# PHYS 222 Assignment Due on 9/9/25

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September 2025

## Second-Order Quadratic Splines

We are given n+1 data points

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), \quad x_0 < x_1 < \cdots < x_n$$

On each interval $[x_i, x_{i+1}]$ we use a quadratic polynomial to approximate the function:

$$S_i(x) = a_i(x - x_i)^2 + b_i(x - x_i) + c_i.$$

We define: $h_i = x_{i+1} - x_i$ And $d_i = y_{j+1} - y_j$

## Constraints

Each spline has 3 unknowns, we have n splines, eventually we have $3n$ unknowns, and so we need $3n$ constraints/equations to solve them.

### Interpolation Conditions

We need to make sure that the spline passes through all the datapoints.

At the left endpoint:

$$S_i(x_i) = c_i = y_i.$$

(n equations)

At the right endpoint:

$$S_i(x_{i+1}) = y_{i+1}$$

$$a_i h_j^2 + b_i h_j + c_i = y_{i+1}.$$

Thus,

$$a_i h_j^2 + b_i h_j = y_{i+1} - y_i.$$

## Continuity of $C^1$: First Derivative Continuity

The derivative is

$$S_i'(x) = 2a_i(x - x_i) + b_i$$

At $x_{i+1}$:

$$S_i'(x_{i+1}) = 2a_i h_j + b_i.$$

We require continuity with the next spline:

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}) \quad \Longrightarrow \quad 2a_i h + b_i = b_{i+1}.$$

## Assumption of Boundary Condition

We have $2n$ unknowns $(a_0, a_1, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$ but only $2n - 1$ equations from interpolation and derivative continuity.

Thus, we impose one boundary condition. A natural choice is $b_0 = \frac{d_0}{h_0} = \frac{y_1 - y_0}{x_1 - x_0}$

## Final Linear System

We now have a system of $(2n - 1)$ equations in $(2n - 1)$ unknowns.

1. $c_i = y_i$
2. $a_i h_j^2 + b_i h_j = y_{i+1} - y_i$
3. $2a_i h + b_i = b_{i+1}$
4. $b_0 = \frac{d_0}{h_0} = \frac{y_1 - y_0}{x_1 - x_0}$

Matrix equation:

$$\left[\begin{array}{ccccccccc}
h_0^2 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & h_1^2 & 0 & \cdots & 0 & h_1 & 0 & \cdots & 0 \\
0 & 0 & h_2^2 & \cdots & 0 & 0 & h_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & h_{n-1}^2 & 0 & 0 & \cdots & h_{n-1} \\
\hline
2h_0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & 2h_1 & 0 & \cdots & 0 & +1 & -1 & \cdots & 0 \\
0 & 0 & 2h_2 & \cdots & 0 & 0 & +1 & \ddots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 2h_{n-2} & 0 & 0 & \cdots & +1
\end{array}\right]
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}
=
\begin{bmatrix} d_0 - h_0 b_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ -b_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{1}$$