Special Functions: Lecture 5

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The Gamma Function

Defn For all $x > 0$, define $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$

Notice this is an improper integral, we need to think of this as:

$$\lim_{\substack{a \to \infty \\ \varepsilon \to 0}} \int_\varepsilon^a t^{x-1} e^{-t}\, dt$$ Does it converge?

Break it into $\int_\varepsilon^1 t^{x-1} e^{-t}\, dt$ and $\int_1^a t^{x-1} e^{-t}\, dt$.

and analyse separately.

Lemma 1 $\lim_{\varepsilon \to 0} \int_\varepsilon^1 t^{x-1} e^{-t}\, dt$ converges if $x > 0$.

Proof: Since $t > 0$, $0 < e^{-t} < 1$ so $0 < t^{x-1} e^{-t} < t^{x-1}$

For $0 < \varepsilon < 1$ $\quad 0 < \int_\varepsilon^1 t^{x-1} e^{-t}\, dt \leq \int_\varepsilon^1 t^{x-1}\, dt$

$$\underset{x>0}{=} \left[\frac{1}{x} t^x\right]_{t=\varepsilon}^{t=1} = \frac{1}{x} - \frac{\varepsilon^x}{x} < \frac{1}{x}$$

So this is bounded as $\varepsilon \to 0 \implies$ converges.

Remark: If $x = 0$, $\lim_{\varepsilon \to 0} \int_\varepsilon^1 t^{-1} e^{-t}\, dt = \infty$

b/c when t is close to 0, $e^{-t} \approx 1$, so

$t^{-1} e^{-t} \approx t^{-1}$ and $\int_0^1 \frac{1}{t} dt = \infty$
diverges.

$\left(O(t^{-1} e^{-t}) \simeq t^{-1} \right.$ ~~as~~ when $t > 0$ is close to $0 \big)$

<u>Lemma 2</u> $\int_1^\infty t^{x-1} e^{-t} dt$ converges for all x..

<u>Proof</u> To find an upper bound for $t^x e^{-t}$, use Taylor series for e^t:

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \dots + \frac{t^n}{n!} + \dots$$

Given x, find $\boxed{n > x+1}$ then $e^t > \frac{t^n}{n!}$

$\Rightarrow e^{-t} < n! \, t^{-n}$ Multiply by t^{x-1}

$$\Rightarrow 0 < t^{x-1} e^{-t} < n! \, t^{x-n-1} < n! \, t^{-2}$$

$$\int_1^a t^{x-1} e^{-t} dt < n! \int_1^a t^{-2} dt = n! \left(1 - \frac{1}{a}\right)$$
$$< n!$$

(n is fixed). Then

$\lim_{a \to \infty} \int_1^a t^{x-1} e^{-t} dt$ converges b/c

positive and bounded above by $n!$
function

__Remark__ Throughout I used implicitly that the improper integral of a positive function converges if it is bounded above.

__Conclusión__ $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$ converges for all $x > 0$.

(it diverges at $x = 0$).

Where does $\Gamma(x)$ come from? The reason is that it generalises the factorial of natural numbers.

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} \, dt \qquad n \in \mathbb{N}$$

. We can compute via integration by parts:

$$\Gamma(n) = \int_0^\infty t^{n-1} \underbrace{e^{-t}}_{= -(e^{-t})'} dt = \underbrace{-t^{n-1} e^{-t} \Big]_{t \to 0}^{t \to \infty}}_{= 0 \text{ if } n > 1} + (n-1) \underbrace{\int_0^\infty t^{n-2} e^{-t} dt}_{= \Gamma(n-1)}$$

if $n > 1$ $\quad t^{n-1} e^{-t} = 0$ when $t = 0$.

always $\lim\limits_{t \to \infty} t^{n-1} e^{-t} = 0$ (because the exponentials beat any polynomial, that is:

$$\lim_{t \to \infty} \frac{P(t)}{e^t} = 0 \quad \text{for any polynomial } P(t))$$

Hence $\Gamma(n) = (n-1)\,\Gamma(n-1)$ for all $n > 1$.

But we didn't really use that n was a natural number, so in fact the same proof works to show:

Lemma For all $x > 0$, $\boxed{\Gamma(x+1) = x\,\Gamma(x)}$.

Back to $x = n \in \mathbb{N}$:

$$\Gamma(n) = (n-1)\,\Gamma(n-1) = (n-1)(n-2)\,\Gamma(n-2)$$

$$= \ldots = \underbrace{(n-1)(n-2)\cdot\ldots\cdot 2\cdot 1}_{(n-1)!}\cdot\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\,dt = \int_0^\infty e^{-t}\,dt$$

$$= -e^{-t}\Big]_{t=0}^{t\to\infty} = 1.$$

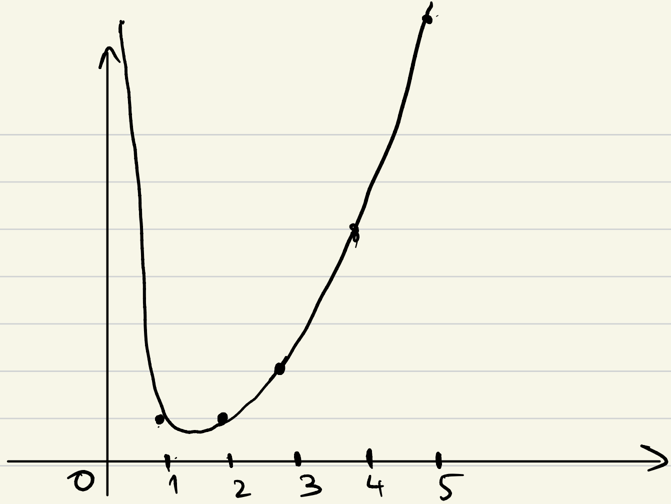
Then $\boxed{\Gamma(n) = (n-1)! \qquad n = 1, 2, 3, \ldots}$

So far we defined $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$

and we know it converges for $x > 0$ and

$\Gamma(n) = (n-1)!$, $n \in \mathbb{N}$ and more generally $\Gamma(x+1) = x\,\Gamma(x)$, $x > 0$.

Graph:



It's a positive convex function for $x > 0$

$\Gamma(1) = 0! = 1$, $\Gamma(2) = 1! = 1$, $\Gamma(3) = 2$, $\Gamma(4) = 6$
$\Gamma(5) = 24 \ldots$

How do we know it's convex? Recall: if $F(x)$ is twice-differentiable $F(x)$ is convex if $F''(x) > 0$.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

Questions

1) Are there other known values of $\Gamma(x)$?
(not just $\Gamma(n)$, $n \in \mathbb{N}$.)

2) Is there a way to define $\Gamma(x)$ for $x < 0$?

3) Why is $\Gamma(x)$ convex for $x > 0$?

Start with 2). The integral definition can't be extended past $x=0$ because of the singularity at 0 ($\Gamma(0)=\infty$)

Instead, we can use the functional equation:

$$\Gamma(x+1) = x\,\Gamma(x) \quad \text{to extend to } x<0.$$

Because: if $-1<x<0$, then $0<x+1<1$ so $\Gamma(x+1)$ is defined.

Set $\boxed{\Gamma(x) := \dfrac{\Gamma(x+1)}{x}}$ for $-1<x<0$

Then if $-2<x<-1$, $-1<x+1<0$ so ~~now~~ we know $\Gamma(x+1)$, set $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$, $-2<x<-1$

keep going.

Inductively, this defines $\Gamma(x)$ for all ~~x~~

$$-n-1 < x < -n, \quad n \in \mathbb{Z}_{\geq 0}.$$

This allows us to define $\Gamma(x)$ for all $x \in \mathbb{R}$ except $x = 0, -1, -2, -3, \ldots$

<u>Example</u> $$\Gamma\left(-\tfrac{5}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}\right)}{-\frac{5}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}$$

so by definition $\Gamma\left(-\tfrac{5}{2}\right) = -\dfrac{8}{15} \cdot \Gamma\left(\tfrac{1}{2}\right)$.

The graph of $\Gamma(x)$, $x \in \mathbb{R}$ looks like:

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \quad -1 < x < 0$$

$\Gamma(x+1) > 0$

$x \leftarrow$ neg.

it alternates between pos/neg and convex/concave

Convexity: If $F(x)$ is continuous twice differentiable $F(x)$ is convex if $F''(x) > 0$.

$$\frac{d}{dx}\left(t^{x-1}\right) = \frac{d}{dx} e^{(x-1)\ln t} = (\ln t) \cdot e^{(x-1)\ln t}$$

$$\left(\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x) \right)$$

twice: $$\frac{d^2}{dx^2}\left(t^{x-1}\right) = (\ln t) \cdot \frac{d}{dx} e^{(x-1)\ln t}$$

$$= (\ln t)^2 \, e^{(x-1)\ln t} = (\ln t)^2 \, t^{x-1}.$$

We use that we can differentiate under the integral sign (that uses a theorem from integration / measure theory)

If $F(x) = \int_A f(x,t)\,dt$ and $f(x,t)$ is diff. w.r.t. x and $|f(x,t)| < g(t)$, such that $\int_A g(t)$ converges integrable

and $\left|\frac{\partial}{\partial x} f(x,t)\right| < h(t)$ (control function)

then $$F'(x) = \int_A \frac{\partial}{\partial x} f(x,t)\,dt.$$

In our case, this can be applied:

$$\Gamma''(x) = \int_0^\infty \frac{d}{dx}\left(t^{x-1}e^{-t}\right)dt$$

$$= \int_0^\infty \underbrace{(\ln t)^2\, t^{x-1} e^{-t}}_{>0}\, dt > 0.$$

(More details in an exercise in the next problem sheet)

What about other values of $\Gamma(x)$?

The most famous is $\Gamma\left(\frac{1}{2}\right)$.

$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t}\, dt$ change of variables $t = u^2,\ u > 0$, $dt = 2u\,du$

$t = 0 \quad u = 0$
$t = \infty \quad u = \infty$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \cancel{u^{-1}}\, e^{-u^2} \cdot 2\cancel{u}\, du = 2\int_0^\infty e^{-u^2} du$$

<u>Remark</u>: $\int_{-\infty}^{\infty} e^{-x^2} dx$ appears in probability (Gauss normal distribution).

How do we compute $\int_0^\infty e^{-x^2} dx = ?$

Call it $I = \int_0^\infty e^{-x^2} dx$. Compute I^2:

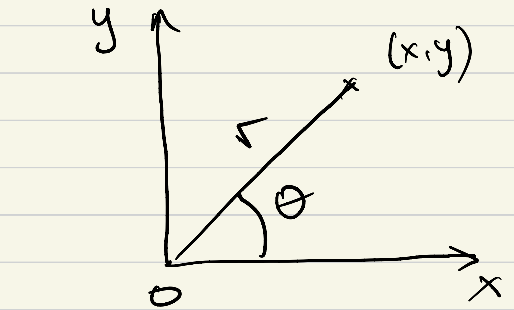
$$I^2 = \left(\int_0^\infty e^{-x^2} dx\right) \cdot \left(\int_0^\infty e^{-y^2} dy\right)$$

$$= \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2}\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

Now this is the double integral of the function $f(x,y) = e^{-(x^2+y^2)}$ over the first quadrant $x > 0,\ y > 0$.

Switch to polar coordinates:



$$x = r\cos\theta$$
$$y = r\sin\theta$$

1st quadrant: $0 < \theta < \frac{\pi}{2}$
$r > 0$.

$$x^2 + y^2 = r^2$$

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \cdot dr\,d\theta$$

Jacobian from change of coords for double integrals

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r(\cos^2\theta + \sin^2\theta) = r$$

Polar coordinates: $dx\,dy = r\,dr\,d\theta$.

Double integral:

$$I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

$$\int_0^\infty e^{-r^2}\,r\,dr = -\frac{1}{2}\,e^{-r^2}\Big]_0^\infty = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$I^2 = \int_0^{\pi/2} \frac{1}{2}\, d\theta = \frac{\pi}{4}.$$ Hence $I = \frac{\sqrt{\pi}}{2}$.

We proved:

$$\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}, \quad \text{so :}$$

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

To summarise:

1. We defined $\Gamma(x)$ for all $x \in \mathbb{R} \setminus \{0, -1, -2, -3, \dots\}$

When $x > 0$, $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$. For $x < 0$, we extend it via:

2. $$\Gamma(x+1) = x\, \Gamma(x), \quad \forall x \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$$

3. $\Gamma(x)$ is a convex function for $x > 0$.
(and the convexity / concavity alternates for $x < 0$).

4. $\Gamma(n) = (n-1)!$, $n \in \mathbb{N}$.

5. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (this is related to the probability integral) for normal distribution.

Then we know $\Gamma(n)$, $n \in \mathbb{N}$ and $\Gamma(x)$ for all $x \in \mathbb{Z} + \frac{1}{2}$.

<u>Example</u> $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

$$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}\right)}{\left(-\frac{5}{2}\right)} = \frac{\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{8}{15}\sqrt{\pi}.$$

But not much more is known in terms of explicit values (not approximations) for $\Gamma(x)$ is x is not a natural number or a half integer.

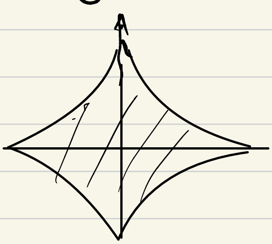
Next time: other identities involving $\Gamma(x)$,

Stirling's formula, reflection formula...

Beta function and applications to Wallis' integrals $\int_0^{\pi/2} \sin^n\theta\, d\theta$, $\int_0^{\pi/2} \cos^n\theta\, d\theta$

and areas for curves like:

$$x^k + y^k = a^k. \qquad a > 0.$$



$(k<1)$