

# **ETC3550/ETC5550**

## **Applied forecasting**

Ch8. Exponential smoothing

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# ETS models

**General notation**    E T S : ExponenTial Smoothing



Error   **T**rend   **S**eason

**Error:** Additive ("A") or multiplicative ("M")

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**Seasonality:** None ("N"), additive ("A") or multiplicative ("M")

# ETS(A,N,N): SES with additive errors

## ETS(A,N,N) model

Observation equation

$$y_t = l_{t-1} + \varepsilon_t$$

State equation

$$l_t = l_{t-1} + \alpha \varepsilon_t$$

where  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

- “innovations” or “single source of error” because equations have the same error process,  $\varepsilon_t$ .
- Measurement equation: relationship between observations and states.
- State equation(s): evolution of the state(s) through time.

# ETS(A,A,N)

Holt's methods method with additive errors.

Forecast equation	$\hat{y}_{t+h t} = \ell_t + hb_t$
Observation equation	$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$
State equations	$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$ $b_t = b_{t-1} + \beta\varepsilon_t$

- Forecast errors:  $\varepsilon_t = y_t - \hat{y}_{t|t-1}$

Holt-Winters additive method with additive errors.

Forecast equation	$\hat{y}_{t+h t} = \ell_t + hb_t + s_{t+h-m(k+1)}$
Observation equation	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$
State equations	$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$
	$b_t = b_{t-1} + \beta\varepsilon_t$
	$s_t = s_{t-m} + \gamma\varepsilon_t$

- Forecast errors:  $\varepsilon_t = y_t - \hat{y}_{t|t-1}$
- $k$  is integer part of  $(h - 1)/m$ .

# ETS(M,A,M)

Holt-Winters multiplicative method with multiplicative errors.

Forecast equation	$\hat{y}_{t+h t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$
Observation equation	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$
State equations	$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

- Forecast errors:  $\varepsilon_t = (y_t - \hat{y}_{t|t-1})/\hat{y}_{t|t-1}$
- $k$  is integer part of  $(h - 1)/m$ .



# ETS model specification

```
ETS(y ~ error("A") + trend("N") + season("N"))
```

By default, optimal values for  $\alpha$ ,  $\beta$ ,  $\gamma$ , and the states at time 0 are used.

The values for  $\alpha$ ,  $\beta$  and  $\gamma$  can be specified:

```
trend("A", alpha = 0.5, beta = 0.2)
trend("A", alpha_range = c(0.2, 0.8), beta_range = c(0.1, 0.4))
season("M", gamma = 0.04)
season("M", gamma_range = c(0, 0.3))
```

# Exponential smoothing methods

		Seasonal Component		
		N	A	M
Trend Component		(None)	(Additive)	(Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A <sub>d</sub>	(Additive damped)	(A <sub>d</sub> ,N)	(A <sub>d</sub> ,A)	(A <sub>d</sub> ,M)

**(N,N):** Simple exponential smoothing

**(A,N):** Holt's linear method

**(A<sub>d</sub>,N):** Additive damped trend method

**(A,A):** Additive Holt-Winters' method

**(A,M):** Multiplicative Holt-Winters' method

**(A<sub>d</sub>,M):** Damped multiplicative Holt-Winters' method

# Exponential smoothing methods

		Seasonal Component		
		N	A	M
Trend Component		(None)	(Additive)	(Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A <sub>d</sub>	(Additive damped)	(A <sub>d</sub> ,N)	(A <sub>d</sub> ,A)	(A <sub>d</sub> ,M)

**(N,N):** Simple exponential smoothing

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**(A<sub>d</sub>,M):** Damped multiplicative Holt-Winters'

There are also multiplicative trend methods (not recommended).

# ETS models

## Additive Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
A	(Additive)	A,A,N	A,A,A	A,A,M
A <sub>d</sub>	(Additive damped)	A,A <sub>d</sub> ,N	A,A <sub>d</sub> ,A	A,A <sub>d</sub> ,M

## Multiplicative Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A <sub>d</sub>	(Additive damped)	M,A <sub>d</sub> ,N	M,A <sub>d</sub> ,A	M,A <sub>d</sub> ,M

# ETS models

## Additive Error

Trend Component		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
N	(None)	A,N,N	A,N,A	<del>A,N,M</del>
A	(Additive)	A,A,N	A,A,A	<del>A,A,M</del>
A <sub>d</sub>	(Additive damped)	A,A <sub>d</sub> ,N	A,A <sub>d</sub> ,A	<del>A,A<sub>d</sub>,M</del>

## Multiplicative Error

Trend Component		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A <sub>d</sub>	(Additive damped)	M,A <sub>d</sub> ,N	M,A <sub>d</sub> ,A	M,A <sub>d</sub> ,M

# AIC and cross-validation

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

# Automatic forecasting

## From Hyndman et al. (IJF, 2002):

- Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE (or some other criterion).
- Select best method using AICc:
- Produce forecasts using best method.
- Obtain forecast intervals using underlying state space model.

Method performed very well in M3 competition.

# Residuals

## Response residuals

$$\hat{e}_t = y_t - \hat{y}_{t|t-1}$$

## Innovation residuals

Additive error model:

$$\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1}$$

Multiplicative error model:

$$\hat{\varepsilon}_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}}$$