Lecture 7 Special functions

10. November 2023

Dan Ciubotaru

Applications of Gamma function

1) Connection with Euler's Beta function (applications to interesting integrals)
2) Stirling's formula.

Beta function Recall the definition of Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \quad , \quad x>0$$

Substitution : $t = u^2 \qquad dt = 2u\, du$

$$\Gamma(x) = 2\int_0^\infty u^{2x-2} e^{-u^2} u\, du = 2\int_0^\infty u^{2x-1} e^{-u^2} du$$

$x, y > 0$

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty \left(u^{2x-1} e^{-u^2} du\right)\left(\int_0^\infty v^{2y-1} e^{-v^2} dv\right)$$

$$= 4\int_0^\infty \int_0^\infty u^{2x-1} v^{2y-1} e^{-(u^2+v^2)} du\, dv$$

Polar coordinates : $u = r\cos\theta, \quad v = r\sin\theta \qquad r>0$

$$0 \le \theta \le \frac{\pi}{2}$$

Jacobian : $du\, dv = r\, dr\, d\theta$.

$$\Gamma(x)\Gamma(y) = 4\int_0^{\pi/2}\int_0^\infty r^{2x-1}\cos^{2x-1}\theta\; r^{2y-1}\sin^{2y-1}\theta \cdot e^{-r^2}\, r\, dr\, d\theta.$$

$$= 4\int_0^{\pi/2}\int_0^\infty (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\, r^{2(x+y)-1} e^{-r^2} dr\, d\theta$$

$$\underset{\text{separate the variables.}}{=} 4 \int_0^{\pi/2} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta \underbrace{\boxed{\int_0^\infty r^{2(x+y)-1} e^{-r^2} dr}}_{\frac{1}{2}\Gamma(x+y)}$$

$$= 2\,\Gamma(x+y) \int_0^{\pi/2} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta.$$

Trigonometric integral:

$$\boxed{2\int_0^{\pi/2} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \quad x,y>0}$$

Particular case: Wallis' integrals W_n, $n\in\mathbb{N}$

$x = \frac{n+1}{2}$, $y=\frac{1}{2}$ or $x=\frac{1}{2}$, $y=\frac{n+1}{2}$

$$W_n = \int_0^{\pi/2} \cos^n\theta \, d\theta = \int_0^{\pi/2} \sin^n\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

We can explicitate this:

Case 1 $\quad n = 2k+1$ odd.

$$W_{2k+1} = \frac{1}{2} \frac{\Gamma(k+1)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)} = \frac{1}{2} \frac{k!\,\cancel{\Gamma\left(\frac{1}{2}\right)}}{\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)\cdots\frac{1}{2}\cdot\cancel{\Gamma\left(\frac{1}{2}\right)}}$$

$$\boxed{W_{2k+1} = \frac{1}{2}\cdot\frac{k!\,2^{k+1}}{1\cdot3\cdot5\cdots(2k+1)} = \frac{k!\,2^k}{1\cdot3\cdot5\cdots(2k+1)} \quad k\geq 0}$$

Case 2 $\quad n = 2k$ even

$$W_{2k} = \frac{1}{2} \frac{\Gamma(k+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(k+1)}$$

$$W_{2k} = \frac{1}{2} \cdot \frac{(k-\frac{1}{2})(k-\frac{3}{2})\cdots\frac{3}{2}\cdot\frac{1}{2}\,\underbrace{\Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2})}_{=\pi}}{k!}$$

$$\boxed{W_{2k} = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^{k+1}\cdot k!}\,\pi} \qquad k \geq 0.$$

Go back to the trigonometric integral:

$$2\int_0^{\pi/2} (\cos\theta)^{2x-1}\,(\sin\theta)^{2y-1}\,d\theta = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \quad x,y>0$$

In LHS: make substitution $t = \cos^2\theta$

$$dt = -2\cos\theta\sin\theta\,d\theta$$

if $\theta = 0 \qquad t = 1$

$\theta = \pi/2 \qquad t = 0$

$$\text{LHS} = \int_0^1 t^{x-1}\cdot(1-t)^{y-1}\underbrace{(-2\sin\theta\cos\theta\,d\theta)}_{dt}$$

$$(\cos\theta)^{2x-1} = (\cos\theta)^{2x-2}\cos\theta = t^{x-1}\cdot\cos\theta$$

$$(\sin\theta)^{2y-1} = (\sin\theta)^{2y-2}\sin\theta = (\sin^2\theta)^{y-1}\sin\theta$$

$$= (1-t)^{y-1}\sin\theta.$$

So the LHS is exactly <u>Euler's Beta function</u>

$$\boxed{B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt} \quad x,y>0$$

<u>Theorem</u> $\quad B(x,y) = \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \quad x,y>0$

<u>Easy properties</u>

1) $B(x,y) = B(y,x)$ (see from Theorem but also directly via the substitution $t \to 1-t$)

2) $B(\frac{1}{2},\frac{1}{2}) = \int_0^1 t^{-1/2}(1-t)^{-1/2}\,dt = \dfrac{\Gamma(\frac{1}{2})^2}{\Gamma(1)} = \pi$

3) If m,n are natural numbers:

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

There are other particular cases that appear in analysis / analytic number theory.

Specialise $x = \frac{m}{n} \in \mathbb{Q}$, $y = \frac{1}{2}$

$$B(\tfrac{m}{n},\tfrac{1}{2}) = \int_0^1 t^{\frac{m}{n}-1}(1-t)^{-\frac{1}{2}}\,dt$$

Substitution
$t = z^n$
$dt = nz^{n-1}dz$

$$= \int_0^1 (z^n)^{\frac{m}{n}-1}(1-z^n)^{-\frac{1}{2}}\,nz^{n-1}\,dz$$

$$B\left(\frac{m}{n},\frac{1}{2}\right) = n\int_0^1 \frac{z^{m-n}}{\sqrt{1-z^n}}\cdot z^{n-1}\,dz$$

"Elliptic Integral"

$$\int_0^1 \frac{z^{m-1}}{\sqrt{1-z^n}}\,dz = \frac{1}{n}B\left(\frac{m}{n},\frac{1}{2}\right)$$

$$= \frac{1}{n}\,\frac{\Gamma\left(\frac{m}{n}\right)\sqrt{\pi}}{\Gamma\left(\frac{m}{n}+\frac{1}{2}\right)}$$

Particular case $m=1$

$$\boxed{\int_0^1 \frac{dz}{\sqrt{1-z^n}} = \frac{\Gamma\left(\frac{1}{n}\right)\sqrt{\pi}}{n\,\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}, \quad n\in\mathbb{N}}$$

Q: What is the geometric meaning of the integral in the left hand side? (say arclength of some curve)

2 Stirling's formula

Starting point: how do we approximate the factorial $n!$ for n large.

Recall the sequences $\left(1+\frac{1}{k}\right)^k \to e$ $k\to\infty$ increasing

and $\left(1+\frac{1}{k}\right)^{k+1} \longrightarrow e$ decreasing.

$$\left(1+\frac{1}{k}\right)^k < e < \left(1+\frac{1}{k}\right)^{k+1} \quad \forall k\in\mathbb{N}$$

$k = 1, 2, 3, \dots, n-1$ multiply them.

$$\prod_{k=1}^{n-1} \left(\frac{k+1}{k}\right)^{k} < e^{n-1} < \prod_{k=1}^{n-1} \left(\frac{k+1}{k}\right)^{k+1}$$

LHS : $\frac{\cancel{2}^{1}}{1^{1}} \cdot \frac{\cancel{3}^{2}}{\cancel{2^{2}}} \cdot \frac{\cancel{4}^{3}}{\cancel{3^{3}}} \cdot \dots \cdot \frac{n^{n-1}}{\cancel{(n-1)^{n-1}}} = \frac{n^{n-1}}{(n-1)!}$

RHS : $\frac{\cancel{2}^{2}}{1^{2}} \cdot \frac{\cancel{3}^{3}}{\cancel{2^{3}}} \cdot \frac{\cancel{4}^{4}}{\cancel{3^{4}}} \cdots \frac{n^{n}}{\cancel{(n-1)^{n}}} = \frac{n^{n}}{(n-1)!}$

$$\frac{n^{n-1}}{(n-1)!} < e^{n-1} < \frac{n^{n}}{(n-1)!}$$

$$\Rightarrow n^{n-1} e^{-(n-1)} < (n-1)! < n^{n} e^{-(n-1)} \quad \Big| \cdot n$$

$$e\, n^{n} e^{-n} < n! < e \cdot n^{n+1} e^{-n}$$

$\Rightarrow$ growth of $n!$ is greater than $n^{n} e^{-n}$ but less then $n^{n+1} e^{-n}$.

In particular, $\underset{(n-1)!}{\overset{\Gamma(n)}{\shortparallel}}$ is between $n^{n-1} e^{-n}$ and $n^{n} e^{-n}$.

We want $\Gamma(x)$. $x > 0$

Take the function:

$$\boxed{f(x) = x^{x-\frac{1}{2}} e^{-x} e^{\mu(x)}}$$ where

$\mu(x)$ is a function that we need to determine.

such $f(x)$ behaves like $\Gamma(x)$.

Recall $\Gamma(x)$ is uniquely determined for $x>0$ by:

a) $\Gamma(1)=1$

① $\Gamma(x+1) = x\,\Gamma(x)$

② $\Gamma(x)$ is log convex.

We'd like $f(x)$ to satisfy ① and ② b/c then it must be a constant multiple of $\Gamma(x)$.

$$x=\frac{f(x+1)}{f(x)} = \frac{(x+1)^{x+\frac{1}{2}}\, e^{-x-1}\, e^{\mu(x+1)}}{x^{x-\frac{1}{2}}\, e^{-x}\, e^{\mu(x)}}$$

$$\cancel{x} = \cancel{x}\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} e^{-1}\, e^{\mu(x+1)-\mu(x)}$$ Take ln

$$\mu(x+1)-\mu(x) + \left(x+\tfrac{1}{2}\right)\ln\left(1+\tfrac{1}{x}\right) - 1 = 0$$

Need: $\mu(x)-\mu(x+1) = \underbrace{\left(x+\tfrac{1}{2}\right)\ln\left(1+\tfrac{1}{x}\right) - 1}_{g(x)}$

Formally, $\mu(x) = \sum_{n=0}^{\infty} g(x+n)$ <u>if</u> this infinite series converges.

Upper bound for $g(x)$?

Taylor series if $|y|<1$

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots$$

$$\ln(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} + \dots$$

difference

$$\ln\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots\right)$$

Set $y = \frac{1}{2x+1}$ $\quad (0 < y < 1 \text{ if } x > 0)$

$$\frac{1+\frac{1}{2x+1}}{1-\frac{1}{2x+1}} = \frac{2x+2}{2x} = 1 + \frac{1}{x}.$$

$$\frac{1}{2}\ln\left(1+\frac{1}{x}\right) = \frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \cdots \quad \Big|\,(2x+1)$$

$$\underbrace{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right) - 1}_{g(x)} = \frac{1}{(2x+1)^2}\left[\frac{1}{3} + \frac{1}{5(2x+1)^2} + \frac{1}{7(2x+1)^4} + \cdots\right]$$

$x > 0$

$$0 < g(x) < \frac{1}{3(2x+1)^2}\Bigg[\underbrace{1 + \frac{1}{(2x+1)^2} + \frac{1}{(2x+1)^4} + \cdots}_{\text{geom. series}}\Bigg]$$

$\left(\frac{1}{2k+1} \le \frac{1}{3},\ k \ge 1\right)$

$$0 < g(x) < \frac{1}{3(2x+1)^2} \cdot \frac{1}{1-\frac{1}{(2x+1)^2}} = \frac{1}{3}\cdot\frac{1}{(2x+1)^2-1}$$

$$= \frac{1}{3}\cdot\frac{1}{4x^2+4x} = \frac{1}{12}\left(\frac{1}{x} - \frac{1}{x+1}\right)$$

So for all $x > 0$ $\quad 0 < g(x) < \frac{1}{12}\left(\frac{1}{x} - \frac{1}{x+1}\right)$

$$\text{Then } 0 \le \sum_{n=0}^{\infty} g(x+n) \le \frac{1}{12}\underbrace{\sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{x+n+1}\right)}_{\text{telescoping series}}$$

$$= \frac{1}{12x}$$

This means that $\forall x > 0$, $\sum_{n=0}^{\infty} g(x+n)$ converges

so we can call the limit $\mu(x)$ and moreover

$$0 < \mu(x) < \frac{1}{12x} . \qquad \mu(x) = \sum_{n=0}^{\infty} g(x+n)$$

Usually we write $\boxed{\mu(x) = \frac{\theta}{12x}}$ where $0 < \theta < 1$ (depends on x).

Recall: $f(x) = x^{x-\frac{1}{2}} e^{-x} e^{\mu(x)}$ this

satisfies $f(x+1) = x f(x)$.

<u>Claim</u> $f(x)$ is log convex.

From last time: a product of log-convex functions is log-convex.

- $x^{x-\frac{1}{2}} e^{-x}$ is log convex. Take log:

$(x - \frac{1}{2}) \log x - x$ diff once

$(x - \frac{1}{2}) \cdot \frac{1}{x} + \log x - 1$ diff twice

$$\frac{1}{x} - (x - \tfrac{1}{2}) \cdot \frac{1}{x^2} + \frac{1}{x} = \cancel{\frac{1}{x}} - \cancel{\frac{1}{x}} + \frac{1}{2x^2} + \frac{1}{x} > 0 \text{ if } x > 0.$$

- $e^{\mu(x)}$ is log convex iff $\mu(x)$ is convex $= \sum_{n=0}^{\infty} g(x+n)$

If $g(x)$ is convex then $\mu(x)$ is (sum of convex)

$$g(x) = \left(x+\tfrac{1}{2}\right)\ln(x+1) - \left(x+\tfrac{1}{2}\right)\ln x - 1$$

$$g'(x) = \underline{\ln(x+1)} + \left(x+\tfrac{1}{2}\right)\frac{1}{x+1} - \underline{\ln x} - \left(x+\tfrac{1}{2}\right)\frac{1}{x}$$

$$g''(x) = \frac{1}{x+1} - \frac{1}{x} + \cdots \qquad \text{exercise } (>0) \text{ for } x>0.$$

In conclusion $\Gamma(x)$ is a constant times $f(x)$:

$$\Gamma(x) = \alpha f(x) = \alpha \cdot x^{x-\frac{1}{2}} e^{-x+\mu(x)}$$

$$= \alpha \cdot x^{x-\frac{1}{2}} e^{-x+\frac{\theta}{12x}}$$

What is the constant α? We'll find α and at the same time determine a multiplication formula for $\Gamma(x)$.

Let p be a positive integer. Consider the function

$$F(x) = p^x\, \Gamma\!\left(\frac{x}{p}\right)\Gamma\!\left(\frac{x+1}{p}\right)\cdots\Gamma\!\left(\frac{x+p-1}{p}\right) \qquad x>0$$

is log convex b/c each factor is.

So if $F(x+1) = x F(x)$ then $F(x)$ is a multiple of $\Gamma(x)$.

$$F(x+1) = p^{x+1}\, \Gamma\!\left(\frac{x+1}{p}\right)\Gamma\!\left(\frac{x+2}{p}\right)\cdots\Gamma\!\left(\frac{x+p-1}{p}\right)\underbrace{\Gamma\!\left(\frac{x}{p}+1\right)}_{=\frac{x}{p}\cdot\Gamma\left(\frac{x}{p}\right)}$$

$$F(x+1) = x\cdot \underbrace{p^x\, \Gamma\!\left(\frac{x}{p}\right)\Gamma\!\left(\frac{x+1}{p}\right)\cdots\Gamma\!\left(\frac{x+p-1}{p}\right)}_{F(x)}$$

so indeed $F(x+1) = x\,F(x)$.

$\Rightarrow$ there exists a constant a_p (it depends on p)

such that $F(x) = a_p \cdot \Gamma(x)$

$$a_p \cdot \Gamma(x) = p^x\, \Gamma\left(\frac{x}{p}\right) \Gamma\left(\frac{x+1}{p}\right) \cdots \Gamma\left(\frac{x+p-1}{p}\right)$$

for $x > 0$.

Take $x = 1$

$$a_p = p\, \Gamma\left(\frac{1}{p}\right) \Gamma\left(\frac{2}{p}\right) \cdots \Gamma\left(\frac{p-1}{p}\right).$$

We know that

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x\, n!}{x(x+1)(x+2)\cdots(x+n)}$$

$$\Gamma\left(\frac{k}{p}\right) = \lim_{n\to\infty} \frac{n^{k/p}\, n!\, p^{n+1}}{k(k+p)(k+2p)\cdots(k+np)}$$

for $k = 1, 2, \dots, p$

Multiply together:

$$a_p = p \lim_{n\to\infty} \frac{n^{\frac{p+1}{2}} (n!)^p\, p^{np+p}}{(np+p)!}$$

$$\underset{\text{Exercise}}{\overset{?}{=}} p \lim_{n\to\infty} \frac{(n!)^p\, n^{\frac{p+1}{2}}\, p^{np+p}}{(np)!\,(np)^p}$$

$$a_p = p \lim_{n\to\infty} \frac{(n!)^p\, p^{np}}{(np)!\, n^{\frac{p-1}{2}}}$$

Apply the approximation to $n! = \Gamma(n+1)$

$$n! = \alpha\, n^{n+\frac{1}{2}}\, e^{-n}\, e^{\theta_1/12n}$$ for some $0 < \theta_1 < 1$

$$(n!)^p = \alpha^p\, n^{pn+\frac{p}{2}}\, e^{-np}\, e^{\theta_1 p/12n}$$

$$(np)! = \alpha\, (np)^{np+\frac{1}{2}}\, e^{-np}\, e^{\theta_2/12np} \qquad 0 < \theta_2 < 1$$

Then $$a_p = p \lim_{n\to\infty} \frac{\alpha^p\, \cancel{n^{np+\frac{p}{2}}}\, \cancel{e^{-np}}\, e^{\frac{\theta_1 p}{12n}} \cdot \cancel{p^{np}}}{\alpha\, \cancel{(np)^{np+\frac{1}{2}}}\, \cancel{e^{-np}}\, e^{\theta_2/12np}\, \cancel{n^{\frac{p-1}{2}}}}$$

$$a_p = \sqrt{p}\, \alpha^{p-1} \lim_{n\to\infty} e^{\theta_1 p/12n - \theta_2/12np}$$

$p^{1/2}$

$e^0 = 1$

Finally!

$$\boxed{a_p = \sqrt{p}\, \alpha^{p-1}}$$ for all p.

$$a_p = p\, \Gamma\left(\tfrac{1}{p}\right) \Gamma\left(\tfrac{2}{p}\right) \cdots \Gamma\left(\tfrac{p-1}{p}\right)$$

$p=2$ $$a_2 = 2\, \Gamma\left(\tfrac{1}{2}\right) \Gamma(1) = 2\sqrt{\pi}$$

$$a_2 = \sqrt{2}\, \alpha$$

Conclusion $$\boxed{\alpha = \sqrt{2\pi}}$$ and $$a_p = p^{1/2} (2\pi)^{\frac{p-1}{2}}$$

To finish we proved:

Thm (Stirling's formulas)

$$\boxed{\begin{aligned} \Gamma(x) &= \sqrt{2\pi}\; x^{x-\frac{1}{2}}\, e^{-x+\frac{\theta}{12x}} \qquad \text{where } 0<\theta<1 \text{ depends on } x \\ n! &= \sqrt{2\pi}\; n^{n+\frac{1}{2}}\, e^{-n+\frac{\theta}{12n}} \end{aligned}}$$

When n is large $\quad n! \simeq \sqrt{2\pi}\; n^{n+\frac{1}{2}} e^{-n}$

$$= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

But we also proved:

Gauss' multiplication formula

$$\Gamma\left(\frac{x}{p}\right)\Gamma\left(\frac{x+1}{p}\right)\cdots\Gamma\left(\frac{x+p-1}{p}\right) = \frac{(2\pi)^{\frac{p-1}{2}}}{p^{x-\frac{1}{2}}}\,\Gamma(x)$$

p pos integer.

If $p=2$

$$\boxed{\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = \frac{\sqrt{\pi}}{2^{x-1}}\,\Gamma(x)}$$

(Legendre's formula)

Exercise Go back to the elliptic integrals and see how you can express $\int_0^1 \frac{dz}{\sqrt{1-z^3}}$ and $\int_0^1 \frac{dz}{\sqrt{1-z^4}}$