Introduction to $\overline{\mathbf{R}}$

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1 Introduction

There's somewhat of an elephant in the room when it comes to mathematical analysis – and that elephant is infinity. Early on we are told to restrain ourselves to understanding ∞ as no more than an abstract symbol, and not something that can be likened to a real number. We usually give up with this when we start studying measure theory, and it becomes convenient to write things like $\int f(x)dx = \infty$, and talk about regions having infinite volume.

The purpose of this document, which in all likelihood only I will see, is to give a "proper" treatment of the extended real numbers. (which is often inferred or handwave) I will take for granted knowledge of topology. The proofs will all be fairly basic, but interesting to explicitly outline since this is often not done in analysis texts. I will take $\mathbf{N}$ to exclude 0, and $\langle x_n \rangle$ to denote a sequence indexed on $\mathbf{N}$.

We start just with the symbols "∞" and "$-\infty$", which we will call *positive infinity* and *negative infinity* respectively. These can represent whatever sets you like, provided they do not lie in $\mathbf{R}$.[1] We will write $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$, which we will call the *extended real number line*. We will call $x \in \overline{\mathbf{R}}$ "finite" if $x \notin \{-\infty, \infty\}$, and infinite if $x \in \{-\infty, \infty\}$.

2 Arithmetic

We define arithmetic on the extended real numbers in a very natural way.

Definition 2.1 (Addition)**.** For each $x \in \mathbf{R}$, we define $x +_{\overline{\mathbf{R}}} \infty = \infty +_{\overline{\mathbf{R}}} x = \infty$ and $x +_{\overline{\mathbf{R}}} (-\infty) = (-\infty) +_{\overline{\mathbf{R}}} x = -\infty$. We also define $\infty +_{\overline{\mathbf{R}}} \infty = \infty$.

From now on we will write $+$ instead of $+_{\overline{\mathbf{R}}}$. Nothing bizarre so far, though I will now make a potentially controversial choice with our definition of multiplication.

Definition 2.2 (Multiplication)**.** For each $x \in \mathbf{R}$, we define:

$$x \times_{\overline{\mathbf{R}}} \infty = \infty \times_{\overline{\mathbf{R}}} x = \begin{cases} 0 & x = 0 \\ \infty & x > 0 \\ -\infty & x < 0 \end{cases}$$

[1] I am not aware of a standard, but in ZF and ZFC, the set $\mathbf{R} \times \{\mathbf{R}\}$ cannot possibly lie in $\mathbf{R}$, so can be used. If we had $\mathbf{R} \in \mathbf{R} \times \{\mathbf{R}\}$, then $\mathbf{R} = (x, \mathbf{R})$ for some $x \in \mathbf{R}$. This would imply $\mathbf{R} \in \{\mathbf{R}\} \in (x, \mathbf{R}) = \mathbf{R}$, using the Kuratowski convention for ordered pairs. Applying **Foundation** to $\{\mathbf{R}, \{\mathbf{R}\}\}$, one of the $\in$ inclusions will be contradicted, so we have $\mathbf{R} \notin \mathbf{R} \times \{\mathbf{R}\}$. In general $x \notin x \times \{x\}$ for any set x.

and:

$$(-\infty) \times_{\overline{\mathbf{R}}} x = x \times_{\overline{\mathbf{R}}} (-\infty) = \begin{cases} 0 & x = 0 \\ -\infty & x > 0 \\ \infty & x < 0 \end{cases}$$

From now on we will write $\times$ instead of $\times_{\overline{\mathbf{R}}}$. The aforementioned "controversial choice" would probably be $0 \times \infty = 0$. Essentially we want lines (or more obtusely "rectangles with infinite width and zero height") such as $\{0\} \times \mathbf{R}$, or $\mathbf{R} \times \{0\}$ to have measure zero. When we are constructing a notion of volume for $\mathbf{R}^2$, we would rather like the volume of a Cartesian product $A \times B$ to simply be the "length" of A times the "length" of B, consistent with ordinary bounded rectangles. For this we will want to write or imply that $0 \times \infty = 0$, since the length of $\{0\}$ will be 0 under the canonical measure for $\mathbf{R}^2$, and the length of $\mathbf{R}$ will be ∞.

Subtraction can now be defined simply by $x -_{\overline{\mathbf{R}}} y = x +_{\overline{\mathbf{R}}} (-y)$. We will largely avoid division, but sometimes we may want to think that $1/\infty = 0$ – in certain embedding results we want to think of $\{p, q\} = \{1, \infty\}$ as satisfying $1/p + 1/q = 1$.

Notice that notably we leave $\infty - \infty$ undefined. We will see why later, but we clearly don't want to set $\infty - \infty = 0$, otherwise we could write, say $0 = \infty - \infty = (1 + \infty) - \infty = 1$, proving $1 = 0$! In a less obviously abusive framing, say we have two extended real numbers α and β and we find that $\alpha = \alpha + \beta$. We *cannot* conclude that $\beta = 0$, since we may have $\alpha = \infty$, in which case β could be any extended real number except $-\infty$. This gets even funkier when we look at orders on $\overline{\mathbf{R}}$, as we will see.

3 Ordering

We will now extend our canonical order on $\mathbf{R}$ to $\overline{\mathbf{R}}$. This will be consistent with our intuition that ∞ should be larger than any real number, and $-\infty$ should be "more negative" than any real number. We define $<_{\overline{\mathbf{R}}}$ by $x <_{\overline{\mathbf{R}}} \infty$ if $x \neq \infty$ and $-\infty <_{\overline{\mathbf{R}}} x$ if $x \neq -\infty$, and $x <_{\overline{\mathbf{R}}} y$ if $x, y \in \mathbf{R}$ and $x < y$ in the usual order. We will now write $<_{\overline{\mathbf{R}}}$ as simply $<$, as we did with addition and multiplication.

The nice thing now is that $(\overline{\mathbf{R}}, <_{\overline{\mathbf{R}}})$ is that every sequence has a convergent subsequence, and that $(\overline{\mathbf{R}}, <_{\overline{\mathbf{R}}})$ is a "complete lattice" – every set has an infimum and supremum. We will now prove this.

Theorem 3.1. *Every set in $\overline{\mathbf{R}}$ has an infimum and supremum.*[2]

Proof. Let $T \subseteq \overline{\mathbf{R}}$. We deal first with the case $T = \emptyset$. Note that $-\infty$ is an upper bound for $\emptyset$, and ∞ is a lower bound for $\emptyset$. No extended real number is less than $-\infty$ or greater than ∞, so $-\infty$ is the least upper bound for $\emptyset$ and ∞ is the greatest lower bound. So we have $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. (weird, right?) Notably, this is the only case where $\inf T > \sup T$, if $T \neq \emptyset$ we have $\inf T \leq \sup T$, with equality holding only for singletons.

[2] Actually, we only need to show that every set in $\overline{\mathbf{R}}$ has an infimum, since we can then just take the infimum of the set of upper bounds, but I do both for slickness.

Now let $T \subseteq \overline{\mathbf{R}}$ be non–empty. Note that ∞ is an upper bound for T. If $\infty \in T$, then we must have $\sup T = \infty$, since any "interior" upper/lower bounds are supremums/infimums. Suppose $\infty \notin T$. If T has a supremum in the usual order of $\mathbf{R}$, great, it is also a supremum in the order of $\overline{\mathbf{R}}$. If not, then for each M there exists $t \in T$ with $t > M$. So no real number can be an upper bound for T, because it will be exceeded by some other element. So the only upper bound is ∞, giving $\sup T = \infty$.

The case of the infimum is similar. Note that $-\infty$ is always a lower bound for T. So if $-\infty \in T$, we have $\inf T = -\infty$. If T has an infimum in the usual order of $\mathbf{R}$, brilliant, we are again done. If not, then for every M there exists $t \in T$ with $t < M$. So no real number is a lower bound for T, so the only lower bound is $-\infty$, giving $\inf T = -\infty$. ■

So we now formally have $\sup(0, \infty) = \infty$ and $\inf(-\infty, 0) = -\infty$, and so on. As a final note, we now need to be careful with inequalities. If we are talking about α in the extended real numbers, we are no longer able to say $\alpha < \alpha + 1$, since this is not true if α is infinite, so often you will have to use non–strict inequalities throughout. (which is usually safer, even in the reals) This is somewhat reminscient of cardinal arithmetic, where we may have $|X| = |X| + 1$ or $|X| < |X| + 1$ when $|X|$ is infinite. (see: *Dedekind infinite*)

4 Topology

Now that we have arithmetic and supremums/infimums set up solidly, we move onto the topology of $\overline{\mathbf{R}}$. We define the *standard topology* on $\overline{\mathbf{R}}$ as the topology generated by the standard topology on $\mathbf{R}$ together with the intervals $(a, \infty] = (a, \infty) \cup \{\infty\}$ and $[-\infty, b) = (-\infty, b) \cup \{-\infty\}$, which we consider neighbourhoods of ∞ and $-\infty$ respectively. This is precisely the order topology of $\overline{\mathbf{R}}$ with respect to $<$, but we will give this bespoke treatment.

A simple check reveals $\overline{\mathbf{R}}$ is Hausdorff (as are all order topologies) with this topology:

(i) $(x - 1/2, x + 1/2)$ and $(x + 1, \infty]$ separate $x \in \mathbf{R}$ from ∞,

(ii) $(x - 1/2, x + 1/2)$ and $[-\infty, x - 1)$ separate $x \in \mathbf{R}$ from $-\infty$,

(iii) $[-\infty, 0)$ and $(0, \infty]$ separate ∞ from $-\infty$.

The separation of other points follows from the fact that $\mathbf{R}$ is Hausdorff with the usual topology. This ensures that limits in $\overline{\mathbf{R}}$ are unique, though we don't yet know how to calculate them. Our goal will now to be to first establish simple criteria for convergence in $\overline{\mathbf{R}}$, and then to show that it is both compact and sequentially compact.

Theorem 4.1. $\overline{\mathbf{R}}$ *is compact with its standard topology.*

Proof. Let $\mathcal{C}$ be an open cover for $\overline{\mathbf{R}}$. Let U_∞ be a set in the cover that contains ∞. We can pull from it an open subset $(M_2, \infty]$. Let $U_{-\infty}$ be a set in the cover that contains $-\infty$. We can pull from it an open subset $[-\infty, M_1)$.

Now consider $\mathcal{C} \cap [M_1, M_2] = \{S \cap [M_1, M_2] : S \in \mathcal{C}\}$. This is an open cover of $[M_1, M_2]$. From the compactness of $[M_1, M_2]$, we have a finite cover $\{S_1 \cap [M_1, M_2], \ldots, S_n \cap [M_1, M_2]\}$ of $[M_1, M_2]$. Since $\overline{\mathbf{R}} = [-\infty, M_1) \cup [M_1, M_2] \cup (M_2, \infty]$, $\{S_1, \ldots, S_n, U_\infty, U_{-\infty}\}$ is then a subcover of $\mathcal{C}$ of size at most[3] $n+2<\infty$. Since the cover $\mathcal{C}$ was arbitrary we have that $\overline{\mathbf{R}}$ is compact. ■

In light of this result, $\overline{\mathbf{R}}$ can be called the *two–point compactification* of $\mathbf{R}$. (there are other constructions that only add one point) We now look at characterising convergence in $\overline{\mathbf{R}}$. First, we verify that a sequence $\langle x_n \rangle \subset \mathbf{R}$ that converges to x in $\mathbf{R}$ still converges to x in $\overline{\mathbf{R}}$. We will frame this a bit more generally.

Theorem 4.2. *Let $\langle x_n \rangle$ be a sequence in $\overline{\mathbf{R}}$ and let $x \in \mathbf{R}$. Then we have $x_n \to x$ in $\overline{\mathbf{R}}$ if and only if:*

(i) there exists $N_1 \in \mathbf{N}$ such that for $n \geq N_1$ we have $x_n \in \mathbf{R}$,

(ii) for each $\varepsilon > 0$ there exists $N_2 \in \mathbf{N}$ such that for $n \geq \max\{N_1, N_2\}$ we have $|x_n - x| < \varepsilon$.

Evidently (i) is satisfied for any real sequence, just taking $N_1 = 1$, so if we can show this we have the desired consistency result.

Proof. Suppose that $x_n \to x$ in $\overline{\mathbf{R}}$. Then for each $\varepsilon > 0$, there exists $N_1 \in \mathbf{N}$ such that we have $x_n \in (x-\varepsilon, x+\varepsilon) \subset \mathbf{R}$ for $n \geq N_1$, since $(x-\varepsilon, x+\varepsilon)$ is an open neighbourhood of x. Fixing ε, we see that $x_n \in \mathbf{R}$ for $n \geq N_1$. Now let $\varepsilon > 0$ be arbitrary, then we can select $N_2 = N_2(\varepsilon) \in \mathbf{N}$ such that $x_n \in (x-\varepsilon, x+\varepsilon)$ for $n \geq N_2$. That is, $|x_n - x| < \varepsilon$. This proves the "only if" direction.

Now suppose that (i) and (ii) hold for a sequence $\langle x_n \rangle$. We aim to show that for each open neighbourhood of x, U, in $\overline{\mathbf{R}}$, there exists $N \in \mathbf{N}$ such that $x_n \in U$ for $n \geq N$. Fix an open neighbourhood of x. We can pull out an open neighbourhood of x that has one of the forms (a,b), $(M,\infty]$ or $[-\infty, m)$. Note that WLOG we can assume the form is (a,b), by looking at $(x-\varepsilon, x+\varepsilon)$ for x sufficiently small. From (ii) we can find $N \in \mathbf{N}$ such that $|x_n - x| < \varepsilon$ for $n \geq N$. In particular, $x_n \in U$ for $n \geq N$. This proves the "if" direction. ■

Now we have our sanity check out of the way, we can move to the more interesting matter of convergence to $\pm\infty$, which will coincide with the typical definition of "divergence to" $\pm\infty$.

Theorem 4.3. *Let $\langle x_n \rangle$ be a sequence in $\overline{\mathbf{R}}$. We have $x_n \to \infty$ if and only if for each $M > 0$ there exists $N \in \mathbf{N}$ such that $x_n > M$ for $n \geq N$.*

Proof. Suppose that $x_n \to \infty$. Then $(M,\infty]$ is an open neighbourhood of ∞, and so there exists $N \in \mathbf{N}$ such that $x_n \in (M,\infty]$ for $n \geq N$. That is, $x_n > M$.

Now to the converse, fix an open neighbourhood U of ∞. From this we can pull a subset $(M,\infty]$. We can find $N \in \mathbf{N}$ such that $x_n > M$ for all $n \geq N$. In particular we have $x_n \in U$ for $n \geq N$. So $x_n \to \infty$. ■

An almost identical procedure shows that:

Theorem 4.4. *Let $\langle x_n \rangle$ be a sequence in $\overline{\mathbf{R}}$. We have $x_n \to -\infty$ if and only if for each $M > 0$ there exists $N \in \mathbf{N}$ such that $x_n < -M$ for $n \geq N$.*

[3] one or more of the S_is may be equal to U_∞ or $U_{-\infty}$

We can now show that $\overline{\mathbf{R}}$ is sequentially compact. We will then go on to prove properties of the limit in $\overline{\mathbf{R}}$. We first need a lemma.

Lemma 4.5. *A monotone sequence in $\overline{\mathbf{R}}$ converges.*

Proof. Let $\langle x_n \rangle$ be a monotone sequence in $\overline{\mathbf{R}}$. First take $\langle x_n \rangle$ to be increasing. If $\sup_n x_n < \infty$, then we know from real analysis that $x_n \to \sup_n x_n$. Now suppose that $\sup_n x_n = \infty$. Then for each $k \in \mathbf{N}$ we can find some n_k such that $x_{n_k} \geq k$. Without loss of generality, pick $\langle n_k \rangle$ to be increasing. Then, for each $M > 0$ we have $x_{n_k} \geq M$ for $k \geq \lfloor M \rfloor + 1$. So we have $x_{n_k} \to \infty$.

The case of decreasing sequences is very similar. If $\infty > \inf_n x_n > -\infty$, (note that the infimum cannot be ∞ since $\{x_n : n \in \mathbf{N}\}$ is not empty) then we know that $x_n \to \inf_n x_n$. Now suppose that $\inf_n x_n = -\infty$. Then for each $k \in \mathbf{N}$ we can find some n_k such that $x_{n_k} \leq -k$. Without loss of generality, pick $\langle n_k \rangle$ to be increasing. Then for each $M > 0$ we have $x_{n_k} \leq -M$ for $k \geq \lfloor M \rfloor + 1$. So we have $x_{n_k} \to -\infty$. ■

To show that $\overline{\mathbf{R}}$ is sequentially compact, it now suffices to show that every sequence in $\overline{\mathbf{R}}$ has a monotone subsequence.

Theorem 4.6. *Every sequence in $\overline{\mathbf{R}}$ has a monotone subsequence.*

Proof. Let $\langle x_n \rangle$ be a sequence in $\overline{\mathbf{R}}$. If $\langle x_n \rangle$ is real–valued, we have the result for real analysis. So $\langle x_n \rangle$ takes infinite values. If $\langle x_n \rangle$ takes infinite values infinitely often, then either $x_n = \infty$ or $x_n = -\infty$, or both. Let $M \in \{\infty, -\infty\}$ be an infinite value $\langle x_n \rangle$ takes infinitely often. Let $\langle n_k \rangle$ be the indices for which $x_n = M$, sorted in increasing order. Then $x_n \to M$. (constant sequences always converge)

Now suppose that $\langle x_n \rangle$ takes infinite values only finitely many times. Let N be the largest index for which $x_N \in \{\infty, -\infty\}$. Then the sequence $\langle x_{n+N} \rangle$ is a real–valued sequence, and so has a monotone subsequence, which is a monotone subsequence of the original sequence $\langle x_n \rangle$. ■

Putting these two theorems together, we find:

Theorem 4.7. *$\overline{\mathbf{R}}$ is sequentially compact.*

We now look at some elementary properties of the limit in $\overline{\mathbf{R}}$.

Theorem 4.8. *Let $\langle x_n \rangle, \langle y_n \rangle$ be sequences such that $x_n \leq y_n$ for each n. Then:*

(i) if $x_n \to \infty$, we have $y_n \to \infty$,

(ii) if $y_n \to -\infty$ then $x_n \to -\infty$.

Proof. First, (i). Let $M > 0$. Pick $N \in \mathbf{N}$ such that $x_n > M$ for all $n \geq N$. Then $y_n > M$ for all $n \geq N$. Since M was arbitrary this shows $y_n \to \infty$.

Now, (ii). Let $M > 0$. Pick $N \in \mathbf{N}$ such that $y_n < -M$ for all $n \geq N$. Then $x_n < -M$ for all $n \geq N$. Since M was arbitrary this shows $x_n \to -\infty$. ■

We can easily replace "for each n" with "for all sufficiently large n" just by truncating the sequence eg. as in Theorem 4.6.

Now, for arithmetic properties.

Theorem 4.9. *Let $\langle x_n \rangle$, $\langle y_n \rangle$ be sequences in $\overline{\mathbf{R}}$ and $\alpha \in \mathbf{R}$. Then:*

(i) if $\langle x_n \rangle$ is bounded below and $y_n \to \infty$ we have $x_n + y_n \to \infty$,

(ii) if $\langle x_n \rangle$ is bounded above and $y_n \to -\infty$ we have $x_n + y_n \to -\infty$,

(iii) if $x_n \to x \in \{\infty, -\infty\}$, we have $\alpha x_n \to \operatorname{sgn}(\alpha) x$.

Proof. First, for (i). Let $M > 0$. Pick $c \in \mathbf{R}$ such that $x_n \geq c$ for all $n \in \mathbf{N}$. Let $N \in \mathbf{N}$ be such that $y_n \geq M - c$ for $n \geq N$. Then we have $x_n + y_n \geq M$ for $n \geq N$, and since M was arbitrary we have $x_n + y_n \to \infty$.

Next to (ii). Let $M > 0$ and pick $C \in \mathbf{R}$ such that $x_n \leq C$ for all $n \in \mathbf{N}$. Pick $N \in \mathbf{N}$ such that $y_n \leq -M - c$ for $n \geq N$. Then $x_n + y_n \leq -M$ for $n \geq N$. Since M was arbitrary we have $x_n + y_n \to -\infty$.

Finally, to (iii). There are several cases:

(1) $\alpha = 0$: We have $\alpha x_n \equiv 0$, and so $\alpha x_n \to 0 = 0 \times x$.

(2a) $\alpha > 0$ and $x = \infty$: Let $M > 0$, then we can find $N \in \mathbf{N}$ such that $x_n > M/\alpha$ for $n \geq N$, then $\alpha x_n > M$ for $n \geq N$. Since M is arbitrary we have $\alpha x_n \to \infty = \operatorname{sgn}(\alpha)\infty$.

(2b) $\alpha > 0$ and $x = -\infty$: Let $M > 0$, then we can find $N \in \mathbf{N}$ such that $x_n < -M/\alpha$ for $n \geq N$, then $\alpha x_n < -M$ for $n \geq N$. Since M is arbitrary we have $\alpha x_n \to -\infty = \operatorname{sgn}(\alpha)(-\infty)$.

(3a) $\alpha < 0$ and $x = \infty$: Let $M > 0$, then $M/(-\alpha) > 0$ and we can find $N \in \mathbf{N}$ such that $x_n > M/(-\alpha)$. Then $\alpha x_n < -M$ for $n \geq N$. Since M is arbitrary we have $\alpha x_n \to -\infty = \operatorname{sgn}(\alpha)\infty$.

(3b) $\alpha < 0$ and $x = -\infty$: Let $M > 0$, then $M/(-\alpha) > 0$ and we can find $N \in \mathbf{N}$ such that $x_n < -M/(-\alpha) = M/\alpha$ for $n \geq N$. Then $\alpha x_n > M$ for $n \geq N$. SInce M was arbitrary we have $\alpha x_n \to \infty = \operatorname{sgn}(\alpha)(-\infty)$.

With that we are done. ∎

I've run out of stuff to say, so here's a concluding remark.

Remark 4.10. Note that it is **not true** in $\overline{\mathbf{R}}$ that if $x_n \to x$ and $y_n \to y$, then $x_n y_n \to xy$. For example, set $x_n = 1/n$ and $y_n = n$ for each $n \in \mathbf{N}$. Then $x_n \to 0$, $y_n \to \infty$ but $x_n y_n \equiv 1 \to 1 \neq 0 = 0 \times \infty$. We cannot let α be infinite in (iii) either, take $x_n = 1/n$ for each $n \in \mathbf{N}$ and $y_n = \infty$ for each $n \in \mathbf{N}$. Then $x_n y_n \equiv \infty \to \infty$, but $x_n \to 0$ so the product of the limits is 0, not ∞. So we do not have a rule of the form $\alpha x_n \to \alpha x$ either when $\alpha \in \{\infty, -\infty\}$.