Linear homogeneous systems with constant coefficients of 2 equations with 2 unknowns

Reduction to a 2nd order equation

Notation for the unknowns: $x(t)$ and $y(t)$

$$(1)\begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases}$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. A is said to be the matrix of the system.

$X \overset{not}{=} \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad (1) \Leftrightarrow X' = A\cdot X$

Remark 1 Any second order LHDE with CC can be written in the form (1) with the unknowns x and $y = x'$. *Indeed, consider*

$(2)\ \ x'' + \alpha_1 x' + \alpha_2 x = 0, \qquad \alpha_1, \alpha_2 \in \mathbb{R}$

$$(2) \Leftrightarrow \begin{cases} x' = y \\ y' = -\alpha_2 x - \alpha_1 y \end{cases}$$

where matrix is $A = \begin{pmatrix} 0 & 1 \\ -\alpha_2 & -\alpha_1 \end{pmatrix}$

Remark 2. $\lambda \in \mathbb{C}$ is an eigenvalue of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$ if and only if $\det(A - \lambda I_2) = 0$ if and only if $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$.

Classification

The system $\begin{cases} x' = a_{11}x \\ y' = a_{22}y \end{cases}$ is called uncoupled system. The general solution of this system is $\begin{cases} x = c_1 e^{a_{11}t} \\ y = c_2 e^{a_{22}t} \end{cases}$, $c_1, c_2 \in \mathbb{R}$.

The other systems are called coupled, that is, any system (1) with $a_{12} \neq 0$ or $a_{21} \neq 0$ is coupled.

$$(1): \begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases}$$

Hypothesis: $a_{12} \neq 0$

Reduction method:

our aim: to find a 2nd order eq. in x

Step 1. first eq. $\Rightarrow y = \frac{1}{a_{12}}(x' - a_{11}x)$

Step 2. first eq. $\Rightarrow x'' = a_{11}x' + a_{12}y'$

Step 3. second eq. and step 2 $\Rightarrow$ $x'' = a_{11}x' + a_{12}(a_{21}x + a_{22}y)$

Step 4. step 3 and step 1 $\Rightarrow$ $x'' = a_{11}x' + a_{12}a_{21}x + \cancel{a_{12}} \cdot a_{22} \cdot \frac{1}{\cancel{a_{12}}}(x' - a_{11}x)$

$\Rightarrow x'' - (a_{11} + a_{22})x' + (a_{11}a_{22} - a_{12}a_{21})x = 0 \quad (3)$

Step 5. we find the general sol of (3), then we find y using step 1.

a) $\begin{cases} x' = x - y \\ y' = x + y \end{cases}$

$y = x - x'$

$x'' = x' - y' = x' - x - y = x' - x - x + x' = 2x' - 2x$

$x'' = 2x' - 2x$

$x'' - 2x' + 2x = 0$

$r^2 - 2r + 2 = 0$

$\Delta = 4 - 4 \cdot 2 = -4$

$r_{1,2} = \frac{2 \pm 2i}{2} = 1 \pm i \longrightarrow e^t\cos t,\ e^t \sin t$

$x = c_1 e^t \cos t + c_2 \cdot e^t \sin t, \quad c_1, c_2 \in \mathbb{R}$

$y = x - x' = \cancel{c_1 e^t \cos t} + \cancel{c_2 e^t \sin t} - (\cancel{c_1 e^t \cos t} - c_1 e^t \sin t + \cancel{c_2 e^t \sin t} + c_2 e^t \cos t)$

$y = c_1 e^t \sin t - c_2 e^t \cos t$

$\begin{cases} x = c_1 e^t \cos t + c_2 e^t \sin t \\ y = c_1 e^t \sin t - c_2 e^t \cos t \end{cases} \quad c_1, c_2 \in \mathbb{R}$

b) $\begin{cases} x' = -y \\ y' = x \end{cases}$

$y = -x'$

$x'' = -y' = -x$

$x'' + x = 0$

$r^2 + 1 = 0$

$r_{1,2} = \pm i \longrightarrow \cos t,\ \sin t$

$\begin{cases} x = c_1 \cos t + c_2 \sin t \\ y = -c_1 \sin t + c_2 \cos t \end{cases}$

c) $\begin{cases} x' = 2y \\ y' = x \end{cases}$

d) $\begin{cases} x' = -5x + 2y \\ y' = -3y \end{cases}$

$y' = -3y \Rightarrow y' + 3y = 0$

$r + 3 = 0 \Rightarrow r = -3 \rightarrow e^{-3t}$

$y = c_1 e^{-3t}, \quad c_1 \in \mathbb{R}$

$x' = -5x + 2c_1 e^{-3t}$

$x' + 5x = 2c_1 e^{-3t}$

$x' + 5x = 0$

$r + 5 = 0 \Rightarrow r = -5$

$x_h = c_2 \cdot e^{-5t}, \quad c_2 \in \mathbb{R}$

$x_p = a \cdot e^{-3t} \qquad a = ?$

$x_p' + 5x_p = 2c_1 e^{-3t}$

$-3a \cdot e^{-3t} + 5a e^{-3t} = 2c_1 e^{-3t}, \quad \forall t \in \mathbb{R} \quad | : e^{-3t}$

$-3a + 5a = 2c_1$

$a = c_1 \Rightarrow x_p = c_1 e^{-3t}$

$x = x_h + x_p$

$\begin{cases} x = c_1 e^{-5t} + c_2 e^{-3t} \\ y = c_1 e^{-3t} \end{cases} \qquad c_1, c_2 \in \mathbb{R}$

Remark 3. The roots of the characteristic eq. of eq. (3) are the eigenvalues of the matrix's system.

The existence and uniqueness theorem

Let $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ be fixed and $A \in \mathcal{M}_2(\mathbb{R})$ be fixed.

We have that the IVP has a unique solution.

$$\begin{cases} \begin{pmatrix} x' \\ y' \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$$

Definitions

A matrix function $U : \mathbb{R} \to \mathcal{M}_2(\mathbb{R})$, $t \mapsto U(t)$ is said to be a matrix solution of system (1) if its columns are solutions of system (1).

A matrix solution of (1) is said to be a fundamental matrix solution if its columns are linearly independent functions.

A matrix solution with $U(0)=I_2$ is called principal matrix sol.

Proposition

(i) U is a matrix solution iff $U'(t)=AU(t)$ $\forall t\in\mathbb{R}$

(ii) Let U be a matrix solution. We have that U is a fundamental matrix solution iff $\det U(0)\neq 0$.

Step 1) Find the solution of the IVPs

$$\begin{cases} x'=-y \\ y'=x \\ x(0)=1 \\ y(0)=0 \end{cases} \quad \text{and} \quad \begin{cases} x'=-y \\ y'=x \\ x(0)=0 \\ y(0)=1 \end{cases}$$

$$\begin{cases} x=c_1\cos t+c_2\sin t \\ y=c_1\sin t-c_2\cos t \end{cases}$$

$x(0)=1 \iff c_1=1$

$y(0)=0 \iff c_2=0$

the unique solution of this ivp is $\begin{cases} x=\cos t \\ y=\sin t \end{cases}$

$x(0)=0 \iff c_1=0$

$y(0)=1 \iff c_2=-1$

$$\begin{cases} x=-\sin t \\ y=\cos t \end{cases}$$

Step 2) Find the principal matrix solution.

$$U(t)=\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$U(0)=I_2 \Rightarrow U$ is the principal matrix sol.

How we find the principal matrix solution of $\begin{cases} x'=-y \\ y'=x \end{cases}$?

How many solutions have the following problems?

a) $\begin{cases} x''+t^2x=0 \\ x(0)=0 \end{cases}$ b) $\begin{cases} x''+t^2x=0 \\ x(0)=0 \\ x'(0)=0 \end{cases} \leftarrow$ IVP c) $\begin{cases} x''+t^2x=0 \\ x(0)=0 \\ x'(0)=0 \\ x''(0)=a \end{cases}$, $a\in\mathbb{R}$ parameter

b) has a unique solution $x=0$ since this is an IVP

a) has infinite solutions: $\forall \eta\in\mathbb{R}$ s.t. $x'(0)=\eta$ $\exists!$ Sol of a)

c) if $a=0$ has a unique solution

if $a\neq 0$ no solution

Let $\omega > 0$ be a fixed parameter. Denote $\varphi(t,\omega)$ the unique sol. of the IVP $x'' + x = \cos(\omega t)$, $x(0) = x'(0) = 0$.

(i) when $\omega \neq 1$ find $x_p = a\cos(\omega t) + b\sin(\omega t)$

(ii) when $\omega = 1$ find $x_p = t(a\cos(\omega t) + b\sin(\omega t))$

(iii) find $\varphi(t,\omega)$

(iv) prove that $\lim_{\omega\to 1} \varphi(t,\omega) = \varphi(t,1)$, $\forall t \in \mathbb{R}$

(i) $x_p'' + x_p = \cos(\omega t)$

$x_p' = -a\omega\sin(\omega t) + b\omega\cos(\omega t)$

$-a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t) + a\cos(\omega t) + b\sin(\omega t) = \cos(\omega t)$, $\forall t \in \mathbb{R}$

$\cos(\omega t)(-a\omega^2 + a - 1) + \sin(\omega t)(-b\omega^2 + b) = 0$

$$\begin{cases} -a\omega^2 + a - 1 = 0 \\ -b\omega^2 + b = 0 \end{cases} \Rightarrow b(-\omega^2+1) = 0 \Rightarrow b = 0$$

$a(-\omega^2+1) = 1 \Rightarrow a = \frac{1}{-\omega^2+1}$

$x_p = \frac{1}{-\omega^2+1}\cos(\omega t)$, $\omega \neq 1$

(ii) $\omega = 1$

$x_p = t(a\cos t + b\sin t)$

...

$\Rightarrow a = 0$, $b = \frac{1}{2}$

$x_p = \frac{1}{2}t\sin t$, $\omega = 1$.

(iii) $\omega \neq 1$

$x'' + x = 0$

$r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i$

$x_h = c_1\sin t + c_2\cos t$

$x = c_1\sin + c_2\cos t + \frac{1}{-\omega^2+1}\cos(\omega t)$

$x(0) = 0 \Rightarrow c_2 + \frac{1}{-\omega^2+1} = 0 \Rightarrow c_2 = \frac{-1}{-\omega^2+1}$

$x'(t) = c_1\cos t - c_2\sin t - \frac{\omega}{\omega^2+1}\sin(\omega t)$

...

$x'(0) = 0 \Rightarrow c_1 = 0$

$\varphi(t,\omega) = \frac{-1}{-\omega^2+1}\cos t + \frac{1}{-\omega^2+1}\cos(\omega t)$, $\omega \neq 1$

$\varphi(t,1) = \frac{1}{2}t\sin t$

(iv) $\lim_{\omega\to 1}\varphi(t,\omega) = \lim_{\omega\to 1}\frac{-\cos t + \cos(\omega t)}{-\omega^2+1} \overset{(\frac{0}{0})}{\underset{L'H}{=}} \lim_{\omega\to 1}\frac{-t\sin(\omega t)}{-2\omega} = \frac{t\sin t}{2} = \varphi(t,1)$.

Linear differential systems
(linear differential equations in $\mathbb{R}^n$)

Let $n \geq 1$ be a fixed integer. We consider:

$$(1)\begin{cases} x_1' = a_{11}(t)x_1 + \dots + a_{1n}(t)x_n + f_1(t) \\ x_2' = a_{21}(t)x_1 + \dots + a_{2n}(t)x_n + f_2(t) \\ \vdots \\ x_n' = a_{n1}(t)x_1 + \dots + a_{nn}(t)x_n + f_n(t) \end{cases} \iff (2)\ X' = A(t)X + F(t)$$

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & \dots & a_{2n}(t) \\ \vdots & & \\ a_{n1}(t) & & a_{nn}(t) \end{pmatrix} \qquad F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Hyp: $A \in C(I, M_n(\mathbb{R}))$, $F \in C(I, \mathbb{R}^n)$, $I \subset \mathbb{R}$ nonempty, open interval

Def. A function $\varphi : I \to \mathbb{R}^n$ is a solution of (2) if $\varphi \in C^1(I, \mathbb{R}^n)$ and $\varphi'(t) = A(t)\varphi(t) + F(t)$, $\forall t \in I$.

Remarks. The system $X' = A(t)X$ is a linear homogeneous system. When $F \not\equiv 0$ the system (2) is linear non-homogeneous. When $A(t)$ is constant we say that system (2) has constant coefficients.

The fundamental theorems

- The existance and uniqueness theorem

Let $t_0 \in I$ be fixed. Let $\eta \in \mathbb{R}^n$ be fixed. We have that the IVP

$$\begin{cases} X' = A(t)X + F(t) \\ X(t_0) = \eta \end{cases}$$ has a unique solution $\varphi^* : I \to \mathbb{R}^n$.

- The fundamental theorem for LHS's

The set of solutions of system $X' = A(t)X$ is a linear space of dimension n. Thus, there exist $X_1, \dots, X_n$ n linearly independent solutions such that the general solution is $X = c_1 X_1 + \dots + c_n X_n$, where $c_1, \dots, c_n \in \mathbb{R}$. Moreover, denoting $U(t) = (X_1(t)\ X_2(t) \dots X_n(t))$ we can write the general solution $X = U(t) \cdot C$, $C \in \mathbb{R}^n$.

Proof. $\mathcal{L}(X)(t) = X'(t) - A(t)\cdot X(t)$, $\forall t\in I$

$X \in C^1(I, \mathbb{R}^n) \longmapsto \mathcal{L}(X) \in C(I, \mathbb{R}^n)$

$C^1(I, \mathbb{R}^n)$ and $C(I, \mathbb{R}^n)$ are linear spaces

$\mathcal{L}$ is a linear map, i.e. $\mathcal{L}(\alpha X + \beta Y) \overset{?}{=} \alpha \mathcal{L}(X) + \beta \mathcal{L}(Y)$, $\forall X, Y \in C^1(I, \mathbb{R}^n)$, $\forall \alpha, \beta \in \mathbb{R}$

$\mathcal{L}(\alpha X + \beta Y)(t) = (\alpha X + \beta Y)'(t) - A(t)(\alpha X + \beta Y)(t) =$

$= \alpha X'(t) + \beta Y'(t) - \alpha A(t) X(t) - \beta A(t) Y(t) = \alpha \mathcal{L}(X)(t) + \beta \mathcal{L}(Y)(t)$, $\forall t \in I$

The set of solutions of the system $X' = A(t)X$ is $\operatorname{Ker}\mathcal{L}$.

$\mathcal{L}$ is a linear map $\Rightarrow$ $\operatorname{Ker}\mathcal{L}$ is a linear space

We know that $\mathbb{R}^n$ is a linear space of dim. n.

We consider $T: \operatorname{Ker}\mathcal{L} \to \mathbb{R}^n$, $T(X) = X(t_0)$, where $t_0 \in I$ is fixed.

T is a linear map (HW). T is bijective $\Leftrightarrow$ $\forall \eta \in \mathbb{R}^n\ \exists!\, X \in \operatorname{Ker}\mathcal{L}$ s.t. $T(X) = \eta$

$\Leftrightarrow$ $\forall \eta \in \mathbb{R}^n\ \exists!$ solution of $\begin{cases} X' = A(t)X \\ X(t_0) = \eta \end{cases}$ this is true by the $\exists!$ Theorem.

- The fundamental theorem for LNS's

The general solution of the system $X' = A(t)X + F(t)$ is $X = X_h + X_p$, where X_h is the general solution of $X' = A(t)X$ and $X_p' = A(t)X_p + F(t)$.
(X_p is a particular solution).

Proof. $\mathcal{L}(X) = F(t)$ $\quad \operatorname{Ker}\mathcal{L} + \{X_p\}$

The Lagrange method (of the variation of constants) to find X_p.

$X_p' - A(t)X_p = F(t)$

Let $U(t)$ be s.t. the general solution of the LHS associated $X' = A(t)X$ is $X_h = U(t)\cdot C$, $C \in \mathbb{R}^n$.

Lagrange: $X_p = U(t)\,\mathcal{C}(t)$, where $\mathcal{C} = ?$ $\quad \mathcal{C} \in C^1(I, \mathbb{R}^n)$

$U'(t)\mathcal{C}(t) + U(t)\cdot \mathcal{C}'(t) - A(t)U(t)\mathcal{C}(t) = F(t)$, $\forall t \in I$

$U'(t) = (X_1'(t) \ldots X_n'(t)) = (A(t)X_1(t) \ldots A(t)X_n(t)) = A(t)\cdot(X_1(t) \ldots X_n(t)) = A(t)U(t)$

$\boxed{U'(t) = A(t)\cdot U(t)}$ $\Leftrightarrow$ U is a matrix solution

$$A(t)\cdot \cancel{U(t)\cdot \varphi(t)} + U(t)\cdot \varphi'(t) - \cancel{A(t)\, U(t)\, \varphi(t)} = F(t)$$

$$\varphi'(t) = U^{-1}(t)\cdot F(t), \quad \forall t\in I$$

$$\varphi(t) = \int_{t_0}^{t} U^{-1}(s)\, F(s)\, ds \quad \forall t\in I$$

$$\Rightarrow \boxed{X_p(t) = U(t)\int_{t_0}^{t} U^{-1}(s)\, F(s)\, ds}$$

The columns of U are linearly independent $\Rightarrow$ $\det U(t) \neq 0$, $\forall t\in I$

LHS with CC

(3) $X' = AX$ where $A\in \mathcal{M}_n(\mathbb{R})$

Denote $E: \mathbb{R}\to \mathcal{M}_n(\mathbb{R})$ such that $E'(t) = A\,E(t)$, $\forall t\in\mathbb{R}$ and $E(0) = I_n$ (the identity matrix). We have that $E(t)$ is said to be the principal matrix solution of $X' = AX$. and that the general of. this system is $X = E(t)C$, $C\in\mathbb{R}^n$.

The exponential matrix

Theorem. Let $A\in\mathcal{M}_n(\mathbb{R})$ be fixed. We have that the series (of matrices) $I_n + \frac{1}{1!}A + \frac{1}{2!}\cdot A^2 + \dots + \frac{1}{k!}A^k + \dots$ is convergent. Denote its sum by e^A or $\exp(A)$.

Examples. 1) $e^{O_n} = I_n$

2) $e^{I_n} = I_n + \frac{1}{1!} I_n + \frac{1}{2!} I_n + \dots + \frac{1}{k!} I_n + \dots = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!} + \dots\right) I_n = e\cdot I_n$

3) $e^{I_n t} = e^t I_n$

$$A = \begin{pmatrix} \lambda_1 t & 0 & \dots & 0 \\ 0 & \lambda_2 t & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \lambda_n t \end{pmatrix}$$

4) $$e^{\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}$$

$$A^k = \begin{pmatrix} \lambda_1^k t^k & 0 & \dots & 0 \\ 0 & \dots & 0 & \dots & \lambda_n^k t^k \end{pmatrix}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \cdot A^n = \mathrm{diag}\left(\sum_{n=0}^{\infty} \frac{1}{n!} \cdot (\lambda_1 t)^n \dots \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda_n t)^n\right) = \mathrm{diag}\left(e^{\lambda_1 t} \dots e^{\lambda_n t}\right)$$

5) $$e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} = I_2 + \frac{1}{1!} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t - \frac{1}{2!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t^2 - \frac{1}{3!} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t^3 + \frac{1}{4!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t^4 + \dots$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t$$

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} t^2 = -I_2 t^2$$

$$A^3 = -I_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t^3 = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t^3$$

$$A^4 = I_2 t^4 \dots$$

$$= \begin{pmatrix} 1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 - \frac{1}{6!} t^6 + \dots & -\frac{1}{1!} t + \frac{1}{3!} t^3 - \frac{1}{5!} t^5 + \dots \\ \frac{1}{1!} t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 + \dots & 1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 + \dots \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ \sin t & \cos t \end{pmatrix}$$

Theorem. $\frac{d}{dt}\left(e^{At}\right) = A e^{At}$ $\quad e^{At}\big|_{t=0} = I_n$ $\quad$ Thus $E(t) = e^{At}$, $\forall t \in \mathbb{R}$.

Remark. The general sol. of $X' = AX$ is $X = e^{At} \cdot C$, $C \in \mathbb{R}^n$.