Combinatorics and Calculus
Dan Ciubotaru
Problem Sheet 3

There are two parts to each problem sheet. The first part is multiple-choice and you should turn it in for marking. The second part consists of more substantial questions that are not to be handed in, but their solutions will be cover in the classes.

1 Part 1

1. The series $\sum_{k=1}^{\infty}(\sqrt{k+1}-\sqrt{k})$

(A) converges?

(B) diverges?

2. Let p be a positive integer. The series $\sum_{k=1}^{\infty}\frac{1}{k(k+p)}$

(A) converges?

(B) diverges?

3. The sum $\sum_{k=1}^{\infty}\frac{\cos k}{2^k}$ equals

(A) $\frac{1}{2}$

(B) $\frac{2\cos 1-1}{5-4\cos 1}$

(C) $\frac{\sin^2 1}{5-4\cos 1}$.

2 Part 2

4. In this exercise, we give a proof that $a_n=(1+\frac{1}{n})^n$ is convergent:

(a) Use the binomial theorem to expand $(1+\frac{1}{n})^n$ and deduce that (a_n) is an increasing sequence.

(b) Prove by induction that $2^{n-1}\leq n!$ for all $n\geq 1$.

(c) Using the binomial theorem from (a), show that

$$(1+\frac{1}{n})^n<1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}.$$

Conclude that $(1+\frac{1}{n})^n<3$, so that the (a_n) is increasing and bounded, hence convergent.

(d) Show that for $n\geq 1$

$$0<e-\sum_{k=1}^{n}\frac{1}{k!}<\frac{1}{n!n}.$$

Deduce that e is irrational.

5. Recall Euler's formula

$$\frac{\sin x}{x}=\prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n\pi)^2}\right).$$

In the lectures, we saw how we can deduce from this and the Taylor series expansion of $\sin x$ that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. Use the same idea to determine just using $frac \sin x x$

$$\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

6. Let P be the probability that two positive integers chosen at random are coprime (meaning their greatest common divisor is 1). Prove that $P = \frac{6}{\pi^2}$.

[*Hint: Consider* P_N=*the probability that two randomly chosen integers in the set* $\{1, 2, \ldots, N\}$ *are coprime so that* $P = \lim_{N\to\infty} P_N$. *Think about the probability that the integers don't have a common prime* 2*, then a common prime* 3*, etc.*]

7. For each of the following statements, give a proof or a counterexample.

(a) If the series $\sum a_k$ with positive terms is divergent, then $\sum \frac{a_k}{a_k+1}$ is also divergent.

(b) Suppose $a_k > 0$ for all k. Let $s_k = a_1 + a_2 + \cdots + a_k$. Then $\sum a_k$ and $\sum \frac{a_k}{s_k}$ either both converge or both diverge.