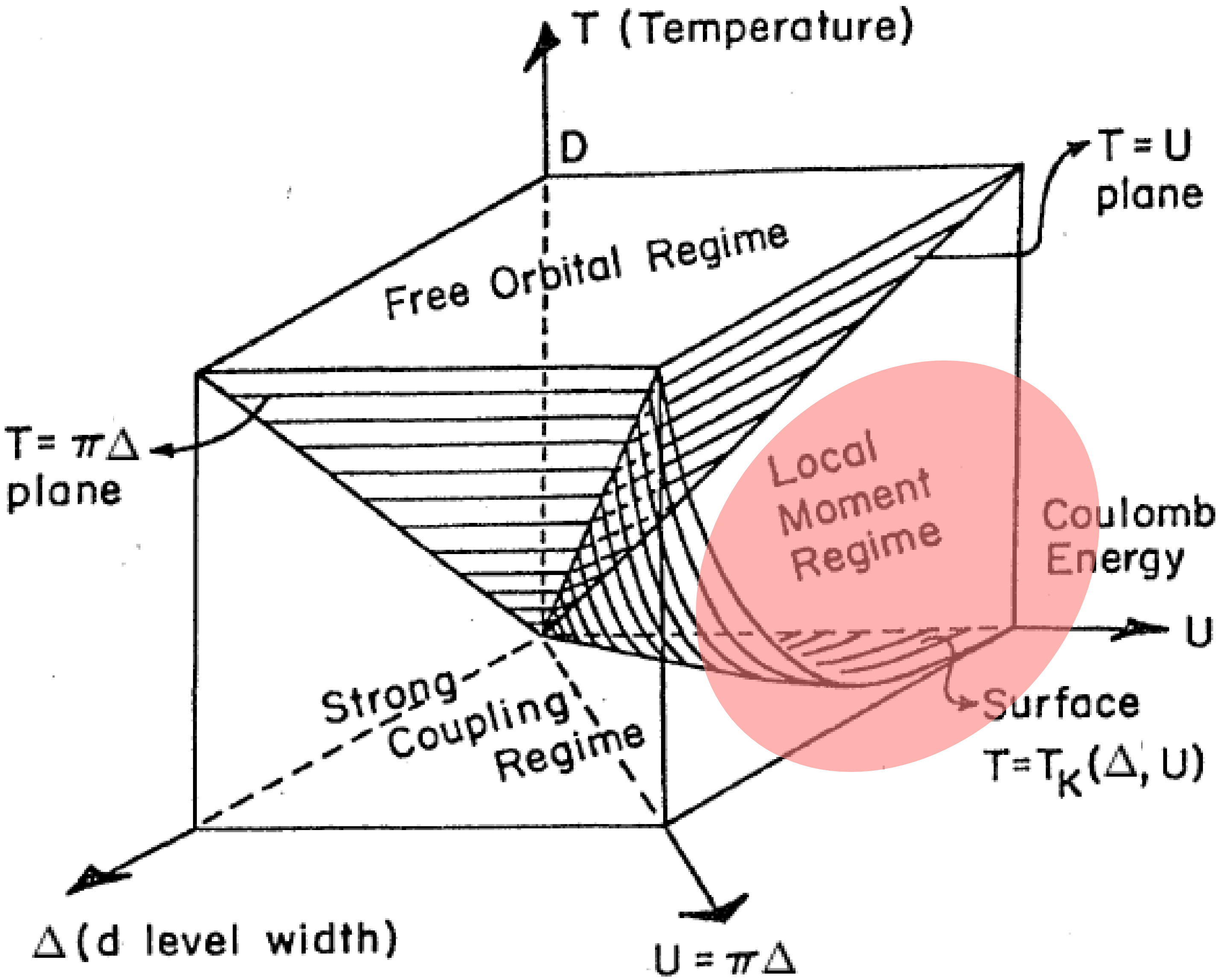




Kondo regime

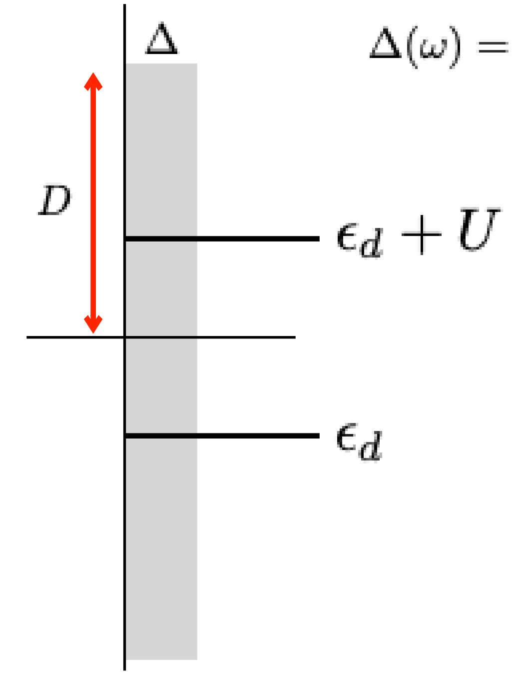
$$\rho J = \frac{4\Delta}{\pi U}$$

$$\Delta, D, T \ll U$$

Krishnamurthy,Wilkins and Wilson, PRB 1980

Anderson/Kondo model 'phase diagram'

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d (n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V^*_k d^\dagger_\alpha c_{k\alpha})$$



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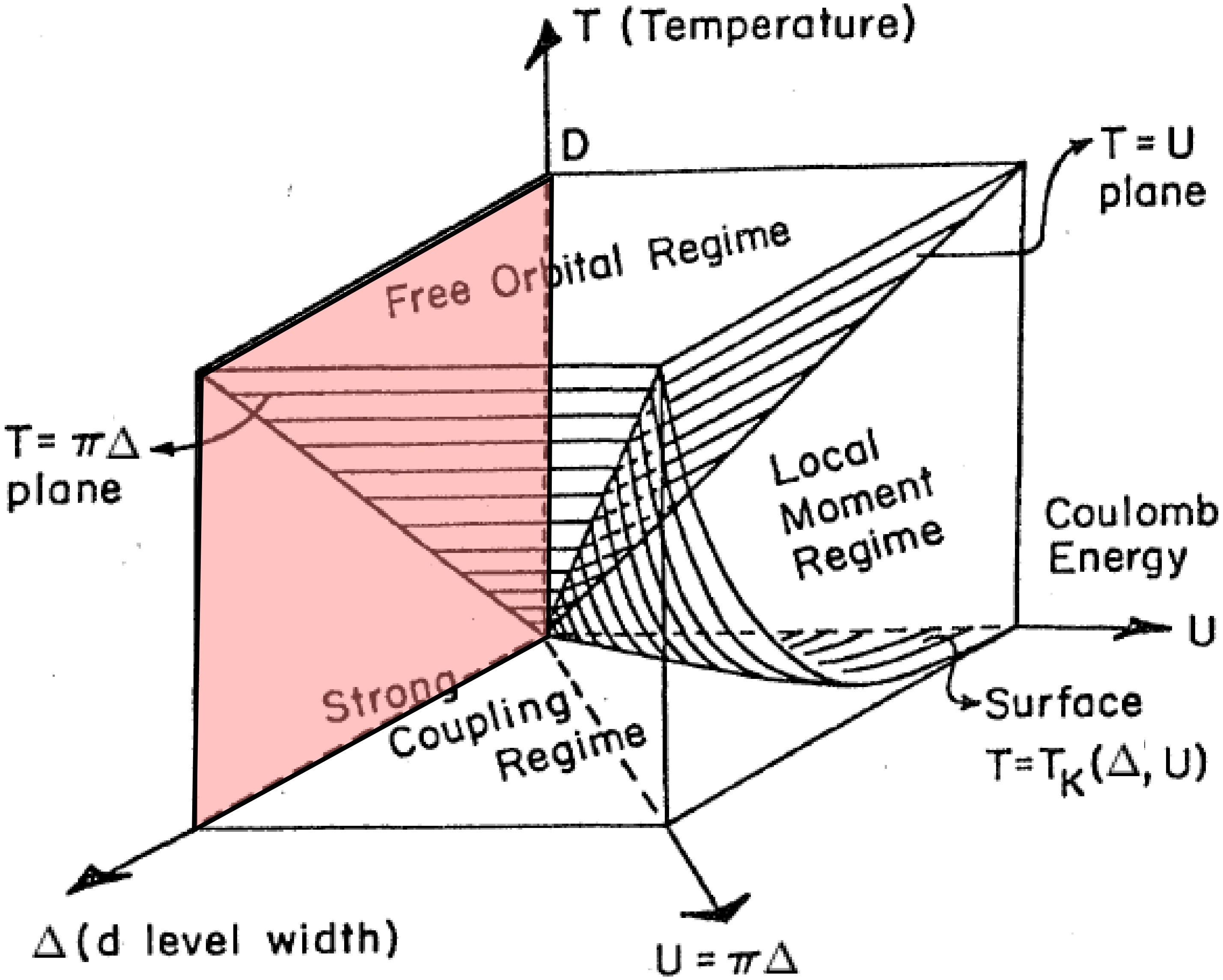


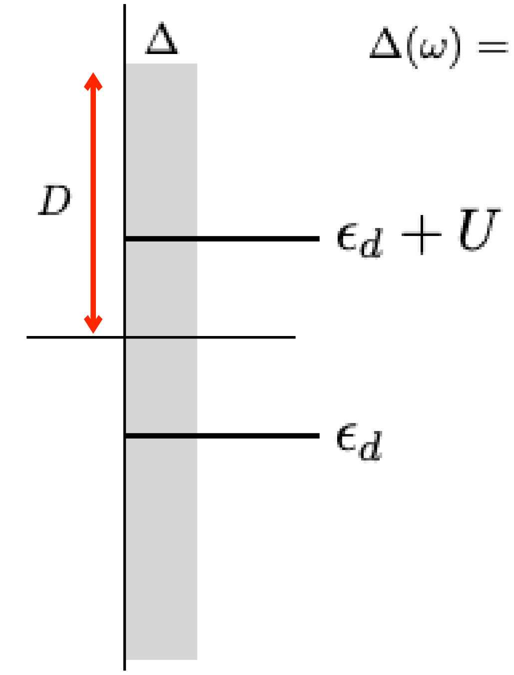
Krishnamurthy,Wilkins and Wilson, PRB 1980

Non-interacting regime

$U = 0$

Anderson/Kondo model 'phase diagram'

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d (n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V_k^* d^\dagger_\alpha c_{k\alpha})$$



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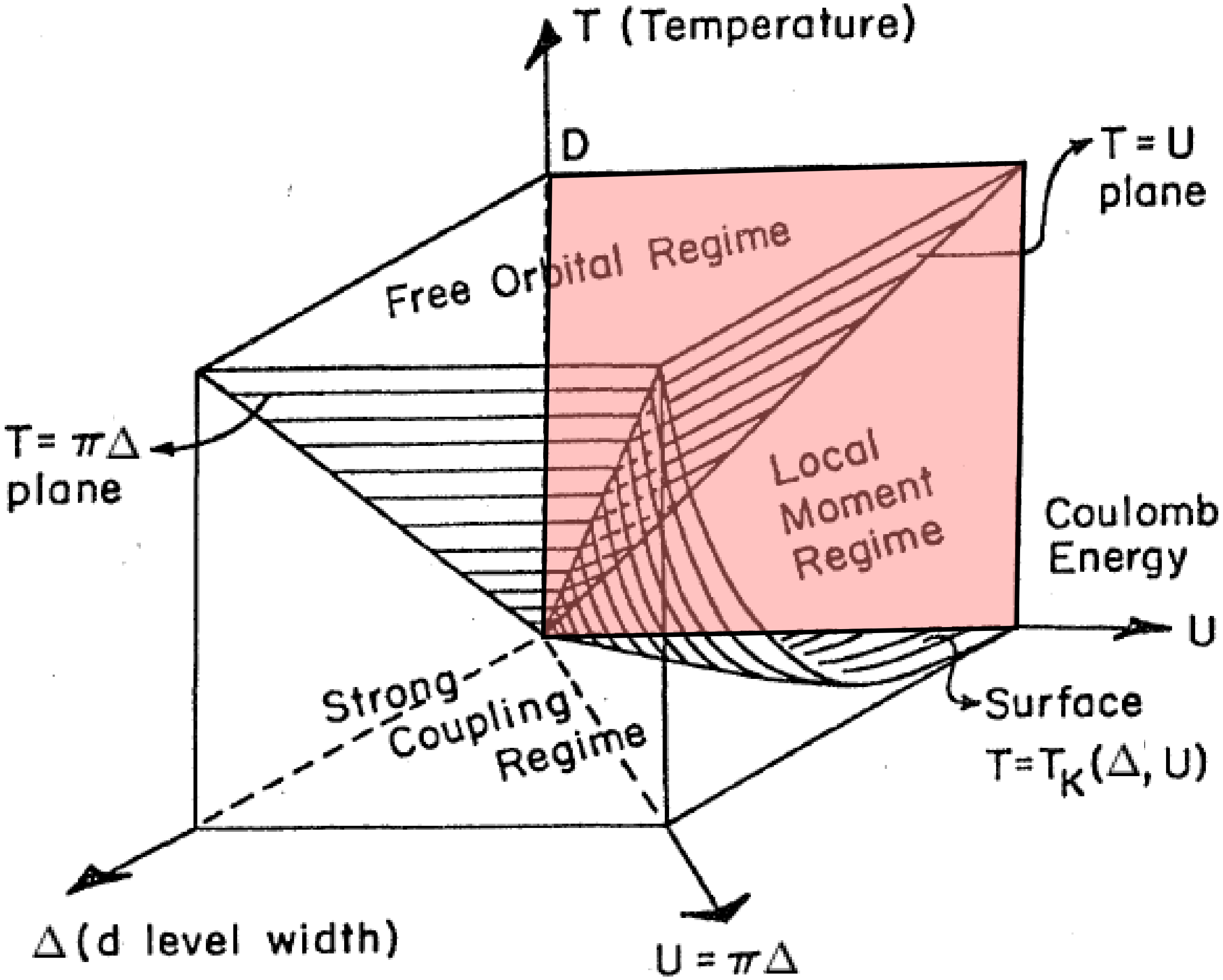
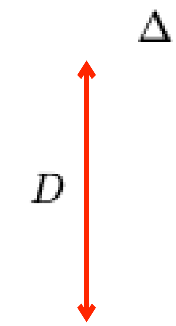


Free impurity
$\Delta = 0$

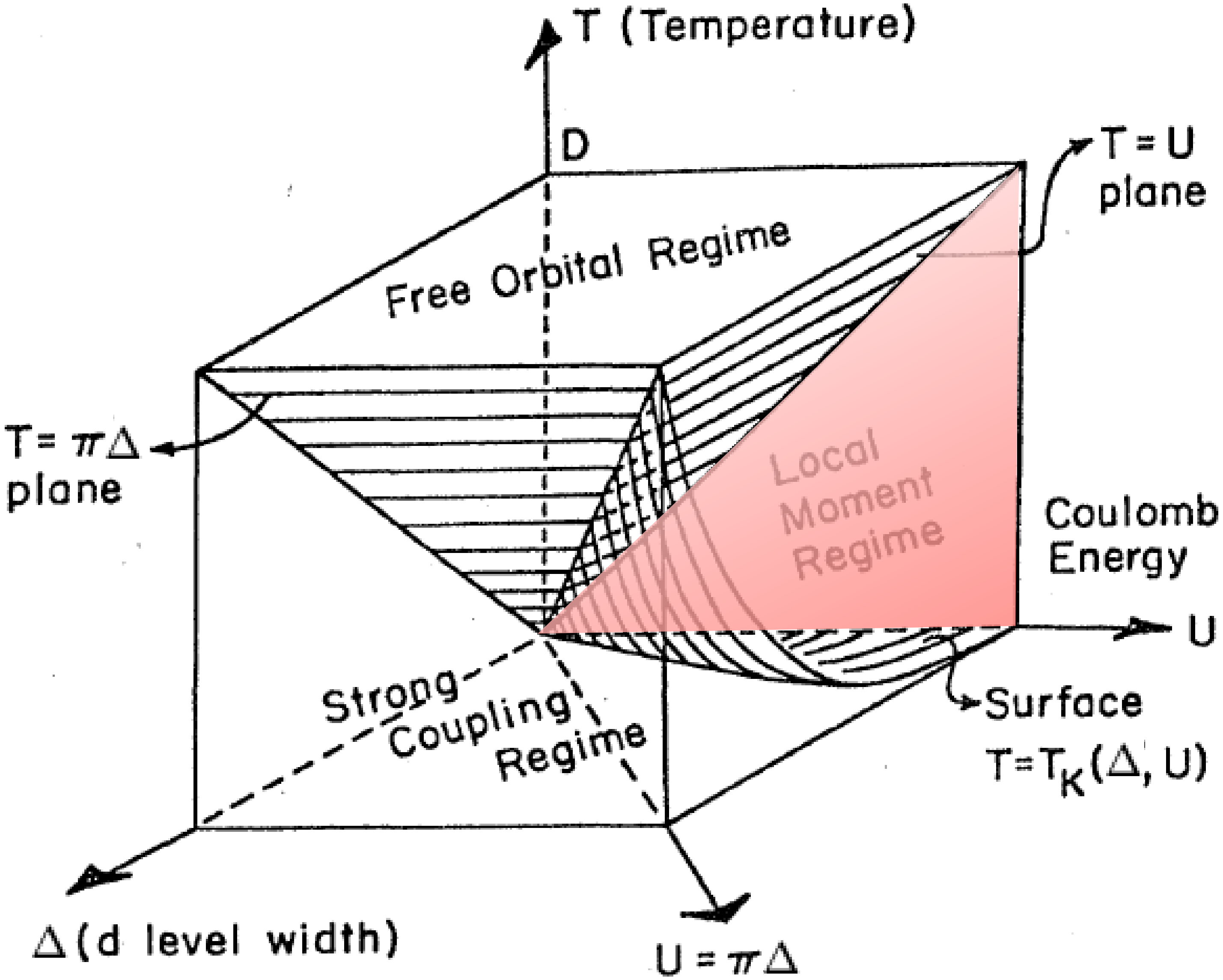
Krishnamurthy,Wilkins and Wilson, PRB 1980

Anderson/Kondo model 'phase diagram'

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d (n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V^*_k d^\dagger_\alpha c_{k\alpha})$$



$$\Delta(\omega) = \pi \sum_k V_k^2 \delta(\omega - \epsilon_k)$$

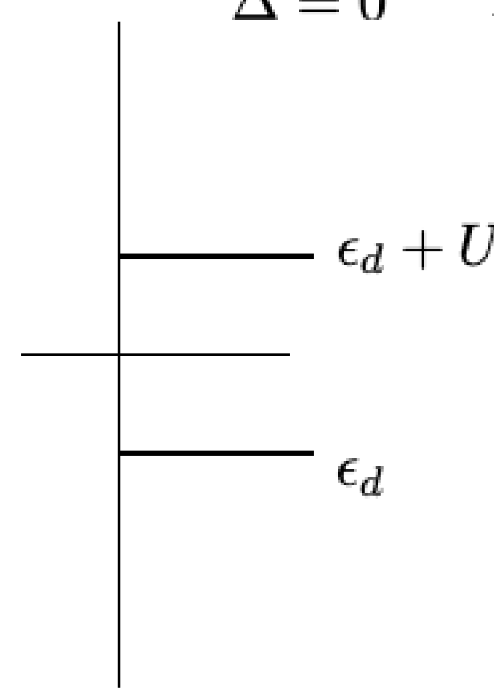


Krishnamurthy,Wilkins and Wilson, PRB 1980

Free impurity

Local moment regime

$\Delta = 0 \quad T \ll U$

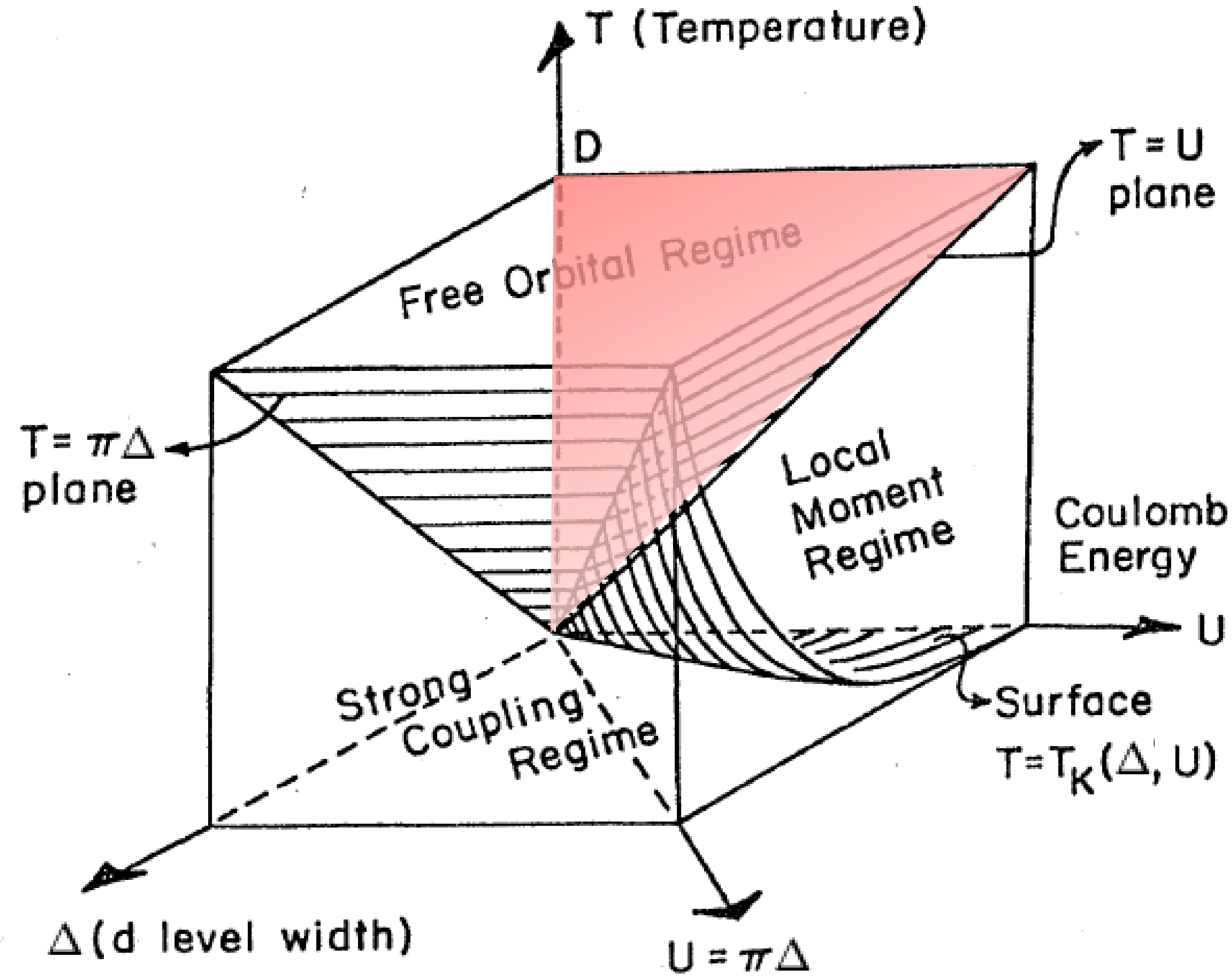


Anderson/Kondo model 'phase diagram'

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d (n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V^*_k d^\dagger_\alpha c_{k\alpha})$$

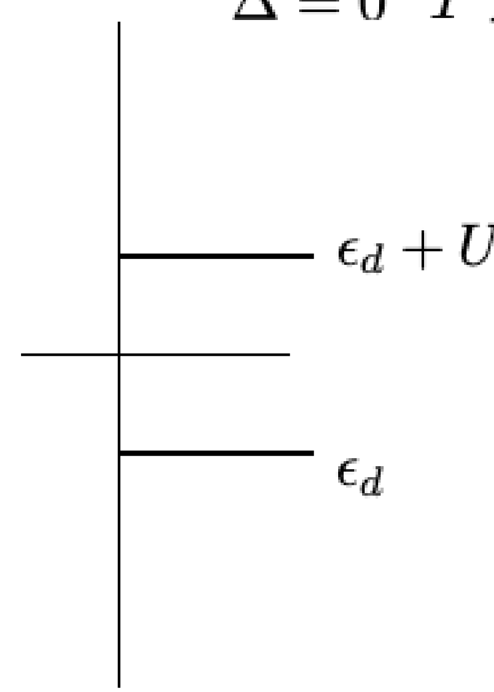
Δ $\quad \Delta(\omega) = \pi \sum_k V_k^2 \delta(\omega - \epsilon_k)$

D



Krishnamurthy,Wilkins and Wilson, PRB 1980

Free impurity
Free orbital regime
$\Delta = 0 \quad T \gg U$



Anderson/Kondo model 'phase diagram'

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d (n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V^*_k d^\dagger_\alpha c_{k\alpha})$$

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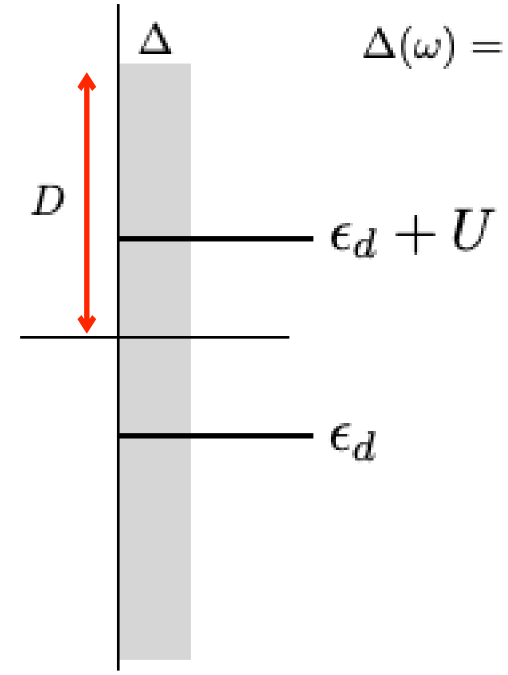
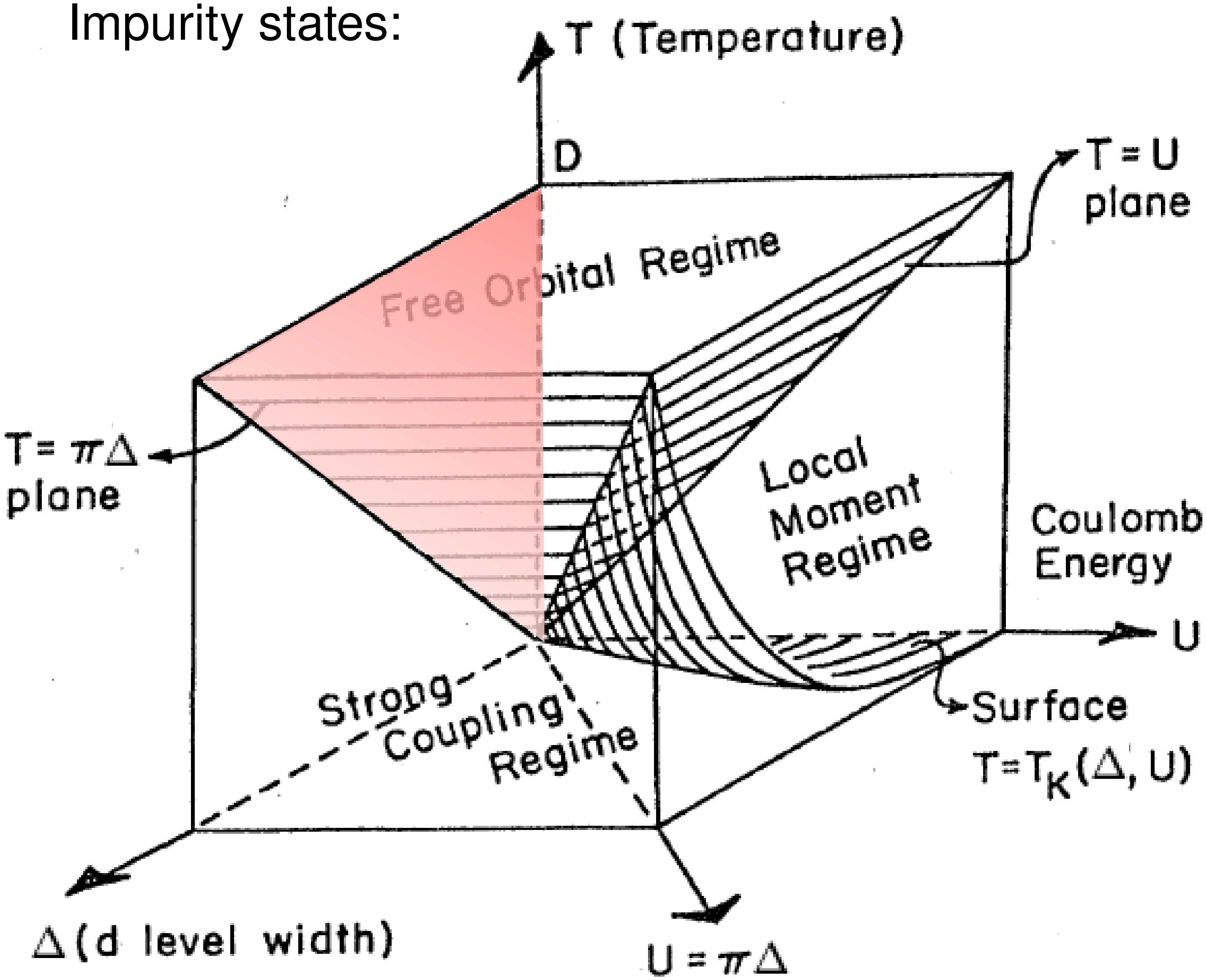


Impurity states:

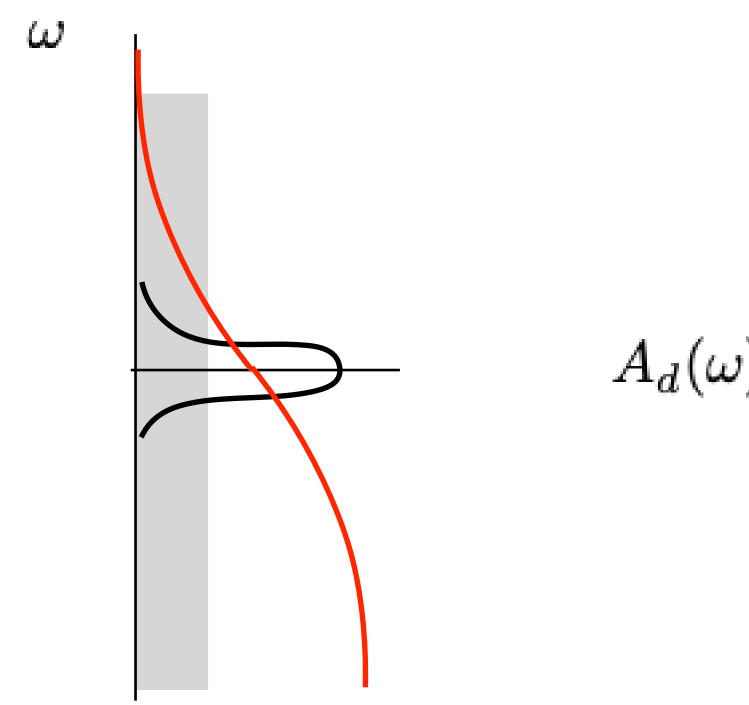


Krishnamurthy,Wilkins and Wilson, PRB 1980

Non-interacting regime

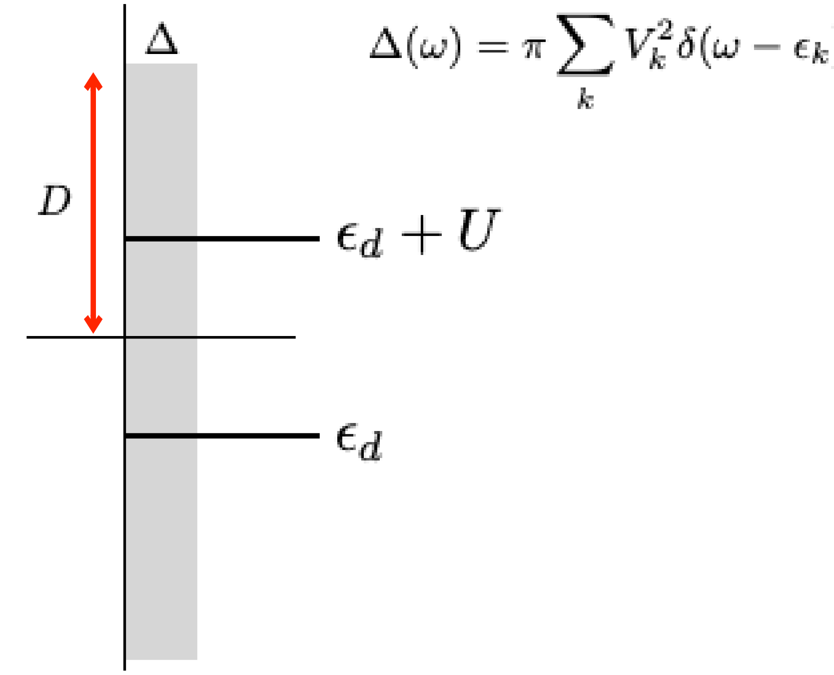
$U = 0$

$T \gg \Delta$

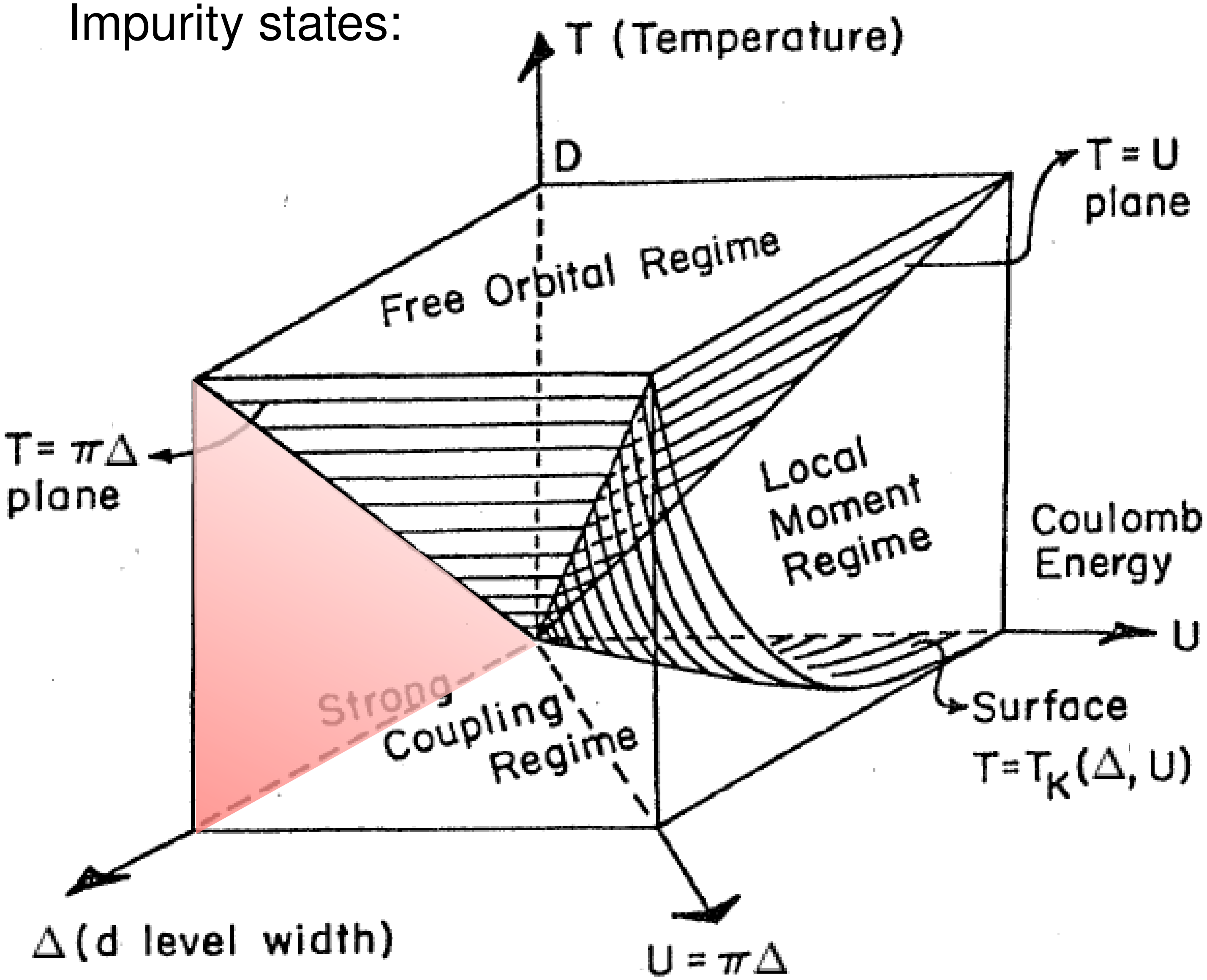


Anderson/Kondo model 'phase diagram'

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d (n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V^*_k d^\dagger_\alpha c_{k\alpha})$$



Impurity states:

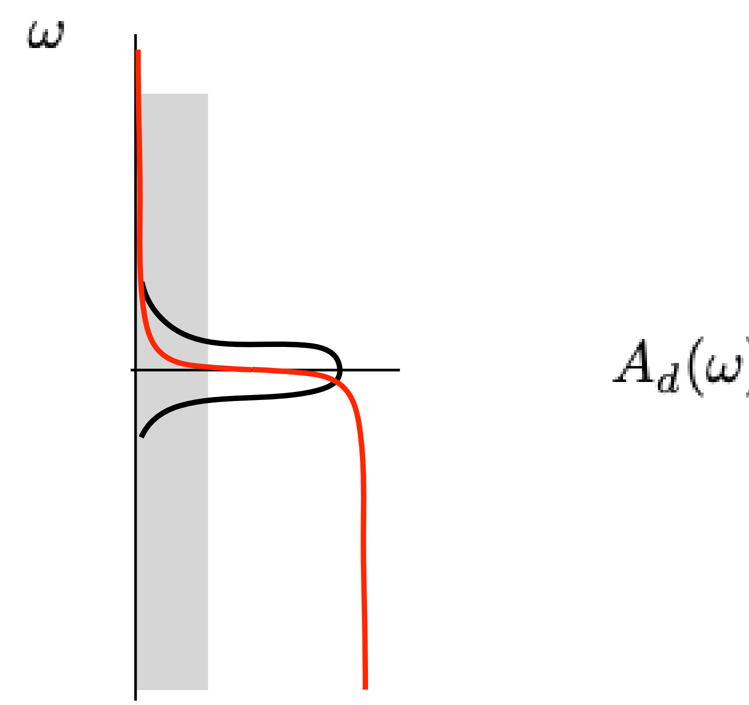


Krishnamurthy,Wilkins and Wilson, PRB 1980

Non-interacting regime

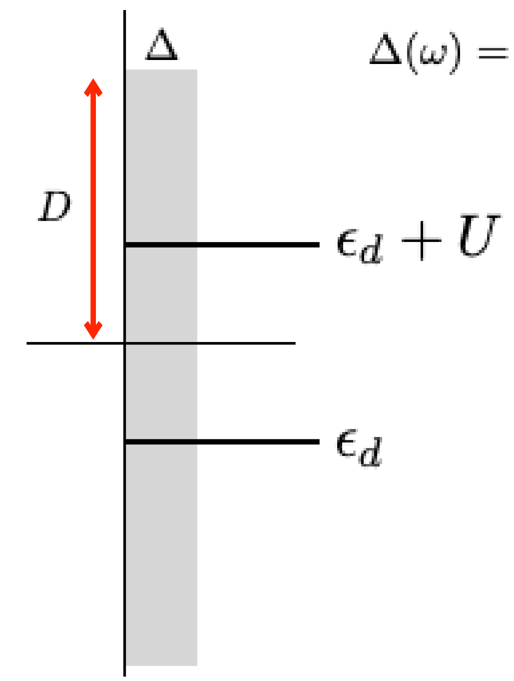
$U = 0$

$T \ll \Delta$

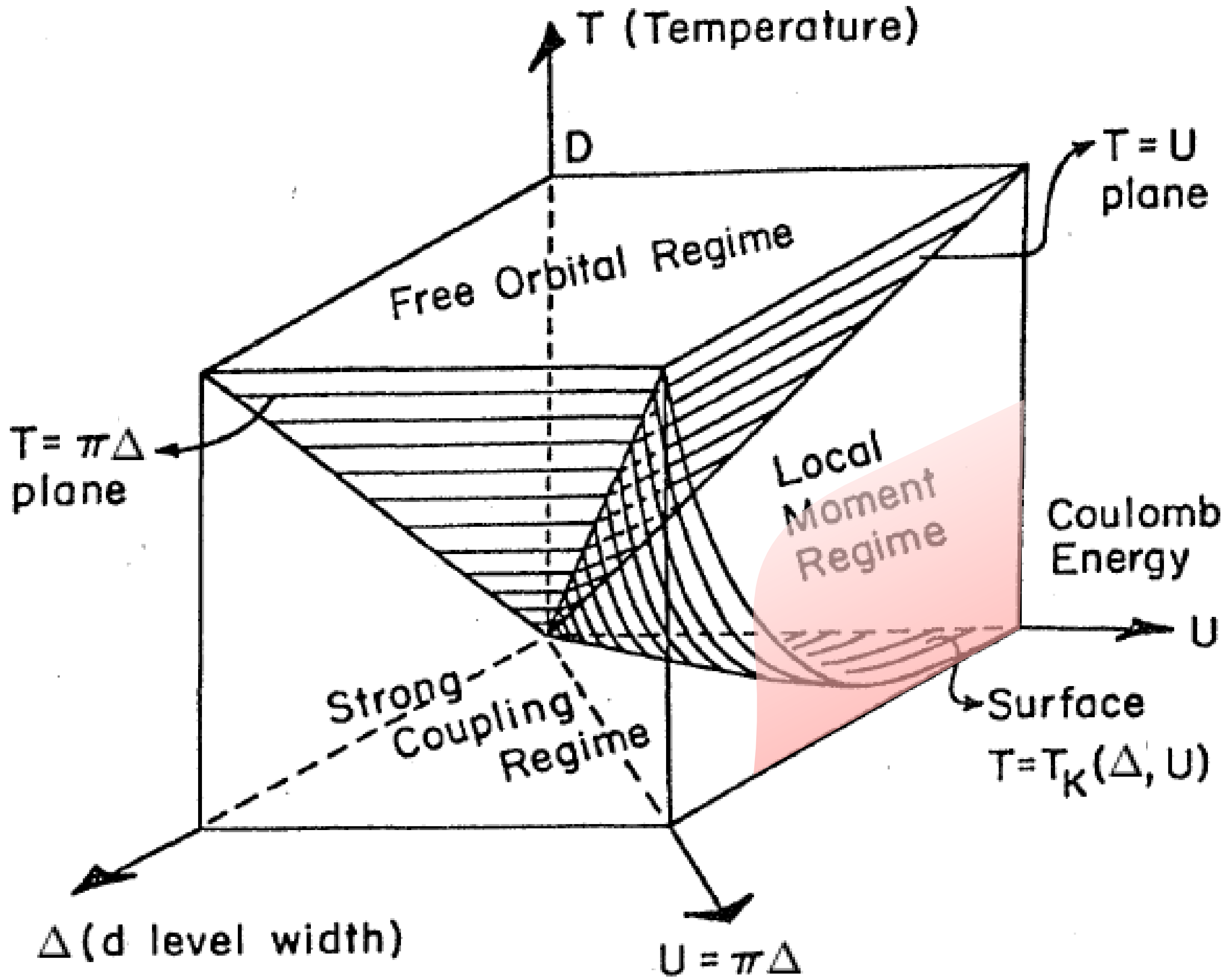


Anderson/Kondo model 'phase diagram'

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d (n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V_k^* d^\dagger_\alpha c_{k\alpha})$$



$$\Delta(\omega) = \pi \sum_k V_k^2 \delta(\omega - \epsilon_k)$$



Krishnamurthy,Wilkins and Wilson, PRB 1980

Kondo regime $\quad \rho J = \frac{4\Delta}{\pi U}$

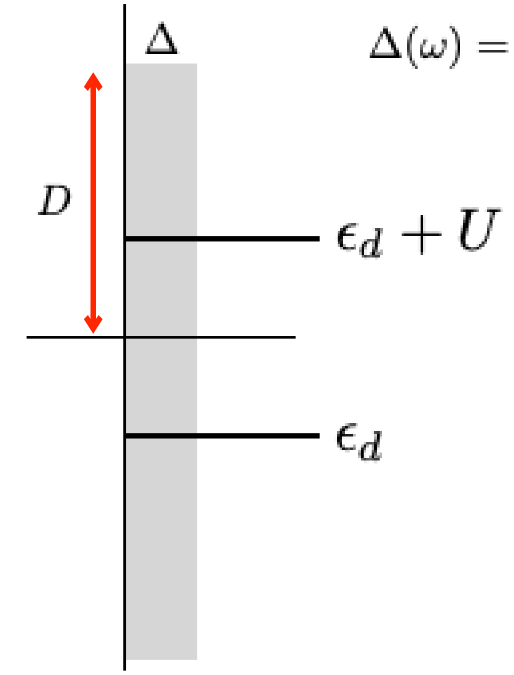
$\Delta, D, T \ll U$

Universal regime

$T \ll D$

Anderson/Kondo model 'phase diagram'

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d (n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V^*_k d^\dagger_\alpha c_{k\alpha})$$



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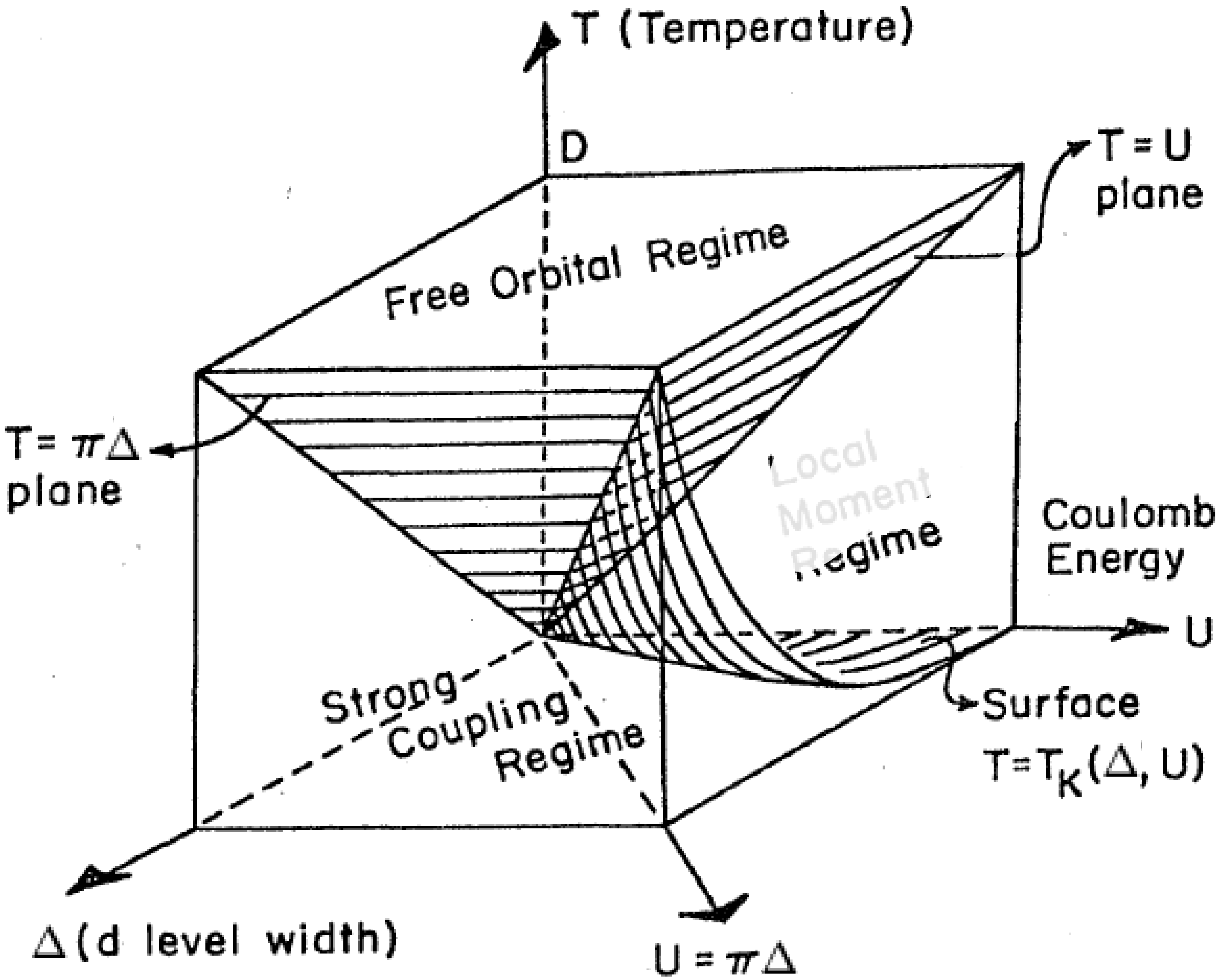

Kondo regime $\quad \rho J = \frac{4\Delta}{\pi U}$

$\Delta, D, T \ll U$

non-universal regime

$T \sim D$

Kondo effect - resistivity minimum

$$\frac{1}{\tau} \propto \left[\rho J + 2(\rho J)^2 \ln \frac{D}{T}\right]^2$$

Krishnamurthy,Wilkins and Wilson, PRB 1980

Anderson/Kondo model 'phase diagram'

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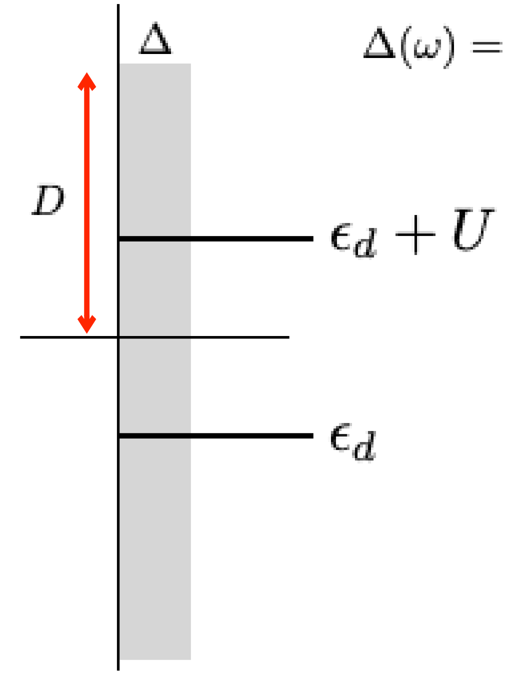

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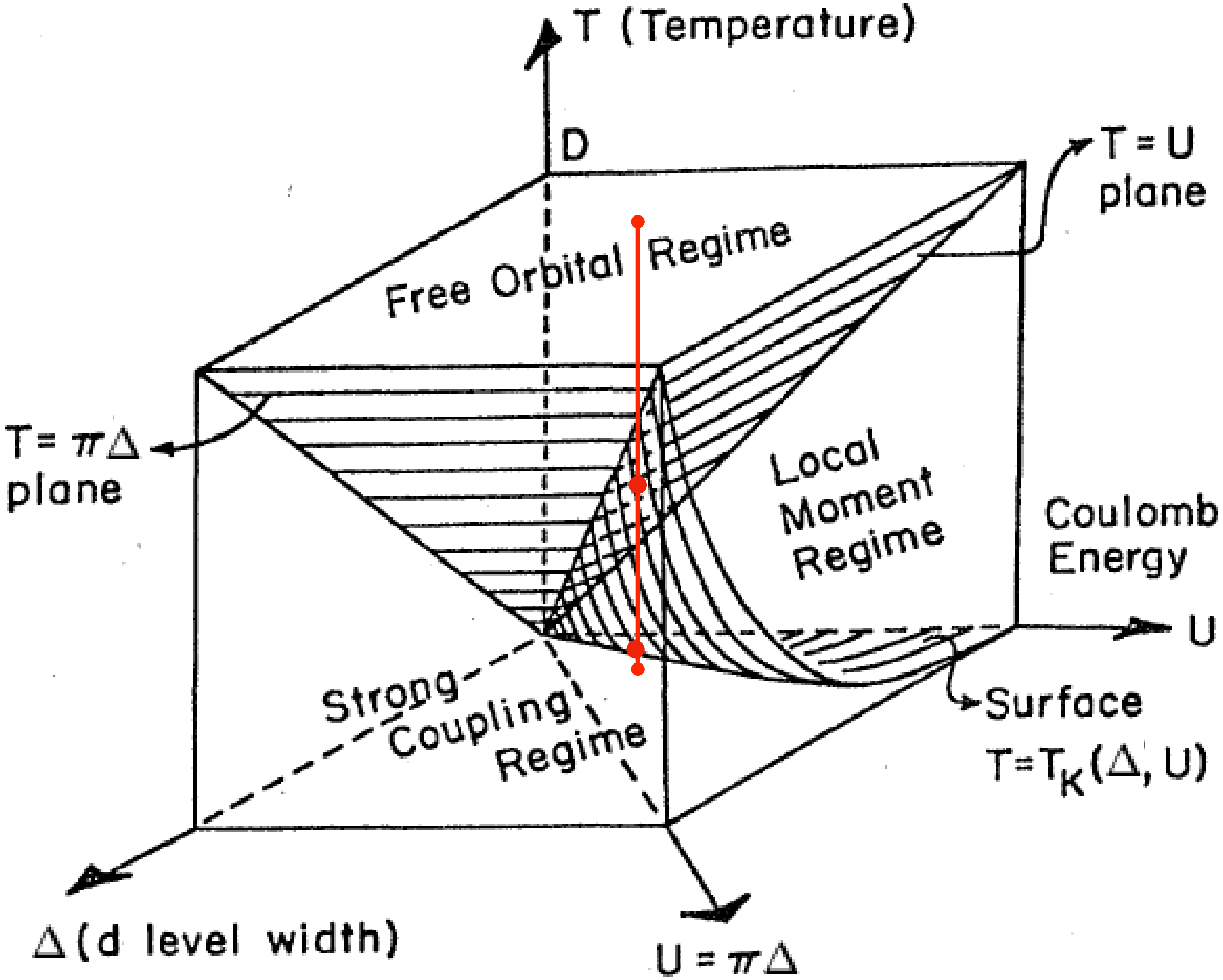


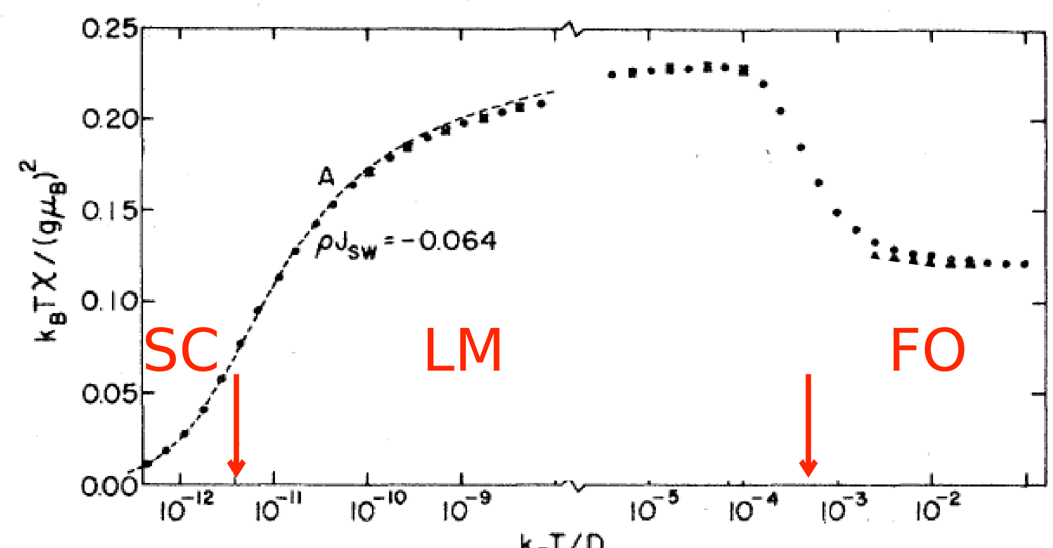
Krishnamurthy,Wilkins and Wilson, PRB 1980

Andeson regime $\rho J = \frac{4\Delta}{\pi U}$

$\Delta \ll U \ll D$

universal regime

$T \lesssim U/10$



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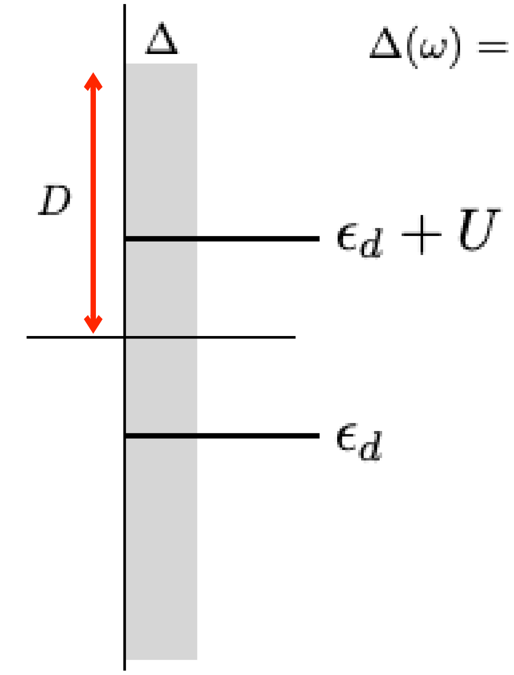
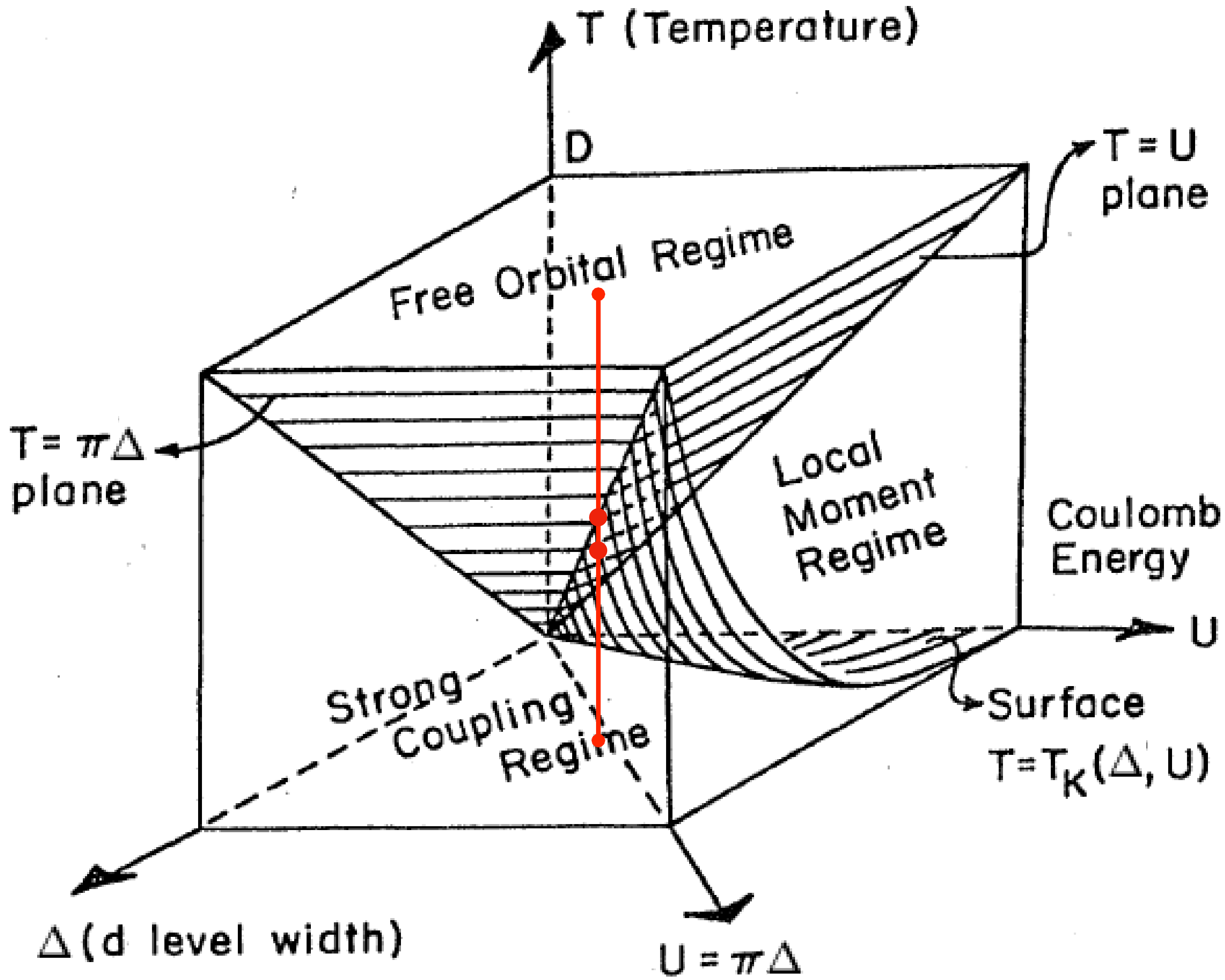




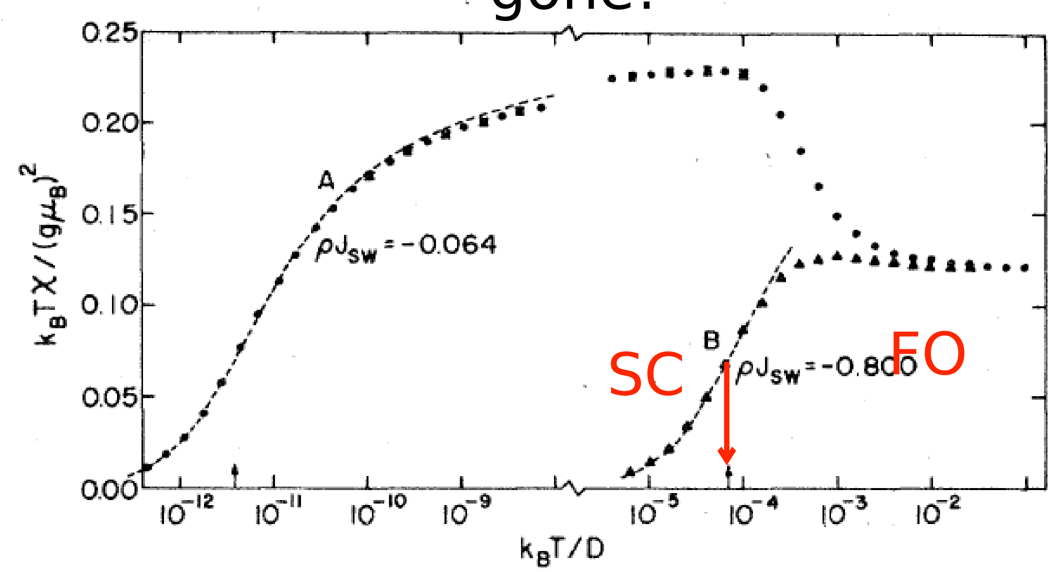
Krishnamurthy,Wilkins and Wilson, PRB 1980

Andeson regime $\rho J = \dfrac{4\Delta}{\pi U}$

$\Delta \sim U \ll D$

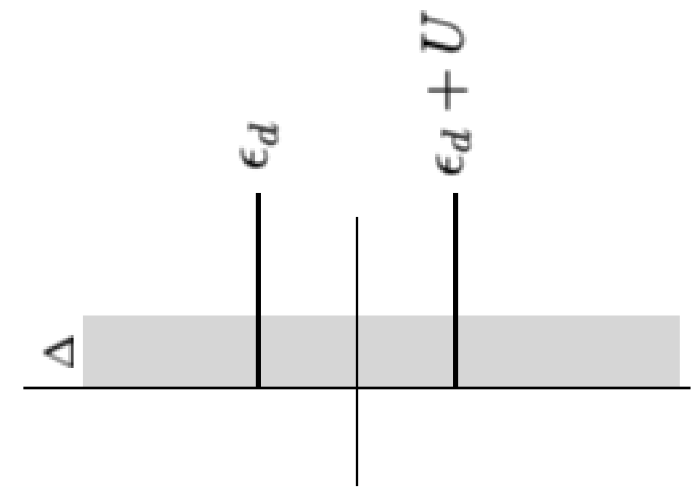
universal regime

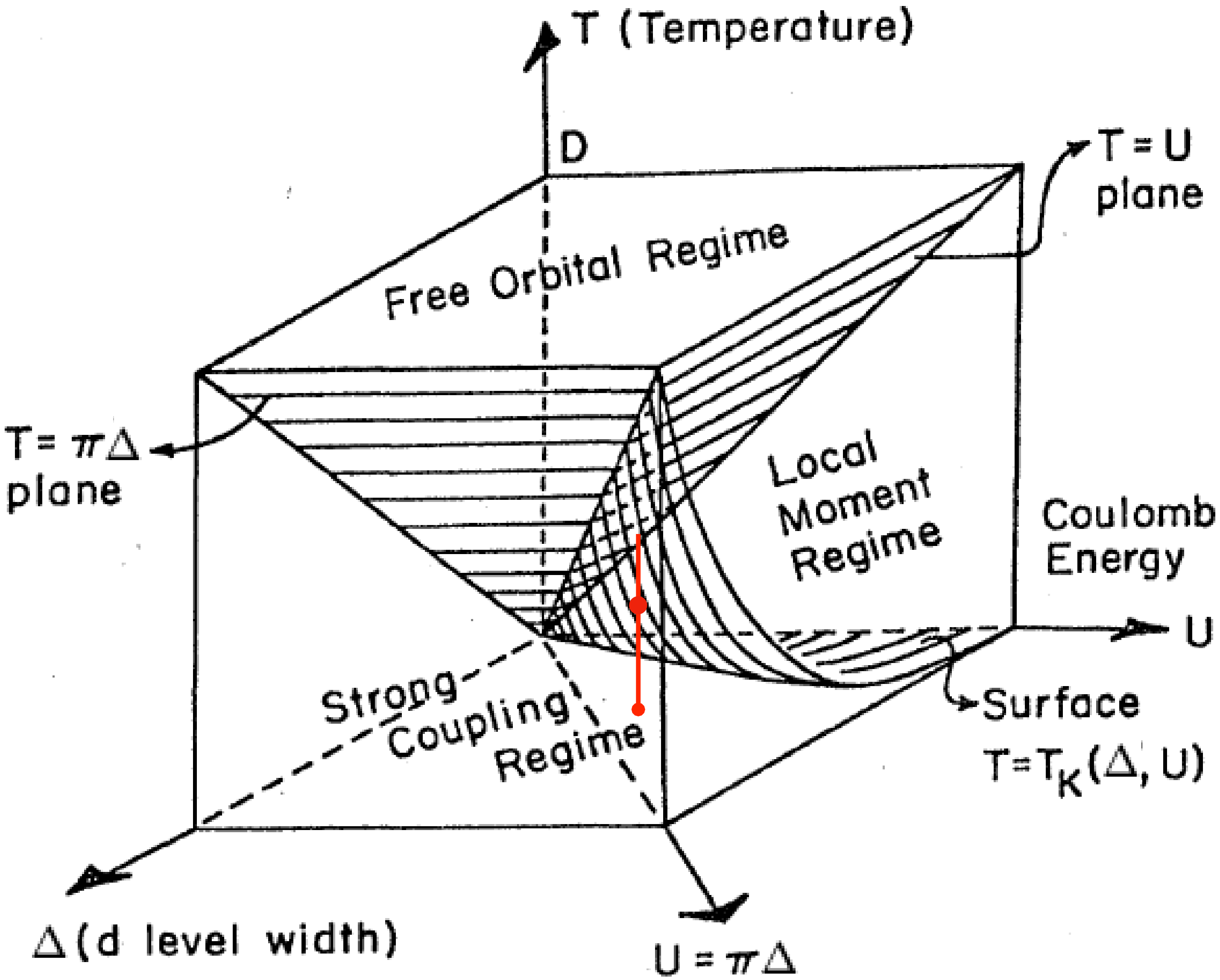
Local moment regime is gone!



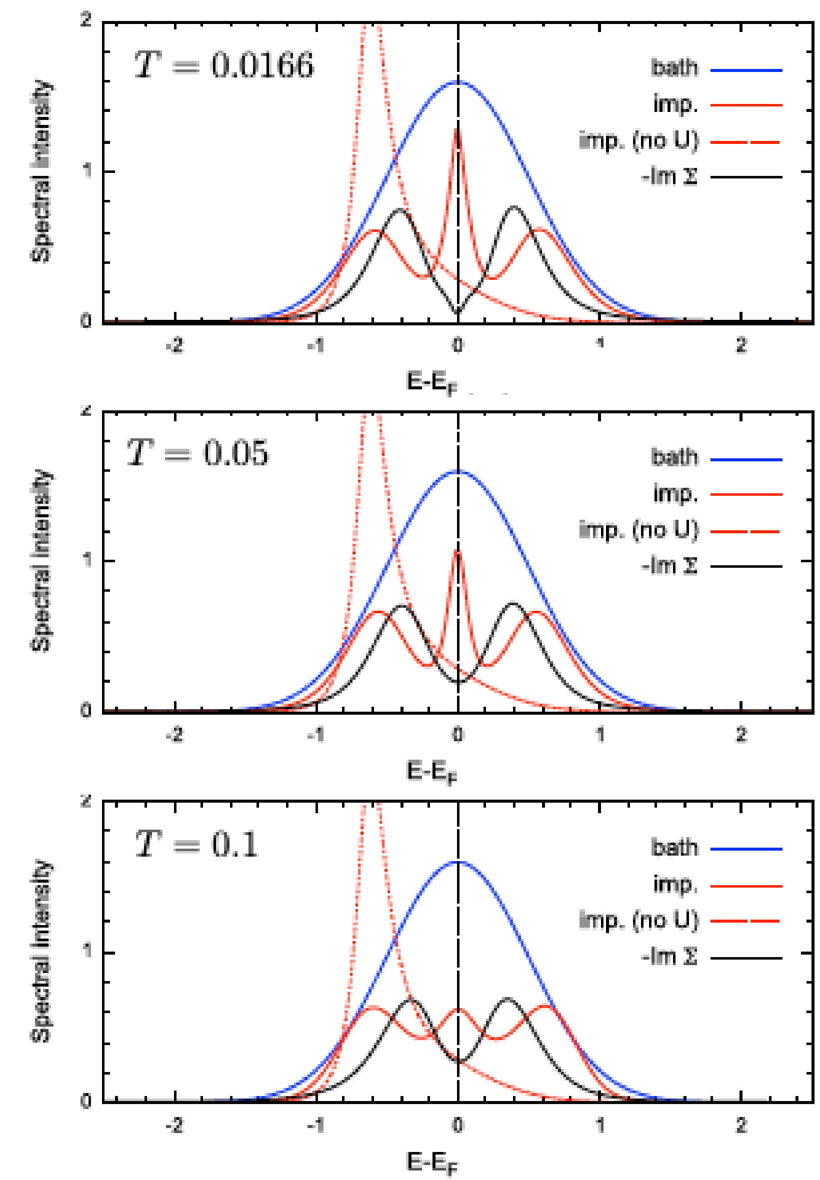
Anderson model impurity spectral function

$$H = \sum_{k,\alpha=\uparrow,\downarrow} \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + \epsilon_d (n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow + \sum_{k,\alpha=\uparrow,\downarrow} (V_k c^\dagger_{k\alpha} d_\alpha + V_k^* d^\dagger_\alpha c_{k\alpha})$$





Krishnamurthy,Wilkins and Wilson, PRB 1980



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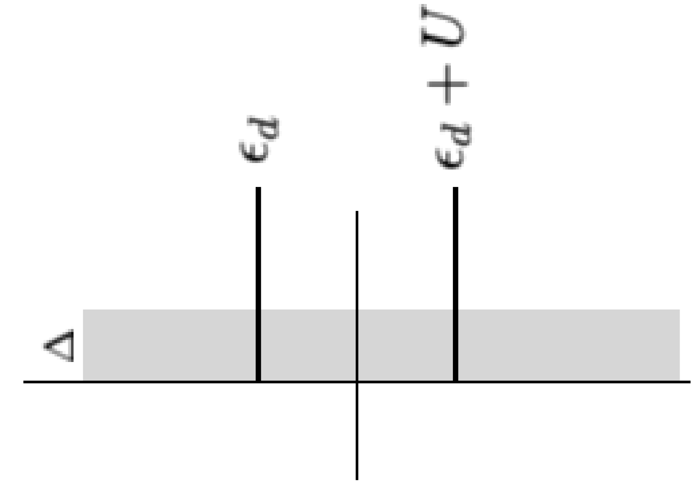
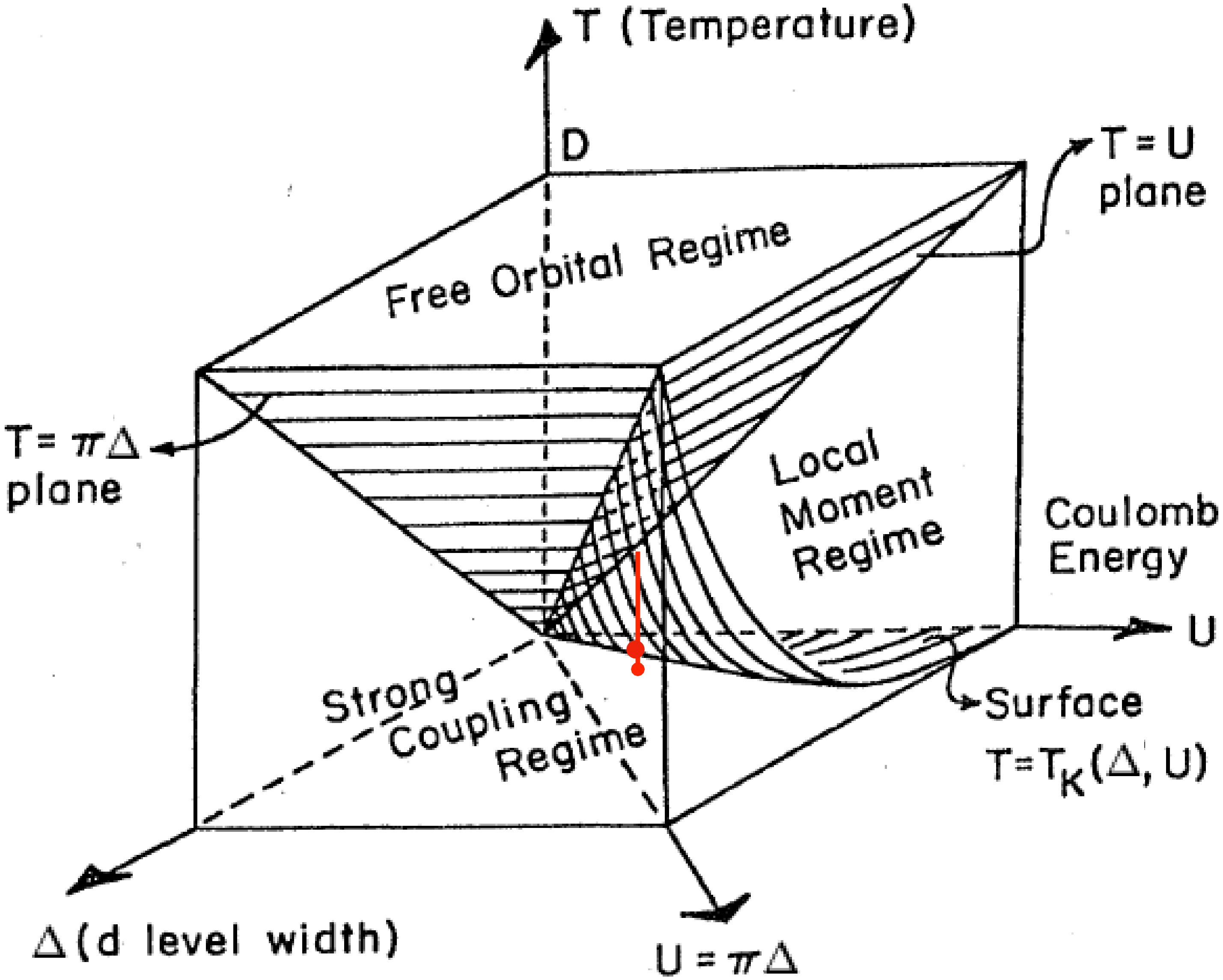
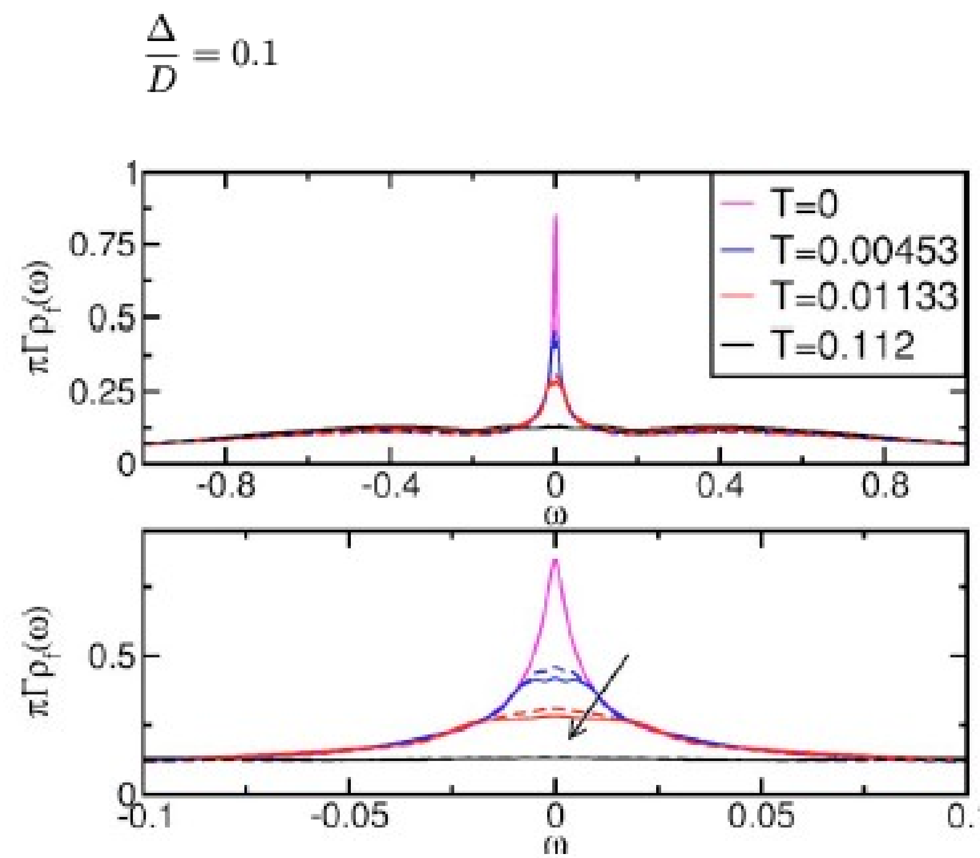




Krishnamurthy,Wilkins and Wilson, PRB 1980



Peters et al., PRB 2006

Anderson model impurity spectral function

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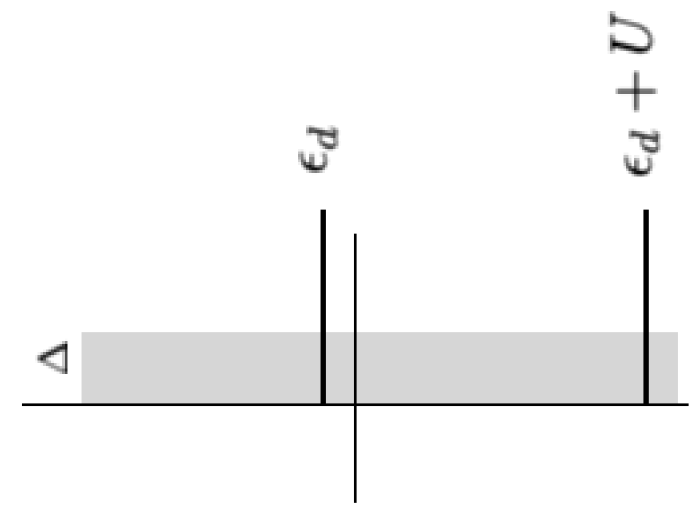
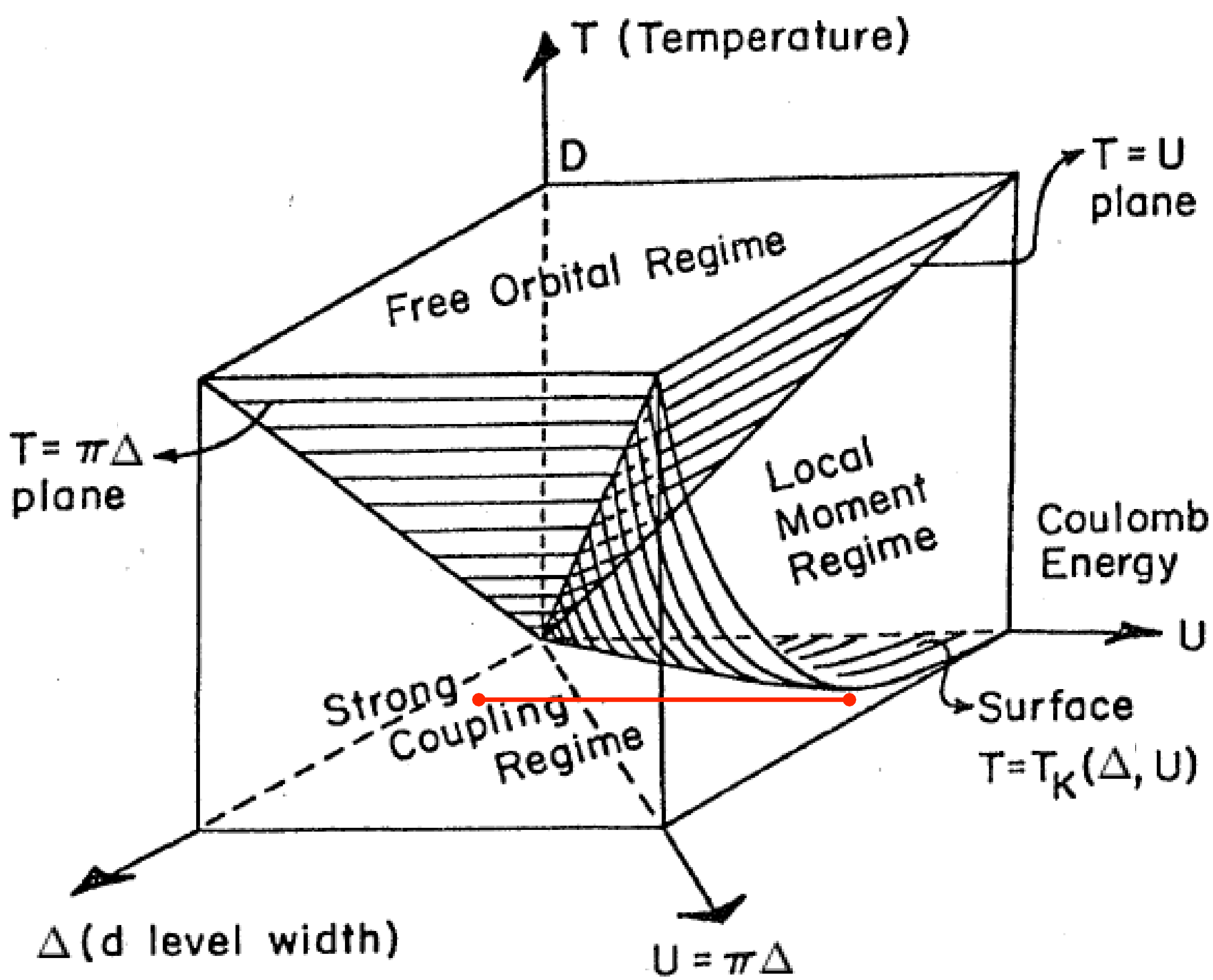
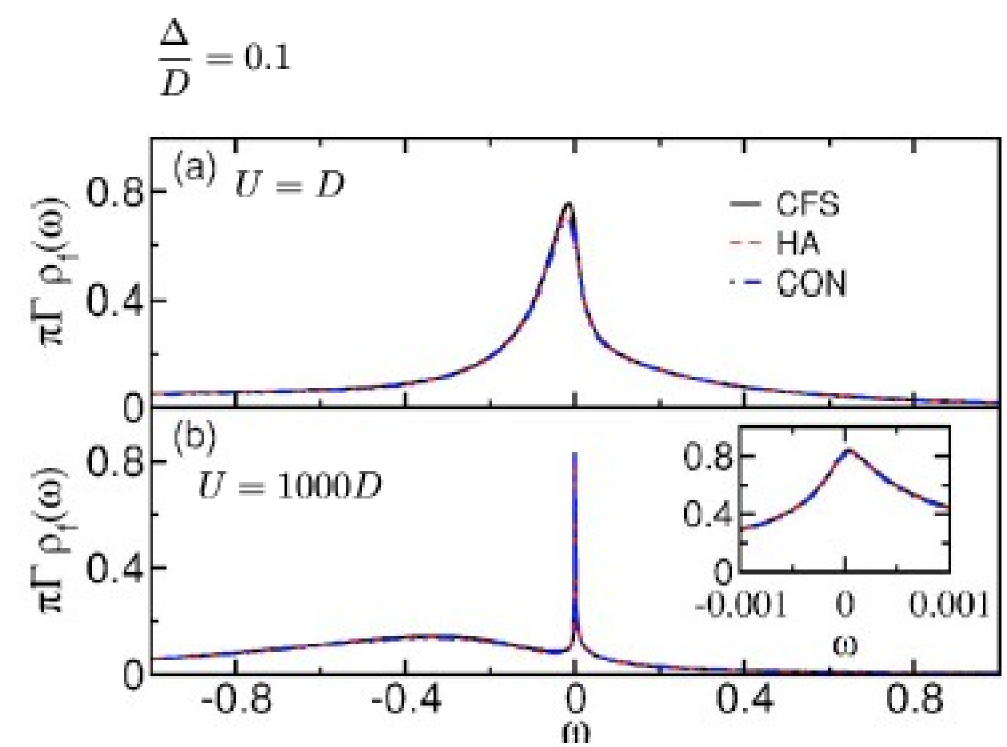
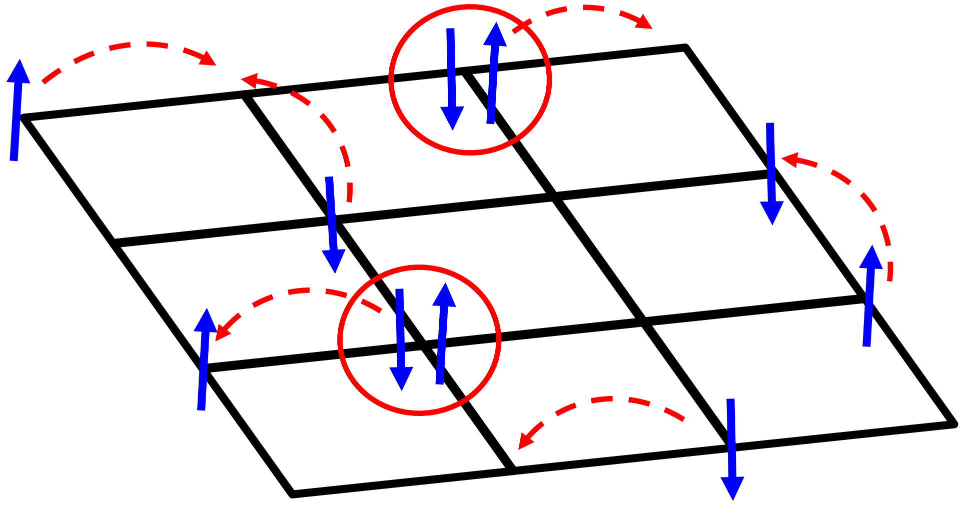


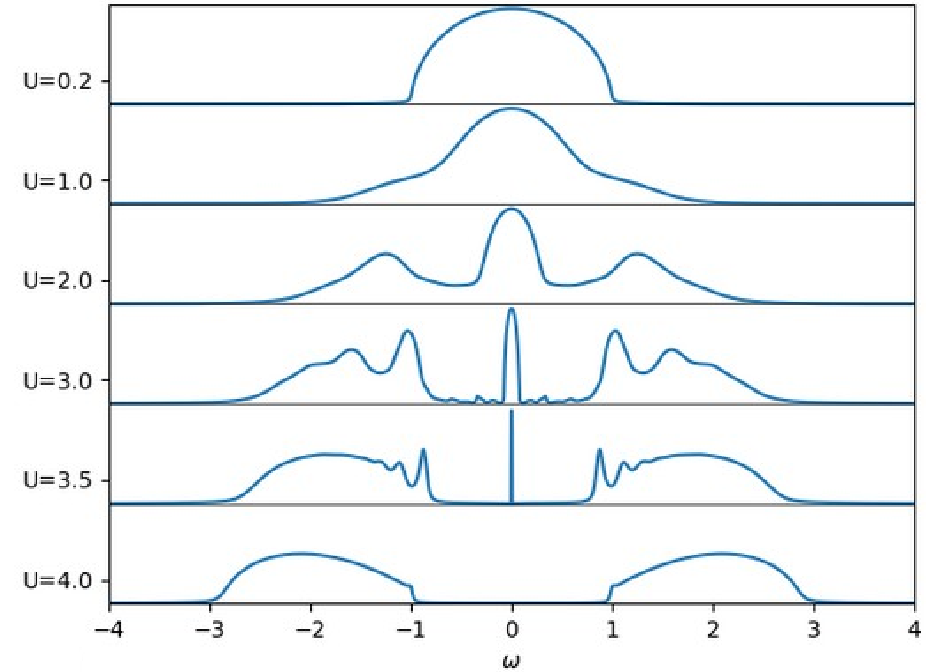




Peters et al., PRB 2006

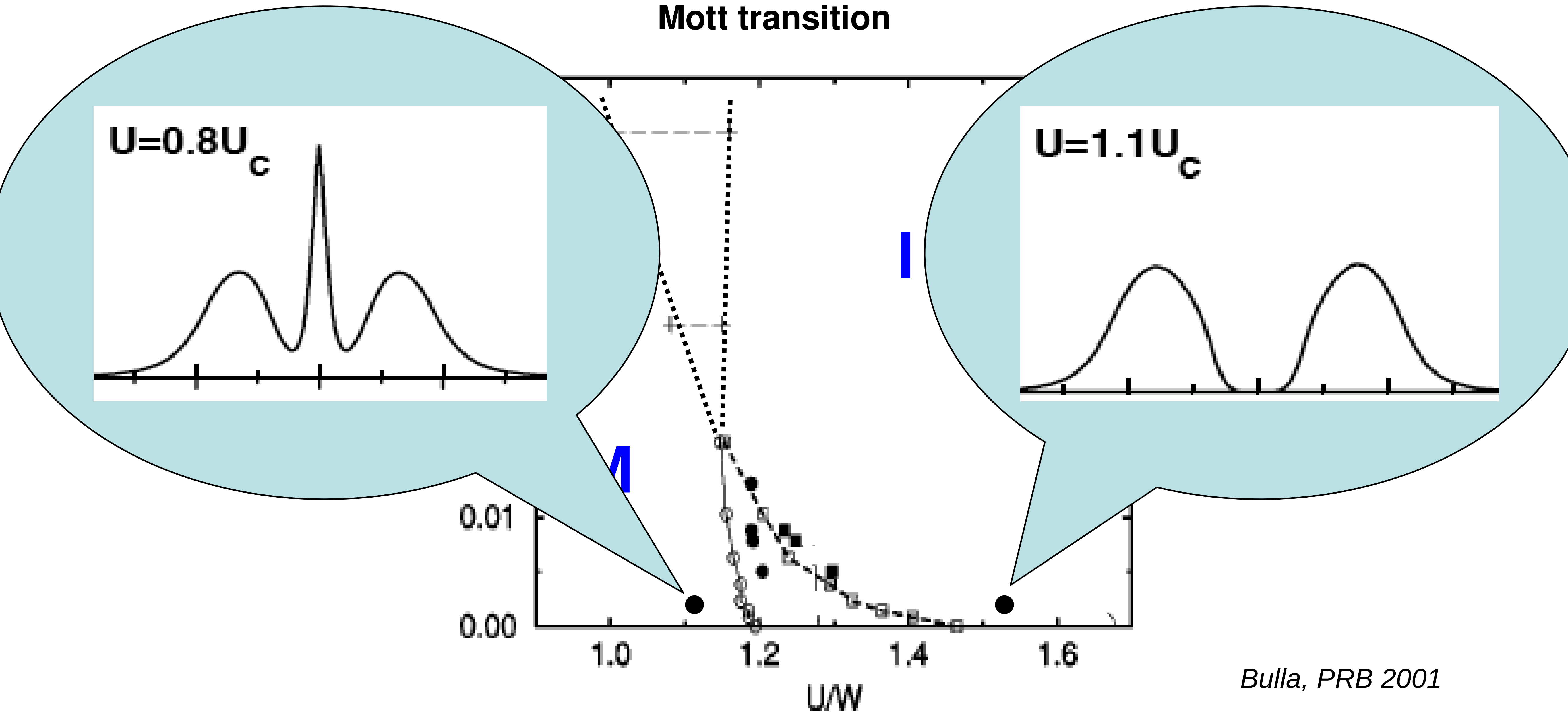
Hubbard model @ n=1: DMFT results





Georges et al., RMP 1996

Mott transition



Bulla, PRB 2001