Lecture 8 : Special functions

17 November 2023

How can we <u>compute values</u> of $\Gamma(x)$ fast and accurately?

(Recall: $\Gamma(n)=(n-1)!$, $\Gamma(\tfrac{1}{2})=\sqrt{\pi}$, $\Gamma(n+\tfrac{1}{2})=(n-\tfrac{1}{2})\Gamma(n-\tfrac{1}{2})$ ...

but there are no explicit values for $\Gamma(x)$ in general)

The starting point is Stirling's formula from last time:

$$\Gamma(x)=\sqrt{2\pi}\; x^{x-\frac{1}{2}}\, e^{-x+\mu(x)}$$

where $$\mu(x)=\sum_{n=0}^{\infty} g(x+n) = \frac{\theta_x}{12x}$$ depends on x. $0<\theta_x<1$

Here $g(x)=(x+\tfrac{1}{2})\log(1+\tfrac{1}{x})-1$.

Goal: Give a better infinite series description of $\mu(x)$ that can be used easily to approximate $\mu(x)$.

Idea: Give an integral representation of $\mu(x)$.

Start by writing $g(x)$ as an integral.

$$\log(1+\tfrac{1}{x}) = \log(x+1)-\log x = \int_0^1 \frac{1}{t+x}\,dt$$

$$g(x)=(x+\tfrac{1}{2})\log(1+\tfrac{1}{x})-1 = (x+\tfrac{1}{2})\int_0^1 \frac{1}{t+x}\,dt - 1$$

$$=\int_0^1\left(\frac{x+\frac{1}{2}}{t+x}-1\right)dt = \int_0^1 \frac{\frac{1}{2}-t}{t+x}\,dt$$

$$\mu(x) = \sum_{n=0}^{\infty} g(x+n) = \sum_{n=0}^{\infty} \int_0^1 \frac{\frac{1}{2}-t}{t+n+x}\, dt$$

<u>Important trick</u>: Define $H(t) = \begin{cases} \frac{1}{2}-t, & 0<t<1 \\ 0, & t=0 \end{cases}$

(on the interval $[0,1)$)

and extend it to a periodic function on $\mathbb{R}$

so that $H(t+n) = H(t) \quad \forall\, n \in \mathbb{Z}$.

The reason for this is:

$$\mu(x) = \sum_{n=0}^{\infty} \int_0^1 \frac{H(t)}{t+n+x}\, dt \overset{\substack{\text{change variable} \\ t+n \to t}}{=\!=} \sum_{n=0}^{\infty} \int_n^{n+1} \frac{H(t-n)}{t+x}\, dt \overset{H \text{ periodic}}{=}$$

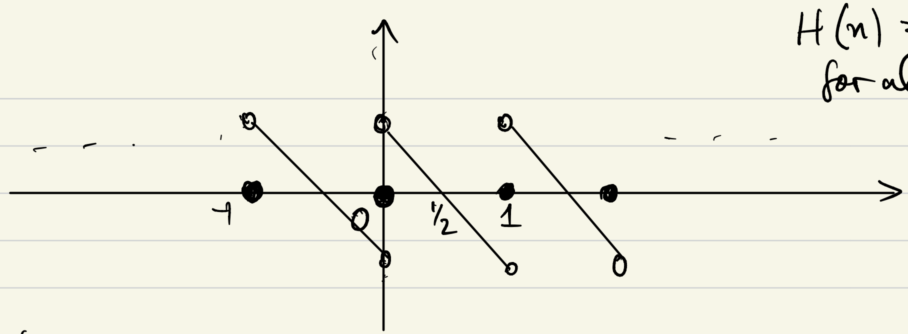
$$= \sum_{n=0}^{\infty} \int_n^{n+1} \frac{H(t)}{t+x}\, dt = \underbrace{\lim_{n\to\infty} \int_0^n \frac{H(t)}{t+x}\, dt}_{\text{converges b/c } \mu(x) \text{ converges}}$$

<u>Conclusion</u>:

$$\boxed{\mu(x) = \int_0^{\infty} \frac{H(t)}{t+x}\, dt}\,,$$ where $H(t)$ is the periodic function with period 1 defined above.

The graph of $H(t)$:

$H(n) = 0$
for all $n \in \mathbb{Z}$



Let's think how one could use the integral representation to produce a series:

$$\mu(x) = \int_0^\infty \frac{H(t)}{t+x}\,dt = \int_0^\infty \frac{K'(t)}{t+x}\,dt$$

(If there exists $K(t)$ s.t. $K'(t) = H(t)$)

Integration by parts

$$= \left.\frac{K(t)}{t+x}\right|_{t=0}^{t\to\infty} + \int_0^\infty \frac{K(t)}{(t+x)^2}\,dt$$

Suppose $\lim_{t\to\infty} \frac{K(t)}{t+x} = 0$ (since $H(t)$ is periodic and bounded $K(t)$ should also be periodic and bounded)

$$\mu(x) = -\frac{K(0)}{x} + \int_0^\infty \frac{K(t)}{(t+x)^2}\,dt$$

If there exists $L(t)$ such that $L'(t) = K(t)$ then you repeat the procedure and get:

$$\mu(x) = -\frac{K(0)}{x} - \frac{L(0)}{x^2} + 2\int_0^\infty \frac{L(t)}{(t+x)^3}\,dt$$

etc.

We need a sequence of succesive antiderivatives.

I want to construct a sequence of nice periodic functions $H_n(t)$ such that

$$H_1(t) = -H(t) \text{ and}$$

$$H_{n+1}'(t) = H_n(t).$$

To do this we use <u>Fourier series</u> (well-suited for periodic functions).

<u>Defn</u> Suppose the function $f(t)$ is defined on the interval $[0, P)$ and extended periodically with period p on $\mathbb{R}$. Define

$$a_n = \frac{2}{P} \int_0^P f(t) \cos\left(\frac{2n\pi t}{P}\right) dt, \quad n \geq 0$$

$$b_n = \frac{2}{P} \int_0^P f(t) \sin\left(\frac{2n\pi t}{P}\right) dt$$

Then $$\boxed{f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi t}{P}\right) + b_n \sin\left(\frac{2n\pi t}{P}\right)}$$

(Fourier series of $f(t)$)

if f is an L^2-function. In particular, if f is a continuous function except finitely many jump discontinuities in $[0, P)$, then f has a Fourier series expansion.

<u>Remark</u> Often f is defined on $[-\pi,\pi]$ and periodic with period 2π, then define Fourier coefficients as $\frac{1}{\pi}\int_{-\pi}^{\pi}$.

<u>Example</u> $H(t)$ is continuous except at the integers (jump discontinuities) so it has a Fourier series.

Notice : $H(t)$ is an odd function $H(-t)=-H(t)$.

so $a_n = 0 \ \forall n \geq 0$. Compute

$$b_n = 2\int_0^1 H(t)\sin(2n\pi t)\,dt \qquad n \geq 1.$$

$$= 2\int_0^1 \left(\frac{1}{2}-t\right) \underbrace{\sin(2n\pi t)}_{= \frac{1}{2n\pi}\left(-\cos(2n\pi t)\right)'} dt \overset{\text{Integration by parts.}}{=}$$

$$= \left[2\cdot\left(\frac{1}{2}-t\right)\cdot\frac{1}{2n\pi}\left(-\cos(2n\pi t)\right)\right]_{t=0}^{t=1}$$

$$- \cancel{2}\int_0^1 (-1)\cdot\frac{1}{\cancel{2}n\pi}(-\cos 2n\pi t)\,dt$$

$$= \cancel{2}\left(-\frac{1}{\cancel{2}}\right)\cdot\frac{1}{2n\pi}\cdot(-1) - \cancel{2}\left(\frac{1}{\cancel{2}}\right)\cdot\frac{1}{2n\pi}\cdot(-1)$$

$$- \cancel{\frac{1}{n\pi}\int_0^1 \cos 2n\pi t}\ dt$$

$$= \frac{1}{2n\pi} + \frac{1}{2n\pi} = \frac{1}{n\pi}$$

We proved that :

$$\boxed{H(t) = \sum_{n=1}^{\infty} \frac{\sin(2n\pi t)}{n\pi}}$$, Fourier series.

$$H_1(t) = -2 \sum_{n=1}^{\infty} \frac{\sin(2n\pi t)}{2n\pi}$$ Integrate term by term

$$H_2(t) = 2 \sum_{n=1}^{\infty} \frac{\cos(2n\pi t)}{(2n\pi)^2}$$

$$H_3(t) = 2 \sum_{n=1}^{\infty} \frac{\sin(2n\pi t)}{(2n\pi)^3}$$

$$H_4(t) = -2 \sum_{n=1}^{\infty} \frac{\cos(2n\pi t)}{(2n\pi)^4}$$

In general, define :

$$\boxed{\begin{aligned} H_{2k-1}(t) &= 2(-1)^k \sum_{n=1}^{\infty} \frac{\sin(2n\pi t)}{(2n\pi)^{2k-1}} \\ H_{2k}(t) &= 2(-1)^{k-1} \sum_{n=1}^{\infty} \frac{\cos(2n\pi t)}{(2n\pi)^{2k}} \end{aligned}}$$

<u>Properties</u> 1) $H_1(t) = -H(t)$.

2) $H_k(0) = H_k(1)$ in fact $H_k(t)$ is periodic with period 1 for all k.

3) $H_{2k-1}(0) = H_{2k-1}(1) = 0$.

4) Because $|\sin(2n\pi t)| \leq 1$ and same for cosine, the n^{t}-th term is $H_k(t)$ with $k \geq 2$ is bounded above by

$$\frac{1}{(2n\pi)^k}$$

But the series $\sum \frac{1}{n^k}$ converges for $k \geq 2$. So by the comparison test the series for $H_k(t)$, $k \geq 2$, converges absolutely and uniformly. This says (some result in basic analysis) that indeed we can differentiate/integrate term by term.

So $H'_{k+1}(t) = H_k(t)$ $k \geq 1$ is a valid formula.

$$\int_0^1 H_k(t)\,dt = H_{k+1}(1) - H_{k+1}(0) = 0$$

$k \geq 1$ b/c of periodicity of $H_{k+1}(t)$.

In conclusion, the functions $H_k(t)$ satisfy:

1) $H_1(t) = -H(t) = t - \frac{1}{2}$ on $(0,1)$.

2) $H'_{k+1}(t) = H_k(t)$

3) $\int_0^1 H_k(t) = 0$, $k \geq 1$.

These look a lot like the <u>Bernoulli polynomials</u>:

$B_k(x)$ are the <u>unique</u> sequence of polynomials on $[0,1]$ such that:

1) $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$

2) $B'_{k+1}(x) = (k+1)B_k(x)$, $k \geq 0$.

3) $\int_0^1 B_k(x) = 0$ for $k \geq 1$.

Therefore, on the interval $(0,1)$ it must be true that

$$\boxed{H_k(t) = \frac{B_k(t)}{k!} \;,\; k \geq 1 \text{ on } (0,1)}$$

In particular

$$H_k(0) = \frac{B_k}{k!}, \; k \geq 2 \text{ where } B_k \text{ is the Bernoulli number.}$$

so $H_{2k+1}(0) = 0$ for all $k \geq 1$.

Back to $H_{2k}(t) = 2(-1)^{k-1} \sum_{n=1}^{\infty} \frac{\cos(2n\pi t)}{(2n\pi)^{2k}}$

Plug in $t = 0$

$$\frac{B_{2k}}{(2k)!} = 2(-1)^{k-1} \sum_{n=1}^{\infty} \frac{1}{(2\pi)^{2k} n^{2k}}$$

Hence $$\boxed{\zeta(2k)=\sum_{n=1}^{\infty}\frac{1}{n^k}=\frac{(-1)^{k-1}B_{2k}(2\pi)^{2k}}{2\cdot(2k)!}}$$

We recovered the formula for $\zeta(2k)$ in terms of Bernoulli numbers!! (This proof used Fourier series.)

Back to $\mu(x)$:

$$\mu(x)=\int_0^\infty \frac{H(t)}{t+x}\,dt=-\int_0^\infty\frac{H_1(t)}{t+x}\,dt$$

$$=-\int_0^\infty \frac{H_2'(t)}{t+x}\,dt \underset{\substack{\text{integration}\\ \text{by parts}}}{=} -\frac{H_2(t)}{t+x}\Big]_0^\infty-\int_0^\infty\frac{H_2(t)}{(t+x)^2}\,dt$$

$$=\frac{H_2(0)}{x}-\int_0^\infty\frac{H_2(t)}{(t+x)^2}\,dt$$ because $H_2(t)$ is bounded hence $\lim_{t\to\infty}\frac{H_2(t)}{t+x}=0$

$$=\frac{H_2(0)}{x}-\int_0^\infty\frac{H_3'(t)}{(t+x)^2}\,dt=\frac{H_2(0)}{x}-\frac{H_3(t)}{(t+x)^2}\Big]_0^\infty-2\int_0^\infty\frac{H_3(t)}{(t+x)^3}\,dt \quad \text{etc.}$$

We find inductively:

$$\mu(x)=\frac{H_2(0)}{x}+\frac{H_3(0)\cdot 1!}{x^2}+\frac{H_4(0)\cdot 2!}{x^3}+\dots+\frac{H_k(0)(k-2)!}{x^{k-1}}$$

$$- \int_0^\infty \frac{H_k(t)\,(k-1)!}{(t+x)^k}\,dt.$$

Rewrite this in terms of Bernoulli numbers:

$$H_k(0) = \frac{B_k}{k!} \qquad k \geq 2.$$

All odd terms vanish:

$$\mu(x) = \frac{B_2}{2\cdot 1\, x} + \frac{B_4}{4\cdot 3\, x^3} + \frac{B_6}{6\cdot 5\cdot x^4} + \dots +$$

$$+ \frac{B_{2k}}{2k\cdot(2k-1)\, x^{2k-1}} + \dots$$

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}$$

$$\underbrace{\mu(x) = \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}}_{\text{good approximation}} - \dots \qquad (*)$$

but as $k \to \infty$ B_{2k} becomes very large

Next problem sheet: Compute the asymptotic behaviour of B_{2k} as $k \to \infty$ (it grows very large)

You can use $(*)$ to estimate $\log \Gamma(x)$ up to five(?) decimal places (As long as k is not too large, this gives a good approximation of $\mu(x)$.)

To finish: We proved Stirling's fomula for $\Gamma(x)$ (and $n!$) and we related the "error" $\mu(x)$ to Bernoulli numbers using Fourier series.