1.

$B \in R^{m \times N}$

$$B = [b_1 \mid b_2 \mid \cdots \mid b_N]_{m \times N}$$

$$C = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_N^T \end{bmatrix}_{N \times n}$$

According to the section 6.4

(i), $P_j = \dfrac{\|b_j\|^2 + \|c_j^T\|^2}{\|B\|_F^2 + \|C\|_F^2}$

(i) $\|\hat{A}\| \leq \max_j \|P_j^{-1} b_j c_j^T\| = \max_j \dfrac{\|b_j\|\|c_j^T\|}{P_j}$

$$\|\hat{A}\| \leq \frac{\|B\|_F^2 + \|C\|_F^2}{\|b_j\|^2 + \|c_j^T\|^2} \max_j \|b_j\|\|c_j^T\|$$

$$\leq \|B\|_F^2 + \|C\|_F^2 \quad \max_j \frac{\|b_j\|\|c_j^T\|}{\|b_j\|^2 + \|c_j^T\|^2}$$

$$\le \frac{1}{2}(\|B\|_F^2 + \|C\|_F^2)$$

$$\mathbb{E}(\hat{A}\hat{A}^*) = \sum_{j=1}^{N} \frac{1}{p_j} \cdot (b_j c_j^T)(b_j c_j^T)^* p_j$$

$$= (\|B\|_F^2 + \|C\|_F^2) \cdot \sum \frac{\|c_j^T\|^2}{\|b_j\|^2 + \|c_j\|^2} \cdot b_j b_j^*$$

$$\preccurlyeq (\|B\|_F^2 + \|C\|_F^2)\, BB^*$$

$$\mathbb{E}(\hat{A}^*\hat{A}) = \sum_{j=1}^{N} \frac{1}{p_j} \cdot (b_j c_j^T)^* (b_j c_j^T) p_j$$

$$= (\|B\|_F^2 + \|C\|_F^2) \cdot \sum \frac{\|b_j\|^2}{\|b_j\|^2 + \|c_j\|^2}\, c_j^* c_j$$

$$\preccurlyeq (\|B\|_F^2 + \|C\|_F^2)\, C^*C$$

$$m_2(\hat{A}) = \max\{\|\mathbb{E}(AA^*)\|, \|\mathbb{E}(A^*A)\|\}$$

$$\le (\|B\|_F^2 + \|C\|_F^2) \max\{\|BB^*\|, \|C^*C\|\}$$

$$= (\|B\|_F^2 + \|C\|_F^2) \max\{\|B\|^2, \|C\|^2\}$$

$$\mathbb{E}\|\hat{A}_r - A\| \le \sqrt{\frac{2 m_2(\hat{A}) \log(d_1+d_2)}{n}} + \frac{2L \log(d_1+d_2)}{3n}$$

$$= \sqrt{2(\|B\|_F^2 + \|C\|_F^2) \max\{\|B\|^2, \|C\|^2\} \log(m+n)} + \frac{(\|B\|_F^2 + \|C\|_F^2) \log(m+n)}{}$$

$$\sqrt{\frac{\cdots}{r}} \;\cdots\; \frac{\cdots}{3r}$$

suppose $\|B\|_F^2 + \|C\|_F^2 = k$

$$\frac{E\|\hat{A}_r - A\|}{\|B\|\|C\|} \le \sqrt{\frac{2k\max\{\frac{1}{\|C\|^2}, \frac{1}{\|B\|^2}\}\log(m,n)}{r}} + \frac{k\log(m+n)}{3r\|B\|\|C\|}$$

(iii')

$$\frac{E\|\hat{A}_r - A\|}{\|B\|\|C\|} \le \epsilon$$

$$\Rightarrow \sqrt{\frac{2k\max\{\frac{1}{\|C\|^2}, \frac{1}{\|B\|^2}\}\log(m+n)}{r}} + \frac{k\log(m+n)}{3r\|B\|\|C\|} \le \epsilon$$

Owing to B.C is known So everything is Constant except for $\hat{A}$

suppose $O = 2k\max\{\frac{1}{\|C\|^2}, \frac{1}{\|B\|^2}\}\log(m+n)$

$$P = \frac{k\log(m+n)}{3\|B\|\|C\|}$$

the inequation $\sqrt{\frac{O}{r}} + \frac{P}{r} \le \epsilon$

s.t. $O \geq 0 \quad P \geq 0 \quad \varepsilon \geq 0$

So According to the solver.

$$r \geq \frac{\varepsilon^2 \sqrt{\frac{O(4\varepsilon P + O)}{\varepsilon^4}} + 2\varepsilon P + O}{2\varepsilon^2}$$

s.t. $O = 2k \max\{\frac{1}{\|C\|^2}, \frac{1}{\|B\|^2}\} \log(m+n)$

$$P = \frac{k \log(m+n)}{3\|B\|\|C\|}$$

(iv) According to the article

$O\left(\frac{\varepsilon^2 \sqrt{\frac{O(4\varepsilon P + O)}{\varepsilon^4}} + 2\varepsilon P + O}{2\varepsilon^2} \cdot mn\right)$ arithmetric operations.