Special functions : Lecture 2

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Bernoulli polynomials

Defn: The sequence of polynomials $B_n(x)$, $n=0,1,2,\dots$ defined by:

(1) $B_0(x) = 1$.

(2) $B_n(x)' = n\, B_{n-1}(x)$, $\quad n \geq 1$.

(3) $\int_0^1 B_n(x) = 0$, $\quad n \geq 1$.

Notice 1) that conditions (1), (2), (3) uniquely determine the set of polynomials.

2) The monomials $1, x, x^2, x^3, \dots, x^n, \dots$ also satisfy (1) and (2), but not (3). Instead $1, x, x^2, \dots, x^n$ is the unique sequence of polynomials $P_n(x)$ satisfying

(1), (2) and

(3') $P_n(0) = 0$ for all $n \geq 1$.

Lemma For every x,

$$\boxed{B_{n+1}(x+1) - B_{n+1}(x) = (n+1)\, x^n, \quad n \geq 0.}$$

Example: $B_1(x) = x - \frac{1}{2}$ $\qquad B_1(x+1) - B_1(x) = (x+\frac{1}{2}) - (x - \frac{1}{2})$
$= 1 = 1 \cdot B_0(x)$

$B_2(x) = x^2 - x + \frac{1}{6}$

$$B_2(x+1) - B_2(x) = (x+1)^2 - (x+1) + \frac{1}{6} - \left(x^2 - x + \frac{1}{6}\right)$$
$$= \cancel{x^2} + 2x \cancel{+1} \cancel{-x} \cancel{-1} \cancel{+\frac{1}{6}} \cancel{-x^2} \cancel{+x} \cancel{-\frac{1}{6}}$$

$= 2x$ ✓

<u>Proof</u> Denote $D_n(x) := \frac{1}{n+1}\left(B_{n+1}(x+1) - B_{n+1}(x)\right)$

Want to show $D_n(x) = x^n$, $n \geq 0$.

We'll check that $D_n(x)$ satisfies conditions (1), (2), (3') from before. Since $P_n(x) = x^n$ is the unique sequence of polynomials that satisfies (1), (2), (3') it follows that

$$D_n(x) = P_n(x) = x^n \quad \forall n.$$

(1) $D_0(x) \stackrel{?}{=} 1$.

$$D_0(x) = B_1(x+1) - B_1(x) = 1 \quad ✓$$

(2) $D_n(x)' \stackrel{?}{=} n\, D_{n-1}(x)$

$$D_n(x)' = \frac{1}{n+1}\left(B_{n+1}(x+1)' - B_{n+1}(x)'\right)$$

$$\underset{\substack{\text{(2) property}\\ \text{of Bernolli}}}{=} \frac{1}{n+1}\left((n+1)\, B_n(x+1) - (n+1)\, B_n(x)\right)$$

$$= B_n(x+1) - B_n(x) = n\, D_{n-1}(x).$$

(3') $D_n(0) \stackrel{?}{=} 0$, $n \geq 1$

$$D_n(0) = \frac{1}{n+1}\left(B_{n+1}(1) - B_{n+1}(0)\right) \underset{\text{Th. Calc.}}{\overset{\text{Fundamental}}{=}} \int_0^1 B_n(x)\,dx$$

b/c $\frac{1}{n+1} B_{n+1}(x)$ is an antiderivative for $B_n(x)$

$\Rightarrow D_n(0) = 0$ using property (3) of Bernoulli polys. $\square$

Application:

Theorem $S_k(n) = 1^k + 2^k + \dots + n^k = \int_0^{n+1} B_k(x)\,dx \quad k \geq 1.$

Proof Use Lemma:

$$B_{k+1}(x+1) - B_{k+1}(x) = (k+1)\,x^k \qquad k \geq 0.$$

Solve for x^k:

$$x^k = \frac{1}{k+1}\left(B_{k+1}(x+1) - B_{k+1}(x)\right)$$

Plug in $x = 1, 2, \dots, n$:

$$1^k = \frac{1}{k+1}\left(\cancel{B_{k+1}(2)} - B_{k+1}(1)\right)$$

$$2^k = \frac{1}{k+1}\left(\cancel{B_{k+1}(3)} - \cancel{B_{k+1}(2)}\right)$$

$$\vdots$$

$$n^k = \frac{1}{k+1}\left(B_{k+1}(n+1) - \cancel{B_{k+1}(n)}\right)$$

add

$$1^k + 2^k + \dots + n^k = \frac{1}{k+1}\left(B_{k+1}(n+1) - B_{k+1}(1)\right)$$

$$= \frac{1}{k+1}\left(B_{k+1}(n+1) - B_{k+1}(0)\right)$$

$B_{k+1}(1) = B_{k+1}(0)$, $k \geq 1$

$$= \int_0^{n+1} B_k(x)\,dx.$$

$$\underbrace{\int_0^1 B_k(x)\,dx}_{= 0} = \frac{1}{k+1}\left(B_{k+1}(1) - B_{k+1}(0)\right), \quad k \geq 1.$$

$B_n(x) =$ polynomial

Defn The n^{th} **Bernoulli number** B_n (= number) is

$$B_n = \mathbb{B}_n(0), \quad n = 0, 1, 2, \ldots$$

(free term of Bernoulli polynomial)

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30} \ldots$$

Lemma (The Bernoulli numbers determine the Bernoulli polynomials)

$$\mathbb{B}_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \cdot x^{n-k}.$$

Example: $$\mathbb{B}_3(x) = \binom{3}{0} \underbrace{B_0}_{=1} \cdot x^3 + \binom{3}{1} \underbrace{B_1}_{=-\frac{1}{2}} \cdot x^2 + \binom{3}{2} \underbrace{B_2}_{=\frac{1}{6}} \cdot x + \binom{3}{3} \underbrace{B_3}_{=0}$$

$$\mathbb{B}_3(x) = x^3 - \frac{3}{2} x^2 + \frac{1}{2} x$$

Proof (by induction) $\quad \mathbb{B}_0(x) = B_0 \cdot x^0 = 1$ ✓

Induction step: Suppose $\mathbb{B}_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \cdot x^{n-k}$

$$\mathbb{B}_{n+1}(x) = (n+1) \int \mathbb{B}_n(x)\, dx + \underbrace{B_{n+1}}_{\text{free term}}$$

$$= (n+1) \sum_{k=0}^{n} \binom{n}{k} B_k \frac{1}{n-k+1} x^{n-k+1} + B_{n+1}$$

$$= \sum_{k=0}^{n} B_k (n+1) \frac{n(n-1)\ldots\cancel{(n-k+1)}}{k!} \cdot \frac{1}{\cancel{n-k+1}} x^{n-k+1}$$

$$+ B_{n+1}$$

$$= \sum_{k=0}^{n} B_k \cdot \binom{n+1}{k} x^{n+1-k} + B_{n+1}$$

$$= \sum_{k=0}^{n+1} B_k \cdot \binom{n+1}{k} x^{n+1-k}$$ exactly the formula for $n+1$.

How do we compute the Bernoulli numbers?

In the Lemma, write it for $n+1$:

$$B_{n+1}(x) = \sum_{k=0}^{n+1} B_k \binom{n+1}{k} x^{n+1-k}.$$ Specialise $x = 1$

$$\underbrace{B_{n+1}(1)}_{\parallel} = \sum_{k=0}^{n+1} B_k \cdot \binom{n+1}{k}$$

$B_{n+1}(0) = B_{n+1}$

$n \geq 1$

$$B_{n+1} = \sum_{k=0}^{n+1} B_k \cdot \binom{n+1}{k}$$

$$B_{n+1} = \sum_{k=0}^{n} B_k \cdot \binom{n+1}{k} + B_{n+1}$$

Hence $$\sum_{k=0}^{n} B_k \cdot \binom{n+1}{k} = 0.$$ Solve for B_n:

<u>Formula</u>:

$$\boxed{B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k, \quad n \geq 1}$$

We can compute the Bernoulli numbers recursively.

<u>Example</u> $$B_4 = -\frac{1}{5} \sum_{k=0}^{3} \binom{5}{k} B_k$$

$B_0 = 1$
$B_1 = -\frac{1}{2}$
$B_2 = \frac{1}{6}, B_3 = 0$

$$B_4 = -\frac{1}{5}\Big(\underset{\binom{5}{0}}{\underset{\shortparallel}{1}} \cdot \underset{B_0}{\underset{\shortparallel}{1}} + 5\cdot\left(-\tfrac{1}{2}\right) + 10\cdot\tfrac{1}{6} + \cancel{10\cdot 0}\Big)$$

$$B_4 = -\frac{1}{5}\left(1 - \frac{5}{2} + \frac{10}{6}\right) = -\frac{1}{5}\cdot\frac{6-15+10}{6}$$

$$B_4 = -\frac{1}{30} \quad ✓$$

<u>Q</u> (open ended) Is there a faster way to compute Bernoulli numbers?

Important tool in combinatorics : <u>generating functions</u>

<u>Idea</u> : If you're given a sequence of numbers $a_0, a_1, a_2, \dots, a_n, \dots$, you can form the infinite series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

then try to find a simple formula for $f(x)$.
($f(x)$ is called <u>generating function</u> of the sequence).

<u>Example</u> <u>Fibonacci numbers</u>: $f_0, f_1, f_2, \dots, f_n, \dots$

defined recursively $\quad f_0 = f_1 = 1$

$$f_{n+1} = f_n + f_{n-1}, \quad n \geq 1.$$

$$f(x) = \sum_{n=0}^{\infty} f_n \cdot x^n$$

$$f(x) = f_0 + f_1\cdot x + f_2\cdot x^2 + f_3\cdot x^3 + \dots$$

$$x f(x) = \qquad f_0\cdot x + f_1\cdot x^2 + f_2\cdot x^3 + \dots$$

$$x^2 f(x) = \qquad\qquad f_0\cdot x^2 + f_1\cdot x^3 + \dots$$

$$(1-x-x^2)f(x) = f_0 + \underbrace{(f_1-f_0)}_{0}x + \underbrace{(f_2-f_1-f_0)}_{0}x^2 + \underbrace{(f_3-f_2-f_1)}_{0}x^3 + \dots$$

$$(1-x-x^2)f(x) = 1 \quad \text{so} \quad f(x) = \frac{1}{1-x-x^2}$$

This is the generating function of Fibonacci sequence.

$$\frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} f_n x^n$$

(this is formal now, but we'll see about convergence next time)

Theorem (Generating function for Bernoulli numbers)

$$\boxed{\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \cdot \frac{x^n}{n!}}$$

Proof Use the recursive formula for Bernoulli numbers.

Multiply both sides by $e^x - 1$. Want to show

$$x \stackrel{?}{=} (e^x-1)\cdot \text{RHS}.$$

Taylor series for $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$

$$x \stackrel{?}{=} \left(B_0 + B_1\cdot x + B_2\cdot\frac{x^2}{2!} + B_3\cdot\frac{x^3}{3!} + \dots\right)\left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)$$

Compare coefficients in both sides:

coeff of x: $\text{LHS} = 1$, $\text{RHS} = B_0 \cdot \frac{1}{1!} = 1$ ✓

coeff of x^n, $n > 1$, $\text{LHS} = 0$

RHS: contribution from x^i in the first bracket and x^{n-i} second bracket. Total contribution is:

$$\sum_{i=0}^{n-1} B_i \cdot \frac{1}{i!} \cdot \frac{1}{(n-i)!} = \frac{1}{n!} \sum_{i=0}^{n-1} B_i \binom{n}{i}$$

$= 0$ by the recursive formula for the Bernoulli numbers.

We proved:

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \cdot \frac{x^n}{n!} \qquad B_0 = 1 \quad B_1 = -\frac{1}{2}$$

$$1 - \frac{x}{2} + \sum_{n=2}^{\infty} B_n \cdot \frac{x^n}{n!} = \frac{x}{e^x-1}$$

$$1 + \sum_{n=2}^{\infty} B_n \cdot \frac{x^n}{n!} = \frac{x}{2} + \frac{x}{e^x-1} = \frac{x}{2}\left(1 + \frac{2}{e^x-1}\right)$$

$$= \frac{x}{2} \cdot \frac{e^x+1}{e^x-1}$$

Divide both numerator & denominator by $e^{x/2}$

$$\boxed{1 + \sum_{n=2}^{\infty} B_n \cdot \frac{x^n}{n!} = \frac{x}{2} \cdot \frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}}}$$

In RHS: change x to $-x$

$$-\frac{x}{2} \cdot \frac{e^{-x/2}+e^{x/2}}{e^{-x/2}-e^{x/2}} = \frac{x}{2} \cdot \frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}}$$

RHS is an <u>even function</u> $(g(x) = g(-x))$

$\Rightarrow$ LHS is even $\Rightarrow$ it can only have even powers.

$$\Rightarrow \boxed{B_n = 0 \text{ for all odd } n \geq 3.}$$

Then rewrite

$$\boxed{1 + \sum_{n=1}^{\infty} B_{2n} \frac{x^{2n}}{(2n)!} = \frac{x}{2} \coth\left(\frac{x}{2}\right)}$$

hyperbolic cotangent.

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

hyperbolic sin & cos

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

so $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

(Can also change x to $2x$ in the formula and get an expansion of $x \coth(x)$ in terms of Bernoulli numbers.)