Improving forecasts via subspace projections

George Athanasopoulos

MONASH University

Outline

1. Improving hierarchical forecasts
2. Improving univariate forecasts
3. Improving cross-temporal forecasts
4. Improving multivariate forecasts
5. Final comments

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1 Improving hierarchical forecasts

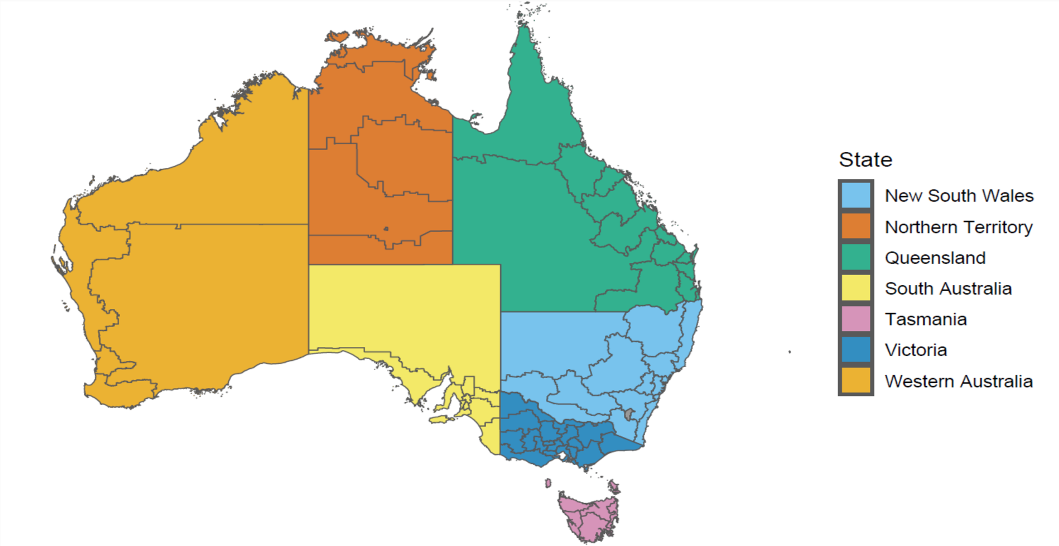
2 Improving univariate forecasts

3 Improving cross-temporal forecasts

4 Improving multivariate forecasts

5 Final comments

Australian tourism regions



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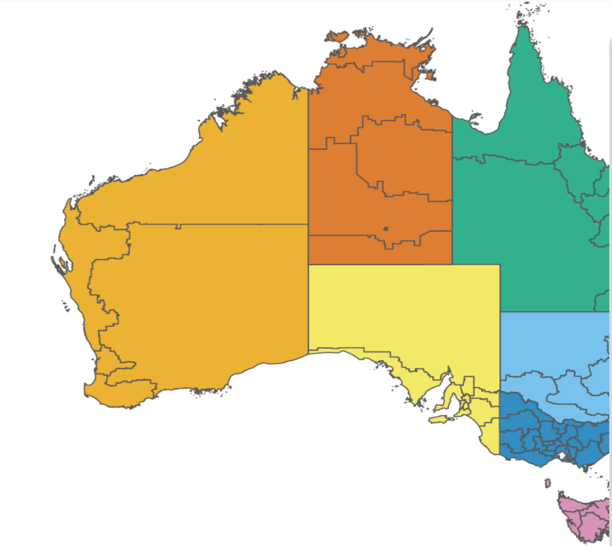
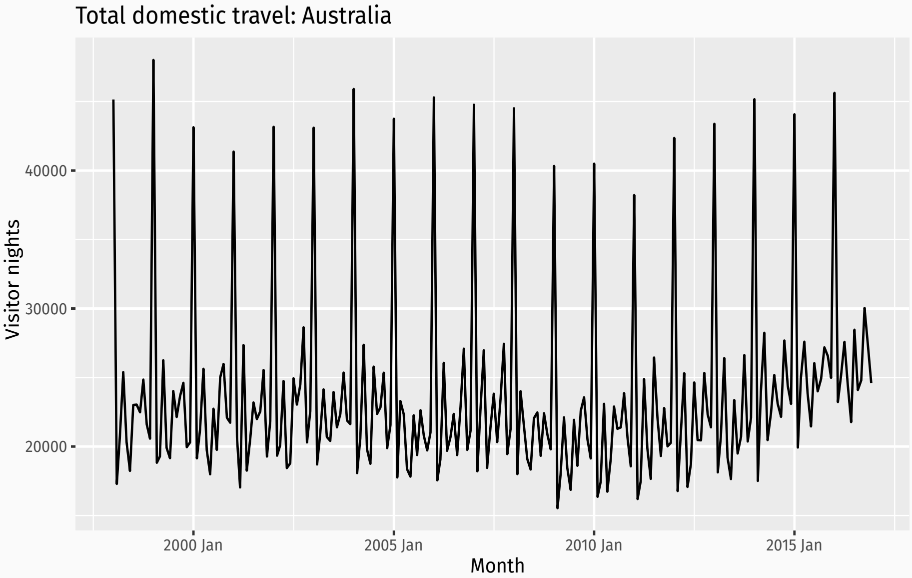


- Monthly data on visitor night: 1998 – 2019
- From *National Visitor Survey*, annual interviews of 120,000 Australians aged 15+.
- Geographical hierarchy split by
  - 7 states
  - 27 zones
  - 75 regions

Australian tourism data



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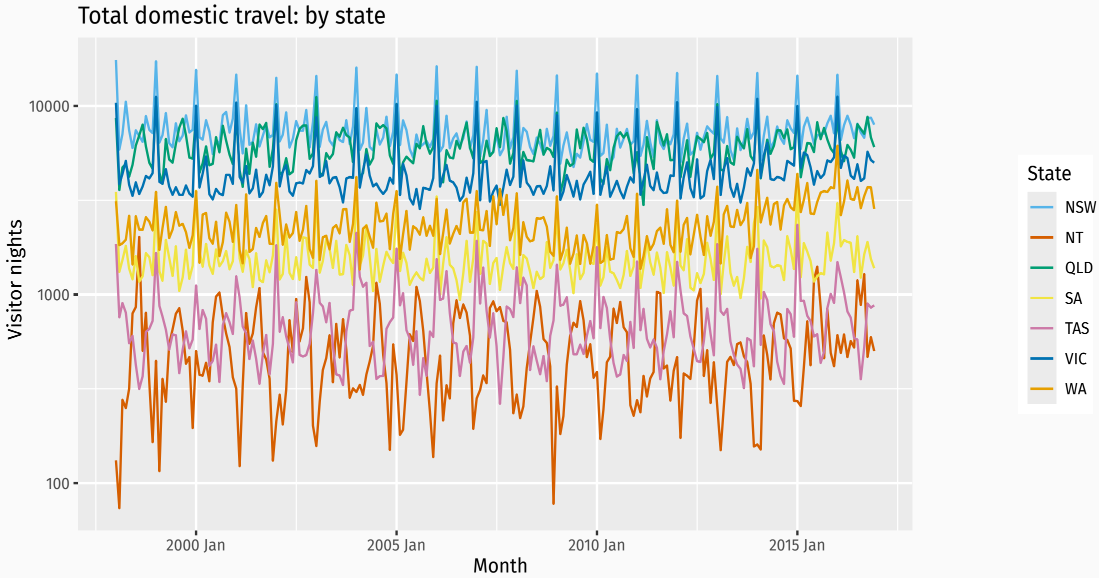

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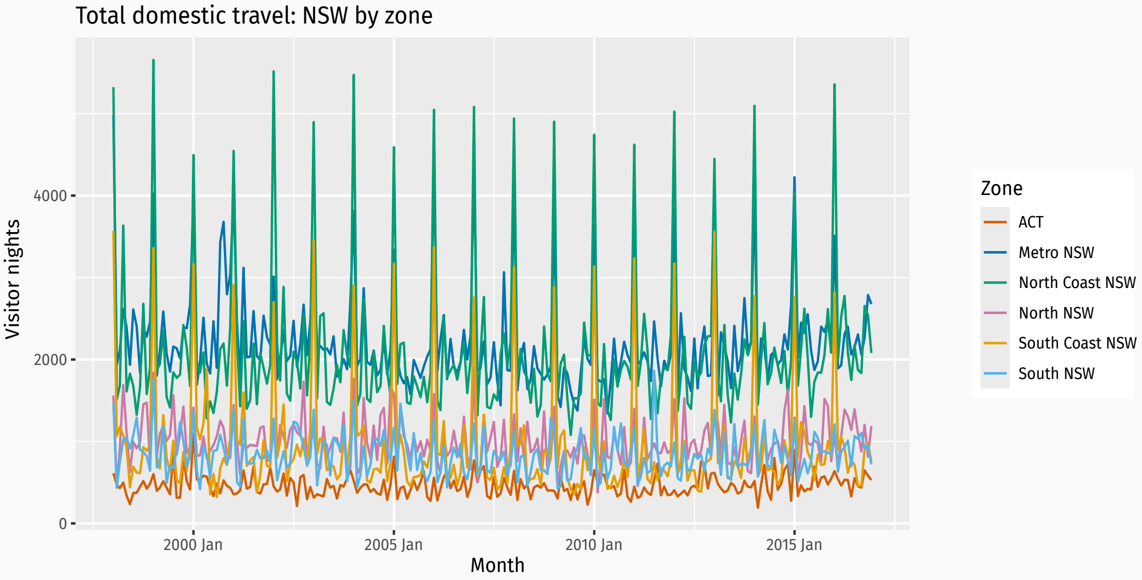

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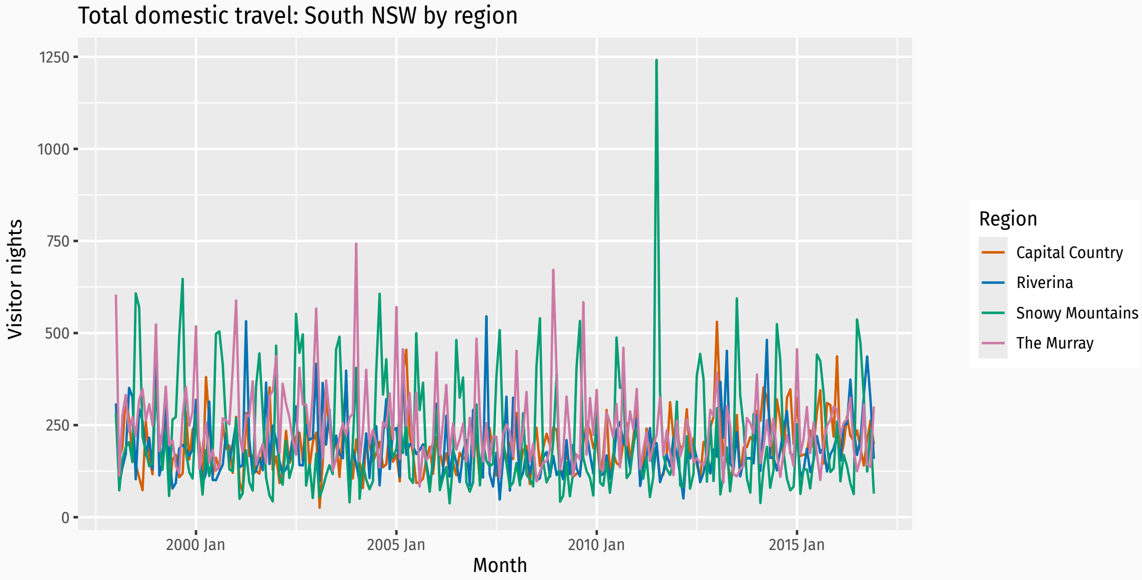

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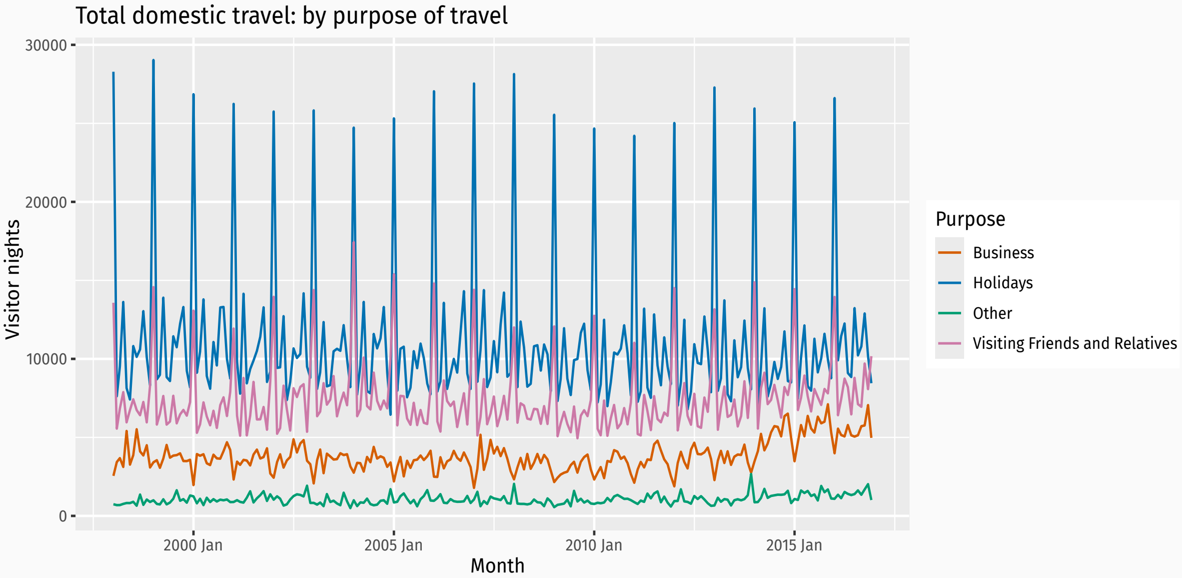

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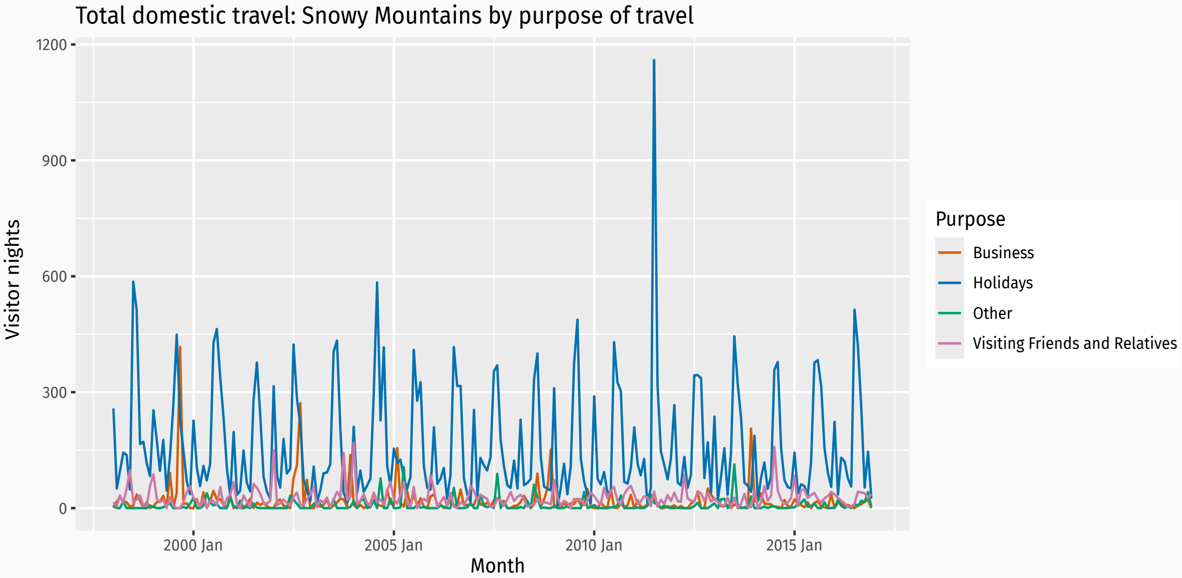

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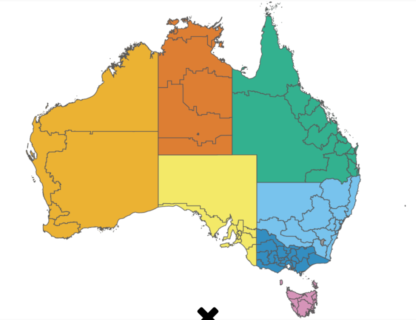
Geographical division



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Purpose of travel

Holiday, Visiting friends & relatives, Business, Other

- **Grouped time series**
  (geographical divisions $\times$ purpose of travel)

	AUS	States	Zones	Regions	Tot
geographical	1	7	21	76	105
purpose	4	28	84	**304**	420
total	5	35	105	380	**525**

m = 304 and **n = 525**

Australian tourism data

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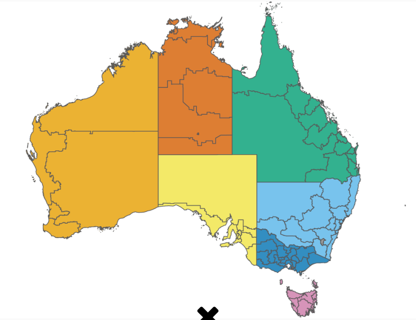

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- Need forecasts at all levels of aggregation.
- Forecasts generated by different agents/models will not adhere to aggregation constraints.

Key idea

- Traditional single level approaches: bottom-up, top-down or middle-out.

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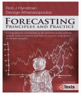
G Athanasopoulos, R Ahmed, and R Hyndman (2009). **“Hierarchical forecasts for Australian domestic tourism”.** In: *International Journal of Forecasting* 25, pp. 146–166. URL: https://doi.org/10.1016/j.ijforecast.2008.07.004

RJ Hyndman, R Ahmed, G Athanasopoulos, and H Shang (2011). **“Optimal combination forecasts for hierarchical time series”.** In: *Computational Statistics & Data Analysis* 55.9, pp. 2579–2589. URL: https://doi.org/10.1016/j.csda.2011.03.006

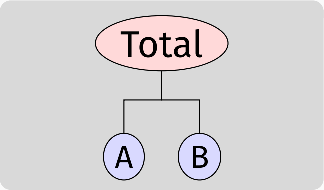


RJ Hyndman and G Athanasopoulos (2021). ***Forecasting: principles and practice.*** 3nd Edn. Melbourne, Australia. URL: http://otexts.com/fpp3/

Notation

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- $\boldsymbol{y}_t$ = vector of all series at time t
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = “structural matrix” containing the linear constraints.



$$\boldsymbol{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

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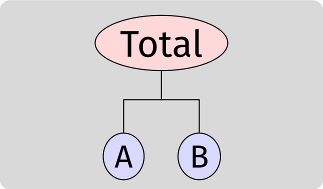
- m – number of bottom-level series
- n – number of all series



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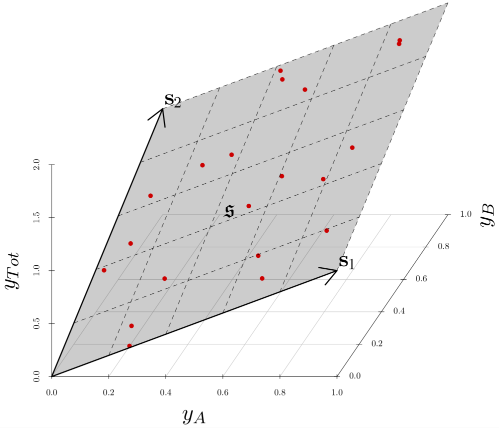
The coherent subspace

Hierarchical time series

Multivariate time series $\boldsymbol{y}_t \in \mathbb{R}^n$, bound by linear contstraints.

Coherent subspace

m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y}_t \in \mathfrak{s}$.



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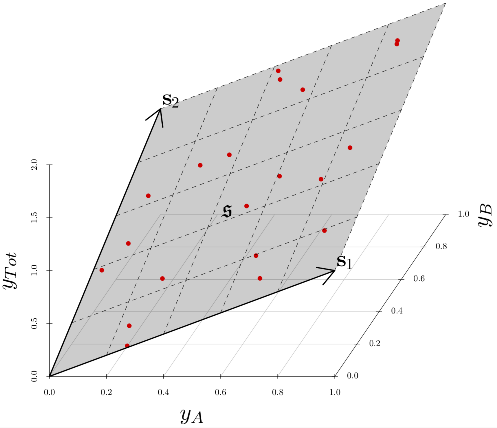
$\tilde{\boldsymbol{y}}_{T+h|T}$ is *coherent* if $\tilde{\boldsymbol{y}}_{T+h|T} \in \mathfrak{s}$.



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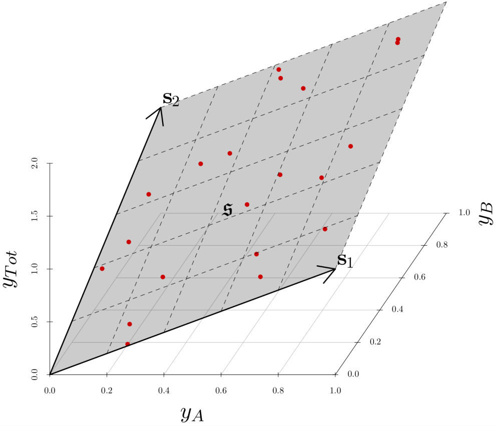
Base forecasts

Let $\hat{\boldsymbol{y}}_{T+h|T}$ be vector of *incoherent* initial h-step forecasts.



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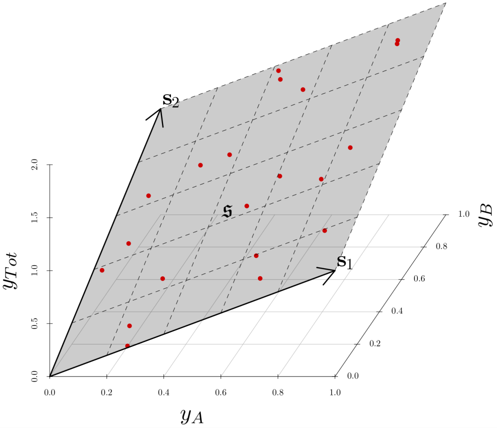

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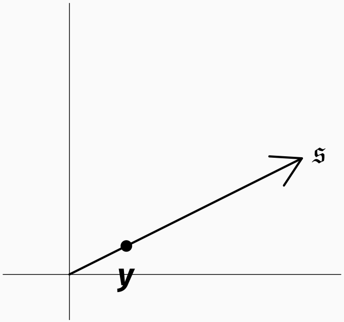
Reconciled forecasts

Let $\boldsymbol{SG}$ be a projection matrix.
$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{SG}\hat{\boldsymbol{y}}_{T+h|T}$ “reconciles” $\hat{\boldsymbol{y}}_{T+h|T}$.

Geometry of forecast reconciliation



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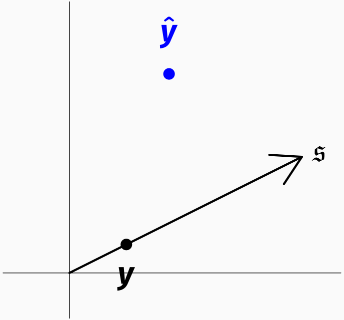
Reconciled point forecasts

Let ψ be a mapping, $\psi : \mathbb{R}^n \rightarrow \mathfrak{s}$. The point forecast $\tilde{\boldsymbol{y}}_{T+h|T} = \psi\left(\hat{\boldsymbol{y}}_{T+h|T}\right)$ "reconciles" a base forecast $\hat{\boldsymbol{y}}_{T+h|T}$ with respect to the mapping $\psi(.)$



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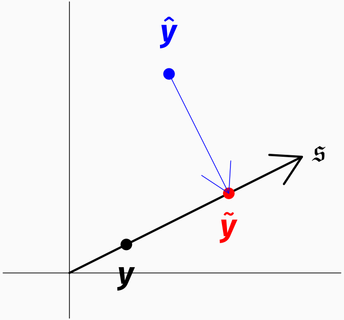

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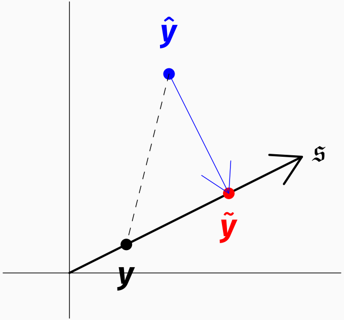

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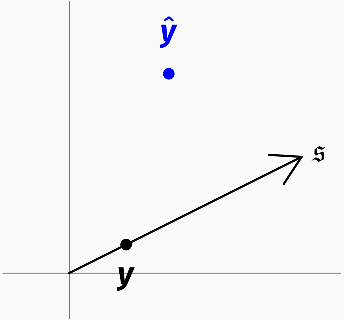

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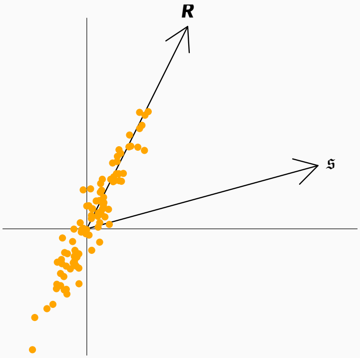

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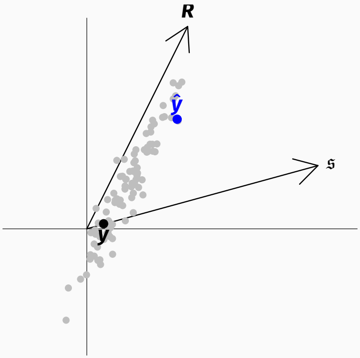

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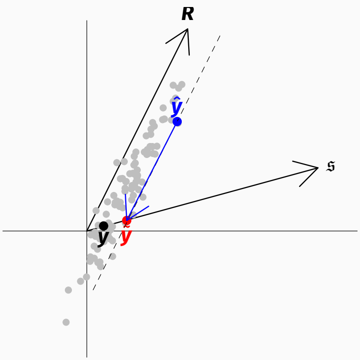

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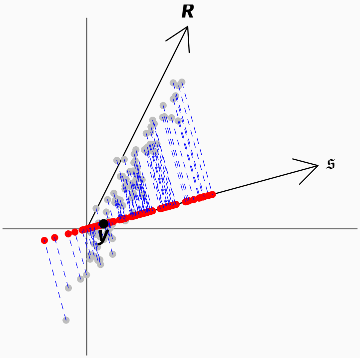

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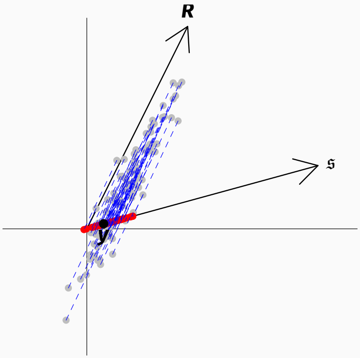

Minimum trace (MinT) reconciliation

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- Let $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]$ be the covariance matrix of the base forecast errors.
- Then $\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]) = \boldsymbol{S}\boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}'\boldsymbol{S}'$ is the covariance matrix of the reconciled forecast errors.

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$$\boldsymbol{S}\boldsymbol{G} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$$

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- MinT is L_2 optimal amongst linear unbiased forecasts.

MinT linear projections $\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$

- How to estimate $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]$?

MinT linear projections

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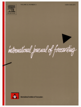
Reconc. method	$\boldsymbol{SG}$
OLS	$\boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
WLS(var)	$\boldsymbol{S}(\boldsymbol{S}'\Lambda_v\boldsymbol{S})^{-1}\boldsymbol{S}'\Lambda_v$ where $\Lambda_v = \text{diag}(\hat{\boldsymbol{W}}_1)^{-1}$
WLS(struct)	$\boldsymbol{S}(\boldsymbol{S}'\Lambda_s\boldsymbol{S})^{-1}\boldsymbol{S}'\Lambda_s$ where $\Lambda_s = \text{diag}(\boldsymbol{S}\boldsymbol{1})^{-1}$
MinT(sample)	$\boldsymbol{S}(\boldsymbol{S}'\hat{\boldsymbol{W}}_1^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_1^{-1}$
MinT(shrink)	$\boldsymbol{S}(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}$

- $\hat{\boldsymbol{W}}_{\text{shr}}$ is shrinkage estimator $\tau\text{diag}(\hat{\boldsymbol{W}}_1) + (1-\tau)\hat{\boldsymbol{W}}_1$ where τ selected optimally.

MinT and Geometry papers

SL Wickramasuriya, G Athanasopoulos, and RJ Hyndman (2019). **“Optimal Forecast Reconciliation for Hierarchical and Grouped Time Series Through Trace Minimization”.** In: *Journal of the American Statistical Association* 114.526, pp. 804–819. URL: https://doi.org/10.1080/01621459.2018.1448825.



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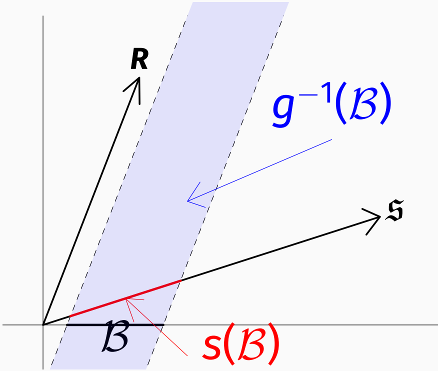
Probabilistic forecast reconciliation

A probability triple $(\mathfrak{s}, \mathscr{F}_{\mathfrak{s}}, \breve{\nu})$ is coherent with the bottom probability triple $(\mathbb{R}^m, \mathscr{F}_{\mathbb{R}^m}, \nu)$, if

$$\breve{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) \quad \forall \mathcal{B} \in \mathscr{F}_{\mathbb{R}^m}$$

- Random draws from coherent distribution must lie on $\mathfrak{s}$.
- The probability of points not on $\mathfrak{s}$ is zero.
- The reconciled distribution is a transformation of the base forecast distribution that is coherent on $\mathfrak{s}$.



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Outline

1 Improving hierarchical forecasts

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4 Improving multivariate forecasts

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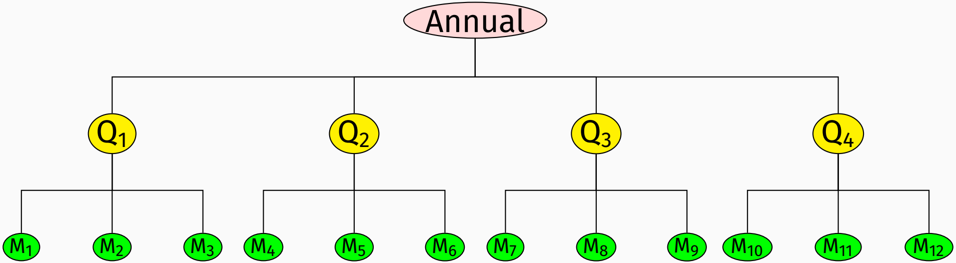
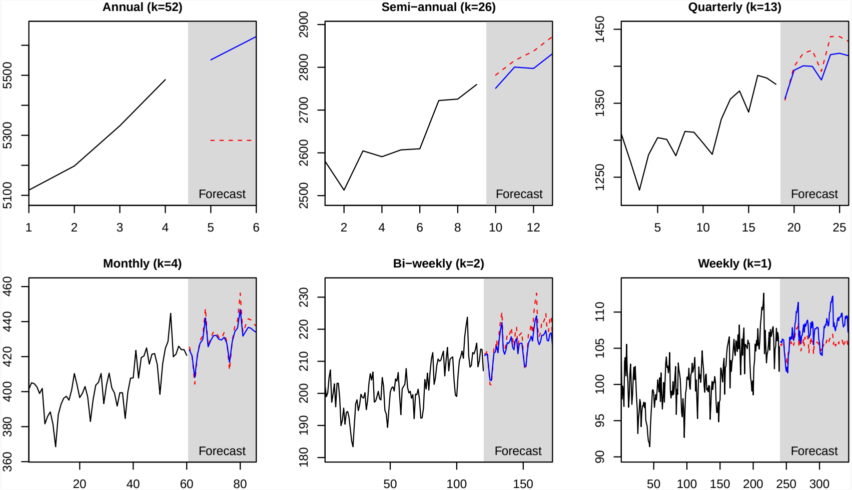
Temporal reconciliation: monthly data



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

Example: Accident & emergency services demand



Temporal Hierarchical Forecasting - THieF

G Athanasopoulos, RJ Hyndman, N Kourentzes, and F Petropoulos (2017). **"Forecasting with Temporal Hierarchies".** In: *European Journal of Operational Research* 262, pp. 60–74. URL: https://doi.org/10.1016/j.ejor.2017.02.046

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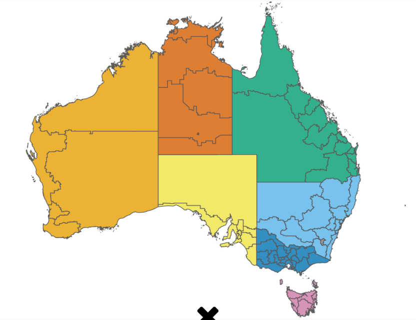
Cross-temporal reconciliation

Geographical division



✖

Purpose of travel

Holiday, Visiting friends & relatives, Business, Other

- **Cross-sectional aggregations**
  (geographical divisions × purpose of travel)

	AUS	States	Zones*	Regions	Tot
geographical	1	7	21	76	105
purpose	4	28	84	**304**	420
total	5	35	105	380	**525**

Cross-temporal reconciliation

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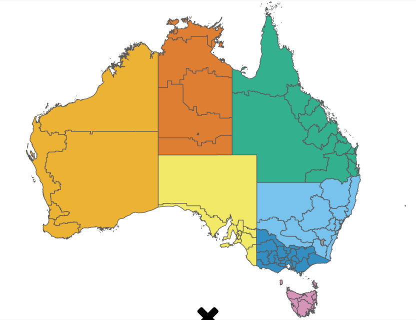

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Cross-temporal reconciliation

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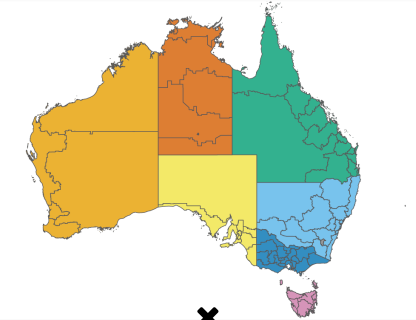

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Purpose of travel

Holiday, Visiting friends & relatives, Business, Other

Total: 3150 Series

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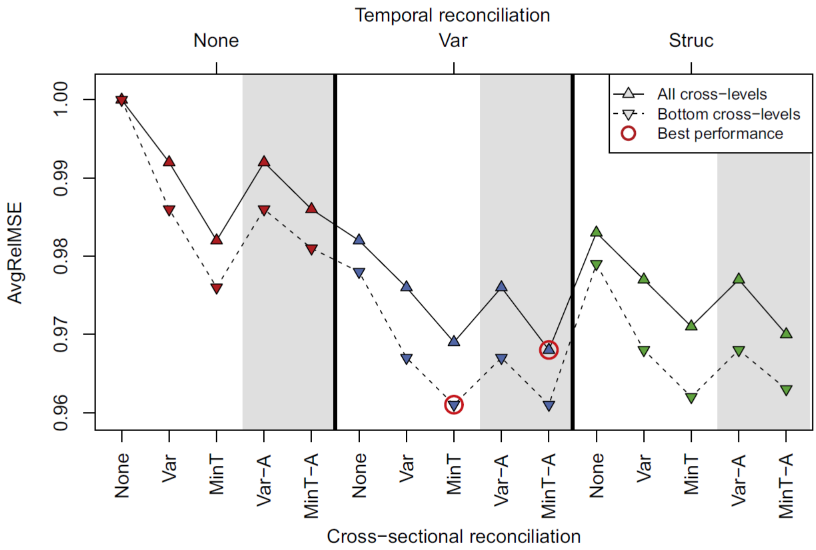

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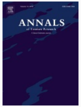

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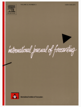

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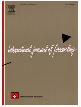

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FLAP (Forecast Linear Augemnted Projection)

Intuition:

- Suppose we are interested in multivariate forecasting but do not have linear (or non-linear) constraints.
- Can reconciliation help?

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- Suppose we are interested in multivariate forecasting but do not have linear (or non-linear) constraints.
- Can reconciliation help?

Can we find linear components that:

1. are easy to forecast (or easier than the original series);
2. can capture possible common signals;
3. can improve forecasts of original series.

Outline of FLAP Implementation

- We want to forecast a multivariate series $\boldsymbol{y}_t \in \mathbb{R}^m$.

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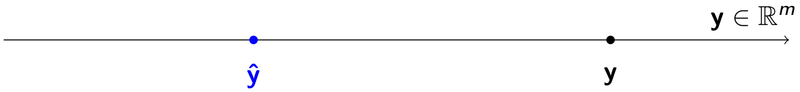
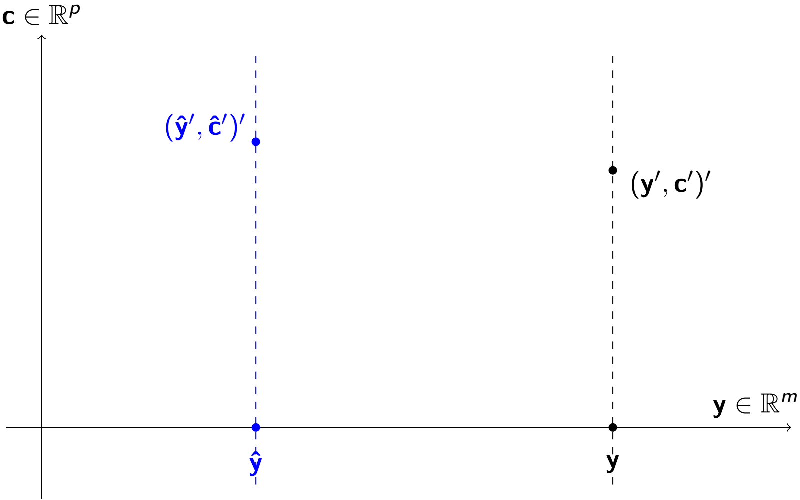
- Produce forecasts for original series and components $\hat{\boldsymbol{y}}_{t+h}$ and $\hat{\boldsymbol{c}}_{t+h}$.
- Project forecasts onto the $\mathbb{R}^m$ coherent subspace using MinT, resulting in $\tilde{\boldsymbol{y}}_{t+h}$.

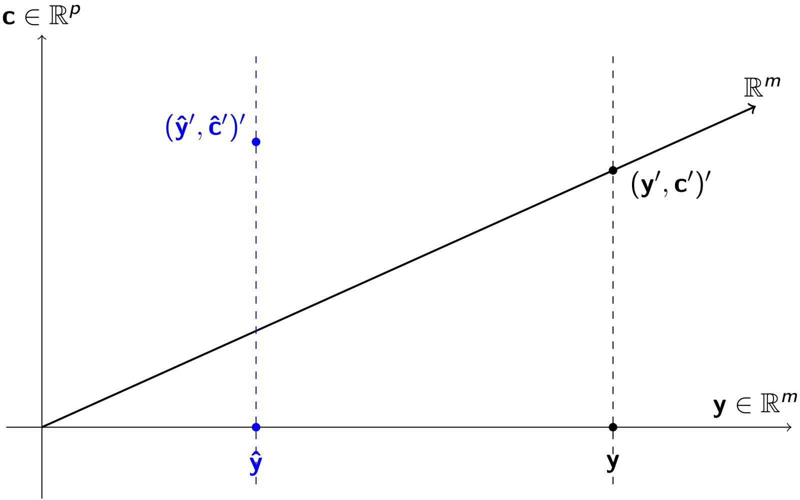
Geometry of FLAP



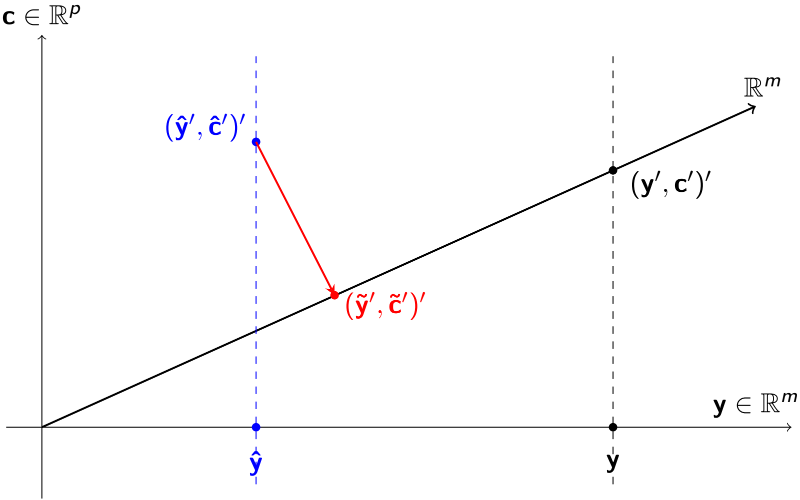
Geometry of FLAP



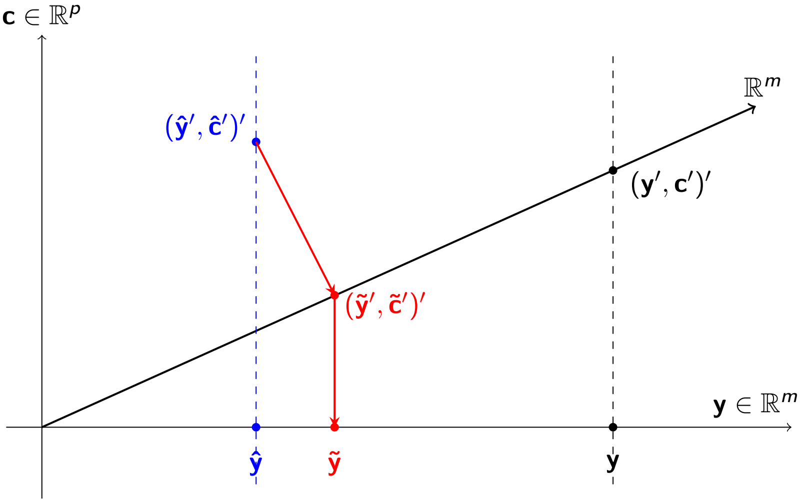
Geometry of FLAP



Geometry of FLAP



Geometry of FLAP



Key results based on MinT

1. The forecast error variance is **reduced** with FLAP
   - $\text{Var}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}) - \text{Var}(\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h})$ is also **positive semi-definite**.

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3. The forecast projection is **optimal** to achieve minimum forecast error variance for each series.

No free lunch

- In practice we need to estimate $\boldsymbol{W}_h = \text{Var}(\boldsymbol{z}_{t+h} - \hat{\boldsymbol{z}}_{t+h})$.
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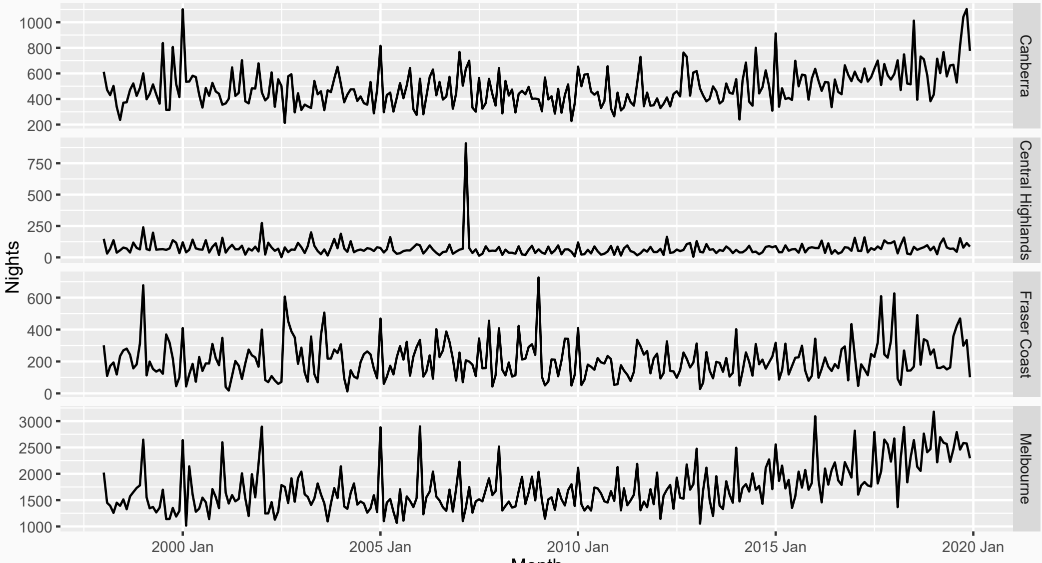
- Theory works with known $\boldsymbol{W}_h$.
- The quality of covariance matrix estimates deteriorate with higher dimension.
- However for finite dimension, the benefit of FLAP outweighs errors in estimating covariance matrix.

Monthly Australian regional tourism

- Monthly Australian tourism data by region giving 77 series, from Jan 1998 to Dec 2019
- Use expanding window time series cross-validation with $T = 84$ observations in first training set, and forecast horizons $h = 1, 2, \dots, 12$.
- Estimate `ets()` models using the `forecast` package.

Monthly Australian regional tourism



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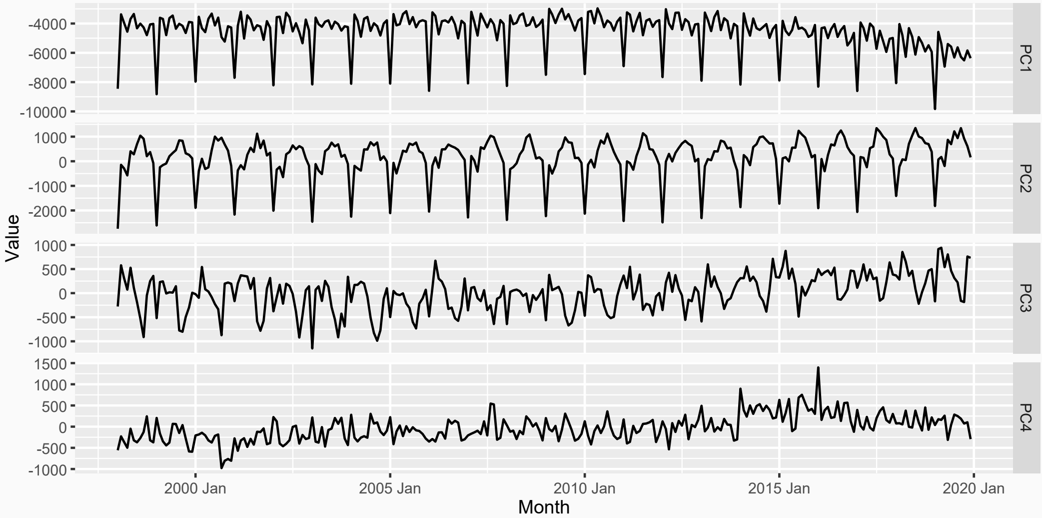
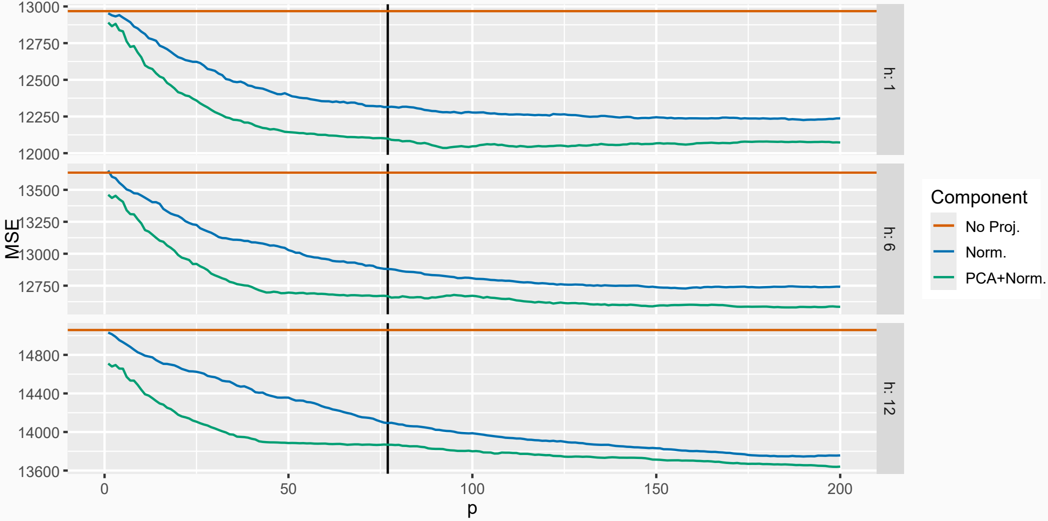


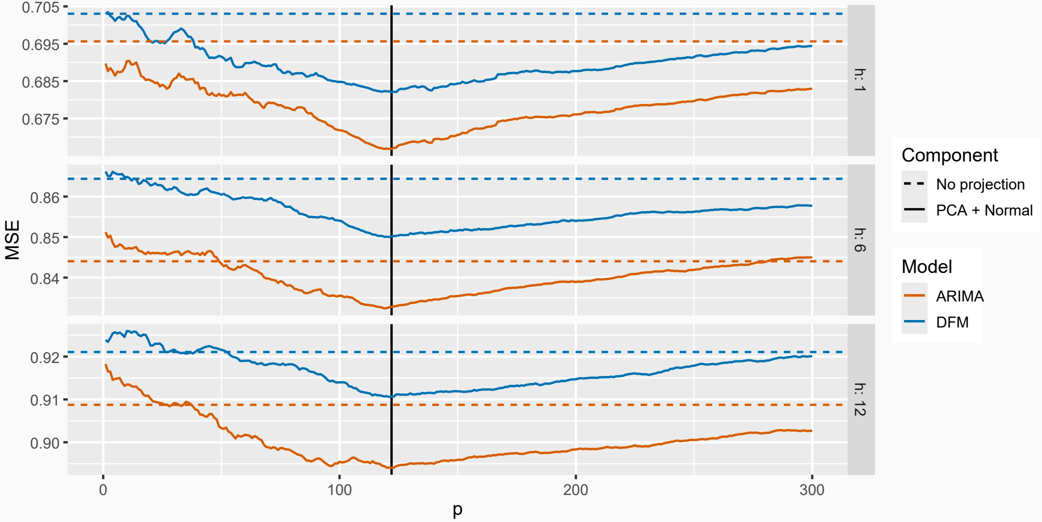
Monthly Australian regional tourism - `ets()`



FRED-MD

- Monthly data of macroeconomic variables (McCracken and Ng, 2016).
- Data from Jan 1959 – Sep 2023. 777 observations on 122 series.
- Same cleaning process as per McCracken and Ng (2016).
- All series scaled to have mean 0 and variance 1.
- Expanding time series cross-validation with initial size of 25 years and forecast horizon 12 months.

FRED-MD



0.705
0.695
0.685
0.675
h: 1
0.86
0.85
0.84
h: 6
0.92
0.91
0.90
h: 12
MSE
0
100
200
300
p
Component
No projection
PCA + Normal
Model
ARIMA
DFM

Working Paper and R Package

YF Yang, G Athanasopoulos, RJ Hyndman, and A Panagiotelis (2024). **"Forecast Linear Augmented Projection (FLAP): A free lunch to reduce forecast error variance".** Department of Econometrics and Business Statistics, Monash University, Working Paper Series 13/24. URL: https://www.monash.edu/business/ebs/research/publications/ebs/2024/wp13-2024.pdf

You can install the stable version from CRAN

```r
## CRAN.R-project.org/package=flap
install.packages("flap")
```

or the development version from Github

```r
## github.com/FinYang/flap
# install.packages("remotes")
remotes::install_github("FinYang/flap")
```

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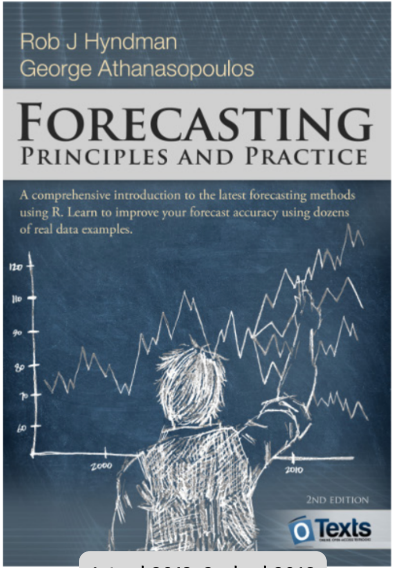
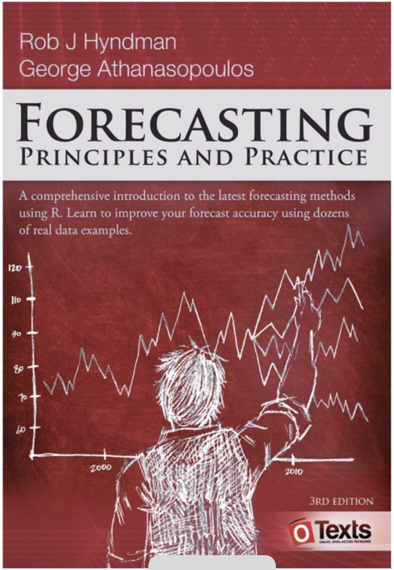
Forecasting: Principles and Practice

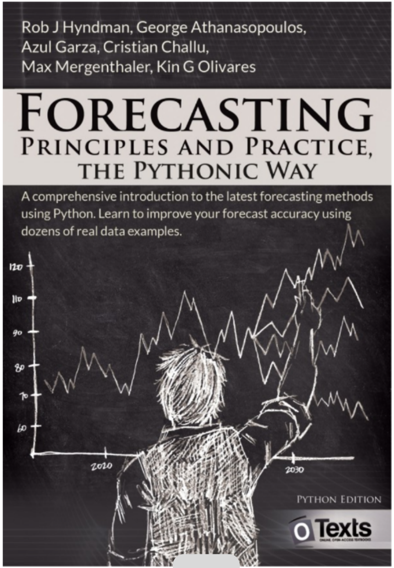
OTexts.com



1st ed 2013; 2nd ed 2018



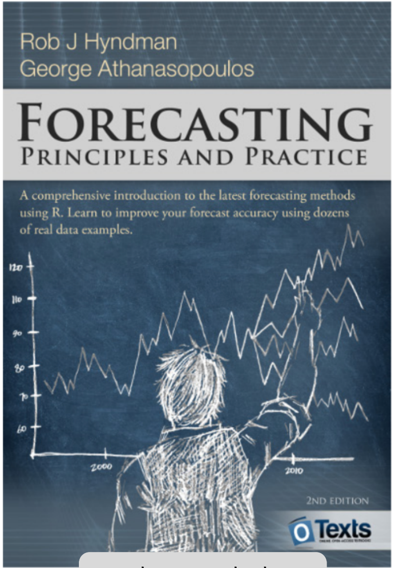
3rd ed 2021



2025

Forecasting: Principles and Practice

OTexts.com



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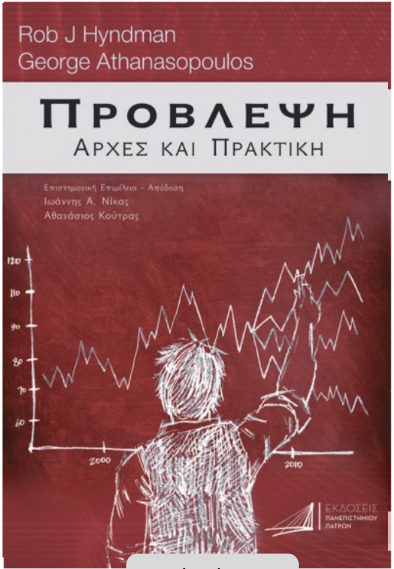
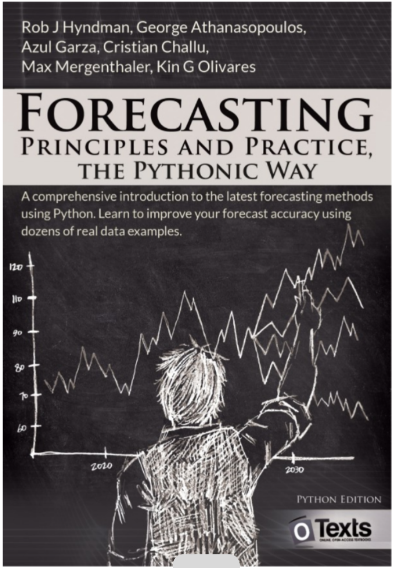


Greek ed 2024



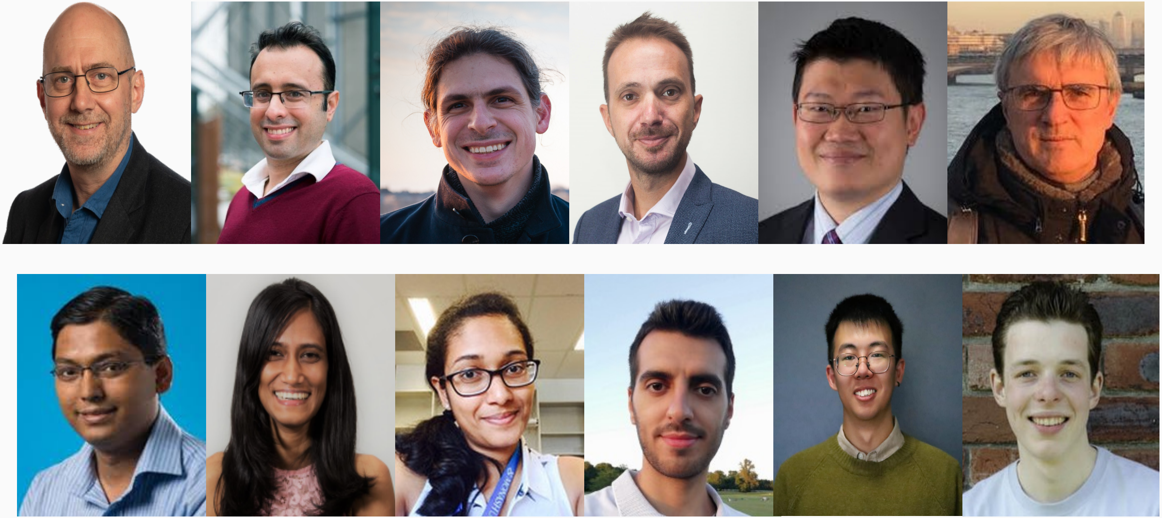
2025

Links

Postdoc opportunity

Link to slides

Thank you to



Other information

Forecasting: Principles and Practice https://otexts.com/fppgr/
(Thank you to Ioannis Nikas and Athanasios Koutras)

Monash webpage
https://research.monash.edu/en/persons/george-athanasopoulos

Thank you!

Athanasopoulos, G, R Ahmed, and R Hyndman (2009). **“Hierarchical forecasts for Australian domestic tourism”.** In: *International Journal of Forecasting* 25, pp. 146–166. URL: https://doi.org/10.1016/j.ijforecast.2008.07.004.

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Hyndman, RJ, R Ahmed, G Athanasopoulos, and H Shang (2011). **“Optimal combination forecasts for hierarchical time series”.** In: *Computational Statistics & Data Analysis* 55.9, pp. 2579–2589. URL: https://doi.org/10.1016/j.csda.2011.03.006.

Hyndman, RJ and G Athanasopoulos (2021). ***Forecasting: principles and practice.*** 3nd Edn. Melbourne, Australia. URL: http://otexts.com/fpp3/.

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