Stochastic Partial Differential Equations

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Abstract

This paper provides an introduction to Stochastic Partial Differential Equations (SPDEs) with a focus on stochastic integration in Hilbert spaces. We discuss the construction of the Itô integral for Hilbert space-valued processes and extend the notion to integration with respect to cylindrical Brownian motion. The theoretical framework is complemented by essential definitions and key results.

1 Introduction

Stochastic Partial Differential Equations arise naturally in various fields such as physics, finance, and engineering, where randomness plays a crucial role. Unlike deterministic PDEs, SPDEs incorporate stochastic forcing terms, typically modeled using Wiener processes or more general martingales. A fundamental aspect in the study of SPDEs is the definition and construction of stochastic integrals in infinite-dimensional spaces.

In this paper, we introduce the necessary mathematical framework, starting with stochastic integration in Hilbert spaces, and then extend it to integration with respect to cylindrical Brownian motion. The development follows closely the classical Itô integral but requires additional structure to handle infinite-dimensional settings.

2 Stochastic Integration in Hilbert Spaces

2.1 L^2 Theory and Cauchy Sequences in Hilbert Spaces

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and let $\mathcal{H}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\| \cdot \|_{\mathcal{H}}$. We consider the space of square-integrable $\mathcal{H}$-valued functions,

$$L^2(X; \mathcal{H}) := \left\{ f : X \to \mathcal{H} \;\middle|\; f \text{ is measurable and } \int_X \|f(x)\|_{\mathcal{H}}^2 \, d\mu(x) < \infty \right\}.$$

equipped with the inner product

$$\langle f, g \rangle_{L^2(X;\mathcal{H})} := \int_X \langle f(x), g(x) \rangle_{\mathcal{H}} \, d\mu(x),$$

and corresponding norm

$$\|f\|_{L^2(X;\mathcal{H})} := \left(\int_X \|f(x)\|_{\mathcal{H}}^2 \, d\mu(x) \right)^{1/2}.$$

This space is a Hilbert space, i.e., a complete inner product space. We recall the definitions and main properties that guarantee this structure.

Definition 2.1. Let $\mathcal{V}$ be a normed vector space. A sequence $(f_n) \subset \mathcal{V}$ is called a *Cauchy sequence* if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \|f_n - f_m\|_{\mathcal{V}} < \varepsilon \quad \forall n, m \geq N.$$

The space $\mathcal{V}$ is said to be *complete* if every Cauchy sequence converges in norm to an element $f \in \mathcal{V}$, i.e.,

$$\lim_{n \to \infty} \|f_n - f\|_{\mathcal{V}} = 0.$$

In the case of $L^2(X; \mathcal{H})$, the norm is induced by the inner product

$$\langle f, g \rangle_{L^2(X;\mathcal{H})} := \int_X \langle f(x), g(x) \rangle_{\mathcal{H}} \, d\mu(x),$$

and the norm is given by

$$\|f\|_{L^2(X;\mathcal{H})} := \left(\int_X \|f(x)\|_{\mathcal{H}}^2 \, d\mu(x) \right)^{1/2}.$$

The space $L^2(X; \mathcal{H})$ is a vector space, and the inner product satisfies the usual Hilbert space properties: linearity in the first argument, symmetry, and positive-definiteness. Moreover, it satisfies the *parallelogram identity*:

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2, \quad \forall f, g \in L^2(X; \mathcal{H}),$$

which characterizes norms induced by inner products, and thus confirms the Hilbert structure.

The completeness of $L^2(X; \mathcal{H})$ follows from the fact that:

(i) The scalar-valued space $L^2(X; \mathbb{R})$ is complete.

(ii) The Hilbert space $\mathcal{H}$ is complete by assumption.

(iii) The Bochner integral preserves completeness: if $f_n : X \to \mathcal{H}$ is a Cauchy sequence in $L^2(X; \mathcal{H})$, then there exists $f \in L^2(X; \mathcal{H})$ such that $\|f_n - f\|_{L^2} \to 0$.

Hence, $L^2(X; \mathcal{H})$ is itself a Hilbert space (see [1, Prop. 3.5], [7, Thm. 5.3]).

The Hilbertian structure of $L^2(X; \mathcal{H})$ also guarantees the *uniqueness of limits* for convergent sequences, a direct consequence of the inner product structure. In particular, one has the identity

$$\|f_n - f_m\|_{L^2}^2 = \|f_n - f\|_{L^2}^2 + \|f_m - f\|_{L^2}^2 - 2\langle f_n - f, f_m - f \rangle_{L^2},$$

which holds for all $f_n, f_m, f \in L^2(X; \mathcal{H})$.

2.2 Itô Stochastic Integral: Definition and Properties

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ be a filtered probability space and $(W_t)_{t \in [0,T]}$ be a standard real-valued Brownian motion adapted to $(\mathcal{F}_t)$. We are interested in defining the stochastic integral

$$\int_0^T \psi_t \, dW_t,$$

where $\psi : [0, T] \times \Omega \to \mathbb{R}$ is a stochastic process.

Step 1: Elementary Processes. We first define the integral for a class of simple, or elementary, processes. A process ψ is called elementary if it can be written in the form

$$\psi_t(\omega) = \sum_{j=0}^{N-1} \psi_j(\omega) \cdot \mathbf{1}_{(t_j, t_{j+1}]}(t),$$

where $0 = t_0 < t_1 < \cdots < t_N = T$ is a fixed partition of $[0, T]$, and each $\psi_j \in L^2(\Omega, \mathcal{F}_{t_j}, \mathbb{P})$. That is, ψ_j is square-integrable and $\mathcal{F}_{t_j}$-measurable (i.e., adapted and non-anticipative).

For such a process ψ, the stochastic integral is defined as

$$\int_0^T \psi_t \, dW_t := \sum_{j=0}^{N-1} \psi_j \cdot \left(W_{t_{j+1}} - W_{t_j}\right).$$

Step 2: Isometry and Extension. For every elementary process ψ, the Itô isometry holds:

$$\mathbb{E}\left[\left(\int_0^T \psi_t \, dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T \psi_t^2 \, dt\right].$$

This identity allows us to uniquely extend the definition of the Itô integral to the closure of the space of elementary processes in $L^2(\Omega \times [0, T])$, i.e., to all processes ψ that are: predictable (i.e., measurable with respect to the predictable σ-algebra), and square-integrable:

$$\mathbb{E}\left[\int_0^T \psi_t^2 \, dt\right] < \infty.$$

Step 3: Definition. Let $\psi \in L^2(\Omega \times [0, T])$ be predictable. Then there exists a sequence $\psi^{(n)}$ of elementary processes such that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |\psi_t^{(n)} - \psi_t|^2 \, dt\right] = 0.$$

We define

$$\int_0^T \psi_t \, dW_t := \lim_{n\to\infty} \int_0^T \psi_t^{(n)} \, dW_t,$$

where the limit is taken in $L^2(\Omega)$, and is well-defined by the Itô isometry.

The construction and theory presented here follow the formal development given in (cf. [3, Lecture 3]).

2.3 Hilbert-Space Valued Itô Integral: Orthonormal Expansion Approach

Let $\mathcal{H}$ be a real separable Hilbert space, and let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Let $\Psi : [0, T] \times \Omega \to \mathcal{H}$ be a predictable process such that

$$\mathbb{E}\left[\int_0^T \|\Psi_t\|_{\mathcal{H}}^2 \, dt\right] < \infty.$$

Definition. We define the stochastic Itô integral of Ψ with respect to a real-valued Brownian motion $(W_t)_{t\in[0,T]}$ as the infinite series

$$\int_0^T \Psi_t \, dW_t := \sum_{k=1}^{\infty} \left(\int_0^T \langle \Psi_t, e_k \rangle_{\mathcal{H}} \, dW_t\right) e_k, \tag{1}$$

provided the series converges in the Hilbert space $L^2(\Omega; \mathcal{H})$.

Well-posedness. The definition (1) is independent of the choice of the orthonormal basis (e_k) and is well-posed due to the following reasons:

- For each $k \in \mathbb{N}$, the process $\langle \Psi_t, e_k \rangle_{\mathcal{H}}$ belongs to $L^2(\Omega \times [0,T])$, and hence the scalar Itô integral $\int_0^T \langle \Psi_t, e_k \rangle \, dW_t$ is well-defined.
- The Parseval identity and Itô isometry together yield

$$\mathbb{E}\left[\left\|\sum_{k=1}^{n}\left(\int_0^T \langle \Psi_t, e_k\rangle \, dW_t\right) e_k\right\|_{\mathcal{H}}^2\right] = \sum_{k=1}^{n}\mathbb{E}\left[\left(\int_0^T \langle \Psi_t, e_k\rangle \, dW_t\right)^2\right] = \sum_{k=1}^{n}\mathbb{E}\left[\int_0^T \langle \Psi_t, e_k\rangle^2 \, dt\right].$$

  Since $\sum_{k=1}^{\infty} \langle \Psi_t, e_k \rangle^2 = \|\Psi_t\|_{\mathcal{H}}^2$ for all t, we obtain

$$\sum_{k=1}^{\infty}\mathbb{E}\left[\int_0^T \langle \Psi_t, e_k\rangle^2 \, dt\right] = \mathbb{E}\left[\int_0^T \|\Psi_t\|_{\mathcal{H}}^2 \, dt\right] < \infty,$$

  which guarantees that the series in (1) converges in $L^2(\Omega; \mathcal{H})$.

Remark. Definition (1) follows the construction outlined in Theorem 2.1 of [4], where the stochastic integral is defined via duality using an orthonormal basis of $\mathcal{H}$. The series representation is not merely a consequence of a limit process, but rather serves as the defining formula for the integral in this Hilbert-space-valued setting. Its well-posedness is guaranteed by the theory developed in the previous subsection: specifically, by the completeness of $L^2(\Omega; \mathcal{H})$, the square integrability of each scalar component $\langle \Psi_t, e_k \rangle$, and the Itô isometry, which ensures convergence of the series in the norm of $L^2(\Omega; \mathcal{H})$.

3 Cylindrical Brownian Motion

In infinite-dimensional settings, a natural generalization of standard Brownian motion is the concept of cylindrical Brownian motion. Let $\mathcal{H}$ be a separable Hilbert space, and let Q be a bounded, positive, self-adjoint operator on $\mathcal{H}$. A Q-cylindrical Brownian motion W^Q over $\mathcal{H}$ is a family of zero-mean Gaussian processes $\{W_h^Q(t)\}_{h \in \mathcal{H}, t \geq 0}$ indexed by $h \in \mathcal{H}$, satisfying for all $h, g \in \mathcal{H}$ and $t, s \geq 0$,

$$\mathbb{E}[W_h^Q(t) W_g^Q(s)] = \langle Qh, g \rangle_{\mathcal{H}} (t \wedge s). \tag{2}$$

A Q-cylindrical Brownian motion W^Q is said to be *regular* if there exists a square integrable $\mathcal{H}$-valued stochastic process $\widetilde{W}^Q = \{\widetilde{W}_t^Q\}_{t \geq 0}$ such that $\widetilde{W}_t^Q \in L^2(\Omega; \mathcal{H})$ for all $t \geq 0$, and for every $h \in \mathcal{H}$, the process W_h^Q has the same law as the real-valued process

$$\left\langle \widetilde{W}^Q, h \right\rangle_{\mathcal{H}} (t, \omega) := \left\langle \widetilde{W}^Q(t, \omega), h \right\rangle_{\mathcal{H}}. \tag{3}$$

That is, for every finite collection of times $0 \leq t_1 < \cdots < t_n$ and for every $h \in \mathcal{H}$, the finite-dimensional distributions coincide:

$$\left(\left\langle \widetilde{W}^Q(t_1), h \right\rangle_{\mathcal{H}}, \ldots, \left\langle \widetilde{W}^Q(t_n), h \right\rangle_{\mathcal{H}}\right) \stackrel{d}{=} \left(W_h^Q(t_1), \ldots, W_h^Q(t_n)\right).$$

A Q-cylindrical Brownian motion W^Q is regular if and only if $Q \in \mathcal{L}^1(\mathcal{H}; \mathcal{H})$, that is, Q is trace-class. In this case, $\tilde{W}^Q$ admits the spectral decomposition

$$\tilde{W}^Q(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i W_t^i, \tag{4}$$

where $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of Q, with corresponding eigenvalues $\{\lambda_i\}$, and (W_t^i) are independent one-dimensional Brownian motions. The convergence of the series holds in $L^2(\Omega;\mathcal{H})$, due to the fact that Q is a trace-class operator, see [2] for background on trace-class properties.

We define the class of integrable processes for stochastic integration with respect to cylindrical Brownian motion.

For our analysis, we will need to make reference to two distinct Hilbert spaces: one over which W is a cylindrical Brownian motion, and the other in which our integrand maps to. Henceforth, we introduce $\mathfrak{U}$ as the Hilbert space over which W is a cylindrical Brownian motion. We shall take (e_i) as an orthonormal basis over $\mathfrak{U}$ and (a_i) as an orthonormal basis over $\mathcal{H}$.

Denote by $\mathcal{I}_T^{\mathcal{H}}(W)$ the class of progressively measurable operator-valued processes B belonging to the space

$$L^2\left(\Omega\times[0,T];\mathcal{L}_2(\mathfrak{U},\mathcal{H})\right),$$

where $\mathcal{L}_2(\mathfrak{U},\mathcal{H})$ denotes the space of Hilbert–Schmidt operators from $\mathfrak{U}$ to $\mathcal{H}$, equipped with the Borel sigma-algebra induced by the Hilbert–Schmidt norm.

Stochastic integration with respect to cylindrical Brownian motion follows naturally. For an operator-valued process $B\in\mathcal{I}_T^{\mathcal{H}}(W)$, we define

$$\int_0^t B(s)\,dW_s := \sum_{i=1}^{\infty}\int_0^t Be_i(s)\,dW_s^i, \tag{5}$$

where each scalar integral is defined according to the Itô expansion in (1), applied to the $\mathcal{H}$-valued process Be_i, and the series converges in $L^2(\Omega;\mathcal{H})$ thanks to the isometry and completeness of the space. This definition corresponds to the special case where the cylindrical Brownian motion is associated with the identity operator $Q=I$. While the formal expression

$$W^I(t)=\sum_{i=1}^{\infty}e_iW_t^i. \tag{6}$$

may serve as a useful heuristic, it does not define a genuine $\mathcal{H}$-valued process because the sum does not converge in $\mathcal{H}$. For a rigorous justification of the convergence in the stochastic integral above, see Definition 2.20 in [4].

4 Well-Posedness of SPDEs

4.1 Gelfand Triples

Let X and Y be Banach spaces. A subspace $X\subset Y$ is said to be *continuously embedded* in Y if the inclusion map $i: X\hookrightarrow Y$ is continuous; that is, there exists a constant $C>0$ such that

$$\|x\|_Y\le C\|x\|_X \quad \text{for all } x\in X.$$

The embedding is called *dense* if the closure of X in Y with respect to the topology induced by $\|\cdot\|_Y$ is equal to Y, i.e., $\overline{X}^{\|\cdot\|_Y}=Y$.

Given a Banach space X, its *(topological) dual space*, denoted X^*, is the set of all continuous linear functionals from X into $\mathbb{R}$ (or $\mathbb{C}$), equipped with the operator norm

$$\|\varphi\|_{X^*} := \sup_{\|x\|_X\le 1}|\varphi(x)|, \quad \varphi\in X^*.$$

In the case where H is a Hilbert space, the *Riesz representation theorem* asserts that every continuous linear functional $\varphi\in H^*$ can be uniquely represented as an inner product with a fixed element $u\in H$; i.e., there exists a unique $u\in H$ such that

$$\varphi(v)=\langle u,v\rangle_H \quad \text{for all } v\in H.$$

This correspondence defines an isometric isomorphism $R : H \to H^*$, called the *Riesz isomorphism*, which allows us to identify $H \cong H^*$ in a canonical way [5, Thm. 5.5].

Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$. Let V be a reflexive Banach space such that $V \subset H$ continuously and densely. Denote by V^* the dual space of V. Then, identifying H with its dual H^* via the Riesz isomorphism, we obtain the Gelfand triple

$$V \subset H \subset V^*,$$

with continuous and dense embeddings.

We now justify rigorously the identification $H \subset V^*$.

Proposition. Given that $V \subset H$ continuously and densely, and identifying $H \cong H^*$ via the Riesz isomorphism, we can regard H as a subspace of V^* by duality. More precisely, for every $h \in H$, we define a linear functional $\ell_h \in V^*$ by

$$\ell_h(v) := \langle h, v \rangle_H \quad \text{for all } v \in V.$$

Proof. First, note that the map $\ell_h : V \to \mathbb{R}$ is linear for any fixed $h \in H$, because the inner product in H is linear in its first argument. We show that $\ell_h \in V^*$, i.e., ℓ_h is continuous on V. Since the embedding $V \hookrightarrow H$ is continuous, there exists a constant $C > 0$ such that

$$\|v\|_H \leq C\|v\|_V \quad \text{for all } v \in V.$$

Then for any $v \in V$,

$$|\ell_h(v)| = |\langle h, v \rangle_H| \leq \|h\|_H \, \|v\|_H \leq C\|h\|_H \, \|v\|_V,$$

so $\|\ell_h\|_{V^*} \leq C\|h\|_H$, which proves that $\ell_h \in V^*$.

Therefore, we obtain a well-defined linear map

$$H \to V^*, \quad h \mapsto \ell_h,$$

which is continuous and injective. Hence, $H \subset V^*$ as a continuously embedded subspace.

□

4.2 Duality Structure and Variational Operators

Let us now analyze the interplay between the Hilbert space H, the Banach space $V \subset H$, and the dual V^* within the Gelfand triple $V \subset H \subset V^*$. Recall that, by construction, each element $z \in H$ is associated with a continuous linear functional $\ell_z \in V^*$ defined by

$$\ell_z(v) := \langle z, v \rangle_H \quad \text{for all } v \in V,$$

so that H can be regarded as a subspace of V^*.

Furthermore, if $\langle \cdot, \cdot \rangle_{V^*,V}$ denotes the duality pairing between V^* and V, we define

$$\langle z, v \rangle_{V^*,V} := z(v), \quad \text{for all } z \in V^*, \ v \in V.$$

Under the above identification of $H \subset V^*$, the duality pairing coincides with the inner product in H; that is, for all $z \in H$, $v \in V$, we have

$$\langle z, v \rangle_{V^*,V} = \langle z, v \rangle_H.$$

The triplet (V, H, V^*) is thus called a *Gelfand triple*. Since $H \subset V^*$ continuously and densely, V^* is also separable, and so is V.

Notation. We occasionally write the duality using the structured notation

$${}_{V^*}\langle z, v \rangle_V$$

which is entirely equivalent to $\langle z, v \rangle_{V^*,V}$. Both express the action of the functional $z \in V^*$ on the element $v \in V$.

Integration by parts. Let $A : V \to V^*$ be an operator. One typically assumes that A admits a variational formulation: there exists a bilinear or nonlinear form $a : V \times V \to \mathbb{R}$ such that

$$\langle Au, v\rangle_{V^*,V} = a(u, v), \quad \text{for all } u, v \in V.$$

In the linear symmetric case, where $A = -\Delta$, this becomes the abstract form of Green's formula:

$$\int_\Lambda \nabla u \cdot \nabla v \, dx = -\int_\Lambda \Delta u \, v \, dx,$$

assuming that $u \in D(\Delta)$, $v \in V$. Such identities allow us to define operators weakly via their action on test functions.

Example. Let $\Lambda \subset \mathbb{R}^d$ be a bounded open set, and consider the following choices:

- $H = L^2(\Lambda)$, the Hilbert space of square-integrable functions;
- $V = H_0^{1,4}(\Lambda)$, the Sobolev space of functions in $L^4(\Lambda)$ with weak gradient in $L^4(\Lambda)^d$, vanishing on the boundary.

Then $V \subset H$ continuously and densely. Define the operator $A : V \to V^*$ by

$$\langle Au, v\rangle_{V^*,V} := \int_\Lambda |\nabla u|^2 \, \nabla u \cdot \nabla v \, dx,$$

which corresponds to the nonlinear p-Laplacian operator (for $p = 4$). The associated variational form is

$$a(u, v) := \int_\Lambda |\nabla u|^2 \nabla u \cdot \nabla v \, dx.$$

This operator is well-defined from V to V^* due to Hölder's inequality and reflexivity of the involved spaces; for a more rigorous treatment, see the example discussed in Section 4.6.

4.3 Operator Conditions

Let

$$A : [0, T] \times V \times \Omega \to V^*, \qquad B : [0, T] \times V \times \Omega \to L_2(U, H),$$

be progressively measurable maps. We impose the following structural conditions on A and B to ensure well-posedness of the SPDE:

(H1) Hemicontinuity. For all $u, v, w \in V$, $\omega \in \Omega$, and $t \in [0, T]$, the map

$$\lambda \mapsto \langle A(t, u + \lambda v, \omega), w\rangle_{V^*,V}$$

is continuous on $\mathbb{R}$.

(H2) Weak monotonicity. There exists $c \in \mathbb{R}$ such that for all $u, v \in V$,

$$2\langle A(\cdot, u) - A(\cdot, v), u - v\rangle_{V^*,V} + \|B(\cdot, u) - B(\cdot, v)\|^2_{L_2(U,H)} \le c\|u - v\|^2_H.$$

(H3) Coercivity. There exist $\alpha > 1$, $c_1 \in \mathbb{R}$, $c_2 > 0$ and an $(\mathcal{F}_t)$-adapted process $f \in L^1([0, T] \times \Omega, dt \otimes \mathbb{P})$ such that for all $v \in V$, $t \in [0, T]$

$$2\,{}_{V^*}\langle A(t, v), v\rangle_V + \|B(t, v)\|^2_{L_2(U,H)} \le c_1\|v\|^2_H - c_2\|v\|^\alpha_V + f(t) \quad \mathbb{P}\text{-a.s.},$$

(H4) Boundedness. There exist $c_3 \ge 0$ and an $(\mathcal{F}_t)$-adapted process $g \in L^{\frac{\alpha}{\alpha-1}}([0, T] \times \Omega, dt \otimes P)$ such that for all $v \in V$, $t \in [0, T]$

$$\|A(t, v)\|_{V^*} \le g(t) + c_3\|v\|^{\alpha-1}_V \qquad \mathbb{P}\text{-a.s.},$$

where α is as in (H3).

4.4 Definition of Solution

We consider the general framework described above and the progressively measurable maps

$$A : [0,T] \times V \times \Omega \to V^*, \qquad B : [0,T] \times V \times \Omega \to L_2(U,H),$$

satisfying assumptions (H1)–(H4).

Let $W(t)$ be a Q-cylindrical Brownian motion on a separable Hilbert space U, with $Q = I$, defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$.

We consider the SPDE:

$$dX(t) = A(t, X(t))\, dt + B(t, X(t))\, dW(t), \qquad X(0) = X_0 \in H. \tag{7}$$

Definition. A continuous H-valued, $(\mathcal{F}_t)$-adapted process $(X(t))_{t\in[0,T]}$ is called a *solution* to equation (7) in H if there exists a V-valued progressively measurable $dt \otimes \mathbb{P}$-version $\tilde{X}$ of X such that

$$\tilde{X} \in L^\alpha([0,T] \times \Omega;\, V) \cap L^2([0,T] \times \Omega;\, H),$$

and for almost every $\omega \in \Omega$,

$$X(t) = X_0 + \int_0^t A(s, \tilde{X}(s))\, ds + \int_0^t B(s, \tilde{X}(s))\, dW(s), \quad \forall t \in [0,T], \tag{8}$$

where the identity holds in V^*.

4.5 Existence and Uniqueness Theorem

Theorem 4.1. *Let A, B satisfy assumptions* (H1)–(H4) *and let $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$. Then there exists a unique solution X to equation* (7)*. Moreover,*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X(t)\|_H^2 \right) < \infty. \tag{9}$$

This result corresponds to [6, Theorem 4.2.4].

4.6 Application to a Specific SPDE

We now verify the conditions of Theorem 4.1 for the following stochastic partial differential equation (SPDE):

$$dX_t = \operatorname{div}\left(|\nabla X_t|^2 \nabla X_t \right) dt + (X_t - X_t^3)\, dt + dW_t^Q, \tag{10}$$

where $\Lambda \subset \mathbb{R}^d$ is a smooth bounded domain and W_t^Q is a Q-cylindrical Brownian motion on $L^2(\Lambda)$, with Q a symmetric, nonnegative and trace-class operator. The noise acts in a Hilbert–Schmidt fashion through the operator $B(t, X_t)$. We aim to show that this SPDE admits a unique variational solution in the sense of Definition 4.4.

We define the Gelfand triple $V \subset H \cong H^* \subset V^*$ as follows:

- $H = L^2(\Lambda)$,
- $V = H_0^{1,4}(\Lambda)$, the Sobolev space of functions $u \in L^4(\Lambda)$ with weak first derivatives in $L^4(\Lambda)$ and vanishing trace on $\partial\Lambda$, i.e.,

  $$H_0^{1,4}(\Lambda) := \overline{C_c^\infty(\Lambda)}^{\|\cdot\|_{W^{1,4}}},$$

  equipped with the norm

  $$\|u\|_{H_0^{1,4}(\Lambda)}^4 := \int_\Lambda |\nabla u(x)|^4\, dx.$$

- The nonlinear operator $A(t,u) : V \to V^*$ is given by

$$A(t,u) = \mathrm{div}(|\nabla u|^2 \nabla u) + u - u^3.$$

Recall that a functional $\ell : V \to \mathbb{R}$ is said to be a *bounded linear functional* (i.e., an element of V^*) if it is linear and there exists a constant $C > 0$ such that

$$|\ell(v)| \leq C\|v\|_V \quad \text{for all } v \in V.$$

We now verify that each term in $A(t,u)$ defines a bounded linear functional on V. Let $u, v \in V = H_0^{1,4}(\Lambda)$.

  - The nonlinear diffusion term is defined via duality:

  $$\left\langle \mathrm{div}(|\nabla u|^2 \nabla u), v \right\rangle := -\int_\Lambda |\nabla u|^2 \nabla u \cdot \nabla v \, dx.$$

  Since $\nabla u \in L^4(\Lambda)^d$, we have $|\nabla u|^2 \nabla u \in L^{4/3}(\Lambda)^d$, and $\nabla v \in L^4(\Lambda)^d$, so the integral is finite by Hölder's inequality:

  $$\left| \int_\Lambda |\nabla u|^2 \nabla u \cdot \nabla v \, dx \right| \leq \||\nabla u|^2 \nabla u\|_{L^{4/3}} \|\nabla v\|_{L^4} \leq C\|u\|_V^3 \|v\|_V.$$

  Thus this term defines a bounded linear functional on V.

  - The linear term u is identified with the functional

  $$\langle u, v \rangle_{L^2} = \int_\Lambda uv \, dx.$$

  Since $u, v \in L^4(\Lambda) \subset L^2(\Lambda)$ (by the embedding $L^4 \subset L^2$ in bounded domains), and $L^4 \hookrightarrow L^2$ is continuous, this pairing is bounded:

  $$|\langle u, v \rangle| \leq \|u\|_{L^2} \|v\|_{L^2} \leq C\|u\|_V \|v\|_V.$$

  - The cubic term $-u^3$ satisfies $u^3 \in L^{4/3}(\Lambda)$ because $u \in L^4(\Lambda)$, and $v \in L^4(\Lambda)$. Therefore,

  $$\left| \int_\Lambda u^3 v \, dx \right| \leq \|u^3\|_{L^{4/3}} \|v\|_{L^4} = \|u\|_{L^4}^3 \|v\|_{L^4} \leq C\|u\|_V^3 \|v\|_V.$$

  Again, this yields a bounded linear functional on V.

Hence, each term in $A(t,u)$ defines a continuous linear functional on V, and we conclude that $A(t,u) \in V^*$.

- The stochastic forcing operator $B(t,u) : H \to \mathcal{L}_2(U_0, H)$ is defined via its action on a complete orthonormal system (e_i) of $L^2(\Lambda)$ consisting of eigenfunctions of Q with corresponding eigenvalues $\lambda_i > 0$:

$$B(t,u)e_i := \sqrt{\lambda_i} e_i, \quad \text{so that} \quad Qe_i = \lambda_i e_i.$$

Since Q is trace-class, we have $\sum_{i=1}^\infty \lambda_i < \infty$, and thus

$$\|B(t,u)\|_{\mathcal{L}_2(U_0,H)}^2 = \sum_{i=1}^\infty \|B(t,u)Q^{-1/2}Q^{1/2}e_i\|_H^2 = \sum_{i=1}^\infty \lambda_i < \infty,$$

which shows that $B(t,u)$ is indeed Hilbert–Schmidt from U_0 into H.

Note that it is sufficient to define B on the whole of $L^2(\Lambda)$ by prescribing its action on the eigenbasis (e_i).

We now construct the so-called *regular representation* of the Q-cylindrical Brownian motion W_t^Q, which is given by:

$$\widetilde{W}_t^Q := \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i W_t^i,$$

where $(W_t^i)_{i\in\mathbb{N}}$ is a sequence of independent standard real-valued Brownian motions. This defines a U-valued Wiener process with the same law as W_t^Q, see e.g. [4, Example 4.1.9].

Under this representation, the SPDE (10) can be equivalently formulated in the variational sense: for every $v \in V$, the process X satisfies the following identity $\mathbb{P}$-almost surely (as processes in $C([0,T];V^*)$):

$$\langle X(t), v\rangle_{V,V^*} = \langle X_0, v\rangle_{V,V^*} + \int_0^t \langle A(s, X(s)), v\rangle_{V,V^*}\, ds + W_v^Q(t),$$

where $W_v^Q(t) := \langle \widetilde{W}_t^Q, v\rangle_H$ is a real-valued Brownian motion adapted to the filtration generated by $\widetilde{W}_t^Q$.

The following result then holds (see also Theorem 4.1):

Theorem 4.2. *Let Q be a trace-class operator on $H = L^2(\Lambda)$, and suppose that the assumptions (H1)–(H4) of Theorem 4.1 hold. Then there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, a regular representation $\widetilde{W}^Q$ of W^Q, and a unique H-valued, progressively measurable process $X \in L^2(\Omega; C([0,T]; V^*))$ such that X satisfies the identity above $\mathbb{P}$-almost surely for all $v \in V$.*

We verify the assumptions (H1)–(H4):

(H1) Hemicontinuity. Let $u, v, w \in V$. We aim to show that the map

$$\lambda \mapsto \langle A(u + \lambda v), w\rangle_{V^*,V}$$

is continuous on $\mathbb{R}$. Recall that

$$\langle A(u + \lambda v), w\rangle_{V^*,V} = \int_\Lambda |\nabla(u + \lambda v)|^2 \nabla(u + \lambda v) \cdot \nabla w\, dx + \int_\Lambda \left[(u + \lambda v) - (u + \lambda v)^3\right] w\, dx.$$

We analyze each term separately.

First, observe that the function $\xi \mapsto |\nabla(u + \lambda v)|^2 \nabla(u + \lambda v)$ is in $L^{4/3}(\Lambda)^d$ for every fixed λ, since $\nabla u, \nabla v \in L^4(\Lambda)^d$ and $|\xi|^2\xi$ maps L^4 into $L^{4/3}$. Moreover, the map $\lambda \mapsto \nabla(u + \lambda v)$ is continuous in L^4, hence the composite is continuous in $L^{4/3}$. Since $\nabla w \in L^4$, the pairing belongs to L^1, and the dominated convergence theorem applies to ensure continuity.

For the algebraic part, note that $u + \lambda v \in L^4(\Lambda)$ implies $(u + \lambda v)^3 \in L^{4/3}(\Lambda)$, and

$$(u + \lambda v - (u + \lambda v)^3) w \in L^1(\Lambda),$$

since $w \in L^4$ and $(u+\lambda v)^3 \in L^{4/3}$. Again, the map $\lambda \mapsto (u+\lambda v)^3$ is continuous in $L^{4/3}$ by Lebesgue's dominated convergence theorem [7, Thm. 1.34], hence the full pairing is continuous in λ.

Thus, the map $\lambda \mapsto \langle A(u + \lambda v), w\rangle_{V^*,V}$ is continuous, and condition (H1) is satisfied.

(H2) Weak Monotonicity. We show that there exists $c \in \mathbb{R}$ such that for all $u, v \in V$,

$$2\langle A(u) - A(v), u - v\rangle_{V^*,V} \leq c\|u - v\|_H^2.$$

We decompose the operator as

$$A(u) = \mathrm{div}(|\nabla u|^2 \nabla u) + u - u^3,$$

and write

$$\langle A(u) - A(v), u - v\rangle_{V^*,V} = I + II,$$

where

$$I := \langle \mathrm{div}(|\nabla u|^2 \nabla u - |\nabla v|^2 \nabla v), u - v\rangle_{V^*,V},$$
$$II := \langle u - v - (u^3 - v^3), u - v\rangle_{V^*,V}.$$

For the first term, define $\Phi(\xi) := |\xi|^2 \xi$. Then, using integration by parts and the fact that $u - v \in H_0^{1,4}(\Lambda)$ vanishes on the boundary, we obtain

$$I = -\int_\Lambda (\Phi(\nabla u) - \Phi(\nabla v)) \cdot \nabla(u - v)\, dx.$$

Since Φ is monotone on $\mathbb{R}^d$, the integrand is non-negative almost everywhere, and we conclude that $I \leq 0$.

For the algebraic part,

$$II = \int_\Lambda (u - v)^2 dx - \int_\Lambda (u^3 - v^3)(u - v) dx.$$

The first term equals $\|u-v\|_H^2$, while the second is non-negative because $x \mapsto x^3$ is monotone increasing. Therefore, $II \leq \|u - v\|_H^2$.

Combining both estimates, we conclude

$$2\langle A(u) - A(v), u - v\rangle_{V^*,V} \leq 2\|u - v\|_H^2,$$

which verifies condition (H2).

(H3) Coercivity. We aim to show that there exist constants $\alpha > 1$, $c_1 \in \mathbb{R}$, $c_2 > 0$, and an $(\mathcal{F}_t)$-adapted process $f \in L^1([0,T] \times \Omega, dt \otimes \mathbb{P})$ such that for all $v \in V$ and $t \in [0,T]$, the following inequality holds:

$$\langle A(v), v\rangle_{V^*,V} \leq c_1\|v\|_H^2 - c_2\|v\|_V^\alpha + f(t).$$

We compute the duality pairing explicitly:

$$\begin{aligned}\langle A(v), v\rangle_{V^*,V} &= \int_\Lambda |\nabla v|^2 \nabla v \cdot \nabla v\, dx + \int_\Lambda v^2\, dx - \int_\Lambda v^4\, dx \\ &= \|v\|_V^4 + \|v\|_H^2 - \|v\|_{L^4}^4.\end{aligned}$$

We isolate the coercive contribution from the first term and rewrite:

$$\langle A(v), v\rangle_{V^*,V} = \|v\|_H^2 - c_2\|v\|_V^4 + \big((1 + c_2)\|v\|_V^4 - \|v\|_{L^4}^4\big).$$

We now define

$$f(t) := (1 + c_2)\|v\|_V^4 - \|v\|_{L^4}^4,$$

which is nonnegative and belongs to $L^1([0,T] \times \Omega)$, since $v \in V \subset L^4(\Lambda)$ and $\nabla v \in L^4(\Lambda)^d$.

Hence, we obtain the desired inequality:

$$\langle A(v), v\rangle_{V^*,V} \leq c_1\|v\|_H^2 - c_2\|v\|_V^4 + f(t),$$

where we take $\alpha := 4$ and $c_1 := 1$.

Therefore, condition (H3) is satisfied, with coercivity provided entirely by the nonlinear 4-Laplacian term $\mathrm{div}(|\nabla v|^2 \nabla v)$, and the dissipative contribution $-v^3$ absorbed in the integrable function $f(t)$.

(H4) Boundedness. We now verify that the operator

$$A(v) := \operatorname{div}(|\nabla v|^2 \nabla v) + v - v^3$$

satisfies the boundedness condition: there exist constants $c_3 \geq 0$ and an $(\mathcal{F}_t)$-adapted process

$$g \in L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega,\, dt \otimes \mathbb{P}),$$

such that for all $v \in V = H_0^{1,4}(\Lambda)$ and $t \in [0,T]$, we have

$$\|A(v)\|_{V^*} \leq g(t) + c_3 \|v\|_V^{\alpha-1},$$

where $\alpha = 4$ as established in (H3).

To estimate the dual norm of $A(v)$, we begin by observing that for any $w \in V$, the term involving the 4-Laplacian satisfies

$$\left|\langle \operatorname{div}(|\nabla v|^2 \nabla v), w\rangle_{V^*,V}\right| = \left|\int_\Lambda |\nabla v|^2 \nabla v \cdot \nabla w \, dx\right| \leq \|\nabla v\|_{L^4}^3 \|\nabla w\|_{L^4} = \|v\|_V^3 \|w\|_V.$$

This implies

$$\|\operatorname{div}(|\nabla v|^2 \nabla v)\|_{V^*} \leq \|v\|_V^3.$$

As for the nonlinear reaction term $v - v^3$, we note that $v \in H_0^{1,4}(\Lambda)$ implies $v \in L^4(\Lambda)$, and thus both v and v^3 belong to $L^{4/3}(\Lambda)$. Since $L^{4/3}(\Lambda) \hookrightarrow V^*$, it follows that

$$\|v - v^3\|_{V^*} \leq \|v\|_{L^{4/3}} + \|v^3\|_{L^{4/3}} = \|v\|_{L^4} + \|v\|_{L^4}^3.$$

Finally, using the continuity of the Sobolev embedding $H_0^{1,4}(\Lambda) \hookrightarrow L^4(\Lambda)$, we conclude

$$\|v - v^3\|_{V^*} \leq C(1 + \|v\|_V^3).$$

Combining both contributions, we obtain

$$\|A(v)\|_{V^*} \leq C(1 + \|v\|_V^3).$$

This verifies the boundedness condition with exponent $\alpha = 4$, setting $c_3 := C$ and $g(t) := C \in L^{4/3}([0,T] \times \Omega)$.

Noise Term. We now verify that the stochastic perturbation satisfies assumptions **(H2)** and **(H4)** in Theorem 4.1. As introduced above, the stochastic forcing operator $B(t,u) : H \to \mathcal{L}_2(U_0, H)$ is defined via the spectral decomposition of the trace-class operator Q, acting on a complete orthonormal system (e_i) of $L^2(\Lambda)$:

$$B(t,u)e_i := \sqrt{\lambda_i} e_i, \quad \text{so that} \quad Qe_i = \lambda_i e_i.$$

Since $\sum_{i=1}^\infty \lambda_i < \infty$, the operator $B(t,u)$ is constant in both t and u, and belongs to $\mathcal{L}_2(U_0, H)$.

For **(H2)**, we need to verify that B is weakly Lipschitz:

$$\|B(u) - B(v)\|_{\mathcal{L}_2(U_0,H)} = 0 \leq c\|u - v\|_H,$$

which is trivially satisfied for all $u, v \in H$, since B does not depend on u.

To verify **(H4)**, observe that

$$\|B(v)\|_{\mathcal{L}_2(U_0,H)}^2 = \sum_{i=1}^\infty \|B(v)e_i\|_H^2 = \sum_{i=1}^\infty \lambda_i =: C < \infty,$$

independently of v. Hence **(H4)** is satisfied with $g(t) \equiv \sqrt{C}$ and $c_3 = 0$, since

$$\|B(v)\|_{\mathcal{L}_2(U_0,H)} \leq \sqrt{C} = g(t) + c_3 \|v\|_V^{\alpha-1}.$$

We conclude that the noise term satisfies the structural conditions required for the variational framework, and therefore does not obstruct the application of Theorem 4.1.

5 Energy Equality for SPDEs

In the analysis of nonlinear stochastic partial differential equations, one of the key tools to derive a priori bounds is the so-called *energy equality*, which generalizes Itô's formula to the infinite-dimensional setting. The following result provides a fundamental identity for the square of the H-norm of a solution process $X(t)$, which is particularly useful when combined with coercivity assumptions and expectation operators to establish integrability and long-time bounds.

We now recall the statement of the energy equality, which holds for solutions of an abstract SPDE in the Gelfand triple framework $V \subset H \subset V^*$:

Theorem 5.1 (Energy Equality, [6, Theorem 4.2.5])**.** *Let $\alpha \in (1, \infty)$ and suppose that*

- $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$*;*
- $A \in L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega,\, dt \otimes \mathbb{P}; V^*)$ *is progressively measurable;*
- $B \in L^2([0,T] \times \Omega,\, dt \otimes \mathbb{P}; L_2(U,H))$ *is progressively measurable;*
- *The process X defined by*

$$X(t) := X_0 + \int_0^t A(s)\, ds + \int_0^t B(s)\, dW(s), \quad t \in [0,T],$$

admits a representative $\widehat{X} \in L^\alpha([0,T] \times \Omega,\, dt \otimes \mathbb{P}; V)$, and

$$\mathbb{E}\left(\|X(t)\|_H^2\right) < \infty \quad \textit{for almost every } t \in [0,T].$$

Then X is a continuous H-valued $(\mathcal{F}_t)$-adapted process, and the following energy equality holds $\mathbb{P}$-almost surely for all $t \in [0,T]$:

$$\|X(t)\|_H^2 = \|X_0\|_H^2 + \int_0^t \left(2\, {}_{V^*}\langle A(s), \bar{X}(s)\rangle_V + \|B(s)\|_{L_2(U,H)}^2\right) ds + 2\int_0^t \langle X(s), B(s)\, dW(s)\rangle_H, \quad (11)$$

for any progressively measurable V-valued $dt \otimes \mathbb{P}$-version $\bar{X}$ of $\widehat{X}$.

5.1 Application to a Specific SPDE

We now verify that the abstract setting of Theorem 11 applies to our SPDE (10). The equation is formulated on a smooth bounded domain $\Lambda \subset \mathbb{R}^d$, and driven by a Q-cylindrical Wiener process W_t^Q on $L^2(\Lambda)$. The noise enters the equation through a Hilbert–Schmidt operator $B(t, X_t)$, and the Gelfand triple is given by

$$V = H_0^{1,4}(\Lambda) \subset H = L^2(\Lambda) \cong H^* \subset V^*.$$

The well-posedness of the SPDE has already been established (see Section 4). In particular, the unique variational solution

$$X \in L^4([0,T] \times \Omega; V) \cap L^2(\Omega; C([0,T]; H))$$

satisfies the integrability and measurability assumptions of Theorem 11.

The abstract framework of Theorem 11 applies to the SPDE (10) under the assumptions established in Section 4. We now verify rigorously that each of its hypotheses is satisfied.

- **Initial condition.** By assumption, the initial datum satisfies

$$X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H).$$

- **Solution regularity.** The variational solution obtained in Theorem 4.1 satisfies

$$X \in L^4([0,T] \times \Omega; V) \cap L^2(\Omega; C([0,T]; H)),$$

hence in particular $X \in L^\alpha([0,T] \times \Omega; V)$ with $\alpha = 4$, as required.

- **Drift integrability and measurability.** Define the drift operator $A : V \to V^*$ by

$$A(u) := \mathrm{div}(|\nabla u|^2 \nabla u) + u - u^3.$$

Previous results (see, e.g., Section 4) show that:

  - A is well-defined from $V = H_0^{1,4}(\Lambda)$ into V^*,
  - $A(X) \in L^{\frac{4}{3}}([0,T] \times \Omega; V^*)$,
  - A is progressively measurable with respect to the filtration generated by X.

- **Diffusion integrability and measurability.** The stochastic perturbation is additive and given by a Q-cylindrical Brownian motion W_t^Q with covariance operator Q of trace class on $H = L^2(\Lambda)$. Thus, the noise enters via a constant diffusion operator

$$B(t,u) \equiv B \in L_2(U,H),$$

independent of both t and u. In particular,

$$B \in L^2([0,T] \times \Omega; L_2(U,H)) \quad \text{and} \quad B \text{ is progressively measurable.}$$

All the hypotheses of Theorem 11 are thus rigorously verified, and we may apply the energy identity to the variational solution of (10).

A Priori Bound on the Expected H-Norm We now observe that the integrand appearing in the energy identity,

$$2\,{}_{V^*}\langle A(t), \bar{X}(t)\rangle_V + \|B(t)\|^2_{L_2(U,H)},$$

coincides precisely with the left-hand side of the coercivity condition (H3), which has been verified in our setting.

Substituting the explicit coercivity inequality verified above into the energy identity (11), we obtain that for all $t \in [0,T]$,

$$\begin{aligned} \|X(t)\|_H^2 &= \|X_0\|_H^2 + \int_0^t \left(2\,{}_{V^*}\langle A(s), \bar{X}(s)\rangle_V + \|B(s)\|^2_{L_2(U,H)}\right) ds + 2\int_0^t \langle X(s), B\,dW(s)\rangle_H \\ &\leq \|X_0\|_H^2 + \int_0^t \left(\|X(s)\|_H^2 - c_2\|X(s)\|_V^4 + f(s)\right) ds + 2\int_0^t \langle X(s), B\,dW(s)\rangle_H, \end{aligned} \tag{12}$$

$\mathbb{P}$-almost surely, where we have taken $\alpha = 4$, $c_1 = 1$, and

$$f(t) := (1 + c_2)\|X(t)\|_V^4 - \|X(t)\|^4_{L^4(\Lambda)} \in L^1([0,T] \times \Omega),$$

as already computed in Section 4.

We now apply the expectation operator $\mathbb{E}$ to both sides of the inequality (12). Recalling that the stochastic integral

$$M(t) := \int_0^t \langle X(s), B\,dW(s)\rangle_H$$

is a real-valued continuous martingale with $M(0) = 0$, we have $\mathbb{E}[M(t)] = 0$. Therefore, taking expectations in (12) yields:

$$\mathbb{E}\|X(t)\|_H^2 \le \mathbb{E}\|X_0\|_H^2 + \int_0^t \big(\mathbb{E}\|X(s)\|_H^2 - c_2\,\mathbb{E}\|X(s)\|_V^4 + \mathbb{E}f(s)\big)\,ds. \tag{13}$$

We rewrite inequality (13) in the form

$$\mathbb{E}\|X(t)\|_H^2 \le \mathbb{E}\|X_0\|_H^2 + \int_0^t \mathbb{E}\|X(s)\|_H^2\,ds + \int_0^t \mathbb{E}f(s)\,ds.$$

Discarding the nonpositive term $-c_2\,\mathbb{E}\|X(s)\|_V^4$, we obtain the inequality

$$\mathbb{E}\|X(t)\|_H^2 \le \mathbb{E}\|X_0\|_H^2 + \int_0^t \mathbb{E}\|X(s)\|_H^2\,ds + \int_0^t \mathbb{E}f(s)\,ds.$$

Let us define the functions:

$$\psi(t) := \mathbb{E}\|X(t)\|_H^2, \qquad g(t) := \mathbb{E}f(t), \qquad C_0 := \mathbb{E}\|X_0\|_H^2.$$

Then the inequality becomes:

$$\psi(t) \le C_0 + \int_0^t \psi(s)\,ds + \int_0^t g(s)\,ds.$$

Since $\psi \in C([0,T])$, $g \in L^1(0,T)$, and both functions are nonnegative, the integral form of Gronwall's inequality applies. It follows that

$$\psi(t) \le \left(C_0 + \int_0^t g(s)\,ds\right)\exp(t), \quad \text{for all } t \in [0,T].$$

In other words, we obtain the a priori bound:

$$\mathbb{E}\|X(t)\|_H^2 \le \left(\mathbb{E}\|X_0\|_H^2 + \int_0^t \mathbb{E}f(s)\,ds\right)e^t.$$

This shows that the expected energy of the solution remains uniformly bounded on finite time intervals, with an explicit upper bound growing at most like e^t.

Integrated Control of the V-Norm We now derive a bound on the expected value of the time integral of the V-norm of the solution. Returning to inequality (13), without discarding the negative term, we have:

$$\mathbb{E}\|X(t)\|_H^2 \le \mathbb{E}\|X_0\|_H^2 + \int_0^t \left(\mathbb{E}\|X(s)\|_H^2 - c_2\,\mathbb{E}\|X(s)\|_V^4 + \mathbb{E}f(s) + \|B\|_{L_2(U,H)}^2\right)ds.$$

Rearranging and integrating in time, we obtain:

$$c_2 \int_0^t \mathbb{E}\|X(s)\|_V^4\,ds \le \mathbb{E}\|X_0\|_H^2 + \int_0^t \left(\mathbb{E}\|X(s)\|_H^2 + \mathbb{E}f(s) + \|B\|_{L_2(U,H)}^2\right)ds.$$

Using the bound already established for $\mathbb{E}\|X(s)\|_H^2$, namely

$$\mathbb{E}\|X(s)\|_H^2 \le \left(\mathbb{E}\|X_0\|_H^2 + \int_0^s \mathbb{E}f(r)\,dr + s\,\|B\|_{L_2(U,H)}^2\right)e^s,$$

and observing that all terms are integrable over finite time intervals, we conclude that there exists a constant $C_T > 0$ such that

$$\int_0^T \mathbb{E}\|X(t)\|_V^4 \, dt \le C_T,$$

with explicit dependence

$$C_T := \frac{1}{c_2}\left(\mathbb{E}\|X_0\|_H^2 + \int_0^T \left(\psi(s) + \mathbb{E}f(s) + \|B\|^2_{L_2(U,H)}\right) ds\right),$$

where $\psi(s) := \mathbb{E}\|X(s)\|_H^2$ satisfies the exponential bound derived earlier.

This estimate provides a global-in-time control on the average dissipation of the solution in the stronger V-norm.

5.2 Long-Time Behaviour under Dissipative Linear Drift: Nonlinear and Linear Cases

5.2.1 General Case: with Nonlinear p-Laplacian Drift

We now consider an alternative SPDE to investigate how the interplay between linear damping and multiplicative noise affects the long-time behaviour of the solution. Specifically, we define the drift and diffusion operators as

$$A(u) := -ku + \operatorname{div}(|\nabla u|^{p-2}\nabla u), \qquad B_i(u) := \frac{u}{2^i}, \quad i \in \mathbb{N},$$

where $k > 0$ and $p > 1$. The nonlinear p-Laplacian term is coercive and dissipative, while the term $-ku$ introduces linear damping. The noise is multiplicative and acts diagonally, with each component $B_i(u)$ proportional to u.

A straightforward computation shows that

$$\|B(u)\|^2_{L_2(U,H)} = \sum_{i=1}^{\infty} \left\|\frac{u}{2^i}\right\|_H^2 = \left(\sum_{i=1}^{\infty} \frac{1}{4^i}\right)\|u\|_H^2 = \frac{1}{3}\|u\|_H^2.$$

To focus on the essential dynamics, we rescale the noise and assume, without loss of generality, that

$$\|B(u)\|^2_{L_2(U,H)} = \|u\|_H^2.$$

We now study the evolution of the expected energy, defined by $\psi(t) := \mathbb{E}\|X(t)\|_H^2$. Applying Itô's formula to the functional $u \mapsto \|u\|_H^2$ and taking expectations, we obtain the energy identity

$$\frac{d}{dt}\mathbb{E}\|X(t)\|_H^2 = 2\,\mathbb{E}\langle A(X(t)), X(t)\rangle + \mathbb{E}\|B(X(t))\|^2_{L_2(U,H)}.$$

For the drift term, we compute

$$\langle A(u), u\rangle = \langle -ku, u\rangle + \int_\Lambda |\nabla u|^p \, dx = -k\|u\|_H^2 + \|\nabla u\|^p_{L^p},$$

so that

$$\mathbb{E}\langle A(X(t)), X(t)\rangle \le -k\,\mathbb{E}\|X(t)\|_H^2.$$

As already discussed, the multiplicative noise defined by $B_i(u) = u/2^i$ satisfies $\|B(u)\|^2_{L_2(U,H)} = \|u\|_H^2$, after rescaling. Therefore, plugging into the energy identity, we obtain:

$$\frac{d}{dt}\mathbb{E}\|X(t)\|_H^2 \le -2k\,\mathbb{E}\|X(t)\|_H^2 + \mathbb{E}\|X(t)\|_H^2 = (1-2k)\,\mathbb{E}\|X(t)\|_H^2.$$

The differential inequality

$$\frac{d}{dt}\psi(t) \le (1-2k)\psi(t), \quad \psi(0) = \mathbb{E}\|X_0\|_H^2,$$

is a linear first-order inequality. By comparison with the equality case, the general solution satisfies

$$\psi(t) \le \psi(0)\, e^{(1-2k)t}.$$

We now distinguish three cases based on the value of k:

Case 1: $k > \frac{1}{2}$. In this case, $1-2k < 0$, so the exponential factor decays to zero. We obtain

$$\mathbb{E}\|X(t)\|_H^2 \le \mathbb{E}\|X_0\|_H^2\, e^{-(2k-1)t} \longrightarrow 0 \quad \text{as } t \to \infty.$$

The solution dissipates energy and stabilizes.

Case 2: $k = \frac{1}{2}$. Here, the exponent vanishes and the bound becomes

$$\mathbb{E}\|X(t)\|_H^2 \le \mathbb{E}\|X_0\|_H^2.$$

That is, the expected energy remains uniformly bounded in time, although it may not remain constant due to the presence of the nonlinear p-Laplacian term.

Case 3: $k < \frac{1}{2}$. In this case, $1-2k > 0$, and the exponential bound grows:

$$\mathbb{E}\|X(t)\|_H^2 \le \mathbb{E}\|X_0\|_H^2\, e^{(1-2k)t}.$$

This suggests that the energy may grow over time, and blow-up in expectation is not excluded.

Note that since we only have an inequality (not an equality), the bounds derived do not necessarily describe the exact growth or decay of the expected energy, but provide upper bounds on its behaviour.

5.2.2 Linear Case: without p-Laplacian

We now consider a simplified version of the model presented above, in which the nonlinear p-Laplacian term is removed. This allows us to isolate the effect of the linear damping and the multiplicative noise on the long-time behaviour of the system.

Specifically, we consider the drift and diffusion operators:

$$A(u) := -ku, \qquad B_i(u) := \frac{u}{2^i}, \quad i \in \mathbb{N},$$

with $k > 0$. As shown previously, this choice of noise leads to

$$\|B(u)\|_{L_2(U,H)}^2 = \|u\|_H^2,$$

after appropriate rescaling.

In this setting, the energy identity becomes an exact differential equation. Letting $\psi(t) := \mathbb{E}\|X(t)\|_H^2$, Itô's formula yields:

$$\frac{d}{dt}\psi(t) = 2\,\mathbb{E}\langle A(X(t)), X(t)\rangle + \mathbb{E}\|B(X(t))\|_{L_2(U,H)}^2 = -2k\psi(t) + \psi(t) = (1-2k)\psi(t).$$

Solving this linear ODE with initial condition $\psi(0) = \mathbb{E}\|X_0\|_H^2$, we obtain:

$$\psi(t) = \mathbb{E}\|X(t)\|_H^2 = \mathbb{E}\|X_0\|_H^2\, e^{(1-2k)t}.$$

We distinguish three regimes depending on the strength of the damping parameter k:

Case 1: $k > \frac{1}{2}$**.** In this case, the exponent $1 - 2k < 0$, so the expected energy decays exponentially to zero:

$$\mathbb{E}\|X(t)\|_H^2 \longrightarrow 0 \quad \text{as } t \to \infty.$$

Case 2: $k = \frac{1}{2}$**.** The exponent vanishes and the energy remains constant:

$$\mathbb{E}\|X(t)\|_H^2 = \mathbb{E}\|X_0\|_H^2 \quad \text{for all } t \geq 0.$$

Case 3: $k < \frac{1}{2}$**.** The exponent is positive and the expected energy grows exponentially:

$$\mathbb{E}\|X(t)\|_H^2 \longrightarrow \infty \quad \text{as } t \to \infty.$$

In this linear setting, the energy evolution is fully determined by the value of k. The threshold $k = \frac{1}{2}$ separates the regimes of dissipation, stationarity, and instability. Compared to the general nonlinear case, here the presence of an exact identity allows for sharp conclusions on the asymptotic behaviour.

6 Conclusion

In this work, we developed a rigorous and self-contained treatment of Stochastic Partial Differential Equations, focusing on the analytical foundations required to formulate and study solutions in infinite-dimensional settings. Starting from the construction of the Itô integral in Hilbert spaces, we extended the theory to cylindrical Brownian motion and established the variational framework based on the Gelfand triple $V \subset H \subset V^*$.

We proved the well-posedness of a class of nonlinear SPDEs under standard structural assumptions—hemicontinuity, weak monotonicity, coercivity, and boundedness—and verified these conditions for a concrete example involving a nonlinear p-Laplacian drift and additive trace-class noise. The regularity of the solution allowed the application of an energy identity, yielding explicit a priori estimates for both the Hilbert norm and the stronger Sobolev norm of the solution.

In the final section, we analyzed the long-time behavior of the solution in the presence of dissipative linear damping and multiplicative noise. By deriving differential inequalities via Itô's formula, we identified a threshold phenomenon governed by the damping parameter k: when $k > \frac{1}{2}$, the expected energy decays exponentially; when $k = \frac{1}{2}$, it remains bounded; and when $k < \frac{1}{2}$, exponential growth may occur. These results illustrate the delicate interplay between deterministic dissipation and stochastic excitation.

The techniques and theoretical framework developed here are directly relevant for the analysis of stochastic models in applied contexts. In particular, SPDEs of the type studied in this work arise naturally in fields such as fluid dynamics, climate modeling, population dynamics, and quantitative finance, where uncertainty interacts with spatially extended systems. The extension of this approach to systems with spatially correlated noise, complex geometries, or coupled multi-physics equations remains a promising direction for future research.

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