Combinatorics and Calculus
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Problem Sheet 1

There are two parts to each problem sheet. The first part is multiple-choice and you should turn it in for marking. The second part consists of more substantial questions that are not to be handed in, but their solutions will be cover in the classes.

1 Part 1

1. Consider n lines in the plane "in general position", that is no two lines are parallel and no three lines go through the same point. The number of regions cut by the lines is

(A) n^2

(B) $\frac{n(n+1)}{2} + 1$

(C) $\frac{n(n+1)}{2}$.

2. Consider n planes "in general position", that is no two planes are parallel, no three planes go through the same line, and no four planes go through the same point. The number of 3-dimensional regions cut by the 4 planes is

(A) 13

(B) 14

(C) 15

(D) 16.

(To discuss in class: what is the general formula?)

3. The local extrema of the Bernoulli polynomial $B_3(x)$ of degree 3 occur at:

(A) $\frac{1}{3}$ and $\frac{2}{3}$

(B) $\frac{1}{2}$

(C) $\frac{\sqrt{3}\pm 1}{2\sqrt{3}}$

(D) $\frac{\sqrt{3}\pm 1}{4}$.

2 Part 2

4. Find simple formulas for the following sums:

(a) $\binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k}$ for $0 \leq k \leq n$, $n \geq 1$.

(b) $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$.

(c) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^k\binom{n}{k}$, for $0 < k \leq n$, $n \geq 1$.

(d) $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2$.

[*Hint: You may want to use the Binomial Theorem for the harder formulas. For example, for (d), start with*

$$(1+x)^n \cdot (1+x)^n = (1+x)^{2n}.]$$

5. Let (f_n) be the Fibonacci sequence: $f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$, $n \geq 1$. Prove by induction that

$$f_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots.$$

What is the interpretation of this formula in terms of the Pascal triangle?

6. Let S_k denote the sum of the k-th powers of the first n natural numbers

$$S_k = 1^k + 2^k + 3^k + \cdots + n^k.$$

(a) Using the formulas for S_1 and S_2, find the formula for S_3, via the binomial theorem method from the lecture. (In the lecture, we found S_4 by applying the binomial theorem to compute $(i+1)^5 - i^5$ and then sum over i; do the similar procedure but for the easier case of S_3 and $(i+1)^4 - i^4$.)

(b) Show that

$$\begin{aligned} n &= S_0 \\ n^2 &= 2S_1 - S_0 \\ n^3 &= 3S_2 - 3S_1 + S_0 \\ n^4 &= 4S_3 - 6S_2 + 4S_1 - S_0. \end{aligned}$$

Guess the general rule for n^k. Can you prove it?

7. The *factorial powers* are the functions:

$$\begin{aligned} x^{\underline{0}} &= 1, \\ x^{\underline{1}} &= x, \\ x^{\underline{2}} &= x(x-1), \\ x^{\underline{3}} &= x(x-1)(x-2), \\ x^{\underline{4}} &= x(x-1)(x-2)(x-3), \text{etc.} \end{aligned}$$

Here $x^{\underline{k}}$ is just a notation for the corresponding k-th factorial power.

(a) Prove the following analogue of the binomial theorem (for example, by induction):

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}.$$

[*Warning: This does not imply that* $(x+1)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}}$*, do you see why?*]

(b) For a function $f(x)$, define the *difference operator* Δ to be:

$$\Delta f(x) = f(x+1) - f(x).$$

Prove that $\Delta x^{\underline{n}} = n x^{\underline{n-1}}$, for all $n \geq 1$.

(c) Find a simple formula for the sum

$$1^{\underline{k}} + 2^{\underline{k}} + \cdots + n^{\underline{k}},$$

where k is a positive integer. Compare the result with

$$\int_0^{n+1} x^k \, dx.$$