Seminar 5

1. Type and stability of linear planar systems
2. Type and stability of the equilibria of nonlinear planar systems using the linearization method

---

① Type and stability of linear planar systems

Consider (1) $\dot{X} = AX$, where $A \in \mathcal{M}_2(\mathbb{R})$, $\det A \neq 0$.

Denote by $\lambda_1, \lambda_2 \in \mathbb{C}$ the two eigenvalues of A.

We know that $\det A = \lambda_1 \lambda_2$.

<u>Remarks</u>

1) $\det A \neq 0 \iff 0 \in \mathbb{R}^2$ is the only equilibrium point of (1).

2) $\det A \neq 0 \iff \lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

<u>Definition 1</u> We say that the equilibrium $0 \in \mathbb{R}^2$ of (1)

is a <u>NODE</u> when $\lambda_1, \lambda_2 \in \mathbb{R}$ and, either

$\lambda_1 \leq \lambda_2 < 0$ or $0 < \lambda_1 \leq \lambda_2$.

<u>SADDLE</u> when $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < 0 < \lambda_2$.

<u>FOCUS</u> when $\lambda_{1,2} = \alpha \pm i\beta$ with $\alpha \neq 0$, $\beta \neq 0$.

<u>CENTER</u> when $\lambda_{1,2} = \pm i\beta$ with $\beta \neq 0$.

Theorem 1 If $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$ then the equilibrium $0 \in \mathbb{R}^2$ of $\dot{X} = AX$ is a global attractor.

If $Re(\lambda_1) > 0$ and $Re(\lambda_2) > 0$ then the equilibrium $0 \in \mathbb{R}^2$ of $\dot{X} = AX$ is a global repeller.

Definition 2. Let $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and $\eta^* \in \mathbb{R}^2$ s.t. $f(\eta^*) = 0$. We say that the equilibrium point η^* of $\dot{X} = f(X)$ is stable if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. whenever $\|\eta - \eta^*\| < \delta$ we have $\|\varphi(t, \eta) - \eta^*\| < \varepsilon, \ \forall t \in (0, \infty)$.

We say that it is unstable if it is not stable.

Theorem 2. Any center is stable.
Any saddle is unstable.

Excercises 1) Decide the type and stability of the equilibrium $0 \in \mathbb{R}^2$ of the linear systems:

a) $\begin{cases} \dot{x} = -y \\ \dot{y} = 5x \end{cases}$ b) $\begin{cases} \dot{x} = -x \\ \dot{y} = 5y \end{cases}$ c) $\begin{cases} \dot{x} = -3x \\ \dot{y} = -2y \end{cases}$

d) $\begin{cases} \dot{x} = x - y \\ \dot{y} = x + y \end{cases}$ e) $\begin{cases} \dot{x} = 4x - 5y \\ \dot{y} = x - 2y \end{cases}$

2) For what values of the real parameter a, the system $\begin{cases} \dot{x} = ax - 5y \\ \dot{y} = x - 2y \end{cases}$ has a center at $0 \in \mathbb{R}^2$?

3) There exists uncoupled linear systems with a center?

<u>Solutions</u>

1) a) $\begin{cases} \dot{x} = -y \\ \dot{y} = 5x \end{cases}$ The matrix's system is $A = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}$

The equation of the eigenvalues of A is $\begin{vmatrix} -\lambda & -1 \\ 5 & -\lambda \end{vmatrix} = 0$

$\Leftrightarrow \lambda^2 + 5 = 0 \Leftrightarrow \lambda_{1,2} = \pm i\sqrt{5}$ is a center;

By Def 1, the equilibrium $0 \in \mathbb{R}^2$ of a) and by Th 2, is stable.

b) $\begin{cases} \dot{x} = -x \\ \dot{y} = 5y \end{cases}$ $A = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$ is a diagonal matrix $\Rightarrow$

$\Rightarrow \lambda_1 = -1$ and $\lambda_2 = 5$. We have $\lambda_1 < 0$ and $\lambda_2 > 0$. is a saddle; and, by Th 2,

Then, by Def 1, 0 of b) it is unstable.

c) $\begin{cases} \dot{x} = -3x \\ \dot{y} = -2y \end{cases}$ $A = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$ diagonal $\Rightarrow \lambda_1 = -3$, $\lambda_2 = -2$

We have $\lambda_1 < 0$ and $\lambda_2 < 0$. is a node; and, by Th. 1,

Then, by Def 1, 0 of c) is a global attractor.

d) and e) Homework

2) $\begin{cases} \dot{x} = ax - 5y \\ \dot{y} = x - 2y \end{cases}$ $\quad A = \begin{pmatrix} a & -5 \\ 1 & -2 \end{pmatrix}$

the equation of the eigenvalues of A is $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$

$\operatorname{tr} A = a + (-2) = a - 2$, $\det A = a\cdot(-2) - (-5)\cdot 1 = 5 - 2a$

(*) $\lambda^2 - (a-2)\lambda + (5-2a) = 0$

We have that, by Def 1, the equilibrium O of this system is a center if and only if $\exists\, \beta \in \mathbb{R}^*$ s.t. $\lambda_{1,2} = \pm i\beta$ are the eigenvalues of A if and only if $\exists\, \beta \in \mathbb{R}^*$ s.t. $\lambda_{1,2} = \pm i\beta$ are the roots of (*) if and only if $\left.\begin{array}{l} \lambda_1 + \lambda_2 = a - 2 \text{ and } \lambda_1\lambda_2 = 5-2a \\ \lambda_{1,2} = \pm i\beta \end{array}\right\} \Leftrightarrow$

$\Leftrightarrow \quad 0 = a - 2$ and $(i\beta)\cdot(-i\beta) = 5 - 2a \Leftrightarrow$

$\Leftrightarrow \quad a = 2$ and $\beta^2 = 5 - 2\cdot 2 \Leftrightarrow a = 2$ and $\beta^2 = 1$

<u>Conclusion</u> O is a center of this system $\Leftrightarrow a = 2$.

3) An uncoupled linear system (planar) has the form $\begin{cases} \dot{x} = ax \\ \dot{y} = by \end{cases}$ with $a, b \in \mathbb{R}$. its matrix is $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

with the eigenvalues $\lambda_1 = a$ and $\lambda_2 = b$.

According to Def 1, since both eigenvalues are real, O can not be a center.

Notice that it can be either a node, or a saddle.

② Stability of the equilibrium points of nonlinear planar systems using the linearization method

Consider (1) $\dot{X} = f(X)$, where $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$.

Recall that $\eta^* \in \mathbb{R}^2$ is an equilibrium of (1) $\Leftrightarrow$ $f(\eta^*) = 0$.

<u>Exercise</u>. Find the equilibria of $\begin{cases} \dot{x} = x(1-x) \\ \dot{y} = y(3-y) \end{cases}$.

Answer: $(0,0)$, $(1,0)$, $(0,3)$, $(1,3)$.

<u>The linearization method</u>

We consider the linear system (2) $\dot{X} = Jf(\eta^*)X$,

which is called the linearization of (1) around η^*.

<u>Recall</u>: For a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$, the <u>jacobian matrix</u> of f computed in (x,y) is

$$Jf(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix}.$$

<u>Definition 3</u> We say that the equilibrium point η^* of $\dot{X} = f(X)$ is <u>hyperbolic</u> if $Re(\lambda_1) \neq 0$ and $Re(\lambda_2) \neq 0$, where $\lambda_{1,2} \in \mathbb{C}$ are the eigenvalues of $Jf(\eta^*)$.

Note: $Re(\lambda)$ denotes the real part of $\lambda \in \mathbb{C}$.

Theorem 3 Let η^* be a hyperbolic equilibrium point of $\dot{X}=f(X)$. If the equilibrium $0\in\mathbb{R}^2$ of $\dot{X}=Jf(\eta^*)X$ is an attractor (repeller) then the equilibrium η^* of $\dot{X}=f(X)$ is also an attractor (repeller). If the equilibrium point $0\in\mathbb{R}^2$ of $\dot{X}=Jf(\eta^*)X$ is a saddle, then the equilibrium point η^* of $\dot{X}=f(X)$ is unstable.

Exercise 4) Study the stability of the equilibria of the system

$$\begin{cases}\dot{x}=x(1-x)\\ \dot{y}=y(3-y)\end{cases}.$$

Solution Let $f:\mathbb{R}^2\to\mathbb{R}^2$, $f(x,y)=\begin{pmatrix}x-x^2\\ 3y-y^2\end{pmatrix}$.

We have that $f(x,y)=\begin{pmatrix}0\\0\end{pmatrix}$ has the roots $(0,0)$, $(1,0)$, $(0,3)$, $(1,3)$. Thus, these 4 points are the equilibria of the given system.

We need to compute $Jf(x,y)=\begin{pmatrix}1-2x & 0\\ 0 & 3-2y\end{pmatrix}$.

In order to study the stability of the equilibrium $(0,0)$, we compute $Jf(0,0)=\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$. Its eigenvalues are $\lambda_1=1$ and $\lambda_2=3$, both real and positive.

By Th 1, the eq. $0\in\mathbb{R}^2$ of $\dot{X}=Jf(0,0)X$ is a repeller.

Thus, by Th3, the eq. $(0,0)\in\mathbb{R}^2$ of the given nonlinear system is also a repeller.

In order to study the stability of the equilibrium $(1,0)$, we compute $Jf(1,0) = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$. Its eigenvalues are

$\lambda_1 = -1 < 0$ and $\lambda_2 = 3 > 0$.

By Def. 1, the eq. $(0,0)$ of $\dot{X} = Jf(1,0)X$ is a saddle.

Thus, by Th 3, the equilibrium $(1,0)$ of the given nonlinear system is unstable.

Homework: study the stability of $(0,3)$ and $(1,3)$.

Important remark In order to study the stability of the equilibria of a second order nonlinear equation (scalar)

$$\ddot{x} = f(x, \dot{x}),$$

we transform it into a planar system with the unknowns x and $y = \dot{x}$. So, the planar system is

$$\begin{cases} \dot{x} = y \\ \dot{y} = f(x,y). \end{cases}$$

Exercise 5) Study the stability of the equilibria of

$\ddot{\theta} + \omega^2 \sin\theta = 0$ (the idealized equation of a pendulum). $\boxed{\omega > 0}$

Solution. Denote $x = \theta$ and $y = \dot{\theta}$. The equivalent planar system is $\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 \sin x \end{cases}$.

In order to find the equilibria, we must solve

$\begin{cases} y = 0 \\ \sin x = 0 \end{cases}$. Thus, the equilibria are $(k\pi, 0)$, $k \in \mathbb{Z}$.

Note that, physically, they corespond to only 2 positions: $(0,0)$ and $(\pi, 0)$.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = \begin{pmatrix} y \\ -w^2 \sin x \end{pmatrix}$. We have

$$Jf(x,y) = \begin{pmatrix} 0 & 1 \\ -w^2 \cos x & 0 \end{pmatrix}$$

In order to study the stability of the equilibrium $(0,0)$, we compute $Jf(0,0) = \begin{pmatrix} 0 & 1 \\ -w^2 & 0 \end{pmatrix}$ whose trace is 0, and determinant is w^2.

Thus, the equation of the eigenvalues is $\lambda^2 - 0 \cdot \lambda + w^2 = 0$
$\iff \lambda^2 + w^2 = 0 \iff \lambda_{1,2} = \pm i w$

Thus, we have $Re(\lambda_1) = Re(\lambda_2) = 0$. By Def. 3, we have that the equilibrium point $(0,0)$ is not hyperbolic. Hence, the main hypothesis of Th 3 is not fulfilled and, for the moment, we do not have another result to study the stability of $(0,0)$. So, we are not able to decide (for the moment).

In order to ~~stt~~ study the stability of the equil. $(\pi, 0)$, we compute $Jf(\pi, 0) = \begin{pmatrix} 0 & 1 \\ w^2 & 0 \end{pmatrix}$ whose trace is 0, and determinant is $-w^2$.

$\lambda^2 - w^2 = 0 \iff \lambda_{1,2} = \pm w$. Thus, the eigenvalues are real, with opposite signs. By Def. 1, the equil. $(0,0)$ of $\dot{X} = Jf(\pi, 0) X$ is a saddle; and, by Th 3, the equil. $(\pi, 0)$ of our system is unstable.