EX4: ORBITS

George Bemrose
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INTRODUCTION

For this exercise, the 4th order Runge-Kutta approach was used to first simulate the orbit of a rocket around a planet and then simulate a rocket orbiting the moon to take a photograph of its surface. The solution required the code to output graphs of the trajectory and the energy of the object. The problem was kept to two dimensions for simplicity.

THEORY

In the previous exercise, Euler's method for solving ordinary differential equations was used. For this exercise, the 4th order Runge-Kutta method is used.
Compared to Euler's method, the Runge-Kutta method performs more evaluations at each step, which increases the accuracy without decreasing the efficiency of the calculation. The Euler method would often overshoot the calculation of gradients, the modified method with a mid step reduced this overshoot. The fourth order Runge-Kutta method evaluates the differential four times every step, allowing for higher values of step sizes to be used.

$$k_1 = f(x_n, y_n) \tag{1}$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}\right) \tag{2}$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}\right) \tag{3}$$

$$k_4 = f\left(x_n + h, y_n + hk_3\right). \tag{4}$$

Equations, 1, 2, 3 and 4 are all general coefficients for the fourth order Runge-Kutta method. Where, $k_{1\rightarrow4}$ are the coefficients, f is the function, x_n is the x value for that step, y_n is the y value for that step and h is the step size.

$$y_{n+1} = y_n + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]. \tag{5}$$

Equation 5 is the general solution for the fourth order Runge-Kutta method.

PART A: PLANETARY ORBITS

Introduction

In this part of the exercise, the problem was to plot the trajectory and energy of a rocket orbiting a planet. A mass of $10,000kg$ was given to the rocket, but this only scales the energy in this section so it is arbitrary.

Theory

Four separate equations had to be calculated with the Runge-Kutta method for part a. The equations came from velocity and acceleration having to be calculated in both dimensions. Simply for velocity,

$$v_x = \frac{dx}{dt}. \tag{6}$$

Where v_x is velocity and t is time. For the y dimension, x is replaced with y.
To calculate the acceleration of the rocket, the following equation is used,

$$a_x = -\frac{GMx}{\sqrt{x^2 + y^2}}. \tag{7}$$

Where a_x is the acceleration in the x direction, G is the gravitational constant, M is the mass of the planet to orbit and $\sqrt{x^2 + y^2}$ is the radius from the Earth. For the y dimension, the x on the numerator is replaced with y with the denominator being left unchanged.

To calculate the kinetic energy of the rocket, the following equation was used,

$$KE = \frac{1}{2}m(v_x^2 + v_y^2). \tag{8}$$

Where m is the mass of the rocket. To calculate the potential energy of the rocket, the following equation was used,

$$PE = -\frac{GMm}{\sqrt{x^2 + y^2}}. \tag{9}$$

The total energy is just the addition of the kinetic and potential energies. For a true circular orbit, with constant kinetic and potential energies, the exact speed had to be calculated. The equation used was,

$$v = \sqrt{\frac{GM}{r}}. \tag{10}$$

Where r is the radius of the rocket from the centre of the planet. A general Runge-Kutta method function was made which returned lists of values to be plotted. This general function worked for both part a and part b. Several other functions such as a plot function and a reset function were made to reduce the length of the code and improve the readability. Conditional if statements were used to test for collisions.

Results

In part a, the user has an option to select three different planets to orbit around and produce graphs of. Figure 1 shows

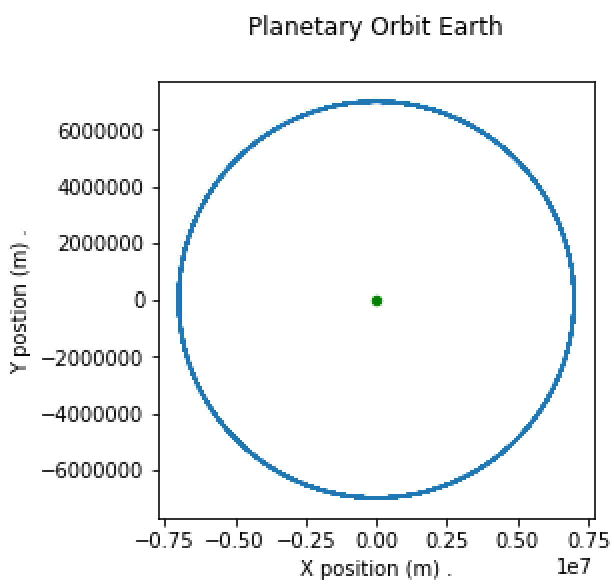

FIG. 1. Trajectory plot of the orbit of a rocket around Earth, with radius $7x10^6 m$, velocity of $7546 m/s$ and time step of 1.

the trajectory of the rocket around Earth. The two other planetary orbit graphs look the same. Equations 8 and 9 were used to create Figure 2, which shows the energies of the rocket. Figure 2 shows the energies being constant for the circular orbit, this is what is expected. By plotting the total energy with different step size values, the true conservation of energy can be seen.

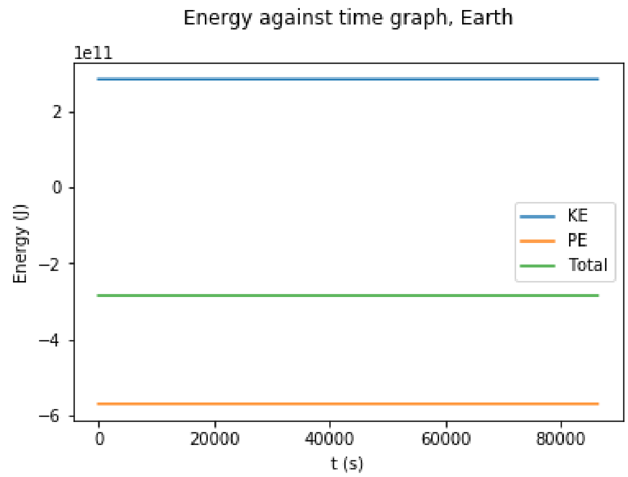

FIG. 2. Energies of the rocket orbiting around Earth, with radius $7x10^6 m$, velocity of $7546 m/s$ and time step of 1.

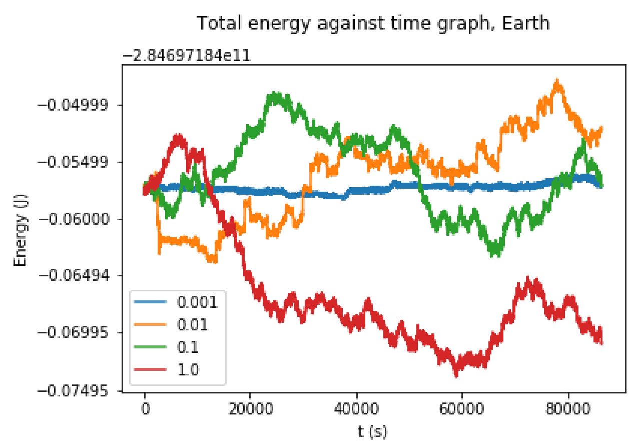

FIG. 3. Total energy of the rocket orbiting around Earth, with radius $7x10^6 m$, velocity of $7546 m/s$ and varying time step shown in the legend.

Figure 3 shows the variation of the total energy over time for multiple values of step size. For the step sizes, $h = 1.0$ the energy fluctuates approximately by $1x10^9 J$, $h = 0.1$ and $h = 0.01$ the energy fluctuates approximately by $5x10^8 J$ and $h = 0.001$ seems to not fluctuate. But if the scale was changed, $h = 0.001$ would show fluctuation visibly. For h being ,0.1 and 0.01 the value of energy is $(1.20 \pm 0.05)x10^{10} J$, with the error being 4% of the value. I would consider a value of $h = 0.001$ to be precise enough for this application. For single orbits, the method is accurate but to calculate multiple orbits a higher precision method should be used.

PART B: FLIGHT TO THE MOON AND BACK

Introduction

For the second part of the exercise, the aim was to simulate the orbit of the rocket going to the moon and back from the Earth in order to take a photo. Similar to the first moon missions during the space race. A rocket of mass $10,000kg$ was used again. This section took a lot of trial and error with the initial velocity to ensure the rocket kept the same orbit every iteration.

Theory

For this section, the velocity equation did not change. The equation for acceleration of the rocket in the x dimension is,

$$a_x = -\frac{GMx}{(x^2+y^2)^{3/2}} - \frac{GMmx}{(x^2+(y_m-y)^2)^{3/2}}. \quad (11)$$

Where y_m is the distance from the Earth to the Moon. The following equation is for acceleration in the y dimension,

$$a_y = -\frac{Gmy}{(x^2+y^2)^{3/2}} + \frac{GMm(y_m-y)}{(x^2+(y_m-y)^2)^{-3/2}}. \quad (12)$$

These equations are for use when the Moon and the Earth are both at $x=0$, with them being a y distance of $384400x10^3m$ apart. The energy equations are the same as for part a.

Conditional if statements were used to test if the rocket collided with the Earth or the Moon. The same general function was used as in part a for the Runge-Kutta method.

Results

As stated before, a large amount of trial and error was involved in getting the right value to make a conserved complete orbit to the moon and back.

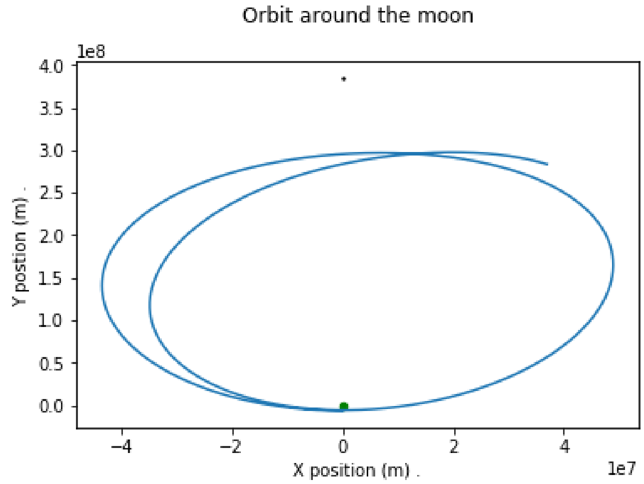

FIG. 4. One attempt at an orbit around the moon, with the initial velocity $= -10.9x10^3ms^{-1}$.

Figure 4 shows an attempt at getting an orbit around the moon. The changing of the initial velocity was repeated multiple times until an orbit was achieved.

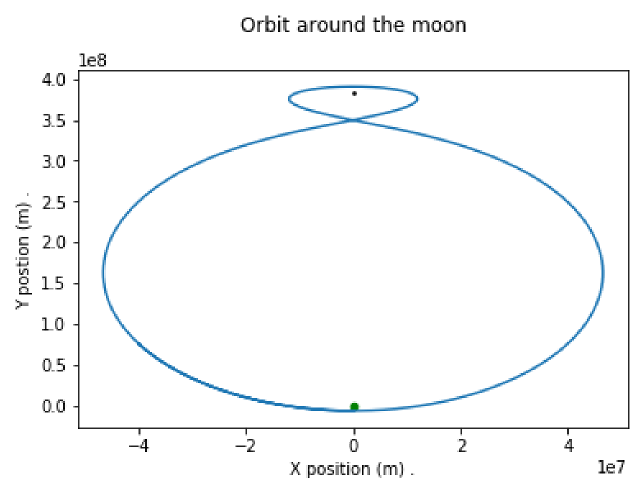

FIG. 5. Orbit to the Moon and back with conserved energy. The velocity used was $-10.91615x10^3m$. The time used was $1x10^6s$.

Figure 5 shows a complete orbit in which the orbit is repeatable. The time taken to go to the moon and back is around $8.5x10^5s$. The closest distance to the moon in this orbit was $4800km$, this is similar to that of the first moon missions.

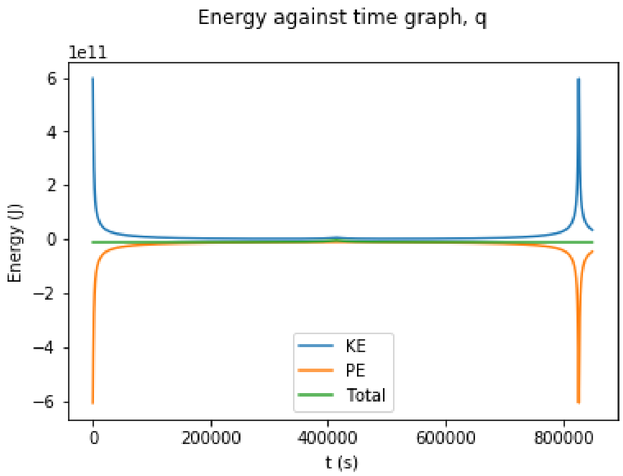

FIG. 6. Energy against time graph for a complete orbit to the Moon and back. The velocity used was $-10.91615x10^3m$. The time used was $1x10^6s$.

Figure 6 shows the energies of the rocket for the complete orbit from the Earth to the Moon and back. In the middle of the graph, around $4x10^5s$ there is a peak where the rocket speeds up around the Moon and is pulled in by its gravitational field. The calculations halt when the rocket crashes into the Moon or the Earth.