Combinatorics and Calculus

Dan Ciubotaru

Problem Sheet 5

1. Use the formulae learned in the course involving the Gamma and Beta functions to deduce the following identities:

(1) $\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{\Gamma(1/4)^2}{\sqrt{32\pi}}$.

(2) $\int_0^1 \frac{dt}{\sqrt{1-t^3}} = \frac{\Gamma(1/3)^3}{\sqrt{3}\sqrt[3]{16\pi}}$.

2. Compute the probability p_n of getting exactly n heads and n tails when flipping a coin $2n$ times. Use Stirling's formula to give an approximation of p_n. Then apply this to the random walk problem: think of the grid of integers $\mathbb{Z}$ on the real line. Start at 0 and at each step, with equal probability 1/2 move one unit to the left or one unit to the right. What is the probability that the random walk returns to the origin infinitely many times?

You can think of a grid $\mathbb{Z}^2$ in two dimensions where the random walk consists of steps front, back, left, right, all with equal probability and ask the same question. More generally one can study the similar random walk in d dimensions. This is a topic that you could explore on your own.

3. Prove that the volume of the unit ball B_n in $\mathbb{R}^n$ equals

$$V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}.$$

Use Stirling's formula to estimate and discuss how the volume changes as $n \to \infty$.

[*Hint:* For the multiple integral computation of the volume you could proceed by proving that $V_n = \frac{2\pi}{n} V_{n-2}$, by choosing two coordinates x_1, x_2 and writing them in polar coordinates, and taking the slice for a fixed (x_1, x_2) for the other $n-2$ coordinates.]

4. What is the number of digits of $n!$ where N is a natural number? Notice that the number of digits of $n!$ equals $\lfloor \log_{10}(n!) \rfloor + 1$, and use Stirling's formula to show that this number lies between $\lfloor M \rfloor$ and $\lfloor M \rfloor + 1$, where

$$M = \frac{(n+1/2)\log n - n + \frac{1}{2}\log(2\pi)}{\log 10}.$$

Deduce that 100! has 157 digits. How many digits does 1000! have.

5. Starting with the formula for $\zeta(2k)$ in terms of the Bernoulli number B_{2k}, use Stirling's formula to estimate the growth of $|B_{2k}|$ as $k \to \infty$. Using this, calculate

$$\lim_{k\to\infty} \frac{1}{k^2} \left| \frac{B_{2k+2}}{B_{2k}} \right|.$$