I Hyperbolic equilibria

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$$\begin{cases} \dot{x} = x - 2xy \\ \dot{y} = \frac{x^2}{2} - y \end{cases}$$

a) all equilibria

$x - 2xy = 0 \Rightarrow x(1-2y) = 0 \Rightarrow x = 0$ or $y = \frac{1}{2}$

$\frac{x^2}{2} - y = 0$

$x = 0 \Rightarrow \frac{0}{2} - y = 0$, $\underline{y = 0}$

$y = \frac{1}{2} \Rightarrow \frac{x^2}{2} - \frac{1}{2} = 0 \Rightarrow x = \pm 1$

$(0,0)$, $(1, \frac{1}{2})$, $(-1, \frac{1}{2})$ are the equilibria

b) The Jacobian ~~[illegible]~~ matrix for the system is

$$Jf(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 2y & -2x \\ x & -1 \end{pmatrix}$$

We study the equilibrium $(0,0)$.

$Jf(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The eigenvalues are 1 and -1, so $(0,0)$ is a saddle for $x' = Jf(0,0) \cdot x$. By theorem 3 if 0 is a saddle for $x' = Jf(0,0)\cdot x$, then $(0,0)$ is an unstable equilibrium point for $\dot{x} = f(x)$.

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$$Jf\left(1,\frac{1}{2}\right) = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$$

$$\det\left(Jf\left(1,\frac{1}{2}\right) - \lambda I_2\right) = 0$$

$$\det\begin{pmatrix} -\lambda & -2 \\ 1 & -1-\lambda \end{pmatrix} = 0$$

$$\lambda(1+\lambda) + 2 = 0$$

$$\lambda^2 + \lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-8}}{2}$$

$$\lambda_1 = \frac{-1}{2} + i\frac{\sqrt{7}}{2}$$

$$\lambda_2 = \frac{-1}{2} - i\frac{\sqrt{7}}{2}$$

By theorem 1 $(0,0)$ is a focus for $x' = Jf\left(1,\frac{1}{2}\right)x$, and the real parts are <0, so it is an attractor. By theorem 3 $\left(1,\frac{1}{2}\right)$ is an attracting focus for $\dot{x} = f(x)$.

$$Jf\left(-1,\frac{1}{2}\right) = \begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix}$$

$$\det\begin{pmatrix} -\lambda & 2 \\ -1 & -1-\lambda \end{pmatrix} = 0$$

$$\lambda(1+\lambda) + 2 = 0$$

$$\lambda_{1,2} = \frac{-1}{2} \pm i\frac{\sqrt{7}}{2}.$$

The eigenvalues are the same as above, ~~so~~ as previously $\left(-1,\frac{1}{2}\right)$ is an attracting focus for $x' = f(x)$.

c) PHASE PORTAIT I.2

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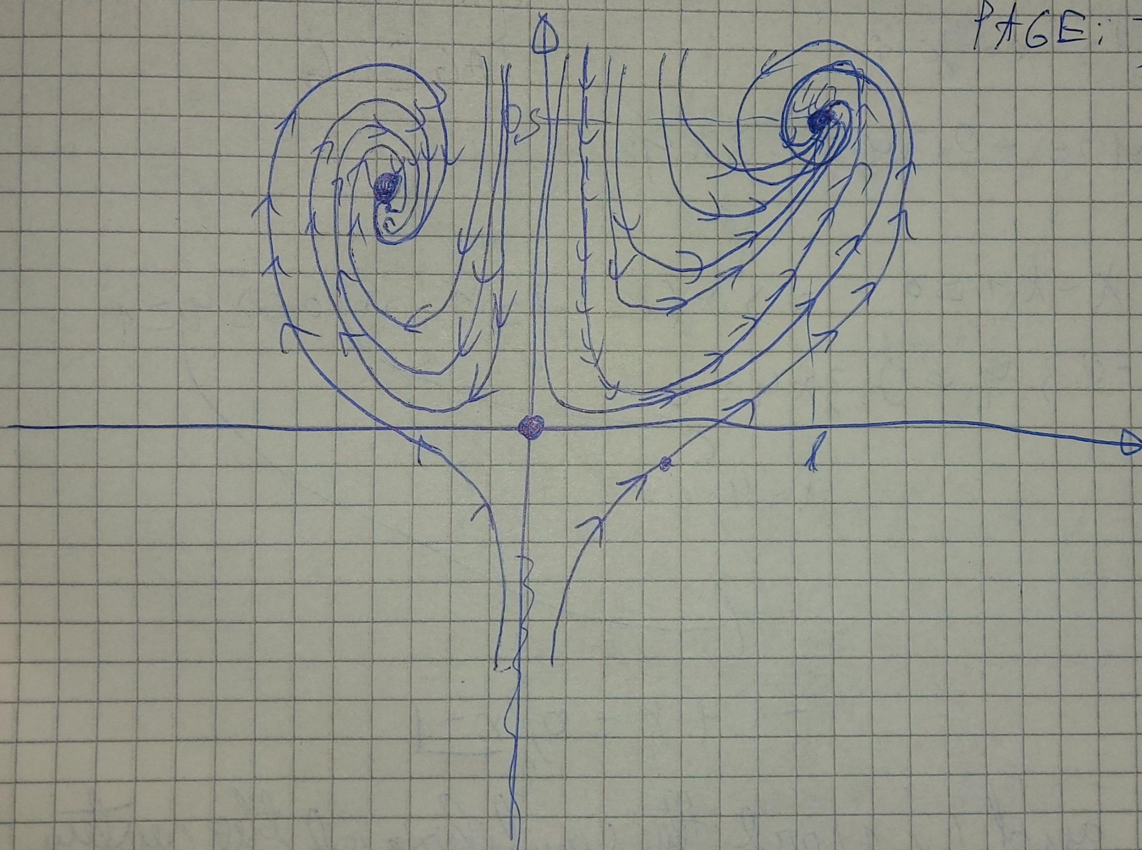

II Non-hyperbolic equilibria

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4.
$$x' = x - xy$$
$$y' = -0.3y + 0.3xy$$

a) $x - xy = 0 \Rightarrow x(1-y) = 0 \Rightarrow x = 0$ or $y = 1$

$-0.3y + 0.3xy = 0$

$x = y = 0$

$-1 + x = 0$, $\underline{x = 1}$

$(0,0)$ and $(1,1)$ are the equilibria of the system

$$Jf(x,y) = \begin{pmatrix} 1-y & -x \\ 0.3y & -0.3+0.3x \end{pmatrix}$$

$$Jf(1,1) = \begin{pmatrix} 0 & -1 \\ 0.3 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & -1 \\ 0.3 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 0.3 = 0 \Rightarrow \lambda = \pm i\sqrt{0.3}$$

The real part of λ is 0, so it is non-hyperbolic

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$\frac{dx}{dt} = $

$\frac{dy}{dt} = $

b) $\frac{dx}{dy} = \frac{x - xy}{-0.3y + 0.3xy}$

$\frac{dx}{dy} = \frac{x(1-y)}{-0.3y(1-x)}$

$\frac{1-x}{x}dx = \frac{1-y}{-0.3y}dy$

$\frac{x}{1-x}dx = -0.3\frac{y}{1-y}dy \mid \int$

$\int\frac{x}{1-x}dx = -0.3\int\frac{y}{1-y}dy$

$\int\frac{x-1+1}{1-x}dx = -0.3\int\frac{y-1+1}{1-y}dy$

$-x - \ln(x-1)$

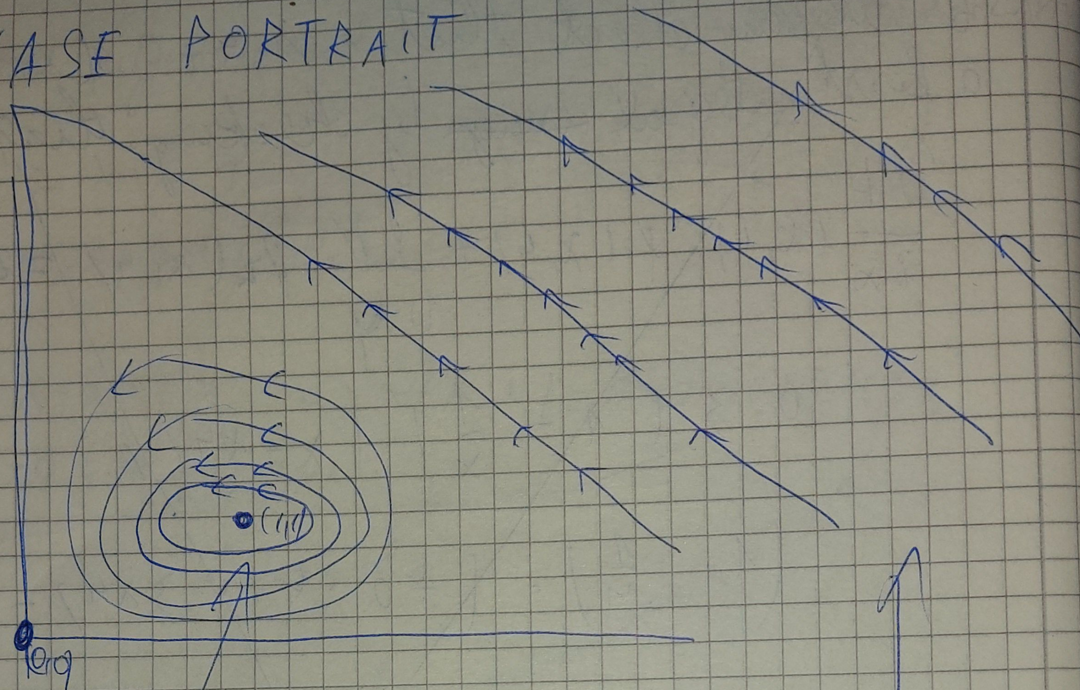
$-0.3\frac{1-x}{x}dx = \frac{1-y}{y}dy \mid \int$

$-0.3\int\frac{1-x}{x}dx = \int\frac{1-y}{y}dy$

$\boxed{-0.3(-x+\ln x) = -y+\ln y}$ first integral

for $\dot{x} = f(x)$

PHASE PORTRAIT



Around 1,1 the orbits are periodic

At much larger values the orbits look like straight lines (almost flat) that drop to 0,0 upon reaching $x = 0$

Explanations are found in the maple sheets :)

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> $eq1 := diff(x(t), t) = x(t) - 2 \cdot x(t) \cdot y(t)$ #Name: Craciun Ioan-Flaviu, Group: 912\/2

$$eq1 := \frac{d}{dt} x(t) = x(t) - 2\, x(t)\, y(t) \quad (1)$$

> $eq2 := diff(y(t), t) = \frac{x(t)^2}{2} - y(t)$

$$eq2 := \frac{d}{dt} y(t) = \frac{x(t)^2}{2} - y(t) \quad (2)$$

> $solve\left(\left\{ x - 2 \cdot x \cdot y = 0, \frac{x^2}{2} - y = 0 \right\}\right)$ #Note that this nonlinear system has three equilibria $\cdot (0, 0)$, $\left(1, \frac{1}{2}\right)$ **and** $\left(-1, \frac{1}{2}\right)$.

$$\{x = 0, y = 0\}, \left\{x = 1, y = \frac{1}{2}\right\}, \left\{x = -1, y = \frac{1}{2}\right\} \quad (3)$$

> $with(linalg) : with(DEtools) : with(VectorCalculus) :$

> $Jm1 := Jacobian\left(\left[x - 2 \cdot x \cdot y, \frac{x^2}{2} - y \right], [x, y]\right)$

$$Jm1 := \begin{bmatrix} -2\,y + 1 & -2\,x \\ x & -1 \end{bmatrix} \quad (4)$$

> $A1 := subs([x = 0, y = 0], Jm1)$
> #So, the linearization of the given nonlinear system around the equilibrium point $\cdot (0, 0)$ is X' = A1 X, that is x'=x, y'= -y.

$$A1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (5)$$

> $eigenvalues(A1)$
> #Note that both eigenvalues are real, different from zero. Then the equilibrium point $\cdot (0, 0)$ is hyperbolic **and** we can apply the Linearization method $\cdot$ (Theorem 3 **from** Sem5). In this case the linear system#=A1 X has a saddle, thus the equil $\cdot (0, 0)$ of the nonlinear system is unstable.

$$1, -1 \quad (6)$$

> $A2 := subs\left(\left[x = 1, y = \frac{1}{2}\right], Jm1\right)$
> # So, the linearization of the given nonlinear system around the equilibrium point $\cdot \left(1, \frac{1}{2}\right)$ is X'= A2 X, that is x'= -2 y, y'=x-y.

$$A2 := \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix} \quad (7)$$

> $eigenvalues(A2)$
> # Note that both eigenvalues are complex conjugate, whose real part is different from zero. Then the equilibrium point $\cdot \left(1, \frac{1}{2}\right)$ is hyperbolic **and** we can apply the Linearization method $\cdot$ (Theorem 3 **from** Sem5)
> # Since the real part of the eigenvalues is stricly negative, the linear system X'=A2 X has an

attracting focus, thus the equilibrium $\left(1, \frac{1}{2}\right)$ *of the nonlinear system is also an attractor.*

$$-\frac{1}{2}+\frac{I\sqrt{7}}{2}, -\frac{1}{2}-\frac{I\sqrt{7}}{2} \quad \textbf{(8)}$$

> $A3 := subs\left(\left[x=-1, y=\frac{1}{2}\right], Jm1\right)$

#the linearization of the given nonlinear system around the equilibrium point $\cdot\left(-1, \frac{1}{2}\right)$ *is X'* $= A3\ X$*, that is* $x' = 2\cdot y$, $y' = -x - y$.

$$A3 := \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} \quad \textbf{(9)}$$

> $eigenvalues(A3)$

#Note that both eigenvalues are complex conjugate, whose real part is different from zero. Then the equilibrium point $\cdot\left(-1, \frac{1}{2}\right)$ *is hyperbolic*

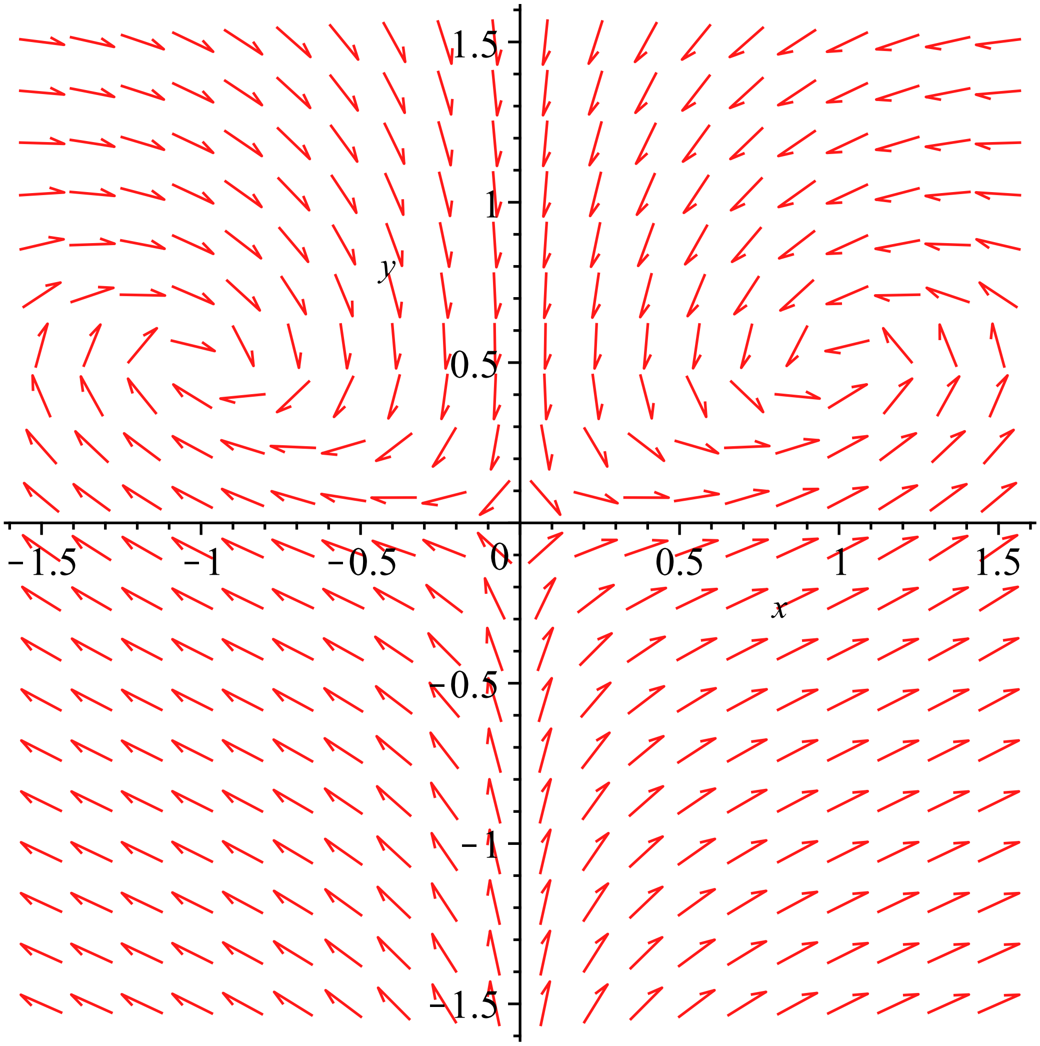
and *we can apply the Linearization method* (*Theorem* 3
#from Sem5) *Since the real part of the eigenvalues is stricly negative, the linear system X'* $= A3\ X$ *has an attracting focus, thus the equilibrium* $\cdot\left(-1, \frac{1}{2}\right)$ *of the nonlinear system is also an attractor.*

$$-\frac{1}{2}+\frac{I\sqrt{7}}{2}, -\frac{1}{2}-\frac{I\sqrt{7}}{2} \quad \textbf{(10)}$$

> $dfieldplot\left([eq1, eq2], [x(t), y(t)], t=0..1, x=-\frac{3}{2}..\frac{3}{2}, y=-\frac{3}{2}..\frac{3}{2}\right)$

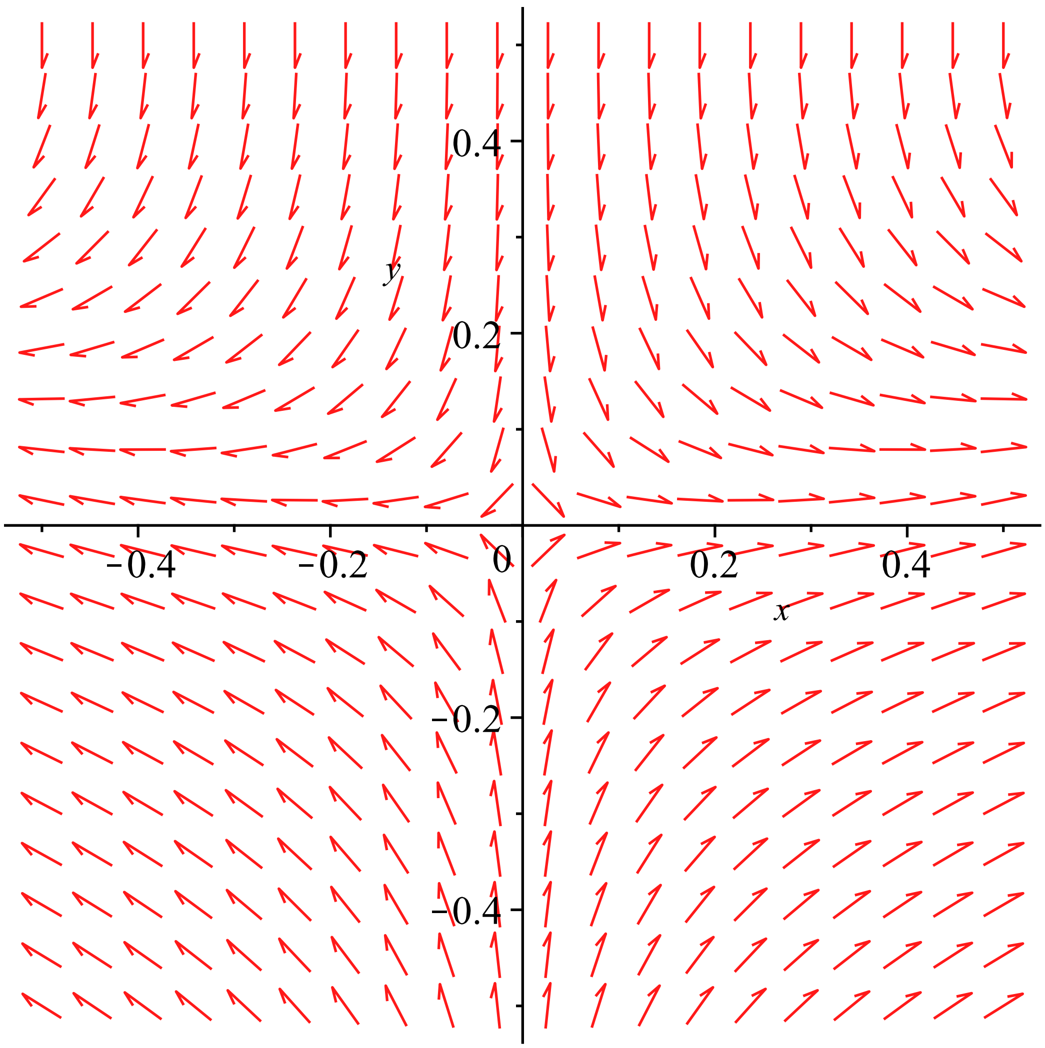
#This is the direction field in the box $\left[-\frac{3}{2}, \frac{3}{2}\right] x \left[-\frac{3}{2}, \frac{3}{2}\right]$

.



> $dfieldplot\left([eq1, eq2], [x(t), y(t)], t=0..1, x=-\frac{1}{2}..\frac{1}{2}, y=-\frac{1}{2}..\frac{1}{2} \right)$

#This is the direction field in the box $\left[-\frac{1}{2}, \frac{1}{2}\right]x\left[-\frac{1}{2}, \frac{1}{2}\right]$. *The only equilibrium point* **in** *this box is* $\cdot(0, 0)$. *It seems that the orbits*
#in this small box looks like the orbits of a linear system with a saddle. In the next figure we will see the direction field of the linearization around $\cdot(0, 0)$, $X' = A1\ X$, *ie.* $x' = x$, $y' = -y$ *which has a saddle. The orbits seem #to look like hyperbolae*



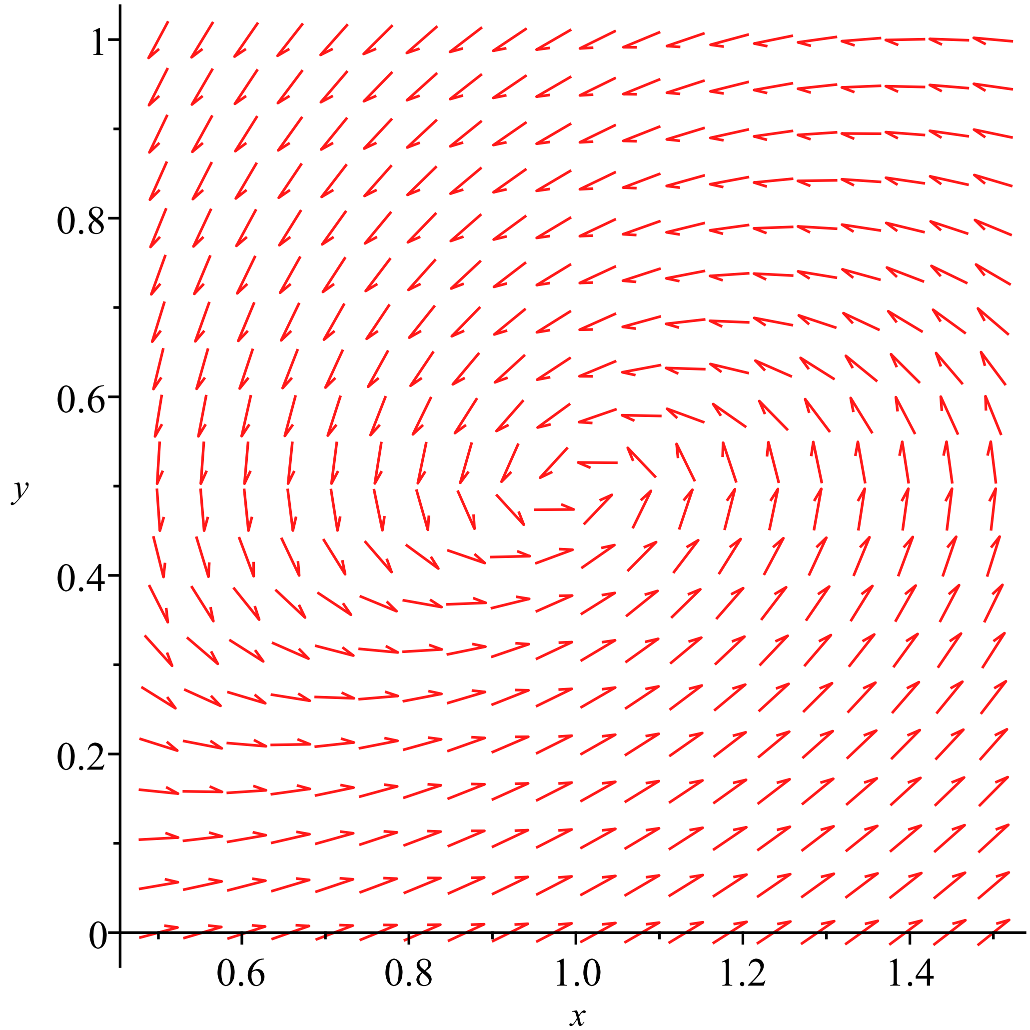
> $dfieldplot\left([eq1, eq2], [x(t), y(t)], t=0..1, x=\frac{1}{2}..\frac{3}{2}, y=0..1 \right)$

#This is the direction field in the box $[0, 1]x\left[\frac{1}{2}, \frac{3}{2}\right]$*. The only equilibrium point*

in *this box is* $\cdot\left(1, \frac{1}{2}\right)$*. It seems that the orbits*

#in this small box looks like the orbits of a linear system with an attracting focus. In the next figure we will see the direction field of the linearization around $\cdot\left(1, \frac{1}{2}\right)$,

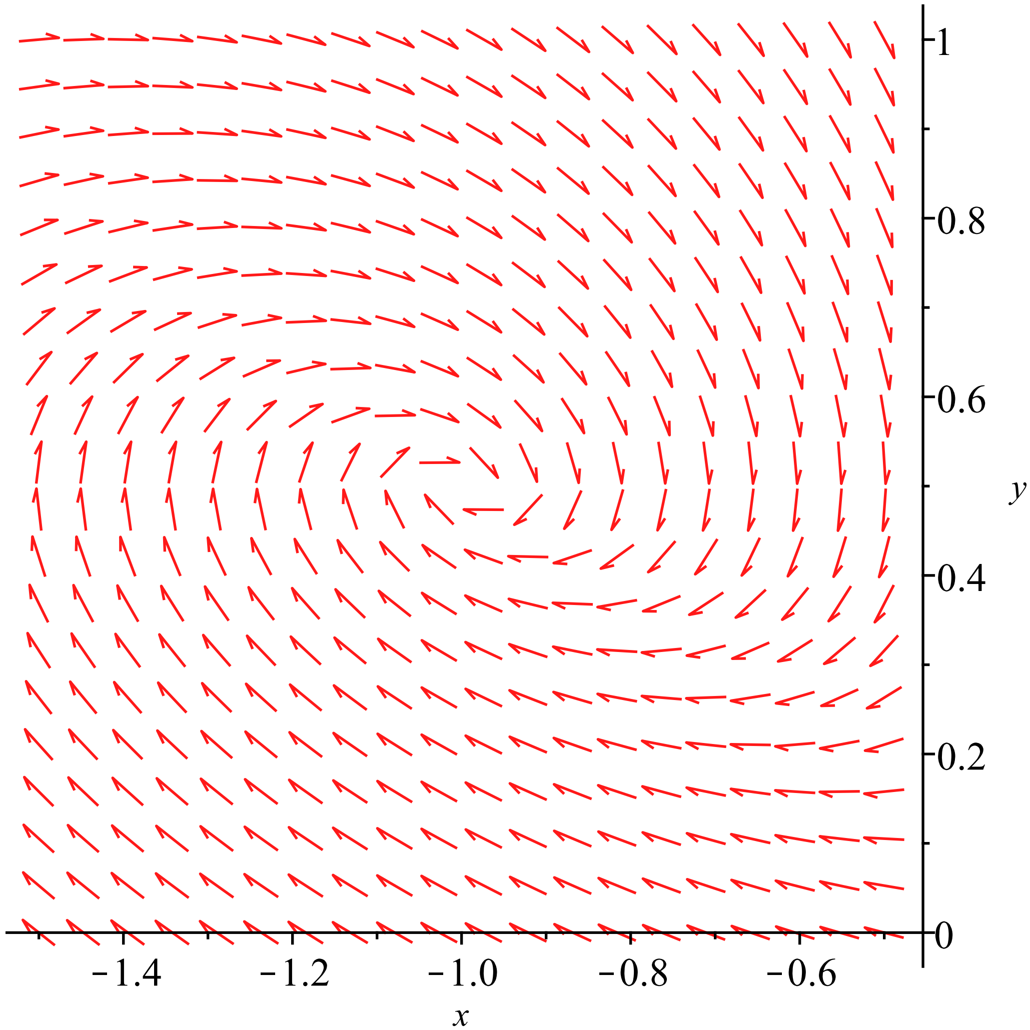
 # X'=A2 X, i.e. $x' = -2\cdot y$, $y' = x - y$ *which has an attracting focus. The orbits seem to look like elipses or spirals*



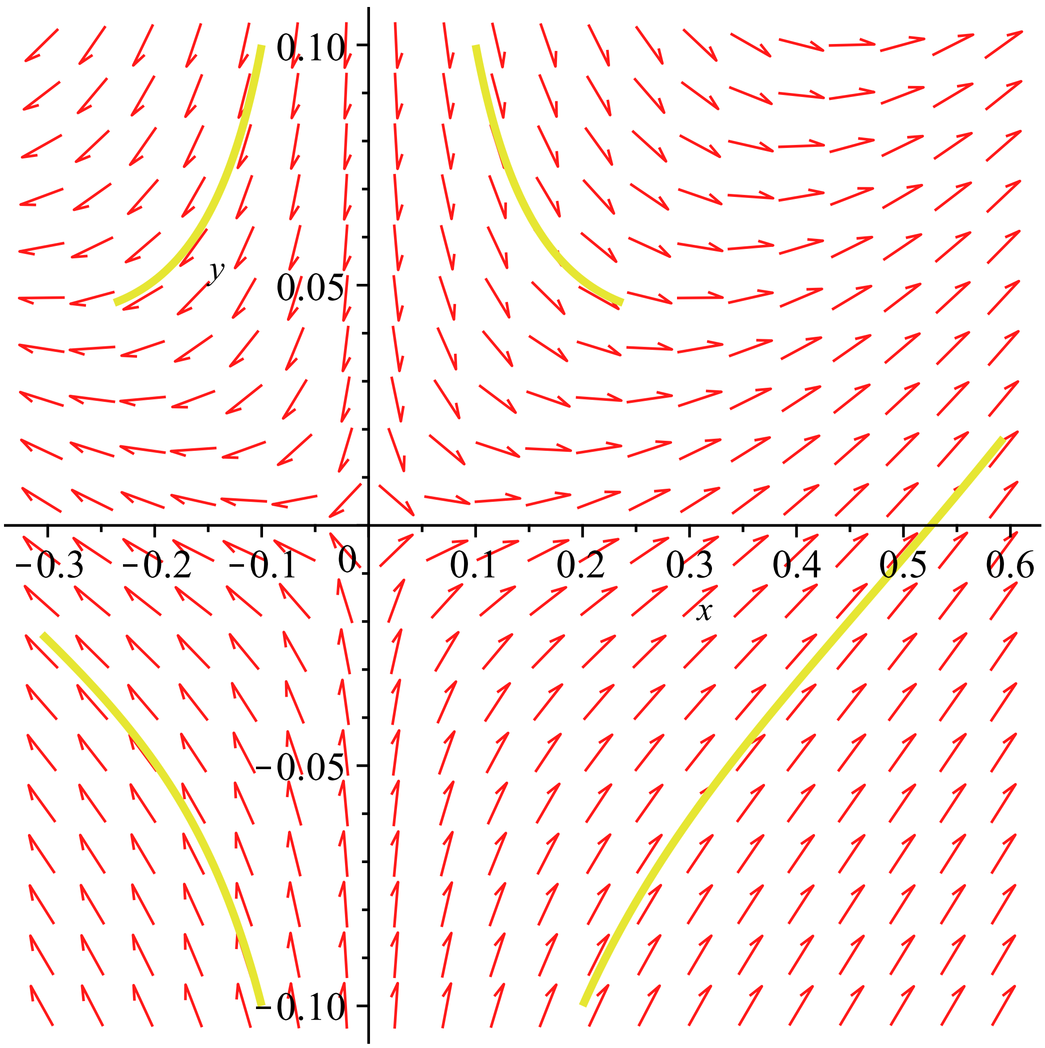
> $dfieldplot\left([eq1, eq2], [x(t), y(t)], t = 0..1, x = \frac{-3}{2}..\frac{-1}{2}, y = 0..1\right)$

#This looks just like the symmetric of image above, same attracting focus. In the next figure we will see the direction field of the linearization around $\cdot\left(-1, \frac{1}{2}\right)$,

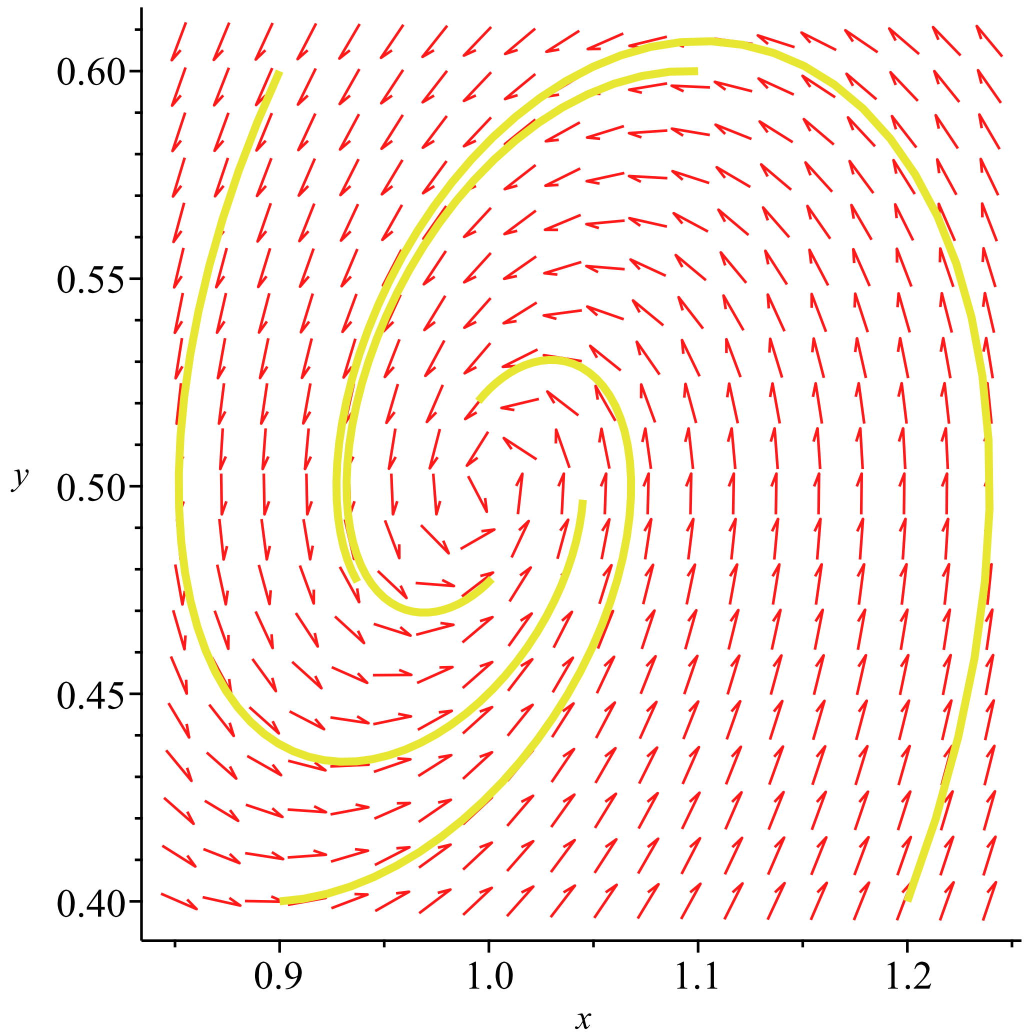
 # X'=A3X. These orbits also look like elipses or spirals



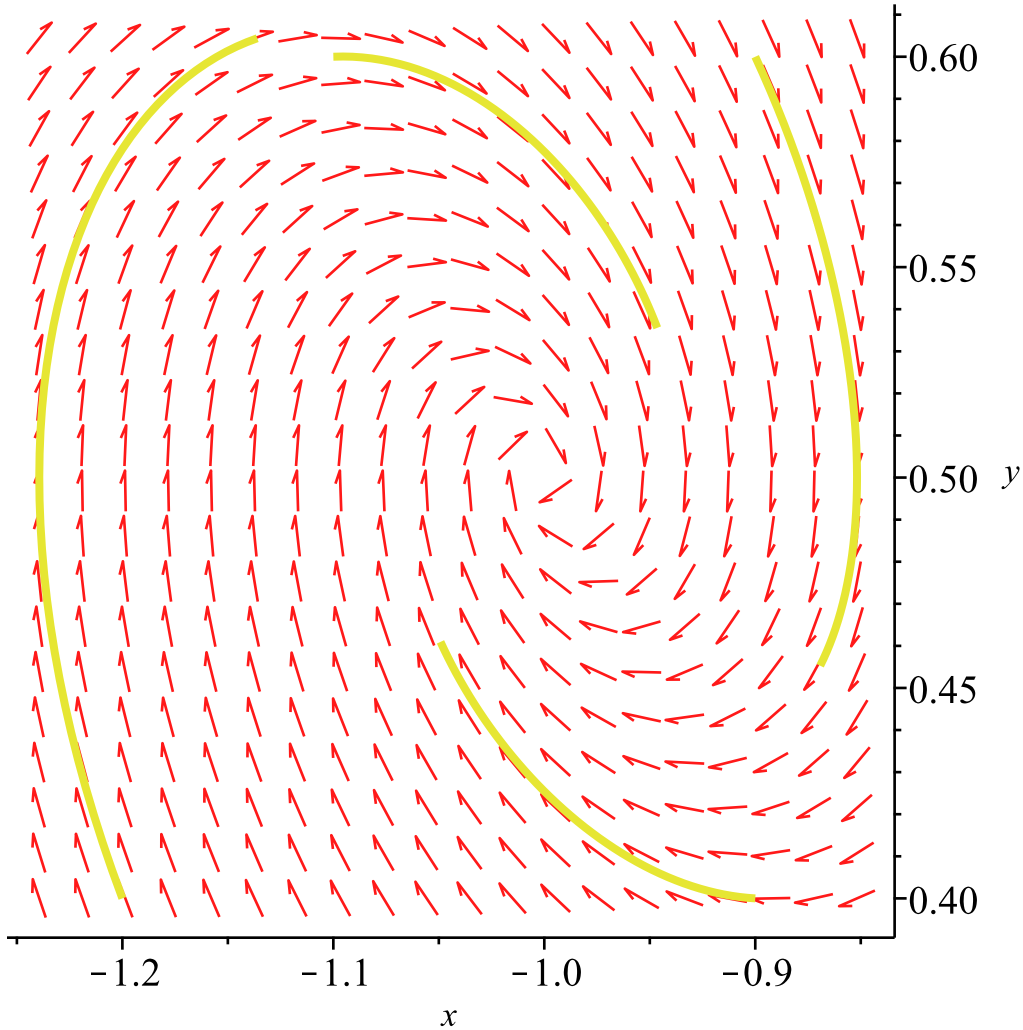
> $DEplot([eq1, eq2], [x(t), y(t)], t=0..1, [[x(0)=0.1, y(0)=0.1], [x(0)=-0.1, y(0)=0.1], [x(0)=0.2, y(0)=-0.1], [x(0)=-0.1, y(0)=-0.1]])$



> $DEplot([eq1, eq2], [x(t), y(t)], t = 0 .. 3, [[x(0) = 1.1, y(0) = 0.6], [x(0) = 0.9, y(0) = 0.6], [x(0) = 1.2, y(0) = 0.4], [x(0) = 0.9, y(0) = 0.4]])$



> $DEplot([eq1, eq2], [x(t), y(t)], t=0..1, [[x(0)=-1.1, y(0)=0.6], [x(0)=-0.9, y(0)=0.6],$
$[x(0)=-1.2, y(0)=0.4], [x(0)=-0.9, y(0)=0.4]])$


0.60
0.55
0.50
0.45
0.40
y
-1.2
-1.1
-1.0
-0.9
x
>

> *eq1* := *diff*(*x*(*t*), *t*) = *x*(*t*) - *x*(*t*)·*y*(*t*)   #Name:Craciun Ioan-Flaviu, group: 912\/2

$$eq1 := \frac{d}{dt}\, x(t) = x(t) - x(t)\, y(t) \tag{1}$$

> *eq2* := *diff*(*y*(*t*), *t*) = -0.3·*y*(*t*) + 0.3 *x*(*t*)·*y*(*t*)

$$eq2 := \frac{d}{dt}\, y(t) = -0.3\, y(t) + 0.3\, x(t)\, y(t) \tag{2}$$

> *solve*({*x* − *x*·*y* = 0, -0.3 *y* + 0.3·*x*·*y* = 0}) #   *x=1,y=1 is an equilibrium point*

$$\{x = 0., y = 0.\}, \{x = 1., y = 1.\} \tag{3}$$

> *with*(*linalg*) : *with*(*DEtools*) : *with*(*VectorCalculus*) :

> *Jm* := *Jacobian*([*x* − *x*·*y*, -0.3·*y* + 0.3·*x*·*y*], [*x*, *y*])

$$Jm := \begin{bmatrix} -y+1 & -x \\ 0.3\, y & -0.3 + 0.3\, x \end{bmatrix} \tag{4}$$

> *A* := *subs*([*x* = 1, *y* = 1], *Jm*)

$$A := \begin{bmatrix} 0 & -1 \\ 0.3 & 0. \end{bmatrix} \tag{5}$$

> *eigenvalues*(*A*)
   #   *We notice that the eigenvalues have real part equal to 0, so they are not hyperbolic*

$$0.547722557505166\, \mathrm{I},\ -0.547722557505166\, \mathrm{I} \tag{6}$$

> *simplify*(*dsolve*({*eq1*, *eq2*, *x*(0) = *eta1*, *y*(0) = *eta2*}, {*x*(*t*), *y*(*t*)}))
   #Very weird solution, without simplify it's huge. Wanted to use this to check that H is a first integral but no chance

$$\Bigg\{ x(t) = RootOf\Bigg(-\Bigg(\int_{\_a}^{\_Z} 1 \Bigg/ \Bigg(\_f \Bigg(\mathrm{LambertW}\Bigg(\_Z3\sim, \tag{7}$$

$$\frac{1}{\_f^{3/10}} \Bigg((-1)^{-\frac{3\,\_Z1\sim}{5}}\, \mathrm{e}^{-1 + \frac{3\,\_f}{10}}\, RootOf\Bigg(\_Z\, \mathrm{e}^{\frac{3\,\eta 1}{10} - \frac{3\,\mathrm{I}\pi\,\_Z1\sim}{5}} \Bigg(\mathrm{LambertW}\Bigg(\_Z3\sim,$$

$$\frac{\mathrm{e}^{-1 + \frac{3\,\eta 1}{10} - \frac{3\,\mathrm{I}\pi\,\_Z1\sim}{5}}\, \_Z}{\eta 1^{3/10}} \Bigg) + \eta 2 \Bigg)\Bigg)\Bigg)\Bigg) + 1 \Bigg)\Bigg)\, d\_f \Bigg) + t + \int_{\_a}^{\eta 1} 1 \Bigg/$$

$$\left(\_f\left(\mathrm{LambertW}\left(\_Z3\sim,\right.\right.\right.$$

$$\frac{1}{\_f^{3/10}}\left((-1)^{-\frac{3\_Z1\sim}{5}}\mathrm{e}^{-1+\frac{3\_f}{10}}\,RootOf\left(\_Z\,\mathrm{e}^{\frac{3\,\eta 1}{10}-\frac{3\,\mathrm{I}\,\pi\_Z1\sim}{5}}\left(\mathrm{LambertW}\left(\_Z3\sim,\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.\left.\left.\left.\frac{\mathrm{e}^{-1+\frac{3\,\eta 1}{10}-\frac{3\,\mathrm{I}\,\pi\_Z1\sim}{5}}\_Z}{\eta 1^{3/10}}\right)+\eta 2\right)\right)\right)\right)+1\right)\right)\mathrm{d}\_f\right), y(t)=0\right\}$$

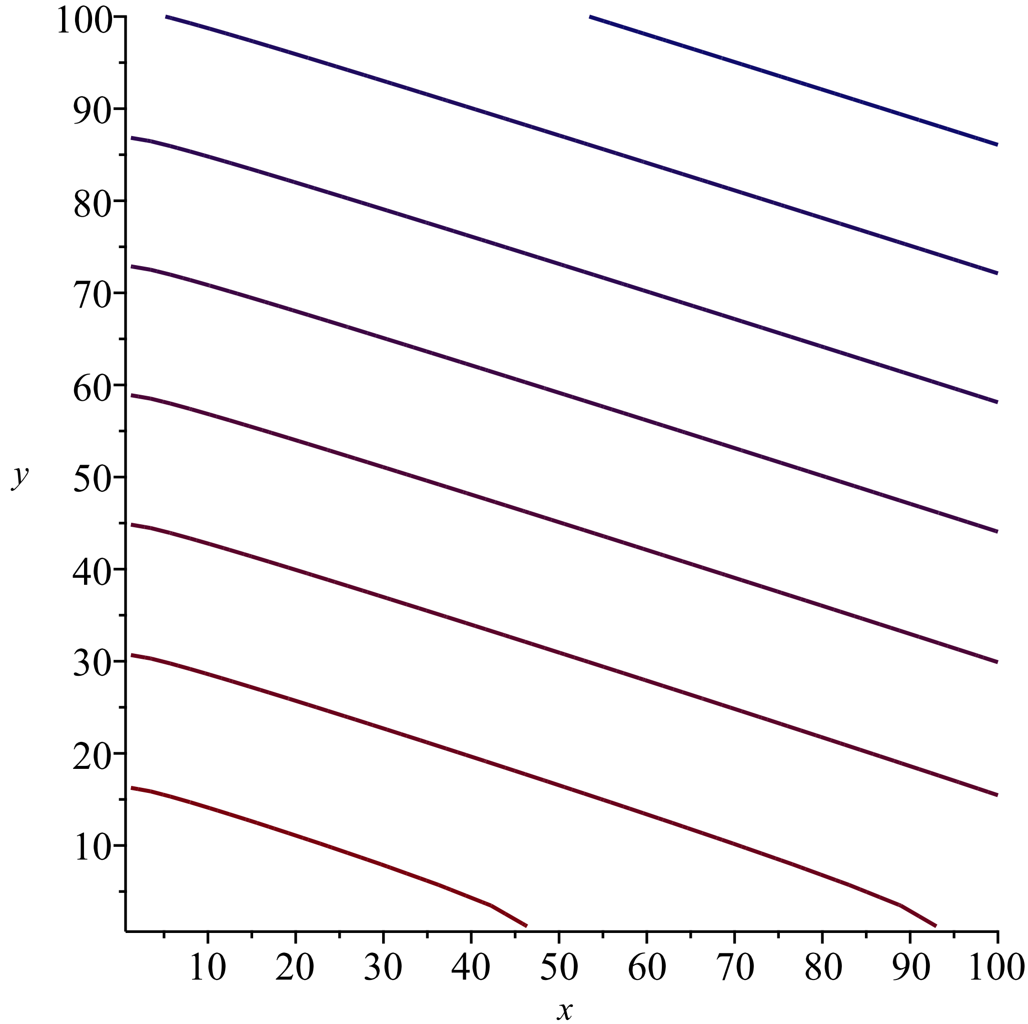
> $H := y - \ln(y) + 0.3\cdot(x - \ln(x))$

$$H := y - \ln(y) + 0.3\,x - 0.3\,\ln(x) \tag{8}$$

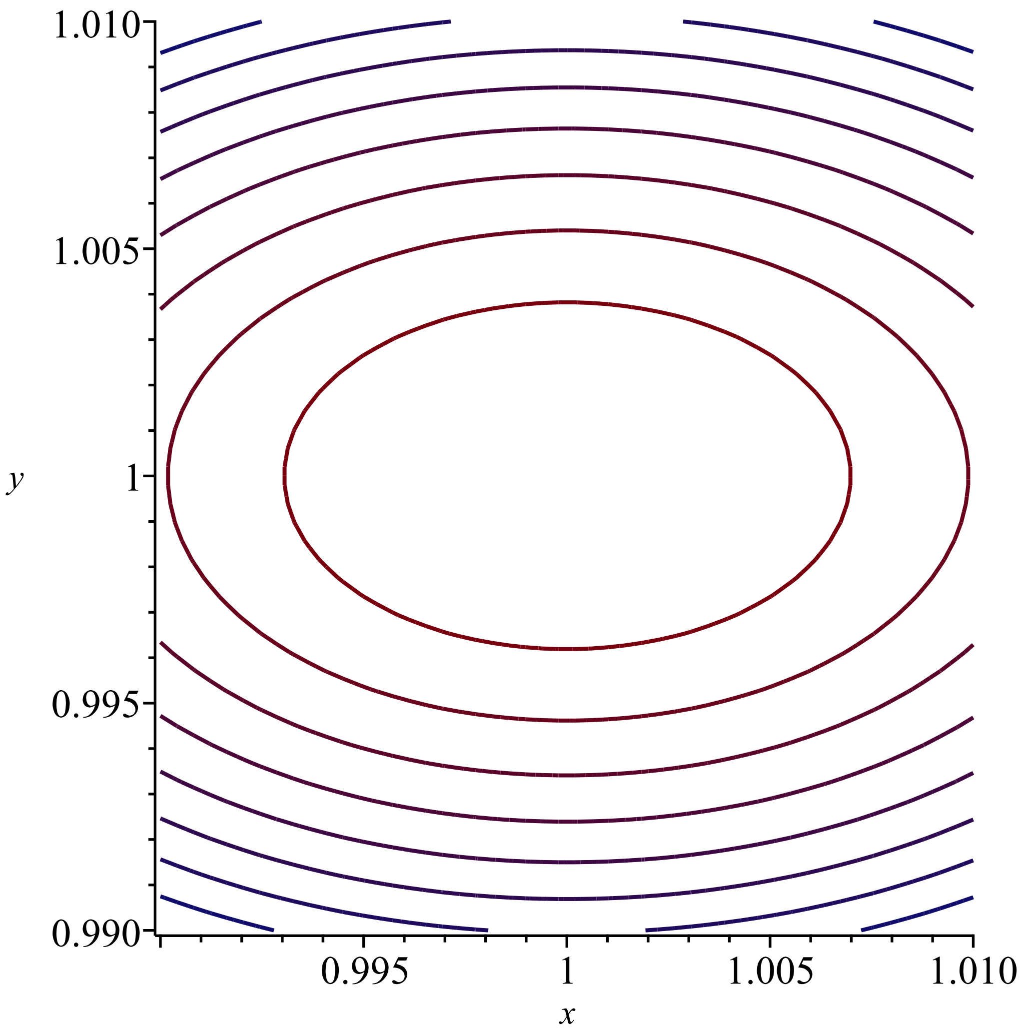
> $simplify(diff(H, x)\cdot(x - x\cdot y) + diff(H, y)\cdot(-0.3\cdot y + 0.3\cdot x\cdot y))$
 # I used the definition from the eighth lecture for the first integral and proved that it is for the given system

$$0. \tag{9}$$

> $with(plots) : contourplot(H, x=-10..100, y=-10..100)$
#The level curves are given by contourplot below which are similar to straight lines because $\frac{1}{y-\ln(y)}(x-\ln(x)) = -\frac{10}{3}$ **and** *x* **and** *y get much larger than lnx* **and** *lny*

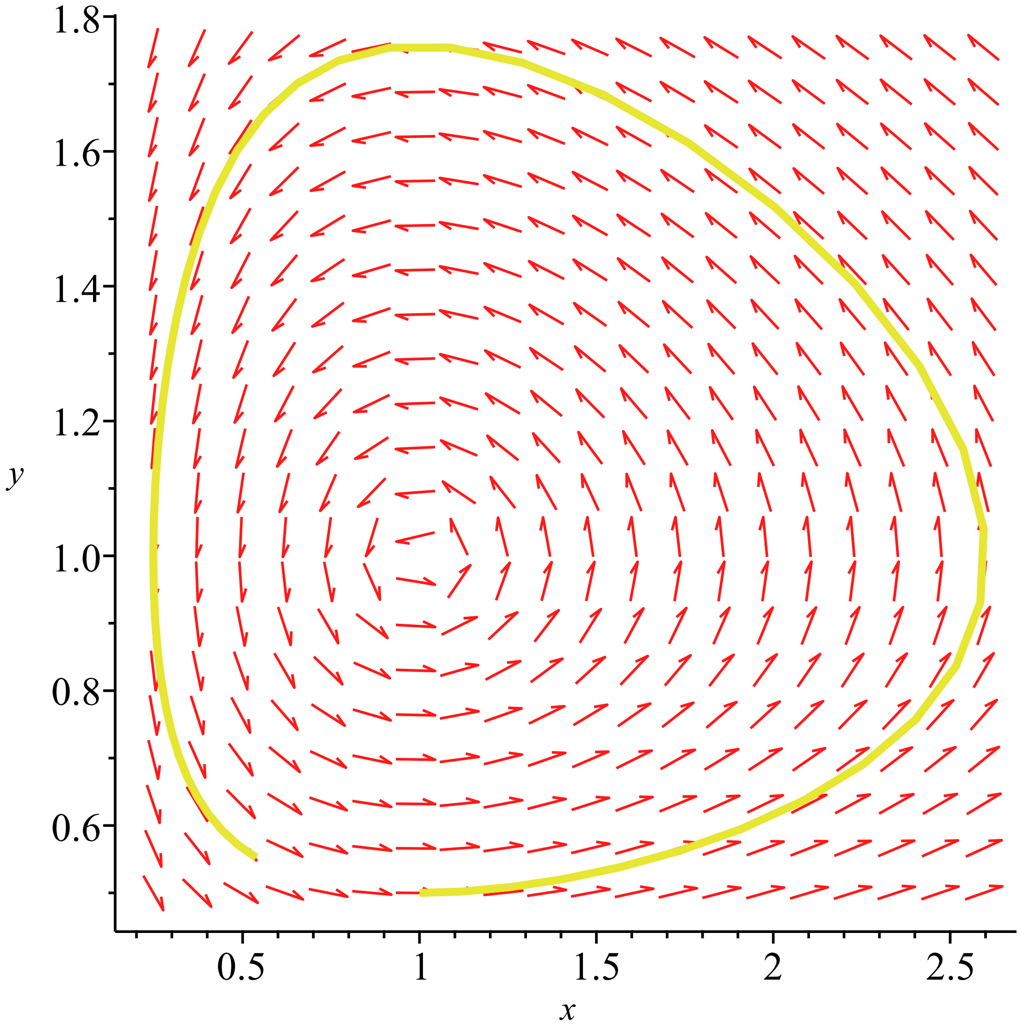


> $with(plots) : contourplot(H, x = 0.99 ..1.01, y = 0.99 ..1.01)$
#Very close to (1,1) the orbits are closed, this means they are periodic

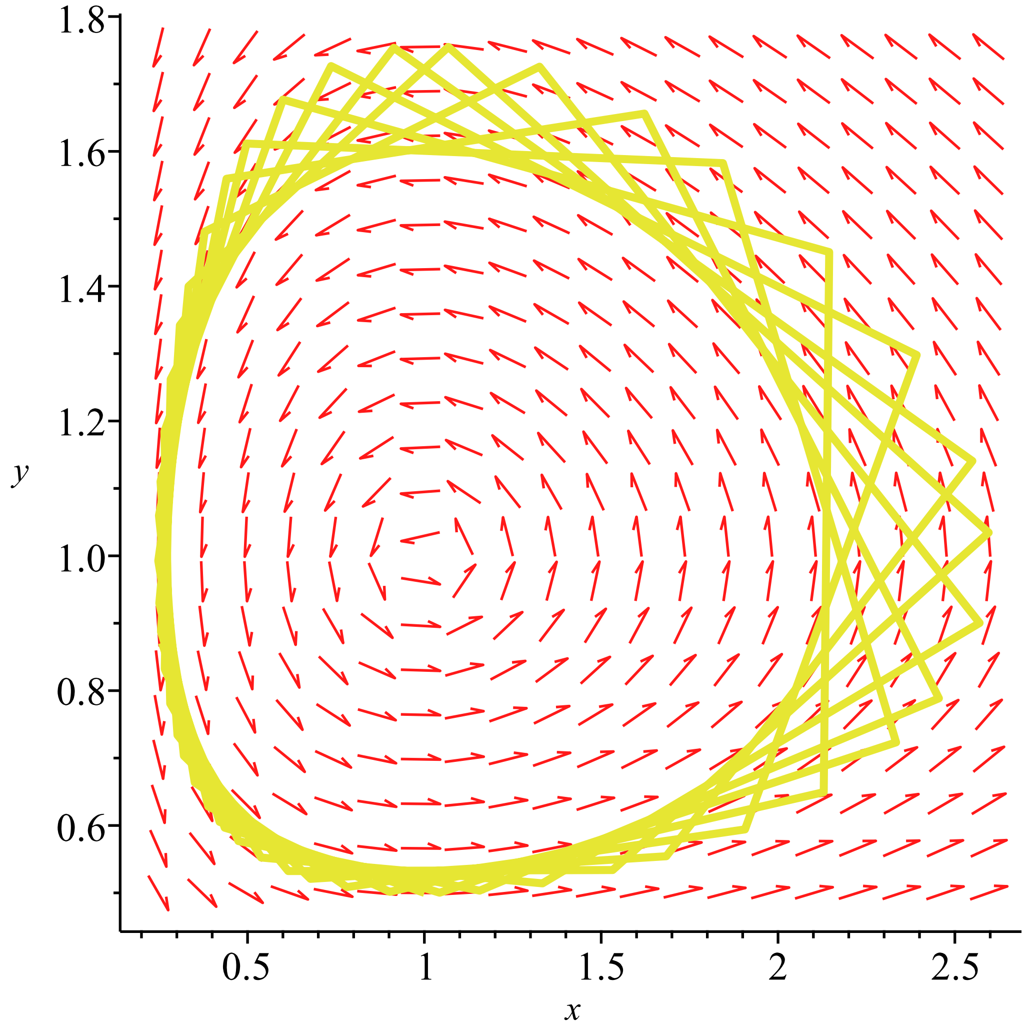


> $DEplot([eq1, eq2], [x(t), y(t)], t=0..11, [[x(0)=1, y(0)=0.5]])$

 # It seems that at the beggining when x=1 and y=0.5 boths populations seem low, then they get larger and larger until there's about 100 foxes and 250 rabbits. Afterwards the population of rabbits goes down because there are too many predators(foxes), so rabbits go down while foxes go up until there's about 180 foxes and 100 rabbits, then both foxes and rabbits go down until there's 100 foxes and 15 rabbits or so. At this point the foxes are going down rapidly but rabbits start multiplying again and it seems to cycle (the orbit is almost closed). But below i will run the simulation for much longer for an interesting result



> $DEplot([eq1, eq2], [x(t), y(t)], t = 0..500, [[x(0) = 1, y(0) = 0.5]])$



> *#The orbit does seem to cycle or at least go through very similar values but the pattern is non −obvious, there are many steep curves in the upper right part showing awesome oscillations.*

> *#In my opinion this does at least somewhat resemble the population progression of a system f oxes and rabbits because they go from one side being more populous than the other and it's a nice back and forth, when there's too few #predators prey thrives and when there's lots of prey predators start to thrive until prey goes down again, it really is close to what i expect to happen in the wild.*