Computational Physics

Project 5: Wave Equation

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Abstract: Here we discuss project #5. We review the context of the problem, its relation to real physical systems, and the methodology by which we tackled its solution. All codes and figures are appended at the end of the paper.

1. Introduction

Here we use iterative methods to find solutions to the Wave equation. The Wave equation appears throughout many different sectors of physics, from fluid mechanics to gravitation to electromagnetism to quantum mechanics. The wave equation is given as

$$\frac{\partial^2 f}{\partial t^2} - \alpha \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) = 0 \tag{1}$$

So that if the field u is time-independent it simply reduces to Laplace's equation.

The wave equation also has tremendous historical significance. Hooke's Law is one of earliest examples of the mathematization of physics that we are familiar with today. Not only was Hooke's law successful in describing spring motion, but it is still plenty useful today. In solid state physics a useful approximation is to model molecules in a solid as springs. From this properties of the bulk material, such as the speed of sound through the solid, can be derived.

2. Analytics

Following the example done in class, we begin with the equation

$$\frac{d^2 y}{dt^2} - \alpha \left(\frac{d^2 y}{dx^2} \right) = 0 \tag{2}$$

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For linear density $\mu = m/\Delta x$, period T, we obtain

$$\mu \Delta x \frac{d^2 y}{dt^2} = T\left(\frac{y_{i+1} - y_i}{\Delta x} - \frac{y_i - y_{i-1}}{\Delta x}\right) \tag{3}$$

yielding the numerical 2nd derivative

$$\frac{d^2 y}{dt^2} = \frac{T}{\mu}\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}\right] \tag{4}$$

Which, in continuous form, is

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} = \nu^2 \frac{\partial^2 y}{\partial x^2} \tag{5}$$

2.1. Electromagnetism Example

The following is an example of the wave equation applied to electromagnetism, an example pulled from project 4.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\rho}{\epsilon_0} = 0 \tag{6}$$

Where

- V is the electrostatic potential
- $\rho = 0$ is the (net) electrostatic charge density, taken here to be zero
- $\epsilon_0 = 8.85418782 \times 10^{-12} Fm^{-1}$, is the vacuum permittivity
- θ is the angle and $\theta_0 \equiv \theta(t = 0)$

So clearly this is just Laplace's equation so that we may also write it as $\nabla^2 V = \Delta V = 0$.

To a rough approximation we have that

$$\frac{\partial V}{\partial x} \approx \frac{\Delta V}{\Delta x} = \frac{V(i+1,j,k) - V(i,j,k)}{\Delta x} \tag{7}$$

$$\frac{\partial V}{\partial y} \approx \frac{\Delta V}{\Delta y} = \frac{V(i,j+1,k) - V(i,j,k)}{\Delta y} \tag{8}$$

$$\frac{\partial V}{\partial z} \approx \frac{\Delta V}{\Delta z} = \frac{V(i,j,k+1) - V(i,j,k)}{\Delta z} \tag{9}$$

Where i is the step for the x coordinate, j for the y, and k for the z. These approximations clearly hold best when V changes only very slowly and when V is monotonic (at least in the applicable regime). These relationships are derived from the definition of derivatives themselves, but rather than a step size $h \to 0$ we use an appropriately small step size. Further we expect that electrostatic potentials obeying Eq. (6) should change fairly smoothly

and slowly (relative to the step size we seek to implement). Obviously if we only used very few steps we would get a significant amount of error.

Now we can write the approximation to the second derivatives

$$\frac{\partial^2 V}{\partial x^2} \approx \frac{V(i+1,j,k) - 2V(i,j,k) + V(i-1,j,k)}{(\Delta x)^2} \tag{10}$$

$$\frac{\partial^2 V}{\partial y^2} \approx \frac{V(i,j+1,k) - 2V(i,j,k) + V(i,j-1,k)}{(\Delta y)^2} \tag{11}$$

$$\frac{\partial^2 V}{\partial z^2} \approx \frac{V(i,j,k+1) - 2V(i,j,k) + V(i,j,k-1)}{(\Delta z)^2} \tag{12}$$

And the indices are arbitrary, so we could make any choice for $n \in \mathbb{Z}$ such that $i \leftrightarrow i+n$, here we just have that $n = 0$ for the center step. From Eq (6) we then have

$$\frac{V(i+1,j,k) - 2V(i,j,k) + V(i-1,j,k)}{(\Delta x)^2} + \frac{V(i,j+1,k) - 2V(i,j,k) + V(i,j-1,k)}{(\Delta y)^2} + \frac{V(i,j,k+1) - 2V(i,j,k) + V(i,j,k-1)}{(\Delta z)^2} = 0 \tag{13}$$

We now assume isotropic steps such that $\Delta x = \Delta y = \Delta z$, giving that

$$V(i,j,k) = \frac{1}{6}\left[V(i+1,j,k) + V(i-1,j,k) + V(i,j+1,k) + V(i,j-1,k) + V(i,j,k+1) + V(i,j,k-1)\right] \tag{14}$$

So each potential is the sum of its nearest neighbors in each direction. For lower dimensional potentials this simply reduces to

$$V(i,j) = \frac{1}{4}\left[V(i+1,j) + V(i-1,j) + V(i,j+1) + V(i,j-1)\right] \text{ (2-dimensional)} \tag{15}$$

$$V(i) = \frac{1}{2}\left[V(i+1) + V(i-1)\right] \text{ (1-dimensional)} \tag{16}$$

3. Computations

The codes used are shown in the Figures appended.

The appended figures are associated with the following parameters

1. Fig. 2 - no time steps
2. Fig. 3 - 50 time steps

3. Fig. 4 - 400 time steps

4. Fig. 5 - 1000 time steps

Note how the scale of the y-axis changes in each plot yet the shape does not. This is because the wave equation causes a decrease in amplitude as we time-evolve the wave further and further. This decrease in amplitude can be carried out until the amplitude becomes to small to discern using our calculations. Of course a real wave dissipates its energy into the environment and into the molecules that make up the material its moving through.

4. Conclusions

These agree well with what we would expect from analytic calculation and simple intuition. We can see that as we increase the number of time steps the wave moves further along the x-axis while decreasing in amplitude, exactly as we expect. An experiment using items as simple as rope or a 'Slinky' can replicate this behavior. So the wave equation can be modelled well with iterative computational methods.

References

[1] S. Stolbov, Computational Physics Class Notes

[2] B. Ydri, *Computational Physics: An Introduction to Monte Carlo Simulations of Matrix Field Theory*, arXiv:1506.02567 [hep-lat] (June 2015)

[3] K. Anagnostopoulos, *Computational Physics Volume 2: A Practical Introduction to Computational Physics and Scientific Computing* pg. 91 (Aug. 2014)

[4] Weisstein, Eric W. *Heat Conduction Equation.* From MathWorld–A Wolfram Web Resource.

Figure 1. **One of the codes used, entitled *string.f*, that we wrote in class.**

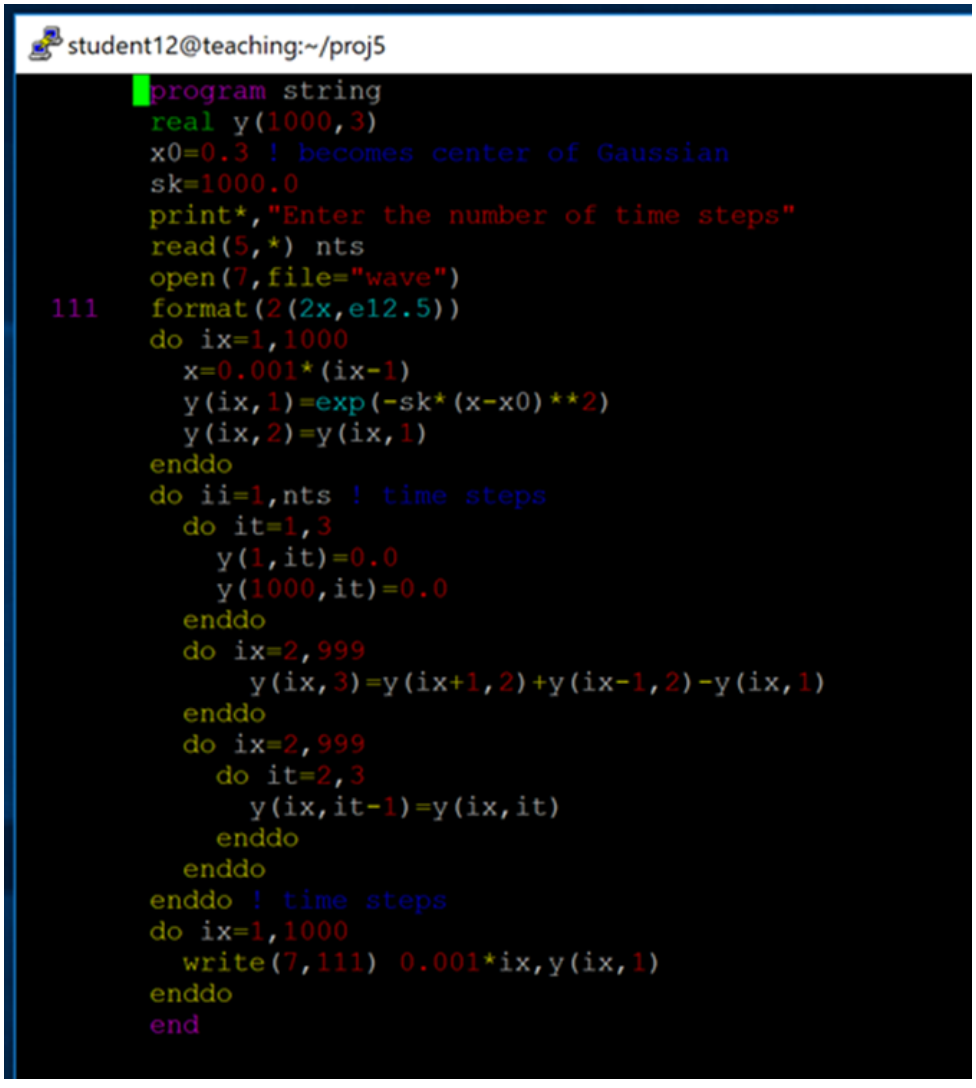

Figure 2. number of time steps = 0

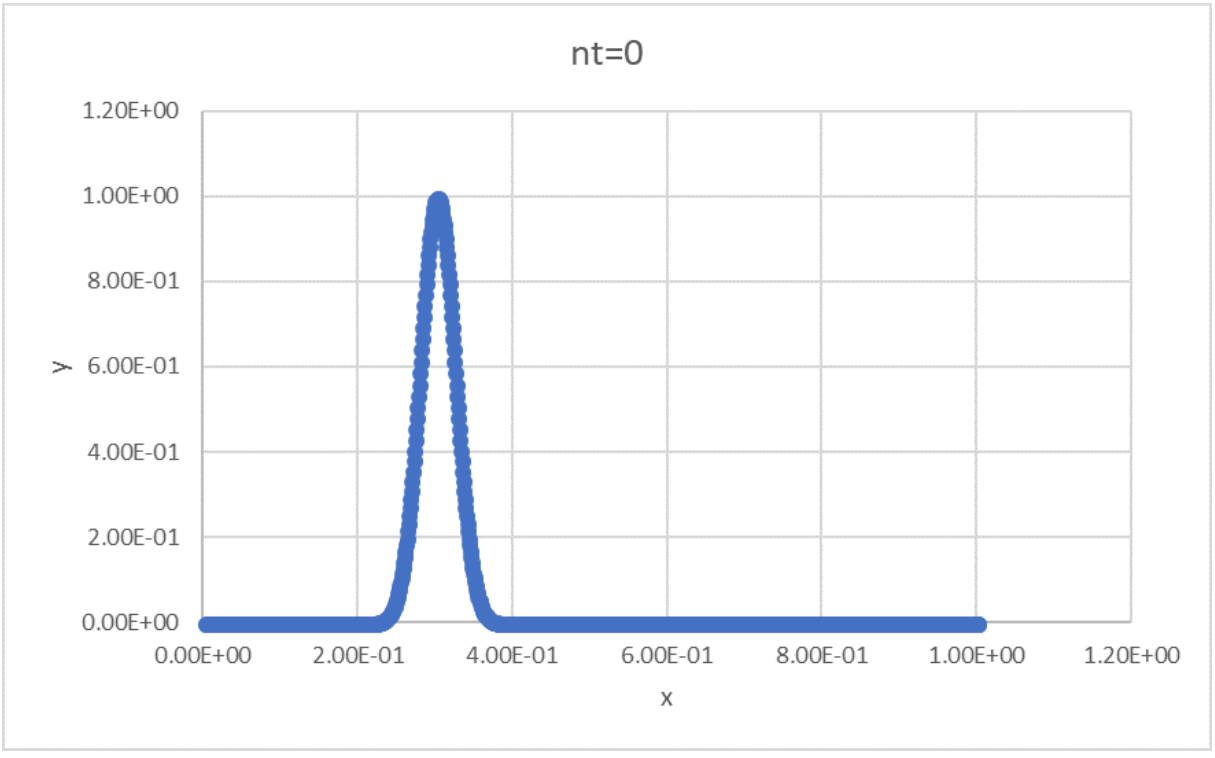

Figure 3. number of time steps = 50

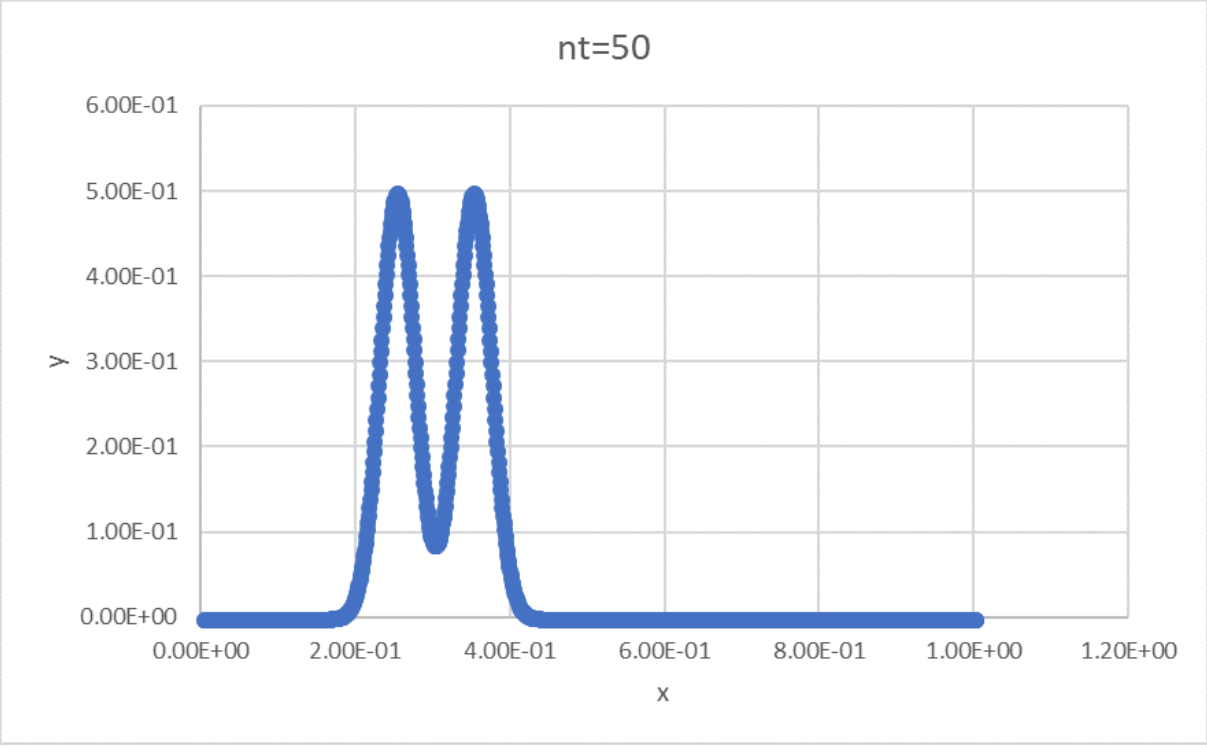

Figure 4. number of time steps = 400

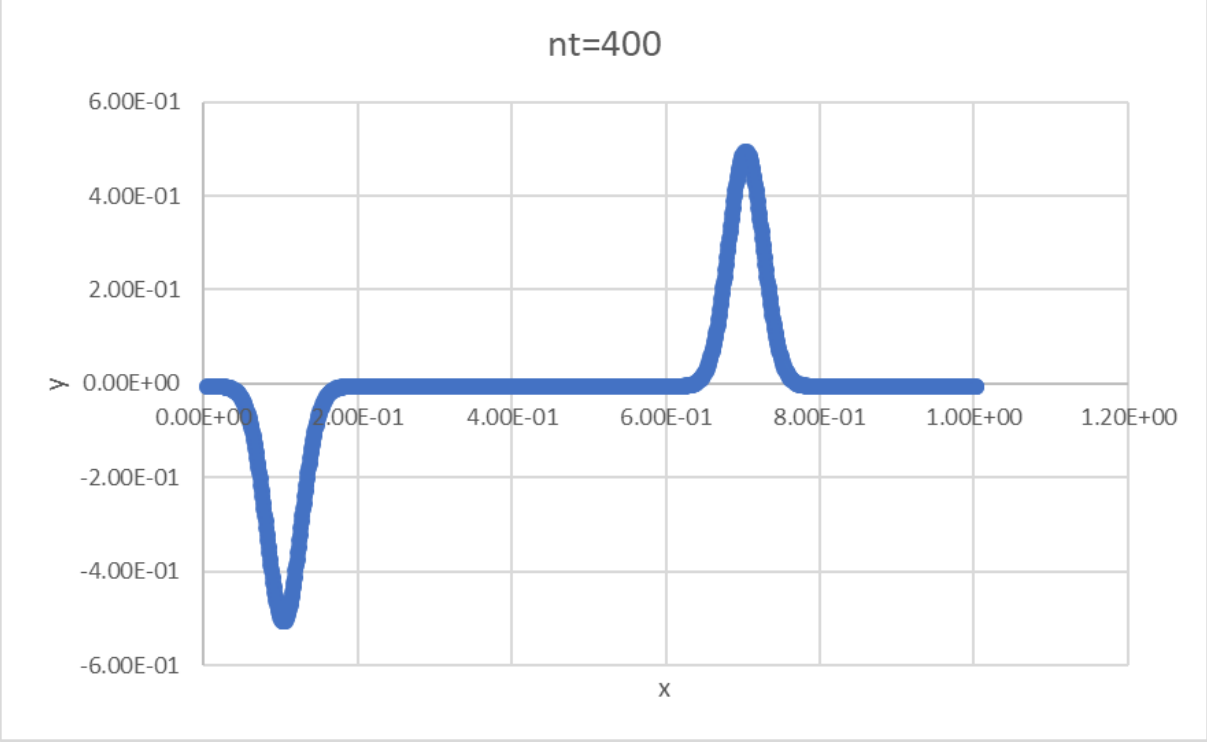

Figure 5. number of time steps = 1000