

# Introduction to $\overline{\mathbf{R}}$

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## 1 Introduction

There's somewhat of an elephant in the room when it comes to mathematical analysis – and that elephant is infinity. Early on we are told to restrain ourselves to understanding  $\infty$  as no more than an abstract symbol, and not something that can be likened to a real number. We usually give up with this when we start studying measure theory, and it becomes convenient to write things like  $\int f(x)dx = \infty$ , and talk about regions having infinite volume.

The purpose of this document, which in all likelihood only I will see, is to give a “proper” treatment of the extended real numbers. (which is often inferred or handwave) I will take for granted knowledge of topology. The proofs will all be fairly basic, but interesting to explicitly outline since this is often not done in analysis texts. I will take  $\mathbf{N}$  to exclude 0, and  $\langle x_n \rangle$  to denote a sequence indexed on  $\mathbf{N}$ .

We start just with the symbols “ $\infty$ ” and “ $-\infty$ ”, which we will call *positive infinity* and *negative infinity* respectively. These can represent whatever sets you like, provided they do not lie in  $\mathbf{R}$ .<sup>1</sup> We will write  $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$ , which we will call the *extended real number line*. We will call  $x \in \overline{\mathbf{R}}$  “finite” if  $x \notin \{-\infty, \infty\}$ , and infinite if  $x \in \{-\infty, \infty\}$ .

## 2 Arithmetic

We define arithmetic on the extended real numbers in a very natural way.

**Definition 2.1** (Addition). For each  $x \in \mathbf{R}$ , we define  $x +_{\overline{\mathbf{R}}} \infty = \infty +_{\overline{\mathbf{R}}} x = \infty$  and  $x +_{\overline{\mathbf{R}}} (-\infty) = (-\infty) +_{\overline{\mathbf{R}}} x = -\infty$ . We also define  $\infty +_{\overline{\mathbf{R}}} \infty = \infty$ .

From now on we will write  $+$  instead of  $+_{\overline{\mathbf{R}}}$ . Nothing bizarre so far, though I will now make a potentially controversial choice with our definition of multiplication.

**Definition 2.2** (Multiplication). For each  $x \in \mathbf{R}$ , we define:

$$x \times_{\overline{\mathbf{R}}} \infty = \infty \times_{\overline{\mathbf{R}}} x = \begin{cases} 0 & x = 0 \\ \infty & x > 0 \\ -\infty & x < 0 \end{cases}$$

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<sup>1</sup>I am not aware of a standard, but in ZF and ZFC, the set  $\mathbf{R} \times \{\mathbf{R}\}$  cannot possibly lie in  $\mathbf{R}$ , so can be used. If we had  $\mathbf{R} \in \mathbf{R} \times \{\mathbf{R}\}$ , then  $\mathbf{R} = (x, \mathbf{R})$  for some  $x \in \mathbf{R}$ . This would imply  $\mathbf{R} \in \{\mathbf{R}\} \in (x, \mathbf{R}) = \mathbf{R}$ , using the Kuratowski convention for ordered pairs. Applying **Foundation** to  $\{\mathbf{R}, \{\mathbf{R}\}\}$ , one of the  $\in$  inclusions will be contradicted, so we have  $\mathbf{R} \notin \mathbf{R} \times \{\mathbf{R}\}$ . In general  $x \notin x \times \{x\}$  for any set  $x$ .

and:

$$(-\infty) \times_{\overline{\mathbf{R}}} x = x \times_{\overline{\mathbf{R}}} (-\infty) = \begin{cases} 0 & x = 0 \\ -\infty & x > 0 \\ \infty & x < 0 \end{cases}$$

From now on we will write  $\times$  instead of  $\times_{\overline{\mathbf{R}}}$ . The aforementioned “controversial choice” would probably be  $0 \times \infty = 0$ . Essentially we want lines (or more obtusely “rectangles with infinite width and zero height”) such as  $\{0\} \times \mathbf{R}$ , or  $\mathbf{R} \times \{0\}$  to have measure zero. When we are constructing a notion of volume for  $\mathbf{R}^2$ , we would rather like the volume of a Cartesian product  $A \times B$  to simply be the “length” of  $A$  times the “length” of  $B$ , consistent with ordinary bounded rectangles. For this we will want to write or imply that  $0 \times \infty = 0$ , since the length of  $\{0\}$  will be 0 under the canonical measure for  $\mathbf{R}^2$ , and the length of  $\mathbf{R}$  will be  $\infty$ .

Subtraction can now be defined simply by  $x -_{\overline{\mathbf{R}}} y = x +_{\overline{\mathbf{R}}} (-y)$ . We will largely avoid division, but sometimes we may want to think that  $1/\infty = 0$  – in certain embedding results we want to think of  $\{p, q\} = \{1, \infty\}$  as satisfying  $1/p + 1/q = 1$ .

Notice that notably we leave  $\infty - \infty$  undefined. We will see why later, but we clearly don’t want to set  $\infty - \infty = 0$ , otherwise we could write, say  $0 = \infty - \infty = (1 + \infty) - \infty = 1$ , proving  $1 = 0$ ! In a less obviously abusive framing, say we have two extended real numbers  $\alpha$  and  $\beta$  and we find that  $\alpha = \alpha + \beta$ . We *cannot* conclude that  $\beta = 0$ , since we may have  $\alpha = \infty$ , in which case  $\beta$  could be any extended real number except  $-\infty$ . This gets even funkier when we look at orders on  $\overline{\mathbf{R}}$ , as we will see.

### 3 Ordering

We will now extend our canonical order on  $\mathbf{R}$  to  $\overline{\mathbf{R}}$ . This will be consistent with our intuition that  $\infty$  should be larger than any real number, and  $-\infty$  should be “more negative” than any real number. We define  $<_{\overline{\mathbf{R}}}$  by  $x <_{\overline{\mathbf{R}}} \infty$  if  $x \neq \infty$  and  $-\infty <_{\overline{\mathbf{R}}} x$  if  $x \neq -\infty$ , and  $x <_{\overline{\mathbf{R}}} y$  if  $x, y \in \mathbf{R}$  and  $x < y$  in the usual order. We will now write  $<_{\overline{\mathbf{R}}}$  as simply  $<$ , as we did with addition and multiplication.

The nice thing now is that  $(\overline{\mathbf{R}}, <_{\overline{\mathbf{R}}})$  is that every sequence has a convergent subsequence, and that  $(\overline{\mathbf{R}}, <_{\overline{\mathbf{R}}})$  is a “complete lattice” – every set has an infimum and supremum. We will now prove this.

**Theorem 3.1.** *Every set in  $\overline{\mathbf{R}}$  has an infimum and supremum.<sup>2</sup>*

*Proof.* Let  $T \subseteq \overline{\mathbf{R}}$ . We deal first with the case  $T = \emptyset$ . Note that  $-\infty$  is an upper bound for  $\emptyset$ , and  $\infty$  is a lower bound for  $\emptyset$ . No extended real number is less than  $-\infty$  or greater than  $\infty$ , so  $-\infty$  is the least upper bound for  $\emptyset$  and  $\infty$  is the greatest lower bound. So we have  $\inf \emptyset = \infty$  and  $\sup \emptyset = -\infty$ . (weird, right?) Notably, this is the only case where  $\inf T > \sup T$ , if  $T \neq \emptyset$  we have  $\inf T \leq \sup T$ , with equality holding only for singletons.

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<sup>2</sup>Actually, we only need to show that every set in  $\overline{\mathbf{R}}$  has an infimum, since we can then just take the infimum of the set of upper bounds, but I do both for slickness.

Now let  $T \subseteq \overline{\mathbf{R}}$  be non-empty. Note that  $\infty$  is an upper bound for  $T$ . If  $\infty \in T$ , then we must have  $\sup T = \infty$ , since any “interior” upper/lower bounds are supremums/infimums. Suppose  $\infty \notin T$ . If  $T$  has a supremum in the usual order of  $\mathbf{R}$ , great, it is also a supremum in the order of  $\overline{\mathbf{R}}$ . If not, then for each  $M$  there exists  $t \in T$  with  $t > M$ . So no real number can be an upper bound for  $T$ , because it will be exceeded by some other element. So the only upper bound is  $\infty$ , giving  $\sup T = \infty$ .

The case of the infimum is similar. Note that  $-\infty$  is always a lower bound for  $T$ . So if  $-\infty \in T$ , we have  $\inf T = -\infty$ . If  $T$  has an infimum in the usual order of  $\mathbf{R}$ , brilliant, we are again done. If not, then for every  $M$  there exists  $t \in T$  with  $t < M$ . So no real number is a lower bound for  $T$ , so the only lower bound is  $-\infty$ , giving  $\inf T = -\infty$ . ■

So we now formally have  $\sup(0, \infty) = \infty$  and  $\inf(-\infty, 0) = -\infty$ , and so on. As a final note, we now need to be careful with inequalities. If we are talking about  $\alpha$  in the extended real numbers, we are no longer able to say  $\alpha < \alpha + 1$ , since this is not true if  $\alpha$  is infinite, so often you will have to use non-strict inequalities throughout. (which is usually safer, even in the reals) This is somewhat reminiscent of cardinal arithmetic, where we may have  $|X| = |X| + 1$  or  $|X| < |X| + 1$  when  $|X|$  is infinite. (see: *Dedekind infinite*)

## 4 Topology

Now that we have arithmetic and supremums/infimums set up solidly, we move onto the topology of  $\overline{\mathbf{R}}$ . We define the *standard topology* on  $\overline{\mathbf{R}}$  as the topology generated by the standard topology on  $\mathbf{R}$  together with the intervals  $(a, \infty] = (a, \infty) \cup \{\infty\}$  and  $[-\infty, b) = (-\infty, b) \cup \{-\infty\}$ , which we consider neighbourhoods of  $\infty$  and  $-\infty$  respectively. A simple check reveals  $\overline{\mathbf{R}}$  is Hausdorff with this topology:

- (i)  $(x - 1/2, x + 1/2)$  and  $(x + 1, \infty]$  separate  $x \in \mathbf{R}$  from  $\infty$ ,
- (ii)  $(x - 1/2, x + 1/2)$  and  $[-\infty, x - 1)$  separate  $x \in \mathbf{R}$  from  $-\infty$ ,
- (iii)  $[-\infty, 0)$  and  $(0, \infty]$  separate  $\infty$  from  $-\infty$ .

The separation of other points follows from the fact that  $\mathbf{R}$  is Hausdorff with the usual topology. This ensures that limits in  $\overline{\mathbf{R}}$  are unique, though we don’t yet know how to calculate them. Our goal will now be to first establish simple criteria for convergence in  $\overline{\mathbf{R}}$ , and then to show that it is both compact and sequentially compact.

**Theorem 4.1.**  $\overline{\mathbf{R}}$  is compact with its standard topology.

*Proof.* Let  $\mathcal{C}$  be an open cover for  $\overline{\mathbf{R}}$ . Let  $U_\infty$  be a set in the cover that contains  $\infty$ . We can pull from it an open subset  $(M_2, \infty]$ . Let  $U_{-\infty}$  be a set in the cover that contains  $-\infty$ . We can pull from it an open subset  $[-\infty, M_1)$ .

Now consider  $\mathcal{C} \cap [M_1, M_2] = \{S \cap [M_1, M_2] : S \in \mathcal{C}\}$ . This is an open cover of  $[M_1, M_2]$ . From the compactness of  $[M_1, M_2]$ , we have a finite cover  $\{S_1 \cap [M_1, M_2], \dots, S_n \cap [M_1, M_2]\}$  of  $[M_1, M_2]$ . Since  $\overline{\mathbf{R}} = [-\infty, M_1) \cup [M_1, M_2] \cup (M_2, \infty]$ ,  $\{S_1, \dots, S_n, U_\infty, U_{-\infty}\}$  is then a subcover of  $\mathcal{C}$  of size  $n + 2 < \infty$ . Since the cover  $\mathcal{C}$  was arbitrary we have that  $\overline{\mathbf{R}}$  is compact. ■

In light of this result,  $\overline{\mathbf{R}}$  can be called the *two-point compactification* of  $\mathbf{R}$ . (there are other constructions that only add one point) We now look at characterising convergence in  $\overline{\mathbf{R}}$ . First, we verify that a sequence  $\langle x_n \rangle \subset \mathbf{R}$  that converges to  $x$  in  $\mathbf{R}$  still converges to  $x$  in  $\overline{\mathbf{R}}$ . We will frame this a bit more generally.

**Theorem 4.2.** *Let  $\langle x_n \rangle$  be a sequence in  $\overline{\mathbf{R}}$  and let  $x \in \mathbf{R}$ . Then we have  $x_n \rightarrow x$  in  $\overline{\mathbf{R}}$  if and only if:*

- (i) *there exists  $N_1 \in \mathbf{N}$  such that for  $n \geq N_1$  we have  $x_n \in \mathbf{R}$ ,*
- (ii) *for each  $\varepsilon > 0$  there exists  $N_2 \in \mathbf{N}$  such that for  $n \geq \max\{N_1, N_2\}$  we have  $|x_n - x| < \varepsilon$ .*

Evidently (i) is satisfied for any real sequence, just taking  $N_1 = 1$ , so if we can show this we have the desired consistency result.

*Proof.* Suppose that  $x_n \rightarrow x$  in  $\overline{\mathbf{R}}$ . Then for each  $\varepsilon > 0$ , there exists  $N_1 \in \mathbf{N}$  such that we have  $x_n \in (x - \varepsilon, x + \varepsilon) \subset \mathbf{R}$  for  $n \geq N_1$ , since  $(x - \varepsilon, x + \varepsilon)$  is an open neighbourhood of  $x$ . Fixing  $\varepsilon$ , we see that  $x_n \in \mathbf{R}$  for  $n \geq N_1$ . Now let  $\varepsilon > 0$  be arbitrary, then we can select  $N_2 = N_2(\varepsilon) \in \mathbf{N}$  such that  $x_n \in (x - \varepsilon, x + \varepsilon)$  for  $n \geq N_2$ . That is,  $|x_n - x| < \varepsilon$ . This proves the “only if” direction.

Now suppose that (i) and (ii) hold for a sequence  $\langle x_n \rangle$ . We aim to show that for each open neighbourhood of  $x$ ,  $U$ , in  $\overline{\mathbf{R}}$ , there exists  $N \in \mathbf{N}$  such that  $x_n \in U$  for  $n \geq N$ . Fix an open neighbourhood of  $x$ . We can pull out an open neighbourhood of  $x$  that has one of the forms  $(a, b)$ ,  $(M, \infty]$  or  $[-\infty, m)$ . Note that WLOG we can assume the form is  $(a, b)$ , by looking at  $(x - \varepsilon, x + \varepsilon)$  for  $x$  sufficiently small. From (ii) we can find  $N \in \mathbf{N}$  such that  $|x_n - x| < \varepsilon$  for  $n \geq N$ . In particular,  $x_n \in U$  for  $n \geq N$ . This proves the “if” direction. ■

Now we have our sanity check out of the way, we can move to the more interesting matter of convergence to  $\pm\infty$ , which will coincide with the typical definition of “divergence to”  $\pm\infty$ .

**Theorem 4.3.** *Let  $\langle x_n \rangle$  be a sequence in  $\overline{\mathbf{R}}$ . We have  $x_n \rightarrow \infty$  if and only if for each  $M > 0$  there exists  $N \in \mathbf{N}$  such that  $x_n > M$  for  $n \geq N$ .*

*Proof.* Suppose that  $x_n \rightarrow \infty$ . Then  $(M, \infty]$  is an open neighbourhood of  $\infty$ , and so there exists  $N \in \mathbf{N}$  such that  $x_n \in (M, \infty]$  for  $n \geq N$ . That is,  $x_n > M$ .

Now to the converse, fix an open neighbourhood  $U$  of  $\infty$ . From this we can pull a subset  $(M, \infty]$ . We can find  $N \in \mathbf{N}$  such that  $x_n > M$  for all  $n \geq N$ . In particular we have  $x_n \in U$  for  $n \geq N$ . So  $x_n \rightarrow \infty$ . ■

An almost identical procedure shows that:

**Theorem 4.4.** *Let  $\langle x_n \rangle$  be a sequence in  $\overline{\mathbf{R}}$ . We have  $x_n \rightarrow -\infty$  if and only if for each  $M > 0$  there exists  $N \in \mathbf{N}$  such that  $x_n < -M$  for  $n \geq N$ .*

We can now show that  $\overline{\mathbf{R}}$  is sequentially compact. We will then go on to prove properties of the limit in  $\overline{\mathbf{R}}$ . We first need a lemma.

**Lemma 4.5.** *A monotone sequence in  $\overline{\mathbf{R}}$  converges.*

*Proof.* Let  $\langle x_n \rangle$  be a monotone sequence in  $\overline{\mathbf{R}}$ . First take  $\langle x_n \rangle$  to be increasing. If  $\sup_n x_n < \infty$ , then we know from real analysis that  $x_n \rightarrow \sup_n x_n$ . Now suppose that  $\sup_n x_n = \infty$ . Then for each  $k \in \mathbf{N}$  we can find some  $n_k$  such that  $x_{n_k} \geq k$ . Without loss of generality, pick  $\langle n_k \rangle$  to be increasing. Then, for each  $M > 0$  we have  $x_{n_k} \geq M$  for  $k \geq \lfloor M \rfloor + 1$ . So we have  $x_{n_k} \rightarrow \infty$ .

The case of decreasing sequences is very similar. If  $\infty > \inf_n x_n > -\infty$ , (note that the infimum cannot be  $\infty$  since  $\{x_n : n \in \mathbf{N}\}$  is not empty) then we know that  $x_n \rightarrow \inf_n x_n$ . Now suppose that  $\inf_n x_n = -\infty$ . Then for each  $k \in \mathbf{N}$  we can find some  $n_k$  such that  $x_{n_k} \leq -k$ . Without loss of generality, pick  $\langle n_k \rangle$  to be increasing. Then for each  $M > 0$  we have  $x_{n_k} \leq -M$  for  $k \geq \lfloor M \rfloor + 1$ . So we have  $x_{n_k} \rightarrow -\infty$ . ■

To show that  $\overline{\mathbf{R}}$  is sequentially compact, it now suffices to show that every sequence in  $\overline{\mathbf{R}}$  has a monotone subsequence.

**Theorem 4.6.** *Every sequence in  $\overline{\mathbf{R}}$  has a monotone subsequence.*

*Proof.* Let  $\langle x_n \rangle$  be a sequence in  $\overline{\mathbf{R}}$ . If  $\langle x_n \rangle$  is real-valued, we have the result for real analysis. So  $\langle x_n \rangle$  takes infinite values. If  $\langle x_n \rangle$  takes infinite values infinitely often, then either  $x_n = \infty$  or  $x_n = -\infty$ , or both. Let  $M \in \{\infty, -\infty\}$  be an infinite value  $\langle x_n \rangle$  takes infinitely often. Let  $\langle n_k \rangle$  be the indices for which  $x_n = M$ , sorted in increasing order. Then  $x_n \rightarrow M$ . (constant sequences always converge)

Now suppose that  $\langle x_n \rangle$  takes infinite values only finitely many times. Let  $N$  be the largest index for which  $x_N \in \{\infty, -\infty\}$ . Then the sequence  $\langle x_{n+N} \rangle$  is a real-valued sequence, and so has a monotone subsequence, which is a monotone subsequence of the original sequence  $\langle x_n \rangle$ . ■

Putting these two theorems together, we find:

**Theorem 4.7.**  *$\overline{\mathbf{R}}$  is sequentially compact.*

We now look at some elementary properties of the limit in  $\overline{\mathbf{R}}$ .

**Theorem 4.8.** *Let  $\langle x_n \rangle, \langle y_n \rangle$  be sequences such that  $x_n \leq y_n$  for each  $n$ . Then:*

- (i) *if  $x_n \rightarrow \infty$ , we have  $y_n \rightarrow \infty$ ,*
- (ii) *if  $y_n \rightarrow -\infty$  then  $x_n \rightarrow -\infty$ .*

*Proof.* First, (i). Let  $M > 0$ . Pick  $N \in \mathbf{N}$  such that  $x_n > M$  for all  $n \geq N$ . Then  $y_n > M$  for all  $n \geq N$ . Since  $M$  was arbitrary this shows  $y_n \rightarrow \infty$ .

Now, (ii). Let  $M > 0$ . Pick  $N \in \mathbf{N}$  such that  $y_n < -M$  for all  $n \geq N$ . Then  $x_n < -M$  for all  $n \geq N$ . Since  $M$  was arbitrary this shows  $x_n \rightarrow -\infty$ . ■

We can easily replace “for each  $n$ ” with “for all sufficiently large  $n$ ” just by truncating the sequence eg. as in Theorem 4.6.

Now, for arithmetic properties.

**Theorem 4.9.** *Let  $\langle x_n \rangle, \langle y_n \rangle$  be sequences in  $\overline{\mathbf{R}}$  and  $\alpha \in \mathbf{R}$ . Then:*

- (i) if  $\langle x_n \rangle$  is bounded below and  $y_n \rightarrow \infty$  we have  $x_n + y_n \rightarrow \infty$ ,
- (ii) if  $\langle x_n \rangle$  is bounded above and  $y_n \rightarrow -\infty$  we have  $x_n + y_n \rightarrow -\infty$ ,
- (iii) if  $x_n \rightarrow x \in \{\infty, -\infty\}$ , we have  $\alpha x_n \rightarrow \text{sgn}(\alpha)x$ .

*Proof.* First, for (i). Let  $M > 0$ . Pick  $c \in \mathbf{R}$  such that  $x_n \geq c$  for all  $n \in \mathbf{N}$ . Let  $N \in \mathbf{N}$  be such that  $y_n \geq M - c$  for  $n \geq N$ . Then we have  $x_n + y_n \geq M$  for  $n \geq N$ , and since  $M$  was arbitrary we have  $x_n + y_n \rightarrow \infty$ .

Next to (ii). Let  $M > 0$  and pick  $C \in \mathbf{R}$  such that  $x_n \leq C$  for all  $n \in \mathbf{N}$ . Pick  $N \in \mathbf{N}$  such that  $y_n \leq -M - c$  for  $n \geq N$ . Then  $x_n + y_n \leq -M$  for  $n \geq N$ . Since  $M$  was arbitrary we have  $x_n + y_n \rightarrow -\infty$ .

Finally, to (iii). There are several cases:

- (1)  $\alpha = 0$ : We have  $\alpha x_n \equiv 0$ , and so  $\alpha x_n \rightarrow 0 = 0 \times x$ .
- (2a)  $\alpha > 0$  and  $x = \infty$ : Let  $M > 0$ , then we can find  $N \in \mathbf{N}$  such that  $x_n > M/\alpha$  for  $n \geq N$ , then  $\alpha x_n > M$  for  $n \geq N$ . Since  $M$  is arbitrary we have  $\alpha x_n \rightarrow \infty = \text{sgn}(\alpha)\infty$ .
- (2b)  $\alpha > 0$  and  $x = -\infty$ : Let  $M > 0$ , then we can find  $N \in \mathbf{N}$  such that  $x_n < -M/\alpha$  for  $n \geq N$ , then  $\alpha x_n < -M$  for  $n \geq N$ . Since  $M$  is arbitrary we have  $\alpha x_n \rightarrow -\infty = \text{sgn}(\alpha)(-\infty)$ .
- (3a)  $\alpha < 0$  and  $x = \infty$ : Let  $M > 0$ , then  $M/(-\alpha) > 0$  and we can find  $N \in \mathbf{N}$  such that  $x_n > M/(-\alpha)$ . Then  $\alpha x_n < -M$  for  $n \geq N$ . Since  $M$  is arbitrary we have  $\alpha x_n \rightarrow -\infty = \text{sgn}(\alpha)\infty$ .
- (3b)  $\alpha < 0$  and  $x = -\infty$ : Let  $M > 0$ , then  $M/(-\alpha) > 0$  and we can find  $N \in \mathbf{N}$  such that  $x_n < -M/(-\alpha) = M/\alpha$  for  $n \geq N$ . Then  $\alpha x_n > M$  for  $n \geq N$ . Since  $M$  was arbitrary we have  $\alpha x_n \rightarrow \infty = \text{sgn}(\alpha)(-\infty)$ .

With that we are done. ■

I've run out of stuff to say, so here's a concluding remark.

**Remark 4.10.** Note that it is **not true** in  $\overline{\mathbf{R}}$  that if  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , then  $x_n y_n \rightarrow xy$ . For example, set  $x_n = 1/n$  and  $y_n = n$  for each  $n \in \mathbf{N}$ . Then  $x_n \rightarrow 0$ ,  $y_n \rightarrow \infty$  but  $x_n y_n \equiv 1 \rightarrow 1 \neq 0 = 0 \times \infty$ . We cannot let  $\alpha$  be infinite in (iii) either, take  $x_n = 1/n$  for each  $n \in \mathbf{N}$  and  $y_n = \infty$  for each  $n \in \mathbf{N}$ . Then  $x_n y_n \equiv \infty \rightarrow \infty$ , but  $x_n \rightarrow 0$  so the product of the limits is 0, not  $\infty$ . So we do not have a rule of the form  $\alpha x_n \rightarrow \alpha x$  either when  $\alpha \in \{\infty, -\infty\}$ .