Analysis of the Koch Snowflake: Perimeter and Area

Introduction

The Koch snowflake is a fractal constructed through the following iterative process:

1. Begin with an equilateral triangle (side length = 1)
2. Replace the middle third of each side with two sides of an equilateral triangle
3. Repeat this process

Analysing the Perimeter

Let P_n be the perimeter after n iterations.
Initial perimeter: $P_0 = 3$
Recursive relation: Each iteration replaces every segment with 4 of $\frac{1}{3}$ length:

$$P_{n+1} = \frac{4}{3} P_n$$

Explicit formula becomes:

$$P_n = 3 \left(\frac{4}{3} \right)^n$$

Now $\frac{4}{3} > 1$ so with $n \to \infty$ the limitof the perimeter will diverge to ∞

$$\lim_{n \to \infty} P_n = \infty$$

Area Analysis

Let A_n be the area after n iterations.
the initial area is: $A_0 = \frac{\sqrt{3}}{4}$
Area added at iteration n:

- the number of new triangles is $3 \cdot 4^{n-1}$
- With side length $\frac{1}{3^n}$
- And Area of each being $\frac{\sqrt{3}}{4 \cdot 9^n}$

- Total area added equals $\Delta A_n = 3 \cdot 4^{n-1} \cdot \frac{\sqrt{3}}{4 \cdot 9^n}$

Total area:

$$
\begin{aligned}
A_\infty &= A_0 + \sum_{n=1} \Delta A_n \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} \sum_{n=1}^{\infty} \frac{4^{n-1}}{9^n} \\
&= \frac{\sqrt{3}}{4} \left(1 + \frac{1}{3} \cdot \frac{1}{1 - \frac{4}{9}} \right) \\
&= \frac{\sqrt{3}}{4} \cdot \frac{8}{5} = \frac{2\sqrt{3}}{5}
\end{aligned}
$$

The area converges to $\frac{2\sqrt{3}}{5} \approx 0.6928$.

Anomaly

Unlike Euclidean shapes where increased perimeter means that there is an increase in area, the Koch snowflake has an infinite perimeter with a finite area.