1. 有限差分

归根结底就是网格宽度不是常数时的差分格式。

考虑方程：

$$-\varepsilon u'' + bu' + cu = f(x)$$

在$x = x_1$点附近对$u(x)$做泰勒展开得：

$$u(x_1 + h) = u(x_1) + u'(x_1)h + \frac{1}{2}u''(x_1)h^2 + \frac{1}{3!}u'''(x_1)h^3 + \frac{1}{4!}u^{(4)}(x_1)h^4 + O(h^5)$$

设$x_0 = x_1 - h_1$，$x_2 = x_1 + h_2$，则有：

$$u(x_0) = u(x_1) - u'(x_1)h_1 + \frac{1}{2}u''(x_1)h_1^2 - \frac{1}{3!}u'''(x_1)h_1^3 + \frac{1}{4!}u^{(4)}(x_1)h_1^4 + O\left(h_1^5\right) \tag{1-1}$$

$$u(x_2) = u(x_1) + u'(x_1)h_2 + \frac{1}{2}u''(x_1)h_2^2 + \frac{1}{3!}u'''(x_1)h_2^3 + \frac{1}{4!}u^{(4)}(x_1)h_2^4 + O\left(h_2^5\right) \tag{1-2}$$

(1-1)、(1-2)两式加权相加消去一阶导数项得二阶导数项的差分格式：

$$\begin{aligned} u''(x_1) &= \frac{2}{h_1 h_2}\left[\frac{h_2 u(x_0)}{(h_1 + h_2)} + \frac{h_1 u(x_2)}{(h_1 + h_2)} - u(x_1)\right] + \frac{1}{3}u'''(x_1)(h_1 - h_2) \\ &\quad - \frac{1}{12}u^{(4)}(x_1)[(h_1 - h_2)^2 + h_1 h_2] + \cdots \\ &= \frac{2}{h_1 h_2}\left[\frac{h_2 u(x_0)}{(h_1 + h_2)} + \frac{h_1 u(x_2)}{(h_1 + h_2)} - u(x_1)\right] + O(h_1 - h_2) + O(h_1 h_2) \end{aligned} \tag{1-3}$$

(1-1)、(1-2)两式加权相减消去二阶导数项得一阶导数项的**中心差分**格式：

$$u'(x_1) = \frac{h_1}{h_2(h_1 + h_2)}u(x_2) - \frac{h_2}{h_1(h_1 + h_2)}u(x_0) + \frac{u(x_1)(h_2 - h_1)}{h_1 h_2} + O(h_1 h_2) \tag{1-4}$$

直接将(1-1)式变形得一阶导数项的**向后差分**格式：

$$u'(x_1) = \frac{u(x_1) - u(x_0)}{h_1} + O(h_1) \tag{1-5}$$

直接将(1-2)式变形得一阶导数项的**向前差分**格式：

$$u'(x_1) = \frac{u(x_2) - u(x_1)}{h_2} + O(h_2) \tag{1-6}$$

在$h_1 \neq h_2$时，(1-3)是一阶精度的，否则是二阶精度的。

2. 有限元

取一维网格$x_0 < x_1 < x_2 < \cdots < x_n$。在此网格上取分段线性函数：

$$\varphi_k(x) \equiv \begin{cases} \dfrac{x - x_{k-1}}{h_k}, & x_{k-1} \le x \le x_k \\ \dfrac{x_{k+1} - x}{h_{k+1}}, & x_k < x \le x_{k+1}, \\ 0, & \text{else} \end{cases} \qquad h_k \equiv x_k - x_{k-1}$$

设方程的解具有以下形式

$$u_h(x) = \sum_{j=1}^{n-1} u_j \varphi_j(x)$$

则可以将原方程写为弱形式

$$\begin{cases} -\varepsilon u'' + bu' + cu = f \\ u(0) = 0 = u(1) \end{cases}, \qquad \varepsilon, b, c \ \text{constant}$$

→

Find $u_h \in V_h = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_{n-1}\}$ s.t.、

$$\varepsilon(u_h', \varphi_k') + b(u_h', \varphi_k) + c(u_h, \varphi_k) = (f, \varphi_k), \qquad k = 1,2,..,n-1 \tag{2-4}$$

→

$$[\varepsilon S + bC + cM]\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{pmatrix}$$

其中

$$\begin{aligned} f_i = (f, \varphi_i) &= \int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)\mathrm{d}x \\ &= h_i \int_{-1}^{0} f(x_i + h_i\tau)(1+\tau)d\tau + h_{i+1}\int_0^1 f(x_i + h_{i+1}\tau)(1-\tau)d\tau \\ &= \int_{-1}^{0} [h_i f(x_i + h_i\tau) + h_{i+1} f(x_i - h_{i+1}\tau)](1+\tau)d\tau \end{aligned}$$

$$S_{ik} = (\varphi_k', \varphi_i') = \int_0^1 \varphi_k'(x)\varphi_i'(x)\mathrm{d}x = \begin{cases} \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right), & i = k \\ -\frac{1}{h_i}, & i = k+1 \\ -\frac{1}{h_{i+1}}, & i = k-1 \end{cases}$$

$$C_{ik} = (\varphi_k', \varphi_i) = \begin{cases} 0, & i = k \\ -\frac{1}{2}, & i = k+1 \\ \frac{1}{2}, & i = k-1 \end{cases}$$

$$M_{ik} = (\varphi_k, \varphi_i) = \begin{cases} \frac{1}{3}(h_i + h_{i+1}), & i = k \\ \frac{h_i}{6}, & i = k+1 \\ \frac{h_{i+1}}{6}, & i = k-1 \end{cases}$$

3. 谱方法(勒让德多项式)

3.1. Legendre 多项式 & Lobatto 多项式

Legendre 多项式$L_k(x)$：

$$L_0 = 1, \qquad L_1 = x, \qquad L_2 = \frac{1}{2}(3x^2 - 1) \dots\dots$$

性质：

(1) 递推公式：$(k+1)L_{k+1}(x) = (2k+1)xL_k(x) - kL_{k-1}(x)$

(2) 正交性：

$$\int_{-1}^{1} L_m(x)L_n(x)\mathrm{d}x = \frac{2}{2n+1}\delta_{mn}$$

(3) 归一化：$L_k(\pm 1) = (\pm 1)^k$

(4) 完备性：$\mathcal{L}_2(-1,1) = \mathrm{span}\{L_k(x)\}_{k=0}^{\infty}$

(5) 微分关系：

$$(2k+1)L_k(x) = \frac{\mathrm{d}}{\mathrm{d}x}[L_{k+1}(x) - L_{k-1}(x)]$$

Lobatto 多项式$Lo_{j+1}(x)$：

$$Lo_{j+1}(x) \equiv \sqrt{\frac{2j+1}{2}}\int_{-1}^{x} L_j(t)\mathrm{d}t, \qquad j = 1,2,\dots$$

(1) $Lo_{j+1}(\pm 1) = 0$，这是由 Legendre 多项式的正交性得到的；

(2) 由 Legendre 多项式的微分关系得：

$$Lo_{j+1}(x) = \sqrt{\frac{2j+1}{2}}\int_{-1}^{x} L_j(t)\mathrm{d}t = \sqrt{\frac{1}{2(2j+1)}}[L_{j+1}(x) - L_{j-1}(x)] \tag{3-3}$$

(3) 导数的正交归一关系：

$$\int_{-1}^{1} Lo'_{m+1}(x)Lo'_{n+1}(x)\mathrm{d}x = \delta_{mn} \tag{3-4}$$

(4) 重叠积分：

$$\begin{aligned}
&\int_{-1}^{1} Lo'_{m+1}(x)Lo_{n+1}(x)\mathrm{d}x \\
&= \sqrt{\frac{2m+1}{2}}\sqrt{\frac{1}{2(2n+1)}}\int_{-1}^{1} L_m(x)[L_{n+1}(x) - L_{n-1}(x)]\mathrm{d}x \\
&= \sqrt{\frac{1}{(2n+1)(2m+1)}}[\delta_{m,n+1} - \delta_{m,n-1}]
\end{aligned} \tag{3-5}$$

$$\begin{aligned}
&\int_{-1}^{1} Lo_{m+1}(x)Lo_{n+1}(x)\mathrm{d}x \\
&= \sqrt{\frac{1}{2(2m+1)}}\sqrt{\frac{1}{2(2n+1)}}\int_{-1}^{1}[L_{m+1}(x) - L_{m-1}(x)][L_{n+1}(x) - L_{n-1}(x)]\mathrm{d}x \\
&= \sqrt{\frac{1}{(2m+1)}\frac{1}{(2n+1)}}\left[\frac{1}{2n+3}\delta_{mn} + \frac{1}{2n-1}\delta_{mn} - \frac{1}{2n-1}\delta_{m,n-2} - \frac{1}{2n+3}\delta_{m,n+2}\right]
\end{aligned} \tag{3-6}$$

3.2. 画网格、展开

取一维网格$x_0 < x_1 < x_2 < \cdots < x_N$。在此网格上取分段线性函数：

$$\varphi_k(x) \equiv \begin{cases} \dfrac{x - x_{k-1}}{h_k}, & x_{k-1} \le x \le x_k \\ \dfrac{x_{k+1} - x}{h_{k+1}}, & x_k < x \le x_{k+1} \\ 0, & \text{else} \end{cases}, \qquad h_k \equiv x_k - x_{k-1}, \tag{3-7}$$

$$k = 1, \dots, N-1$$

此外，在每一段网格$[x_{k-1}, x_k]$上，我们有一组 Lobatto 多项式：

$$\psi_{k,j}(x) \equiv \begin{cases} Lo_{j+1}\left(\frac{2}{h_k}(x - x_{k-1}) - 1\right), & x_{k-1} \le x \le x_k \\ 0, & \text{else} \end{cases}, \quad j = 1,2,\dots,M_k, \quad k = 1,\dots,N \tag{3-8}$$

这里的M_k是截断阶数。显然有$\psi_{k,j}(x_{k-1}) = \psi_{k,j}(x_k) = 0$。

我们可以将待求函数在上述$(N-1+\sum_k M_k)$个基函数上展开（注意：它们并不是互相正交的）：

$$u^h(x) = \sum_{k=1}^{N}\sum_{j=1}^{M_k} c_j^{(k)}\psi_{k,j}(x) + \sum_{k=1}^{N-1} u_k\varphi_k(x) \tag{3-9}$$

我们可以将(2-4)式改写为：

Find $u_h \in V_h = \mathrm{span}\{\varphi_k, \psi_{k,j}\}$ s.t.

$$\varepsilon(u_h', \varphi_l') + b(u_h', \varphi_l) + c(u_h, \varphi_l) = (f, \varphi_l), \quad l = 1,2,..,N-1 \tag{3-10a}$$

$$\varepsilon(u_h', \psi_{l,m}') + b(u_h', \psi_{l,m}) + c(u_h, \psi_{l,m}) = (f, \psi_{l,m}), \quad m = 1,2,..,M_l, \quad l = 1,\dots,N \tag{3-10b}$$

代入(3-9)式得：

$$\begin{aligned}(f, \varphi_l) = &\sum_{k=1}^{N}\sum_{j=1}^{M_k}\{\varepsilon(\psi_{k,j}', \varphi_l') + b(\psi_{k,j}', \varphi_l) + c(\psi_{k,j}, \varphi_l)\}c_j^{(k)} \\ &+ \sum_{k=1}^{N-1}\{\varepsilon(\varphi_k', \varphi_l') + b(\varphi_k', \varphi_l) + c(\varphi_k, \varphi_l)\}u_k, \quad l = 1,2,..,N-1\end{aligned} \tag{3-11a}$$

$$\begin{aligned}(f, \psi_{l,m}) = &\sum_{k=1}^{N}\sum_{j=1}^{M_k}\{\varepsilon(\psi_{k,j}', \psi_{l,m}') + b(\psi_{k,j}', \psi_{l,m}) + c(\psi_{k,j}, \psi_{l,m})\}c_j^{(k)} \\ &+ \sum_{k=1}^{N-1}\{\varepsilon(\varphi_k', \psi_{l,m}') + b(\varphi_k', \psi_{l,m}) + c(\varphi_k, \psi_{l,m})\}u_k \\ &m = 1,2,..,M_l, \quad l = 1,\dots,N\end{aligned} \tag{3-11b}$$

(3-11)两式共$(N-1+K)$条方程可以写成 $H\vec{v} = \vec{f}$ 的形式，其中：

$$K = \sum_{k=1}^{N} M_k, \quad \vec{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_{N-1} \\ \hline c_1^{(1)} \\ \vdots \\ c_{M_1}^{(1)} \\ \hline c_1^{(2)} \\ \vdots \\ c_{M_N}^{(N)} \end{pmatrix}, \quad \vec{f} = \begin{pmatrix} (f, \varphi_1) \\ \vdots \\ (f, \varphi_{N-1}) \\ \hline (f, \psi_{1,1}) \\ \vdots \\ (f, \psi_{1,M_1}) \\ \hline (f, \psi_{2,1}) \\ \vdots \\ (f, \psi_{N,M_N}) \end{pmatrix}$$

$$H=\begin{pmatrix} H^{(00)}_{(N-1)\times(N-1)} & H^{(01)}_{(N-1)\times M_1} & \cdots & H^{(0N)}_{(N-1)\times M_N} \\ H^{(10)}_{M_1\times(N-1)} & H^{(11)}_{M_1\times M_1} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ H^{(N0)}_{M_N\times(N-1)} & 0 & 0 & H^{(NN)}_{M_N\times M_N} \end{pmatrix}$$

$$=\varepsilon\begin{pmatrix} S^{(00)} & S^{(01)} & \cdots & S^{(0N)} \\ S^{(10)} & S^{(11)} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ S^{(N0)} & 0 & 0 & S^{(NN)} \end{pmatrix}+b\begin{pmatrix} C^{(00)} & C^{(01)} & \cdots & C^{(0N)} \\ C^{(10)} & C^{(11)} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ C^{(N0)} & 0 & 0 & C^{(NN)} \end{pmatrix}+c\begin{pmatrix} M^{(00)} & M^{(01)} & \cdots & M^{(0N)} \\ M^{(10)} & M^{(11)} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ M^{(N0)} & 0 & 0 & M^{(NN)} \end{pmatrix}$$

上述矩阵中的零是因为两段不同网格上的 Lobatto 多项式$\psi_{k,j}$在空间上没有任何重合。

$$S^{(00)}_{ik}=(\varphi'_k,\varphi'_i),\qquad C^{(00)}_{ik}=(\varphi'_k,\varphi_i),\qquad M^{(00)}_{ik}=(\varphi_k,\varphi_i)$$

$$S^{(0m)}_{ik}=(\psi'_{m,k},\varphi'_i),\qquad C^{(0m)}_{ik}=(\psi'_{m,k},\varphi_i),\qquad M^{(0m)}_{ik}=(\psi_{m,k},\varphi_i),\qquad m=1,2,\dots,N$$

$$S^{(m0)}_{ik}=(\varphi'_k,\psi'_{m,i}),\qquad C^{(m0)}_{ik}=(\varphi'_k,\psi_{m,i}),\qquad M^{(m0)}_{ik}=(\varphi_k,\psi_{m,i}),\qquad m=1,2,\dots,N$$

$$S^{(mm)}_{ik}=(\psi'_{m,k},\psi'_{m,i}),\qquad C^{(mm)}_{ik}=(\psi'_{m,k},\psi_{m,i}),\qquad M^{(mm)}_{ik}=(\psi_{m,k},\psi_{m,i})$$

求解出系数$\vec{v}$后代入(3-9)式即可得到解u_h。

3.3. 系数矩阵的性质

由于所有基函数在边界上都是零，所以$(\varphi'_k,\psi_{m,i})=-(\varphi_k,\psi'_{m,i})$，所以有：

$$S^{(lm)}_{ik}=S^{(ml)}_{ki},\qquad C^{(lm)}_{ik}=-C^{(ml)}_{ki},\qquad M^{(lm)}_{ik}=M^{(ml)}_{ki}$$

也就是说S,M矩阵时对称的，C矩阵是反对称的。此外，S,M矩阵是正定的，因为它们都可以写成以下形式：

$$A_{ik}=(F_i,F_k),\qquad i,k=1,2,\dots(N-1+K)$$

对于S，

$$F=\left(\frac{\{\varphi'_i\}}{\{\psi'_{k,i}\}}\right)$$

对于M，

$$F=\left(\frac{\{\varphi_i\}}{\{\psi_{k,i}\}}\right)$$

矩阵正定的要求是，对于任意非零向量$\vec{v}$均有$\vec{v}^{\mathrm{T}}A\vec{v}>0$。然而在这里：

$$\vec{v}^{\mathrm{T}}A\vec{v}=\sum_{ik}v_iA_{ik}v_k=\sum_{ik}v_i(F_i,F_k)v_k=\left(\sum_i v_iF_i,\sum_k v_kF_k\right)$$

这里选取的基$\{\psi_{k,i},\varphi_i\}$是线性独立的，所以对于M，$\sum_i v_iF_i\neq 0$，M一定是正定的。如果基函数的一阶导数也是线性独立的，那么S也是正定的，否则S是半正定的。

3.4. 计算系数矩阵

$S^{(00)},C^{(00)},M^{(00)}$的计算见第二节末尾：

$$S^{(00)}_{ik}=(\varphi'_k,\varphi'_i)=-\frac{1}{h_i}\delta_{k,i-1}+\left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)\delta_{k,i}-\frac{1}{h_{i+1}}\delta_{k,i+1}$$

$$C_{ik}^{(00)}=(\varphi_k',\varphi_i)=-\frac{1}{2}\delta_{k,i-1}+\frac{1}{2}\delta_{k,i+1}$$

$$M_{ik}^{(00)}=(\varphi_k,\varphi_i)=\frac{h_i}{6}\delta_{k,i-1}+\frac{1}{3}(h_i+h_{i+1})\delta_{k,i}+\frac{h_{i+1}}{6}\delta_{k,i+1}$$

$S_{ik}^{(mm)},C_{ik}^{(mm)},M_{ik}^{(mm)}$由(3-4)~(3-6)式以及(3-8)式给出：

$$\begin{aligned}S_{ik}^{(mm)}&=(\psi_{m,k}',\psi_{m,i}')=\int_{x_{m-1}}^{x_m}\left[\frac{\mathrm{d}}{\mathrm{d}x}Lo_{k+1}\left(\frac{2}{h_m}(x-x_{m-1})-1\right)\right]\left[\frac{\mathrm{d}}{\mathrm{d}x}Lo_{i+1}\left(\frac{2}{h_m}(x-x_{m-1})-1\right)\right]\mathrm{d}x\\&=\frac{2}{h_m}\int_{-1}^{1}\left[\frac{\mathrm{d}}{\mathrm{d}t}Lo_{k+1}(t)\right]\left[\frac{\mathrm{d}}{\mathrm{d}t}Lo_{i+1}(t)\right]\mathrm{d}t\\&=\frac{2}{h_m}\delta_{k,i}\end{aligned}$$

$$\begin{aligned}C_{ik}^{(mm)}&=(\psi_{m,k}',\psi_{m,i})=\int_{x_{m-1}}^{x_m}\left[\frac{\mathrm{d}}{\mathrm{d}x}Lo_{k+1}\left(\frac{2}{h_m}(x-x_{m-1})-1\right)\right]Lo_{i+1}\left(\frac{2}{h_m}(x-x_{m-1})-1\right)\mathrm{d}x\\&=\int_{-1}^{1}\left[\frac{\mathrm{d}}{\mathrm{d}t}Lo_{k+1}(t)\right]Lo_{i+1}(t)\mathrm{d}t\\&=-\sqrt{\frac{1}{(2i+1)(2i-1)}}\delta_{k,i-1}+\sqrt{\frac{1}{(2i+1)(2i+3)}}\delta_{k,i+1}\end{aligned}$$

$$\begin{aligned}M_{ik}^{(mm)}&=(\psi_{m,k},\psi_{m,i})=\int_{x_{m-1}}^{x_m}Lo_{k+1}\left(\frac{2}{h_m}(x-x_{m-1})-1\right)Lo_{i+1}\left(\frac{2}{h_m}(x-x_{m-1})-1\right)\mathrm{d}x\\&=\frac{h_m}{2}\int_{-1}^{1}Lo_{k+1}(t)Lo_{i+1}(t)\mathrm{d}t\\&=\frac{h_m}{2}\sqrt{\frac{1}{(2i+1)}}\left[-\sqrt{\frac{1}{(2i-1)^2(2i-3)}}\delta_{k,i-2}+\sqrt{\frac{(4i+2)^2}{(2i+1)(2i+3)^2(2i-1)^2}}\delta_{k,i}-\sqrt{\frac{1}{(2i+3)^2(2i+5)}}\delta_{k,i+2}\right]\end{aligned}$$

$S^{(0m)},C^{(0m)},M^{(0m)}$的计算结果如下：

$$\begin{aligned}S_{ik}^{(0m)}=(\psi_{m,k}',\varphi_i')&=\begin{cases}\displaystyle\int_{x_{m-1}}^{x_m}\left[\frac{\mathrm{d}}{\mathrm{d}x}Lo_{k+1}\left(\frac{2}{h_m}(x-x_{m-1})-1\right)\right]\frac{1}{h_m}\mathrm{d}x, & i=m\\ \displaystyle-\int_{x_{m-1}}^{x_m}\left[\frac{\mathrm{d}}{\mathrm{d}x}Lo_{k+1}\left(\frac{2}{h_m}(x-x_{m-1})-1\right)\right]\frac{1}{h_m}\mathrm{d}x, & i=m-1\end{cases}\\&=\begin{cases}\displaystyle\frac{1}{h_m}\int_{-1}^{1}\left[\frac{\mathrm{d}}{\mathrm{d}t}Lo_{k+1}(t)\right]\mathrm{d}t, & i=m\\ \displaystyle-\frac{1}{h_m}\int_{-1}^{1}\left[\frac{\mathrm{d}}{\mathrm{d}t}Lo_{k+1}(t)\right]\mathrm{d}t, & i=m-1\end{cases}\\&=0\end{aligned}$$

$$C_{ik}^{(0m)} = \left(\psi'_{m,k}, \varphi_i\right) = \begin{cases} \displaystyle\int_{x_{m-1}}^{x_m} \left[\frac{\mathrm{d}}{\mathrm{d}x} Lo_{k+1}\left(\frac{2}{h_m}(x - x_{m-1}) - 1\right)\right] \frac{x - x_{m-1}}{h_m} \mathrm{d}x\,, & i = m \\ \displaystyle\int_{x_{m-1}}^{x_m} \left[\frac{\mathrm{d}}{\mathrm{d}x} Lo_{k+1}\left(\frac{2}{h_m}(x - x_{m-1}) - 1\right)\right] \frac{x_m - x}{h_m} \mathrm{d}x\,, & i = m-1 \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{2}\int_{-1}^{1} \left[\frac{\mathrm{d}}{\mathrm{dt}} Lo_{k+1}(t)\right](1+t)\mathrm{d}t\,, & i = m \\ \displaystyle\frac{1}{2}\int_{-1}^{1} \left[\frac{\mathrm{d}}{\mathrm{d}t} Lo_{k+1}(t)\right](1-t)\mathrm{d}t\,, & i = m-1 \end{cases}$$

$$= \sqrt{\frac{2k+1}{2}} \begin{cases} \displaystyle\frac{1}{2}\int_{-1}^{1} L_k(t)(1+t)\mathrm{d}t\,, & i = m \\ \displaystyle\frac{1}{2}\int_{-1}^{1} L_k(t)(1-t)\mathrm{d}t\,, & i = m-1 \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{\sqrt{6}}\delta_{k1}, & i = m \\ \displaystyle-\frac{1}{\sqrt{6}}\delta_{k1}, & i = m-1 \end{cases} \qquad (k = 1, \dots, M_m)$$

其中：$L_0 = 1$，$L_1 = x$。

$$M_{ik}^{(0m)} = \left(\psi_{m,k}, \varphi_i\right) = \begin{cases} \displaystyle\int_{x_{m-1}}^{x_m} Lo_{k+1}\left(\frac{2}{h_m}(x - x_{m-1}) - 1\right) \frac{x - x_{m-1}}{h_m} \mathrm{d}x\,, & i = m \\ \displaystyle\int_{x_{m-1}}^{x_m} Lo_{k+1}\left(\frac{2}{h_m}(x - x_{m-1}) - 1\right) \frac{x_m - x}{h_m} \mathrm{d}x\,, & i = m-1 \end{cases}$$

$$= \frac{h_m}{4} \begin{cases} \displaystyle\int_{-1}^{1} Lo_{k+1}(t)(1+t)\mathrm{d}t\,, & i = m \\ \displaystyle\int_{-1}^{1} Lo_{k+1}(t)(1-t)\mathrm{d}t\,, & i = m-1 \end{cases}$$

$$= \frac{h_m}{4}\sqrt{\frac{1}{2(2k+1)}} \begin{cases} \displaystyle\int_{-1}^{1} [L_{k+1}(t) - L_{k-1}(t)](1+t)\mathrm{d}t\,, & i = m \\ \displaystyle\int_{-1}^{1} [L_{k+1}(t) - L_{k-1}(t)](1-t)\mathrm{d}t\,, & i = m-1 \end{cases}$$

$$= -\frac{h_m}{4}\sqrt{\frac{2}{3}} \begin{cases} \delta_{k1} + \dfrac{1}{\sqrt{15}}\delta_{k2}, & i = m \\ \delta_{k1} - \dfrac{1}{\sqrt{15}}\delta_{k2}, & i = m-1 \end{cases} \qquad (k = 1, \dots, M_m)$$