Definitional sections (those before ' GR ') below come from http : // web.physics.ucsb.edu / ~gravitybook / mathematica.html

Christoffel Symbols and Geodesic Equation

THE SHAPE OF ORBITS IN THE SCHWARZSCHILD GEOMETRY

FRW Models

Curvature and the Einstein Equation

GR

```
Tr[inversemetric.metric // FullSimplify]
Tr[metric.inversemetric // FullSimplify] (*Should both =n for GR metric *)
```

```
4
```

```
4
```

Metric

Note this notebook uses (+++−) form for metric, so time comp at end of metric
Need to define a metric,
then this program tests if that metric obeys the Einstein Field Equations
Each time metric is redefined, need to re − run tenor definitions section

```
coord = {r, θ, ϕ, t};
n = 4;
```

coord = {r[t], θ[t], ϕ[t], t};
/. r → r[t] /. θ → θ[t] /. ϕ → ϕ[t]
Using one of this makes all coods time − dependent
Be sure to apply second one to the metric too

```
metric = {{(1 - 2 m / r) ^ (-1), 0, 0, 0}, {0, r^2, 0, 0},
   {0, 0, r^2 Sin[θ]^2, 0}, {0, 0, 0, - (1 - 2 m / r)}}; (* Schwarzchild *)
inversemetric = Simplify[Inverse[metric]];
```

```
coord = {r[t], θ[t], ϕ[t], t};
metric = {{(1 - 2 m / r) ^ (-1), 0, 0, 0}, {0, r^2, 0, 0}, {0, 0, r^2 Sin[θ]^2, 0},
      {0, 0, 0, - (1 - 2 m / r)}} /. r → r[t] /. θ → θ[t] /. ϕ → ϕ[t];
(* Time-dependent Coods and Schwarzchild metric *)
inversemetric = Simplify[Inverse[metric]];
```

```
metric = {{1, 0, 0, 0}, {0, r^2 + b^2, 0, 0}, {0, 0, (r^2 + b^2) Sin[θ]^2, 0}, {0, 0, 0, -1}};
(* Sch w/ pertubation *)
inversemetric = Simplify[Inverse[metric]];
```

```
metric // MatrixForm
```

$$\begin{pmatrix} \frac{1}{1-\frac{2m}{r[t]}} & 0 & 0 & 0 \\ 0 & r[t]^2 & 0 & 0 \\ 0 & 0 & r[t]^2 \,\mathrm{Sin}[\theta[t]]^2 & 0 \\ 0 & 0 & 0 & -1+\frac{2m}{r[t]} \end{pmatrix}$$

Key Question :
Can all metrics with Time − dependent spatial coordinates be made isomorphic to some spacetime ? Or some finite set of spacetimes ?
i.e. For a metric with coordinates r[t], θ[t], ϕ[t], t,
can we always perform a change of coods so that it looks like a minkowski spacetime,
as we can do for a metric with time − independent coordinates ?
Note that the tensors of GR all involve partial derivatives −
i.e. coods assumed to be independent of aone another. Can we reparametrize in terms of only time and alter these tensors ?
This would change the formalism of GR, but how so ?

Tensor Definitions

```
affine := affine = Simplify[Table[(1/2) * Sum[(inversemetric[[i, s]]) *
        (D[metric[[s, j]], coord[[k]]] +
          D[metric[[s, k]], coord[[j]]] - D[metric[[j, k]], coord[[s]]]), {s, 1, n}],
    {i, 1, n}, {j, 1, n}, {k, 1, n}]]
```

```
riemann := riemann = Simplify[Table[
    D[affine[[i, j, l]], coord[[k]]] - D[affine[[i, j, k]], coord[[l]]] +
     Sum[affine[[s, j, l]] affine[[i, k, s]] - affine[[s, j, k]] affine[[i, l, s]],
      {s, 1, n}],
    {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, n}]]
```

```
ricci := ricci = Simplify[Table[Sum[riemann[[i, j, i, l]], {i, 1, n}], {j, 1, n}, {l, 1, n}]]
```

```
scalar = Simplify[Sum[inversemetric[[i, j]] ricci[[i, j]], {i, 1, n}, {j, 1, n}]]
```

$$-\frac{1}{(2m-r[t])^3\, r[t]} 2\Big(\big(12m^2-6m\,r[t]+r[t]^2\big)\,r'[t]^2+\mathrm{Cot}[\theta[t]]\,r[t]\,\big(16m^2-14m\,r[t]+3r[t]^2\big)\,r'[t]\,\theta'[t]+ r[t]\,(-2m+r[t])\,\big(-5m\,r''[t]+r[t]^2\,(-\theta'[t]^2+\mathrm{Cot}[\theta[t]]\,\theta''[t])+ 2r[t]\,(m\,\theta'[t]^2+r''[t]-m\,\mathrm{Cot}[\theta[t]]\,\theta''[t])\big)\Big)$$

```
einstein := einstein = Simplify[ricci - (1/2) scalar * metric]
```

Test of Metric Validity

Tensors defined above give below
Note that theyre tensors, not necessarily perfect matrices
Although the Ricci and Einstein Tensors are matrices the way theyre expressed here,
since theyre rank − 2

```
affine // MatrixForm;
ricci // MatrixForm;
riemann // MatrixForm;
einstein // MatrixForm;
```

$Aborted

```
listeinstein := Table[If[UnsameQ[einstein[[j, l]], 0],
   {ToString[G[j, l]], einstein[[j, l]]}] , {j, 1, n}, {l, 1, j}]
```

```
TableForm[Partition[DeleteCases[Flatten[listeinstein], Null], 2], TableSpacing → {2, 2}]
```

G[1, 1]	$\frac{f_{22}[r[t],\theta[t],\phi[t],t]\; f_{33}[r[t],\theta[t],\phi[t],t]\; f_{44}[r[t],\theta[t],\phi[t],t]\; \left(f_{44}[r[t],\theta[t],\phi[t],t]\; f_{22}{}^{(1,0,0,0)}[r[t],\theta[t],\phi[t]\right.}{}$
G[2, 1]	$\frac{1}{4}\left(\frac{\left(\frac{f_{33}{}^{(0,1,0,0)}[r[t],\theta[t],\phi[t],t]}{f_{33}[r[t],\theta[t],\phi[t],t]}+\frac{f_{44}{}^{(0,1,0,0)}[r[t],\theta[t],\phi[t],t]}{f_{44}[r[t],\theta[t],\phi[t],t]}\right) f_{22}{}^{(1,0,0,0)}[r[t],\theta[t],\phi[t],t]}{f_{22}[r[t],\theta[t],\phi[t],t]}+\frac{f_{33}{}^{(0,1,0,0)}[r[t],\theta[t],\phi[t],t]\; f_{33}}{f_{33}[r[t],\theta[t],\phi}\right.$
G[2, 2]	$\frac{f_{11}[r[t],\theta[t],\phi[t],t]\; f_{33}[r[t],\theta[t],\phi[t],t]\; f_{44}[r[t],\theta[t],\phi[t],t]\; \left(f_{44}[r[t],\theta[t],\phi[t],t]\; f_{11}{}^{(0,1,0,0)}[r[t],\theta[t],\phi[t]\right.}{}$
G[3, 1]	$\frac{1}{4}\left(\frac{f_{22}{}^{(0,0,1,0)}[r[t],\theta[t],\phi[t],t]\; f_{22}{}^{(1,0,0,0)}[r[t],\theta[t],\phi[t],t]}{f_{22}[r[t],\theta[t],\phi[t],t]^2}+\frac{f_{11}{}^{(0,0,1,0)}[r[t],\theta[t],\phi[t],t]\; \left(\frac{f_{22}{}^{(1,0,0,0)}[r[t],\theta[t],\phi[t],t}{f_{22}[r[t],\theta[t],\phi[t],t]}\right.}{f_{11}[r[t],\theta[t],\phi[t],t]}\right.$
G[3, 2]	$\frac{1}{4}\left(\frac{f_{11}{}^{(0,0,1,0)}[r[t],\theta[t],\phi[t],t]\; \left(f_{11}{}^{(0,1,0,0)}[r[t],\theta[t],\phi[t],t]+\frac{f_{11}[r[t],\theta[t],\phi[t],t]\; f_{33}{}^{(0,1,0,0)}[r[t],\theta[t],\phi[t],t]}{f_{33}[r[t],\theta[t],\phi[t],t]}\right)+f_{11}[r[t],\theta[t}{f_{11}[r[t],\theta[t],\phi[t],t]^2}\right.$
G[3, 3]	$-\frac{f_{11}[r[t],\theta[t],\phi[t],t]\; f_{33}[r[t],\theta[t],\phi[t],t]\; f_{44}[r[t],\theta[t],\phi[t],t]\; \left(f_{44}[r[t],\theta[t],\phi[t],t]\; \left(f_{11}{}^{(0,1,0,0)}[r[t],\theta[t],\phi\right.\right.}{}$
G[4, 1]	$\frac{1}{4}\left(\frac{f_{22}[r[t],\theta[t],\phi[t],t]\; \left(f_{11}{}^{(0,0,0,1)}[r[t],\theta[t],\phi[t],t]+\phi'[t]\; f_{11}{}^{(0,0,1,0)}[r[t],\theta[t],\phi[t],t]+\theta'[t]\; f_{11}{}^{(0,1,0,0)}[r[t],\theta[t],\phi[t],t]+r'[t]\; f_{11}{}^{(1,0,0,0)}[r[t],\theta[t],\phi[t\right.}{f_{11}[r[t],\theta[t],\phi[t],t]}\right.$
G[4, 2]	$\frac{1}{4}\left(\frac{f_{11}{}^{(0,1,0,0)}[r[t],\theta[t],\phi[t],t]\; \left(f_{11}{}^{(0,0,0,1)}[r[t],\theta[t],\phi[t],t]+\phi'[t]\; f_{11}{}^{(0,0,1,0)}[r[t],\theta[t],\phi[t],t]+\theta'[t]\; f_{11}{}^{(0,1,0,0)}[r\right.}{}\right.$
G[4, 3]	$\frac{1}{4}\left(\frac{f_{11}{}^{(0,0,1,0)}[r[t],\theta[t],\phi[t],t]\; \left(f_{11}{}^{(0,0,0,1)}[r[t],\theta[t],\phi[t],t]+\phi'[t]\; f_{11}{}^{(0,0,1,0)}[r[t],\theta[t],\phi[t],t]+\theta'[t]\; f_{11}{}^{(0,1,0,0)}[r\right.}{}\right.$
G[4, 4]	$\frac{-f_{22}[r[t],\theta[t],\phi[t],t]\; f_{33}[r[t],\theta[t],\phi[t],t]\; f_{44}[r[t],\theta[t],\phi[t],t]\; \left(f_{33}[r[t],\theta[t],\phi[t],t]\; \left(f_{11}{}^{(0,1,0,0)}[r[t],\theta[t],\phi\right.\right.}{}$

If the above yields an empty list {}, then the metric satisfies Einstein ' s Eqns
If it ' s not empty but the left over terms → 0 under some condition,
then EFE ' s satisfied under that condition

Stuff

```
ricci
{{(m (4 m r′[t]^2 + Cot[θ[t]] (2 m - r[t]) r[t] r′[t] θ′[t] + (2 m - r[t]) r[t] r″[t])) /
   (-2 m + r[t])^4, 0, 0, (Cot[θ[t]] (3 m - r[t]) θ′[t]) / (r[t] (-2 m + r[t]))},
 {0, (r[t] ((-4 m + r[t]) r′[t]^2 - Cot[θ[t]] (2 m - r[t]) r[t] r′[t] θ′[t] +
        r[t] (-2 m + r[t]) r″[t])) / (-2 m + r[t])^2, 0, θ′[t]}, {0, 0, - 1/(-2 m + r[t])^2
     r[t] Sin[θ[t]] ((4 m - r[t]) Sin[θ[t]] r′[t]^2 + Cos[θ[t]] (8 m - 3 r[t]) r[t] r′[t] θ′[t] +
        r[t] (-2 m + r[t]) (-Sin[θ[t]] r″[t] + r[t] (Sin[θ[t]] θ′[t]^2 - Cos[θ[t]] θ″[t]))),
  0}, {(Cot[θ[t]] (3 m - r[t]) θ′[t]) / (r[t] (-2 m + r[t])), θ′[t], 0, 1/(r[t]^2 (-2 m + r[t])^2)
    (-4 m^2 r′[t]^2 - Cot[θ[t]] r[t] (10 m^2 - 9 m r[t] + 2 r[t]^2) r′[t] θ′[t] +
      r[t] (-2 m + r[t]) (5 m r″[t] + r[t]^2 (θ′[t]^2 - Cot[θ[t]] θ″[t]) -
         2 r[t] (m θ′[t]^2 + r″[t] - m Cot[θ[t]] θ″[t])))}}

Sum[Sum[Laplacian[ricci[[i, j]], {r[t], θ[t], ϕ[t]}, "Spherical"], {i, 1, 4}], {j, 1, 4}] //
  Simplify;
```

Time evolution expressions

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab}\, R \;=\; 8\pi\, T_{ab}$$

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu\, g_{\sigma\nu} + \partial_\nu\, g_{\sigma\mu} - \partial_\sigma\, g_{\mu\nu})$$

$$\Rightarrow\; \partial_t\left(\Gamma^{\lambda}{}_{\mu\nu}\right) =$$

$$\frac{1}{2}\,\partial_t\left[g^{\lambda\sigma}(\partial_\mu\, g_{\sigma\nu} + \partial_\nu\, g_{\sigma\mu} - \partial_\sigma\, g_{\mu\nu})\right] = \frac{1}{2}\left(\partial_t g^{\lambda\sigma}\right)(\partial_\mu\, g_{\sigma\nu} + \partial_\nu\, g_{\sigma\mu} - \partial_\sigma\, g_{\mu\nu}) \;+\; \frac{1}{2} g^{\lambda\sigma}[\partial_t\,(\partial_\mu\, g_{\sigma\nu} + \partial_\nu\, g_{\sigma\mu} - \partial_\sigma\, g_{\mu\nu})]$$

$$\partial_t\,(\partial_\mu\, g_{\sigma\nu} + \partial_\nu\, g_{\sigma\mu} - \partial_\sigma\, g_{\mu\nu}) = \partial_\mu\;\partial_t\,(g_{\sigma\nu}) + \partial_\nu\;\partial_t\,(g_{\sigma\mu}) - \partial_\sigma\;\partial_t\,(g_{\mu\nu})$$

$$\partial_t\left(\Gamma^{\lambda}{}_{\mu\nu}\right) \;=\; \frac{1}{2}\left(\partial_t g^{\lambda\sigma}\right)(\partial_\mu\, g_{\sigma\nu} + \partial_\nu\, g_{\sigma\mu} - \partial_\sigma\, g_{\mu\nu}) \;+\; \frac{1}{2} g^{\lambda\sigma}[\partial_\mu\;\partial_t\,(g_{\sigma\nu}) + \partial_\nu\;\partial_t\,(g_{\sigma\mu}) - \partial_\sigma\;\partial_t\,(g_{\mu\nu})]$$

$$R^{\lambda}{}_{\mu\sigma\nu} = \partial_\sigma\;\Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu\;\Gamma^{\lambda}{}_{\mu\sigma} + \Gamma^{\eta}{}_{\mu\nu}\,\Gamma^{\lambda}{}_{\eta\sigma} - \Gamma^{\eta}{}_{\mu\sigma}\,\Gamma^{\lambda}{}_{\eta\nu}$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda\;\Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu\;\Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\eta}{}_{\mu\nu}\,\Gamma^{\lambda}{}_{\eta\lambda} - \Gamma^{\eta}{}_{\mu\lambda}\,\Gamma^{\lambda}{}_{\eta\nu}$$

$$R = g^{\mu\nu}\, R_{\mu\nu} \;\Rightarrow\; \partial_t R \;=\; \partial_t\left(g^{\mu\nu}\, R_{\mu\nu}\right) \;=\; \left(\partial_t g^{\mu\nu}\right) R_{\mu\nu} + g^{\mu\nu}(\partial_t R_{\mu\nu})$$

$$\partial_t R_{\mu\nu} \;=\; \partial_t\left(\partial_\lambda\;\Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu\;\Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\eta}{}_{\mu\nu}\,\Gamma^{\lambda}{}_{\eta\lambda} - \Gamma^{\eta}{}_{\mu\lambda}\,\Gamma^{\lambda}{}_{\eta\nu}\right) =$$
$$\partial_t\left(\partial_\lambda\;\Gamma^{\lambda}{}_{\mu\nu}\right) \;-\; \partial_t\left(\partial_\nu\;\Gamma^{\lambda}{}_{\mu\lambda}\right) + \;\partial_t\left(\Gamma^{\eta}{}_{\mu\nu}\right)\Gamma^{\lambda}{}_{\eta\lambda} \;+\; \Gamma^{\eta}{}_{\mu\nu}\,\partial_t\left(\Gamma^{\lambda}{}_{\eta\lambda}\right) \;-\; \partial_t\left(\Gamma^{\eta}{}_{\mu\lambda}\right)\Gamma^{\lambda}{}_{\eta\nu} - \;\Gamma^{\eta}{}_{\mu\lambda}\,\partial_t\left(\Gamma^{\lambda}{}_{\eta\nu}\right)$$

$$\partial_t R_{\mu\nu} \;=\; \partial_\lambda\left(\,\partial_t\Gamma^{\lambda}{}_{\mu\nu}\right) \;-\; \partial_\nu\left(\partial_t\Gamma^{\lambda}{}_{\mu\lambda}\right) + \;\left(\partial_t\Gamma^{\eta}{}_{\mu\nu}\right)\Gamma^{\lambda}{}_{\eta\lambda} \;+\; \Gamma^{\eta}{}_{\mu\nu}\left(\partial_t\Gamma^{\lambda}{}_{\eta\lambda}\right) \;-\left(\,\partial_t\Gamma^{\eta}{}_{\mu\lambda}\right)\Gamma^{\lambda}{}_{\eta\nu} - \;\Gamma^{\eta}{}_{\mu\lambda}\left(\partial_t\Gamma^{\lambda}{}_{\eta\nu}\right)$$

$$\partial_t\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}\left(\partial_t g^{\lambda\sigma}\right)(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) + \frac{1}{2} g^{\lambda\sigma}[\partial_\mu \partial_t(g_{\sigma\nu}) + \partial_\nu \partial_t(g_{\sigma\mu}) - \partial_\sigma \partial_t(g_{\mu\nu})]$$

$$\partial_t\Gamma^{\lambda}{}_{\mu\lambda} = \frac{1}{2}\left(\partial_t g^{\lambda\sigma}\right)(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\mu\lambda}) + \frac{1}{2} g^{\lambda\sigma}[\partial_\mu \partial_t(g_{\sigma\lambda}) + \partial_\lambda \partial_t(g_{\sigma\mu}) - \partial_\sigma \partial_t(g_{\mu\lambda})]$$

$$\partial_t\Gamma^{\eta}{}_{\mu\nu} = \frac{1}{2}\left(\partial_t g^{\eta\sigma}\right)(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) + \frac{1}{2} g^{\eta\sigma}[\partial_\mu \partial_t(g_{\sigma\nu}) + \partial_\nu \partial_t(g_{\sigma\mu}) - \partial_\sigma \partial_t(g_{\mu\nu})]$$

$$\partial_t\Gamma^{\lambda}{}_{\eta\lambda} = \frac{1}{2}\left(\partial_t g^{\lambda\sigma}\right)(\partial_\eta g_{\sigma\lambda} + \partial_\lambda g_{\sigma\eta} - \partial_\sigma g_{\eta\lambda}) + \frac{1}{2} g^{\lambda\sigma}[\partial_\eta \partial_t(g_{\sigma\lambda}) + \partial_\lambda \partial_t(g_{\sigma\eta}) - \partial_\sigma \partial_t(g_{\eta\lambda})]$$

$$\partial_t\Gamma^{\eta}{}_{\mu\lambda} = \frac{1}{2}\left(\partial_t g^{\eta\sigma}\right)(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\mu\lambda}) + \frac{1}{2} g^{\eta\sigma}[\partial_\mu \partial_t(g_{\sigma\lambda}) + \partial_\lambda \partial_t(g_{\sigma\mu}) - \partial_\sigma \partial_t(g_{\mu\lambda})]$$

$$\partial_t\Gamma^{\lambda}{}_{\eta\nu} = \frac{1}{2}\left(\partial_t g^{\lambda\sigma}\right)(\partial_\eta g_{\sigma\nu} + \partial_\nu g_{\sigma\eta} - \partial_\sigma g_{\eta\nu}) + \frac{1}{2} g^{\lambda\sigma}[\partial_\eta \partial_t(g_{\sigma\nu}) + \partial_\nu \partial_t(g_{\sigma\eta}) - \partial_\sigma \partial_t(g_{\eta\nu})]$$

$$\begin{aligned}
\partial_t R_{\mu\nu} = {} & \partial_\lambda\left(\frac{1}{2}\left(\partial_t g^{\lambda\sigma}\right)(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) + \frac{1}{2} g^{\lambda\sigma}[\partial_\mu \partial_t(g_{\sigma\nu}) + \partial_\nu \partial_t(g_{\sigma\mu}) - \partial_\sigma \partial_t(g_{\mu\nu})]\right) \\
& - \partial_\nu\left(\frac{1}{2}\left(\partial_t g^{\lambda\sigma}\right)(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\mu\lambda}) + \frac{1}{2} g^{\lambda\sigma}[\partial_\mu \partial_t(g_{\sigma\lambda}) + \partial_\lambda \partial_t(g_{\sigma\mu}) - \partial_\sigma \partial_t(g_{\mu\lambda})]\right) \\
& + \left(\frac{1}{2}\left(\partial_t g^{\eta\sigma}\right)(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) + \frac{1}{2} g^{\eta\sigma}[\partial_\mu \partial_t(g_{\sigma\nu}) + \partial_\nu \partial_t(g_{\sigma\mu}) - \partial_\sigma \partial_t(g_{\mu\nu})]\right)\Gamma^{\lambda}{}_{\eta\lambda} \\
& + \Gamma^{\eta}{}_{\mu\nu}\left(\frac{1}{2}\left(\partial_t g^{\lambda\sigma}\right)(\partial_\eta g_{\sigma\lambda} + \partial_\lambda g_{\sigma\eta} - \partial_\sigma g_{\eta\lambda}) + \frac{1}{2} g^{\lambda\sigma}[\partial_\eta \partial_t(g_{\sigma\lambda}) + \partial_\lambda \partial_t(g_{\sigma\eta}) - \partial_\sigma \partial_t(g_{\eta\lambda})]\right) \\
& - \left(\frac{1}{2}\left(\partial_t g^{\eta\sigma}\right)(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\mu\lambda}) + \frac{1}{2} g^{\eta\sigma}[\partial_\mu \partial_t(g_{\sigma\lambda}) + \partial_\lambda \partial_t(g_{\sigma\mu}) - \partial_\sigma \partial_t(g_{\mu\lambda})]\right)\Gamma^{\lambda}{}_{\eta\nu} \\
& - \Gamma^{\eta}{}_{\mu\lambda}\left(\frac{1}{2}\left(\partial_t g^{\lambda\sigma}\right)(\partial_\eta g_{\sigma\nu} + \partial_\nu g_{\sigma\eta} - \partial_\sigma g_{\eta\nu}) + \frac{1}{2} g^{\lambda\sigma}[\partial_\eta \partial_t(g_{\sigma\nu}) + \partial_\nu \partial_t(g_{\sigma\eta}) - \partial_\sigma \partial_t(g_{\eta\nu})]\right)
\end{aligned}$$

Use the identity : $\nabla_l R^l{}_m = \frac{1}{2}\nabla_m R$ (https://en.wikipedia.org/wiki/List_of_formulas_in_Riemannian_geometry)

$$\nabla_l R^l{}_0 = \frac{1}{2}\nabla_0 R = -\frac{1}{2}\partial_t R, \quad \text{since} \quad \nabla_0 = -\partial_t$$

Normal Cood System

In a normal coordinate system based at p, at the point p we have that

$\Delta \equiv$ Laplace – Beltrami Operator

$$\Delta f = \nabla^2 f = \frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}\, g^{ab}\,\partial_b f\right) \Rightarrow \Delta g_{ab} = \frac{1}{\sqrt{|g|}}\partial_a\left[\sqrt{|g|}\, g^{ab}\,(\partial_b g_{ab})\right]$$

$$R_{ab} = -\frac{3}{2}\Delta(g_{ab}) = -\frac{3}{2}\frac{1}{\sqrt{|g|}}\partial_a\left[\sqrt{|g|}\, g^{ab}\,(\partial_b g_{ab})\right]$$

$$\Delta g_{ab} = \frac{1}{\sqrt{|g|}}\, \partial_a\left[\sqrt{|g|}\; g^{ab}\,(\partial_b g_{ab})\right] =$$

$$\frac{1}{\sqrt{|g|}}\Big[\partial_0\left(\sqrt{|g|}\; g^{00}\,\partial_0 g_{00}\right) + \partial_0\left(\sqrt{|g|}\; g^{01}\,\partial_1 g_{01}\right) + \partial_0\left(\sqrt{|g|}\; g^{02}\,\partial_2 g_{02}\right) + \partial_0\left(\sqrt{|g|}\; g^{03}\,\partial_3 g_{03}\right) +$$
$$\partial_1\left(\sqrt{|g|}\; g^{10}\,\partial_0 g_{10}\right) + \partial_1\left(\sqrt{|g|}\; g^{11}\,\partial_1 g_{11}\right) + \partial_1\left(\sqrt{|g|}\; g^{12}\,\partial_2 g_{12}\right) + \partial_1\left(\sqrt{|g|}\; g^{13}\,\partial_3 g_{13}\right) +$$
$$\partial_2\left(\sqrt{|g|}\; g^{20}\,\partial_0 g_{20}\right) + \partial_2\left(\sqrt{|g|}\; g^{21}\,\partial_1 g_{21}\right) + \partial_2\left(\sqrt{|g|}\; g^{22}\,\partial_2 g_{22}\right) + \partial_2\left(\sqrt{|g|}\; g^{23}\,\partial_3 g_{23}\right) +$$
$$\partial_3\left(\sqrt{|g|}\; g^{30}\,\partial_0 g_{30}\right) + \partial_3\left(\sqrt{|g|}\; g^{31}\,\partial_1 g_{31}\right) + \partial_3\left(\sqrt{|g|}\; g^{32}\,\partial_2 g_{32}\right) + \partial_3\left(\sqrt{|g|}\; g^{33}\,\partial_3 g_{33}\right)\Big]$$

$$\Rightarrow\; R = g^{ab} R_{ab} = -\frac{3}{2} g^{ab}\,\Delta g_{ab}$$

$$\Rightarrow\; \left[\partial_t(R_{ab}) - \frac{1}{2} g_{ab}\,(\partial_t R)\right] = \left[\partial_t(R_{ab}) - \frac{1}{2} g_{ab}\,(\partial_t R)\right],\ \text{also use } \partial_t R = \left(\partial_t g^{ab}\right) R_{ab} + g^{ab}(\partial_t R_{ab})$$

$$\Rightarrow\; \partial_t(R_{ab}) - \frac{1}{2} g_{ab}\,(\partial_t R) =$$

$$\partial_t(R_{ab}) - \frac{1}{2} g_{ab}\left[\left(\partial_t g^{ab}\right) R_{ab}\right] - \frac{1}{2} g_{ab}\, g^{ab}(\partial_t R_{ab}) = \left[1 - \frac{1}{2} g_{ab}\, g^{ab}\right]\partial_t(R_{ab}) - \frac{1}{2} g_{ab}\left(\partial_t g^{ab}\right) R_{ab}$$

$$\Rightarrow\; \left[\partial_t(R_{ab}) - \frac{1}{2} g_{ab}\,(\partial_t R)\right] = -\frac{3}{2}\left[1 - \frac{1}{2} g_{ab}\, g^{ab}\right]\partial_t(\Delta g_{ab}) + \frac{3}{4} g_{ab}\left(\partial_t g^{ab}\right)\Delta g_{ab}$$

$$(\partial_t g_{ab})\,R - 2\left[\partial_t(R_{ab}) - \frac{1}{2} g_{ab}\,(\partial_t R)\right] = (\partial_t g_{ab})\,R + 3\left[1 - \frac{1}{2} g_{ab}\, g^{ab}\right]\partial_t(\Delta g_{ab}) - \frac{3}{2} g_{ab}\left(\partial_t g^{ab}\right)\Delta g_{ab}$$

$$= -\frac{3}{2}(\partial_t g_{ab})\, g^{ab}\,\Delta g_{ab} + 3\left[1 - \frac{1}{2} g_{ab}\, g^{ab}\right]\partial_t(\Delta g_{ab}) - \frac{3}{2} g_{ab}\left(\partial_t g^{ab}\right)\Delta g_{ab}$$

since $\partial_t\left(g_{ab}\, g^{ab}\right) = (\partial_t g_{ab})\, g^{ab} + g_{ab}\left(\partial_t g^{ab}\right) \;\Rightarrow\; g_{ab}\left(\partial_t g^{ab}\right) = \partial_t\left(g_{ab}\, g^{ab}\right) - (\partial_t g_{ab})\, g^{ab}$ we get

$$= -\frac{3}{2}(\partial_t g_{ab})\, g^{ab}\,\Delta g_{ab} + 3\left[1 - \frac{1}{2} g_{ab}\, g^{ab}\right]\partial_t(\Delta g_{ab}) - \frac{3}{2}\left[\partial_t\left(g_{ab}\, g^{ab}\right) - (\partial_t g_{ab})\, g^{ab}\right]\Delta g_{ab}$$

$$= -\frac{3}{2}(\partial_t g_{ab})\, g^{ab}\,\Delta g_{ab} + 3\left[1 - \frac{1}{2} g_{ab}\, g^{ab}\right]\partial_t(\Delta g_{ab}) - \frac{3}{2}\partial_t\left(g_{ab}\, g^{ab}\right)\Delta g_{ab} + \frac{3}{2}(\partial_t g_{ab})\, g^{ab}\,\Delta g_{ab}$$

And $g_{ab}\, g^{ab} = n\,(=4)$ for all metrics in GR, so $\partial_t\left(g_{ab}\, g^{ab}\right) = 0$

So we now have that

$$(\partial_t g_{ab})\,R - 2\left[\partial_t(R_{ab}) - \frac{1}{2} g_{ab}\,(\partial_t R)\right] =$$

$$3\left[1 - \frac{1}{2} g_{ab}\, g^{ab}\right]\partial_t(\Delta g_{ab}) = 3\left[1 - \frac{1}{2} n\right]\partial_t(\Delta g_{ab}) = -16\,\pi\,\partial_t(T_{ab})$$

In *a* normal coordinate system based at *p*, at the point *p*, we have that $R_{ab} = -\frac{3}{2}\Delta(g_{ab})$,

(Normal coordinates always exist for the Levi –

Civita connection of *a* Riemannian or Pseudo − Riemannian manifold)

this gives us

$$(\partial_t g_{ab})R - 2\left[\partial_t(R_{ab}) - \frac{1}{2} g_{ab}(\partial_t R)\right] =$$

$$3\left[1 - \frac{1}{2} g_{ab} g^{ab}\right]\partial_t(\Delta g_{ab}) = 3\left[1 - \frac{1}{2} n\right]\partial_t(\Delta g_{ab}) = -16\pi\,\partial_t(T_{ab})$$

$$\frac{3}{2}[2-n]\;\partial_t(\Delta g_{ab}) = -16\pi\,\partial_t(T_{ab})$$

$$\Rightarrow \left[-\frac{3}{2}\partial_t(\Delta g_{ab})\right](2-n) = 16\pi\,\partial_t(T_{ab})$$

$$\Rightarrow [\partial_t(R_{ab})](2-n) = 16\pi\,\partial_t(T_{ab})$$

$$\Rightarrow \partial_t R_{ab} = \frac{16\pi}{(2-n)}\partial_t T_{ab} = -\frac{3}{2} k\,\partial_t T_{ab}$$

Also note that we can switch out *t* for any spatial coordinate,
as *t* in this derivativation is just some parameter. And we can commute the Laplace −
Beltrami operator and partials since this operator is just *a* sum of partials.

$$\left[-\frac{3}{2}(2-n)\right]\partial_t(\Delta g_{ab}) = \left[3\left(\frac{1}{2} n - 1\right)\right]\partial_t(\Delta g_{ab}) = 16\pi\,\partial_t(T_{ab})$$

$$\Rightarrow \partial_t(\Delta g_{ab}) =$$

$$\Delta(\partial_t g_{ab}) = \frac{16\pi}{3\left(\frac{1}{2} n - 1\right)}\partial_t T_{ab} = k\,\partial_t T_{ab} \qquad \text{letting } k = \frac{16\pi}{3\left(\frac{1}{2} n - 1\right)} = \frac{16\pi}{3} \text{ (for } n = 4)$$

$$R_{ab} = -\frac{3}{2}\Delta(g_{ab})$$

$$\Rightarrow \Delta(\partial_t g_{ab}) = -\frac{2}{3}\partial_t R_{ab}$$

So for the case $\partial_t T_{ab} = 0$

$$\Delta(\partial_t g_{ab}) = 0$$

$$\left(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\right)\partial_t g_{ab} = 0,$$

or for the case $\partial_c T_{ab} = 0$

$\left(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\right)\partial_c g_{ab} = 0$ **for any coordinate or sum of coods *c***

***i.e.* evolution of the metric satisfies the 4 − wave equation**
Fact that evolution terms obey these nice relations might hint that flow of st might be easier to understand

***t* not constrained, *i.e.* same formula holds replacing *t* with spatial coods,**
so for index μ *a* sum over all spatial coods and *c* over all space and time coods we can see that

$$\Gamma^m{}_{cb} = \frac{1}{2} g^{ma}(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{cb}) \Rightarrow g_{ma}\Gamma^m{}_{cb} =$$

$$\frac{1}{2} g_{ma}\, g^{ma}(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb}) = \frac{n}{2}(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb})$$

$$(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb}) = \frac{2}{n} g_{ma}\, \Gamma^m{}_{cb}$$

$$\Delta\Gamma^m{}_{cb} = \Delta\left[\frac{1}{2} g^{ma}(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb})\right] = \frac{1}{2}\left(\Delta g^{ma}\right)(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb}) + \frac{1}{2} g^{ma}\, \Delta(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb})$$

$$-\Delta(\partial_t g_{ab}) + \Delta(\partial_\mu g_{ab}) = \Delta(\partial_c g_{ab}) = k\, \partial_c T_{ab}$$

$$\Rightarrow \Delta(\partial_c\, g_{ab}) + \Delta(\partial_b\, g_{ac}) - \Delta(\partial_a\, g_{cb}) = k\, \partial_c T_{ab} + k\, \partial_b T_{ac} - k\, \partial_a T_{cb}$$

$$\Rightarrow \Delta(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb}) = k(\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb})$$

$$\Rightarrow \frac{1}{2} g^{ma}\, \Delta(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb}) = k \frac{1}{2} g^{ma}(\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb})$$

$$\frac{1}{2} g^{ma}\, \Delta(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb}) = \Delta\left[\frac{1}{2} g^{ma}(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb})\right] - \frac{1}{2}\left(\Delta g^{ma}\right)(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb})$$

$$\frac{1}{2} g^{ma}\, \Delta(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb}) = \Delta\Gamma^m{}_{cb} - \frac{1}{2}\left(\Delta g^{ma}\right)(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb}) = \Delta\Gamma^m{}_{cb} - \frac{1}{n}\left(\Delta g^{ma}\right) g_{ma}\, \Gamma^m{}_{cb}$$

since, from defn of Christoffel symbol, $(\partial_c\, g_{ab} + \partial_b\, g_{ac} - \partial_a\, g_{cb}) = \frac{2}{n} g_{ma}\, \Gamma^m{}_{cb}$

$$\Rightarrow \Delta\Gamma^m{}_{cb} - \frac{1}{n}\left(\Delta g^{ma}\right) g_{ma}\, \Gamma^m{}_{cb} = k \frac{1}{2} g^{ma}(\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb})$$

$$\Rightarrow \Delta\Gamma^m{}_{cb} = k \frac{1}{2} g^{ma}(\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb}) + \frac{1}{n}\left(\Delta g^{ma}\right) g_{ma}\, \Gamma^m{}_{cb}$$

$$\Rightarrow \frac{1}{n}\left(\Delta g^{ma}\right) g_{ma}\, \Gamma^m{}_{cb} = \Delta\Gamma^m{}_{cb} - k \frac{1}{2} g^{ma}(\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb})$$

$$\Rightarrow \frac{1}{n} g^{ma}(\Delta g_{ma})\left(\Delta g^{ma}\right) g_{ma}\, \Gamma^m{}_{cb} = g^{ma}(\Delta g_{ma})\left[\Delta\Gamma^m{}_{cb} - k \frac{1}{2} g^{ma}(\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb})\right]$$

$(\Delta g_{ma})\left(\Delta g^{ma}\right) = \alpha =$ **some scalar constant, although the inverse exists iff det**$\left[\Delta g^{ma}\right] \neq 0$**,**
but in this case would also be necessary to have $\Delta g^{ma} =$
0 for metric in GR (since it's symmetric about p **and non − degenerate),**
so simplifications would happen sooner
Probably often have that $(\Delta g_{ma})\left(\Delta g^{ma}\right) = n$ **or 0, if not always (box below this checks this)**

$$\Rightarrow g^{ma}(\Delta g_{ma})\left(\Delta g^{ma}\right) g_{ma}\, \Gamma^m{}_{cb} = \alpha\, g^{ma}\, g_{ma}\, \Gamma^m{}_{cb} = \alpha\, n\, \Gamma^m{}_{cb} = n\, g^{ma}(\Delta g_{ma})\left[\Delta\Gamma^m{}_{cb} - k \frac{1}{2} g^{ma}(\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb})\right]$$

$$\Rightarrow \Gamma^{m}{}_{cb} = \frac{1}{\alpha} g^{ma} (\Delta g_{ma}) \left[\Delta \Gamma^{m}{}_{cb} - k \frac{1}{2} g^{ma} (\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb}) \right]$$

∗Note that the term $k \frac{1}{2} g^{ma} (\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb})$ is analogous to the affine connection, and is proportional to $\Gamma^{m}{}_{cb}$ for T_{ab} propartional to the metric tensor

$$-\Delta \left(\frac{d u^m}{d\tau} \right) = \Delta \left(\Gamma^{m}{}_{cb} u^b u^c \right) = \left(\Delta \Gamma^{m}{}_{cb} \right) u^b u^c + \Gamma^{m}{}_{cb} \Delta \left(u^b u^c \right)$$

$$\Rightarrow -\Delta \left(\frac{d u^m}{d\tau} \right) = \left(k \frac{1}{2} g^{ma} (\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb}) + \frac{1}{n} \left(\Delta g^{ma} \right) g_{ma} \Gamma^{m}{}_{cb} \right) u^b u^c + \Gamma^{m}{}_{cb} \Delta \left(u^b u^c \right)$$

$$\Rightarrow -\Delta \left(\frac{d u^m}{d\tau} \right) = k \frac{1}{2} g^{ma} (\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb}) u^b u^c + \frac{1}{n} \left(\Delta g^{ma} \right) g_{ma} \Gamma^{m}{}_{cb} u^b u^c + \Gamma^{m}{}_{cb} \Delta \left(u^b u^c \right)$$

$$\Rightarrow -\Delta \left(\frac{d u^m}{d\tau} \right) = k \frac{1}{2} g^{ma} (\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb}) u^b u^c + \\ \frac{1}{n} \left(\Delta g^{ma} \right) g_{ma} \left[\frac{1}{2} g^{ma} (\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{cb}) \right] u^b u^c + \left[\frac{1}{2} g^{ma} (\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{cb}) \right] \Delta \left(u^b u^c \right)$$

$$\Rightarrow -2 \Delta \left(\frac{d u^m}{d\tau} \right) = k g^{ma} (\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb}) u^b u^c + \\ \frac{1}{n} \left(\Delta g^{ma} \right) \left(g_{ma} g^{ma} \right) (\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{cb}) u^b u^c + g^{ma} (\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{cb}) \Delta \left(u^b u^c \right)$$

$$\Rightarrow -2 \Delta \left(\frac{d u^m}{d\tau} \right) = \left[k g^{ma} (\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb}) + \left(\Delta g^{ma} \right) (\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{cb}) \right] u^b u^c + \\ g^{ma} (\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{cb}) \Delta \left(u^b u^c \right)$$

$$\frac{d u^m}{d\tau} = -\Gamma^{m}{}_{cb} u^b u^c = -\frac{1}{\alpha} g^{ma} (\Delta g_{ma}) \left[\Delta \Gamma^{m}{}_{cb} - k \frac{1}{2} g^{ma} (\partial_c T_{ab} + \partial_b T_{ac} - \partial_a T_{cb}) \right] u^b u^c$$

```
(* Lapofmetric=
 D[metric,{r,2}]+D[metric,{θ,2}]+D[metric,{t,2}]+D[metric,{ϕ,2}]//FullSimplify;
(*Laplacian of some metric in flat background, spherical coods *)
inverseLapmetric=Simplify[Inverse[Lapofmetric]];
Tr[inverseLapmetric.Lapofmetric//FullSimplify]
 Tr[Lapofmetric.inverseLapmetric//FullSimplify]
 lapmet=Sum[D[√-Det[metric] metric.D[metric,cood[[i]]],cood[[i]]]/(√-Det[metric]),
    {i,1,4}]//FullSimplify; (*Laplacian in the metrics own background *)
invlapmet=Simplify[Inverse[lapmet]];
Tr[invlapmet.lapmet//FullSimplify]
 Tr[lapmet.invlapmet//FullSimplify]  *)
```

For *a* null geodesic

$g_{cb}\, u^b\, u^c = 0$, **so that** $\dfrac{d\, u^m}{d\tau} =$

0 is obtained. This describes the motion of light (*i.e.* light never accelarates according to an inertial observer, although it can soort of be said to for *a* non − inertial one, for instance severe grav lensing may be seen as slowing down *a* bit from the right angle)

Now we would like to relate this to the geodesic equation $d\, u^\alpha / d\tau = -\Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma$, **so find Christoffel symbol version of this relation :**

$$R_{ab} = R^c{}_{acb} = \partial_c\, \Gamma^c{}_{ab} - \partial_b\, \Gamma^c{}_{ac} + \Gamma^d{}_{ab}\, \Gamma^c{}_{dc} - \Gamma^d{}_{ac}\, \Gamma^c{}_{db}$$

$$\Rightarrow\ \partial_t R_{ab} = \partial_t\left(\partial_c\, \Gamma^c{}_{ab} - \partial_b\, \Gamma^c{}_{ac} + \Gamma^d{}_{ab}\, \Gamma^c{}_{dc} - \Gamma^d{}_{ac}\, \Gamma^c{}_{db}\right)$$

Or, by Palatini' *s* Identity for the variation of the Ricci Tensor, (For − +++)

$$\delta R_{ab} = \nabla_c\left(\delta\Gamma^c{}_{ba}\right) - \nabla_b\left(\delta\Gamma^c{}_{ca}\right)$$

$$\partial_t(\delta R_{ab}) = \delta(\partial_t R_{ab}) = \partial_t\left[\nabla_c\left(\delta\Gamma^c{}_{ba}\right) - \nabla_b\left(\delta\Gamma^c{}_{ca}\right)\right]$$

$$\Rightarrow\ \delta(\partial_t R_{ab}) = \frac{16\,\pi}{(2-n)}\,\delta(\partial_t T_{ab}) = \partial_t\left[\nabla_c\left(\delta\Gamma^c{}_{ba}\right) - \nabla_b\left(\delta\Gamma^c{}_{ca}\right)\right]$$

$$\Rightarrow\ \partial_t\left[\nabla_c\left(\delta\Gamma^c{}_{ba}\right)\right] = \frac{16\,\pi}{(2-n)}\,\delta(\partial_t T_{ab}) + \partial_t\left[\nabla_b\left(\delta\Gamma^c{}_{ca}\right)\right]$$

$$\nabla_c\left(\frac{d\, u^c}{d\tau}\right) = -\nabla_c\left(\Gamma^c{}_{ba}\, u^b\, u^a\right) = -\left(\nabla_c\ \Gamma^c{}_{ba}\right) u^b\, u^a - \Gamma^c{}_{ba}\left(\nabla_c\ \left(u^b\, u^a\right)\right)$$

```
(* ∂θ(Sum[∂a(d binv[a,b]( ∂bg[a,b])),{a,0,3},{b,0,3}]) *)
```

```
(* cood={t,r,θ,ϕ};
∂t(Sum[1/d ∂cood[[a]] (d binv[a,b]( ∂cood[[b]] g[a,b])),{a,0,3},{b,0,3}])//FullSimplify;*)
cood = {t, r, θ, ϕ};
Sum[1/d ∂cood[[a]] (d binv[t, r, θ, ϕ][a, b] (∂cood[[b]] g[t, r, θ, ϕ][a, b])),
   {a, 0, 3}, {b, 0, 3}] // FullSimplify; (* = Δgab *)
∂t (Sum[1/d ∂cood[[a]] (d binv[t, r, θ, ϕ][a, b] (∂cood[[b]] g[t, r, θ, ϕ][a, b])),
      {a, 0, 3}, {b, 0, 3}]) // FullSimplify; (* = ∂t(Δgab) *)
Sum[∂t g[t, r, θ, ϕ][a, b], {a, 0, 3}, {b, 0, 3} ];   (* = ∂tgab *)

(* Sum[∂a(d ginvab ( ∂bgab )),{a,0,3},{b,0,3}]//MatrixForm *)
```

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}$$

$$\Rightarrow \partial_t G_{ab} = \partial_t\left(R_{ab} - \frac{1}{2} g_{ab} R\right) = 8\pi \partial_t T_{ab}$$

$$\Rightarrow \partial_t R_{ab} - \frac{1}{2} (\partial_t g_{ab}) R - \frac{1}{2} g_{ab} (\partial_t R) = 8\pi \partial_t T_{ab}$$

$$\Rightarrow -\frac{1}{2} (\partial_t g_{ab}) R = 8\pi \partial_t T_{ab} - \partial_t R_{ab} + \frac{1}{2} g_{ab} (\partial_t R)$$

$$\Rightarrow -(\partial_t g_{ab}) = \frac{2}{R} \partial_t (8\pi T_{ab} - R_{ab}) + \frac{1}{R} g_{ab} (\partial_t R)$$

$$\Rightarrow \partial_t g_{ab} = -\frac{2}{R}[\partial_t (8\pi T_{ab} - R_{ab})] - \frac{1}{R} g_{ab} (\partial_t R)$$

$$\Rightarrow \partial_t g_{ab} =$$
$$-\frac{16\pi}{R} \partial_t (T_{ab}) + \frac{2}{R}\left[\partial_t (R_{ab}) - \frac{1}{2} g_{ab} (\partial_t R)\right]$$ **In general for any metric & T_{ab} satisying EFE ' s**

$$\Rightarrow \partial_t g_{ab} = \frac{2}{R}\left[\partial_t (R_{ab}) - \frac{1}{2} g_{ab} (\partial_t R)\right]$$ **In $\partial_t T_{ab} = 0$ (Static) case**

$$\partial_t g_{ab} = \frac{2}{R}\left[\partial_t R_{ab} - \frac{1}{2} g_{ab} (\partial_t R)\right] - \frac{16\pi}{R} \partial_t T_{ab}$$

$$R = g^{\mu\nu} R_{\mu\nu} \Rightarrow \partial_t R = \partial_t\left(g^{\mu\nu} R_{\mu\nu}\right) = \left(\partial_t g^{\mu\nu}\right) R_{\mu\nu} + g^{\mu\nu}(\partial_t R_{\mu\nu})$$

$$\Rightarrow \partial_t g_{ab} = \frac{2}{R}\left[\partial_t R_{ab} - \frac{1}{2} g_{ab}\left\{\left(\partial_t g^{ab}\right) R_{ab} + g^{ab}(\partial_t R_{ab})\right\}\right] - \frac{16\pi}{R} \partial_t T_{ab}$$

$$\Rightarrow \partial_t g_{ab} = \frac{2}{R}\left[\partial_t R_{ab} - \left\{\frac{1}{2} g_{ab}\left(\partial_t g^{ab}\right) R_{ab} + \frac{1}{2} g_{ab} g^{ab}(\partial_t R_{ab})\right\}\right] - \frac{16\pi}{R} \partial_t T_{ab} \qquad g_{ab} g^{ab} = n$$

$$\Rightarrow \partial_t g_{ab} = \frac{2}{R}\left[\partial_t R_{ab} - \left\{\frac{1}{2} g_{ab}\left(\partial_t g^{ab}\right) R_{ab} + \frac{n}{2} (\partial_t R_{ab})\right\}\right] - \frac{16\pi}{R} \partial_t T_{ab}$$

$$\Rightarrow \partial_t g_{ab} = \frac{2}{R}\left[\left(1 - \frac{n}{2}\right) \partial_t R_{ab} - \frac{1}{2} g_{ab}\left(\partial_t g^{ab}\right) R_{ab}\right] - \frac{16\pi}{R} \partial_t T_{ab}$$

$$\Rightarrow (\partial_t g_{ab}) R = (2 - n) \partial_t R_{ab} - g_{ab}\left(\partial_t g^{ab}\right) R_{ab} - 16\pi \partial_t T_{ab}$$

$$\Rightarrow (\partial_t g_{ab})\, R + g_{ab}\left(\partial_t g^{ab}\right) R_{ab} = (2-n)\,\partial_t R_{ab} - 16\,\pi\,\partial_t T_{ab}$$

$$\Rightarrow (\partial_t g_{ab})\, g^{ab} R_{ab} + g_{ab}\left(\partial_t g^{ab}\right) R_{ab} = (2-n)\,\partial_t R_{ab} - 16\,\pi\,\partial_t T_{ab}$$

$$\Rightarrow \left[(\partial_t g_{ab})\, g^{ab} + g_{ab}\left(\partial_t g^{ab}\right)\right] R_{ab} = (2-n)\,\partial_t R_{ab} - 16\,\pi\,\partial_t T_{ab}$$

$$\Rightarrow \left[\partial_t\left(g_{ab}\, g^{ab}\right)\right] R_{ab} = (2-n)\,\partial_t R_{ab} - 16\,\pi\,\partial_t T_{ab}$$

$$\Rightarrow [\partial_t (n)]\, R_{ab} = (2-n)\,\partial_t R_{ab} - 16\,\pi\,\partial_t T_{ab}$$

$$\Rightarrow 0 = (2-n)\,\partial_t R_{ab} - 16\,\pi\,\partial_t T_{ab}$$

$$\Rightarrow (2-n)\,\partial_t R_{ab} = 16\,\pi\,\partial_t T_{ab}$$

$$\Rightarrow \partial_t R_{ab} = \frac{16\,\pi}{(2-n)}\,\partial_t T_{ab}$$

This is precisely the solution we obtained using normal coordinates,

but here we didn' *t* have to use normal coordinates or other such contraints!

⇒ D[metric, t] = $\frac{2}{\text{scalar}}\left(\text{D[ricci, t]} - \left(\frac{1}{2}\,(\text{metric})\,(\text{D[scalar, t]})\right)\right)$ In $\partial_t T_{ab}$ = 0 (Static) case

⇒ D[metric, t] − $\frac{2}{\text{scalar}}\left(\text{D[ricci, t]} - \left(\frac{1}{2}\,(\text{metric})\,(\text{D[scalar, t]})\right)\right)$ = 0 (= 0 I_4 = Zero matrix), In $\partial_t T_{ab}$ = 0 (Static) case

⇒ (scalar * D[metric, t]) − 2 $\left(\text{D[ricci, t]} - \left(\frac{1}{2}\,(\text{metric})\,(\text{D[scalar, t]})\right)\right)$ = 0 (= 0 I_4 = Zero matrix), In $\partial_t T_{ab}$ = 0 (Static) case

Move scalar to top so we don ' t have singularities in case R = 0

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\,\pi\,T_{ab}$$

$$\Rightarrow \partial_t G_{ab} = \partial_t\left(R_{ab} - \frac{1}{2} g_{ab} R\right) = 8\,\pi\,\partial_t T_{ab}$$

$$\Rightarrow \partial_t R_{ab} - \frac{1}{2}(\partial_t g_{ab})\, R - \frac{1}{2} g_{ab}\,(\partial_t R) = 8\,\pi\,\partial_t T_{ab}$$

$$\Rightarrow -\frac{1}{2}(\partial_t g_{ab})\, R = 8\,\pi\,\partial_t T_{ab} - \partial_t R_{ab} + \frac{1}{2} g_{ab}\,(\partial_t R)$$

$$\Rightarrow -(\partial_t g_{ab}) = \frac{2}{R}\,\partial_t (8\,\pi\,T_{ab} - R_{ab}) + \frac{1}{R} g_{ab}\,(\partial_t R)$$

$$\Rightarrow \partial_t g_{ab} = -\frac{2}{R}[\,\partial_t (8\,\pi\,T_{ab} - R_{ab})\,] - \frac{1}{R} g_{ab}\,(\partial_t R)$$

$$\Rightarrow \partial_t g_{ab} = -\frac{16\,\pi}{R}\,\partial_t(T_{ab}) + \frac{2}{R}\left[\partial_t(R_{ab}) - \frac{1}{2} g_{ab}\,(\partial_t R)\right]$$

In general for any metric & T_{ab} satisying EFE' *s*

$$(\partial_t g_{ab})\, R = 2\left[\partial_t R_{ab} - \frac{1}{2} g_{ab}\,(\partial_t R)\right] - 16\pi\, \partial_t T_{ab}$$

$$R = g^{\mu\nu} R_{\mu\nu} \Rightarrow \partial_t R = \partial_t\left(g^{\mu\nu} R_{\mu\nu}\right) = \left(\partial_t g^{\mu\nu}\right) R_{\mu\nu} + g^{\mu\nu}(\partial_t R_{\mu\nu})$$

$$\Rightarrow (\partial_t g_{ab})\, R = 2\left[\partial_t R_{ab} - \left\{\frac{1}{2} g_{ab}\left(\partial_t g^{ab}\right) R_{ab} + \frac{1}{2} g_{ab}\, g^{ab}(\partial_t R_{ab})\right\}\right] - 16\pi\, \partial_t T_{ab} \qquad g_{ab}\, g^{ab} = n$$

$$\Rightarrow (\partial_t g_{ab})\, R = 2\left[\partial_t R_{ab} - \left\{\frac{1}{2} g_{ab}\left(\partial_t g^{ab}\right) R_{ab} + \frac{n}{2}(\partial_t R_{ab})\right\}\right] - 16\pi\, \partial_t T_{ab}$$

$$\Rightarrow (\partial_t g_{ab})\, R = 2\left[\left(1 - \frac{n}{2}\right)\partial_t R_{ab} - \frac{1}{2} g_{ab}\left(\partial_t g^{ab}\right) R_{ab}\right] - 16\pi\, \partial_t T_{ab}$$

$$\Rightarrow (\partial_t g_{ab})\, R = (2 - n)\,\partial_t R_{ab} - g_{ab}\left(\partial_t g^{ab}\right) R_{ab} - 16\pi\, \partial_t T_{ab}$$

$$\Rightarrow (\partial_t g_{ab})\, R + g_{ab}\left(\partial_t g^{ab}\right) R_{ab} = (2 - n)\,\partial_t R_{ab} - 16\pi\, \partial_t T_{ab}$$

$$\Rightarrow (\partial_t g_{ab})\, g^{ab} R_{ab} + g_{ab}\left(\partial_t g^{ab}\right) R_{ab} = (2 - n)\,\partial_t R_{ab} - 16\pi\, \partial_t T_{ab}$$

$$\Rightarrow \left[(\partial_t g_{ab})\, g^{ab} + g_{ab}\left(\partial_t g^{ab}\right)\right] R_{ab} = \left[\partial_t\left(g_{ab}\, g^{ab}\right)\right] R_{ab} = [\partial_t(n)]\, R_{ab} = (2 - n)\,\partial_t R_{ab} - 16\pi\, \partial_t T_{ab}$$

$$\Rightarrow 0 = (2 - n)\,\partial_t R_{ab} - 16\pi\, \partial_t T_{ab}$$

$$\Rightarrow (2 - n)\,\partial_t R_{ab} = 16\pi\, \partial_t T_{ab}$$

$$\Rightarrow \partial_t R_{ab} = \frac{16\pi}{(2 - n)}\,\partial_t T_{ab}$$

This is precisely the solution we obtained using normal coordinates,
but here we didn' *t* have to use normal coordinates or other such contraints!

$$(\partial_t g_{ab})\, R = 2\left[\partial_t R_{ab} - \frac{1}{2} g_{ab}\,(\partial_t R)\right] - 16\pi\, \partial_t T_{ab}$$

$$R = g^{\mu\nu} R_{\mu\nu} \Rightarrow \partial_t R = \partial_t\left(g^{\mu\nu} R_{\mu\nu}\right) = \left(\partial_t g^{\mu\nu}\right) R_{\mu\nu} + g^{\mu\nu}(\partial_t R_{\mu\nu})$$

$$(\partial_t g_{ab})\, R = 2\left[\partial_t R_{ab} - \frac{1}{2} g_{ab}\left\{\left(\partial_t g^{\mu\nu}\right) R_{\mu\nu} + g^{\mu\nu}(\partial_t R_{\mu\nu})\right\}\right] - 16\pi\, \partial_t T_{ab}$$

$$(\partial_t g_{ab})\, R = 2\,\partial_t R_{ab} - g_{ab}\left(\partial_t g^{\mu\nu}\right) R_{\mu\nu} + g_{ab}\, g^{\mu\nu}(\partial_t R_{\mu\nu}) - 16\pi\, \partial_t T_{ab}$$

$$(\partial_t g_{ab})\, R + g_{ab}\left(\partial_t g^{\mu\nu}\right) R_{\mu\nu} = 2\,\partial_t R_{ab} + g_{ab}\, g^{\mu\nu}(\partial_t R_{\mu\nu}) - 16\pi\, \partial_t T_{ab}$$

$$(\partial_t g_{ab})\, g^{\mu\nu} R_{\mu\nu} + g_{ab}\left(\partial_t g^{\mu\nu}\right) R_{\mu\nu} = 2\,\partial_t R_{ab} + g_{ab}\, g^{\mu\nu}(\partial_t R_{\mu\nu}) - 16\pi\, \partial_t T_{ab}$$

Normalize $(\partial_t g_{ab} = 0)$

$$(\partial_t g_{ab})\,R = 2\left[\partial_t R_{ab} - \frac{1}{2} g_{ab}\,(\partial_t R)\right] - 16\pi\,\partial_t T_{ab}$$

$$(\partial_t g_{ab})\,R = 0 = 2\left[\partial_t R_{ab} - \frac{1}{2} g_{ab}\,(\partial_t R)\right] - 16\pi\,\partial_t T_{ab}$$

$$\Rightarrow\; 8\pi\,\partial_t T_{ab} = \partial_t R_{ab} - \frac{1}{2} g_{ab}\,(\partial_t R) = \left(1 - \frac{n}{2}\right)\partial_t R_{ab}$$

Time dependent Schwarzchild Metric with time dependent spatial coods

```
coord = {r[t], θ[t], ϕ[t], t};
metric = {{(1 - 2 m / r) ^ (-1), 0, 0, 0}, {0, r^2, 0, 0}, {0, 0, r^2 Sin[θ]^2, 0},
    {0, 0, 0, -(1 - 2 m / r)}} /. r → r[t] /. θ → θ[t] /. ϕ → ϕ[t];
inversemetric = Simplify[Inverse[metric]];
```

```
TimedependentSchFlowSolution = ((scalar) (D[metric, t])) -
    2 (D[ricci, t] - (1/2 (metric) (D[scalar, t]))) // FullSimplify;
```

```
TimedependentSchFlowSolution /. θ′[t] → 0 /. θ′′[t] → 0 // MatrixForm
TimedependentSchFlowSolution /. θ′[t] → 0 /. θ′′[t] → 0 /. r′′[t] → 0 // MatrixForm
TimedependentSchFlowSolution /. θ′[t] → 0 /. θ′′[t] → 0 /. r′[t] → 0 /. r′′[t] → 0 // MatrixForm
```

$$\left(\begin{array}{ll} \frac{2\left(2\,(5m-r[t])\,r'[t]^3+12m\,(2m-r[t])\,r'[t]\,r''[t]+(2m-r[t])^2\,r[t]\,\left(2\,r^{(3)}[t]+\mathrm{Cot}[\theta[t]]\,r[t]\,\theta^{(3)}[t]\right)\right)}{(-2m+r[t])^4} & \\ 0 & -\frac{2\left(8m^2\,(m+r[t])\,r'[t]^3+(2\mathrm{r}\right.}{} \\ 0 & \\ 0 & \end{array}\right.$$

$$\left(\begin{array}{ll} \frac{2\left(2\,(5m-r[t])\,r'[t]^3+(2m-r[t])^2\,r[t]\,\left(2\,r^{(3)}[t]+\mathrm{Cot}[\theta[t]]\,r[t]\,\theta^{(3)}[t]\right)\right)}{(-2m+r[t])^4} & 0 \\ 0 & -\frac{2\left(8m^2\,(m+r[t])\,r'[t]^3+r[t]^2\,(-2m+r[t])^2\,\left((3m-r[t])\right.\right.}{(-2m+r[t]} \\ 0 & 0 \\ 0 & 0 \end{array}\right.$$

$$\left(\begin{array}{lll} \frac{2\,(2m-r[t])^2\,r[t]\,\left(2\,r^{(3)}[t]+\mathrm{Cot}[\theta[t]]\,r[t]\,\theta^{(3)}[t]\right)}{(-2m+r[t])^4} & 0 & \\ 0 & -\frac{2\,r[t]^2\,\left((3m-r[t])\,r^{(3)}[t]+\mathrm{Cot}[\theta[t]]\,(2m-r[t])\,r[t]\,\theta^{(3)}[t]\right)}{(-2m+r[t])^2} & \\ 0 & 0 & \frac{2\,r[t]^2\,(-3}{} \\ 0 & 0 & \end{array}\right.$$

```
TimedependentSchFlowSolution /. θ′[t] → 0 /. θ′′[t] → 0 /. θ^(3)[t] → 0 /. r′[t] → 0 /.
   r′′[t] → 0 /. r^(3)[t] → 0 // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
einstein // MatrixForm
einstein /. θ′[t] → 0 // MatrixForm
einstein /. θ′[t] → 0 /. θ′′[t] → 0 // MatrixForm
einstein /. θ′[t] → 0 /. θ′′[t] → 0 /. r′′[t] → 0 // MatrixForm
einstein /. θ′[t] → 0 /. θ′′[t] → 0 /. r′[t] → 0 /. r′′[t] → 0 // MatrixForm
einstein /. r′[t] → 0 /. θ′[t] → 0 // MatrixForm
```

$$\left(\begin{array}{ll}
\frac{(-4m+r[t])\, r'[t]^2-\mathrm{Cot}[\theta[t]]\,(7m-3r[t])\, r[t]\, r'[t]\, \theta'[t]+r[t]\,(-2m+r[t])\,\left(2r''[t]+r[t]\,\left(-\theta'[t]^2+\mathrm{Cot}[\theta[t]]\,\theta''[t]\right)\right)}{(2m-r[t])^3} & \\
0 & -\frac{r[t]\,(4m^2}{} \\
0 & \\
\frac{\mathrm{Cot}[\theta[t]]\,(3m-r[t])\,\theta'[t]}{r[t]\,(-2m+r[t])} &
\end{array}\right.$$

$$\left(\begin{array}{ll}
\frac{(-4m+r[t])\, r'[t]^2+r[t]\,(-2m+r[t])\,(2r''[t]+\mathrm{Cot}[\theta[t]]\, r[t]\,\theta''[t])}{(2m-r[t])^3} & 0 \\
0 & -\frac{r[t]\,\left(4m^2 r'[t]^2+r[t]\,(-2m+r[t])\,\left(-3m\, r''[t]+\mathrm{Cot}[\theta[t]]\, r[t\right.\right.}{(-2m+r[t])^3} \\
0 & 0 \\
0 & 0
\end{array}\right.$$

$$\left(\begin{array}{lll}
\frac{(-4m+r[t])\, r'[t]^2+2r[t]\,(-2m+r[t])\, r''[t]}{(2m-r[t])^3} & 0 & \\
0 & -\frac{r[t]\,\left(4m^2 r'[t]^2+r[t]\,(-2m+r[t])\,(-3m\, r''[t]+r[t]\, r''[t])\right)}{(-2m+r[t])^3} & \\
0 & 0 & -\frac{r[t]\,\mathrm{Sin}[\theta[t]]^2\,(4m^2}{} \\
0 & 0 &
\end{array}\right.$$

$$\begin{pmatrix}
\frac{(-4m+r[t])\, r'[t]^2}{(2m-r[t])^3} & 0 & 0 & 0 \\
0 & -\frac{4m^2 r[t]\, r'[t]^2}{(-2m+r[t])^3} & 0 & 0 \\
0 & 0 & -\frac{4m^2 r[t]\,\mathrm{Sin}[\theta[t]]^2\, r'[t]^2}{(-2m+r[t])^3} & 0 \\
0 & 0 & 0 & \frac{(-4m+r[t])\, r'[t]^2}{r[t]^2\,(-2m+r[t])}
\end{pmatrix}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

$$\left(\begin{array}{lll}
\frac{r[t]\,(-2m+r[t])\,(2r''[t]+\mathrm{Cot}[\theta[t]]\, r[t]\,\theta''[t])}{(2m-r[t])^3} & 0 & \\
0 & -\frac{r[t]^2\left(-3m\, r''[t]+\mathrm{Cot}[\theta[t]]\, r[t]^2\,\theta''[t]+r[t]\,(r''[t]-2m\,\mathrm{Cot}[\theta[t]]\,\theta''[t])\right)}{(-2m+r[t])^2} & \\
0 & 0 & - \\
0 & 0 &
\end{array}\right.$$

So for this metric only need to let $\theta'[t] \to 0$ to get the Einstein tensor to be diagonal
Also, simply letting velocities go to 0 doesnt make the Einstein tensor go to 0

General Diag Metric with time dependent spatial coordinates

```
(* Table[f_ij[r[t],θ[t],ϕ[t],t],{i,1,n},{j,1,4}]//MatrixForm *)
(*To index it like this need to do two separate ctrl _ 's *)
```

Of course, this is just the most general such matrix,
an actual metric has some constraints such as
Symmetry (in certain coods, which we can always choose)
Can typically diagonalize, if not always
Apply these to reduce compuatation time

```
coord = {r[t], θ[t], ϕ[t], t};
metric = Table[If[i == j, f_ij[r[t], θ[t], ϕ[t], t], 0], {i, 1, n}, {j, 1, 4}];
(*This makes the expression above diagonal *)
inversemetric = Simplify[Inverse[metric]];
```

```
metric // MatrixForm
```

$$\begin{pmatrix} f_{11}[r[t], \theta[t], \phi[t], t] & 0 & 0 & 0 \\ 0 & f_{22}[r[t], \theta[t], \phi[t], t] & 0 & 0 \\ 0 & 0 & f_{33}[r[t], \theta[t], \phi[t], t] & 0 \\ 0 & 0 & 0 & f_{44}[r[t], \theta[t], \end{pmatrix}$$

```
TimedependentGeneralFlowSolution =
  ((scalar) (D[metric, t])) - 2 (D[ricci, t] - (1/2 (metric) (D[scalar, t])));
```

```
TimedependentGeneralFlowSolution /. θ′[t] → 0 /. θ′′[t] → 0 // MatrixForm
TimedependentGeneralFlowSolution /. θ′[t] → 0 /. θ′′[t] → 0 /. r′′[t] → 0 // MatrixForm
TimedependentGeneralFlowSolution /. θ′[t] → 0 /. θ′′[t] → 0 /. r′[t] → 0 /. r′′[t] → 0 //
 MatrixForm
```

$\left(\frac{\left(f_{11}{}^{(0,0,0,1)}[r[t],\theta[t],\phi[t],t]+\phi'[t]\, f_{11}{}^{(0,0,1,0)}[r[t],\theta[t],\phi[t],t]+r'[t]\, f_{11}{}^{(1,0,0,0)}[r[t],\theta[t],\phi[t],t]\right)\left(f_{11}[r[t],\theta[t],\phi[t],t]\, f_{33}\right.}{}\right.$

$\left(\frac{\left(f_{11}{}^{(0,0,0,1)}[r[t],\theta[t],\phi[t],t]+\phi'[t]\, f_{11}{}^{(0,0,1,0)}[r[t],\theta[t],\phi[t],t]+r'[t]\, f_{11}{}^{(1,0,0,0)}[r[t],\theta[t],\phi[t],t]\right)\left(f_{11}[r[t],\theta[t],\phi[t],t]\, f_{33}\right.}{}\right.$

(⋯ 1 ⋯)

large output | show less | show more | show all | set size limit...

```
TimedependentGeneralFlowSolution /. θ′[t] → 0 /. θ′′[t] → 0 /. θ^(3)[t] → 0 /. ϕ′[t] → 0 /.
    ϕ′′[t] → 0 /. ϕ^(3)[t] → 0 /. r′[t] → 0 /. r′′[t] → 0 /. r^(3)[t] → 0 // MatrixForm
```

$\left(\frac{f_{11}{}^{(0,0,0,1)}[r[t],\theta[t],\phi[t],t]\left(f_{11}[r[t],\theta[t],\phi[t],t]\, f_{33}[r[t],\theta[t],\phi[t],t]\, f_{44}[r[t],\theta[t],\phi[t],t]\left(f_{11}[r[t],\theta[t],\phi[t],t]\left(f_{44}[r[\right.\right.\right.}{}\right.$

```
2 * f44[r[t], θ[t], ϕ[t], t] /.
 f44[r[t], θ[t], ϕ[t], t] → C (* So can do this with an entire functional *)
```

2 C

```
f11^(0,1,0,0)[r[t], θ[t], ϕ[t], t]
D[f11[r[t], θ[t], ϕ[t], t], θ[t]]
```

$f_{11}{}^{(0,1,0,0)}[r[t], \theta[t], \phi[t], t]$

$f_{11}{}^{(0,1,0,0)}[r[t], \theta[t], \phi[t], t]$

```
Series[f11[r[t], θ[t], ϕ[t], t], {r[t], 0, 2}, {θ[t], 0, 2}, {ϕ[t], 0, 2}, {t, 0, 2}] //
  FullSimplify; (*Quadratic expansion in all variables*)
```

```
TimedependentGeneralFlowSolution /. θ'[t] → 0 /. θ''[t] → 0 /. θ^(3)[t] → 0 /. ϕ'[t] → 0 /.
       ϕ''[t] → 0 /. ϕ^(3)[t] → 0 /. r'[t] → 0 /. r''[t] → 0 /. r^(3)[t] → 0 /.
  f44[r[t], θ[t], ϕ[t], t] → -1 // MatrixForm(* letting time comp = -1 *)
```

$\left(\frac{f_{11}{}^{(0,0,0,1)}[r[t],\theta[t],\phi[t],t] \left(-f_{11}[r[t],\theta[t],\phi[t],t]\, f_{33}[r[t],\theta[t],\phi[t],t] \left(f_{11}[r[t],\theta[t],\phi[t],t] \left(-f_{22}{}^{(0,0,1,0)}[r[t],\theta[t],\phi[t]\right.\right.\right.}{} \right.$

```
einstein // MatrixForm
einstein /. θ'[t] → 0 // MatrixForm
einstein /. θ'[t] → 0 /. θ''[t] → 0 // MatrixForm
einstein /. θ'[t] → 0 /. θ''[t] → 0 /. r''[t] → 0 // MatrixForm
einstein /. θ'[t] → 0 /. θ''[t] → 0 /. r'[t] → 0 /. r''[t] → 0 // MatrixForm
einstein /. r'[t] → 0 /. θ'[t] → 0 // MatrixForm
```

$\left(\frac{f_{22}[r[t],\theta[t],\phi[t],t]\, f_{33}[r[t],\theta[t],\phi[t],t]\, f_{44}[r[t],\theta[t],\phi[t],t] \left(f_{44}[r[t],\theta[t],\phi[t],t]\, f_{22}{}^{(1,0,0,0)}[r[t],\theta[t],\phi[t],t]\, f_{33}{}^{(1,0,0,}\right.}{} \right.$

(⋯ 1 ⋯)

large output | show less | show more | show all | set size limit...

(⋯ 1 ⋯)

large output | show less | show more | show all | set size limit...

(⋯ 1 ⋯)

large output | show less | show more | show all | set size limit...

$$-f_{11}[r[t],\theta[t],\phi[t],t]\ \big(f_{22}[r[t],\theta[t],\phi[t],t]\ f_{44}[r[t],\theta[t],\phi[t],t]\ \big(f_{22}[r[t],\theta[t],\phi[t],t]\ \big(f_{33}{}^{(0,0,0,1)}[r[t],\theta[t],\phi[t],t]^2+2\,\phi'$$

$$-f_{11}[r[t],\theta[t],\phi[t],t]\ \big(f_{22}[r[t],\theta[t],\phi[t],t]\ f_{44}[r[t],\theta[t],\phi[t],t]\ \big(f_{22}[r[t],\theta[t],\phi[t],t]\ \big(f_{33}{}^{(0,0,0,1)}[r[t],\theta[t],\phi[t],t]^2+2\,\phi'$$

Key Question :
Can all metrics with Time – dependent spatial coordinates be made isomorphic to some spacetime ? Or some finite set of spacetimes ?
i.e. For a metric with coordinates r[t], θ[t], ϕ[t], t,
can we always perform a change of coods so that it looks like a minkowski spacetime,
as we can do for a metric with time – independent coordinates ?
Note that the tensors of GR all involve partial derivatives –
i.e. coods assumed to be independent of aone another. Can we
reparametrize in terms of only time and alter these tensors ?
This would change the formalism of GR, but how so ?

Finding Solutions

$$\partial_t g_{ab} = -\frac{16\pi}{R}\partial_t(T_{ab}) + \frac{2}{R}\left[\partial_t(R_{ab}) - \frac{1}{2}g_{ab}(\partial_t R)\right]$$

In general for any metric & T_{ab} satisying EFE' s

$$(\partial_t g_{ab})R - 2\left[\partial_t(R_{ab}) - \frac{1}{2}g_{ab}(\partial_t R)\right] = -16\pi\,\partial_t(T_{ab})$$

Keep Ricci Scalar on top so we dont have singularities in case $R = 0$

For a perfect fluid $T_{ab} = (\rho + p)u_a u_b + p\,g_{ab}$ where $u^a u_a = 1$ and there' s no flux, so the above becomes

$$R(\partial_t g_{ab}) - 2\left[\partial_t(R_{ab}) - \frac{1}{2}g_{ab}(\partial_t R)\right] =$$

$$-16\pi\,\partial_t[(\rho + p)u_a u_b + p\,g_{ab}] = -16\pi(\rho + p)[\partial_t(u_a u_b)] - 16\pi\,p\,\partial_t g_{ab}$$

$$\Rightarrow (R + 16\pi p)\partial_t g_{ab} - 2\left[\partial_t(R_{ab}) - \frac{1}{2}g_{ab}(\partial_t R)\right] = -16\pi(\rho + p)[\partial_t(u_a u_b)]$$

$$\Rightarrow (\texttt{scalar} * \texttt{D[metric, t]}) - 2\left(\texttt{D[ricci, t]} - \left(\frac{1}{2}(\texttt{metric})(\texttt{D[scalar, t]})\right)\right) = -16\pi D[T_{ab}, t]$$