FORWARD KINEMATICS SOLUTION OF A STEWART PLATFORM ACTUATED BY ROTARY MOTORS (DRAFT)

NICOLAS JEANMONOD, OUILOGIQUE.COM, MAY 2019

Part 1. Abstract

This paper presents an algorithm for the calculation of the forward kinematics solution of a Stewart platform actuated by rotary motors. It is based on the following paper by an unknown author from the Wokingham U3A Math Group: https://bit.ly/2FU3rUJ. This algorithm has been implemented on an ESP32 microcontroller: https://github.com/NicHub/stewart-platform-esp32.

Part 2. Positions of the platform joints P_i relative to the servo pivots B_i

Let's consider an orthogonal system of axis xyz with origin in the middle of the moving platform and the z axis pointing downwards. The coordinates of the platform joints are called P_i where i = motor index = 0..5. Note that at home position $Pz_i = 0$.

$$P_i = \{Px_i, Py_i, 0, 1\}$$

The platform has six degrees of freedom, three rotations and three translations with the following transformation matrices:

$$Rx = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos A & \sin A & 0 \\ 0 & -\sin A & \cos A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad Ry = \begin{pmatrix} \cos B & 0 & -\sin B & 0 \\ 0 & 1 & 0 & 0 \\ \sin B & 0 & \cos B & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Rz = \begin{pmatrix} \cos C & \sin C & 0 & 0 \\ -\sin C & \cos C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad Txyz = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ Tx & Ty & Tz & 1 \end{pmatrix}$$

The servo pivots have the following coordinates.

$$B_i = \{Bx_i, By_i, Zhome, 1\}$$

Translation of servo pivots B_i relative to the origin:

$$TB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -Bx & -By & -Zhome & 1 \end{pmatrix}$$

The positions of the platform joints P_i relative to the servo pivots B_i after movement are called BP_i and can be found by calculating the dot product below. Note that the result is the solution of the forward kinematics of platforms actuated by linear motors.

$$BP_i = P_i \cdot Rx \cdot Ry \cdot Rz \cdot Txyz \cdot TB =$$

$$\begin{pmatrix} BPx_i \\ BPy_i \\ BPz_i \\ 1 \end{pmatrix}^T = \begin{pmatrix} Px_i \cos B \cos C & + & Py_i (\sin A \sin B \cos C - \cos A \sin C) & + & Tx & - & Bx_i \\ Px_i \cos B \sin C & + & Py_i (\sin A \sin B \sin C + \cos A \cos C) & + & Ty & - & By_i \\ -Px_i \sin B & + & Py_i \sin A \cos B & + & Tz & - & Zhome \\ & & 1 & & & & \end{pmatrix}^T \tag{1}$$

Part 3. Rotation of the vector BP in the plane of the servo arm

The arm plane is rotated by a negative angle Θ_s and it defines a new system of coordinates that is also rotated. Let's find the coordinates of BP in this new reference system:

$$\begin{pmatrix} BPx_i \\ BPy_i \\ BPz_i \\ 1 \end{pmatrix}^T \cdot \begin{pmatrix} \cos\Theta_s & \sin\Theta_s & 0 & 0 \\ -\sin\Theta_s & \cos\Theta_s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Theta_s BPx_i - \sin\Theta_s BPy_i \\ \sin\Theta_s BPx_i + \cos\Theta_s BPy_i \\ BPz_i \\ 1 \end{pmatrix}^T = \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix}^T$$

In the original algorithm, Θ_s is positive, so we can rewrite:

$$\begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix}^T = \begin{pmatrix} \cos\Theta_s BPx_i + \sin\Theta_s BPy_i \\ -\sin\Theta_s BPx_i + \cos\Theta_s BPy_i \\ BPz_i \\ 1 \end{pmatrix}^T \tag{2}$$

Part 4. Intersection between servo arm circle and rod sphere in the plane of the servo arm circle

The end of the rod is somewhere on a sphere. This sphere will intersect the circle of the servo arm 0, 1 or 2 times. This circle lies in a plane which has two axis x and z, so $y = 0$. This plane is rotated by an angle Θ_s along the z axis. We applied the rotation to the BP vector in the previous section to find the values of a, b and c. The rotation implies that BP is in the negative region of y and that explains the sign of the term $+b^2$ in the equation of the sphere below.

Now we will find the intersection of the rod sphere and the arm circle. The circle is expressed in parametric coordinates, but is also possible to solve this problem by expressing it in Cartesian coordinates.

$$\begin{cases} (x-a)^2 + b^2 + (z-c)^2 & = rod^2 \\ arm\ \cos\varphi & = x \\ arm\ \sin\varphi & = z \end{cases}$$

$$\begin{aligned}
(arm\ \cos\varphi - a)^2 + b^2 + (arm\ \sin\varphi - c)^2 &= rod^2 \\
\left(arm^2\cos^2\varphi - 2\,arm\ \cos\varphi\, a + a^2\right) &+ \\
\left(arm^2\sin^2\varphi - 2\,arm\ \sin\varphi\, c + c^2\right) &= rod^2 - b^2 \\
\left(\cos^2\varphi - 2\frac{a}{arm}\cos\varphi + \frac{a^2}{arm^2}\right) &+ \\
\left(\sin^2\varphi - 2\frac{c}{arm}\sin\varphi + \frac{c^2}{arm^2}\right) &= \frac{rod^2 - b^2}{arm^2} \\
d &= \frac{a}{arm} \\
e &= \frac{c}{arm} \\
\left(\cos^2\varphi - 2d\cos\varphi + d^2\right) &+ \\
\left(\sin^2\varphi - 2e\sin\varphi + e^2\right) &= \frac{rod^2 + b^2}{arm^2} \\
\cos^2\varphi + \sin^2\varphi &= 1 \\
d\cos\varphi + e\sin\varphi &= \frac{1}{2}\left(1 + d^2 + e^2 - \frac{rod^2 - b^2}{arm^2}\right) \\
d\cos\varphi + e\sin\varphi &= \frac{1}{2}\left(\frac{arm^2}{arm^2} + \frac{a^2}{arm^2} + \frac{c^2}{arm^2} - \frac{rod^2 - b^2}{arm^2}\right) \\
d\cos\varphi + e\sin\varphi &= \frac{a^2 + b^2 + c^2 + arm^2 - rod^2}{2\,arm^2} = f
\end{aligned}$$

Identity

$$m\sin\alpha + n\cos\alpha = o\sin(\alpha + p)$$
$$o = \sqrt{m^2 + n^2}$$
$$\tan p = \frac{n}{m}$$
$$n\cos\alpha + m\sin\alpha = \pm\sqrt{m^2 + n^2}\sin\left(\alpha + \arctan\frac{n}{m}\right)$$
$$if\ -\frac{\pi}{2} < \arctan\frac{n}{m} < \frac{\pi}{2}\ then$$
$$n\cos\alpha + m\sin\alpha = \sqrt{m^2 + n^2}\sin\left(\alpha + \arctan\frac{n}{m}\right) = f$$

Thus

$$f = \sqrt{d^2 + e^2}\sin\left(\varphi + \arctan\frac{d}{e}\right)$$
$$\varphi = \arcsin\frac{f}{\sqrt{d^2 + e^2}} - \arctan\frac{d}{e}$$
$$\varphi = \arcsin\frac{\frac{a^2+b^2+c^2+arm^2-rod^2}{2\,arm^2}}{\sqrt{\frac{a^2}{arm^2} + \frac{c^2}{arm^2}}} - \arctan\frac{a}{c}$$
$$\varphi = \arcsin\frac{a^2 + b^2 + c^2 + arm^2 - rod^2}{2\,arm\sqrt{a^2 + c^2}} - \arctan\frac{a}{c}$$
$$\varphi = \arcsin\frac{a^2 + b^2 + c^2 + arm^2 - rod^2}{2 \cdot arm \cdot c \cdot \sqrt{1 + \left(\frac{a}{c}\right)^2}} - \arctan\frac{a}{c}$$

The length of the vector BP is the same in both systems of coordinates (arm and platform). So we calculate it in the platform system because this allows us to drop the calculation of b:

$$a^2 + b^2 + c^2 = BP_x^2 + BP_y^2 + BP_z^2 = BP^2 \tag{3}$$

$$\varphi = \arcsin\frac{BP^2 + arm^2 - rod^2}{2 \cdot arm \cdot c \cdot \sqrt{1 + \left(\frac{a}{c}\right)^2}} - \arctan\frac{a}{c} \tag{4}$$

The angle φ equals 0 when the arm is horizontal, but we want it to be 0 when the arm is at half of its full angular range:

$$\varphi_{final} = \varphi + \frac{servo\,full\,angular\,range}{2} \tag{5}$$

Part 5. Conversion from radian to PWM microseconds and mirroring

The conversion of the angle in radian to an angle expressed in PWM microseconds is done with a simple linear relation that is different for odd and even servomotors:

$$\varphi_{PWM\mu s} = \begin{cases} gain \cdot \varphi_{final} + servo\,min\,pwm\,\mu s + offset \\ -gain \cdot \varphi_{final} + servo\,max\,pwm\,\mu s + offset \end{cases} \tag{6}$$

where

$$gain = \frac{servo\,max\,pwm\,\mu s - servo\,min\,pwm\,\mu s}{servo\,full\,angular\,range\,rad} \tag{7}$$

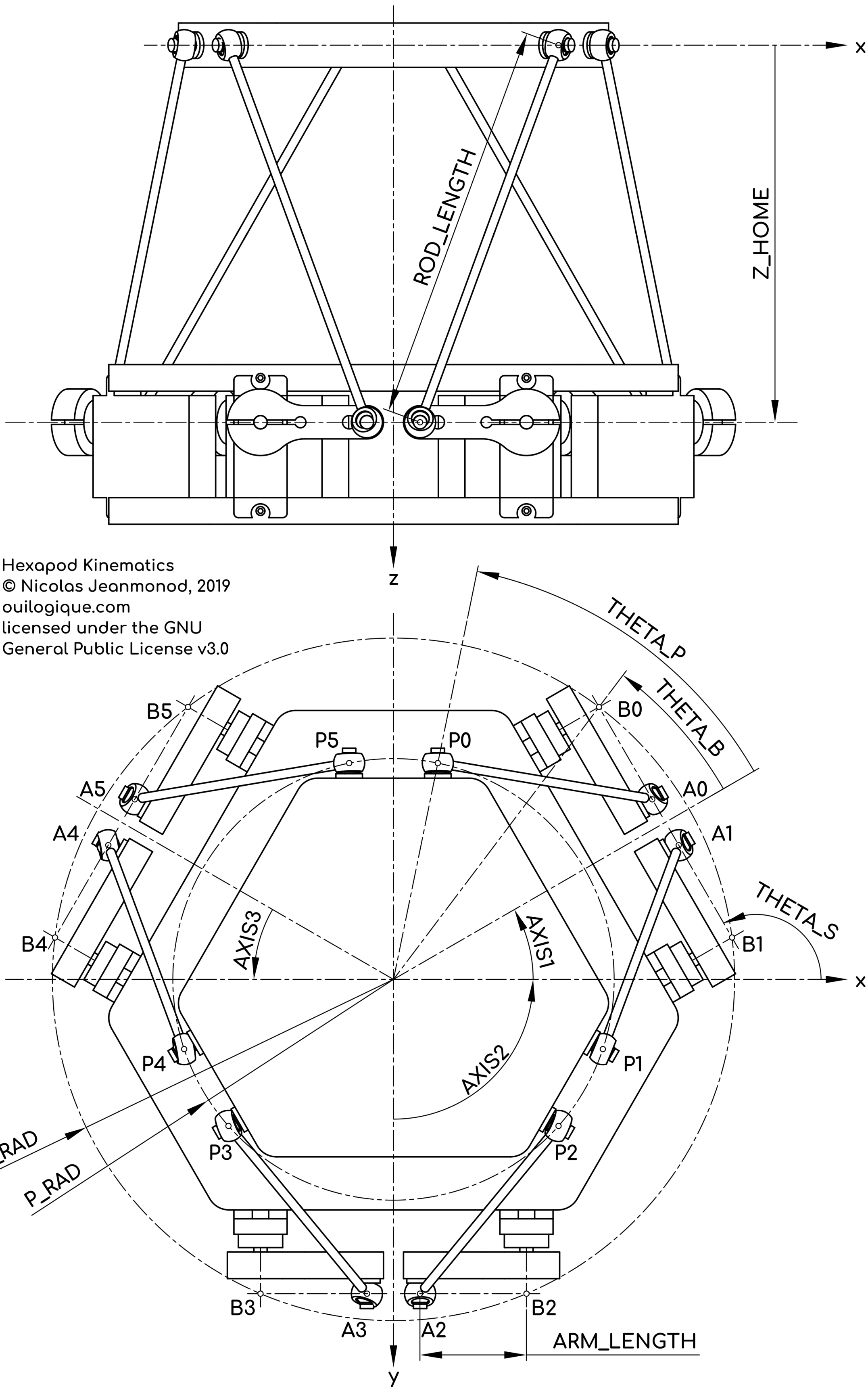
A typical value for the gain is $(2000 - 1000)/\pi \approx 318\,\mu s/rad \approx 5.6\,\mu s/^\circ$.

The $offset$ will be typically 0, but can be adjusted to take into account that the arms cannot be mounted exactly at the correct angle. The sign of $gain$ takes into account the fact that the odd and even arms are a reflection of each other. Thus, odd and even servomotors must have opposite gain values.

Part 6. Calculation of $Zhome$

$Zhome$ must be calculated for each platform geometry so that when the platform is at home position, i.e. with no translation and no rotation, the arms are horizontal or tilted at any value that makes sense.

TO DO: calculation of $Zhome$.


x
ROD_LENGTH
Z_HOME
z
THETA_P
THETA_B
THETA_S
B5
B0
P5
P0
A5
A0
A4
A1
B4
B1
AXIS3
AXIS1
x
AXIS2
P4
P1
B_RAD
P_RAD
P3
P2
B3
B2
A3
A2
ARM_LENGTH
y