# Combinatorics and Calculus

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**Problem Sheet 2**

There are two parts to each problem sheet. The first part is multiple-choice and you should turn it in for marking. The second part consists of more substantial questions that are not to be handed in, but their solutions will be cover in the classes.

## 1 Part 1

**1.** Do the following series converge?

(i) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

(A) Yes

(B) No

(ii) $\sum_{n=1}^{\infty} n \sin^2(\frac{1}{n})$

(A) Yes

(B) No

(iii) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

(A) Yes

(B) No

(iv) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

(A) Yes

(B) No

**2.** A correct infinite expansion for $\pi$ is ?

(A) $\pi = 2(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots)$

(B) $\pi = 4(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots)$

(C) $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots)$

**3.** The degree 3 Taylor polynomial around 0 for the function $\ln(1 + 2x)$ is?

(A) $x - \frac{x^2}{2} + \frac{x^3}{3}$

(B) $2x - 2x^2 + \frac{4}{3}x^3$

(C) $2x - 2x^2 + \frac{8}{3}x^3$.

## 2 Part 2

**4.** Recall the Bernoulli polynomials $\mathbf{B}_n(x)$ and the Bernoulli numbers $B_n$.

(a) Prove that $\mathbf{B}_n(1 - x) = (-1)^n \mathbf{B}_n(x)$.

(b) Prove that $B_{2n+1} = 0$ for all $n > 1$.

(c) Investigate the patterns of signs of $\mathbf{B}_n(x)$ for $0 \leq x \leq 1$. In particular, prove that the signs of $B_{2n}$ and $B_{2n+2}$ are opposite for $n > 0$.

**5.** Let $(f_n)$ be the Fibonacci sequence: $f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$, $n \geq 1$. Set $\tau_n = \frac{f_{n+1}}{f_n}$, $n \geq 1$. Prove that

$$\lim_{n\to\infty} \tau_n = \frac{1+\sqrt{5}}{2}.$$

[*Hint: Show that* $\tau_1 < \tau_3 < \tau_5 < \dots$ *and* $\tau_2 > \tau_4 > \tau_6 > \dots$ *and* $\tau_1 < \tau_2$, $\tau_2 > \tau_3$, $\tau_3 < \tau_4$ *etc. Deduce that the limit of* $(\tau_n)$ *exists and then use the recursive formula to find it.*]

**6.** Prove the expansion

$$\tan x = \sum_{n=1}^{\infty} (-1)^{n-1} 4^n (4^n - 1) B_{2n} \frac{x^{2n-1}}{(2n)!}.$$

Can you deduce any new information about $B_{2n}$ from this expansion of $\tan x$?

[*Hint:* Use the generating function for the Bernoulli numbers

$$\frac{x}{e^x - 1} = \sum_{n=2}^{\infty} B_n \frac{x^n}{n!}$$

and complex numbers to first prove the expansion

$$x \cot x = \sum_{n=0}^{\infty} (-4)^n B_{2n} \frac{x^{2n}}{(2n)!}.]$$

**7.** The $n$-th harmonic number $H_n$ is defined by

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}, \quad n = 1, 2, 3, \dots.$$

Show that the generating function for the sequence $\{H_n\}$ is given by

$$\sum_{n=1}^{\infty} H_n x^n = \frac{1}{1-x} \ln\left(\frac{1}{1-x}\right).$$

**8.** A partition of a number $n$ is a representation of $n$ as a sum of any number of positive integers. The order of the terms is irrelevant. For example, the five partitions of the number 4 are

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

How many partitions of 6 are there?

(a) If $p(n)$ denotes the number of partitions of $n$, show that the generating function for $\{p(n)\}$ is given by

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = 1 + \sum_{n=1}^{\infty} p(n) x^n.$$

(b) Show that the generating function for the number of partitions of $n$ into *distinct* positive integers is

$$(1+x)(1+x^2)(1+x^3)\cdots.$$

**9.** Functions defined by power series are often useful for solving differential equations. For example, for any real numbers $a, b, c$, the *hypergeometric series/function* is defined to be

$$F(a, b; c; x) = 1 + \frac{a \cdot b}{c} \frac{x}{1!} + \frac{a(a+1) \cdot b(b+1)}{c(c+1)} \frac{x^2}{2!} + \dots.$$

For example,

$$F(1, b; b; x) = 1 + x + x^2 + \dots \text{ is the usual geometric series.}$$

Show that:

(a) the radius of convergence of the hypergeometric series is $R = 1$ (for all $a, b, c$).

(b) $F(1, 1; 2; -x) = \frac{\ln(1+x)}{x}$.

(c) $F(a, b; b; x) = (1-x)^{-a}$. [*Hint: think about the generalized binomial theorem.*]

(d) $F(a, b; c; x)$ is a solution of Gauss' hypergeometric differential equation

$$x(1-x)F'' + (c - (a+b+1)x)\, F' - abF = 0.$$