Special functions Lecture 6

3/11/2023

Dan Ciubotaru

Recall: Gamma function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt$ $\quad x>0$

and then extended to all $x \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$ via the functional equation:

$$\boxed{\Gamma(x+1) = x\,\Gamma(x)}$$

In particular, $\Gamma(n) = (n-1)!$ for all integers $n \geq 1$.

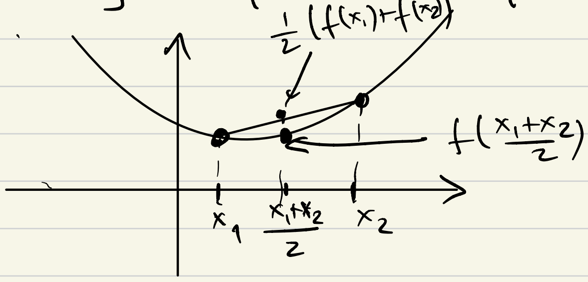
<u>Q</u>: Is this the only way to generalise the factorial?

<u>Convex functions</u>

<u>Defn</u> A function $f:(a,b) \to \mathbb{R}$ is called <u>convex</u> if it is continuous and

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{f(x_1)+f(x_2)}{2} \quad \forall\, x_1, x_2 \in (a,b)$$

A function that only satisfies this inequality is called <u>weakly convex</u>.



<u>Jensen's inequality</u> If f is weakly convex, then

$$(*) \quad f\left(\frac{x_1+x_2+\dots+x_n}{n}\right) \leq \frac{f(x_1)+\dots+f(x_n)}{n} \quad \forall n \geq 1.$$

Proof **Step 1** If $(*)$ holds for n, then it holds for $2n$.

$x_1, x_2, \dots, x_{2n}$

$$f\left(\frac{x_1+x_2+\dots+x_{2n}}{2n}\right) = f\left(\frac{\frac{x_1+\dots+x_n}{n} + \frac{x_{n+1}+\dots+x_{2n}}{n}}{2}\right)$$

$$\underset{\text{weakly convex}}{\leq} \frac{1}{2}\left(f\left(\frac{x_1+\dots+x_n}{n}\right) + f\left(\frac{x_{n+1}+\dots+x_{2n}}{n}\right)\right)$$

$$\underset{(*)\text{ for } n}{\leq} \frac{1}{2}\left(\frac{f(x_1)+\dots+f(x_n)}{n} + \frac{f(x_{n+1})+\dots+f(x_{2n})}{n}\right)$$

$$= \frac{1}{2n}\left(f(x_1)+\dots+f(x_{2n})\right)$$

Step 2 If $(*)$ holds for $n+1$, then it holds for n.

$x_1, \dots, x_n$, take $x_{n+1} = \frac{x_1+\dots+x_n}{n}$

$$f\left(\frac{x_1+x_2+\dots+x_{n+1}}{n+1}\right) \underset{(*)\text{ for } n+1}{\leq} \frac{1}{n+1}\left(f(x_1)+\dots+f(x_{n+1})\right)$$

$$\|$$

$$f\left(\frac{\overbrace{(x_1+\dots+x_n)}^{=nx_{n+1}} + x_{n+1}}{n+1}\right)$$

$$\|$$

$$f(x_{n+1}) \leq \frac{1}{n+1}\left(f(x_1)+\dots+f(x_{n+1})\right)$$

$$n\, f(x_{n+1}) \leq f(x_1)+\dots+f(x_n)$$

$$f\left(\frac{x_1+\dots+x_n}{n}\right) \leq \frac{1}{n}\left(f(x_1)+\dots+f(x_n)\right)$$

The proof goes by induction on n

- prove it for $n=1$ obvious
- suppose it holds for n $\xRightarrow{\text{Step 1}}$ holds for $2n$

$\xRightarrow{\text{Step 2}}$ holds for $2n-1, 2n-2, \ldots, \underline{n+1}$
done.

<u>Criterion (exercise)</u> If f is twice differentiable, then f is convex iff $f'' \geq 0$.

<u>Defn</u> A positive function $f(x)$ is called <u>log-convex</u> if $\log f(x)$ is convex.

This means $\log f(x)$ is continuous ($\Leftrightarrow$ $f(x)$ is continuous)

and $$\log f\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{2} \underbrace{\left(\log f(x_1) + \log f(x_2)\right)}_{\log\left(\left(f(x_1) f(x_2)\right)^{1/2}\right)}$$

$\Longleftrightarrow$ ($\log$ is increasing)

$$\boxed{f\left(\frac{x_1+x_2}{2}\right)^2 \leq f(x_1) f(x_2)}$$

log convexity.

<u>Easy properties</u> 1) A sum of convex functions is convex. Hence a product of log-convex functions is log-convex.

2) Suppose $(f_n(x))_{n\geq 1}$ is a sequence of log convex functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists and it is positive. Then $f(x)$ is log convex.

3) Suppose $f(x)$ is twice differentiable and positive. Then f is log-convex iff

$$f(x)\cdot f''(x) - f'(x)^2 \geq 0.$$

(This follows by computing the second derivative of $\log f(x)$.)

<u>Proposition</u> The sum of two log convex functions is log convex.

<u>Pf</u> f, g are log convex ; then

$$\underbrace{f\left(\tfrac{x_1+x_2}{2}\right)^2}_{b_1^2} \leq \underbrace{f(x_1)}_{a_1}\,\underbrace{f(x_2)}_{c_1} \quad \text{and} \quad \underbrace{g\left(\tfrac{x_1+x_2}{2}\right)^2}_{b_2^2} \leq \underbrace{g(x_1)}_{a_2}\,\underbrace{g(x_2)}_{c_2}$$

We'd like to show:

$$\left(f\left(\tfrac{x_1+x_2}{2}\right) + g\left(\tfrac{x_1+x_2}{2}\right)\right)^2 \overset{?}{\leq} \left(f(x_1)+g(x_1)\right)\left(f(x_2)+g(x_2)\right)$$

$$(b_1+b_2)^2 \leq (a_1+a_2)(c_1+c_2)$$

Consider the quadratic polynomial: $a_1x^2 + 2b_1x + c_1$

Discriminant is $b_1^2 - a_1c_1 \leq 0$

$\implies a_1x^2+2b_1x+c_1$ has the same sign as a_1 for all x.

In fact we know f, g are positive so $a_1 > 0$

$\Rightarrow \quad a_1 x^2 + 2b_1 x + c_1 \geq 0 \quad \forall x$

Similarly $\quad a_2 x^2 + 2b_2 x + c_2 \geq 0 \quad \forall x$

add

$$\underbrace{(a_1+a_2)}_{>0} x^2 + 2(b_1+b_2)x + (c_1+c_2) \geq 0 \quad \forall x$$

$\Rightarrow$ discriminant is ≤ 0 $\Rightarrow$ desired inequality.

Example $f(x) = x^k$ for some $k<0$ is log convex for $x>0$.

$\log f(x) = k \log x$

$\frac{d}{dx^2}(\log f(x)) = -\frac{k}{x^2} \geq 0 \Rightarrow \log f(x)$ is convex.

Next: $\Gamma(x)$ is log convex.

Corollary Suppose $f(t,x)$ is a function of two variables $a \leq t \leq b$ and x such that $f(t,x)$ is log convex in x. Then

$$F(x) = \int_a^b f(t,x)\, dt \quad \text{is also log convex.}$$

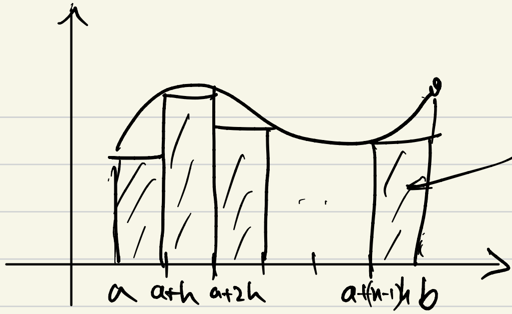
Pf Remember Riemann sums:

log convex in x

$$F_n(x) = h\left(f(a,x) + f(a+h,x) + \dots + f(a+(n-1)h,x)\right)$$

$h = \frac{b-a}{n}$ $\qquad \lim_{n\to\infty} F_n(x) = F(x)$

$F_n(x)$ is log convex (sum of log convex) $\Rightarrow$ limit is also log convex



<u>Remark</u> The corollary also works for improper integrals if the integral converges.

<u>Application</u>: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, $x > 0$ is log convex.

<u>Proof</u> Apply the previous corollary to $f(t,x) = \underset{\text{cont \& pos.}}{e^{-t} t^{x-1}}$

$\log f(t,x) = -t + (x-1)\ln t$.

$\frac{d}{dx^2} \log f(t,x) = 0 \Rightarrow f(t,x)$ is log-convex in x.

$\underset{\text{Cor.}}{\Longrightarrow}$ the integral is as well.

<u>Theorem</u> (Unique characterisation of Gamma function)
If $f(x)$ satisfies the following three conditions then it equals the Gamma function:

(0) $f(1) = 1$

(1) $f(x+1) = x f(x)$

(2) The domain of $f(x)$ contains $x > 0$ and $f(x)$ is log convex for $x > 0$.

Proof (0) + (1) $\Longrightarrow$ $f(n) = (n-1)!$ for all integers $n \geq 1$.

It is enough to show $f(x) = \Gamma(x)$ for $0 < x \leq 1$ (because after that we can apply (1) and get equality on all other intervals).

Easy fact (exercise) If $g(x)$ is convex then for all $x_1 < x_2$ and x_0

$$\frac{g(x_1) - g(x_0)}{x_1 - x_0} \leq \frac{g(x_2) - g(x_0)}{x_2 - x_0}$$

We say: the difference quotients are increasing.

Take $g(x) = \log f(x)$ and apply it to:

$n \geq 2$ $\quad \underbrace{-1+n}_{x_1} < \underbrace{x+n}_{x_2} \leq \underbrace{1+n}_{x_3}$ and $x_0 = n$

$$\frac{\log f(-1+n) - \log f(n)}{(-1+n) - n} \leq \frac{\log f(x+n) - \log f(n)}{(x+n) - n} \leq \frac{\log f(1+n) - \log f(n)}{(1+n) - n}$$

Left: $\dfrac{\log \frac{f(n-1)}{f(n)}}{-1} = \dfrac{\log \frac{1}{n-1}}{-1} = \log(n-1)$ $\qquad f(n) = (n-1)!$, $\quad \dfrac{f(n-1)}{f(n)} = \dfrac{1}{n-1}$

Right: $\dfrac{\log \frac{f(n+1)}{f(n)}}{1} = \log n$

Middle: $\dfrac{\log f(x+n) - \log (n-1)!}{x}$ $\qquad$ Inequality:

$$\log(n-1) \leq \frac{\log f(x+n) - \log (n-1)!}{x} \leq \log n$$

$$\log\left((n-1)^x (n-1)!\right) \leq \log f(x+n) \leq \log\left(n^x (n-1)!\right)$$

$\Longrightarrow$ log is increasing

$$(n-1)^x (n-1)! \leq f(x+n) \leq n^x (n-1)!$$

$\forall n \geq 2$

Using (1) : $f(x+n) = x(x+1)\dots(x+n-1) f(x)$

$$\Rightarrow \frac{(n-1)^x (n-1)!}{x(x+1)\dots(x+n-1)} \leq f(x) \leq \frac{n^x (n-1)!}{x(x+1)\dots(x+n-1)}$$

$\forall n \geq 2$

In the left inequality change n to $n+1$

$$\frac{n^x\, n!}{x(x+1)\dots(x+n)} \leq f(x)$$

In the right inequality:

$$f(x) \leq \frac{n^x\, n!}{x(x+1)\dots(x+n)} \cdot \frac{x+n}{n}$$

Denote $f_n(x) = \dfrac{n^x \cdot n!}{x(x+1)\cdot\ldots\cdot(x+n)}$ $\quad n \geq 1$

We proved $f(x) \cdot \dfrac{n}{x+n} \leq f_n(x) \leq f(x)$ for all $n \geq 2$

and all $0 < x \leq 1$

As $n \to \infty$ (and x is fixed) $\frac{n}{x+n} \to 1$

$\Longrightarrow$ $\lim_{n\to\infty} f_n(x)$ exists and it equals $f(x)$.

Conclusion

$$f(x) = \lim_{n\to\infty} \frac{n^x \, n!}{x(x+1)\cdots(x+n)} \qquad 0 < x \leq 1$$

but this means that $f(x)$ is unique

Since $\Gamma(x)$ satisfies the conditions of the theorem

$\Longrightarrow$ $f(x) = \Gamma(x)$ on $0 < x \leq 1$ (hence everywhere)

<u>Corollary</u>

$$\boxed{\Gamma(x) = \lim_{n\to\infty} \frac{n^x \, n!}{x(x+1)\cdots(x+n)} \quad \text{for all } x \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}.}$$

Sanity check: $x = 3$, $\Gamma(3) = \lim_{n\to\infty} \frac{n^3 \, n!}{3 \cdot 4 \cdots (n+3)}$

$$= \lim_{n\to\infty} 1 \cdot 2 \cdot \frac{n^3}{(n+1)(n+2)(n+3)}$$

$$= 1 \cdot 2 = 2!$$

Similarly, you can verify it for $x = k$ pos integer.

Recall we proved $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

$$\sqrt{\pi} = \Gamma\left(\tfrac{1}{2}\right) = \lim_{n\to\infty} \frac{n^{1/2}\; n!}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n+1}{2}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\cdot n!}{1\cdot 3\cdot 5\cdots(2n+1)}\cdot 2^{n+1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\; 2^{n+1}\; n!\; \left(2\cdot 4\cdot 6\cdots(2n)\right)}{1\cdot 2\cdot 3\cdot 4\cdots(2n)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\cdot 2^{n+1}\; n!\; 2^{n}\cdot n!}{(2n+1)!}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\cdot 2^{2n+1}}{2n+1}\cdot\frac{1}{\binom{2n}{n}}$$

$$\Rightarrow \boxed{\sqrt{\pi} = \lim_{n\to\infty} \frac{4^{n}}{\sqrt{n}\cdot\binom{2n}{n}}}$$

$$\Gamma_n(x) := \frac{n^{x}\; n!}{x(x+1)\cdots(x+n)} \quad \text{so} \quad \Gamma(x) = \lim_{n\to\infty}\Gamma_n(x)$$

$$\Gamma_n(x) = e^{x\log n}\,\frac{1}{x}\cdot\frac{1}{(1+x)}\cdot\frac{1}{\left(1+\frac{x}{2}\right)}\cdots\frac{1}{\left(1+\frac{x}{n}\right)}$$

$$\Gamma_n(x) = e^{x\left(\log n - 1 - \frac{1}{2} - \cdots - \frac{1}{n}\right)}\,\frac{1}{x}\cdot\frac{e^{x}}{1+x}\cdot\frac{e^{x/2}}{1+x/2}\cdots\frac{e^{x/n}}{1+\frac{x}{n}}$$

Exercise The sequence $a_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n$ converges as $n \to \infty$. The limit is γ called the Euler constant $\quad \gamma \simeq 0.577\dots$

As $n \to \infty$:

$$\Gamma(x) = \lim_{n\to\infty} \Gamma_n(x) = e^{-\gamma x} \cdot \frac{1}{x} \lim_{n\to\infty} \prod_{i=1}^{n} \frac{e^{x/i}}{1 + x/i}$$

Formula:

$$\boxed{\Gamma(x) = e^{-\gamma x} \cdot \frac{1}{x} \cdot \prod_{i=1}^{\infty} \frac{e^{x/i}}{1 + x/i}}$$

~~for~~ all $x \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$

Suppose $x \in \mathbb{R} \setminus \mathbb{Z}$: change x to $-x$ here:

$$\Gamma(-x) = e^{\gamma x} \cdot \frac{1}{-x} \prod_{i=1}^{\infty} \frac{e^{-x/i}}{1 - \frac{x}{i}}$$

$$\Gamma(x) \cdot \Gamma(-x) = -\frac{1}{x^2} \prod_{i=1}^{\infty} \frac{1}{1 - \frac{x^2}{i^2}}$$

$$\frac{1}{\Gamma(x)\,\Gamma(-x)} = -x^2 \prod_{i=1}^{\infty} \left(1 - \frac{x^2}{i^2}\right)$$

Use $\Gamma(-x) = \dfrac{\Gamma(-x+1)}{-x} = -\dfrac{\Gamma(1-x)}{x}$

$$\frac{1}{\Gamma(x)\,\Gamma(1-x)} = x \prod_{i=1}^{\infty}\left(1-\frac{x^2}{i^2}\right)$$

Recall Euler's formula for $\sin x$:

$$\sin x = x \prod_{i=1}^{\infty}\left(1-\frac{x^2}{\pi^2 i^2}\right)$$

$$\sin \pi x = \pi x \prod_{i=1}^{\infty}\left(1-\frac{x^2}{i^2}\right)$$

So $$\frac{1}{\Gamma(x)\,\Gamma(1-x)} = \frac{\sin \pi x}{\pi} \qquad x \in \mathbb{R}\setminus\mathbb{Z}$$

Reflection formula $$\boxed{\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}}$$

for $x \in \mathbb{R}\setminus\mathbb{Z}$

Example $x=\frac{1}{2}$ $$\Gamma\left(\tfrac{1}{2}\right)^2 = \frac{\pi}{\sin\frac{\pi}{2}} = \pi \Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$ (we knew)

$x = \frac{1}{3}$ $$\Gamma\left(\tfrac{1}{3}\right)\cdot\Gamma\left(\tfrac{2}{3}\right) = \frac{\pi}{\sin \pi/3} = \frac{2\pi}{\sqrt{3}}$$