Combinatorics and Calculus

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Problem Sheet 4

There are two parts to each problem sheet. The first part is multiple-choice and you should turn it in for marking. The second part consists of more substantial questions that are not to be handed in, but their solutions will be cover in the classes.

1 Part 1

1. Let $a, b > 0$. By using double integrals, compute the value of the integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx.$$

The result is:

(A) $\ln(\frac{b}{a})$

(B) 0

(C) $\frac{b-a}{ba}$.

2. Let $a \geq 0$. By applying double integrals to $e^{-yx}\sin x$ compute the integral

$$\int_0^\infty (1 - e^{-ax})\frac{\sin x}{x}\, dx.$$

The result is

(A) 0

(B) $\arcsin a$

(C) $\arctan a$.

3. Use the formulae for the Gamma and Beta integrals to determine the value of the integral

$$\int_0^\infty \frac{t^{x-1}}{1+t}\, dt$$

for $0 < x < 1$. The answer is

(A) π

(B) $\frac{\pi}{\sin(\pi x)}$

(C) $\Gamma(x)^2$.

2 Part 2

4. Let $a > 0$. Sketch the curve

$$x^{2/3} + y^{2/3} = a^{2/3}$$

in the quadrant $x, y > 0$. The variables u and v are given in terms of x and y by

$$x = u\cos^3 v, \quad y = u\sin^3 v.$$

What is the equation of the curve in terms of the new coordinates u and v?

Calculate the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ and hence find the area of the region bounded by the curve and the positive x- and y-axes.

5. Generalise the previous problem to compute the area of

$$x^k + y^k = a^k$$

in the quadrant $x, y > 0$, where $k > 0$ is a constant. The answer should be in terms of values of the Gamma function.

6. Stirling's formula gives an approximation of the factorial:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n .$$

(a) Using a computer/calculator, make a table of the factorials and Stirling's approximations for $n = 1, 2, \ldots, 10$ and of the ratio $\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n!}$. What do you notice?

(b) From Stirling's formula, deduce an approximation for the binomial coefficient $\binom{2n}{n}$.

(c) Find $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}}$.

7. Let $f(x)$ be a real-valued function defined on the interval (a, b). For $x_1 \neq x_2$ in (a, b) define the difference quotient

$$\varphi(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2},$$

and for each triple of distinct numbers x_1, x_2, x_3 define

$$\Psi(x_1, x_2, x_3) = \frac{\varphi(x_1, x_3) - \varphi(x_2, x_3)}{x_1 - x_2}.$$

Prove that if f is a continuous function, then f is convex if and only if $\Psi(x_1, x_2, x_3) \geq 0$ for all distinct x_1, x_2, x_3.

8. (if you need more practice with double integrals). The parabolic coordinates in the plane are

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv, \qquad u, v \geq 0.$$

Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$. Then use parabolic coordinates to calculate the area in the upper half-plane bounded by the curves

$$2x = 1 - y^2, \quad 2x = y^2 - 1, \quad 8x = 16 - y^2, \quad 8x = y^2 - 16.$$