Deep Learning

Perceptron

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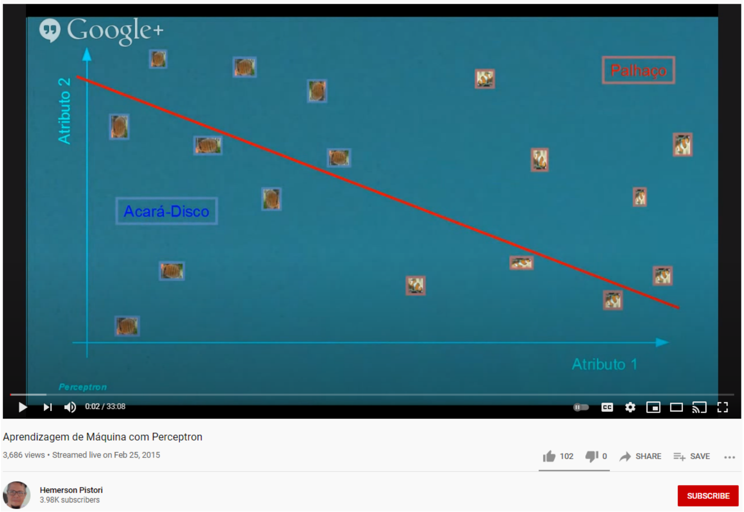
Motivation – ML w/ Perceptron



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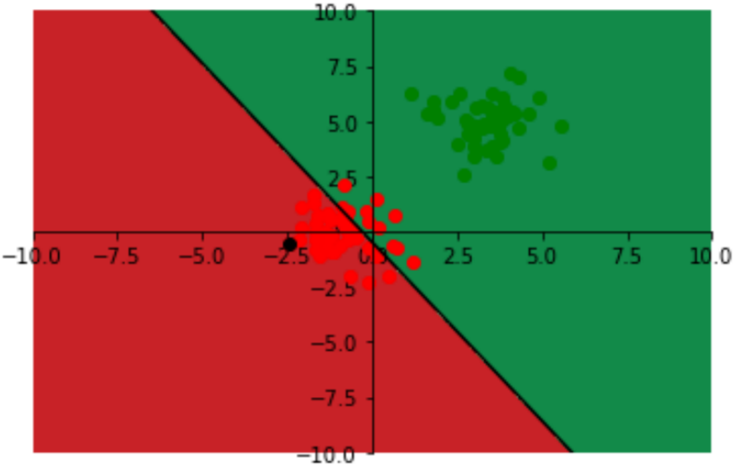
[1] https://www.youtube.com/watch?v=-C07ansuc-8

Motivation



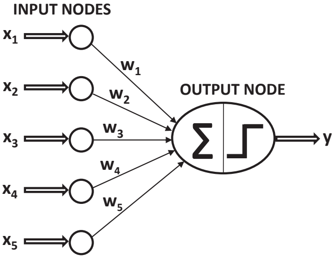
[2]

[2] Simple perceptron example using NumPy

Binary Classification and Linear Regression Problems

- In the binary classification problem, each training pair $(\overline{X}, y)$ contains feature variables $\overline{X} = (x_1, \ldots x_d)$, and label y drawn from $\{-1, +1\}$.
  - Example: Feature variables might be frequencies of words in an email, and the class variable might be an indicator of spam.
  - Given labeled emails, recognize incoming spam.
- In linear regression, the *dependent* variable y is real-valued.
  - Feature variables are frequencies of words in a Web page, and the dependent variable is a prediction of the number of accesses in a fixed period.
- Perceptron is designed for the binary setting.

The Perceptron: Earliest Historical Architecture



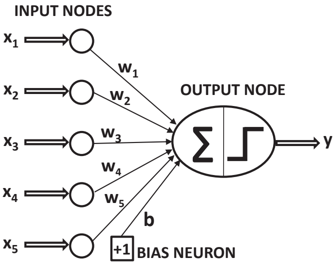
- The d nodes in the input layer only transmit the d features $\overline{X} = [x_1 \ldots x_d]$ without performing any computation.
- Output node multiplies input with weights $\overline{W} = [w_1 \ldots w_d]$ on incoming edges, aggregates them, and applies *sign activation*:

$$\hat{y} = \text{sign}\{\overline{W} \cdot \overline{X}\} = \text{sign}\{\sum_{j=1}^{d} w_j x_j\}$$

What is the Perceptron Doing?

- Tries to find a *linear separator* $\overline{W} \cdot \overline{X} = 0$ between the two classes.
- Ideally, all positive instances $(y = 1)$ should be on the side of the separator satisfying $\overline{W} \cdot \overline{X} > 0$.
- All negative instances $(y = -1)$ should be on the side of the separator satisfying $\overline{W} \cdot \overline{X} < 0$.

Bias Neurons



- In many settings (e.g., skewed class distribution) we need an invariant part of the prediction with bias variable b:

$$\hat{y} = \text{sign}\{\overline{W} \cdot \overline{X} + b\} = \text{sign}\{\sum_{j=1}^{d} w_j x_j + b\} = \text{sign}\{\sum_{j=1}^{d+1} w_j x_j\}$$

- On setting $w_{d+1} = b$ and x_{d+1} as the input from the bias neuron, it makes little difference to learning procedures $\Rightarrow$ Often implicit in architectural diagrams

Training a Perceptron

- Go through the input-output pairs $(\overline{X}, y)$ one by one and make updates, if predicted value $\hat{y}$ is different from observed value $y \Rightarrow$ Biological readjustment of synaptic weights.

$$\overline{W} \Leftarrow \overline{W} + \alpha \underbrace{(y - \hat{y})}_{\text{Error}} \overline{X}$$

$$\overline{W} \Leftarrow \overline{W} + (2\alpha) y \overline{X} \text{ [For misclassified instances } y - \hat{y} = 2y\text{]}$$

- Parameter α is the learning rate $\Rightarrow$ Turns out to be irrelevant in the special case of the perceptron
- One cycle through the entire training data set is referred to as an *epoch* $\Rightarrow$ Multiple epochs required
- How did we derive these updates?

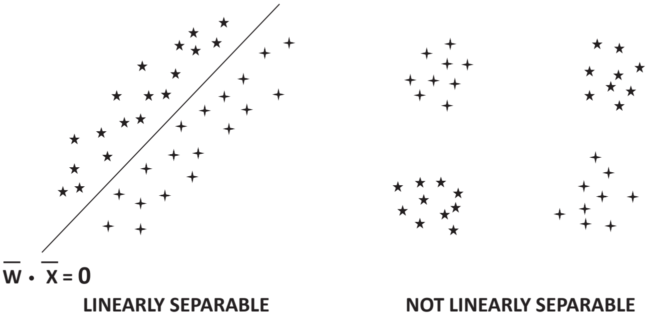
What Objective Function is the Perceptron Optimizing?

- At the time, the perceptron was proposed, the notion of loss function was not popular $\Rightarrow$ Updates were heuristic
- Perceptron optimizes the perceptron criterion for ith training instance:

$$L_i = \max\{-y_i(\overline{W} \cdot \overline{X_i}), 0\}$$

  - Loss function tells us how far we are from a desired solution $\Rightarrow$ Perceptron criterion is 0 when $\overline{W} \cdot \overline{X_i}$ has same sign as y_i.
- Perceptron updates use *stochastic gradient descent* to optimize the loss function and reach the desired outcome.
  - Updates are equivalent to $\overline{W} \Leftarrow \overline{W} - \alpha \left(\frac{\partial L_i}{\partial w_1} \ldots \frac{\partial L_i}{\partial w_d} \right)$

Where does the Perceptron Fail?



- The perceptron fails at similar problems as a linear SVM
  - **Classical solution:** Feature engineering with Radial Basis Function network $\Rightarrow$ Similar to kernel SVM and good for noisy data
  - **Deep learning solution:** Multilayer networks with nonlinear activations $\Rightarrow$ Good for data with a lot of structure

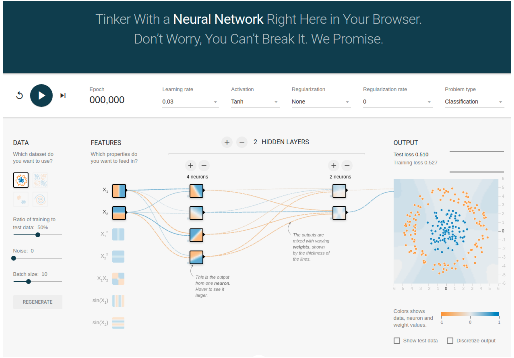
Historical Origins

- The first model of a computational unit was the *perceptron* (1958).
  - Was roughly inspired by the biological model of a neuron.
  - Was implemented using a large piece of hardware.
  - Generated great excitement but failed to live up to inflated expectations.
- Was not any more powerful than a simple linear model that can be implemented in a few lines of code today.

Perceptron Tutorial

Perceptron tutorial (`simple_perceptron.py`).

Tensorflow Playground[3]

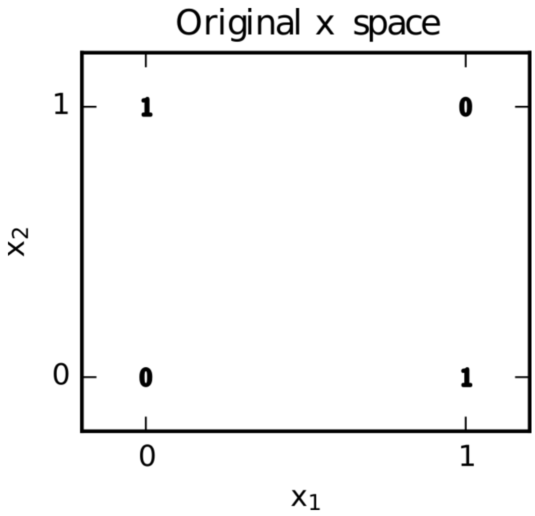


[3] https://playground.tensorflow.org/

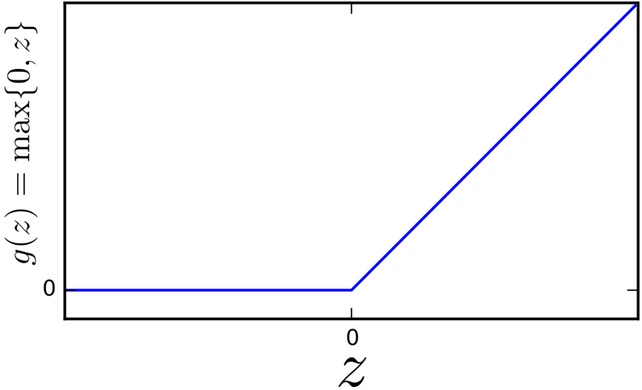
The XOR Problem

- "*Perceptrons*" by Marvin Minsky and Seymour Papert (1969).
- Perceptrons cannot solve the XOR problem.
- Significant decline in interest and funding of neural network research.

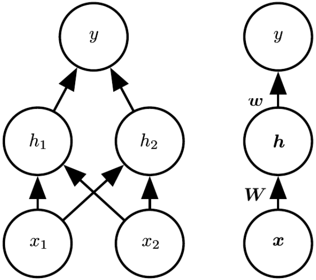
The XOR Problem



Rectified Linear Activation

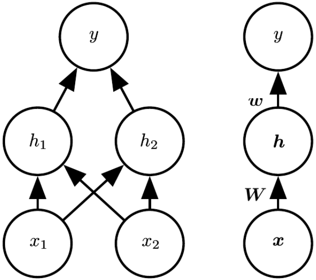


Network Diagrams



$$\mathbf{h} = \max(0, \mathbf{W}^T\mathbf{x} + \mathbf{c})$$
$$f(\mathbf{x}; (\mathbf{W}; \mathbf{c}); (\mathbf{w}, b)) = \mathbf{w}^T\mathbf{h} + b$$

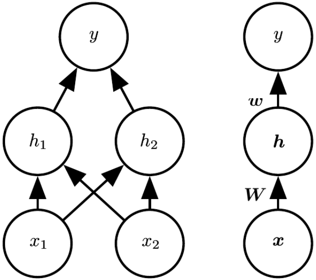
Solving XOR



$$\mathbf{h} = \max(0, \mathbf{W}^T\mathbf{x} + \mathbf{c})$$
$$f(\mathbf{x}; (\mathbf{W}; \mathbf{c}); (\mathbf{w}, b)) = \mathbf{w}^T\mathbf{h} + b$$

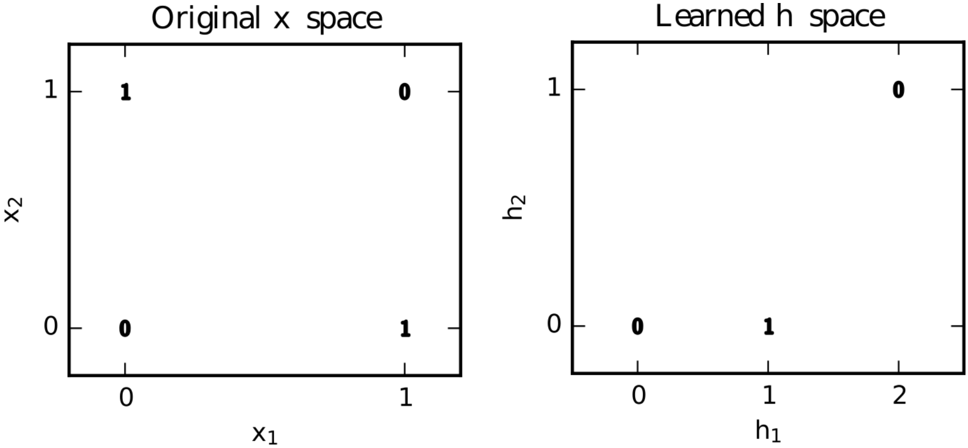
$$X = [\mathbf{x}]_{i=1}^{4} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$\mathbf{c} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$\mathbf{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$b = 0$$

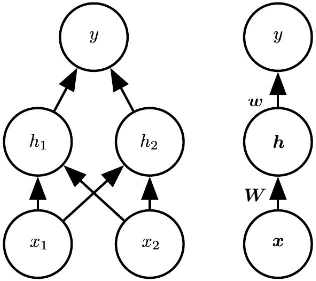
Solving XOR



$$\mathbf{h} = \max(0, \mathbf{W}^T\mathbf{x} + \mathbf{c})$$
$$f(\mathbf{x}; (\mathbf{W}; \mathbf{c}); (\mathbf{w}, b)) = \mathbf{w}^T\mathbf{h} + b$$

$$H = \max\left(0, \mathbf{W}^T X + \mathbf{c}\right)$$
$$H = \max\left(0, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$
$$H = \max\left(0, \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$
$$H = \max\left(0, \begin{bmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 0 & 1 \end{bmatrix}\right)$$
$$H = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$


Original x space
Learned h space
x1
x2
h1
h2
0
1
2



$$Y = \max\left(0, \mathbf{w}^T H + \mathbf{b}\right)$$

$$Y = \max\left(\begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$Y = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{h} = \max(0, \mathbf{W}^T \mathbf{x} + \mathbf{c})$$
$$f(\mathbf{x}; (\mathbf{W}; \mathbf{c}); (\mathbf{w}, b)) = \mathbf{w}^T \mathbf{h} + b$$

Thank you!

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