Special functions Lecture 4

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Dan Ciubotaru

<u>Zeta function</u> : For every real number $k \geq 1$ define $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$. We'll see soon that $\zeta(k) < \infty$ if $k > 1$ and $= \infty$ if $k = 1$.

(In fact, one may define $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for all complex number s such that $Re(s) > 1$.)

$k = 1$ harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \dots$ $(= \infty)$ divergent.

You may know that for example:

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$$

<u>Goal</u> : find an explicit formula for $\zeta(2k)$, $k \geq 1$ (natural).

this will involve Bernoulli numbers.

<u>Brief interlude on convergence of series</u>

(a_n) sequence of real (or complex) numbers:

series $\sum_{n=1}^{\infty} a_n$. Define the <u>sequence of partial sums</u>:

$$s_n = a_1 + a_2 + \dots + a_n, \quad n \geq 1.$$

We say $\sum_{n=1}^{\infty} a_n$ converges (diverges) if $(s_n)_n$ converges (diverges)

A sequence (s_n) converges to L if $\forall \varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$
$$|s_n - L| < \varepsilon.$$

<u>Example</u> The geometric series $\sum_{n=0}^{\infty} x^n$ converges for all $|x| < 1$.

$$s_n = 1 + x + \ldots + x^{n-1} = \frac{1-x^n}{1-x} \longrightarrow \frac{1}{1-x} = L$$
if $|x| < 1$.

Take $\varepsilon > 0$ $\quad \left| s_n - \frac{1}{1-x} \right| = \left| \frac{1-x^n}{1-x} - \frac{1}{1-x} \right| = |x^n| \cdot \frac{1}{|1-x|}$

if $|x| < 1$ then $|x|^n \to 0$ so there exists N s.t. $|x|^n < \varepsilon |1-x|$ for all $n \geq N$

Then $\left| s_n - \frac{1}{1-x} \right| < \varepsilon |1-x| \cdot \frac{1}{|1-x|} = \varepsilon$ for all $n \geq N$.

But in general, we don't prove convergence of series directly from the definition. Instead, we have several criteria for convergence.

<u>Tests for convergence</u> (I will state them for series of real numbers, but they also work for complex numbers by replacing a_n with $|a_n|$ in the appropriate places)

(1) If $\sum_{n=1}^{\infty} |a_n|$ converges (we say $\sum a_n$ converges <u>absolutely</u>)

then $\sum_{n=1}^{\infty} a_n$ converges.

(2) <u>Comparison test</u>: If $|a_n| \leq b_n$ for all n (or for all $n \geq n_0$), if $\sum b_n$ converges then $\sum a_n$ converges absolutely. If $\sum |a_n|$ diverges, then $\sum b_n$ diverges.

(<u>Example</u>: $\sum_{n\geq 0} \frac{1}{n+2^n}$ $a_n = \frac{1}{n+2^n}$, $b_n = \frac{1}{2^n}$

$a_n \leq b_n$

and $\sum_{n\geq 0} \frac{1}{2^n}$ converges as a geometric series,

so $\sum_{n\geq 0} \frac{1}{n+2^n}$ converges.)

(3) <u>Comparison Test</u> (limit form) Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. If $0 < L < \infty$ then $\sum a_n$ converges if and only if $\sum b_n$ converges.

- If $L = 0$ and $\sum b_n$ converges then $\sum a_n$ converges
- If $L = \infty$ and $\sum b_n$ diverges then $\sum a_n$ diverges.

(4) <u>Ratio Test</u> (to prove this, you compare your series to a geometric series)

If $\sum a_n$ has positive terms and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

then:

(a) if $0 \le L < 1$, $\sum a_n$ converges

(b) If $L > 1$, $\sum a_n$ diverges

(If $L = 1$, test is inconclusive.)

<u>Non-example</u> $\sum_{n=1}^{\infty} \frac{1}{n^k}$ "the test is inconclusive"

$$\frac{a_{n+1}}{a_n} = \frac{n^k}{(n+1)^k} = \left(\frac{n}{n+1}\right)^k \longrightarrow 1 \text{ as } n \to \infty$$

<u>Example</u>: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}$$

$$= \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= (n+1) \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e} \text{ as } n \to \infty$$

<u>Fact</u> $\left(1+\frac{1}{n}\right)^n \to e$ as $n \to \infty$. (For us, the definition of e is $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots$)

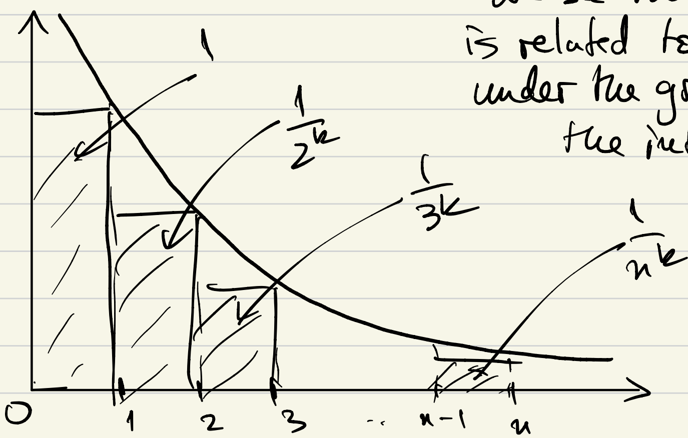
(5) <u>Integral test</u> If $f: [1,\infty) \to [0,\infty)$ is a decreasing continuous function then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the improper integral

$\int_1^\infty f(x)\,dx$ converges. $\left(= \lim_{a\to\infty} \int_1^a f(x)\,dx \right)$

<u>Idea</u>: (Riemann sums)

$f(x) = \frac{1}{x^k}$ $\quad k > 1$

we see the series is related to the area under the graph, hence the integral $\int_1^\infty$



<u>Example</u> The series $\sum_{n=1}^{\infty} \frac{1}{n^k}$, $k \in \mathbb{R}$ converges if and only if $k > 1$.

<u>Pf</u>: Easy $\sum_{n=1}^{\infty} \frac{1}{n^k}$ diverges if $k \leq 0$ (look at the n^{th} term $\to \infty$ or 1)

Suppose $k > 0$.

$f(x) = \frac{1}{x^k} : [1, \infty) \longrightarrow [0, \infty)$ $\qquad x^{-k}$

$$\int_1^\infty f(x)\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x^k}\,dx$$

$$= \lim_{a\to\infty} \begin{cases} \ln a - \ln 1, & k = 1 \\ \left[\frac{1}{-k+1} x^{-k+1} \right]_1^a, & k \neq 1. \end{cases}$$

$$= \lim_{a\to\infty} \begin{cases} \ln a, & k=1 \\ \frac{1}{k-1} - \frac{a^{-k+1}}{k-1}, & k \neq 1 \end{cases}$$

$$= \begin{cases} \infty, & 0<k \leq 1 \\ \frac{1}{k-1}, & k>1. \end{cases}$$

Then applying the Integral test $\sum \frac{1}{n^k}$ conv. iff $k>1$

<u>Calculate</u> $\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$, $k \geq 1$ integer.

<u>Idea</u>: give two power series expansions for the function $\cot x$ and compare them. $(x \cot x)$

<u>First expansion</u> Recall the generating function for Bernoulli numbers:

$$\frac{x}{2} \frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} x^{2k}, \quad |x|<2\pi$$

(true for $x \in \mathbb{C}$ not just $x \in \mathbb{R}$)

Hyperbolic sine: $\sinh x = \frac{e^x - e^{-x}}{2}$

cosine: $\cosh x = \frac{e^x + e^{-x}}{2}$

LHS: $\frac{x}{2} \cdot \coth \frac{x}{2}$.

Substitute $2x$ for x:

$$(*) \quad x \coth x = \sum_{k=0}^{\infty} \frac{B_{2k} \cdot 2^{2k}}{(2k)!} x^{2k} \qquad |x|<\pi$$

Trick: substitute ix for x $\quad (|ix| = |x| < \pi)$

$$\coth(ix) = \frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}} = \frac{2\cos x}{2i \sin x} = -i \cot x$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i} ; \quad \cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\coth(ix) = -i \cot x$$

$$ix \coth(ix) = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} 2^{2k} (ix)^{2k}$$

$$\boxed{x \cot(x) = \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k} \cdot 2^{2k}}{(2k)!} x^{2k}, \quad |x|<\pi}$$

(Taylor series expansion at 0 of $x \cot x$)

Second expansion (Idea goes back to Euler)

$$\sin x = x \cdot \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(n\pi)^2}\right)$$

Notice: the right hand side has the same zeros as the left hand side: $\pm n\pi$, $n \in \mathbb{N}$

moreover $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (the same for the RHS).

Skip a rigorous proof for now. (later or references)

Warm up: $\dfrac{\sin x}{x} = \displaystyle\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(n\pi)^2}\right)$

What is the coefficient of x^2 if we expand the RHS as a series: pick 1 in all but one bracket

$$x^2: \quad -\frac{1}{\pi^2} - \frac{1}{(2\pi)^2} - \frac{1}{(3\pi)^2} - \dots = -\frac{1}{\pi^2}\zeta(2)$$

On the other hand:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

Taylor expansion of $\sin x$

coeff of x^2 in LHS: $-\dfrac{1}{6}$ Therefore $\boxed{\zeta(2) = \dfrac{\pi^2}{6}}$

To do this in general:

$$\cot x = \frac{\cos x}{\sin x} = \frac{(\sin x)'}{\sin x} = (\ln(\sin x))'$$

In Euler's product formula, take $\ln$:

$$\ln \sin x = \ln x + \sum_{n=1}^{\infty} \ln\left(1 - \frac{x^2}{(n\pi)^2}\right)$$

Differentiate:

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - (n\pi)^2}$$

$$\frac{d}{dx} \ln\left(1 - \frac{x^2}{(n\pi)^2}\right) = \frac{-\dfrac{2x}{(n\pi)^2}}{1 - \dfrac{x^2}{(n\pi)^2}} = \frac{2x}{x^2 - (n\pi)^2}$$

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{(x - n\pi)(x + n\pi)}$$

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x+n\pi} + \frac{1}{x-n\pi}\right)$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left(\frac{1}{1+\frac{x}{n\pi}} - \frac{1}{1-\frac{x}{n\pi}}\right)$$

Geometric series:

$$\frac{1}{1-y} = 1+y+y^2+y^3+\dots \qquad |y|<1$$

$$\frac{1}{1+y} = 1-y+y^2-y^3+\dots$$

$$\frac{1}{1+y} - \frac{1}{1-y} = -2\left(y+y^3+y^5+\dots\right) = -2\sum_{k=1}^{\infty} y^{2k-1}$$

$$|y|<1.$$

We assume $|x|<\pi$ so that $\left|\frac{x}{n\pi}\right|<1$ for all $n\in\mathbb{N}$

Then

$$\cot x = \frac{1}{x} - \sum_{n=}^{\infty}\frac{2}{n\pi}\sum_{k=1}^{\infty}\frac{x^{2k-1}}{(n\pi)^{2k-1}} \qquad |x|<\pi$$

$$x\cot x = 1-2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^{2k}}{(n\pi)^{2k}} \qquad |x|<\pi$$

One more step: we need to change the order of summation; this can be done b/c the series converges absolutely.

$$x\cot x = 1-2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^{2k}}{(n\pi)^{2k}}$$

$$(1)\quad x\cot x = 1-2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{\pi^{2k}}\,x^{2k} \qquad |x|<\pi$$

So we have two expansions for $x\cot x$, $|x|<\pi$

formula (1) and:

$$x\cot x = 1 + \sum_{k=1}^{\infty} \frac{B_{2k}\, 2^{2k}}{(2k)!}(-1)^k x^{2k}$$

So the coefficients must be equal:

<u>Theorem</u>

$$\frac{B_{2k}\, 2^{2k}}{(2k)!}(-1)^k = -2\,\frac{\zeta(2k)}{\pi^{2k}}$$

$$\boxed{\zeta(2k) = (-1)^{k-1}\frac{B_{2k}\,(2\pi)^{2k}}{2\cdot(2k)!}, \quad k\in\mathbb{N}}$$

Notice that this also implies that $(-1)^{k-1}B_{2k}>0$

(B_{2k} have alternating signs, we knew this.)

<u>Example</u> $k=2$, $B_4 = -\frac{1}{30}$

$$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\frac{1}{30}\cdot(2\pi)^4}{2\cdot 4!} = \frac{\cancel{16}\pi^4}{30\cdot\cancel{2\cdot 24}_{3}}$$

$$\boxed{\zeta(4) = \frac{\pi^4}{90}}$$

In general $\zeta(2k) = (\text{rational number})$ times π^{2k}.

Remark: $\zeta(k)$, $k \geq 3$ is odd, are unknown.

(Try to compute $\zeta(4)$ directly from $\frac{\sin x}{x}$ Euler's formula)

Why the interest in the zeta function? It related to prime numbers.

The origin is Euler's proof of the infinitude of prime numbers:

Elementary proof: suppose by contradiction that there are finitely many prime numbers $p_1, \dots, p_n$. Form $N = p_1 \cdots p_n + 1$. Then N is not divisible by any $p_1, \dots, p_n$, so it's prime, but $N >$ all p_i, so contradiction.

Euler's proof For all $k \geq 1$

$$\boxed{\zeta(k) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^k}}}$$

If there were finitely many primes, then RHS is finite for $k=1$ as well, but then contradiction since $\zeta(1) = \infty$.

Start with geom. series again $|x| < 1$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

Plug in $x = \frac{1}{2^k}, \frac{1}{3^k}, \frac{1}{5^k}, \ldots, \frac{1}{p^k}, \ldots$ Since $k \geq 1$ they're all < 1.

$$\frac{1}{1-\frac{1}{2^k}} = 1 + \frac{1}{2^k} + \frac{1}{2^{2k}} + \cdots$$

$$\frac{1}{1-\frac{1}{3^k}} = 1 + \frac{1}{3^k} + \frac{1}{3^{2k}} + \cdots$$

$$\vdots$$

$$\frac{1}{1-\frac{1}{p^k}} = 1 + \frac{1}{p^k} + \frac{1}{p^{2k}} + \cdots$$

Then

$$\prod_p \frac{1}{1-\frac{1}{p^k}} = \left(1+\frac{1}{2^k}+\frac{1}{2^{2k}}+\cdots\right)\left(1+\frac{1}{3^k}+\frac{1}{3^{2k}}+\cdots\right)\cdots$$

If we multiply the RHS, a typical element in the resulting sum is:

$$\frac{1}{p_1^{m_1 k} p_2^{m_2 k} \cdots p_\ell^{m_\ell k}} = \left(\frac{1}{p_1^{m_1} p_2^{m_2} \cdots p_\ell^{m_\ell}}\right)^k$$

By the Fundamental Thm of Arithmetic, every n can be written uniquely as:

$$n = p_1^{m_1} \cdots p_\ell^{m_\ell} \quad \text{for distinct primes}$$

$p_1, \ldots, p_\ell$ and positive integers $m_1, \ldots, m_\ell$.

Hence $\text{RHS} = \sum_{n=1}^{\infty} \frac{1}{n^k}$.

<u>Remark</u> Euler's formula holds for $\zeta(s)$, $\operatorname{Re} s > 1$ (b/c $|p^s| = p^{\operatorname{Re} s}$ so $\left|\frac{1}{p^s}\right| < 1$ if $\operatorname{Re} s > 1$)

Now this can be generalised to <u>Dirichlet series</u>

$$f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$$

If $a_n = 1$ for all n this Riemann zeta function

You want $f(s)$ to have a product formula over primes, just like the zeta function.

Dirichlet: $a_n = \chi_4(n) = \begin{cases} 0, & n \text{ is even} \\ 1, & n \equiv 1 \ (4) \\ -1, & n \equiv 3 \ (4) \end{cases}$

I will give some reference on the next problem sheet if you want to read about Dirichlet series and how to prove that there infinitely many primes in certain arithmetic progressions:

<u>Thm (Dirichlet)</u> If q and l are relatively prime pos numbers, then there are infinitely many primes of the form $l + kq$, $k \in \mathbb{Z}$.

Idea: define a character $\chi(n)$, set $a_n = \chi(n)$ in terms of l and q

and prove

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{\chi(p)}{p^s}} \quad \operatorname{Re} s > 1$$

and show LHS diverges when $s = 1$.