Hexapod kinematics

This document is a draft

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Displacements of the servo pivots B_i and the platform joints $\mathrm{P1}_i$

The coordinates of the platform joints at home position are called $\mathrm{P0}_i$.

Note that $\mathrm{P0z}_i = 0$ and $i = \text{motor index} = 0..5$.

$$\mathrm{P0}_i = \{P0x_i, P0y_i, 0, 1\}$$

The platform has six degrees of freedom, three rotations and three translations with the following transformation matrices:

$$\mathrm{Rx} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos A & \sin A & 0 \\ 0 & -\sin A & \cos A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathrm{Ry} = \begin{pmatrix} \cos B & 0 & -\sin B & 0 \\ 0 & 1 & 0 & 0 \\ \sin B & 0 & \cos B & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{Rz} = \begin{pmatrix} \cos C & \sin C & 0 & 0 \\ -\sin C & \cos C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathrm{Txyz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{Tx} & \mathrm{Ty} & \mathrm{Tz} & 1 \end{pmatrix}$$

The coordinates of the platform joints after movement are called $\mathrm{P1}_i$ and can be found by calculating the following dot product:

$$\mathrm{P1}_i = (\mathrm{P0}_i \cdot \mathrm{Rx} \cdot \mathrm{Ry} \cdot \mathrm{Rz} \cdot \mathrm{Txyz}) =$$

$$\begin{pmatrix} \mathrm{P1x}_i \\ \mathrm{P1y}_i \\ \mathrm{P1z}_i \\ 1 \end{pmatrix}^T = \begin{pmatrix} P0x_i \cos B \cos C + P0y_i(\sin A \sin B \cos C - \cos A \sin C) + \mathrm{Tx} \\ P0x_i \cos B \sin C + P0y_i(\sin A \sin B \sin C + \cos A \cos C) + \mathrm{Ty} \\ -P0x_i \sin B + P0y_i \sin A \cos B + \mathrm{Tz} \\ 1 \end{pmatrix}^T$$

We can now calculate the distances between the servo pivots B_i and the platform joints $\mathrm{P1}_i$:

$$\mathrm{dPB}_i = (\mathrm{P1}_i - \mathrm{B}_i) = (\mathrm{P1}_i - \{Bx_i, By_i, -Z_{home}, 1\}) =$$

$$\boxed{\begin{pmatrix} \mathrm{dPBx}_i \\ \mathrm{dPBy}_i \\ \mathrm{dPBz}_i \\ 1 \end{pmatrix}^T = \begin{pmatrix} P0x_i \cos B \cos C + P0y_i(\sin A \sin B \cos C - \cos A \sin C) + \mathrm{Tx} - \mathrm{Bx}_i \\ P0x_i \cos B \sin C + P0y_i(\sin A \sin B \sin C + \cos A \cos C) + \mathrm{Ty} - \mathrm{By}_i \\ -P0x_i \sin B + P0y_i \sin A \cos B + \mathrm{Tz} + \mathrm{Z}_{home} \\ 1 \end{pmatrix}^T} \text{(eq 1)}$$

And the squares of the length of the vectors are:

$$\boxed{d_i^2 = \mathrm{dPBx}_i^2 + \mathrm{dPBy}_i^2 + \mathrm{dPBz}_i^2} \text{ (eq 2)}$$

For a platform using linear motors, the calculation can be done with $\text{eq } 1$ and $\text{eq } 2$. In our case we have rotational motors so we need to continue. Note that it is not a good idea to compute the square root of d_i^2 because it is a computer intensive operation and we don't need the value of d.

Now, we need to check if the arm of length a and the rod of length s are long enoug to actualy go to the target position:

$$d_i^2 \le (a+s)^2 \text{ (eq 3)}$$

Projection of the rod sphere on the arm plane

The calculation below works when point B is between P' and R. It doesn't work when P' is between B and R.

P is the projection of P_i on the xy plane.

$\overline{PR}$ = rod length

$\overline{BR}$ is a segment that belongs to the plane formed by point A (joint at the end of servo arm) during rotation

P' is the projection of P on $\overline{BR}$

$\overline{PB'} = \mathrm{dPBx}_i$

$\overline{BB'} = \mathrm{dPBy}_i$

Angles calculation

$$\widehat{P'PB'} = \frac{\pi}{2} - \theta s_i$$

$$\widehat{BPB'} = atan\frac{\mathrm{dPBy}_i}{\mathrm{dPBx}_i}$$

$$\widehat{P'PB} = \widehat{P'PB'} - \widehat{BPB'} = \frac{\pi}{2} - \theta s_i - atan\frac{\mathrm{dPBy}_i}{\mathrm{dPBx}_i}$$

Length calculation

Projection of P on the servo arm plane.

$$\overline{PB} = \sqrt{\mathrm{dPBx}_i^2 + \mathrm{dPBy}_i^2}$$

$$\overline{P'P} = \overline{PB} \cdot cos\widehat{P'PB}$$

$$\boxed{\overline{P'P} = \sqrt{\mathrm{dPBx}_i^2 + \mathrm{dPBy}_i^2} \cdot cos\left(\frac{\pi}{2} - \theta s_i - atan\frac{\mathrm{dPBy}_i}{\mathrm{dPBx}_i}\right)}$$

The intersection of the sphere formed by the rod and the plane of the rotation of the servo arm joint is a circle of radius r_{cs}.

$$r_{cs} = \overline{P'R} = \sqrt{\overline{PR}^2 - \overline{P'P}^2}$$

$$\boxed{r_{cs} = \sqrt{\overline{\text{Rod Length}}^2 - \overline{P'P}^2}}$$

$$r_{cs} = \sqrt{\text{Rod Length}^2 - \left(\text{dPBx}_i^2 + \text{dPBy}_i^2\right) \cdot cos^2\left(\frac{\pi}{2} - \theta s_i - atan\frac{\text{dPBy}_i}{\text{dPBx}_i}\right)}$$

In the arm plane

The center of this circle **in the plane of the servo arm** is at coordinates c_x, c_z when the origin $0, 0$ is set to point B, i.e. on the projection of the servo shaft.

$$c_z = \text{dPBz}_i$$

$$c_x = \overline{P'B_{\square\text{xy}}} = \sqrt{\overline{PB}^2 - \overline{P'P}^2}$$

$$\boxed{c_x = \sqrt{\text{dPBx}_i^2 + \text{dPBy}_i^2 - \overline{P'P}^2}}$$

$$c_x = \sqrt{\text{dPBx}_i^2 + \text{dPBy}_i^2 - \left(\text{dPBx}_i^2 + \text{dPBy}_i^2\right) \cdot cos^2\left(\frac{\pi}{2} - \theta s_i - atan\frac{\text{dPBy}_i}{\text{dPBx}_i}\right)}$$

$$\boxed{\alpha = atan\left(\frac{\text{dPBz}_i}{c_x}\right)}$$

Cosinus theorem

$$r_{\text{cs}} = \text{arm}^2 + \overline{P'B_{\square\text{arm}}}^2 - 2\ \text{arm}\ \overline{P'B_{\square\text{arm}}}\ cos\beta$$

$$\boxed{\beta = acos\left(\frac{\text{arm}^2 + \overline{P'B_{\square\text{arm}}}^2 - r_{\text{cs}}^2}{2\ \text{arm}\ \overline{P'B_{\square\text{arm}}}}\right)}$$

$$\boxed{\overline{P'B_{\square\text{arm}}} = \sqrt{\overline{P'B_{\square\text{xy}}}^2 + \text{dPBz}_i^2}}$$

$$\boxed{\text{servo angle} = \pi - \alpha - \beta}$$