$$\frac{1}{\pi}\,\mathrm{Sum}\left[\frac{\mathrm{Sin}[2\,\pi\,k\,x]}{k},\ \{k,\ 1,\ \infty\}\right]$$

$$\frac{i\,\left(\mathrm{Log}\left[1-e^{2\,i\,\pi\,x}\right]-\mathrm{Log}\left[e^{-2\,i\,\pi\,x}\left(-1+e^{2\,i\,\pi\,x}\right)\right]\right)}{2\,\pi}$$

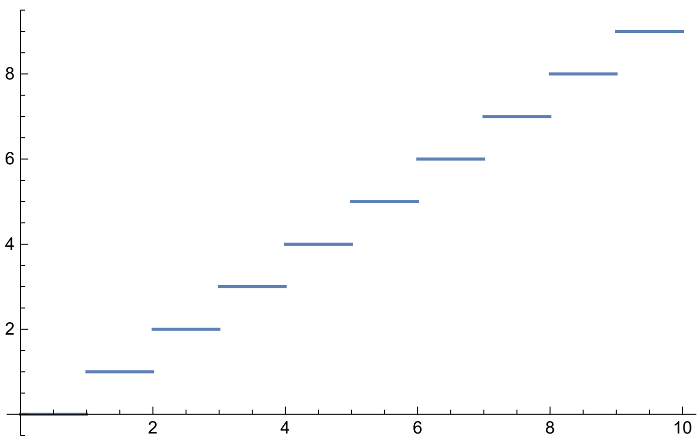
Note that $\mathrm{Floor}[x] = \left(x-\frac{1}{2}\right)+\frac{1}{\pi}\,\mathrm{Sum}\left[\frac{\mathrm{Sin}[2\,\pi\,k\,x]}{k},\ \{k,\ 1,\ \infty\}\right]$

from defn of sawtooth fn

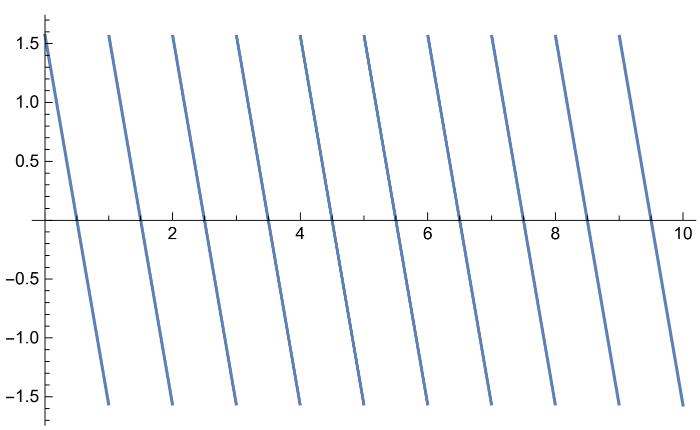
$$\left(x-\frac{1}{2}\right)+\frac{1}{\pi}\,\mathrm{Sum}\left[\frac{\mathrm{Sin}[2\,\pi\,k\,x]}{k},\ \{k,\ 1,\ \infty\}\right]\ //\ \mathrm{FullSimplify}$$

$$-\frac{\pi-2\,\pi\,x+i\,\mathrm{Log}\left[1-e^{-2\,i\,\pi\,x}\right]-i\,\mathrm{Log}\left[1-e^{2\,i\,\pi\,x}\right]}{2\,\pi}$$

$$\mathrm{Plot}\left[\left(x-\frac{1}{2}\right)+\frac{1}{\pi}\,\mathrm{Sum}\left[\frac{\mathrm{Sin}[2\,\pi\,k\,x]}{k},\ \{k,\ 1,\ \infty\}\right],\ \{x,\ 0,\ 10\}\right]$$



$$\mathrm{Plot}\left[\mathrm{Sum}\left[\frac{\mathrm{Sin}[2\,\pi\,k\,x]}{k},\ \{k,\ 1,\ \infty\}\right],\ \{x,\ 0,\ 10\}\right]$$



$$\frac{i\,\left(\mathrm{Log}\left[1-e^{2\,i\,\pi\,x}\right]-\mathrm{Log}\left[e^{-2\,i\,\pi\,x}\left(-1+e^{2\,i\,\pi\,x}\right)\right]\right)}{2\,\pi} = \frac{i}{2\,\pi}\left(\mathrm{Log}\left[1-e^{2\,i\,\pi\,x}\right]-\mathrm{Log}\left[\left(-e^{-2\,i\,\pi\,x}+1\right)\right]\right)$$

$$= \frac{i}{2\pi}\left(\text{Log}\left[1-e^{2 i \pi x}\right]-\text{Log}\left[1-e^{-2 i \pi x}\right]\right) = = \frac{i}{2\pi}\text{Log}\left[\frac{1-e^{2 i \pi x}}{1-e^{-2 i \pi x}}\right]$$

$\frac{1}{\pi}$ Sum[$\frac{\text{Cos}[2\pi k x]}{k}$, {k, 1, ∞}]

$$\frac{-\text{Log}\left[1-e^{2 i \pi x}\right]-\text{Log}\left[e^{-2 i \pi x}\left(-1+e^{2 i \pi x}\right)\right]}{2\pi}$$

$\frac{1}{\pi}$ Sum[$\frac{\text{Exp}[2\pi k y]\,\text{Cos}[2\pi k x]}{k}$, {k, 1, ∞}]

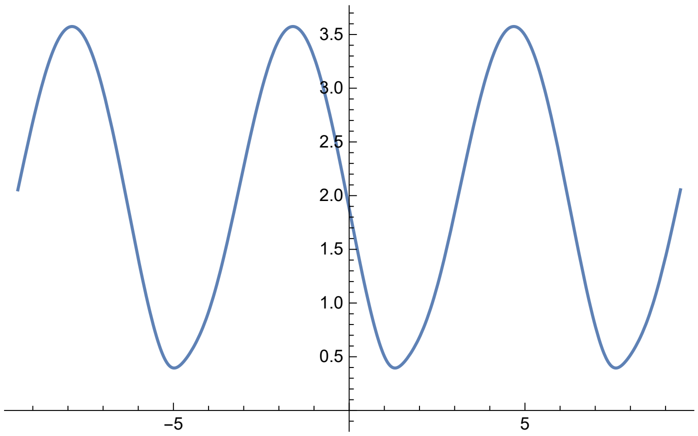
$$\frac{-\text{Log}\left[1-e^{2\pi(-i x+y)}\right]-\text{Log}\left[1-e^{2\pi(i x+y)}\right]}{2\pi}$$

Series[LogIntegral[x], {x, 0, 2}] // FullSimplify

$$\begin{cases}
\frac{1}{\text{Log}[x]^6}\left(120+\text{Log}[x]\left(24+\text{Log}[x]\left(6+\text{Log}[x]\left(2+\text{Log}[x]+\text{Log}[x]^2\right)\right)\right)\right)x+O[x]^3 \\
\frac{1}{(-i\pi+\text{Log}[x])^6}\left(120-6\left(\pi+i\,\text{Log}[x]\right)^2+\left(\pi+i\,\text{Log}[x]\right)^4+24\left(-i\pi+\text{Log}[x]\right)+2\left(-i\pi+\text{Log}[x]\right)^3+\right. \\
-i\pi+\frac{1}{\text{Log}[x]^6}\left(120+\text{Log}[x]\left(24+\text{Log}[x]\left(6+\text{Log}[x]\left(2+\text{Log}[x]+\text{Log}[x]^2\right)\right)\right)\right)x+O[x]^3 \\
i\pi+\frac{1}{\text{Log}[x]^6}\left(120+\text{Log}[x]\left(24+\text{Log}[x]\left(6+\text{Log}[x]\left(2+\text{Log}[x]+\text{Log}[x]^2\right)\right)\right)\right)x+O[x]^3 \\
-i\pi+\frac{1}{(-i\pi+\text{Log}[x])^6}\left(120-6\left(\pi+i\,\text{Log}[x]\right)^2+\left(\pi+i\,\text{Log}[x]\right)^4+24\left(-i\pi+\text{Log}[x]\right)+2\left(-i\pi+\text{Log}[\right.\right. \\
i\pi+\frac{1}{(-i\pi+\text{Log}[x])^6}\left(120-6\left(\pi+i\,\text{Log}[x]\right)^2+\left(\pi+i\,\text{Log}[x]\right)^4+24\left(-i\pi+\text{Log}[x]\right)+2\left(-i\pi+\text{Log}[x]\right.\right.
\end{cases}$$

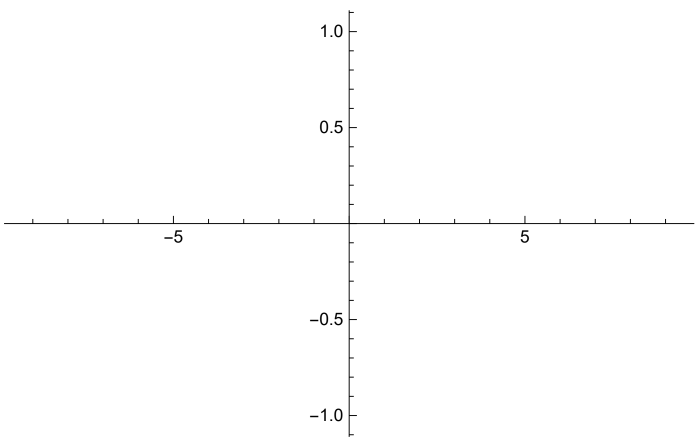
```
FourierSeries[Log[x], x, 1]
Plot[Abs[%], {x, -3 Pi, 3 Pi}]
```

$$-1 + \frac{i\pi}{2} + \text{Log}[\pi] - \frac{e^{ix}\left(\pi + \text{SinIntegral}[\pi]\right)}{\pi} + e^{-ix}\left(1 - \frac{\text{SinIntegral}[\pi]}{\pi}\right)$$



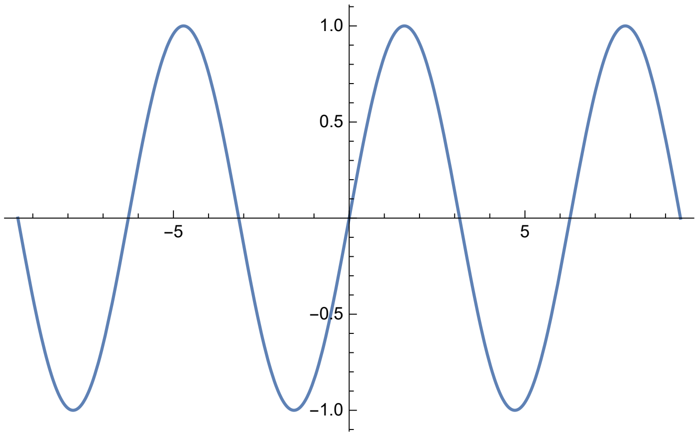
```
FourierSeries[1/(x - 1) + x, x, 2]
Plot[Abs[%], {x, -3 Pi, 3 Pi}]
```

$$i\, e^{-ix} - i\, e^{ix} - \frac{1}{2}\, i\, e^{-2ix} + \frac{1}{2}\, i\, e^{2ix} + \text{FourierSeries}\left[\frac{1}{-1+x}, x, 2\right]$$



```
FourierSeries[t/2, t, 1]
Plot[%, {t, -3 Pi, 3 Pi}]
```

$$\frac{1}{2} i e^{-i t} - \frac{1}{2} i e^{i t}$$




```
Series[LogIntegral[x], {x, 1, 5}, Assumptions -> x > 1]
```

$$(\text{EulerGamma} + \text{Log}[-1+x]) + \frac{x-1}{2} - \frac{1}{24}(x-1)^2 + \frac{1}{72}(x-1)^3 - \frac{19(x-1)^4}{2880} + \frac{3}{800}(x-1)^5 + O[x-1]^6$$

$$(\text{EulerGamma} + \text{Log}[-1+x]) + \frac{x-1}{2} - \frac{1}{24}(x-1)^2 + \frac{1}{72}(x-1)^3 - \frac{19(x-1)^4}{2880} + \frac{3}{800}(x-1)^5 \text{ // FullSimplify}$$

$$\frac{1}{14400}(14400\,\text{EulerGamma} + (-1+x)(8149 + x(-1501 + x(809 + x(-311 + 54x))))) + \text{Log}[-1+x]$$

$$(\text{EulerGamma} + \text{Log}[-1+x]) + \frac{x-1}{2} - \frac{1}{24}(x-1)^2 - \left(\frac{1}{2}\left((\text{EulerGamma} + \text{Log}[-1+x]) + \frac{x-1}{2} - \frac{1}{24}(x-1)^2\right)\right) \text{ // FullSimplify}$$

$$\frac{1}{48}(24\,\text{EulerGamma} - (-13+x)(-1+x) + 24\,\text{Log}[-1+x])$$

```
FindRoot[LogIntegral[x] == 0, {x, 2}]
```

{x → 1.45137}

```
Series[LogIntegral[1.451369234883381` x], {x, 1, 8}, Assumptions -> x > 1]
```

$$1.56853\times10^{-16} + 3.89622(x-1) - 5.22972(x-1)^2 + 11.1027(x-1)^3 - 26.7355(x-1)^4 + 68.7538(x-1)^5 - 184.219(x-1)^6 + 507.722(x-1)^7 - 1428.49(x-1)^8 + O[x-1]^9$$

```
1428.493422745596` ^(1/8)
```

2.47948

```
FnseriesofLI[multconstant_, power_] :=
 Normal[Series[LogIntegral[(multconstant) x], {x, 1, (power)}, Assumptions -> x > 1]]
FnCoefficients[multconstant_, power_] :=
 Table[SeriesCoefficient[FnseriesofLI[multconstant, power], {x, 1, i}], {i, 1, power}]
```

```
FnseriesofLI[1, 12]
```

$$\text{EulerGamma} + \frac{1}{2}(-1+x) - \frac{1}{24}(-1+x)^2 + \frac{1}{72}(-1+x)^3 - \frac{19(-1+x)^4}{2880} + \frac{3}{800}(-1+x)^5 - \frac{863(-1+x)^6}{362\,880} + \frac{275(-1+x)^7}{169\,344} - \frac{33\,953(-1+x)^8}{29\,030\,400} + \frac{8183(-1+x)^9}{9\,331\,200} - \frac{3\,250\,433(-1+x)^{10}}{4\,790\,016\,000} + \frac{4671(-1+x)^{11}}{8\,673\,280} - \frac{13\,695\,779\,093(-1+x)^{12}}{31\,384\,184\,832\,000} + \text{Log}[-1+x]$$

```
FnseriesofLI[1, 12] // N
FnseriesofLI[2, 12] // N
FnseriesofLI[3, 12] // N
```

$$0.577216 + 0.5(-1.+x) - 0.0416667(-1.+x)^2 + 0.0138889(-1.+x)^3 - 0.00659722(-1.+x)^4 + 0.00375(-1.+x)^5 - 0.0023782(-1.+x)^6 + 0.00162391(-1.+x)^7 - 0.00116957(-1.+x)^8 + 0.00087695(-1.+x)^9 - 0.000678585(-1.+x)^{10} + 0.000538551(-1.+x)^{11} - 0.000436391(-1.+x)^{12} + \text{Log}[-1.+x]$$

$$1.04516 + 2.88539(-1.+x) - 2.08137(-1.+x)^2 + 2.69564(-1.+x)^3 - 4.01433(-1.+x)^4 + 6.40837(-1.+x)^5 - 10.6721(-1.+x)^6 + 18.2895(-1.+x)^7 - 32.0027(-1.+x)^8 + 56.891(-1.+x)^9 - 102.402(-1.+x)^{10} + 186.183(-1.+x)^{11} - 341.334(-1.+x)^{12}$$

$$2.16359 + 2.73072(-1.+x) - 1.2428(-1.+x)^2 + 1.16843(-1.+x)^3 - 1.28761(-1.+x)^4 + 1.53181(-1.+x)^5 - 1.90699(-1.+x)^6 + 2.44683(-1.+x)^7 - 3.208(-1.+x)^8 + 4.27483(-1.+x)^9 - 5.76912(-1.+x)^{10} + 7.86551(-1.+x)^{11} - 10.8139(-1.+x)^{12}$$

```
FactorInteger[33953]
```

{{19, 1}, {1787, 1}}

```
FnCoefficients[1, 20]
FnCoefficients[1, 20] // N
```

$$\left\{\frac{1}{2}, -\frac{1}{24}, \frac{1}{72}, -\frac{19}{2880}, \frac{3}{800}, -\frac{863}{362\,880}, \frac{275}{169\,344}, -\frac{33\,953}{29\,030\,400}, \frac{8183}{9\,331\,200}, -\frac{3\,250\,433}{4\,790\,016\,000}, \frac{4671}{8\,673\,280}, -\frac{13\,695\,779\,093}{31\,384\,184\,832\,000}, \frac{2\,224\,234\,463}{6\,181\,733\,376\,000}, -\frac{132\,282\,840\,127}{439\,378\,587\,648\,000}, \frac{2\,639\,651\,053}{10\,346\,434\,560\,000}, -\frac{111\,956\,703\,448\,001}{512\,189\,896\,458\,240\,000}, \frac{50\,188\,465}{265\,423\,814\,656}, -\frac{2\,334\,028\,946\,344\,463}{14\,148\,260\,909\,088\,768\,000}, \frac{301\,124\,035\,185\,049}{2\,076\,423\,318\,208\,512\,000}, -\frac{12\,365\,722\,323\,469\,980\,029}{96\,342\,919\,523\,794\,944\,000\,000}\right\}$$

{0.5, -0.0416667, 0.0138889, -0.00659722, 0.00375, -0.0023782, 0.00162391, -0.00116957, 0.00087695, -0.000678585, 0.000538551, -0.000436391, 0.000359808, -0.000301068, 0.000255127, -0.000218584, 0.000189088, -0.000164969, 0.000145021, -0.000128351}

```
FactorInteger[Numerator[FnCoefficients[1, 20]]] // MatrixForm
```

```
{{1, 1}}
{{-1, 1}}
{{1, 1}}
{{-1, 1}, {19, 1}}
{{3, 1}}
{{-1, 1}, {863, 1}}
{{5, 2}, {11, 1}}
{{-1, 1}, {19, 1}, {1787, 1}}
{{7, 2}, {167, 1}}
{{-1, 1}, {3 250 433, 1}}
{{3, 3}, {173, 1}}
{{-1, 1}, {541, 1}, {4801, 1}, {5273, 1}}
{{11, 2}, {2207, 1}, {8329, 1}}
{{-1, 1}, {132 282 840 127, 1}}
{{13, 2}, {41, 1}, {380 957, 1}}
{{-1, 1}, {6427, 1}, {17 419 745 363, 1}}
{{5, 1}, {37, 1}, {271 289, 1}}
{{-1, 1}, {23, 1}, {1 606 897, 1}, {63 152 473, 1}}
{{17, 2}, {109 321, 1}, {9 531 121, 1}}
{{-1, 1}, {12 365 722 323 469 980 029, 1}}
```

```
Sum[FnCoefficients[k, 20], {k, 1, 20}]
```

```
NSum[FnCoefficients[1, 20]]
```

```
FnCoefficients[1, 12] // N
FnCoefficients[2, 12] // N
FnCoefficients[3, 12] // N
FnCoefficients[4, 12] // N
```

```
{0.5, -0.0416667, 0.0138889, -0.00659722, 0.00375, -0.0023782,
 0.00162391, -0.00116957, 0.00087695, -0.000678585, 0.000538551, -0.000436391}
```

```
{2.88539, -2.08137, 2.69564, -4.01433, 6.40837,
 -10.6721, 18.2895, -32.0027, 56.891, -102.402, 186.183, -341.334}
```

```
{2.73072, -1.2428, 1.16843, -1.28761, 1.53181,
 -1.90699, 2.44683, -3.208, 4.27483, -5.76912, 7.86551, -10.8139}
```

```
{2.88539, -1.04068, 0.847358, -0.819551, 0.860478,
 -0.948124, 1.07845, -1.25467, 1.48446, -1.77943, 2.1554, -2.63319}
```

```
ListPlot[Table[FnCoefficients[k, 12], {k, 1, 8}]]
```