# Sums

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**Abstract**

There are 3,628,800 ways to sum up the first nine whole numbers, but as summation is commutative, the means of such computation is not often given attention. If one sums the first nine whole numbers in ascending order, one will find that the last digit of the partial sum occasionally repeats itself. These last digits for ascending order would be $\{0, 1, 3, 6, 0, 5, 1, 8, 6, 5\}$. In fact, 99.99% of all orders of summing the first nine whole numbers fail the same way. Which orders, if any, have unique last digits at every partial sum?

## 1 Existence Theorem

**Definition 1.** Suppose $N \in \mathbb{N}$. Let $[N] := \{0, 1, \dots, N-1\}$.

**Definition 2.** Suppose $(a_n)$ is a finite sequence with domain $M$. We call $(a_n)$ a **permutation sequence** of a finite set $S$ if $a_n : M \to S$ is bijective.

**Definition 3.** Suppose $N \in \mathbb{N}$ and let $(a_n)$ be a finite sequence with domain $M$. We define the **remainder sequence** of $(a_n)$ for $N$ as $\varphi : M \to [N]$ with

$$\varphi_m := \left( \sum_{i=n}^{m} a_n \right) (\mathrm{mod} N).$$

**Definition 4.** We call a sequence $(a_n)$ **perfectly summed modulo** $N$ if the remainder sequence of $(a_n)$ for $N$ is injective over its domain. That is, each value of the sequence is unique over its domain. Thus, a permutation sequence of $[N]$ is perfectly summed modulo $N$ if each possible remainder occurs exactly once in its remainder sequence.

**Theorem 1.1.** *Let $N \in \mathbb{N}^+$ and $(a_n)$ be a permutation sequence of $[N]$. If $(a_n)$ is perfectly summed modulo $N$, then $a_0 = 0$.*

*Proof.* Suppose not, implying $a_k = 0$ for some $k > 0$. Then $\sum_{i=0}^{k-1} a_i = \sum_{i=0}^{k} a_i$ and hence $\varphi(k-1) = \varphi(k)$, implying $a_n$ is not perfectly summed modulo $N$. Thus we reach contradiction and $a_0 = 0$. $\square$

**Theorem 1.2** (Existence Theorem)**.** *Let $N \in \mathbb{N}^+$. There exists a permutation sequence of $[N]$ perfectly summed modulo $N$ if and only if $N$ is even. Namely,*

$$a_n = \begin{cases} n & n \text{ is even} \\ N - n & n \text{ is odd} \end{cases}.$$

*Proof.* In the case $N$ is not even, $N$ is odd. Then $\frac{(N-1)}{2} = k$ for some $k \in \mathbb{N}$ so

$$\sum_{i=0}^{N-1} a_n = N \cdot \frac{(N-1)}{2} = N \cdot k$$

Hence, $\left(\sum_{i=0}^{N-1} a_n\right) \bmod N = 0$. By Theorem 1.1, $a_0 = 0$. Thus, the remainder sequence is not injective and in turn $(a_n)$ is not perfectly summed modulo $N$.

Otherwise, we assume $N$ is even. First we will first show $a_n$ is a permutation sequence. Suppose $n_0, n_1 \in [N]$ are distinct. In the case they are both even, trivially $a_{n_0} \neq a_{n_1}$. In the case they are both odd, observe that $N - n_0 \neq N - n_1$, implying $a_{n_0} \neq a_{n_1}$. Otherwise, without loss of generality $n_0$ is even and $n_1$ odd. Then since $N$ is even, $N - n$ is odd, implying $N - n_1 \neq n_0$ and in turn $a_{n_0} \neq a_{n_1}$. Hence, $a_n$ is injective on $[N]$. Now suppose $k \in [n]$. If $k$ is even, $a_k = k$. Otherwise, $a_{N-k} = N - (N - k) = k$. Hence, $a_n$ is surjective onto $[N]$. Thus, $a_n$ is in fact a permutation sequence of $[N]$.

We now define

$$S_n := \frac{(N + (-1)^n)(2n + 1 - (-1)^n)}{4}.$$

We will show by induction that $S_n = \sum_{i=0}^{n} a_n$. In the base case of $n = 0$, $S_n = \frac{(N+1)(2(0)+1-1)}{4} = 0$, and $a_0 = 0$ by Theorem 1.1, so the base case holds. To prove the inductive step, we assume $\sum_{i=1}^{n} a_i = S_n$ for some $n \in \mathbb{N}$. Note that, since $\frac{1-(-1)^n}{2}$ is 1 for odd $n$ and 0 for even, $a_n = \frac{1-(-1)^n}{2} N + (-1)^n n$. It suffices to show that $S_{n+1} - S_n = a_{n+1}$. Observe that

$$\begin{aligned}
S_{n+1} - S_n &= \frac{(N + (-1)^{n+1})(2(n+1) + 1 - (-1)^{n+1})}{4} - \frac{(N + (-1)^n)(2n + 1 - (-1)^n)}{4} \\
&= \frac{(N - (-1)^n)(2n + 3 + (-1)^n)}{4} - \frac{(N + (-1)^n)(2n + 1 - (-1)^n)}{4}.
\end{aligned}$$

Now call $\sigma := (-1)^n$. Then, equivalently,

$$\begin{aligned}
S_{n+1} - S_n &= \frac{(N - \sigma)(2n + 3 + \sigma)}{4} - \frac{(N + \sigma)(2n + 1 - \sigma)}{4} \\
&= \frac{(N + \sigma)(2n + 3 + \sigma)}{4} - \frac{(N + \sigma)(2n + 1 - \sigma)}{4} - \left(\frac{(2\sigma)(2n + 3 + \sigma)}{4}\right) \\
&= \frac{(N + \sigma)(2n + 1 - \sigma)}{4} - \frac{(N + \sigma)(2n + 1 - \sigma)}{4} - \left(\frac{(2\sigma)(2n + 3 + \sigma)}{4}\right) + \left(\frac{(N + \sigma)(2 + 2\sigma)}{4}\right) \\
&= 0 + \left(\frac{-\sigma(2n + 3 + \sigma) + (N + \sigma)(1 + \sigma)}{2}\right) \\
&= \left(\frac{-2n\sigma - 3\sigma - 1 + N + \sigma + \sigma N + 1}{2}\right) \\
&= \frac{1 + \sigma}{2} N - \sigma(n + 1) \\
&= \frac{1 - (-1)^{n+1}}{2} N + (-1)^{n+1}(n + 1) \\
&= a_{n+1}.
\end{aligned}$$

Thus by induction, $S_n = \sum_{i=1}^{n} a_n$ for all $n \in \mathbb{N}$.

Now suppose for some $n_0, n_1 \in [N]$. There are three cases.
In the case they are both even,

$$\begin{aligned}
(S_{n_1} - S_{n_0}) \bmod N &= \left(\frac{(N+1)(2n_1)}{4} - \frac{(N+1)(2n_0)}{4}\right) \bmod N \\
&= \left(\left(N\frac{n_1}{2}\right) + \left(\frac{n_1}{2}\right) - \left(N\frac{n_0}{2}\right) - \left(\frac{n_0}{2}\right)\right) \bmod N \\
&= \left(\frac{n_1}{2} - \frac{n_0}{2}\right) \bmod N.
\end{aligned}$$

Since $0 \le n_0, n_1 < N$, if $S_{n_1} \bmod N = S_{n_0} \bmod N$, such implies $n_0 = n_1$.
In the case they are both odd,

$$\begin{aligned}
(S_{n_1} - S_{n_0}) \bmod N &= \left(\frac{(N-1)(2n_1+2)}{4} - \frac{(N-1)(2n_0+2)}{4}\right) \bmod N \\
&= \left(\left(N\frac{(n_1+1)}{2}\right) - \left(\frac{(n_1+1)}{2}\right) - \left(N\frac{n_0+1}{2}\right) + \left(\frac{(n_0+1)}{2}\right)\right) \bmod N \\
&= \left(-\frac{(n_1)}{2} + \frac{(n_0)}{2}\right) \bmod N.
\end{aligned}$$

Since $0 \le n_0, n_1 < N$, if $S_{n_1} \bmod N = S_{n_0} \bmod N$, such implies $n_0 = n_1$.
Otherwise, without loss of generality $n_0$ is even and $n_1$ is odd, so

$$\begin{aligned}
(S_{n_1} - S_{n_0}) \bmod N &= \left(\frac{(N-1)(2n_1+2)}{4} - \frac{(N+1)(2n_0)}{4}\right) \bmod N \\
&= \left(\left(N\frac{(n_1+1)}{2}\right) - \left(\frac{(n_1+1)}{2}\right) - \left(N\frac{n_0}{2}\right) - \left(\frac{n_0}{2}\right)\right) \bmod N \\
&= \left(\frac{(n_1+1)}{2} + \frac{n_0}{2}\right) \bmod N.
\end{aligned}$$

Suppose for the sake of contradiction $S_{n_1} \bmod N = S_{n_0} \bmod N$. Then $n_1 + 1 + n_0 = 2kN$ for some $k \in \mathbb{N}$, and since $0 \le n_0, n_1 < N$, $k > 0$ and $k < 1$. Hence, we reach contradiction and there $S_{n_1} \bmod N \ne S_{n_0} \bmod N$.

Hence, the remainder sequence of $(a_n)$ is injective and in turn $(a_n)$ is perfectly summed modulo $N$. ◻

**Corollary 1.3.** *The remainder sequence of this solution $(a_n)$ is*

$$\varphi(n) = \frac{(N + (-1)^n)(2n + 1 - (-1)^n)}{4} \bmod N.$$

*The inverse $\varphi^{-1} : [N] \to [N]$ also exists. Namely,*

$$\varphi^{-1}(v) = \begin{cases} 2(N-v) - 1 & v \ge \frac{N}{2} \\ 2v & v < \frac{N}{2} \end{cases},$$

*or equivalently,*

$$\varphi^{-1}(v) = \frac{|4v - (2N-1)| + (2N-1)}{2}.$$

*Proof.* We define the function $n^* : [N] \to [N]$ as

$$n^*(v) := \begin{cases} 2(N-v)-1 & v \geq \frac{N}{2} \\ 2v & v < \frac{N}{2} \end{cases}.$$

Suppose $v^* \in [N]$. Then there are two cases. If $v^* \geq \frac{N}{2}$, $n^*(v)$ is odd, so

$$\begin{aligned} S_{n^*(v^*)} \bmod N &= \frac{(N-1)(4(N-v)-2+1-(-1))}{4} \bmod N \\ &= (N-1)(N-v) \bmod N \\ &= N(N-v)+(v-N) \bmod N \\ &= v. \end{aligned}$$

Otherwise, $v^* < \frac{N}{2}$ and $n^*(v)$ is even. Then,

$$\begin{aligned} S_{n^*(v^*)} \bmod N &= \frac{(N+1)(4v+1-(-1))}{4} \bmod N \\ &= (N+1)v \bmod N \\ &= Nv + v \bmod N \\ &= v. \end{aligned}$$

In all cases, $S_{n^*(v)} \bmod N = v$. Hence, $\varphi(x)$ is surjective, and since it is injective, we have found its inverse $n^*$. It is a less important exercise to prove the given inverses are equivalent on $[N]$. $\square$

## 2 Characterization of Solution Set

**Theorem 2.1.** *(Equivalent Definition) A sequence $(a_n)$ is perfeclty summed modulo $N$ if no consecutive subsequence that excludes the first term sums to a multiple of $N$.*

*Proof.* We first assume $(a_n)$ is not perfectly summed modulo $N$. Then its remainder sequence is not injective, implying that there exists $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$ such that

$$\left(\sum_{i=0}^{k_2} a_i\right) \bmod N = \left(\sum_{i=0}^{k_1} a_i\right) \bmod N$$

$$\left(\sum_{i=0}^{k_2} a_i\right) \bmod N = \left(\sum_{i=0}^{k_1} a_i\right) \bmod N + \left(\sum_{i=k_1+1}^{k_2} a_i\right) \bmod N$$

$$0 = \left(\sum_{i=k_1+1}^{k_2} a_i\right) \bmod N.$$

Thus there exists a consecutive subsequence which sums to a multiple of $N$ with first term beyond $a_0$. Hence, the contrapositive is proven.

We now assume $(a_n)$ is perfectly summed modulo $N$. Then suppose for the sake of contradiction there exists a consecutive subsequence $a_k, a_{k+1}, \ldots, a_{k+\delta}$ which sums to a multiple of $N$ such that $k > 0$. Then,

$$\begin{aligned}\left(\sum_{i=0}^{k+\delta} a_i\right) \bmod N &= \left(\sum_{i=0}^{k-1} a_i\right) \bmod N + \left(\sum_{i=k}^{k+\delta} a_i\right) \bmod N \\ &= \left(\sum_{i=0}^{k-1} a_i\right) \bmod N,\end{aligned}$$

implying the remainder sequence is not injective and in turn reaching contradiction. Hence, there are no such subsequences. □

**Corollary 2.2.** *A permutation sequence of $[N]$ is perfectly summed modulo $N$ if and only if no consecutive subsequence sums to a multiple of $N$.*

*Proof.* Indeed, by Theorem 1.1, $a_0 = 0$. Since the first term contributes nothing to a sum, if and only if there is a consecutive subsequence including the first term that sums to a multiple of $N$, then there exists one excluding it. □

**Theorem 2.3.** *(Equivalent Definition) A sequence $(a_n)$ is perfectly summed modulo $N$ if and only if every consecutive subsequence is perfectly summed modulo $N$.*

*Proof.* In the case every consecutive subsequence of $(a_n)$ is perfectly summed modulo $N$, the subsequence $(a_n)$ itself must be. Thus the first direction is trivial.

We will now assume a sequence $(a_n)$ is perfectly summed modulo $N$. Then suppose for the sake of contradiction there exists a consecutive subsequence $a_k, a_{k+1}, \ldots, a_{k+\delta}$ which is not perfectly summed modulo $N$. By Theorem 2.1, there exists a consecutive subsequence $a_j, a_{j+1}, \ldots, a_{j+\gamma}$ of this subsequence which sums to a multiple of $N$ and such that $j > k$. Since $j > k$, then $j > 0$, so by Theorem 2.1 $(a_n)$ must not be perfectly summed modulo $N$, which reaches contradiction. Thus there are no such consecutive subsequences. □

**Theorem 2.4.** *Let $N \in \mathbb{N}^+$. If a sequence is perfectly summed modulo $N$, then, fixing the first term, its reverse is perfectly summed modulo $N$.*

*Proof.* We will show the contrapositive. Thus we assume the reverse sequence excluding the first term is not perfectly summed modulo $N$. Then by Theorem 2.1 there exists a consecutive subequence of the reverse sequence excluding the first term which sums to a multiple of $N$. By the commutative property of addition, the reverse of this subsequence, which is a consecutive subsequence of the initial sequence, sums to a multiple of $N$. Since this subsequence of course excludes the firm term, by Theorem 2.1 the initial sequence must not be perfectly summed modulo $N$. □

**Conjecture 2.1.** *Let $N \in \mathbb{N}^+$. The sequence $a_0 = N, a_1 = N, \ldots, a_{N-1} = N$ is perfectly summed modulo $M$ for any $M > N$.*

**Question 2.1.** *How many permutation sequences perfectly summed modulo $N$ are there for a given $N$?*

**Theorem 2.5.** *The ascending permutation sequence $a_i = i$ is perfectly summed modulo $N$ if and only if $N = 2^k$ for some $k \in \mathbb{N}$ with $k > 2$. Repeats $N/2$.*

*Proof.* We first assume $N = 2^k$ for some $k \in \mathbb{N}$ with $k > 2$. Then suppose for some $x, y \in [N]$ that

$$\begin{aligned}
\left(\sum_{i=0}^{x} i\right)(\bmod N) &= \left(\sum_{i=0}^{y} i\right)(\bmod N) \\
\frac{x^2+x}{2}(\bmod N) &= \frac{y^2+y}{2}(\bmod N) \\
\left(\frac{x^2+x}{2} - \frac{y^2+y}{2}\right)(\bmod N) &= 0 \\
\frac{(x-y)(x+y+1)}{2}(\bmod N) &= 0 \\
(x-y)(x+y+1) &= n \cdot 2^{k+1}
\end{aligned}$$

for some $n \in \mathbb{Z}$. Observe that either $(x-y)$ or $(x+y+1)$ must be odd. In the case $(x-y)$ is odd, $x-y = n_0$ and $x+y+1 = n_1 2^{k+1}$ for some $n_0 n_1 = n$. But $x+y+1 < 2^{k+1}$, so $n = 0$ and in turn $x = y$. In the case $(x+y+1)$ is odd, $x - y = n' 2^{k+1}$ for some $n' \in \mathbb{Z}$. But $-2^k \geq x - y \leq 2^k$ so $n' = 0$ and in turn $x = y$. In either case, $x = y$ and $a_i = i$ is perfectly summed modulo $2^k$. □

## 3 Applications

The primary motivation of this investigation was actually a very practical one, and I will describe its utility for me now. Consider the composer interested in exercising his ability to modulate between keys. He thus devises an exercise for himself to write a piece proceeding through the ascending modulation intervals $1 - 11$ one after another. Beginning at $C$ (pitch 0), he modulates to $C\#$ (pitch $0+1$), and then to $D\#$ (pitch $0+1+2$) and so on. But he will quickly realize that an ascending order of summation will actually repeat a few keys, and his piece will sound a lot richer if he instead makes use of all 12 keys. That is, the ascending order permutation sequence is not perfectly summed modulo 12. Thus, he may make use of these theorems to quickly construct a modulation pattern that does not repeat keys.

The principle purpose of this generalization is that brute-forcing any solution is probably about $O(n \log(n))$. The Existence Theorem provides one in $O(n)$ time. Corollary 2.2 certainly cuts back on computation time for the number of possible permutation seqeunces of $[N]$ perfectly summed modulo $N$.