Explore Expansions of LogIntegral Fn & Many Related Fns/Concepts

```
Series[ⅇ^x, {x, 0, 3}]
```

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O[x]^4$$

```
LogIntegral[2 + x + x^2/2 + O[x]^3]
```

$$\text{LogIntegral}[2] + \frac{x}{\text{Log}[2]} + \left(\frac{1}{4\,\text{Log}[2]} + \frac{-2 + \text{Log}[4]}{8\,\text{Log}[2]^2}\right) x^2 + O[x]^3$$

```
Series[LogIntegral[2 + x + x^2/2], {x, 0, 2}] // FullSimplify
```

$$\text{LogIntegral}[2] + \frac{x}{\text{Log}[2]} + \frac{(-1 + \text{Log}[4])\, x^2}{4\,\text{Log}[2]^2} + O[x]^3$$

```
Series[ⅇ^x, {x, 0, 3}]
```

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O[x]^4$$

```
LogIntegral[1 + x + x^2/2 + x^3/6 + O[x]^4] // FullSimplify
```

$$-i\pi\left(\text{Floor}\left[\frac{\pi + \text{Arg}[x] - \text{Arg}\left[\frac{1}{6+x\,(3+x)}\right]}{2\pi}\right] - \text{Floor}\left[-\frac{-\pi + \text{Arg}[x] + \text{Arg}\left[6 + x\,(3 + x)\right]}{2\pi}\right]\right) + \left((\text{EulerGamma} + \text{Log}[x]) + x + \frac{x^2}{4} + \frac{x^3}{72} + O[x]^4\right)$$

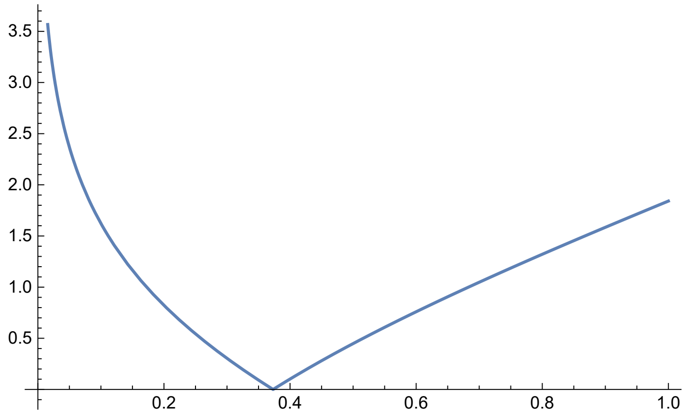
```
F[x_] := -ⅈ π (Floor[(π + Arg[x] - Arg[1/(6+x (3+x))])/(2 π)] - Floor[-(-π + Arg[x] + Arg[6 + x (3 + x)])/(2 π)]) +
   ((EulerGamma + Log[x]) + x + x^2/4 + x^3/72);
```

```
F[x]
```

$$\text{EulerGamma} + x + \frac{x^2}{4} + \frac{x^3}{72} - i\pi\left(\text{Floor}\left[\frac{\pi + \text{Arg}[x] - \text{Arg}\left[\frac{1}{6+x\,(3+x)}\right]}{2\pi}\right] - \text{Floor}\left[-\frac{-\pi + \text{Arg}[x] + \text{Arg}\left[6 + x\,(3 + x)\right]}{2\pi}\right]\right) + \text{Log}[x]$$

```
Abs[F[x]]; (* Use in plots since this has an imag part *)
```

```
Plot[Abs[F[x]], {x, 0, 1}]
```




```
FindRoot[Abs[F[x]] == 0, {x, 0.4}]
```

FindRoot::lstol : The line search decreased the step size to within tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the merit function. You may need more than MachinePrecision digits of working precision to meet these tolerances. >>

```
{x → 0.373118}
```

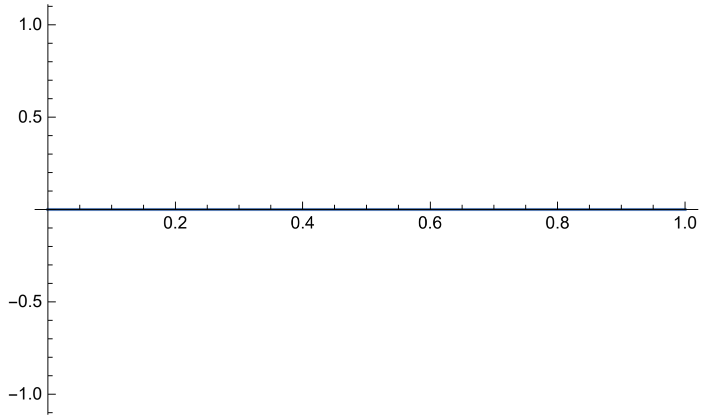
```
B[x_, n_] := Normal[LogIntegral[Evaluate[Series[ⅇ^y, {y, 0, n}]]]] /. y → x
B[x, 10] // FullSimplify
```

$$\text{EulerGamma} + x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} + \frac{x^6}{4320} + \frac{x^7}{35\,280} + \frac{x^8}{322\,560} + \frac{x^9}{3\,265\,920} + \frac{x^{10}}{399\,168\,000} - i\,\pi\left(\text{Floor}\left[\frac{1}{2\pi}\left(\pi + \text{Arg}[x] - \text{Arg}\left[1\big/\left(3\,628\,800 + x\left(1\,814\,400 + x\left(604\,800 + x\left(151\,200 + x\left(30\,240 + x\left(5040 + x\left(720 + x\left(90 + x\left(10 + x\right)\right)\right)\right)\right)\right)\right)\right)\right)\right]\right)\right] - \text{Floor}\left[-\frac{1}{2\pi}\left(-\pi + \text{Arg}[x] + \text{Arg}\left[3\,628\,800 + x\left(1\,814\,400 + x\left(604\,800 + x\left(151\,200 + x\left(30\,240 + x\left(5040 + x\left(720 + x\left(90 + x\left(10 + x\right)\right)\right)\right)\right)\right)\right)\right)\right]\right)\right]\right) + \text{Log}[x]$$

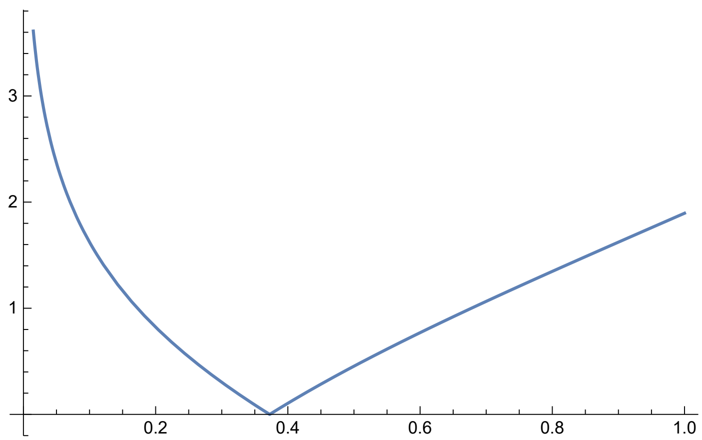
```
B[1, 2] // FullSimplify // N
B[1, 3] // FullSimplify // N
B[1, 5] // FullSimplify // N
B[1, 10] // FullSimplify // N
B[1, 15] // FullSimplify // N
Abs[B[ⅈ, 15]] // FullSimplify // N
```

```
1.66055

1.8411

1.89347

1.89512

1.89512

2.53939
```

```
Plot[{Im[B[x, 10]]}, {x, 0, 1}]
```




```
Plot[Abs[B[x, 10]], {x, 0, 1}]
```




```
FindRoot[Abs[B[x, 15]] == 0, {x, 0.4}, WorkingPrecision → 50]
```

FindRoot::lstol : The line search decreased the step size to within tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the merit function. You may need more than 50.` digits of working precision to meet these tolerances. ≫

```
{x → 0.37250741078129390800455626413917840922263025049649}
```

```
          1
---------------------
(0.37250741078129024`)
N[Exp[0.372507410781293908004556264139178409222630250496490636672528228440704747225], 15]
```

```
2.68451

1.45136923488328
```

This is the Ramanujan–Soldner constant, makes sense since this is an appx of Li

```
Zeta[2]
```

$$\frac{\pi^2}{6}$$

```
Z[x_, n_] := Normal[LogIntegral[Evaluate[Series[Zeta[y], {y, 0, n}]]]] /. y → x
```

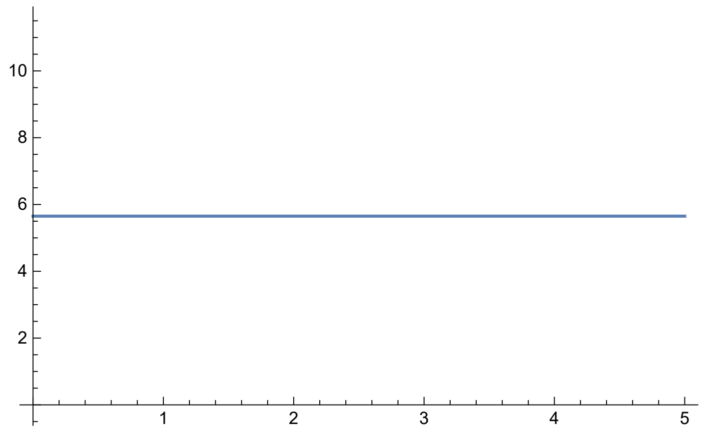
```
Z[x, 4]
```

```
$Aborted
```

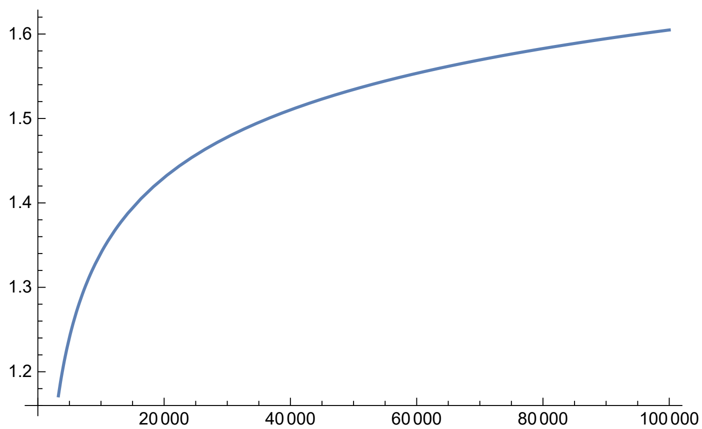
```
Z[2, 4] // N
```

$-2.03454 + 12.8445\, i$

```
Plot[Abs[Z[1, 8]], {x, 0, 5}]
```




```
Plot[LogIntegral[Log[Log[x]]], {x, 0, 100000}]
```




```
Sum[(PrimePi[x])^n, {n, 0, k}]
```

$$\frac{-1 + \text{PrimePi}[x]^{1+k}}{-1 + \text{PrimePi}[x]}$$

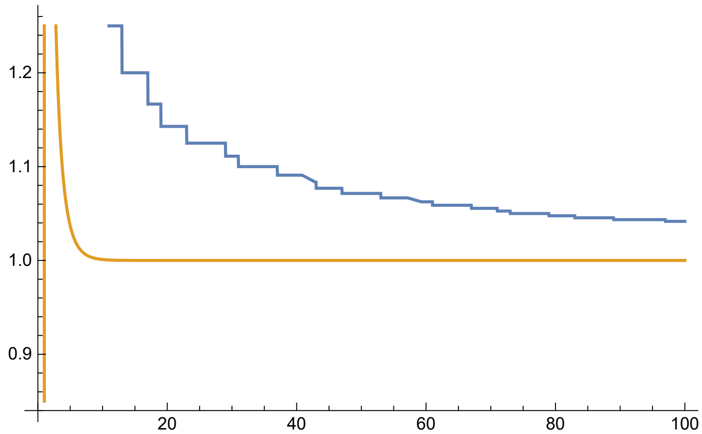
```
Sum[(PrimePi[x])^n, {n, 0, ∞}]
```

$$\frac{1}{1 - \text{PrimePi}[x]}$$

```
Sum[(PrimePi[x])^(-n), {n, 0, ∞}] // FullSimplify
```

$$1 + \frac{1}{-1 + \text{PrimePi}[x]}$$

```
Plot[{1 + 1/(-1 + PrimePi[x]), Zeta[x]}, {x, 0, 100}]
```



Hypothesize that

$1+\frac{1}{-1+\text{PrimePi}[x]}-\text{Zeta}[x]>0$ For all x>1

Sum[(PrimePi[x])^(-n),{n,0,∞}] = $1+\frac{1}{-1+\text{PrimePi}[x]}$

```
Sum[1 + 1/(-1 + PrimePi[x]), {x, 3, 200}] // N
```

212.827

```
SumConvergence[1 + 1/(-1 + (x/Log[x+1])), x]
```

False

```
Series[Log[x + 1], {x, 0, 4}]
```

$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+O[x]^5$

```
Union[Table[1 + 1/(-1 + PrimePi[x]), {x, 3, 200}]]
```

$\left\{\frac{46}{45},\frac{45}{44},\frac{44}{43},\frac{43}{42},\frac{42}{41},\frac{41}{40},\frac{40}{39},\frac{39}{38},\frac{38}{37},\frac{37}{36},\frac{36}{35},\frac{35}{34},\frac{34}{33},\frac{33}{32},\frac{32}{31},\frac{31}{30},\frac{30}{29},\frac{29}{28},\frac{28}{27},\frac{27}{26},\frac{26}{25},\frac{25}{24},\frac{24}{23},\frac{23}{22},\frac{22}{21},\frac{21}{20},\frac{20}{19},\frac{19}{18},\frac{18}{17},\frac{17}{16},\frac{16}{15},\frac{15}{14},\frac{14}{13},\frac{13}{12},\frac{12}{11},\frac{11}{10},\frac{10}{9},\frac{9}{8},\frac{8}{7},\frac{7}{6},\frac{6}{5},\frac{5}{4},\frac{4}{3},\frac{3}{2},2\right\}$

```
Sum[(x + 1)/x, {x, 1, k}]
```

k + HarmonicNumber[k]

```
Sum[1 + 1/(-1 + PrimePi[x]), {x, 3, 20}] // N
Sum[1 + 1/(-1 + PrimePi[x]), {x, 3, 2000}] // N
Sum[1 + 1/(-1 + PrimePi[x]), {x, 3, 20000}] // N
```

24.2524

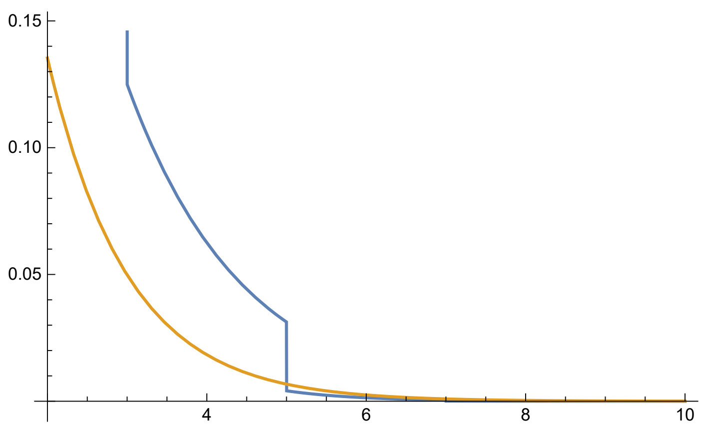
2025.65

20043.4

```
xmax = 2000;
(Sum[(1 + 1/(-1+PrimePi[x])), {x, 3, xmax}] - xmax) / Sum[(1 + 1/(-1+PrimePi[x])), {x, 3, xmax}] // N (*relative error*)
```

0.0126634

```
Plot[{(PrimePi[x])^(-x), Exp[-x]}, {x, 2, 10}]
```




```
Sum[(PrimePi[x])^(-n), {n, 0, ∞}] // FullSimplify
```

$$1 + \frac{1}{-1 + \text{PrimePi}[x]}$$

Sum[(PrimePi[x])^(-n), {n, 0, ∞}] $= \sum_{n=0}^{\infty} \pi(x)^{-n} = 1 + \frac{1}{-1 + \text{PrimePi}[x]}$

Sum$\left[\left(1 + \frac{1}{-1 + \text{PrimePi}[x]}\right), \{x, 3, k\}\right] =$

$$\left(1 + \frac{1}{-1 + \text{PrimePi}[3]}\right) + \left(1 + \frac{1}{-1 + \text{PrimePi}[4]}\right) + \left(1 + \frac{1}{-1 + \text{PrimePi}[5]}\right) + \ldots + \left(1 + \frac{1}{-1 + \text{PrimePi}[k]}\right)$$

So we have that

$\left(\text{Sum}\left[\left(1+\frac{1}{-1+\text{PrimePi}[x]}\right),\{x,3,k\}\right]\right)\to k \text{ as } k\to\infty$

So sort of like the integral is a sum of harmonic fns
With the # of times each fractional term appears related to the length of the prime gap with respect to the integers
In a space where 1 prime gap = 1, the intgral is just k+HarmonicNumber[k]

Also define a Zeta - like function :

$$\text{Sum}[(\text{PrimePi}[n])\,\hat{}\,(-s),\{n,2,\infty\}] = \sum_{n=2}^{\infty}\pi(n)^{-s}$$

$$\sum_{n=2}^{\infty}\frac{1}{\pi(n)^s} = \frac{1}{\pi(2)^s}+\frac{1}{\pi(3)^s}+\frac{1}{\pi(4)^s}+\frac{1}{\pi(5)^s}+\frac{1}{\pi(6)^s}+\frac{1}{\pi(7)^s}+\frac{1}{\pi(8)^s}+\frac{1}{\pi(9)^s}+\ldots =$$
$$\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{4^s}+\frac{1}{4^s}+\ldots$$

So we can rearrange

$$=\frac{1}{1^s}+\frac{1}{2^s}+0+\frac{1}{3^s}+0+\frac{1}{4^s}+0+0+\left(\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{4^s}\right)+\ldots =$$
$$\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}\right)+\left(\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}\right)+\left(\frac{1}{4^s}\right)+\ldots$$

And every prime after 2 is odd, so for any $k > 2$,
$\frac{1}{k^s}$ appears at least twice, so rewrite this sum as

$$\sum_{n=2}^{\infty}\frac{1}{\pi(n)^s} = \text{Zeta}[s] + (\text{Zeta}[s]-1) + \left(\frac{1}{4^s}\right) + \ldots = 2\,\text{Zeta}[s]-1+\ldots$$

And we can see numerically that this relationship does seem to hold very nearly, especially for larger s, so we expect the sum of extra terms to be relatively small

```
Zeta[-1]
```

$-\frac{1}{12}$

```
PiZeta[s_, k_] := Sum[(PrimePi[n])^(-s), {n, 2, k}]
(* s determines power of denominator, k determines number of iterations,
ideally want k → ∞ to recover analytic expressions *)
```

```
PiZeta[1, 9]   (*This checks that the fn works, they should be equal *)
```

$\frac{1}{1^1}+\frac{1}{2^1}+\frac{1}{2^1}+\frac{1}{3^1}+\frac{1}{3^1}+\frac{1}{4^1}+\frac{1}{4^1}+\frac{1}{4^1}$

$\frac{41}{12}$

$\frac{41}{12}$

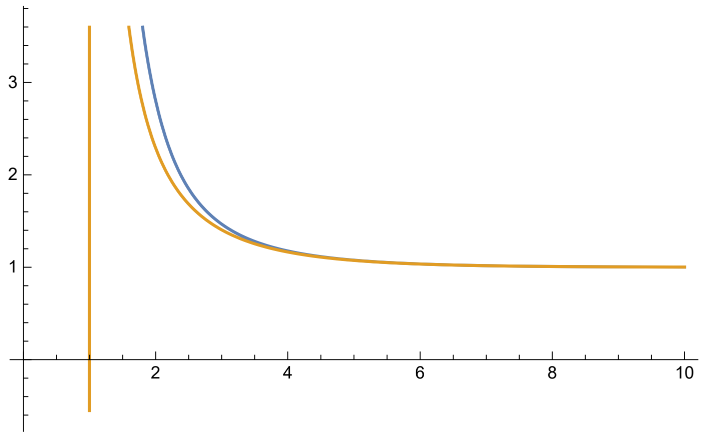
```
PiZeta[-1, 100] // N
```

1465.

```
PiZeta[5, 1000] // N
```

1.07625

```
Plot[{PiZeta[s, 1000], 2 Zeta[s] - 1}, {s, 0, 10}]
```




```
Plot[{PiZeta[Prime[s], 1000]}, {s, 0, 100}] // N
```

$Aborted

```
LogZeta[s_, k_] := Sum[(LogIntegral[n])^(-s), {n, 2, k}]
```

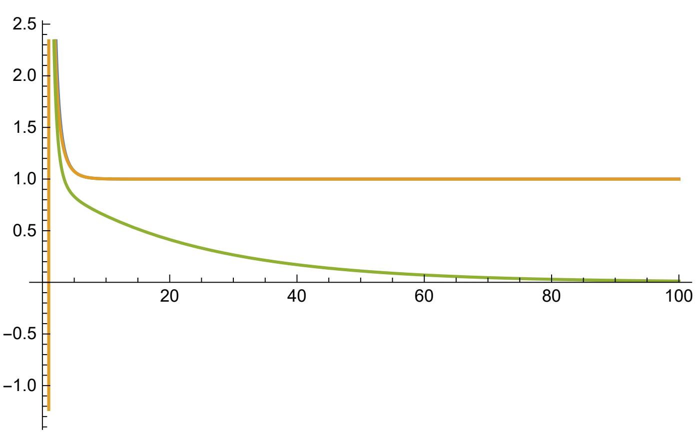
```
PiZeta[11, 100] // N
LogZeta[11, 100] // N
LogZeta[1000, 1000] // N
```

1.00099

0.61535

6.54107×10^{-20}

```
Plot[{PiZeta[s, 100], 2 Zeta[s] - 1, LogZeta[s, 100]}, {s, 0, 100}]
```

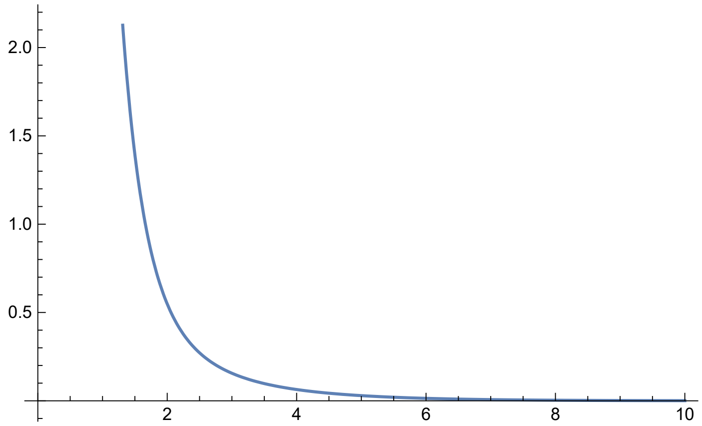



```
LogPiZeta[s_, k_] := Sum[(LogIntegral[n] + PrimePi[n])^(-s), {n, 2, k}]
```

```
LogPiZeta[5, 1000] // N
```

0.0292859

```
Plot[{LogPiZeta[s, 100]}, {s, 0, 10}]
```




```
Integrate[PrimePi[x]/(x ((x^s) - 1)), {x, 0, ∞}]
```

$$\int_0^\infty \frac{\text{PrimePi}[x]}{x\left(-1+x^s\right)}\,dx$$

$\pi(x) = \sum_{p\in\text{Primes}} \theta(x-p)$ is unique in that it spikes at the primes and only the primes

So, an exact analytical expression for $\frac{d\pi(x)}{dx} =$

$\sum_{p\in\text{Primes}} \delta(x-p)$ would have poles at the primes and only the primes

$\text{Log}[\text{Zeta}[s]] = s\int_0^\infty \frac{\text{PrimePi}[x]}{x(-1+x^s)}\,dx$ for $\text{Re}[s] > 1$ implies that

$$\frac{1}{\zeta[s]}\frac{d}{ds}\zeta[s] =$$

$$\int_0^\infty \frac{\text{PrimePi}[x]}{x(-1+x^s)}\,dx - s\int_0^\infty \frac{x^{-1+s}\,\text{Log}[x]\,\text{PrimePi}[x]}{(-1+x^s)^2}\,dx \qquad \text{For Re}[s] > 1$$

$$= \sum_{p\in\text{Primes}}\left(\int_0^\infty \frac{\theta(x-p)}{x(-1+x^s)}\,dx - s\int_0^\infty \frac{x^{-1+s}\,\text{Log}[x]\,\theta(x-p)}{(-1+x^s)^2}\,dx\right)$$

$$= \sum_{p\in\text{Primes}}\left(-\frac{\text{Log}[1-p^{-s}]}{s} - \frac{\frac{p^s\,s\,\text{Log}[p]}{-1+p^s} - \text{Log}[-1+p^s]}{s}\right)$$

$$= \sum_{p\in\text{Primes}}\left(-\frac{\text{Log}[1-p^{-s}]}{s} + \frac{+\text{Log}[-1+p^s]}{s} - \frac{\frac{p^s\,s\,\text{Log}[p]}{-1+p^s}}{s}\right) = \sum_{p\in\text{Primes}}\left(\frac{\text{Log}[p^s]}{s} - \frac{\frac{p^s\,s\,\text{Log}[p]}{-1+p^s}}{s}\right)$$

$$= \sum_{p\in\text{Primes}}\left(\frac{s\,\text{Log}[p]}{s} - \frac{s\,\text{Log}[p]}{s}\,\frac{p^s}{(-1+p^s)}\right) = \sum_{p\in\text{Primes}}\left(1 - \frac{p^s}{(-1+p^s)}\right)\text{Log}[p]$$

$$= \sum_{p \in Primes} \left(1 - \frac{1}{(1 - p^{-s})}\right) Log[p]$$

$$= \sum_{p \in Primes} \frac{1}{(1 - p^{s})} Log[p] \quad = \frac{1}{\zeta[s]} \frac{d}{ds} \zeta[s]$$

And

$$Log[Zeta[s]] = s \int_0^\infty \frac{PrimePi[x]}{x(-1 + x^s)} dx = \sum_{p \in Primes} \int_0^\infty \frac{s\,\theta\,(x - p)}{x(-1 + x^s)} dx = \sum_{p \in Primes} -Log[1 - p^{-s}]$$

$$Log[Zeta[s]] = \sum_{p \in Primes} (Log[p^s] - Log[p^s - 1]) = \sum_{p \in Primes} Log\left[\frac{p^s}{(p^s - 1)}\right]$$

$$Zeta[s] = Exp\left[\sum_{p \in Primes} Log\left[\frac{p^s}{(p^s - 1)}\right]\right] = \prod_{n=1}^{\infty} \frac{p^s}{(p^s - 1)}$$

Which is the Euler product form

No poles of $\zeta[s]$ are expected for $Re[s] > 1$, only for $Re[s] = \frac{1}{2}$,

but this does give a hint to the structure of the Zeta fn poles

$$\textbf{Fpder}[s\_, k\_] := Sum\left[\frac{1}{(1 - Prime[n]^s)} Log[Prime[n]], \{n, 1, k\}\right]$$

$$D\left[\frac{(s\,PrimePi[x])}{x(-1 + x^s)}, s\right]$$

$$\frac{PrimePi[x]}{x(-1 + x^s)} - \frac{s\,x^{-1+s} Log[x]\,PrimePi[x]}{(-1 + x^s)^2}$$

$$Integrate\left[HeavisideTheta[x - p] \frac{(s\,x^{-1+s} Log[x])}{(-1 + x^s)^2}, \{x, 0, Infinity\}\right]$$

$$ConditionalExpression\left[\frac{\frac{p^s\,s\,Log[p]}{-1+p^s} - Log[-1 + p^s]}{s}, \right.$$
$$\left. p^s \in Reals \,\&\&\, Re[s] > 0 \,\&\&\, Im[s] == 0 \,\&\&\, Re[p] > 1 \,\&\&\, Im[p] == 0\right]$$

$$Integrate\left[HeavisideTheta[x - p] \frac{s}{x(-1 + x^s)}, \{x, 0, Infinity\}\right]$$

$$ConditionalExpression\left[-Log\left[1 - p^{-s}\right], \right.$$
$$\left. p^s \in Reals \,\&\&\, Re[s] > 0 \,\&\&\, Im[s] == 0 \,\&\&\, Re[p] > 1 \,\&\&\, Im[p] == 0\right]$$

$$-\frac{\text{Log}[1-p^{-s}]}{s}+\frac{\text{Log}[-1+p^{s}]}{s}\ \text{// FullSimplify}$$

$$\frac{\text{Log}[p^{s}]}{s}$$

$$\text{Integrate}\left[\text{HeavisideTheta}[x-p]\ \frac{1}{x\left(-1+x^{s}\right)},\ \{x,\ 0,\ \text{Infinity}\}\right]$$

$$\text{NIntegrate}\left[(2)\ \frac{\text{LogIntegral}[x]}{(x\ ((x\text{^}2)-1))},\ \{x,\ 1.45,\ 100000\}\right]$$

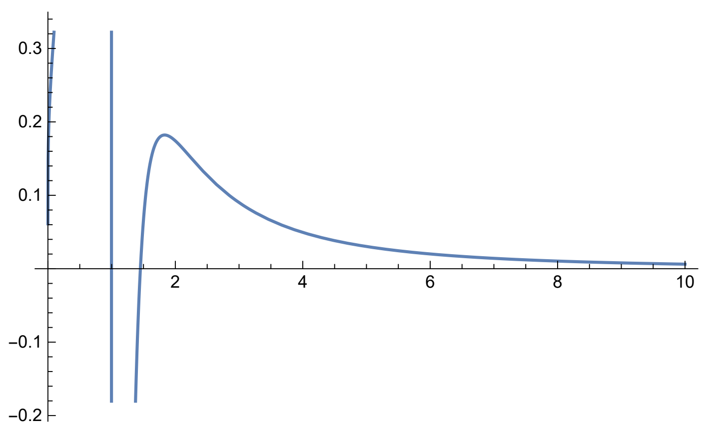
```
(*Gives numerical appx to this integral *)
```

$$\text{NIntegrate}\left[(10)\ \frac{\text{LogIntegral}[x]}{(x\ ((x\text{^}10)-1))},\ \{x,\ 1.45,\ 100000\}\right]$$

0.868869

0.00842422

$$\text{Plot}\left[\frac{\text{LogIntegral}[x]}{(x\ ((x\text{^}2)-1))},\ \{x,\ 0,\ 10\}\right]$$



Analogous to $\pi(x) = \sum_{p\in\text{Primes}} \theta(x-p)$, Define an integer counting fn

$Z(x) = \text{IntPi}[x] = \sum_{n\in\text{Integers}} \theta(x-n)$, $n > 1$ (* = Floor[x] *)

BUT START FROM 2 to avoid singularity,
also prime counting fn starts from 2, so it ' s good

(*Perhaps by analog PrimePi[x] is a floor fn in some prime space*)

Then, if analogous to $\text{Log}[\text{Zeta}[s]] = s\int_0^\infty \frac{\text{PrimePi}[x]}{x\,(-1+x^s)}\,dx$ for $\text{Re}[s] > 1$,

we have some IntZeta[s] such that

$\text{Log}[\text{IntZeta}[s]] = s\int_0^\infty \frac{\text{IntPi}[x]}{x\,(-1+x^s)}\,dx$ for $\text{Re}[s] > 1$, Then

(IntPi[x] := "integer counting" function = Floor[x] = actually)

$$\text{Log}[\text{IntZeta}[s]] = s\int_0^\infty \frac{\text{IntPi}[x]}{x\,(-1+x^s)}\,dx =$$

$$\sum_{n\in\text{Integers}} \int_0^\infty \frac{s\,\theta\,(x-n)}{x\,(-1+x^s)}\,dx = \int_0^\infty \frac{s\,\theta\,(x-1)}{x\,(-1+x^s)}\,dx + \sum_{n=2} \int_0^\infty \frac{s\,\theta\,(x-n)}{x\,(-1+x^s)}\,dx$$

$$\int_0^\infty \frac{s\,\theta\,(x-1)}{x\,(-1+x^s)}\,dx = 0 + \int_1^\infty \frac{s\,\theta\,(x-1)}{x\,(-1+x^s)}\,dx$$

Took out n = 1 term since the above is only exactly integrable if Re[n] > 1,
so deal with n = 1 term separately

$$= \sum_{n\in\text{Integers}} -\text{Log}[1-n^{-s}]$$

$$\text{Log}[\text{IntZeta}[s]] = \sum_{n\in\text{Integers}} (\text{Log}[n^s] - \text{Log}[n^s-1]) = \sum_{n\in\text{Integers}} \text{Log}\left[\frac{n^s}{(n^s-1)}\right]$$

$$\text{IntZeta}[s] = \text{Exp}\left[\sum_{n\in\text{Integers}} \text{Log}\left[\frac{n^s}{(n^s-1)}\right]\right] = \prod_{n=2}^{\infty} \frac{n^s}{(n^s-1)}$$

Starting from n = 2 as decribed above

```
Plot[
 {Sum[HeavisideTheta[(x - n)], {n, 2, 100}] + HeavisideTheta[(x - 1)], Floor[x]}, {x, 0, 10}]
Plot[{Sum[HeavisideTheta[(x - n)], {n, 2, 100}] + 1, Floor[x]}, {x, 0, 10}]
```

Note that in the plots

Plot[{Sum[HeavisideTheta[(x − n)], {n, 2, 100}] + HeavisideTheta[(x − 1)], Floor[x]},
 {x, 0, 10}]

Plot[{Sum[HeavisideTheta[(x − n)], {n, 2, 100}] + 1, Floor[x]}, {x, 0, 10}]

letting HeavisideTheta[(x − 1)] → 1 only makes a difference for x < 1
This difference doesn't matter for our purposes
 since we only need these functions to be equivalent for n ≥ 2
So even if we did include the n = 1 term, this still wouldn't make
 the definition different than if we had used the floor funtion

```
Integrate[HeavisideTheta[(x - n)] s/(x (-1 + x^s)), {x, 0, Infinity}]
(*Note that one condition is that Re[n]>1,i.e. we must require the n start from n=2 *)
```

$$\text{ConditionalExpression}\left[-\text{Log}\left[1-n^{-s}\right],\ n^s \in \text{Reals} \,\&\&\, \text{Re}[s] > 0 \,\&\&\, \text{Im}[s] == 0 \,\&\&\, \text{Re}[n] > 1 \,\&\&\, \text{Im}[n] == 0\right]$$

$$\text{IntZeta}[s\_, k\_] := \prod_{n=2}^{k} \frac{n^s}{(n^s-1)}$$

```
IntZeta[2, 10] // N
IntZeta[2, 100] // N
IntZeta[2, 10000] // N
```

1.81818

1.9802

1.9998

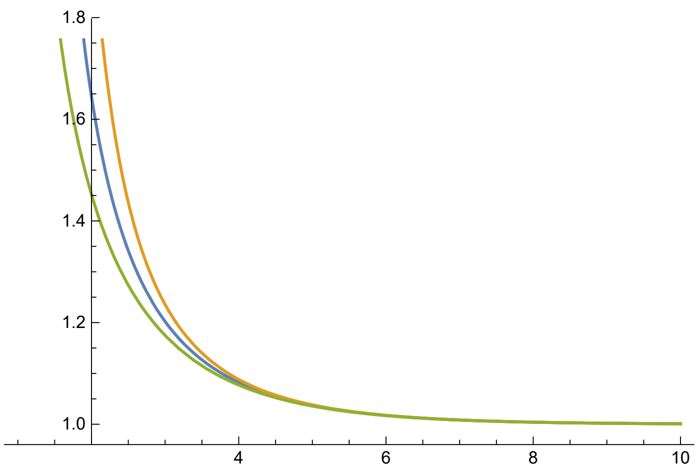
```
IntZeta[3, 10000] // N
```

1.23549

```
IntZeta[π, 100] // N
```

1.20177

```
Plot[{Zeta[s], IntZeta[s, 1000], 1 + PrimeZetaP[s]}, {s, 1, 10}]
```



For Re[s] > 1

$$\text{Log}[\text{IntZeta}[s]] = s \int_0^\infty \frac{\text{Floor}[x]}{x\,(-1 + x^s)}\, dx$$

$$\Rightarrow \text{IntZeta}[s] = \text{Exp}\left[s \int_0^\infty \frac{\text{Floor}[x]}{x\,(-1 + x^s)}\, dx\right]$$

And

$$\text{IntZeta}[s] = \prod_{n=2}^{\infty} \frac{1}{(1 - n^{-s})}$$

Therefore

$$\text{IntZeta}[s] = \prod_{n=2}^{\infty} \frac{1}{(1 - n^{-s})} = \frac{1}{(1 - 2^{-s})} \frac{1}{(1 - 3^{-s})} \frac{1}{(1 - 4^{-s})} \frac{1}{(1 - 5^{-s})} \frac{1}{(1 - 6^{-s})} \ldots$$

$$\text{IntZeta}[s] = \prod_{n=2}^{\infty} \frac{1}{(1 - n^{-s})} = \prod_{\text{prime}} \frac{1}{(1 - p^{-s})} \prod_{\text{Compound}} \frac{1}{(1 - c^{-s})} = \text{Zeta}[s] \prod_{\text{Compound}} \frac{1}{(1 - c^{-s})}$$

$$\Rightarrow \text{IntZeta}[s] = \text{Zeta}[s] \prod_{\text{Compound}} \frac{1}{(1 - c^{-s})}$$

$$\frac{\text{IntZeta}[s]}{\text{Zeta}[s]} = \prod_{\text{Compound}} \frac{1}{(1 - c^{-s})}$$

Following the pattern we could note that $\prod_{\text{Compound}} \frac{1}{(1 - c^{-s})}$ is a zeta – like function,

since both $\prod_{n=2}^{\infty} \frac{1}{(1 - n^{-s})}$ and $\prod_{\text{prime}} \frac{1}{(1 - p^{-s})}$ give zeta fns. Call this CompoundZeta[s] =

$\prod_{\text{Compound}} \frac{1}{(1 - c^{-s})}$. Although note that CompoundZeta[s]

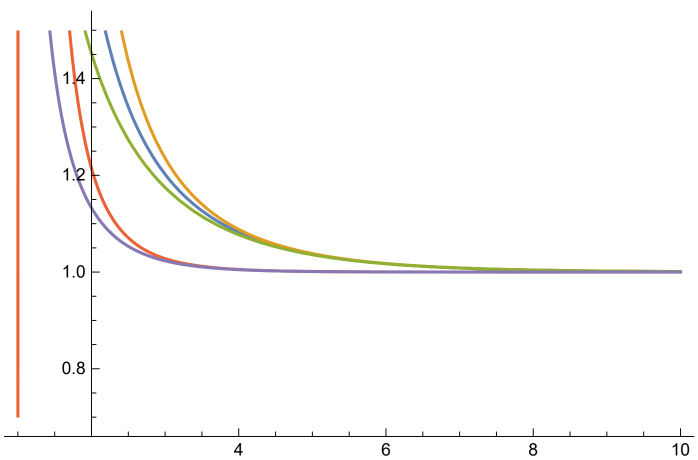
differs much more from Zeta[s] than IntZeta[s] does

$$\frac{\text{Zeta}[s]}{\text{IntZeta}[s]} = \prod_{\text{Compound}} (1 - c^{-s}) =$$

$$(1 - 4^{-s})(1 - 6^{-s})(1 - 8^{-s})(1 - 9^{-s})(1 - 10^{-s})(1 - 12^{-s})(1 - 14^{-s})(1 - 15^{-s})\ldots,$$

$$(1 - c^{-s}) = 1 - \frac{1}{c^s} = \frac{c^s - 1}{c^s} =$$

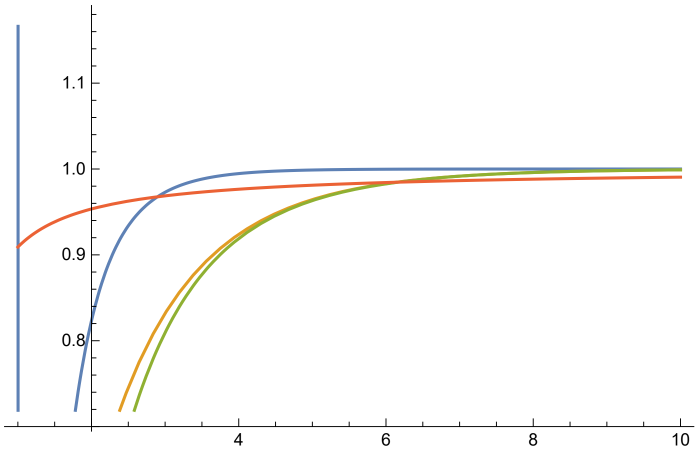
```
Plot[{Zeta[s], IntZeta[s, 1000], 1 + PrimeZetaP[s],
  IntZeta[s, 1000]/Zeta[s], Zeta[s]/(1 + PrimeZetaP[s])}, {s, 1, 10}]
```



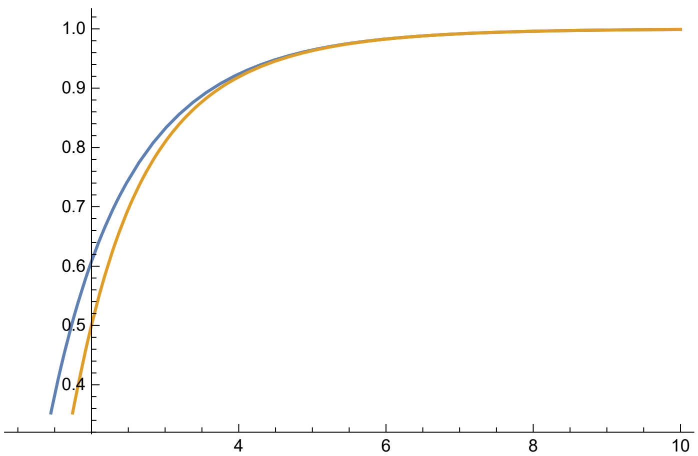

```
2^(-10)
```

$$\frac{1}{1024}$$

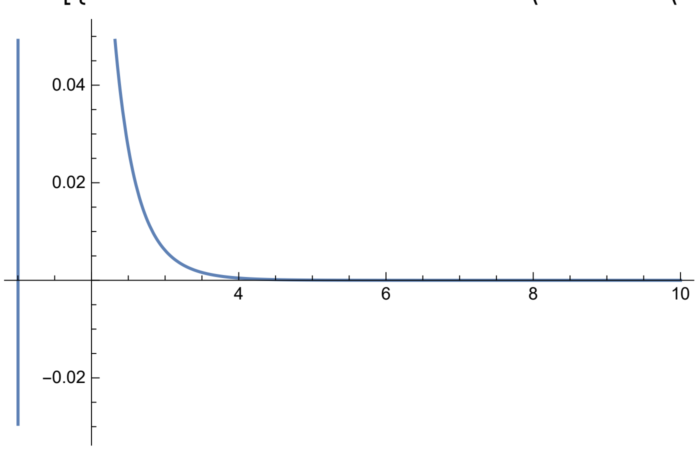
```
Plot[{Zeta[s]/IntZeta[s, 1000], 1/Zeta[s], 1/IntZeta[s, 1000], 1.1^(-1/s)}, {s, 1, 10}]
```




```
Plot[{1/Zeta[s], 1/IntZeta[s, 1000]}, {s, 1, 10}]
```




```
Plot[{Abs[IntZeta[s, 1000] - Zeta[s]] - (Zeta[s] - (1 + PrimeZetaP[s]))}, {s, 1, 10}]
```




```
s12 = 4;
Abs[IntZeta[s12, 1000] - Zeta[s12]] - (Zeta[s12] - (1 + PrimeZetaP[s12]));
```

```
s12 = 20;
IntZeta[s12, 1000] // N
Abs[IntZeta[s12, 1000] - Zeta[s12]] - (Zeta[s12] - (1 + PrimeZetaP[s12])) // N
```

1.

6.74753×10^{-16}

```
s12 = 6;
Zeta[s12]
Zeta[s12] // N
IntZeta[s12, 10000] // N
N[(π^s12) / (IntZeta[s12, 10000]), 20]
```

$\frac{\pi^6}{945}$

1.01734

1.01762

944.74204531769695243

```
s22 = 9;
IntZeta[s22, 1000] - Zeta[s22] // N
Zeta[s22] - (1 + PrimeZetaP[s22]) // N
```

3.93315×10^{-6}

3.92525×10^{-6}

$\text{Log}[\text{IntZeta}[s]] = s \int_0^{\infty} \frac{\text{IntPi}[x]}{x\,(-1 + x^s)}\, dx$ for Re[s] > 1 , Then

$\text{IntZeta}[s] = \text{Exp}\left[\sum_{n \in \text{Integers}} \text{Log}\left[\frac{n^s}{(n^s - 1)}\right]\right] = \prod_{n=2}^{\infty} \frac{n^s}{(n^s - 1)}$

IntZeta[s] → Zeta[s] as $s \to \infty$, converges very rapidly too

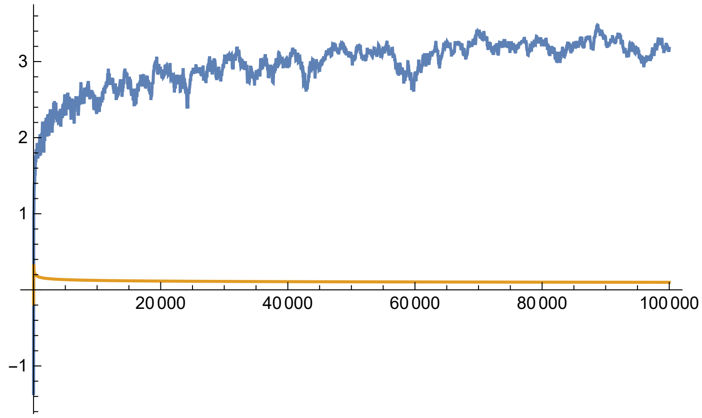
But note that

$\text{IntPi}[x] = \sum_{n \in \text{Integers}, n > 1} \theta\,(x - n) = \text{Floor}[x]$, So

$\text{Log}[\text{IntZeta}[s]] = s \int_0^{\infty} \frac{\text{Floor}[x]}{x\,(-1 + x^s)}\, dx$

So have evidence of a relationship between the floor fn and prime - counting / Zeta fns.

```
Plot[{Log[π, LogIntegral[x] - PrimePi[x]], Log[x, π]}, {x, 0, 100000}]
```




```
Table[Log[π, LogIntegral[x] - PrimePi[x]], {x, 2, 100000, 10000}] // N
```

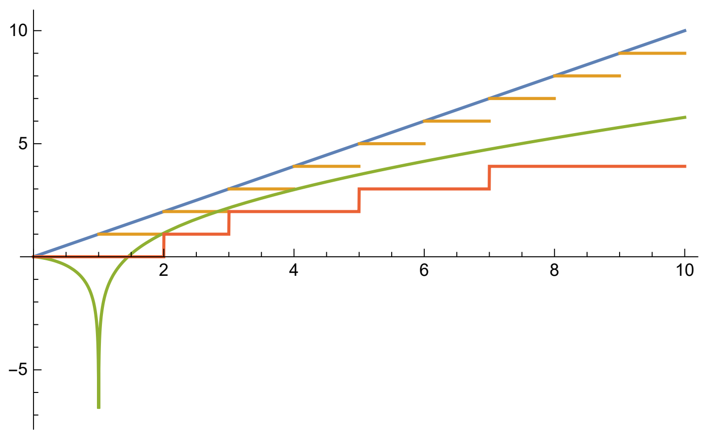
```
{-2.70584, 2.49303, 2.87317, 3.0301, 2.97698, 3.07359, 2.84714, 3.40818, 3.20907, 3.2951}
```

```
Table[Log[π, LogIntegral[x] - PrimePi[x]], {x, 1000000000, 1000000010}] // N
```

```
{6.49843, 6.49845, 6.49848, 6.4985, 6.49853,
 6.49855, 6.49858, 6.49809, 6.49811, 6.49762, 6.49765}
```

Conj :
  A^Log[b, LogIntegral[x] - PrimePi[x]] = constant
for some A (perhaps modulating) and a Logarithm of some base b

```
Plot[{x, Floor[x], LogIntegral[x], PrimePi[x]}, {x, 0, 10}]
```



So there should exist a fn B ~ LogIntegral such that B[Floor[x]] = PrimePi[x]

twist prime spae ti integers, find x in tht space, then twist back

Zeta Decomposition

```
Sum[(2 x)^(-s), {x, 1, ∞}] (* Even Zeta *)
Sum[(2 x + 1)^(-s), {x, 1, ∞}] (* Odd Zeta *)
(* So these + 1 = Zeta[s] *)
```

2^{-s} Zeta[s]

2^{-s} Zeta$\left[s, \frac{3}{2}\right]$

```
2^-s Zeta[s] + 2^-s Zeta[s, 3/2] // FullSimplify
```

$-1 +$ Zeta[s]

```
Sum[(3 x)^(-s), {x, 1, ∞}] - Sum[(6 x)^(-s), {x, 1, ∞}]
```

3^{-s} Zeta[s] $- 6^{-s}$ Zeta[s]

3^{-s} Zeta[s] $- 6^{-s}$ Zeta[s] $= 3^{-s}$ $(1 - 2^{-s})$ Zeta[s]

So for the n-Zeta to take out Sum[(GCD(n,m) x)^(-s),{x,1,∞}] , where m is the previous prime.

```
(* Can break odd Zetas down further *)
Sum[(6 x + 3)^(-s), {x, 0, ∞}] (*Non-even multiples of 3 *)
Sum[(5 x)^(-s), {x, 1, ∞}] - Sum[(10 x)^(-s), {x, 1, ∞}] -
 Sum[(15 x)^(-s), {x, 1, ∞}] (*Non-even multiples of 5, not multiples of 3 either *)
```

6^{-s} $(-1 + 2^{s})$ Zeta[s]

5^{-s} Zeta[s] $- 10^{-s}$ Zeta[s] $- 15^{-s}$ Zeta[s]

```
5^-s Zeta[s] - 10^-s Zeta[s] - 15^-s Zeta[s] // FullSimplify
```

30^{-s} $(-2^{s} - 3^{s} + 6^{s})$ Zeta[s]

```
5^-s (1 - 2^-s - 3^-s) Zeta[s];
```

So the Zeta function can be written as

$1 +$ $(2^{-s}$ Zeta[s]$)$ $+ 3^{-s}$ $(1 - 2^{-s})$ Zeta[s] $+ 5^{-s}$ $(1 - 2^{-s} - 3^{-s})$ Zeta[s] $+ ...=$
$1 +$ $((2^{-s})$ $+ 3^{-s}$ $(1 - 2^{-s})$ $+ 5^{-s}$ $(1 - 2^{-s} - 3^{-s})$ $+ ...)$ Zeta[s]

```
((2^-s) + 3^-s (1 - 2^-s) + 5^-s (1 - 2^-s - 3^-s)) Zeta[s];

(*
(2^-s ) + 3^-s(1 -2^-s ) + 5^-s (1-2^-s -3^-s ) + 7^-s (1-2^-s -3^-s -5^-s)
  Prime[1]^-s (1) + Prime[2]^-s(1 -Prime[1]^-s ) + Prime[3]^-s (1-Prime[1]^-s -Prime[2]^-s ) +
 Prime[4]^-s (1-Prime[1]^-s -Prime[2]^-s -Prime[3]^-s)
  Prime[n] (  1-Sum[((Prime[k])^(-s)),{k,1,n-1}])
*)

Sum[((Prime[k])^(-s)), {k, 1, 0}] (*So code used below works for first term too *)
```

0

```
Sum[((Prime[n])^(-s)) (1 - Sum[((Prime[k])^(-s)), {k, 1, n - 1}]), {n, 1, 4}]
```

$$2^{-s} + 3^{-s} \left(1 - 2^{-s}\right) + 5^{-s} \left(1 - 2^{-s} - 3^{-s}\right) + 7^{-s} \left(1 - 2^{-s} - 3^{-s} - 5^{-s}\right)$$

```
(* So Zeta[s] - 1 is equivalent to
  Sum[((Prime[n])^(-s))(  1-Sum[((Prime[k])^(-s)),{k,1,n-1}]),{n,1,∞}] Zeta[s]
  have the minus 1 with the Zeta since 1 wasn't incorporated into the sum here *)
```

```
Zeta[s] - 1 =
 Sum[((Prime[n])^(-s)) ( 1 - Sum[((Prime[k])^(-s)), {k, 1, n - 1}]), {n, 1, ∞}]
  Zeta[s]
For Zeta[s] ≠ 0, we have
1 - (Zeta[s])^(-1) =
 Sum[((Prime[n])^(-s)) ( 1 - Sum[((Prime[k])^(-s)), {k, 1, n - 1}]), {n, 1, ∞}]
(Zeta[s])^(-1) = 1 -
  Sum[((Prime[n])^(-s)) ( 1 - Sum[((Prime[k])^(-s)), {k, 1, n - 1}]), {n, 1, ∞}]
(Zeta[s])^(-1) = 1 - G[s, a]
```

$$\text{Zeta}[s] = \frac{1}{1 - G[s, a]}$$

```
G[s_, a_] := Sum[((Prime[n])^(-s)) ( 1 - Sum[((Prime[k])^(-s)), {k, 1, n - 1}]), {n, 1, a}]
```

```
G[s, a]
```

$$\sum_{n=1}^{a} \text{Prime}[n]^{-s} \left(1 - \sum_{k=1}^{-1+n} \text{Prime}[k]^{-s}\right)$$

```
G[s, 5]
```

$$2^{-s} + 3^{-s} \left(1 - 2^{-s}\right) + 5^{-s} \left(1 - 2^{-s} - 3^{-s}\right) + 7^{-s} \left(1 - 2^{-s} - 3^{-s} - 5^{-s}\right) + 11^{-s} \left(1 - 2^{-s} - 3^{-s} - 5^{-s} - 7^{-s}\right)$$

So we can separate G[s, a] into sums over primes and **non - square** semiprimes

$$G[s, a] = \left(2^{-s} + 3^{-s} + 5^{-s} + 7^{-s} + 11^{-s} + \ldots\right) - \left(3^{-s}\left(2^{-s}\right) + 5^{-s}\left(2^{-s} + 3^{-s}\right) + 7^{-s}\left(2^{-s} + 3^{-s} + 5^{-s}\right) + 11^{-s}\left(2^{-s} + 3^{-s} + 5^{-s} + 7^{-s}\right) + \ldots\right)$$

All the terms on the right are semiprime, but don't include all semiprimes, missing squares such as 9.

In the $a \to \infty$ limit (where the Zeta equality actually holds), this becomes

```
G[s, a → ∞] = PrimeZetaP[s] - (SPZeta[s] - (-1 + Zeta[2 s]))
G[s, a → ∞] = (PrimeZetaP[s] + Zeta[2 s] - 1) - SPZeta[s]
```

For some semiprime Zeta SPZeta, which now includes the squares since we removed their term

```
Zeta[s] = 1/(1 - (PrimeZetaP[s] + Zeta[2 s] - 1) - SPZeta[s]) =
  1/(2 - (PrimeZetaP[s] + Zeta[2 s]) - SPZeta[s])
```

Apparently SPZeta[s] ≈ 0.14076043434902338822275 =
C = Constant (sequence A117543 in the OEIS)
APPX (https : // en.wikipedia.org / wiki / Prime_zeta _function)

$$\text{Zeta}[s] = \frac{1}{(2 - C) - (\text{PrimeZetaP}[s] + \text{Zeta}[2 s])}$$

$$(2 - C) - (\text{PrimeZetaP}[s] + \text{Zeta}[2 s]) = \frac{1}{\text{Zeta}[s]}$$

$$\mathbf{(2 - C) = \frac{1}{\text{Zeta}[s]} + (\text{PrimeZetaP}[s] + \text{Zeta}[2 s])}$$

RHS converges to 2, but is not 2 for s closer to 1. Although doesn't seem to go to 2 − C
Note the SPZeta only appx,
which may explain why it doesn't go to 2 − C and why it's not constant for low s.
Maybe 2 − C equality only holds for s → ∞ limit?

Also gives us

$$\mathbf{\text{PrimeZetaP}[s] = (2 - C) - \text{Zeta}[2 s] - \frac{1}{\text{Zeta}[s]}} = (2 - C) - \frac{\text{Zeta}[2 s]\, \text{Zeta}[s] - 1}{\text{Zeta}[s]}$$

For s → 1 we have PrimeZetaP[s] ≈ Log[Zeta[s]]

$$\text{Log}[\text{Zeta}[s]] \approx (2 - C) - \text{Zeta}[2 s] - \frac{1}{\text{Zeta}[s]}$$

$$\text{Zeta}[s] \approx \text{Exp}[(2 - C)]\, \text{Exp}[-\text{Zeta}[2 s]]\, \text{Exp}\left[-\frac{1}{\text{Zeta}[s]}\right]$$

```
Sum[(k^2)^(-s), {k, 2, a}]
(*Gives the Zeta fn for just the squares that are greater than 1 *)
Sum[(k^2)^(-s), {k, 2, ∞}]
```

−1 − HurwitzZeta[2 s, 1 + a] + Zeta[2 s]

−1 + Zeta[2 s]

```
G[2, 10] // N
G[2, 100] // N
G[2, 1000] // N (*Converges pretty fast *)
```

0.384488

0.388341

0.388473

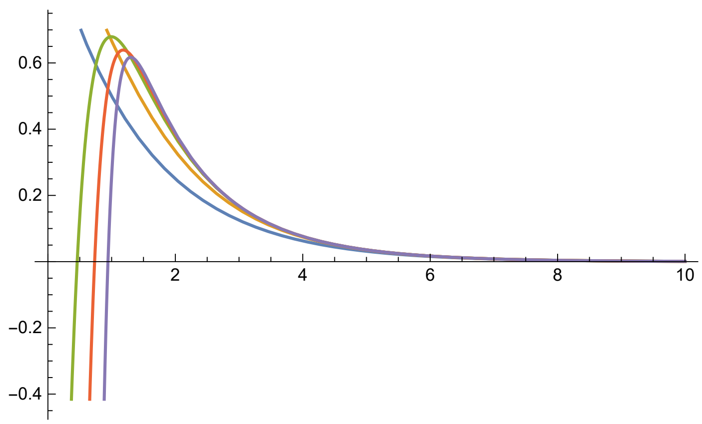
```
G[1, 1] // N
G[1, 2] // N
G[1, 5] // N
G[1, 10] // N
G[1, 900] // N
```

```
0.5

0.666667

0.679221

0.580227

-0.316474
```

```
Table[G[1, a], {a, 1, 10}] // N
```

```
{0.5, 0.666667, 0.7, 0.695238, 0.679221, 0.658675, 0.638438, 0.617236, 0.597432, 0.580227}
```

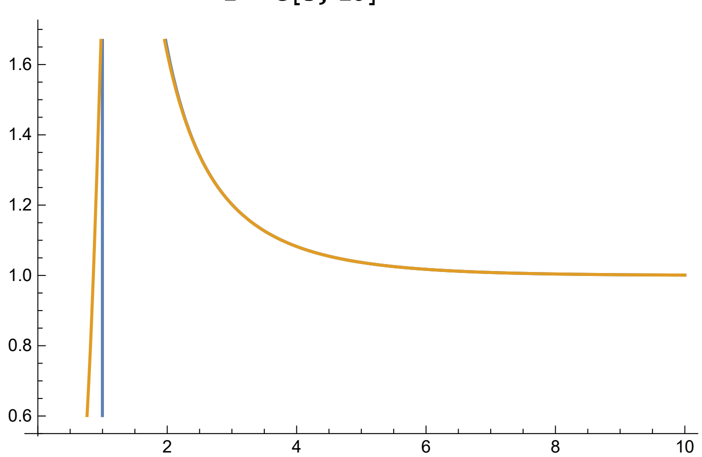
```
Plot[{G[s, 1], G[s, 2], G[s, 5], G[s, 10], G[s, 50]}, {s, 0, 10}]
```




```
FindRoot[G[s, 500], {s, 1}]
```

```
{s → 1.02635}
```

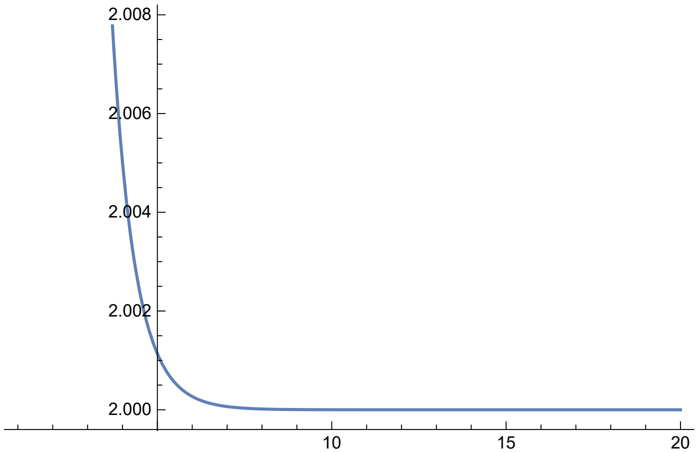
Plot$\left[\left\{\text{Zeta}[s], \dfrac{1}{1 - G[s, 20]}\right\}, \{s, 0, 10\}\right]$



```
Table[1/Zeta[s] + (PrimeZetaP[s] + Zeta[2 s]), {s, 0, 20}] // N
```

```
{-2.5 + PrimeZetaP[0.], ComplexInfinity, 2.1425, 2.02401, 2.00501, 2.00114,
 2.00027, 2.00006, 2.00002, 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2.}
```

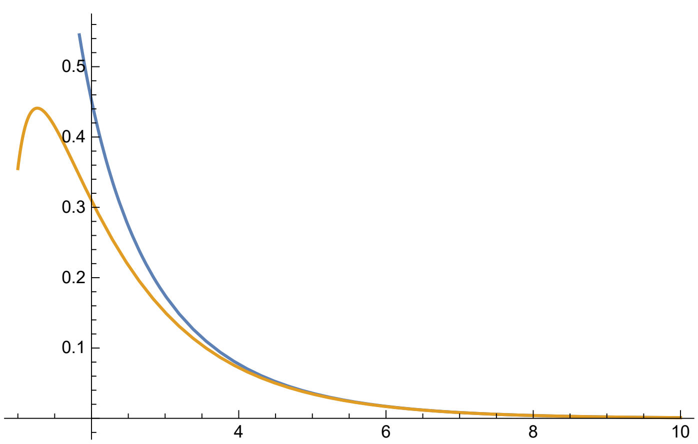
```
Plot[1/Zeta[s] + (PrimeZetaP[s] + Zeta[2 s]), {s, 1, 20}]
```




```
N[1/Zeta[1000] + (PrimeZetaP[1000] + Zeta[2 × 1000]), 100]
```

```
2.00000000000000000000000000000000000000000000000000000000000000000000000000000000000000\
 0000000000000
```

```
Plot[{PrimeZetaP[s], 2 - Zeta[2 s] - 1/Zeta[s]}, {s, 1, 10}]
```



Side Stuff on even Zetas and Volume

Floor Fn

$$x - \frac{1}{2} + \frac{1}{\pi}\,\mathrm{Sum}\left[\frac{\mathrm{Sin}[2\,\pi\,k\,x]}{k},\ \{k,\ 1,\ \infty\}\right]$$

$$-\frac{1}{2} + x + \frac{i\left(\mathrm{Log}\left[1 - e^{2\,i\,\pi\,x}\right] - \mathrm{Log}\left[e^{-2\,i\,\pi\,x}\left(-1 + e^{2\,i\,\pi\,x}\right)\right]\right)}{2\,\pi}$$

$$\frac{1}{2\,\pi}\,\mathrm{Sum}\left[\frac{\mathrm{Sin}[2\,\pi\,k\,x]}{k},\ \{k,\ 1,\ \infty\}\right]\ //\ \mathrm{FullSimplify}$$

$$\frac{1}{\pi}\,\mathrm{Sum}\left[\frac{\mathrm{Sin}[2\,\pi\,k\,x]}{k},\ \{k,\ 0,\ \infty\}\right]\ //\ \mathrm{FullSimplify}$$

$$\frac{i\left(-\mathrm{Log}\left[1 - e^{-2\,i\,\pi\,x}\right] + \mathrm{Log}\left[1 - e^{2\,i\,\pi\,x}\right]\right)}{4\,\pi}$$

$$\frac{i\left(-4\,i\,\pi\,x - \mathrm{Log}\left[1 - e^{-2\,i\,\pi\,x}\right] + \mathrm{Log}\left[1 - e^{2\,i\,\pi\,x}\right]\right)}{2\,\pi}$$

$$\mathrm{Im}\left[\frac{i\left(-4\,i\,\pi\,x - \mathrm{Log}\left[1 - e^{-2\,i\,\pi\,x}\right] + \mathrm{Log}\left[1 - e^{2\,i\,\pi\,x}\right]\right)}{2\,\pi}\right]$$

$$\frac{4\,\pi\,\mathrm{Im}[x] + \mathrm{Re}\left[-\mathrm{Log}\left[1 - e^{-2\,i\,\pi\,x}\right] + \mathrm{Log}\left[1 - e^{2\,i\,\pi\,x}\right]\right]}{2\,\pi}$$

$$\mathrm{Fn}[x\_] := \frac{i\left(-\mathrm{Log}\left[1 - e^{-2\,i\,\pi\,x}\right] + \mathrm{Log}\left[1 - e^{2\,i\,\pi\,x}\right]\right)}{2\,\pi}$$

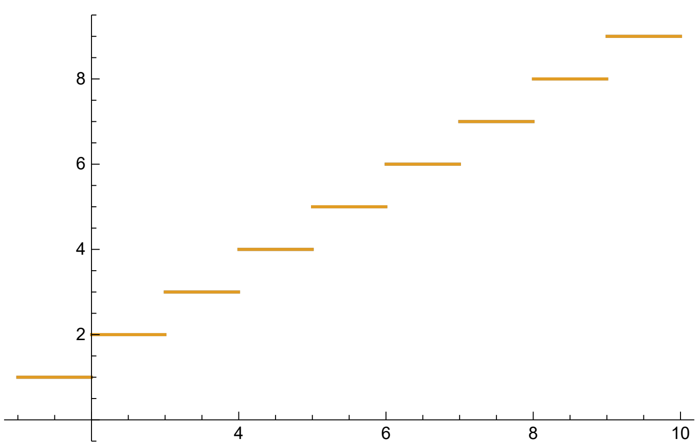
```
Fn[5/2 + 1/2 ⅈ] // N
Fn[3 + 3 ⅈ] // N
Fn[1/2] // N
Fn[1/2 + 1/2 ⅈ] // N
Fn[27 + 17 ⅈ] // N
Fn[E - 17 ⅈ] // N
Fn[π - 17 ⅈ] // N
Fn[2 π - 17 ⅈ] // N
Fn[3 π - 17 ⅈ] // N
```

```
0. - 0.5 ⅈ

0.5 - 3. ⅈ

0.

0. - 0.5 ⅈ

0.5 - 17. ⅈ

-0.218282 + 17. ⅈ

0.358407 + 17. ⅈ

0.216815 + 17. ⅈ

0.075222 + 17. ⅈ
```

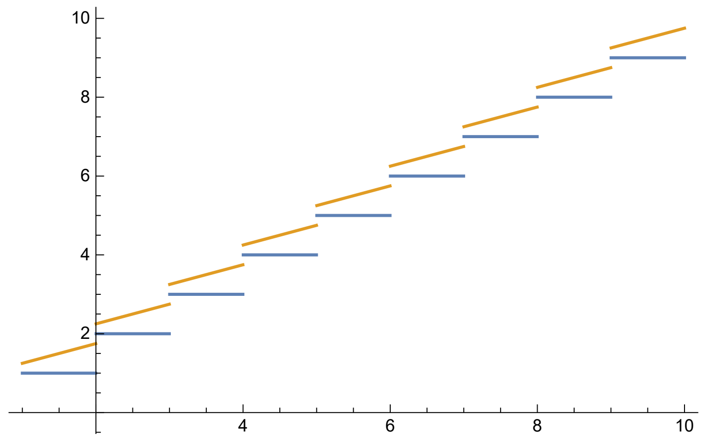
$$\text{Plot}\left[\left\{\frac{i\left(-\text{Log}\left[1-e^{-2 i \pi x}\right]+\text{Log}\left[1-e^{2 i \pi x}\right]\right)}{2\pi}\right\}, \{x, 1, 10\}\right]$$

$$\text{ContourPlot}\left[\left\{\text{Im}\left[-\text{Log}\left[1-e^{-2 i \pi (a+b i)}\right]+\text{Log}\left[1-e^{2 i \pi (a+b i)}\right]\right]\right\}, \{a, -1, 1\}, \{b, -1, 1\}\right];$$

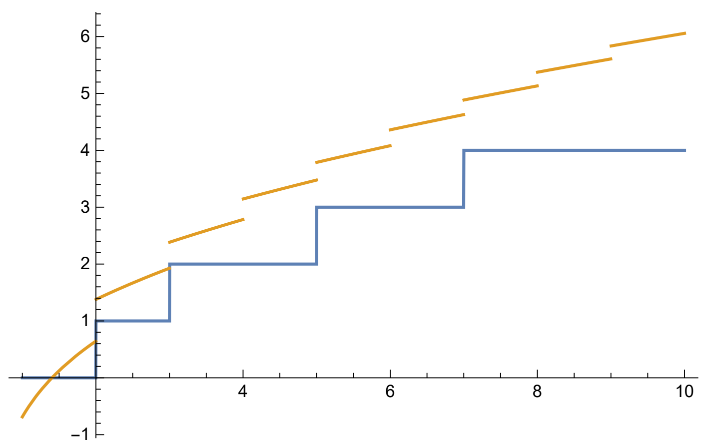
$$\text{Plot}\left[\left\{\text{Floor}[x], x-\frac{1}{2}+\frac{1}{\pi}\text{Sum}\left[\frac{\text{Sin}[2\pi k x]}{k}, \{k, 1, \infty\}\right]\right\}, \{x, 1, 10\}\right]$$



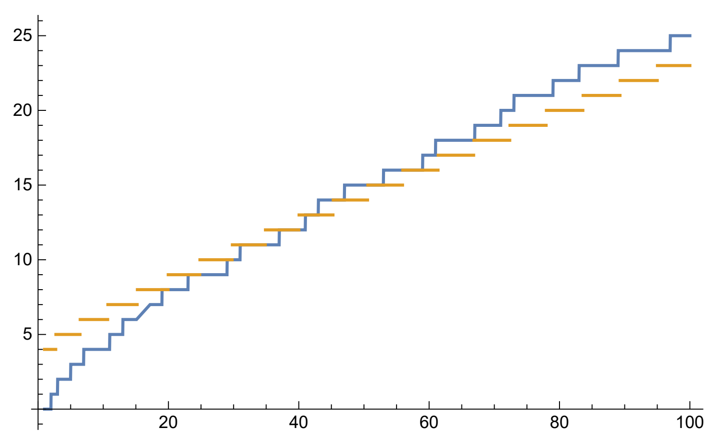
```
Plot[{Floor[x], 1/(2π) Sum[Sin[2 π k x]/k, {k, 0, ∞}]}, {x, 1, 10}]
```



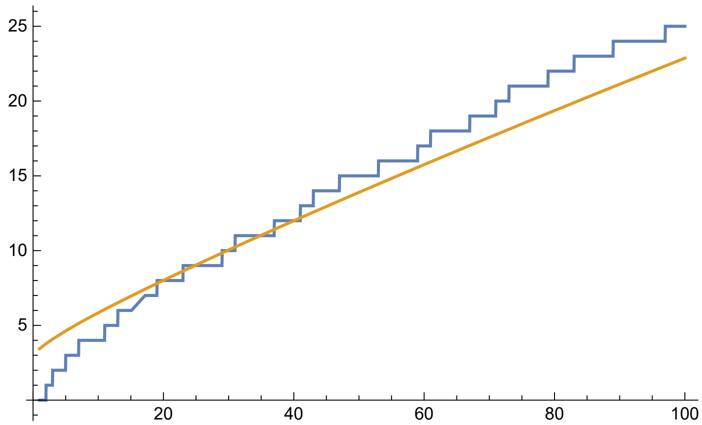

```
Plot[{PrimePi[x], LogIntegral[1/(2π) Sum[Sin[2 π k x]/k, {k, 0, ∞}]]}, {x, 1, 10}]
```




```
Plot[{PrimePi[x], Ceiling[Abs[LogIntegral[-Log[1 - e^(2 π x)]/(2 π)]]]}, {x, 1, 100}]
```



```
Plot[{PrimePi[x], Abs[LogIntegral[PolyLog[1, ℯ^(2 π x)]/(2 π)]]}, {x, 1, 100}]
```




```
1/(2 π) Sum[Cos[2 π k x]/(k^s), {k, 1, ∞}] // FullSimplify
```

```
(PolyLog[s, ℯ^(-2 ⅈ π x)] + PolyLog[s, ℯ^(2 ⅈ π x)])/(4 π)
```

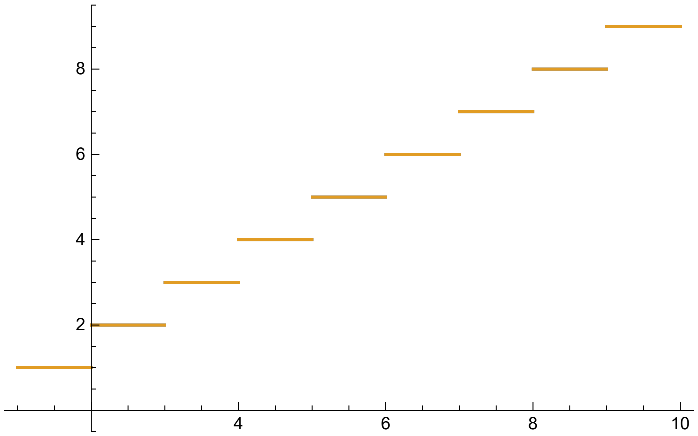
```
1/(2 π) Sum[Exp[2 π k x]/(k^s), {k, 1, ∞}] // FullSimplify
```

```
PolyLog[s, ℯ^(2 π x)]/(2 π)
```

```
Gb[s_, x_] := LogIntegral[PolyLog[s, ℯ^(2 π x)]/(2 π)] // N

GbAbs[s_, x_] := LogIntegral[PolyLog[s, ℯ^(2 π x)]/(2 π)] // N

Abs[Gb[2, 2]]
Abs[GbAbs[2, 2]]
```

```
7.1861

7.1861
```

$$\text{Plot}\left[\left\{\text{Floor}[x],\ x-\frac{1}{2}+\frac{1}{\pi}\text{Sum}\left[\frac{\text{Sin}[2\pi k x]}{k},\ \{k,1,\infty\}\right]\right\},\ \{x,1,10\}\right]$$



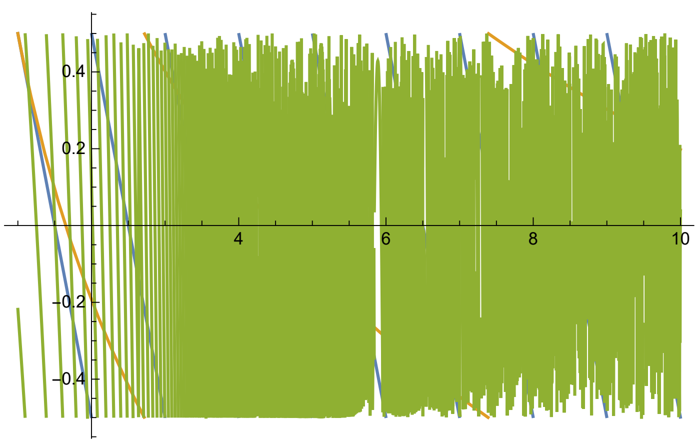
$$\text{Series}\left[\frac{1}{\pi}\text{Sum}\left[\frac{\text{Sin}[2\pi k x]}{k},\ \{k,1,\infty\}\right],\ \{x,0,6\}\right]$$

$$\frac{i\,(\text{Log}[-2\,i\,\pi]-\text{Log}[2\,i\,\pi])}{2\pi}-x+O[x]^7$$

$$\frac{i\,(\text{Log}[-2\,i\,\pi]-\text{Log}[2\,i\,\pi])}{2\pi}\ //\ N$$

$$0.5+0.\,i$$

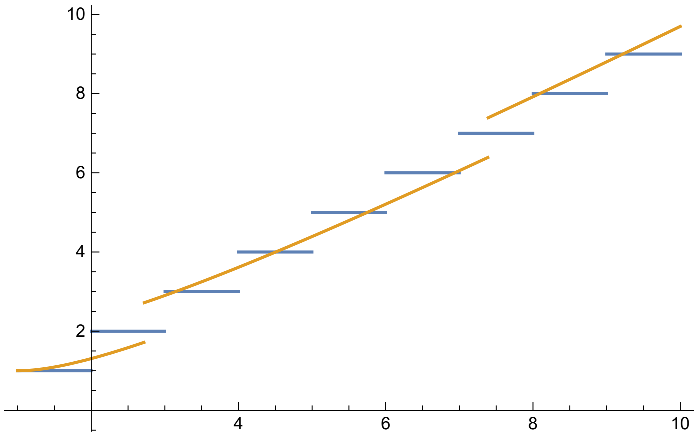
$$\text{Plot}\left[\left\{\frac{1}{\pi}\text{Sum}\left[\frac{\text{Sin}[2\pi k x]}{k},\ \{k,1,\infty\}\right],\ \frac{1}{\pi}\text{Sum}\left[\frac{\text{Sin}[2\pi k\,\text{Log}[x]]}{k},\ \{k,1,\infty\}\right],\ \frac{1}{\pi}\text{Sum}\left[\frac{\text{Sin}[2\pi k\,\text{Exp}[x]]}{k},\ \{k,1,\infty\}\right]\right\},\ \{x,1,10\}\right]$$



$$\frac{1}{\pi}\text{Sum}\left[\frac{\text{Sin}[2\pi k\,\text{Log}[x]]}{k},\ \{k,1,\infty\}\right]$$

$$\frac{i\,\left(\text{Log}\left[1-x^{2\,i\,\pi}\right]-\text{Log}\left[x^{-2\,i\,\pi}\left(-1+x^{2\,i\,\pi}\right)\right]\right)}{2\pi}$$

$$\text{Plot}\left[\left\{\text{Floor}[x],\ x-\frac{1}{2}+\frac{1}{\pi}\text{Sum}\left[\frac{\text{Sin}[2\,\pi\,k\,\text{Log}[x]]}{k},\ \{k,\ 1,\ \infty\}\right]\right\},\ \{x,\ 1,\ 10\}\right]$$



$$\text{Plot}\left[\left\{\text{PrimePi}[x],\ \text{LogIntegral}\left[x-\frac{1}{2}\text{Sum}\left[\frac{\text{Sin}[2\,\pi\,k\,\text{Log}[x]]}{k},\ \{k,\ 1,\ \infty\}\right]\right]\right\},\ \{x,\ 1,\ 100\}\right]$$