Fractal Dimension Analysis of the Sierpinski Triangle

Constructing the Sierpinski Triangle

The Sierpinski Triangle, denoted $\mathcal{S}$, is constructed iteratively:

- Start with a solid equilateral triangle (S_0).
- For each iteration $n+1$, subdivide every black triangle in S_n into four equilateral triangles and remove the central one.
- The fractal is defined as: $\mathcal{S} = \lim_{n\to\infty} S_n$.

Box Counting Dimension

Self-Similarity Property: The Sierpinski Triangle exhibits exact self-similarity where scaling down by a factor of $1/2$ produces exactly 3 identical copies of the original structure.

Let $N(\varepsilon)$ denote the number of boxes of size ε required to cover the Sierpinski Triangle. The self-similarity property gives the following realtion:

$$N(\varepsilon/2) = 3 \times N(\varepsilon) \tag{1}$$

Deriving the general form of $N(\varepsilon)$ for Sierpinski Triangle:
Let ε_0 be a reference box size. Then by repeating the iteration:

$$\begin{align}
N(\varepsilon_0/2) &= 3 \times N(\varepsilon_0) \tag{2}\\
N(\varepsilon_0/4) &= 3 \times N(\varepsilon_0/2) = 3^2 \times N(\varepsilon_0) \tag{3}\\
N(\varepsilon_0/8) &= 3 \times N(\varepsilon_0/4) = 3^3 \times N(\varepsilon_0) \tag{4}\\
&\vdots \tag{5}\\
N(\varepsilon_0/2^n) &= 3^n \times N(\varepsilon_0) \tag{6}
\end{align}$$

For any box size $\varepsilon = \varepsilon_0/2^n$, we can express n as:

$$n = \log_2\left(\frac{\varepsilon_0}{\varepsilon}\right) \tag{7}$$

Substitute:

$$N(\varepsilon) = N(\varepsilon_0) \times 3^{\log_2(\varepsilon_0/\varepsilon)} = N(\varepsilon_0) \times \left(\frac{\varepsilon_0}{\varepsilon}\right)^{\log_2(3)} \tag{8}$$

This can be written in the general power-law form:

$$N(\varepsilon) = C \times \varepsilon^{-\log_2(3)} \tag{9}$$

where $C = N(\varepsilon_0) \times \varepsilon_0^{\log_2(3)}$ is a constant.

Box Counting Dimension:

The box counting dimension is defined as:

$$D_{\text{box}} = \lim_{\varepsilon \to 0^+} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \tag{10}$$

Substituting Equation 9:

$$\begin{aligned} D_{\text{box}} &= \lim_{\varepsilon \to 0^+} \frac{\log(C \times \varepsilon^{-\log_2(3)})}{\log(1/\varepsilon)} & (11) \\ &= \lim_{\varepsilon \to 0^+} \frac{\log(C) - \log_2(3) \times \log(\varepsilon)}{\log(1/\varepsilon)} & (12) \\ &= \lim_{\varepsilon \to 0^+} \frac{\log(C) + \log_2(3) \times \log(1/\varepsilon)}{\log(1/\varepsilon)} & (13) \\ &= \log_2(3) & (14) \end{aligned}$$

So:

$$\boxed{D_{\text{box}} = \log_2(3) = \frac{\log(3)}{\log(2)} \approx 1.584962500721156} \tag{15}$$

Hausdorff Dimension for Self-Similar Fractals

Theoretical Background:

Theorem 1 (Dimension of Self-Similar Fractals)**.** *If a fractal set F can be expressed as the union of N non-overlapping copies of itself, each scaled by a factor $r < 1$, then its Hausdorff dimension D_H is given by:*

$$N \cdot r^{D_H} = 1 \quad \Rightarrow \quad D_H = \frac{\log N}{\log(1/r)} \tag{16}$$

The Sierpinski Triangle is self-similar:

- It consists of $N = 3$ copies of itself.
- Each copy is scaled by a factor of $r = \frac{1}{2}$.

Substituting into the formula:

$$D_H = \frac{\log 3}{\log(1/(1/2))} = \frac{\log 3}{\log 2}$$

The Hausdorff dimension is:

$$\boxed{D_H = \frac{\log 3}{\log 2} \approx 1.585}$$

conclusion

As we can see both methods gave approximatly the same answer for the dimension $D_H \approx 1.585$