Special functions: Lecture 1

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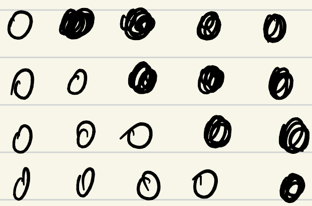
Bernoulli polynomials

Motivating question: What is the sum of k powers of the first n natural numbers?

Examples

$k=1$ $\quad 1+2+3+\ldots+n = \frac{n(n+1)}{2}$ To visualise the proof



4 rows & 5 columns so a total of 4×5

$\Rightarrow 1+2+3+4 = \frac{4\times 5}{2}$

(same for n)

$k=2$ $\quad 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

I don't know a geometric way of proving it, but we can prove by induction.

Base case: $n=1$ $\quad 1^2 = \frac{1\cdot 2\cdot 3}{6} = 1$ ✓

Inductive step: We assume the proposition $P(n)$ is true and we show it follows that

$P(n+1)$ is true as well $\quad (P(n) \Rightarrow P(n+1))$.

Assume $1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$. Add $(n+1)^2$ on both sides:

$$1^2+2^2+\ldots+n^2+(n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n+1}{6}\Big(\underbrace{n(2n+1)+6(n+1)}_{2n^2+7n+6}\Big) = \frac{(n+1)(n+2)(2n+3)}{6} \checkmark$$

What about cubes?

$$1^3+2^3+\ldots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$$

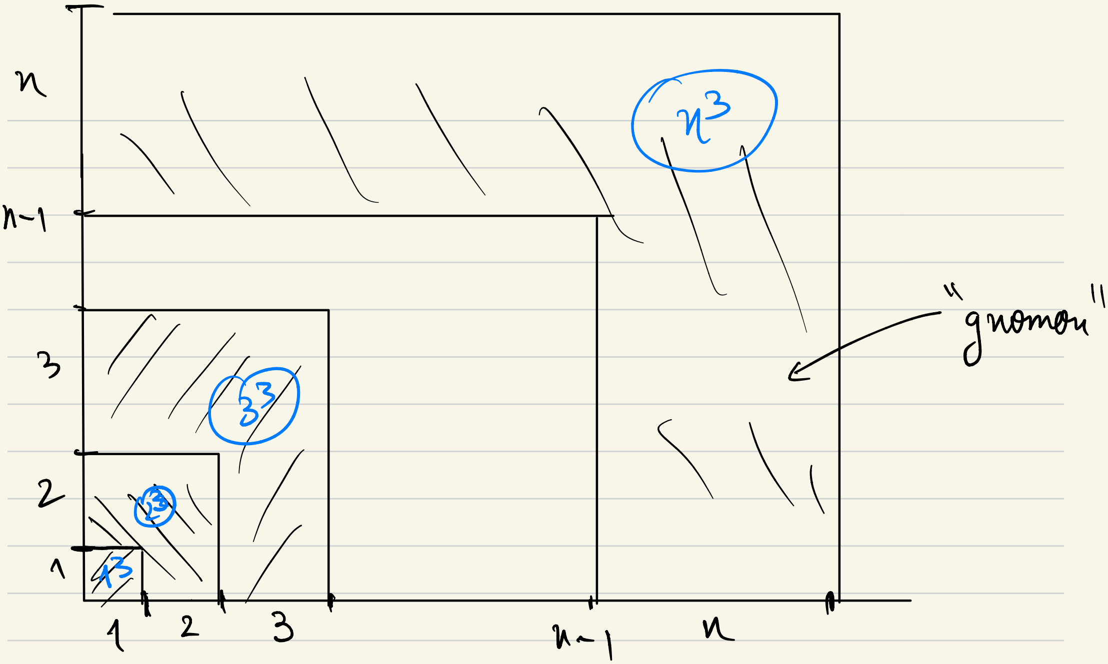
This has a geometric meaning

$$1^3+2^3+\ldots+n^3 = (1+2+\ldots+n)^2$$

Draw a square of side

$$1+2+\ldots+n$$

and compute the area in two different ways.



Area of a gnomon:

$$(1+2+\dots+n)^2 - (1+2+\dots+n-1)^2 = \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{(n-1)n}{2}\right)^2$$

$$= \frac{n^2}{4}\left((n+1)^2 - (n-1)^2\right) = \frac{n^2}{4}(4n) = n^3. \qquad \square$$

What about a general method?

One solution involves the Binomial formula.

Defn The number of ways to choose k objects out of n is the binomial coefficient $\binom{n}{k}$

(the order doesn't matter) "n choose k"

Some other notations are C^n_k or ${}^nC_k$

Formula: $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ $\qquad n! = 1\cdot 2\cdot 3 \cdots n$

$$= \frac{n(n-1)\dots(n-k+1)}{1\cdot 2\cdot \dots \cdot k}$$

(How do you prove this?)

$$\binom{4}{2} = \frac{4\cdot 3}{1\cdot 2} = 6.$$

The way to visualise binomial coefficients is using Pascal's triangle

$n=0$ $\qquad \binom{0}{0} = 1$

$n=1$ $\qquad 1 = \binom{1}{0} \qquad \binom{1}{1} = 1$

$n=2$ $\qquad 1 = \binom{2}{0} \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$

$n=3$ $\qquad 1 = \binom{3}{0} \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = 3 \qquad \binom{3}{3} = 1$

The sides are all 1 and any other entry is the sum of the two entries above it.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

etc.

This is based on the following basic properties:

1) $\binom{n}{k} = \binom{n}{n-k}$ symmetry about the middle vertical line.

2) $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

Proof of 2) You can use the factorial formula to prove it...

Combinatorially: $n+1$ objects. Think of n being black and one blue.

$$\bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet$$

To choose k objects: $= \binom{n+1}{k}$

- either choose k black $\binom{n}{k}$
- choose the blue one and then $k-1$ black $\binom{n}{k-1}$.

Binomial Theorem n natural number

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{n} x^n$$

Proof: use induction on n and the basic identity of binomial coeffs.

Here's how we may apply the Binomial Thm to compute the sum of k^{th} powers.

Example: $1^4+2^4+\dots+n^4$. Start with the Binomial Theorem for one degree higher

$$(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$$

$$(1+x)^5-x^5 = 1+5x+10x^2+10x^3+5x^4$$

Plug in $x=1,2,3,\dots,n$

$$\begin{aligned}
x=1 \quad & \cancel{2^5}-1^5 = 1+5\cdot 1+10\cdot 1^2+10\cdot 1^3+5\cdot 1^4 \\
x=2 \quad & \cancel{3^5}-\cancel{2^5} = 1+5\cdot 2+10\cdot 2^2+10\cdot 2^3+5\cdot 2^4 \\
x=3 \quad & \cancel{4^5}-\cancel{3^5} = 1+5\cdot 3+10\cdot 3^2+10\cdot 3^3+5\cdot 3^4 \\
& \vdots \\
x=n \quad & (n+1)^5-\cancel{n^5} = 1+5\cdot n+10\cdot n^2+10\cdot n^3+5\cdot n^4
\end{aligned}$$

add

$$(n+1)^5-1 = \underbrace{n}_{=S_0(n)}+5\cdot S_1(n)+10\cdot S_2(n)+10\,S_3(n)+5\,S_4(n)$$

where $S_k(n) = 1^k+2^k+\dots+n^k$

Then

$$5\,S_4(n) = (n+1)^5-1-n-5\,S_1(n)-10\,S_2(n)-10\,S_3(n)$$

Inductively, we know $S_1(n)$, $S_2(n)$, $S_3(n)$

Do the algebra:

$$S_4(n) = \frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}.$$

Notice This method gives us a recursive way to compute $S_k(n)$.

Is there a different way?

Look at the formulas:

$$S_1(n) = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

$$S_2(n) = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \dots$$

$$S_3(n) = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

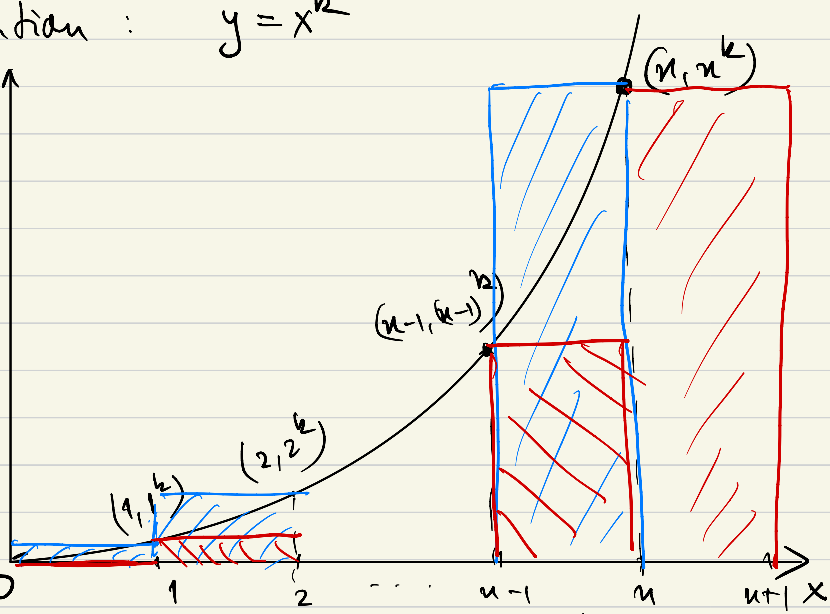
$$S_4(n) = \frac{n^5}{5} + \dots$$

In general $S_k(n) = \dfrac{n^{k+1}}{k+1}$ + smaller powers of n.

This reminds us of Calculus:

$$\int_0^n x^k\,dx = \frac{x^{k+1}}{k+1}\Bigg]_0^n = \frac{n^{k+1}}{k+1}$$

The explanation: $y = x^k$



The sum of the areas of the blue rectangles is $S_k(n)$ (This is a Riemann sum)

$>$ area between 0 and n $= \int_0^n x^k\,dx$.

Red rect. area is also a Riemann sum.

$S_k(n) <$ area between 0 and $n+1$ $= \int_0^{n+1} x^k\,dx$.

$$\int_0^n x^k\,dx < S_k(n) < \int_0^{n+1} x^k\,dx$$

$$\int_0^n x^k\,dx = \frac{n^{k+1}}{k+1}, \qquad \int_0^{n+1} x^k\,dx = \frac{(n+1)^{k+1}}{k+1}$$

(in fact upper bound is $\int_1^{n+1} x^k\,dx$)

Is there a polynomial $B_k(x)$ of degree k such that

$$S_k(n) = \int_0^{n+1} B_k(x)\,dx$$

x^k doesn't quite work it gives too much

so $B_k(x) = x^k$ + smaller terms

(would like coeffs of $B_k(x)$ to be independent of n)

Example $\underline{k=1}$:

$$\frac{n(n+1)}{2} = \int_0^{n+1} (x+a)\,dx$$

$$= \frac{x^2}{2} + ax \Big]_{x=0}^{x=n+1}$$

$$\frac{n\cancel{(n+1)}}{2} = \frac{(n+1)^{\cancel{2}}}{2} + a\cancel{(n+1)} \qquad | : (n+1)$$

$$a = -\frac{1}{2} \qquad\qquad B_1(x) = x - \frac{1}{2}$$

$\underline{k=2}$

$$S_2(n) = \frac{n(n+1)(2n+1)}{6} = \int_0^{n+1} (x^2+ax+b)\,dx$$

It's possible to do the calculation as before and solve a linear system of equations, but there is another trick:

Write LHS as a polynomial in $(n+1)$:

$$\text{LHS} = \frac{((n+1)-1)(n+1)(2(n+1)-1)}{6}$$

$$= \frac{((n+1)^2-(n+1))(2(n+1)-1)}{6} = \frac{2(n+1)^3-3(n+1)^2+(n+1)}{6}$$

$$= \frac{(n+1)^3}{3} - \frac{(n+1)^2}{2} + \frac{(n+1)}{6} = \int_0^{n+1} (x^2+ax+b)\,dx$$

$$\Rightarrow \quad a=-1, \quad b=\frac{1}{6}$$

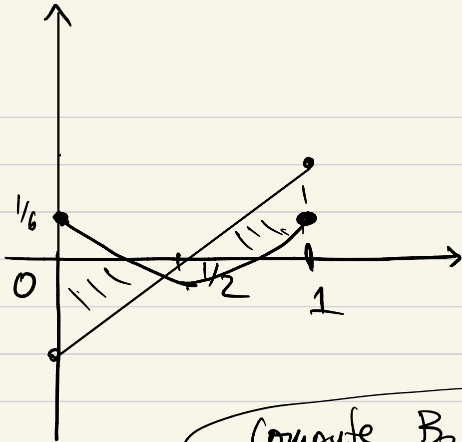
Get $\quad B_2(x) = x^2 - x + \frac{1}{6} \qquad B_1(x) = x - \frac{1}{2}$

Can try $B_3(x)$ ... $\qquad$ These are called <u>Bernoulli polynomials</u>

<u>Defn</u> (Bernoulli polynomials) This is the set of polynomials $B_0(x), B_1(x), B_2(x), \dots$, defined by:

1) $B_0(x) = 1$.
2) $B'_{k+1}(x) = (k+1)\,B_k(x), \quad k \geq 0$.
3) $\displaystyle\int_0^1 B_k(x)\,dx = 0, \quad k \geq 1$.

$B_1(x) = x - \frac{1}{2}$



$B_2(x) = x^2 - x + \frac{1}{6}$

it has a min at $x = \frac{1}{2}$

$$\int_0^1 B_2(x)\,dx = \left. \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{6} \right]_{x=0}^{x=1}$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{6} = 0$$

Compute $B_3(x)$, $B_4(x)$ on your own and graph them.

Next time: We'll prove:

Theorem For all $n,\ k \geq 1$:

$$\underbrace{1^k + 2^k + \cdots + n^k}_{S_k(n)} = \int_0^{n+1} B_k(x)\,dx.$$

Then look other properties and applications of Bernoulli polynomials.