Cubic Spline Interpolation

Given data points:

$$(x_0, y_0),\ (x_1, y_1),\ \ldots,\ (x_n, y_n) \quad [n+1 \text{ points}]$$

For each interval, we define

$$S_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i,$$

with $4n$ unknowns.

Constraints

1. Interpolation at the nodes:

$$S_i(x_i) = y_i, \quad S_i(x_{i+1}) = y_{i+1} \quad (2n \text{ constraints})$$

2. First derivative continuity:

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}) \quad (n-1 \text{ constraints})$$

3. Second derivative continuity:

$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}) \quad (n-1 \text{ constraints})$$

so we have $4n$ unknowns, $4n - 2$ constraints.

Definitions

$$h_i = x_{i+1} - x_i, \qquad \eta_i = y_{i+1} - y_i$$

From the constraints, we get:

$$\begin{align} d_i &= y_i \tag{1} \\ a_i h_i^3 + b_i h_i^2 + c_i h_i &= \eta_i \tag{2} \\ 3a_i h_i^2 + 2b_i h_i + c_i &= c_{i+1} \tag{3} \\ 3a_i h_i^2 + b_i &= b_{i+1} \tag{4} \end{align}$$

Reduction

Solve (3) for a_i in terms of b_i. Solve (2) for c_i in terms of b_i.

Then (4) becomes:

$$\frac{1}{3}h_i b_i + \frac{2}{3}(h_i + h_{i+1})b_{i+1} + \frac{1}{3}h_{i+1}b_{i+2} = \frac{\eta_{i+1}}{h_{i+1}} - \frac{\eta_i}{h_i}$$

Boundary Conditions

We implement the not-a-knot condition for the two missing constraints:

$$S_0'''(x_1) = S_1'''(x_1) \quad \Rightarrow \quad a_0 = a_1$$

$$S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1}) \quad \Rightarrow \quad a_{n-2} = a_{n-1}$$

Express a_i

To express a_i in terms of b_i we use the 2nd order derivative of the polynomial and the relation

$$S''(x_i) = 2b_i$$
$$S_i''(x_{i+1}) = 6a_ih_i + 2b_i$$

now we compute

$$S_i(x_{i+1})S_i(x_i) = (6a_ih_i + 2b_i) - 2b_i = 6a_ih_i$$

$a_i = \frac{2b_{i+1}-2b_i}{6h_i} = \frac{b_{i+1}-b_i}{3h_i}$

$$a_i = \frac{b_{i+1} - b_i}{3h_i} \qquad (i = 0, \ldots, n-1),$$

On the left boundary

$$\frac{b_1 - b_0}{3h_0} = \frac{b_2 - b_1}{3h_1} \quad \Longrightarrow \quad -\frac{1}{h_0}b_0 + \left(\frac{1}{h_0} + \frac{1}{h_1}\right)b_1 - \frac{1}{h_1}b_2 = 0.$$

on the right boundary

$$\frac{b_n - b_{n-1}}{3h_{n-1}} = \frac{b_{n-1} - b_{n-2}}{3h_{n-2}} \quad \Longrightarrow \quad -\frac{1}{h_{n-2}}b_{n-2} + \left(\frac{1}{h_{n-2}} + \frac{1}{h_{n-1}}\right)b_{n-1} - \frac{1}{h_{n-1}}b_n = 0.$$

Matrix Equation $AX = B$

$$A = \begin{pmatrix} -\frac{1}{h_0} & \frac{1}{h_0} + \frac{1}{h_1} & -\frac{1}{h_1} & 0 & \cdots & 0 \\ \frac{1}{3}h_0 & \frac{2}{3}(h_0 + h_1) & \frac{1}{3}h_1 & 0 & \cdots & 0 \\ 0 & \frac{1}{3}h_1 & \frac{2}{3}(h_1 + h_2) & \frac{1}{3}h_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{3}h_{n-2} & \frac{2}{3}(h_{n-2} + h_{n-1}) & \frac{1}{3}h_{n-1} \\ 0 & \cdots & 0 & -\frac{1}{h_{n-2}} & \frac{1}{h_{n-2}} + \frac{1}{h_{n-1}} & -\frac{1}{h_{n-1}} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ \frac{n_1}{h_1} - \frac{n_0}{h_0} \\ \frac{n_2}{h_2} - \frac{n_1}{h_1} \\ \vdots \\ \frac{n_{n-1}}{h_{n-1}} - \frac{n_{n-2}}{h_{n-2}} \\ 0 \end{pmatrix}$$

After solving for b, we find a_i, c_i, d_i from the identities and the constraints.