Special functions : Lecture 3

13.10.23

Dan Ciubotaru

Taylor polynomials and Taylor series

f function that has derivatives of order at least n.

Taylor polynomial of degree n around $x=a$ is

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$f^{(k)}(a) = k^{th}$ derivative of $f(x)$ at $x=a$.

Most often we take $a=0$ (sometimes these are called MacLaurin polynomials)

$T_n(x)$ gives a good approximation of $f(x)$ around $x=a$.

Why is that?

$$f(a) = T_n(a) \; ; \; f'(a) = T_n'(a) ; \dots ; f^{(k)}(a) = T_n^{(k)}(a)$$

for all $0 \le k \le n$.

<u>Theorem</u> (Taylor's Thm) There exists an error function $E_n(x)$ such that

$$f(x) = T_n(x) + (x-a)^n \cdot E_n(x)$$

and $\lim\limits_{x \to a} E_n(x) = 0$.

If we know that $f(x)$ is differentiable $(n+1)$ times and the derivatives are continuous, then for every

x there exists ξ between a and x (depends on x) such that

$$f(x) = T_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

(in other words, $E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)$)

Proof : some hints in the lecture notes.

Examples

① $f(x) = e^x$, $f^{(k)}(x) = e^x$, $f^{(k)}(0) = 1$

$$T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \quad \text{around } 0$$

Error term : For every x, there is ξ between 0 and x such that

$$f(x) = T_n(x) + \frac{e^{\xi}}{(n+1)!}x^{n+1} \quad \text{error term}$$

(ξ depends on x).

If $n \to \infty$ $\quad \frac{e^{\xi}}{(n+1)!}x^{n+1} \longrightarrow 0$ no matter what x is.

$e^{\xi} \leq 1$ if $x < 0$
$\leq e^x$ if $x > 0$.

and $\dfrac{x^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$ for all x.

This means that as $n \to \infty$

$$e^x = \underbrace{1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots}_{\text{infinite series} \quad \underline{\text{Taylor series}}}$$

for all x.

<u>Remark</u> The same analysis works if x is complex.

(2) $f(x) = \dfrac{1}{1+x} = (1+x)^{-1}$

$f'(x) = -(1+x)^{-2}$; $f''(x) = 2(1+x)^{-3}$

$f'''(x) = 2(-3)(1+x)^{-4} \dots$

$f^{(n)}(x) = (-1)^n \, n! \, (1+x)^{-(n+1)}$

$f^{(n)}(0) = (-1)^n \cdot n!$ $\qquad \dfrac{f^{(n)}(0)}{n!} = (-1)^n$

$T_n(x) = 1 - x + x^2 - \dots + (-1)^n x^n$

Error term : $\dfrac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} = \dfrac{(-1)^{n+1} \cancel{(n+1)!} (1+\xi)^{-(n+2)}}{\cancel{(n+1)!}} \cdot x^{n+1}$

$$|\text{Error}| = \frac{|x|^{n+1}}{|1+\xi|^{n+2}}$$ ξ is between 0 and x

If $|x| < 1$ then $|x|^{n+1} \to 0$ as $n \to \infty$

For example if $0 < x < 1$ then $0 < \xi < 1$

so $1 < 1+\xi < 2$ so $|1+\xi|^{n+2} \to \infty$ as $n \to \infty$

we have $\frac{0}{\infty} = 0$ ✓

<u>Exercise</u>: Check this calculation and complete the proof.

The Taylor series for $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ converges for $|x| < 1$.

Generalization is the <u>Binomial Theorem</u>

α real number. $f(x) = (1+x)^\alpha$

$$f^{(k)}(x) = \alpha(\alpha-1)\cdots(\alpha-k+1)\,(1+x)^{\alpha-k}$$

$$\frac{f^{(k)}(0)}{k!} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

<u>Denote</u> $$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$ generalized binomial coefficient

Thm $\boxed{(1+x)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} x^k \quad \text{for } |x|<1.}$

Remark If $\alpha \in \mathbb{N}$ (natural) then the Binomial Theorem holds for all x (not just $|x|<1$) because the RHS is a finite sum.

Double-check $\alpha = -1$

$$\binom{-1}{k} = \frac{(-1)(-2)\cdots(-k)}{k!} = (-1)^k.$$

$\alpha = -\frac{1}{2}$

$$\binom{-\frac{1}{2}}{k} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2k-1}{2}\right)}{k!}$$

$$= (-1)^k \cdot \frac{1\cdot 3\cdot 5\cdots(2k-1)}{k!\, 2^k}$$

$$= (-1)^k \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2k-1)(2k)}{k!\, 2^k\, (2\cdot 4\cdots 2k)}$$

$$= (-1)^k \frac{(2k)!}{k!\, 2^k \cdot 2^k\, k!} = \frac{(-1)^k}{4^k} \frac{(2k)!}{k!\, k!}$$

$$\binom{-1/2}{k} = \frac{(-1)^k}{4^k} \binom{2k}{k}.$$

Formula: $\alpha = -\frac{1}{2}$ $f(x) = \frac{1}{\sqrt{1+x}}$

$$\boxed{\frac{1}{\sqrt{1+x}} = \sum_{k\geq 0} \frac{(-1)^k}{4^k}\binom{2k}{k} x^k, \quad |x|<1}$$

If I want to get an approx of $\sqrt{2}$, plug in $x = -\frac{1}{2}$

$$\sqrt{2} = \sum_{k\geq 0} \frac{\cancel{(-1)^k}}{4^k}\binom{2k}{k}\frac{\cancel{(-1)^k}}{2^k}$$

$$\sqrt{2} = \sum_{k\geq 0}\binom{2k}{k}\cdot\frac{1}{8^k}$$

You can investigate how many terms we need to get a good approx of $\sqrt{2}$ (up to 4 decimal places)

Or, more clever,

take $x = -\frac{1}{9}$ $\frac{1}{\sqrt{1+x}} = \frac{1}{\sqrt{\frac{8}{9}}} = \frac{3}{2\sqrt{2}}$

get a faster approx for $\frac{3}{2\sqrt{2}}$ and then for $\sqrt{2}$ from this.

$$\frac{3}{2\sqrt{2}} = \sum_{k\geq 0}\binom{2k}{k}\frac{1}{36^k}$$

this converges a lot faster.

In general, to obtain Taylor series, we don't use the definition, but rather we start with known Taylor series and use some tricks, including differentiation and integration.

If $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

for $|x| < R$, then

- $f'(x) = a_1 + 2a_2 x + \dots + n a_n x^{n-1} + \dots$

  for $|x| < R$, and

- $$\int_0^x f(t)\,dt = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \dots$$

  $|x| < R$

in other words, we can differentiate and integrate term by term.

<u>Example</u> $\quad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \qquad |x|<1$

Integrate

$$\underbrace{\int_0^x \frac{1}{1+t}\,dt}_{\ln(1+x) - \cancel{\ln 1}_{0}} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad |x|<1$$

Taylor series: $\quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad |x|<1$

Ex Find the Taylor series for $\arctan x$ (inverse tangent), $\arcsin x$ (inverse sine)

Taylor series are examples of power series:

$$a_0 + a_1 x + a_2 x^2 + \dots = \sum_{n \geq 0} a_n x^n .$$

For every power series, we have the radius of convergence $R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$ if this limit exists.

Fact. $\sum_{n \geq 0} a_n x^n$ converges absolutely (even in absolute values) if $|x| < R$

and diverges if $|x| > R$.

(When $|x| = R$ check case by case)

We can recover the previous results:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\left| \frac{a_n}{a_{n+1}} \right| = 1 \longrightarrow 1 \quad \text{as } n \to \infty$$

so $R = 1$ in this case

A more interesting example:

Generating function for Fibonacci numbers

$f_0 = f_1 = 1, \quad f_{n+1} = f_n + f_{n-1} \quad n \geq 1.$

$$\frac{1}{1-x-x^2} = \sum_{n \geq 0} f_n x^n$$ (Taylor series for $\frac{1}{1-x-x^2}$)

What is the radius of convergence:

$$R = \lim_{n\to\infty} \left|\frac{f_n}{f_{n+1}}\right| = \lim_{n\to\infty} \frac{f_n}{f_{n+1}}$$

New problem sheet: show $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}$
(golden ratio)

Then $$R = \frac{1}{\frac{1+\sqrt{5}}{2}} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$$

Notice that $\frac{\sqrt{5}-1}{2}$ is the smallest absolute value of a root of $1-x-x^2$

Roots $1-x-x^2 = 0$

$x^2+x-1=0 \qquad x = \frac{-1\pm\sqrt{5}}{2}$

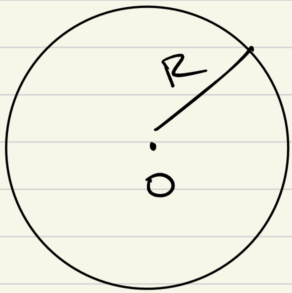
$$\left|\frac{-1-\sqrt{5}}{2}\right| = \frac{\sqrt{5}+1}{2} \qquad \left|\frac{-1+\sqrt{5}}{2}\right| = \frac{\sqrt{5}-1}{2}$$

closest to 0.

This makes sense if you know about
holomorphic function (complex functions)
$\Updownarrow$
complex differentiable.

$F(z) = \dfrac{1}{1-z-z^2}$ $\quad z \in \mathbb{C}$ is holomorphic whenever $1-z-z^2 \neq 0$.

in particular in the disk $|z| < \dfrac{\sqrt{5}-1}{2}$



$$R = \frac{\sqrt{5}-1}{2}$$

General theory of holomorphic functions says:
if $f(z)$ is holomorphic in the disk $D(\underset{\text{centre}}{a}, \underset{\text{radius}}{r})$

then $f(z)$ has a (convergent) Taylor series expansion in $D(a, r)$.

Something really interesting: use the same idea for the generating function of Bernoulli numbers.

Recall: Generating function

$$1+\sum_{n=1}^{\infty} B_{2n} \frac{x^{2n}}{(2n)!} = \frac{x}{2} \cdot \frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}}$$

Think of $x=z$ as complex: $z \in \mathbb{C}$

RHS is a holomorphic function whenever

$$e^{z/2}-e^{-z/2} \neq 0 \iff e^{z/2} \neq e^{-z/2} \,\Big|\, e^{z/2}$$

$$\iff e^{z} \neq 1.$$

$e^z = 1$ if and only if $z = 2\pi i n$, $n \in \mathbb{Z}$

$|2\pi i n| = 2\pi n$

At $n=0$ $\quad \frac{z}{2} \cdot \frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}} = \frac{z}{2} \cdot \frac{e^z+1}{e^z-1}$

$\lim_{z\to 0} \frac{z}{e^z-1} = 1$ $\quad$ (b/c $e^z-1 = z+\frac{z^2}{2}+\dots$

At $z=0$ the function is still holomorphic $\quad \frac{z}{e^z-1} = \frac{1}{1+\frac{z}{2}+\frac{z^2}{6}+\dots}$

$\longrightarrow 1$ as $z\to 0$)

The closest point to 0 where $f(z)$ is not holomorphic (it has a simple pole) is

$z = \pm 2\pi i$

So if we take a disk of radius $R = 2\pi$

Inside $D(0, 2\pi)$, the RHS is holomorphic and it's not holom. at $z = 2\pi i$

$\Longrightarrow$ radius of convergence is 2π.
for the generating function.

But we said $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

In this case:

$$LHS = 1 + \sum_{n=1}^{\infty} B_{2n} \frac{x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty} \frac{|B_{2n+2}|}{|B_{2n}|} \cdot \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \left|\frac{B_{2n+2}}{B_{2n}}\right| \cdot \frac{1}{(2n+2)\cdot(2n+1)}$$

$$= \frac{1}{R^2} = \frac{1}{4\pi^2}.$$

(squared because the powers in the series go up by two).

This tells us the growth of $\frac{B_{2n}}{(2n)!}$ as $n \to \infty$.