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Fuzzy data envelopment analysis (DEA): Model and ranking method

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ABSTRACT

Data Envelopment Analysis (DEA) is a very effective method to evaluate the relative efficiency of decision-making units (DMUs). Since the data of production processes cannot be precisely measured in some cases, the uncertain theory has played an important role in DEA. This paper attempts to extend the traditional DEA models to a fuzzy framework, thus producing a fuzzy DEA model based on credibility measure. Following is a method of ranking all the DMUs. In order to solve the fuzzy model, we have designed the hybrid algorithm combined with fuzzy simulation and genetic algorithm. When the inputs and outputs are all trapezoidal or triangular fuzzy variables, the model can be transformed to linear programming. Finally, a numerical example is presented to illustrate the fuzzy DEA model and the method of ranking all the DMUs.

1. Introduction

Data envelopment analysis (DEA), which initially proposed by Charnes et al. [5], is a nonparametric method for evaluating the relative efficiency of decision-making units (DMUs) on the basis of multiple inputs and outputs. Since DEA was proposed in 1978, it has been got comprehensive attention both in theory and application. Now DEA becomes the important analysis tool and research way in management science, operational research, system engineering, decision analysis and so on. Based on the original DEA model [5], various theoretical extensions have been developed [3,4,25,28,8]. More DEA papers can refer to Seiford [26] in which 500 references are documented.

Often decision makers are interested in a complete ranking beyond the dichotomized classification. The researches on ranking have come up. By now many papers on ranking have been published over the last decade within the DEA context. The cross-efficiency ranking method was first developed by Sexton et al. [27]. By evaluating DMUs through both self and peer pressure, one can attain a more balanced view of the decision-making units. Andersen & Petersen [2] developed the super-efficiency approach, in which the efficient units can receive a score greater than one, through the unit's exclusion from the column being scored in the linear program. However, each unit is evaluated by its own weights as opposed to the cross-efficiency concept in which all units are compared using the same sets of weights. In the benchmarking ranking method [29], a DMU is highly ranked if it is chosen as a useful target for many other DMUs. This is of substantial use when looking to benchmark industries. For a review of ranking methods see [1].

Most methods of ranking DMUs assume that all inputs and outputs data are exactly known. But in more general cases, the data for evaluation are often collected from investigation to decide the natural language such as good, medium and bad rather than a specific case. That is, the inputs and outputs are fuzzy. We can find several fuzzy approaches to the assessment of efficiency in the DEA literature. Cooper et al. [6,7] were the first, to the best of our knowledge, to study how to deal with imprecise data such as bounded data, ordinal data and ratio bounded data in DEA. Kao and Liu [11] develop a method to

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find the membership functions of the fuzzy efficiency scores when some observations are fuzzy numbers. Entani et al. [9] propose an interval efficiency obtained from the pessimistic and the optimistic viewpoints. Since Zadeh [32,33] initiated the possibility measure, many researchers have introduced it into DEA [10,12]. Although possibility measure has been widely used, it has no self-duality property. However, a self-dual measure is absolutely needed in both theory and practice. In order to define a self-dual measure, Liu and Liu [20] presented the concept of credibility measure in 2002. An axiomatic foundation of credibility theory was given by Liu [22] in 2004. In this paper, the credibility measure is employed to the fuzzy DEA models.

This paper is organized as follows: some basic concept and results on credibility measure will be introduced in Section 2; In Section 3, we will give some introduction about CCR model; Section 4 will give the fuzzy DEA model and the ranking method; In order to solve the fuzzy DEA model, a hybrid algorithm is designed in Section 5. Section 6 will predigest the fuzzy model when the inputs and outputs are all trapezoidal fuzzy variables. Finally, a numerical example will be given to illustrate the fuzzy DEA model and the method of ranking all the DMUs in Section 7.

2. Credibility measure

In this section, we will state some basic concepts and results on fuzzy variables. These results are crucial for the remainder of this paper. For the up-to-date credibility theory, the interested reader can consult to Liu [22,24].

Let Θ be a nonempty set, and $\mathcal{P}(\Theta)$ the power set of Θ. For any $A \in \mathcal{P}(\Theta)$, Liu and Liu [20] presented a credibility measure $\mathrm{Cr}\{A\}$ to express the chance that fuzzy event A occurs. Li and Liu [13] proved that a set function $\mathrm{Cr}\{\cdot\}$ is a credibility measure if and only if

(i) $\mathrm{Cr}\{\Theta\} = 1$;
(ii) $\mathrm{Cr}\{A\} \le \mathrm{Cr}\{B\}$, whenever $A \subset B$;
(iii) Cr is self-dual, i.e., $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$, for any $A \in \mathcal{P}(\Theta)$;
(iv) $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}(\Theta)$ with $\sup_i \mathrm{Cr}\{A_i\} < 0.5$.

The triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is called a credibility space and fuzzy variable is defined as a function from this space to the set of real numbers [23]. Furthermore, the credibility theory was developed by Liu [22] as a branch of mathematics for studying the behavior of fuzzy phenomena.

Suppose that ξ is a fuzzy variable defined on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. Then its membership function is derived from the credibility measure by

$$\mu(x) = (2\mathrm{Cr}\{\xi = x\}) \wedge 1, \quad x \in \mathcal{R}. \tag{1}$$

Conversely, let ξ be a fuzzy variable with membership function μ. Then for any set B of real numbers, we have

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x\in B}\mu(x) + 1 - \sup_{x\in B^c}\mu(x)\right). \tag{2}$$

The credibility distribution $\Phi : \Re \to [0, 1]$ and the expected value of a fuzzy variable ξ are defined by

$$\Phi(x) = \mathrm{Cr}\{\theta \in \Theta | \xi(\theta) \le x\}$$

and

$$\mathrm{E}[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi \le r\}\mathrm{d}r.$$

3. DEA model

The most frequently used DEA model is CCR model, which is proposed by Charnes et al. [5]. Firstly let us review some symbols and variables:

DMU_i: the ith DMU, $i = 1, 2, \ldots, n$;
DMU_0: the target DMU;
$\boldsymbol{x}_i \in R^{m\times 1}$: the inputs vector of DMU_i, $i = 1, 2, \ldots, n$;
$\boldsymbol{x}_0 \in R^{m\times 1}$: the inputs vector of the target DMU_0;
$\boldsymbol{y}_i \in R^{m\times 1}$: the outputs vector of DMU_i, $i = 1, 2, \ldots, n$;
$\boldsymbol{y}_0 \in R^{m\times 1}$: the outputs vector of the target DMU_0;
$\boldsymbol{u} \in R^{m\times 1}$: the vector of input weights;
$\boldsymbol{v} \in R^{r\times 1}$: the vector of output weights.

In this model, the efficiency of entity evaluated is obtained as a ratio of the weighted output to the weighted input subject to the condition that the ratio for every entity is not larger than 1. Mathematically, it is described as follows:

$$\begin{array}{l} \max\limits_{\boldsymbol{u},\boldsymbol{v}} \ \theta = \dfrac{\boldsymbol{v}^T\boldsymbol{y}_0}{\boldsymbol{u}^T\boldsymbol{x}_0} \\ \text{subject to:} \\ \qquad \boldsymbol{v}^T\boldsymbol{y}_j \le \boldsymbol{u}^T\boldsymbol{x}_j, \quad j = 1, 2, \ldots, n \\ \qquad \boldsymbol{u} \ge 0 \\ \qquad \boldsymbol{v} \ge 0. \end{array} \tag{3}$$

Definition 1 (*Efficiency*). DMU_0 is efficient if $\theta^* = 1$, where θ^* is the optimal value of (3).

4. Fuzzy DEA model

In recent years, fuzzy set theory has been proposed as a way to quantify imprecise and vague data in DEA models. Different to the traditional DEA model introduced in Section 2, this paper considers that the inputs $\tilde{\boldsymbol{x}}_j$ and the outputs $\tilde{\boldsymbol{y}}_j$ of the DMU_j are fuzzy variables, $j = 1, 2, \ldots, n$.

Since the fuzzy constraints $\boldsymbol{v}^T\tilde{\boldsymbol{y}}_j \le \boldsymbol{u}^T\tilde{\boldsymbol{x}}_j$ do not define a deterministic feasible set, a natural idea is to provide a confidence level $1-\alpha$ at which it is desired that the fuzzy constraints hold. In other words, the constraints will be violated at most α. Thus we have some chance constraints as follows:

$$\mathrm{Cr}\{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_j \le \boldsymbol{u}^T\tilde{\boldsymbol{x}}_j\} \ge 1-\alpha, \quad j = 1, 2, \ldots, n. \tag{4}$$

The purpose of the model in Section 3 is to maximize the ratio of $\frac{\boldsymbol{v}^T\boldsymbol{y}}{\boldsymbol{u}^T\boldsymbol{x}}$. The larger the value is, the more efficient the DMU is. When it gets to 1, the DMU is efficient. However, it is meaningless in fuzzy environment. From Definition 1, we know that the event $\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0} \ge 1$ is efficient. Thus we want to maximize the credibility of the event $\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0} \ge 1$. Considering the chance constraints (4), the fuzzy DEA model can be written as follows:

$$\begin{cases} \max\limits_{\boldsymbol{u},\boldsymbol{v}} \ \theta = \mathrm{Cr}\left\{\dfrac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0} \ge 1\right\} \\ \text{subject to:} \\ \qquad \mathrm{Cr}\{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_j \le \boldsymbol{u}^T\tilde{\boldsymbol{x}}_j\} \ge 1-\alpha, \quad j = 1, 2, \ldots, n \\ \qquad \boldsymbol{u} \ge 0 \\ \qquad \boldsymbol{v} \ge 0 \end{cases} \tag{5}$$

in which $\alpha \in (0, 0.5]$.

The greater the optimal objective is, the more efficient DMU_0 is ranked. If there exists at least one DMU with the optimal value $\theta^* \ge \alpha$, we can give the following definition:

Definition 2 (α*-Efficiency*). DMU_0 is α-efficient if $\theta^* \ge \alpha$, where θ^* is the optimal value of (5).

5. Hybrid intelligent algorithm

As the coefficient is fuzzy, the fuzzy DEA model has become an uncertain programming model which is difficult to solve by traditional methods due to its complexity. A good way is to design some hybrid intelligent algorithms for solving them [14,17,18,21,15,16,19]. In this paper, we integrate the fuzzy simulations and genetic algorithms to produce a hybrid intelligent algorithm for solving fuzzy DEA model.

5.1. Estimating uncertain functions

Uncertain functions are defined as functions with uncertain parameters. Due to the complexity, we design some fuzzy simulations to estimate the uncertain functions. We write $f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\xi}) = \frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0}$, in which $\boldsymbol{\xi}$ is the fuzzy vector. The uncertain function is:

$$U : (\boldsymbol{u}, \boldsymbol{v}) \to \mathrm{Cr}\left\{f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\xi}) \ge 1\right\}. \tag{6}$$

We randomly generate θ_k from Θ, write $\nu(k) = (2\mathrm{Cr}\{\theta_k\}) \wedge 1$ and produce $\boldsymbol{\xi}(\theta_k)$, $k = 1, 2, \ldots, N$, respectively. Equivalently, we randomly generate $\boldsymbol{\xi}(\theta_k)$ and write $\nu_k = \mu(\boldsymbol{\xi}(\theta_k))$ for $k = 1, 2, \ldots, N$, where μ is the membership function of $\boldsymbol{\xi}$. Then U can be estimated by the formula,

$$\frac{1}{2}\left(\max_{1\le k\le N}\{\nu_k | f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\xi}(\theta_k)) \ge 1\} + \min_{1\le k\le N}\{1-\nu_k | f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\xi}(\theta_k)) < 1\}\right). \tag{7}$$

We summarize this process as follows:

Step 1. Randomly generate θ_k from Θ, write $\nu(k) = (2\mathrm{Cr}\{\theta_k\}) \wedge 1$ and produce $\boldsymbol{\xi}(\theta_k)$, $k = 1, 2, \dots, N$, respectively. Equivalently, we randomly generate $\boldsymbol{\xi}(\theta_k)$ and write $\nu_k = \mu(\boldsymbol{\xi}(\theta_k))$ for $k = 1, 2, \dots, N$, where μ is the membership function of $\boldsymbol{\xi}$.

Step 2. Return U via the estimation formula.

5.2. Genetic algorithm

Heuristic methods have been shown to be the best way to tackle complex problems. Modern heuristics such as simulated annealing, tabu search, genetic algorithm, variable neighborhood search, and ant systems increase the chance of avoiding local optimality. Since genetic algorithm has solved many uncertain programming models successfully [29–31], here we use it to compute the fuzzy DEA model.

Chromosome representation: We use a nonnegative vector $\boldsymbol{x} = (v_1, v_2, \dots, v_q, u_1, u_2, \dots, u_p)$ to express a decision, in which v_i is the ith coefficient of output and u_j is the jth coefficient of input, $i = 1, 2, \dots, q, j = 1, 2, \dots, p$.

Initialization process: We randomly generate $\boldsymbol{x} = (v_1, v_2, \dots, v_q, u_1, u_2, \dots, u_p)$, in which v_i and u_j are nonnegative numbers, $i = 1, 2, \dots, q, j = 1, 2, \dots, p$. The feasibility of $\boldsymbol{x}$ can be verifies by the fuzzy simulations. If it is feasible, then it will be accepted as a chromosome. If not, then we regenerate a point randomly until a feasible one is obtained. We can make *pop_size* initial feasible chromosomes $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{pop\_size}$ by repeating the above process *pop_size* times.

Evaluation function: Evaluation function, denoted by $Eval(\boldsymbol{x})$, is to assign a probability of reproduction to each chromosome $\boldsymbol{x}$ so that its likelihood of being selected is proportional to its fitness relative to the other chromosomes in the population. That is, the chromosomes with higher fitness will have more chance to produce offspring by using roulette wheel selection.

Firstly we compute the objective value by fuzzy simulation for chromosomes $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{pop\_size}$. According to these values, we can give an order relationship among them such that the *pop_size* chromosomes can be rearranged from good to bad. Rewrite them as $\boldsymbol{x}'_1, \boldsymbol{x}'_2, \dots, \boldsymbol{x}'_{pop\_size}$.

Now let a parameter $a \in (0, 1)$ in the genetic system be given. We can define the rank-based evaluation function as follows,

$$Eval(\boldsymbol{x}'_i) = a(1-a)^{i-1}, \quad i = 1, 2, \dots, pop\_size.$$

Selection process: The selection process is based on spinning the roulette wheel pop size times. Each time we select a single chromosome for a new population. The roulette wheel is a fitness-proportional selection. The process is always stated as follows:

Step 1 Calculate the cumulative probability q_i for each chromosome $\boldsymbol{x}_i$,

$$\begin{cases} q_0 = 0, \\ q_i = \displaystyle\sum_{j=1}^{i} eval(\boldsymbol{x}_j), \quad i = 1, 2, \dots, pop\_size. \end{cases}$$

Step 2 Generate a random number s in $(0, q_{pop\_size}]$.
Step 3 Select the chromosome $\boldsymbol{x}_i$ such that $q_{i-1} < s \le q_i$.
Step 4 Repeat the second and third steps pop size times and obtain *pop_size* copies of chromosome.

Cross-over operation: We define a parameter P_c as the probability of cross-over. Generating a random number r from the interval [0, 1], the chromosome x_i is selected if $r < P_c$. We denote the selected parents by $\boldsymbol{x}'_1, \boldsymbol{x}'_2, \dots$. The children of $\boldsymbol{x}'_1$ and $\boldsymbol{x}'_2$ are

$$\boldsymbol{x}''_1 = (x_1^{(1)} \times u + x_1^{(2)} \times (1-u), x_2^{(1)} \times u + x_2^{(2)} \times (1-u), \dots, x_n^{(1)} \times u + x_n^{(2)} \times (1-u))$$

and

$$\boldsymbol{x}''_2 = (x_1^{(2)} \times u + x_1^{(1)} \times (1-u), x_2^{(2)} \times u + x_2^{(1)} \times (1-u), \dots, x_n^{(2)} \times u + x_n^{(1)} \times (1-u)),$$

in which $c \in (0, 1)$. The feasibility of x''_1 and x''_2 can be obtained by fuzzy simulation.

Mutation operation: We define a parameter P_m as the probability of mutation. Generating a random number r from the interval [0, 1], the chromosome x_i is selected if $r < P_m$. Let W be an appropriate large positive number. The child of x_1 is

$$\boldsymbol{x}' = (x_1 + d_1 \times w, x_2 + d_2 \times w, \dots, x_n + d_n \times w)$$

in which $d_i \in [-1, 1], w \in [0, W]$. If $\boldsymbol{x}'$ is not feasible, then we set W as a random number between 0 and W until it is feasible.

5.3. Hybrid intelligent algorithm

In order to solve the fuzzy DEA model, we integrate the fuzzy simulations and genetic algorithm to produce a hybrid intelligent algorithm. We describe the algorithm as the following procedure:

Step 1. Initialize *pop_size* chromosomes $V_k = (\boldsymbol{u}^k, \boldsymbol{v}^k)$, $k = 1, 2, \dots, pop\_size$.

Step 2. Calculate the objective values U^k for all chromosomes V_k, $k = 1, 2, \dots, pop\_size$ by fuzzy simulations respectively.

Table 1
DMUs with two fuzzy inputs and two fuzzy outputs

DMU_i	1	2	3	4	5
Input 1	(3.5, 4.0, 4.5)	(2.9, 2.9, 2.9)	(4.4, 4.9, 5.4)	(3.4, 4.1, 4.8)	(5.9, 6.5, 7.1)
Input 2	(1.9, 2.1, 2.3)	(1.4, 1.5, 1.6)	(2.2, 2.6, 3.0)	(2.1, 2.3, 2.5)	(3.6, 4.1, 4.6)
Output 1	(2.4, 2.6, 2.8)	(2.2, 2.2, 2.2)	(2.7, 3.2, 3.7)	(2.5, 2.9, 3.3)	(4.4, 5.1, 5.8)
Output 2	(3.8, 4.1, 4.4)	(3.3, 3.5, 3.7)	(4.3, 5.1, 5.9)	(5.5, 5.7, 5.9)	(6.5, 7.4, 8.3)

Table 2
Results of ranking DMUs with $\alpha = 0.4$

DMUs	$(v_1^*, v_2^*, u_1^*, u_2^*)$	θ^*
DMU_1	(0.7990, 0.0994, 0.1668, 1.1348)	0.08
DMU_2	(1.6689, 0.5428, 0.0000, 3.9143)	0.40
DMU_3	(0.4602, 0.0000, 0.0370, 0.6285)	0.20
DMU_4	(0.5104, 0.1769, 0.0523, 1.1192)	0.40
DMU_5	(0.3321, 0.0527, 0.3668, 0.0000)	0.40

Step 3. Compute the fitness of all chromosomes V_k, $k = 1, 2, \ldots, pop\_size$.
Step 4. Select the chromosomes for a new population.
Step 5. Renew the chromosomes V_k, $k = 1, 2, \ldots, pop\_size$ by cross-over and mutation operations.
Step 6. Repeat the second to the fifth steps for a given number of cycles.
Step 7. Return the best value.

6. A special case

When the inputs and outputs are trapezoidal fuzzy variables denoted by (r_1, r_2, r_3, r_4). Following Liu's book [21], we can easily obtain the next theorem:

Theorem 1. *Let ξ_i be trapezoidal fuzzy variables with $\xi_i = (r_1^i, r_2^i, r_3^i, r_4^i)$, $i = 1, 2, \ldots, n$, then*

(a) $\mathrm{Cr}\{\sum_{i=1}^n k_i \cdot \xi_i \le b\} \ge \alpha$ *if and only if* $(2\alpha - 1)\sum_{i=1}^n k_i r_4^i + 2(1-\alpha)\sum_{i=1}^n k_2 r_3^i \le b$
(b) $\mathrm{Cr}\{\sum_{i=1}^n k_i \cdot \xi_i \ge b\} \ge \alpha$ *if and only if* $(2\alpha - 1)\sum_{i=1}^n k_i r_1^i + 2(1-\alpha)\sum_{i=1}^n k_2 r_2^i \ge b$

in which k_i are positive numbers and $0.5 \le \alpha \le 1$.

For simplicity, we give this symbol $\boldsymbol{x}^{r_i} = (x_1^{r_i}, x_2^{r_i}, \ldots, x_n^{r_i})$. When all the inputs and outputs are trapezoidal fuzzy variables, the fuzzy DEA model (5) becomes the following fractional programming:

$$\begin{cases} \max\limits_{\boldsymbol{u},\boldsymbol{v}} \ \dfrac{\boldsymbol{v}^T\boldsymbol{y}_0^{r_4} - \boldsymbol{u}^T\boldsymbol{x}_0^{r_1}}{2(\boldsymbol{u}^T\boldsymbol{x}_0^{r_2} - \boldsymbol{v}^T\boldsymbol{y}_0^{r_3} - \boldsymbol{u}^T\boldsymbol{x}_0^{r_1} + \boldsymbol{v}^T\boldsymbol{y}_0^{r_4})} \\ \text{subject to :} \\ \quad (1-2\alpha)(\boldsymbol{v}^T\boldsymbol{y}_j^{r_4} - \boldsymbol{u}^T\boldsymbol{x}_j^{r_1}) + 2\alpha(\boldsymbol{v}^T\boldsymbol{y}_j^{r_3} - \boldsymbol{u}^T\boldsymbol{x}_j^{r_2}) \le 0, \quad j = 1, 2, \ldots, n \\ \quad \boldsymbol{u} \ge 0 \\ \quad \boldsymbol{v} \ge 0 \end{cases} \tag{8}$$

which is equivalent to the following linear programming model:

$$\begin{cases} \max\limits_{\boldsymbol{u},\boldsymbol{v}} \quad \boldsymbol{v}^T\boldsymbol{y}_0^{r_4} - \boldsymbol{u}^T\boldsymbol{x}_0^{r_1} \\ \text{subject to :} \\ \quad 2\boldsymbol{v}^T(\boldsymbol{y}_0^{r_4} - \boldsymbol{y}_0^{r_3}) + 2\boldsymbol{u}^T(\boldsymbol{x}_0^{r_2} - \boldsymbol{x}_0^{r_1}) = 1 \\ \quad \boldsymbol{v}^T\left[(1-2\alpha)\boldsymbol{y}_j^{r_4} + 2\alpha\boldsymbol{y}_j^{r_3}\right] - \boldsymbol{u}^T\left[(1-2\alpha)\boldsymbol{x}_j^{r_1} + 2\alpha\boldsymbol{x}_j^{r_2}\right] \le 0, \quad j = 1, 2, \ldots, n \\ \quad \boldsymbol{u} \ge 0 \\ \quad \boldsymbol{v} \ge 0. \end{cases} \tag{9}$$

7. A numerical example

In this section, a numerical example is presented to illustrate the fuzzy DEA model. The example is taken from [10]. Table 1 provides the information of the DMUs. There are two fuzzy inputs and two fuzzy outputs which are all triangular variables.

Table 3
Results of ranking DMUs with different α

Credibility level	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
0.50	0.12	0.50	0.22	0.50	0.50
0.40	0.08	0.40	0.20	0.40	0.40
0.30	0.02	0.30	0.19	0.30	0.30
0.20	0	0.20	0.16	0.20	0.20
0.10	0	0.08	0.10	0.10	0.10

Table 4
Results of evaluating the DMUs with different α

Credibility level	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
0.5	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency
0.4	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency
0.3	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency
0.2	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency
0.1	Inefficiency	Inefficiency	Efficiency	Efficiency	Efficiency

Table 5
Evaluating results with different h from Guo & Tanaka [10]

h	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
0	Inefficiency	Efficiency	Efficiency	Efficiency	Efficiency
0.5	Inefficiency	Efficiency	Inefficiency	Inefficiency	Efficiency
0.75	Inefficiency	Efficiency	Inefficiency	Inefficiency	Efficiency
0.1	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency

Table 2 shows the results of evaluating DMUs with $\alpha = 0.4$. According to the ranking method, the DMUs can be ranked as follows: DMU_2, DMU_4, DMU_5, DMU_3, DMU_1. Moreover DMU_2, DMU_4, DMU_5 are efficient and DMU_1, DMU_3 are inefficient according to Definition 2.

Table 3 gives the results of ranking DMUs with different α. The greater the value is, the more efficient DMU_0 is ranked. The ranking results is varying with different α. When $\alpha = 0.10$, the DMUs are ranked: DMU_3, DMU_4, DMU_5, DMU_2, DMU_1. At other α, the DMUs are ranked: DMU_2, DMU_4, DMU_5, DMU_3, DMU_1. This phenomena indicates that the ranking method in fuzzy environment is more complex than the traditional ranking methods because of the inherent fuzziness contained in inputs and outputs.

Table 4 gives the results of evaluating DMUs with different α according to Definition 2. The evaluating results in Table 5 come from [10]. From the comparison of these two tables, we can see that the results of evaluating DMU_1 and DMU_5 are same at all levels. However, DMU_2, DMU_3 and DMU_4 have the same results at some levels and have different results at other levels. In general, the results in this paper are coincident with the results from [10].

8. Conclusion

Due to its widely used practical background, data envelopment analysis (DEA) has become a pop area of research. This paper gave a new model with fuzzy inputs and outputs based on credibility measure, and proposed a method of ranking all the DMUs. In order to solve the fuzzy model, we have designed the hybrid algorithm combined with fuzzy simulation and genetic algorithm to compute it. When the inputs and outputs are trapezoidal or triangular fuzzy variables, the model can be transformed to linear programming. The numerical example illustrated the fuzzy DEA model and the method of ranking all the DMUs.

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References

[1] N. Adler, L. Friedman, Z. Sinuany-Stern, Review of ranking methods in data envelopment analysis context, European Journal of Operational Research 140 (2002) 249–265.
[2] P. Andersen, N.C. Petersen, A procedure for ranking efficient units in data envelopment analysis, Management Science 39 (10) (1993) 1261–1294.
[3] R.D. Banker, A. Charnes, W.W. Cooper, Some models for estimating technical and scale efficiencies in data envelopment analysis, Management Science 30 (1984) 1078–1092.

[4] A. Charnes, W.W. Cooper, L. Seiford, J. Stutz, A multiplicative model for efficiency analysis, Socio-Economic Planning Sciences 6 (1982) 223–224.
[5] A. Charnes, W.W. Cooper, E. Rhodes, Measuring the efficiency of decision making units, European Journal of Operational Research 2 (1978) 429–444.
[6] W.W. Cooper, K.S. Park, G. Yu, IDEA and AR-IDEA: Models for dealing with imprecise data in DEA, Management Science 45 (1999) 597–607.
[7] W.W. Cooper, K.S. Park, J.T. Pastor, RAM: a range adjusted measure of inefficiency for use with additive models, and relations to other models and measures in DEA, The Journal of Productivity Analysis 11 (1999) 5–24.
[8] W.W. Cooper, L.M. Seiford, K. Tone, Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References, and DEA-Solver Software, Kluwer Academic, Boston, 2000.
[9] T. Entani, Y. Maeda, H. Tanaka, Dual models of interval DEA and its extension to interval data, European Journal of Operational Research 136 (2002) 32–45.
[10] P. Guo, H. Tanaka, Fuzzy DEA: A perceptual evaluation method, Fuzzy Sets and Systems 119 (2001) 149–160.
[11] C. Kao, S.T. Liu, Fuzzy efficiency measures in data envelopment analysis, Fuzzy Sets and Systems 119 (2000) 149–160.
[12] S. Lertworasirikul, S.C. Fang, J.A. Joines, H.L.W. Nuttle, Fuzzy data envelopment analysis (DEA): A possibility approach, Fuzzy Sets and Systems 139 (2003) 379–394.
[13] X. Li, B. Liu, A sufficient and necessary condition for credibility measures, International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems 14 (5) (2006) 527–535.
[14] B. Liu, Minimax chance constrained programming models for fuzzy decision systems, Information Sciences 112 (1998) 25–38.
[15] B. Liu, K. Iwamura, Chance constrained programming with fuzzy parameters, Fuzzy Sets and Systems 94 (1998) 227–237.
[16] B. Liu, K. Iwamura, A note on chance constrained programming with fuzzy coefficients, Fuzzy Sets and Systems 100 (1998) 229–233.
[17] B. Liu, Dependent-chance programming with fuzzy decisions, IEEE Transactions on Fuzzy Systems 7 (1999) 354–360.
[18] B. Liu, Uncertain Programming, Wiley, New York, 1999.
[19] B. Liu, K. Iwamura, Fuzzy programming with fuzzy decisions and fuzzy simulation-based genetic algorithm, Fuzzy Sets and Systems 122 (2001) 253–262.
[20] B. Liu, Y.K. Liu, Expected value of fuzzy variable and fuzzy expected value models, IEEE Transactions on Fuzzy Systems 10 (4) (2002) 445–450.
[21] B. Liu, Theory and Practice of Uncertain Programming, Physica-Verlag, Heidelberg, 2002.
[22] B. Liu, Uncertainty Theory: An Introduction to its Axiomatic Foundations, Springer-Verlag, Berlin, 2004.
[23] B. Liu, A survey of credibility theory, Fuzzy Optimization and Decision Making 5 (4) (2006) 387–408.
[24] B. Liu, Uncertainty Theory, 2nd edition, Springer-Verlag, Berlin, 2007.
[25] N.C. Petersen, Data envelopment analysis on a relaxed set of assumptions, Management Science 36 (3) (1990) 305–313.
[26] L.M. Seiford, A DEA bibliography (1978–1992), in: A. Charnes, W.W. Cooper, A. Lewin, L. Seiford (Eds.), Data Envelopment Analysis: Theory, Methodology and Applications, Kluwer Academic Publishers, Boston, 1994.
[27] T.R. Sexton, R.H. Silkman, A.J. Hogan, Data envelopment analysis: Critique and extensions, in: R.H. Silkman (Ed.), Measuring Efficiency: An Assessment of Data Envelopment Analysis, Jossey-Bass, San Francisco, CA, 1986, pp. 73–105.
[28] K. Tone, A slack-based measure of efficiency in data envelopment analysis, European Journal of Operational Research 130 (2001) 498–509.
[29] A.M. Torgersen, F.R. Forsund, S.A.C. Kittelsen, Slack-adjusted efficiency measures and ranking of efficient units, The Journal of Productivity Analysis 7 (1996) 379–398.
[30] M.L. Wen, K. Iwamura, Fuzzy facility location-allocation problem under the Hurwicz criterion, European Journal of Operational Research 184 (2008) 627–635.
[31] M.L. Wen, K. Iwamura, Facility location-allocation problem in random fuzzy environment: Using (alpha, beta)-cost minimization model under Hurwicz criterion, Computers and Mathematics with Applications 55 (4) (2008) 704–713.
[32] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[33] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1 (1978) 3–28.