When the Bisection Method Fails or Becomes Problematic

The bisection method requires that f is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs ($f(a)f(b) < 0$)

The method may fail or become unreliable when:

1. **Function is Not Continuous on $[a, b]$.**

   If f has discontinuities within $[a, b]$, the intermediate value theorem does not guarantee a root exists.

   Example:

   $$f(x) = \frac{1}{x - 0.5} \quad \text{on } [0, 1]$$

   $$f(0) = -2 < 0, \quad f(1) = 2 > 0, \quad \text{but } f \text{ is discontinuous at } x = 0.5$$

2. **Multiple Roots with the Same Sign at Endpoints**

   When there are an even number of roots in $[a, b]$, $f(a)$ and $f(b)$ may have the same sign, the bisection method fail when the used step size is large, and passes the interval $[a, b]$

   Example:

   $$f(x) = (x - 0.3)(x - 0.7) = x^2 - x + 0.21 \quad \text{on } [0, 1]$$

   $$f(0) = 0.21 > 0, \quad f(1) = 0.21 > 0 \quad \text{(no sign change)}$$

   So it will assume no roots exist in the that interval.

3. **Root at an Endpoint**

   If $f(a) = 0$ or $f(b) = 0$, then the endpoint itself is a root. This is not a true failure, but some implementations may overlook it if only the sign test is used.

   Example:

   $$f(x) = x \quad \text{on } [0, 1]$$

   $$f(0) = 0 \quad \text{(root at endpoint)}$$

4. **Numerical Precision Limitations**

   If the interval becomes so small, smaller than the machines epsilon, because to the finite limitations of machine precision .

   Example:

   $$f(x) = x - 0.123456789012345 \quad \text{on } [0, 1]$$

   When the interval width approaches machine epsilon ($\approx 2.2 \times 10^{-16}$ for double precision), and bisection after that becomes impossible.