Modeling asset pricing via SDEs

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Introduction

This project focuses on modeling asset prices using stochastic differential equations, a fundamental tool for capturing the uncertain and dynamic nature of financial markets. Starting from a stochastic description of the price $S(t)$, the interest rate $I(t)$, and the volatility $\sigma(t)$, the study addresses two key aspects: the theoretical analysis of the processes' properties and their numerical verification through computational simulations.
The first part of the project is dedicated to analytical studies, investigating essential properties of the processes, such as their martingale nature, positivity, and stationary distributions for specific models. The numerical analysis then focuses on studying the convergence orders of approximation schemes, such as the Euler-Maruyama method and the stochastic-θ method. A central role in the numerical analysis is played by Monte Carlo simulations, which are essential for computing expected values necessary for convergence studies and for validating numerical schemes. However, these simulations often introduce significant errors due to their intrinsic variance, which must be carefully analyzed and interpreted, especially when assessing the convergence of temporal discretization errors. The project leverages MATLAB for numerical implementations, applying the described methods to evaluate both theoretical accuracy and practical numerical errors.

Q1.1) We aim to solve the following stochastic differential equation (SDE):

$$dS(t) = IS(t)\,dt + \sigma S(t)\,dW(t) \tag{1}$$

Formally the equation yields:

$$\frac{dS(t)}{S(t)} = I\,dt + \sigma\,dW(t) \tag{2}$$

Applying the Itô formula to $Y(t) = \ln S(t)$

$$dY(t) = \frac{dS(t)}{S(t)} - \frac{1}{2}\frac{1}{S(t)^2}\sigma^2 S(t)^2 dt = \left(I - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t) \tag{3}$$

$$\ln S(t) = \ln S(0) + \int_0^t \left(I - \frac{1}{2}\sigma^2\right)ds + \int_0^t \sigma dW(s) \tag{4}$$

$$S(t) = S_0 e^{\left(I - \frac{1}{2}\sigma^2\right)t + \sigma W(t)} \tag{5}$$

Then, we state that $S(t)$ is a strong solution of the SDE, knowing that:

1. $t \mapsto S(t)$ is continuous
2. $S(t)$ is $\mathcal{F}(t)$-adapted
3. $(t,\omega) \mapsto IS(t,\omega)$ is in $\mathcal{M}^1(0,T)$ and $(t,\omega) \mapsto \sigma S(t,\omega)$ is in $\mathcal{M}^2(0,T)$
4. the equation holds almost surely for each t

Since $IS(t)$ and $\sigma S(t)$ are continuous, then 3 is verified because the mappings are progressively measurable.

Q1.2) We aim to show that the process $G(t) = e^{\sigma W(t)}$, where $W(t)$ is a standard B.M. and σ is a constant, is a submartingale but not a martingale.
We rewrite:

$$G(t) = e^{\sigma W(t)} = e^{\sigma W(s)} e^{\sigma(W(t) - W(s))} \tag{6}$$

Since $W(t) - W(s)$ is independent of $\mathcal{F}_s$ and $e^{\sigma W(s)}$ is $\mathcal{F}_s$-measurable:

$$\mathbb{E}[G(t) \mid \mathcal{F}_s] = e^{\sigma W(s)} \mathbb{E}[e^{\sigma(W(t)-W(s))}] \tag{7}$$

Now, being $W(t) - W(s) \sim \mathcal{N}(0, t-s)$, we compute the mgf:

$$\mathbb{E}[e^{\sigma(W(t)-W(s))}] = e^{\frac{1}{2}\sigma^2(t-s)} \tag{8}$$

Thus, we have:

$$\mathbb{E}[G(t) \mid \mathcal{F}_s] = G(s)e^{(\frac{1}{2}\sigma^2(t-s))} \tag{9}$$

Since $e^{\frac{1}{2}\sigma^2(t-s)} \geq 1$ for all $t \geq s$, it follows that $\mathbb{E}[G(t) \mid \mathcal{F}_s] \geq G(s)$, proving that $G(t)$ is a submartingale. Additionally, since $e^{\frac{1}{2}\sigma^2(t-s)} > 1$ for $t > s$, $G(t)$ is not a martingale.

Q1.3) The process $S(t)$ is a martingale if

$$\mathbb{E}[S(t) \mid \mathcal{F}_s] = S(s) \quad \text{for all } 0 \leq s \leq t \tag{10}$$

Taking s as the initial time, from the solution of the SDE we have

$$S(t) = S(s)e^{(I-\frac{1}{2}\sigma^2)(t-s)+\sigma(W(t)-W(s))} \tag{11}$$

Taking the conditional expectation:

$$\mathbb{E}[S(t) \mid \mathcal{F}_s] = S(s)e^{(I-\frac{1}{2}\sigma^2)(t-s)}\mathbb{E}\left[e^{\sigma(W(t)-W(s))}\right] = S(s)e^{(I-\frac{1}{2}\sigma^2)(t-s)}e^{\frac{1}{2}\sigma^2(t-s)} \tag{12}$$

$$\mathbb{E}[S(t) \mid \mathcal{F}_s] = S(s)e^{I(t-s)} \tag{13}$$

For $S(t)$ to be a martingale, we require $S(s)e^{I(t-s)} = S(s)$ which holds iff $I = 0$.

Q1.4) The procedure to solve the SDE with time-varying $I(t)$ and $\sigma(t)$ follows the same steps as in the case with constant parameters. We apply Itô's Lemma to $\ln S(t)$, and we obtain an equation for $\ln S(t)$. The difference is that when $I(t)$ and $\sigma(t)$ are time-dependent, we cannot simplify the terms as in the constant case.
Formally, the closed-form solution is:

$$S(t) = S(0)\exp\left(\int_0^t \left(I(u) - \frac{1}{2}\sigma(u)^2\right) du + \int_0^t \sigma(u)\, dW(u)\right) \tag{14}$$

Then, it is obvious that conditions 1, 2, and 4 of Q1.1 are satisfied. However, we shall ask $I(t)$ and $\sigma(t)$ to satisfy

$$(t,\omega) \mapsto I(t)S(t,\omega) \in \mathcal{M}^1(0,T) \quad \text{and} \quad (t,\omega) \mapsto \sigma(t)S(t,\omega) \in \mathcal{M}^2(0,T)$$

Q2.1) The given SDE is:

$$dI(t) = (a - bI(t))\, dt + c\, dW(t) \tag{15}$$

We apply the variation of constants method multiplying by the integrating factor e^{bt}:

$$e^{bt}\, dI(t) = e^{bt}(a - bI(t))\, dt + e^{bt}c\, dW(t) \tag{16}$$

$$d\left(I(t)e^{bt}\right) = ae^{bt}\, dt + ce^{bt}\, dW(t) \tag{17}$$

$$I(t)e^{bt} - I(0) = a\int_0^t e^{bu}\, du + c\int_0^t e^{bu}\, dW(u) \tag{18}$$

$$I(t) = I(0)e^{-bt} + \frac{a}{b}\left(1 - e^{-bt}\right) + ce^{-bt}\int_0^t e^{bu}\, dW(u) \tag{19}$$

Q2.2) $I(t)$ is gaussian because it consists of a deterministic part:

$$\mu(t) = I(0)e^{-bt} + \frac{a}{b}\left(1 - e^{-bt}\right) \tag{20}$$

and a stochastic integral term that is gaussian. To see why the stochastic integral is gaussian, note that by the definition of the Itô integral, it is the limit in L^2 of the Itô integrals of a sequence of simple processes converging to the integrand in L^2. But, for each step function, the Itô integral is simply a sum of scaled Brownian increments, making it gaussian. Since convergence in L^2 implies convergence in distribution, the stochastic integral is gaussian as the distributional limit of these gaussian sums.
The expectation of $I(t)$ is just $\mu(t)$, as defined above, because the mean of the stochastic integral term is zero by its properties. Now, let's compute the variance, using Itô's Isometry we get:

$$\mathrm{Var}(I(t)) = c^2e^{-2bt}\mathbb{E}\left[(\int_0^t e^{bu}dW_u)^2\right] = c^2e^{-2bt}\int_0^t e^{2bu}du = \frac{c^2}{2b}(1 - e^{-2t}) \tag{21}$$

Thus:

$$I(t) \sim \mathcal{N}\left(I(0)e^{-bt} + \frac{a}{b}\left(1 - e^{-bt}\right), \frac{c^2}{2b}(1 - e^{-2t})\right) \tag{22}$$

Let $I(0) = a = b = c = 1$. The probability of $I(t) < 0$ is:

$$\mathbb{P}(I(t) < 0) = \mathbb{P}\left(\frac{I(t) - \mu(t)}{\sigma(t)} < \frac{0 - 1}{\sigma(t)}\right) = \mathbb{P}\left(Z < -\frac{1}{\sigma(t)}\right) = \Phi\left(-\frac{1}{\sigma(t)}\right) \tag{23}$$

The values for different t are:

t	1	5	10	15	20
$\mathbb{P}(I(t) < 0)$	0.064	0.0786	0.0786	0.0786	0.0786

Since I(t) is intended to model the interest rate, it is expected to remain positive in most cases, as interest rates represent the expected return on an investment. Investors are unlikely to buy an asset if it offers no profit, which underscores the need for a positive rate. While this is an oversimplification, since negative interest rates can occur, ensuring positivity might be essential in some models. This SDE, however, does not enforce positivity, making it unsuitable for accurately modeling interest rates in most typical scenarios.

Q2.3) Consider the following auxiliary process, which also has a unique strong solution:

$$dI(t) = (a - bI(t))\,dt + c\sqrt{(I(t) \vee 0)}\,dW_t \tag{24}$$

Let $\epsilon > 0$ and define $\tau_\epsilon = \inf\{t > 0 : I_t = -\epsilon\}$, which is the first time the process I_t reaches $-\epsilon$.
We want to show that $P(\tau_\epsilon = \infty) = 1$, so let assume for contradiction that this is not the case, i.e., suppose there exists a sample path $\omega \in \Omega$ such that $\tau(\omega) < \infty$. Since I_t is continuous a.s. for B.M. properties, there must exist an interval $(\tau(\omega) - h, \tau(\omega))$ for some $h > 0$ in which $I_s(\omega) < 0$ for all s in this interval. Meaning that I_t is strictly negative immediately before it reaches the value $-\epsilon$ at time $\tau(\omega)$.
On this interval (24) can locally be expressed as

$$dI_t = a(b - I_t)\,dt \tag{25}$$

But since $a > 0$ and $b > 0$ for hypothesis and $I_t(\omega) < 0$ in $(\tau(\omega) - h, \tau(\omega))$, we have $a(b - I_t(\omega)) > 0$, meaning I_t is strictly increasing immediately before it reaches $-\epsilon$. However, this situation contradicts the assumption that I_t could reach the value $-\epsilon$: if I_t were indeed approaching $-\epsilon$ for the first time, the process should be decreasing in that interval. This contradiction implies that our initial assumption $P(\tau_\epsilon < \infty) > 0$ must be false. Therefore, we conclude that $P(\tau_\epsilon = \infty) = 1$.
Taking the limit as ϵ approaches zero and noting that the auxiliary process coincides with the original process for $I(t) \geq 0$, and since we have shown that the auxiliary is non-negative almost surely, the original process I_t is also almost surely non-negative.

Q2.4) Disclaimer: (2) and (3) which I refer are of the text of the project.
A mean-reverting process is a process where the SDE has a drift term like (2), i.e., there exists a value, $\frac{a}{b}$, such that if $I > \frac{a}{b}$, the drift pulls the process down to $\frac{a}{b}$, and if $I < \frac{a}{b}$, the drift pushes the process up to $\frac{a}{b}$. Plugging $I(0) = \frac{a}{b}$ into (20), we obtain $\mu(t) = \mu = \frac{a}{b}$, i.e., the expected value of the process is always $\frac{a}{b}$. On the other hand, if $I(0) \neq \frac{a}{b}$, this holds true only for the long-term behavior:

$$\lim_{t\to\infty} \mathbb{E}[I(t)] = \frac{a}{b} \tag{26}$$

(3) is mean reverting too, since its drift is the same of of (2), furthermore its diffusion is not sufficiently strong to make the process explode and it can be shown with Fokker-Planck equation that the process admits a stationary distribution with mean $\frac{a}{b}$.

Q3.1) The given SDE is:

$$d\tilde{\sigma}(t) + \lambda\tilde{\sigma}(t)\,dt = f\,dW(t) \tag{27}$$

We apply the variation of constants method multiplying by the integrating factor $e^{\lambda t}$:

$$e^{\lambda t} d\tilde{\sigma}(t) + \lambda e^{\lambda t}\tilde{\sigma}(t)\,dt = f e^{\lambda t}\,dW(t) \tag{28}$$

$$d\left(\tilde{\sigma}(t)e^{\lambda t}\right) = f e^{\lambda t}\,dW(t) \tag{29}$$

$$\tilde{\sigma}(t)e^{\lambda t} - \tilde{\sigma}(0) = f\int_0^t e^{\lambda u}\,dW(u) \tag{30}$$

$$\tilde{\sigma}(t) = \tilde{\sigma}(0)e^{-\lambda t} + f\int_0^t e^{-\lambda(t-u)}\,dW(u) \tag{31}$$

Q3.2) Following the same argument of Q2.2 we deduce that:

$$\tilde{\sigma}(t) \sim \mathcal{N}\left(\tilde{\sigma}(0)e^{-\lambda t}, \frac{f^2}{2\lambda}\left(1 - e^{-2\lambda t}\right)\right) \tag{32}$$

So taking the $t \to \infty$ distributional limit we have:

$$\tilde{\sigma}(\infty) \sim \mathcal{N}\left(0, \frac{f^2}{2\lambda}\right) \quad \rho_\infty(\tilde{\sigma}) = \frac{\sqrt{\lambda}}{f\sqrt{\pi}} \exp\left(-\frac{\lambda\tilde{\sigma}^2}{f^2}\right) \tag{33}$$

The Fokker-Planck equation for the probability density $\rho(\tilde{\sigma}, t)$ associated with this SDE is:

$$\frac{\partial \rho(\tilde{\sigma}, t)}{\partial t} = \frac{\partial}{\partial \tilde{\sigma}}\left[\lambda \tilde{\sigma} \rho(\tilde{\sigma}, t)\right] + \frac{f^2}{2}\frac{\partial^2}{\partial \tilde{\sigma}^2}\rho(\tilde{\sigma}, t) \tag{34}$$

Plugging in the expression of $\rho_\infty(\tilde{\sigma})$ and noting that $\frac{\partial \rho_\infty(\tilde{\sigma})}{\partial t} = 0$ and $\frac{f^2}{2}\frac{\partial}{\partial \tilde{\sigma}}\rho_\infty(\tilde{\sigma}) = -\lambda\tilde{\sigma}\rho_\infty(\tilde{\sigma})$:

$$0 = \frac{\partial}{\partial \tilde{\sigma}}\left[\lambda\tilde{\sigma}\rho_\infty(\tilde{\sigma}) - \lambda\tilde{\sigma}\rho_\infty(\tilde{\sigma})\right] = 0 \tag{35}$$

Which is consistent with the fact that in order to derive the ergodic distribution we could also set the time derivative to zero in (34) and solve it.

Q3.3) Let $v(t) = \tilde{\sigma}^2(t)$. Itô's Lemma gives:

$$dv(t) = 2\tilde{\sigma}(t)\left(-\lambda\tilde{\sigma}(t)\,dt + f\,dW(t)\right) + f^2\,dt \tag{36}$$

$$dv(t) = -2\lambda v(t)\,dt + 2f\sqrt{v(t)}\,dW(t) + f^2\,dt \tag{37}$$

$$dv(t) = 2\lambda[\frac{f^2}{2\lambda} - v(t)]\,dt + 2f\sqrt{v(t)}\,dW(t) \tag{38}$$

$$dv(t) = k[\mu - v(t)]\,dt + \eta\sqrt{v(t)}\,dW(t) \tag{39}$$

Thus, the parameters are:

$$k = 2\lambda, \quad \mu = \frac{f^2}{2\lambda}, \quad \eta = 2f \tag{40}$$

This new SDE can be considered a good model in certain contexts because its volatility term ensures the positivity of the process (contrary to (15)). This, as said before, could be necessary in most cases where negative values are not realistic. However, the model has some limitations: not all processes exhibit mean-reverting behavior, and in certain scenarios, we might prefer a model where the speed of mean reversion adapt dynamically to how far the process is from its mean, while in the case of (39) it is constant.

Q4.1) We do not report the proof, which we built in class, but under the following assumptions it can be shown that:

$$\sqrt{\mathbb{E}\left[\sup_{0 \le t \le T}(S(t) - \hat{S}(t))^2\right]} \le C_p\sqrt{\Delta t}$$

Assumptions:

1. $S(0)$ is measurable, $S(0) \in L^2(\Omega)$
2. b and σ are Lipschitz in the first argument:

   $$|b(x,t) - b(y,t)| + |\sigma(x,t) - \sigma(y,t)| \le L|x - y|$$

3. b and σ satisfy a linear growth bound:

   $$|b(x,t)| + |\sigma(x,t)| \le K(1 + |x|)$$

4. b and σ have local Hölder continuity with $\lambda \ge 1/2$:

   $$|b(x,t) - b(x,s)| + |\sigma(x,t) - \sigma(x,s)| \le K(1+|x|)|t-s|^\lambda$$

Verification:

1. $S(0) = S_0 > 0$ is constant, so it is measurable and $\mathbb{E}[S(0)^2] = S_0^2 < \infty$
2. $b(x,t) = Ix$ and $\sigma(x,t) = \sigma x$:
   - $|b(x,t) - b(y,t)| = |Ix - Iy| = |I| \cdot |x - y|$
   - $|\sigma(x,t) - \sigma(y,t)| = |\sigma x - \sigma y| = |\sigma| \cdot |x - y|$
3. $|b(x,t)| + |\sigma(x,t)| = |Ix| + |\sigma x| \le (|I| + |\sigma|)|x| \le K(1 + |x|)$
4. b and σ do not depend on t, so $|b(x,t) - b(x,s)| = 0$ and $|\sigma(x,t) - \sigma(x,s)| = 0$. The Hölder condition is trivially satisfied.

Q4.2 and Q4.3) Let $dS(t) = \phi_M(I(t))S(t)\,dt + \phi_M(\sigma(t))S(t)\,dt$, and $\hat{S}(t)$ its *Euler-Maruyama* approximation. To better understand the impact of discretizing $I(t)$ and $\sigma(t)$, we introduce an intermediate process $S^q(t)$, which is the *Euler-Maruyama* approximation of the system with $I(t)$ and $\sigma(t)$ kept continuous. This intermediate process $S^q(t)$ bridges the gap between the fully discretized approximation $\hat{S}(t)$ and the true solution $S(t)$:

$$\mathbb{E}\left[\sup_{0\le t\le T}|S(t)-\hat{S}(t)|^2\right]=\mathbb{E}\left[\sup_{0\le t\le T}|S(t)-S^q(t)+S^q(t)-\hat{S}(t)|^2\right]\le \tag{41}$$

$$\le\mathbb{E}\left[\sup_{0\le t\le T}|S(t)-S^q(t)|^2\right]+\mathbb{E}\left[\sup_{0\le t\le T}|S^q(t)-\hat{S}(t)|^2\right]=A+B$$

Where now we already know the order of convergence of A, which is simply the one described in Q4.1, so we shall focus only on B. The subscript n_s indicates that the processes are frozen at the most recent value t_{n_s} step of the EM discretization. The process $S^q(t)-\hat{S}(t)$ satisfies:

$$|S^q(t)-\hat{S}(t)|=\left|\int_0^t\left(\phi_M(I(s))S^q(s)-\phi_M(I_{n_s})\hat{S}_{n_s}\right)ds+\int_0^t\left(\phi_M(\sigma(s))S^q(s)-\phi_M(\sigma_{n_s})\hat{S}_{n_s}\right)dW_s\right| \tag{42}$$

By triangular inequality and introducing $\phi_M(I(s))\hat{S}_{n_s}$ in the first term let us exploit the boundedness and Lipschitz condition of ϕ_M:

$$\left|\int_0^t\left(\phi_M(I(s))S^q(s)-\phi_M(I(s))\hat{S}_q(s)+\phi_M(I(s))\hat{S}_{n_s}-\phi_M(I_{n_s})\hat{S}_{n_s}\right)ds\right| \tag{43}$$

$$\le\int_0^t|\phi_M(I(s))|\left(|S^q(s)-\hat{S}_{n_s}|\right)ds+|\hat{S}_{n_s}|\left(|\phi_M(I(s))-\phi_M(I_{n_s}|)\right)ds\le M\int_0^t\left(|S^q(s)-\hat{S}_{n_s}|\right)ds+\int_0^t|I(s)-I_{n_s}|\,|\hat{S}_{n_s}|ds$$

Same for the second term:

$$\left|\int_0^t\left(\phi_M(\sigma(s))S^q(s)-\phi_M(\sigma_{n_s})\hat{S}_{n_s}\right)dW_s\right|\le M\int_0^t\left(|S^q(s)-\hat{S}_{n_s}|\right)dW_s+\int_0^t|\sigma(s)-\sigma_{n_s}||\hat{S}_{n_s}|dW_s \tag{44}$$

Combining:

$$|S^q(t)-\hat{S}(t)|\le M\int_0^t\left(|S^q(s)-\hat{S}_{n_s}|\right)ds+M\int_0^t\left(|S^q(s)-\hat{S}_{n_s}|\right)dW_s+\int_0^t|I(s)-I_{n_s}|\,|\hat{S}_{n_s}|ds+\int_0^t|\sigma(s)-\sigma_{n_s}|\,|\hat{S}_{n_s}|dW_s$$

Now taking the sup and the expectation and applying Young's inequality, we get:

$$\mathbb{E}\left[\sup_{0\le t\le T}|S^q(t)-\hat{S}(t)|^2\right]\le 4\mathbb{E}\left[\sup_{0\le t\le T}|M\int_0^t\left(|S^q(s)-\hat{S}_{n_s}|\right)|^2ds\right]$$

$$+4\mathbb{E}\left[\sup_{0\le t\le T}\left|M\int_0^t\left(|S^q(s)-\hat{S}_{n_s}|\right)dW_s\right|^2\right]+4\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t\left(|I(s)-I_{n_s}|\right)|\hat{S}_{n_s}|ds\right|^2\right]$$

$$+4\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t\left(|\sigma(s)-\sigma_{n_s}|\right)|\hat{S}_{n_s}|dW_s\right|^2\right]=C+D+E+F$$

Let's bound first $\mathbb{E}[\hat{S}_{n_s}^2]$ and then delve with the terms.

$$\mathbb{E}[S_{n+1}^2]=\mathbb{E}[S_n^2]\left[1+\phi_M(I_n)\Delta t+\phi_M(\sigma_n)\Delta W_n\right]^2=\mathbb{E}[S_n^2]\mathbb{E}[(1+\phi_M(I_n)\Delta t+\phi_M(\sigma_n)\Delta W_n)^2] \tag{45}$$

Now for BM properties $\mathbb{E}[\Delta W_n]=0$ and $\mathbb{E}[\Delta W_n^2]=\Delta t$

$$\begin{aligned}\mathbb{E}[S_{n+1}^2]&=\mathbb{E}[S_n^2]\left(1+2\phi_M(I_n)\Delta t+(\phi_M(I_n))^2\Delta t^2+(\phi_M(\sigma_n))^2\Delta t\right)\\&\le\mathbb{E}[S_n^2]\left(1+(2M+M^2)\Delta t+M^2\Delta t^2\right)\\&\le\mathbb{E}[S_n^2]e^{(2M+M^2)\Delta t+M^2\Delta t^2}\end{aligned} \tag{46}$$

Iterating, we get an upper bound with the second moment of S_0 which is bounded being S_0 a costant

$$\mathbb{E}[S_n^2]\le\mathbb{E}[S_0^2]\prod_{k=0}^{n-1}e^{(2M+M^2)\Delta t+M^2\Delta t^2}\le\mathbb{E}[S_0^2]e^{(2M+M^2)T+M^2T^2}=C_T \tag{47}$$

Now for C we use Cauchy-Schwarz to get:

$$C\le 4M^2\mathbb{E}\left[\sup_{0\le t\le T}t\int_0^t\left|S^q(s)-\hat{S}_{n_s}\right|^2ds\right]\le 4M^2T\int_0^T\mathbb{E}\left[\left|S^q(s)-\hat{S}_{n_s}\right|^2\right]ds\le 4M^2T\int_0^T\mathbb{E}\left[\sup_{0\le\tau\le s}\left|S^q(\tau)-\hat{S}_{n_\tau}\right|^2\right]ds \tag{48}$$

For D by Doob's inequality and Itô's isometry:

$$D \leq 16M^2 \mathbb{E}\left[\left|\int_0^T |S^q(s) - \hat{S}_{n_s}|ds\right|^2\right] = 16M^2 \int_0^T \mathbb{E}\left[\left|S^q(s) - \hat{S}_{n_s}\right|^2 ds\right] \leq 16M^2 \int_0^T \mathbb{E}\left[\sup_{0\leq\tau\leq s}\left|S^q(\tau) - \hat{S}_{n_\tau}\right|^2\right] ds \tag{49}$$

For E by Holder inequality and sup properties:

$$E \leq 4\mathbb{E}\left[\sup_{0\leq t\leq T}\left(\int_0^t |I(s) - I_{n_s}|^2\, ds\right)\left(\int_0^t |\hat{S}_{n_s}|^2\, ds\right)\right] \leq 4\mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^t |I(s) - I_{n_s}|^2\, ds\right]\mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^t |\hat{S}_{n_s}|^2\, ds\right] \tag{50}$$

$$\leq 4\mathbb{E}\left[\int_0^T \sup_{0\leq u\leq s}|I(u) - I_{n_u}|^2\, ds\right]\cdot TC_T = 4TC_T \cdot \int_0^T \mathbb{E}\left[\sup_{0\leq u\leq s}|I(u) - I_{n_u}|^2\right] ds \leq C_1 \Delta t$$

Where in the last passage we applied strong convergence of the error of EM as presented in Q4.1 since the SDE of I(t) satisfies the hypothesis and incorporated all costants in C_1. Now, to answer Q4.2, since the F term would be zero being σ just a constant we would finish with Gronwall's lemma: calling $e_s = \mathbb{E}\left[\sup_{0\leq\tau\leq s}\left|S^q(\tau) - \hat{S}_{n_\tau}\right|^2\right]$:

$$e_T = \mathbb{E}\left[\sup_{0\leq t\leq T}|S^q(t) - \hat{S}(t)|^2\right] \leq C_1\Delta t + (4M^2T + 16M^2)\int_0^T e_s\, ds \implies e_T \leq C_2 \Delta t\, e^{C_3 T} \tag{51}$$

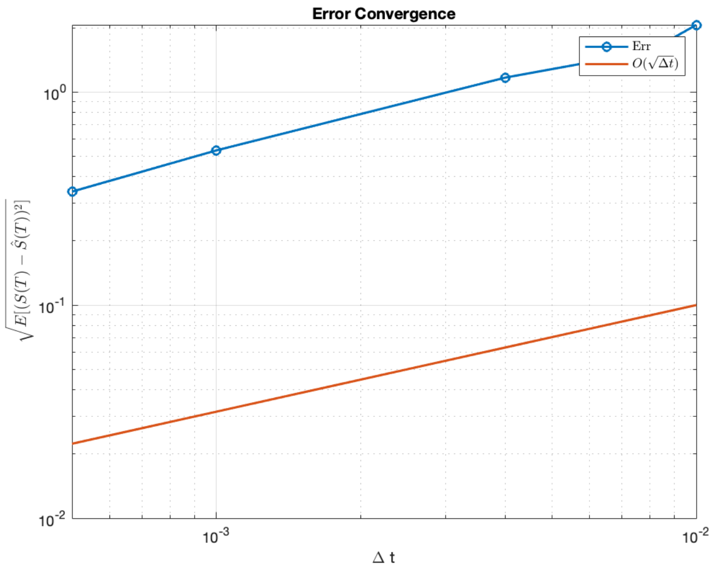
Thus, we proved that B converge as $O(\Delta t)$ and for A it holds the same for Q4.1, so their sum converge as $O(\Delta t)$ too and taking the square root yields the final order of convergence $O(\sqrt{\Delta t})$.
In Q4.3 we have to bound similarly F, using Doob's inequality and Itô's isometry:

$$F = 4\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t |\sigma(s) - \sigma_{n_s}||\hat{S}_{n_s}|dW_s\right|^2\right] \leq 16\mathbb{E}\left[\left|\int_0^T |\sigma(s) - \sigma_{n_s}||\hat{S}_{n_s}|dW_s\right|^2\right] = 16\mathbb{E}\left[\int_0^T |\sigma(s) - \sigma_{n_s}|^2|\hat{S}_{n_s}|^2 dW_s\right]$$

$$\leq 16\int_0^T \mathbb{E}\left[\sup_{0\leq\tau\leq s}|\sigma(\tau) - \sigma_{n_\tau}|^2\right] C_T \leq C_4\Delta t \tag{52}$$

Where in the last passage we applied strong convergence of the error of EM as presented in Q4.1 since the SDE of $\sigma(t)$ satisfies the hypothesis. So we proved that also F converges as $O(\Delta t)$, thus E + F converges as $O(\Delta t)$ and we conclude in the same way as before with Gronwall's lemma and obtain that also in this case the order of convergence of e_T is $O(\Delta t)$, thus $A + B = O(\Delta t)$ and $\sqrt{\mathbb{E}\left[\sup_{0\leq t\leq T}(S(t) - \hat{S}(t))^2\right]}$ is $O(\sqrt{\Delta t})$.

Q4.4) We observe a clear $O(\sqrt{\Delta T})$ convergence, consistent with our previous theoretical results and the strong error of the Euler-Maruyama (EM) discretization scheme.



Q5.1) We have:

$$\lim_{t\to+\infty}\frac{1}{t}\log(S(t)) = \lim_{t\to+\infty}\frac{\log S(0)}{t} + \frac{\left(I - \frac{1}{2}\sigma^2\right)t}{t} + \frac{\sigma W(t)}{t} = I - \frac{1}{2}\sigma^2 \quad \text{a.s. if } I \neq \frac{\sigma^2}{2} \tag{53}$$

since

$$\lim_{t \to +\infty} \frac{\ln S(0)}{t} = 0 \quad \text{and} \lim_{t \to +\infty} \frac{\sigma W(t)}{t} = \delta_0 \text{ a.s.} \tag{54}$$

since it is a random variable with zero mean and variance $\frac{\sigma^2}{t}$.
From above, if $I = \frac{\sigma^2}{2}$, we have $\log(S(t)) = \log(S(0)) + \sigma W(t)$, thus

$$\limsup_{t \to +\infty} \frac{\log(S(t))}{\sqrt{2t \log \log t}} = \limsup_{t \to +\infty} \sigma \cdot \frac{W(t)}{\sqrt{2t \log \log t}} = \sigma \quad \text{a.s.,} \tag{55}$$

$$\liminf_{t \to +\infty} \frac{\log(S(t))}{\sqrt{2t \log \log t}} = \liminf_{t \to +\infty} \sigma \cdot \frac{W(t)}{\sqrt{2t \log \log t}} = -\sigma \quad \text{a.s.,} \tag{56}$$

as by the Law of Iterated Logarithms,

$$\limsup_{t \to +\infty} \frac{W(t)}{\sqrt{2t \log \log t}} \to 1 \quad \text{and} \ \liminf_{t \to +\infty} \frac{W(t)}{\sqrt{2t \log \log t}} \to -1 \quad \text{a.s..} \tag{57}$$

For what concerns mean-square stability of the geometric B.M. we have:

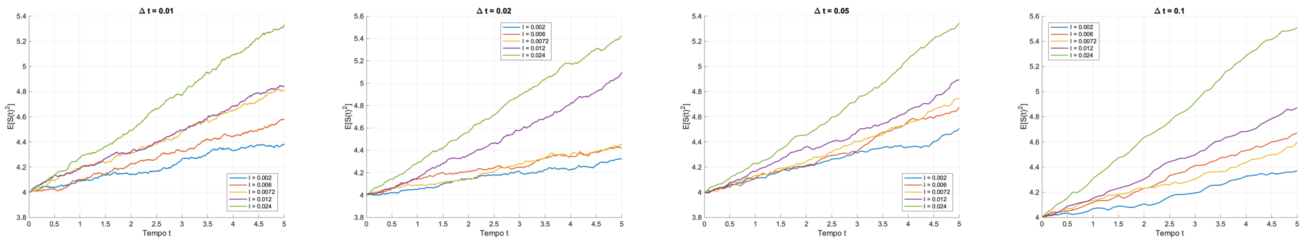
$$\mathbb{E}[S(t)^2] = S(0)^2 \left[e^{2\left(I - \frac{\sigma^2}{2}\right)t} \mathbb{E}\left[e^{2\sigma W(t)}\right] \right] \tag{58}$$

Using the moment generating function of a gaussian random variable $\mathbb{E}\left[e^{2\sigma W(t)}\right] = e^{2\sigma^2 t}$:

$$\mathbb{E}[S(t)^2] = S(0)^2 e^{2\left(I - \frac{\sigma^2}{2}\right)t} \cdot e^{2\sigma^2 t} = S(0)^2 e^{2\left(I + \frac{\sigma^2}{2}\right)t} \tag{59}$$

For mean-square stability, we require that $\lim_{t \to \infty} \mathbb{E}[S(t)^2] = 0$ which happens iff $I < -\frac{\sigma^2}{2}$.

Q5.2) As demonstrated earlier, the SDE (1) is mean square unstable for every simulated value of I. Consequently, since the SDE itself exhibits mean square instability, the forward Euler-Maruyama scheme also inherits this mean square instability. This observation aligns with the behavior observed in the plots, further confirming the consistency of the results.
This is a realistic asset behaviour according to the following interpretation: mean square instability implies that the variance of the process explodes as $t \to +\infty$ and this is consistent with the fact that for long time intervals it should be impossible to predict the asset price behaviour.
On the other hand, an infinite variance is not realistic for asset prices because real-world constraints limit their behavior. On the upside, prices are tied to fundamental factors like economic value or market capacity, which cannot grow indefinitely. On the downside, prices have natural floors, such as zero for most assets. Additionally, regulations and investor behavior act to stabilize markets, preventing excessive fluctuations. These structural limits mean that variance cannot grow without bounds in practice, making such behavior an oversimplification of reality.



(a) $\Delta t = 0.01$ (b) $\Delta t = 0.02$ (c) $\Delta t = 0.05$ (d) $\Delta t = 0.1$

Q5.3) The stochastic theta method yields:

$$S_{\theta,n+1} = S_{\theta,n} + \theta I S_{\theta,n+1} \Delta t + (1 - \theta) I S_{\theta,n} \Delta t + \sigma S_{\theta,n} \Delta W_n \tag{60}$$

$$S_{\theta,n+1} = \frac{1 + (1 - \theta) I \Delta t}{1 - \theta I \Delta t} S_{\theta,n} + \frac{\sigma}{1 - \theta I \Delta t} S_{\theta,n} \Delta W_n \tag{61}$$

Define the coefficients:

$$A = \frac{1 + (1 - \theta) I \Delta t}{1 - \theta I \Delta t}, \quad B = \frac{\sigma}{1 - \theta I \Delta t} \tag{62}$$

In order to get:

$$S_{\theta,n+1} = A S_{\theta,n} + B S_{\theta,n} \Delta W_n \tag{63}$$

Now we compute the second moment:

$$\mathbb{E}[|S_{\theta,n+1}|^2] = \mathbb{E}\left[(A + B\Delta W_n)^2\right]\mathbb{E}[|S_{\theta,n}|^2] \tag{64}$$

Since $\mathbb{E}[\Delta W_n] = 0$ and $\mathbb{E}[(\Delta W_n)^2] = \Delta t$, we get

$$\mathbb{E}[|S_{\theta,n+1}|^2] = (A^2 + B^2\Delta t)\mathbb{E}[|S_{\theta,n}|^2] \tag{65}$$

Which is a recursive equation, so to ensure stability we have to ask that the coefficient is smaller than one:

$$(A^2 + B^2\Delta t) = \frac{(1 + (1-\theta)I\Delta t)^2 + \sigma^2\Delta t}{(1-\theta I\Delta t)^2} < 1 \tag{66}$$

Simplifying we get:

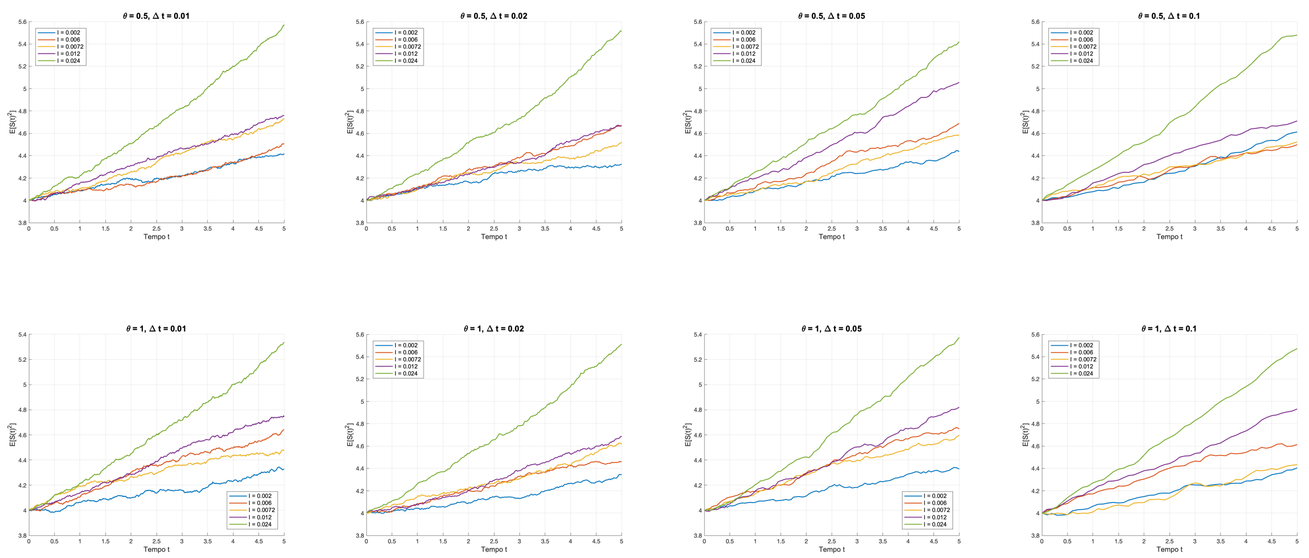
$$2I + \sigma^2 + (1 - 2\theta)I^2\Delta t < 0 \tag{67}$$

$\theta = \mathbf{0} \implies 2I + \sigma^2 + I^2\Delta t < 0$
M.S. SDE stability $\Rightarrow$ the theta method is stable for $\Delta t < \frac{2\left|I+\frac{\sigma^2}{2}\right|}{I^2}$
M.S. SDE instability $\Rightarrow$ the method is always M.S. unstable for any $\Delta t > 0$

$\theta = \frac{\mathbf{1}}{\mathbf{2}} \implies 2I + \sigma^2 < 0$
M.S. SDE stability $\Rightarrow$ the theta method is always M.S. stable
M.S. SDE instability $\Rightarrow$ the theta method is always M.S. unstable

$\theta = \mathbf{1} \implies 2I + \sigma^2 - I^2\Delta t < 0$
M.S. SDE stability $\Rightarrow$ the theta method is always M.S. stable.
M.S. SDE instability $\Rightarrow$ the theta method is stable if $\Delta t > \frac{2I+\sigma^2}{I^2}$

For $\theta = 0, \frac{1}{2}$, we already know that since the SDE is unstable, the theta method is also unstable. For $\theta = 1$, the Δt required to achieve stability is unrealistic and much larger than the time span of the simulation (T = 5). Thus for the data in Q5.2 we always have instability for every theta and the plots all look the same. We report only $\theta = \frac{1}{2}, 1$ as in Q5.2 we have already reported $\theta = 0$:



(a) $\Delta t = 0.01$ (b) $\Delta t = 0.02$ (c) $\Delta t = 0.05$ (d) $\Delta t = 0.1$

Q5.4) The Euler-Maruyama discretization of (27) is given by:

$$\sigma_{n+1} = (1 - \lambda\Delta t)\sigma_n + f\Delta W_n \tag{68}$$

Since $\mathbb{E}[\Delta W_n] = 0$ and $\mathbb{E}[(\Delta W_n)^2] = \Delta t$, we get

$$\mathbb{E}[\sigma_{n+1}^2] = (1 - \lambda\Delta t)^2\mathbb{E}[\sigma_n^2] + f^2\Delta t \tag{69}$$

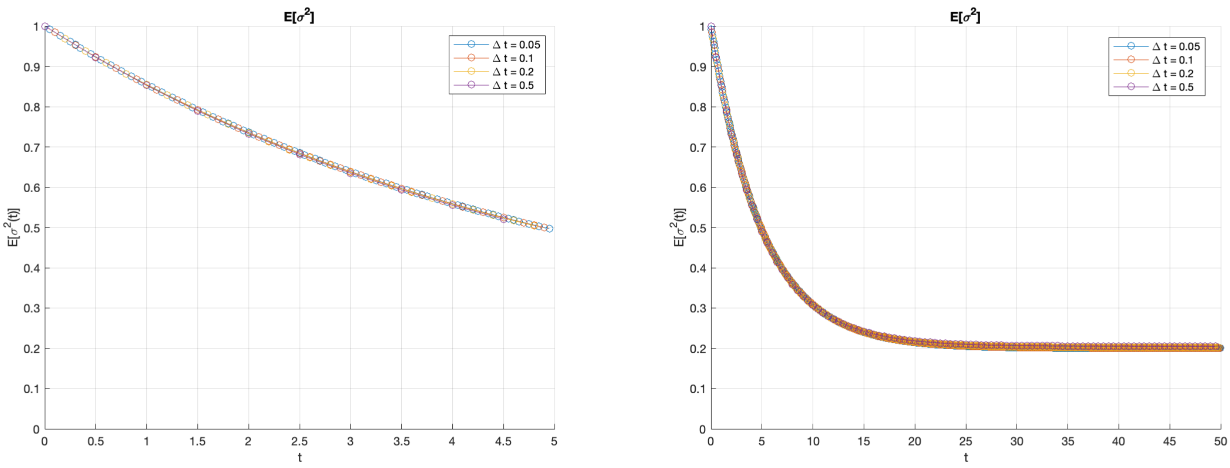
Similarly as before to ensure that the second moment is bounded we require the coefficient to be smaller or equal than one, otherwise it would explode.

$$|1 - \lambda \Delta t| \leq 1 \implies \begin{cases} \Delta t \leq \frac{2}{\lambda} \\ \lambda \geq 0 \end{cases} \tag{70}$$

With $\lambda = 0.1$, achieving divergence of the second moment would require $\Delta t = 20$, which is much larger than the time span considered in the simulation. Therefore, we expect the second moment to remain bounded for all relevant values of Δt. From the first plot ($T = 5$), we observe that the second moment is bounded and exhibits a geometric decay. However, in the second plot ($T = 50$), the time span is extended to demonstrate that the system is not mean-square stable under these conditions. Indeed, the steady-state solution for the second moment in this scenario is given by:

$$\mathbb{E}[\sigma_\infty^2] = \frac{f^2 \Delta t}{1 - (1 - \lambda \Delta t)^2} \tag{71}$$

and this cannot be zero since for hypothesis f and Δt are positive.



(a) $T = 5$

(b) $T = 50$

Q6.1) Itô's Lemma for $S_i(t)S_j(t)$ yields:

$$d(S_i(t)S_j(t)) = S_j(t)dS_i(t) + S_i(t)dS_j(t) + dS_i(t)dS_j(t) \tag{72}$$

Substituting:

$$dS_i(t) = \sum_{k=1}^{n} I_{ik} S_k(t)\, dt + \sigma \sum_{k=1}^{n} K_{ik} S_k(t) dW_k(t) \quad dS_j(t) = \sum_{l=1}^{n} I_{jl} S_l(t)\, dt + \sigma \sum_{l=1}^{n} K_{jl} S_l(t) dW_l(t) \tag{73}$$

So we have:

$$d(S_i(t)S_j(t)) = S_j(t) \left(\sum_{k=1}^{n} I_{ik} S_k(t)\, dt + \sigma \sum_{k=1}^{n} K_{ik} S_k(t) dW_k(t) \right) \tag{74}$$

$$+ S_i(t) \left(\sum_{l=1}^{n} I_{jl} S_l(t)\, dt + \sigma \sum_{l=1}^{n} K_{jl} S_l(t) dW_l(t) \right) + dS_i(t) dS_j(t) \tag{75}$$

The last term is:

$$dS_i(t) dS_j(t) = \sum_{k,\ell} I_{ik} S_k(t) S_\ell(t) I_{\ell j}\ dt^2 + \sigma \sum_{k,\ell} I_{ik} S_k(t) S_\ell(t) K_{\ell j}\, dt\, dW_\ell(t) \tag{76}$$

$$+ \sigma \sum_{k,\ell} K_{ik} S_k(t) S_\ell(t) I_{\ell j}\, dW_k(t)\, dt + \sigma^2 \sum_{k,\ell} K_{ik} S_k(t) S_\ell(t) K_{\ell j}\, dW_k(t)\, dW_\ell(t) \tag{77}$$

At this point, I take the expected value and drop all the terms containing only one $dW(t)$, as the expectation of the Itô integral is zero, I neglect the higher-order dt^2 terms, substitute $\mathbb{E}[dW(t) dW(t)^\top] = \mathbb{I} dt$ and use Fubini's theorem to exchange integrals to get:

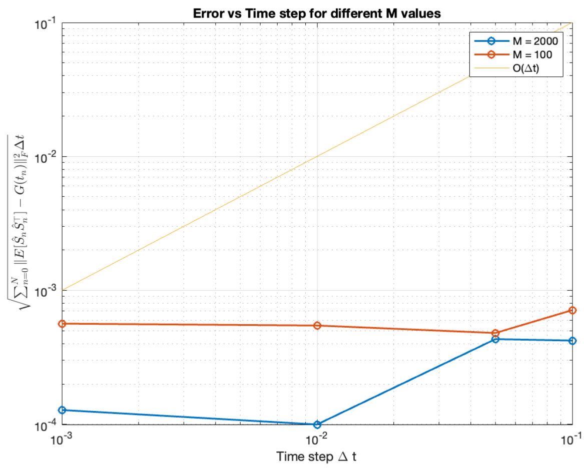
$$\big(dG(t)\big)_{ij} = \mathbb{E}[S_i(t) dS_j(t)] + \mathbb{E}[dS_i(t) S_j(t)] + \mathbb{E}[dS_i(t) dS_j(t)] \tag{78}$$

$$\big(dG(t)\big)_{ij} = \sum_{\ell} I_{j\ell} G_{i\ell}(t)\,dt + \sum_{\ell} I_{i\ell} G_{\ell j}(t)\,dt + \sigma^2 \sum_{k} K_{ik} K_{jk} G_{kk}(t)\,dt \tag{79}$$

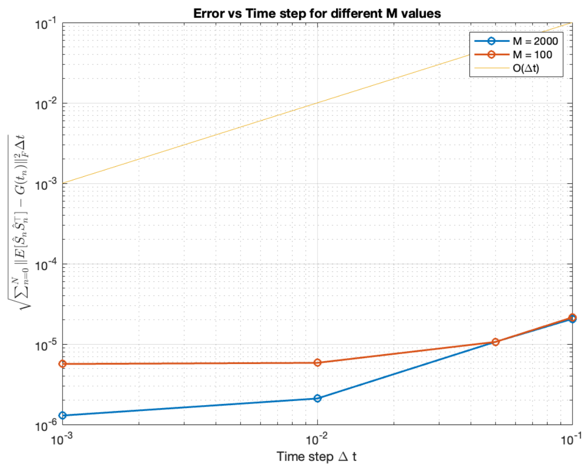
Which, written in matrix form, corresponds to the following system of ODEs

$$\frac{dG(t)}{dt} = IG(t) + G(t)I + \sigma^2 K \,\mathrm{diag}(G(t))\, K \tag{80}$$

Q6.2) The quantity $\mathbb{E}[\hat{S}_n \hat{S}_n^\top] - G(t_n)$ represents the weak error of the Euler-Maruyama (EM) scheme due to time discretization. This error is of order $\mathcal{O}(\Delta t)$. Squaring this, the error becomes $\mathcal{O}(\Delta t^2)$, and summing over $T/\Delta t$ intervals simplifies with the additional scaling Δt outisde the frobenius norm, resulting in a total error of order Δt^2. Taking the square root brings it back to $\mathcal{O}(\Delta t)$. If the expectation is computed analytically, a convergence rate of $\mathcal{O}(\Delta t)$ is expected. However, using Monte Carlo introduces an additional error of order $\mathcal{O}(1/\sqrt{M})$, where M is the number of simulated paths. In the implementation, $G(t_n)$ has been computed analytically by solving the ODE system (see MATLAB code for a brief explanation), ensuring no error in this term. In the simulation with the given data the MC error dominates, matching the time discretization error even at the largest Δt, thereby obscuring the linear convergence of the time discretization error, which is not evident. Even increasing the number of simulated paths to 2000 does not reduce the MC error enough to either confirm or refute the expected order of convergence. Therefore, we present two plots: on the left the results with the original parameters, and on the right the results from the same simulation but with $\sigma = 10^{-5}$.
In principle, we could have achieved the same reduction in MC variance by further increasing the number of paths, but due to computational limitations, we instead chose to decrease σ to lower the variance of the random variable and thus the MC error. As observed in the plots, reducing σ makes the Monte Carlo (MC) error smaller than



(a) $\sigma = 10^{-3}$



(b) $\sigma = 10^{-5}$

the time discretization error, at least for larger Δt, allowing us to clearly observe the alignment with the expected $\mathcal{O}(\Delta t)$ behavior. Furthermore, in the right plot, we can see the influence of the number of paths: with $M = 100$, the alignment is visible only for the two largest Δt, whereas increasing to $M = 2000$ reduces the MC error further. This reduction makes the MC error comparable to the time discretization error at smaller Δt, allowing us to clearly observe the alignment with the expected convergence rate for the first three time steps, then at the fourth the two errors become again comparable.

Q7.1) Let's consider again (39). If the process is non-negative, which happens for $2k\mu > \eta^2$, we can apply the transformation $Y_t = \sqrt{v(t)}$ and move the non-linearity from the diffusion coefficient into the drift coefficient. Itô's lemma yields:

$$dY_t = \frac{1}{2Y_t}\big(k(\mu - Y_t^2)\,dt + \eta Y_t\, dW(t)\big) - \frac{1}{4Y_t^3}\frac{\eta^2 Y_t^2}{2}\,dt \tag{81}$$

$$dY_t = \frac{4k\mu - \eta^2}{8Y_t}\,dt - \frac{k}{2} Y_t\,dt + \frac{\eta}{2}\,dW_t \tag{82}$$

Now, calling $\alpha = \frac{4k\mu - \eta^2}{8}$, $\beta = -\frac{k}{2}$, and $\gamma = \frac{\eta}{2}$, we can discretize the SDE with the stochastic theta method with $\theta = 1$, giving:

$$y_{k+1} = y_k + \left(\frac{\alpha}{y_{k+1}} + \beta y_{k+1}\right)\Delta t + \gamma(W_{k+1} - W_k) \quad k = 0, 1, \ldots \tag{83}$$

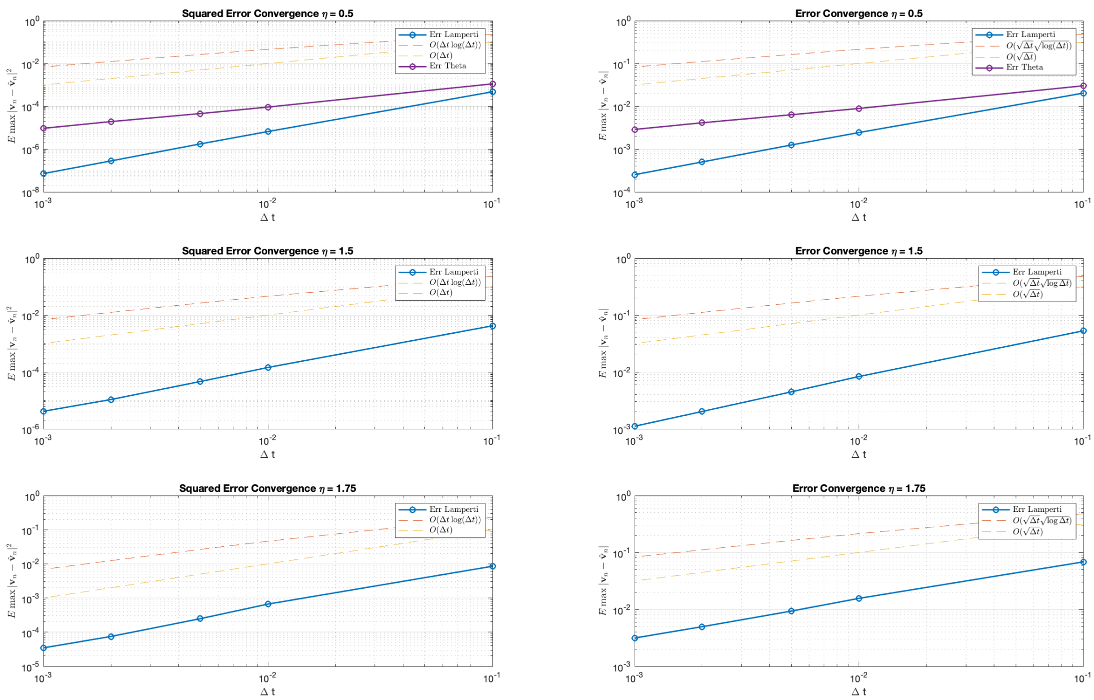
Recalling that $\alpha, \gamma > 0$ and $\beta < 0$, we have the following positive solution:

$$y_{k+1} = \frac{y_k + \gamma \Delta_k W}{2(1-\beta\Delta t)} + \sqrt{\frac{(y_k + \gamma\Delta_k W)^2}{4(1-\beta\Delta t)^2} + \frac{\alpha \Delta t}{1-\beta\Delta t}} \tag{84}$$

We recover the original solution by applying the transformation $x_k = y_k^2$. This approach will henceforth be referred to as the "Lamperti method."
For $\eta = 1.5$ and $\eta = 1.75$, the process is not guaranteed to remain positive because $2k\mu < \eta^2$. As a result, the basic theta method, which does not ensure the positivity of the numerical solution, may produce nonsensical complex values. Conversely, while the Lamperti method ensures that the numerical solution remains positive, it does not guarantee convergence to the true process (which may itself become negative). Consequently, the bounds established by the theorem are not valid in this scenario.
For these reasons, we compare both methods only for $\eta = 0.5$, where positivity is preserved and the bounds remain applicable. For $\eta = 1.5$ and $\eta = 1.75$, we focus exclusively on analyzing the Lamperti method.
From the plots we can observe the convergence of the strong notion of the error for p = 2 (left) and p = 1 (right) against the discretization step Δt.
We observe that the Lamperti method exhibits a strong convergence of order Δt^2, instead of the expected $\Delta t \log(\Delta t)$. We attribute this behavior to three main factors:

1. The values of Δt used in the simulation may be too large, making the effects of $\log(\Delta t)$ negligible.
2. The chosen parameters ($v(0) = k = \mu = 1$) simplify the inital time SDE, effectively canceling the drift term. This might lead the numerical method to behave in an overly optimistic way.
3. The "true solution" is, in practice, just the Lamperti method evaluated on a finer grid.

While the theta method goes as expected as $O(\Delta t)$ for p = 2 and as $O(\sqrt{\Delta t})$ for p = 1, the Lamperti method for increasing η degrades his order of convergence behaving like $O(\Delta t)$ and $O(\sqrt{\Delta t})$ in the last case.



Comparison of methods: Lamperti and theta for $\eta = 0.5$, Lamperti only for $\eta = 1.5, 1.75$.

Q7.2) Affirming that $v(t)$ cannot have the same law as $\sigma^2(t)$, and thus that it makes no sense to square the solution (31), is equivalent to saying that (82) and (27) cannot have the same solution. This is because (82) is obtained by applying Itô's Lemma "in reverse" to retrieve $\tilde{\sigma}$. If we plug into (82) the relations defined by (40), we recover exactly (27) and hence the solutions would be the same. However, in Q7.1, it is assumed that $\mu = 1$, $k = 1$, and $\eta = 0.5, 1.5, 1.75$. But from the relations in (40), we have:

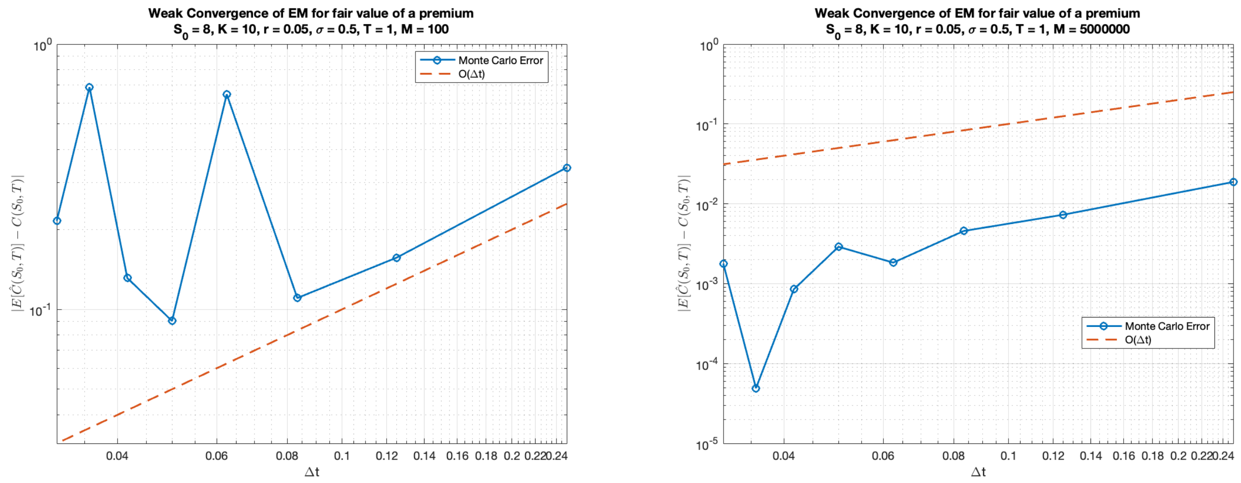
$$f^2 = 2\lambda\mu \quad f = \frac{\eta}{2} \tag{85}$$

With $k = 1$, we have $\lambda = \frac{k}{2} = \frac{1}{2}$. Since $\mu = 1$, it follows that:

$$f^2 = 2\lambda\mu = 2 \cdot \frac{1}{2} \cdot 1 = 1 \quad \text{so } f = 1 \tag{86}$$

Thus, $\eta = 2f = 2 \cdot 1 = 2$. However, the values of η considered in Q7.1 are $\eta = 0.5, 1.5, 1.75$, which do not match $\eta = 2$, therefore with these parameters, SDEs (82) and (27) cannot match and do not share the same solution.

Q8) Verifying the expected weak convergence rate ($O(\Delta t)$) for the Euler-Maruyama scheme in Monte Carlo simulations is challenging with only $M = 100$. This difficulty arises due to the interplay between discretization error and Monte Carlo sampling error, thus to explore convergence we present two plots: the first with $M = 100$ paths, and the second with $M = 5 \cdot 10^6$ paths.
The total error can be broken down into two main components. The first component is the time discretization error, which arises from approximating the SDE with discrete time steps. In this case this error is expected to decrease linearly with the time step size Δt, consistent with the weak convergence of EM. The second component is the MC sampling error, which arises from approximating the expected value with the sample mean using a finite number of simulated paths. According to the CLT, this error scales as $\mathcal{O}(1/\sqrt{M})$, leading to an error floor of approximately 10^{-1} when $M = 100$.
Furthermore having a high volatility-to-drift ratio ($\sigma/r = 10$) and the lack of smoothness of the payoff lead to increased variability in the MC simulation, making it increasingly harder to reduce the total error when M is small.



(a) $M = 100$

(b) $M = 5 \cdot 10^6$

For $M = 100$, the Monte Carlo sampling error dominates, creating an error floor of $\mathcal{O}(10^{-1})$. Even if the discretization error decreases for smaller Δt, the total error does not decrease due to the dominant sampling error. Thus, the first plot does not show convergence, not because it is absent, but because it is masked by the sampling error.
With $M = 5 \cdot 10^6$, the sampling error is much smaller ($\mathcal{O}(1/\sqrt{M}) \approx 4 \cdot 10^{-4}$), allowing the linear convergence of the discretization error to become visible. The second plot aligns with the theoretical weak convergence rate ($O(\Delta t)$) up to the point where the discretization error and the sampling error become again comparable.

Conclusion

This project highlighted the intrinsic challenges of performing numerical simulations of stochastic processes, a key aspect of financial modeling. In particular, the use of "crude" Monte Carlo simulation methods, without variance reduction techniques or Multi-Level Monte Carlo, proved to be a significant challenge. While these simulations are essential in the numerical analysis for estimating expected values and verifying the accuracy of numerical schemes, they can introduce high variance in the estimates and fail to converge accurately to the process's expected value. This can undermine the analysis of the convergence of errors caused by temporal discretization, making it difficult to distinguish the effects of Monte Carlo simulations from those of the numerical schemes themselves.
This work therefore underscores the need to consider advanced Monte Carlo simulation methods to improve the efficiency and accuracy of estimates, particularly in financial contexts where model uncertainty is critical.