Examines mathematical structure of ‘Cornell Potential’ (**arXiv:hep-ph/0701157** or just see wiki for an easy summary https://en.wikipedia.org/wiki/Quarkonium#QCD_and_quarkonia) , a phenomenologically determined equation for describing nuclear strong force interactions. Here I explore relationship to number theoretic function - Log integral function which relates to the density of prime numbers.

First lets look at the expression: $V[r\_] := p\left(-\frac{\kappa 0}{r} + (\sigma 0\ (r))\right)$ The inverse term represents the ‘Strong’ nature of the aptly named Nuclear strong force, thats the term that keeps quarks close to each other most of the time. The linear term is the ‘Confinement’ term, what makes it energetically favorable for the creation of two new quark pairs when quarks are separated, which is why we dont see quarks alone. Doesn’t at all exclude presence of higher order terms and the AdS/QCD works actually explicitly introduce higher order terms.

Cornell potential itself has yet to be assigned a precise mathematical description (2017). Being phenomenological the equation of the cornell potential is only a leading-order approximation. Some interesting work in field of the Anti-de Sitter/ Conformal Field Theory correspondence pertaining to the ‘conformal window’ in Quantum ChromoDynamics, hence the use of the term AdS/QCD, (**arXiv:1706.04335 [hep-th]**) has actually reproduced Cornell potential, a success for AdS/QCD theory. *Of course numerical coefficients have to be fitted. But, as I describe below, both Cornell potential and the AdS/QCD expression turn out to have polynomial expansions very closely related to a rare and very unique number theory function. Chances of this similarity happening by some ‘accident’ of theory very low in my opinion. Cant think of many places in fundamental physics or even physics in general where similar expressions occur , as the associated force and physics it creates is very unique to the strong force. Some approximations may invoke similar leading terms, especially those for density (whether it be galxies, air density, particle density etc.) (think in the case approximation that uses r^6 and r^12 in some condensed matter thing, forget what its called, some other appxs for same quantity use similar leading terms) but those are very inexact macroscopic approximations, not fundamental physics.*

I noticed the relevant integral equations were similar in form to number-theoretic ones. The leading order terms (ie the actual Cornell Potential), do in fact turn out to be closely related to the log integral, a famous number theory function, and its variations. Main difference seems to be of the coefficients. Log intergal fn has integer coefficients, with integer coefficients in the asymptotic expansions of modifications to it (e.g. li(x) / [ln(x) *x])as well. And these coefficients are usually very nicely proportioned, e.g. li(x) / [ln(x) *x] $\sim 1 + \frac{1}{\ln(x)} + \frac{2}{[\ln(x)]^2} + \frac{6}{[\ln(x)]^3} + \ldots$..

To the contrary, the coefficients determined phenomenolgically in the form of Cornell Potential are valued thusly:

```
V[r_] := p (-κ0/r + (σ0 (r))) (* + higher order terms *)
(*^ Small r appx*)
p = 0.94;
κ0 = 0.23;
σ0 = 0.16;(* GeV^2 *)
σ = 0.19; (* GeV^2 *)
c = 0.9; (* GeV^2 *)
(* Appx numerical values *)
```

With the sigmas being two possible values, and the values being somewhat inexact, ie might be irrational have more digits or could even turn out to be some nicer fraction or whatever . Would need more experiments to constrain those.

We can see that the coefficients of these leading terms have a proportion of either $\sim \frac{.23}{.16} = 1.4375$ or $\sim \frac{.23}{.19} = 1.21052631579$, not as nice (but could turn out to be somewhat nice fractions if by some good fortune (basically a miracle in particle physics) these coefficients turns out to be fractions) but not being a fraction or a ‘nice number’ doesn’t really mean much as far as the physics goes, it would just be nice if they were. Of course there’s also that p = 0.94 in front so these proportions may not be at all helpful in determining a generative fn for the expansion (ie a fn or series fn that would determine the polynomial coefficients).

Finally: fact that its so similar is **very important**. Not only would it an awesome intersection of particle physics with number theory, it could allow us to try and make an exact expression for the actual expansion and try the **predict theoretically the exact values of the coefficients**. Then we could compare those to experiments and it would make such experiments more worth doing.
Also interesting for number theory, does provide a bit of a link of number theory fns to these complex AdS geometries.

Maybe an equation with such leading order terms can, aside from the AdS/QCD expressions, be ascribed by or closely to a one of the generalizations/variations of logarithmic integral function. See some here https://en.wikipedia.org/wiki/List_of_integrals_of_logarithmic_functions

As a distribution

```
Expectation[Abs[2 x - 1], x ≈ ExponentialDistribution[λ]]
```

$$\frac{e^{-\lambda/2}\left(4 - 2\, e^{\lambda/2} + e^{\lambda/2}\,\lambda\right)}{\lambda}$$

```
ExponentialDistribution[λ]
```

```
ExponentialDistribution[λ]
```

```
Expectation[Abs[2 Log[x] - 1], x ≈ ExponentialDistribution[λ]]
```

$1 + 2\,\text{EulerGamma} + 2\,\text{Gamma}\left[0, \sqrt{e}\,\lambda\right] + 2\,\text{Log}[\lambda] + 2\,\text{MeijerG}\left[\{\{\}, \{1\}\}, \{\{0, 0\}, \{\}\}, \sqrt{e}\,\lambda\right]$

```
Expectation[Abs[2 x - 1], x ≈ NormalDistribution[]]
```

$$\frac{e^{1/8} + 4\sqrt{\frac{2}{\pi}} + e^{1/8}\,\text{Erf}\left[\frac{1}{2\sqrt{2}}\right] - e^{1/8}\,\text{Erfc}\left[\frac{1}{2\sqrt{2}}\right]}{2\,e^{1/8}}$$

```
Expectation[Log[x], x ≈ ExponentialDistribution[λ]]
```

$-\text{EulerGamma} - \text{Log}[\lambda]$

```
Erf[x] // N
```

Erf[x]

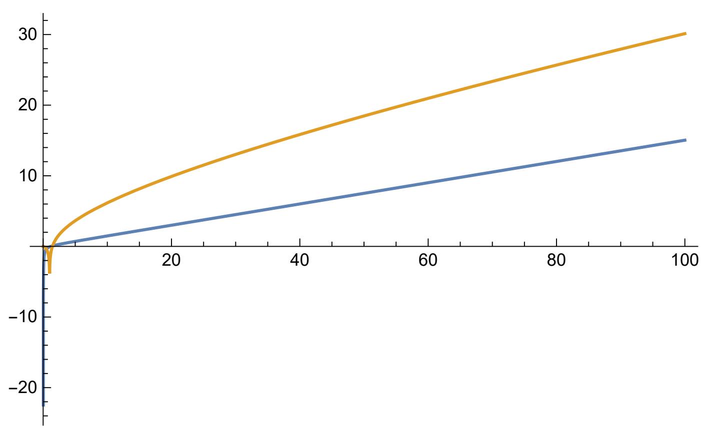
```
Plot[Beta[s, 2, 3], {s, 0, 10}]
```

Cornell Potential

$$V[r\_] := p\left(-\frac{\kappa 0}{r} + (\sigma 0\,(r))\right)\ (*\ +\ \text{higher order terms}\ *)$$

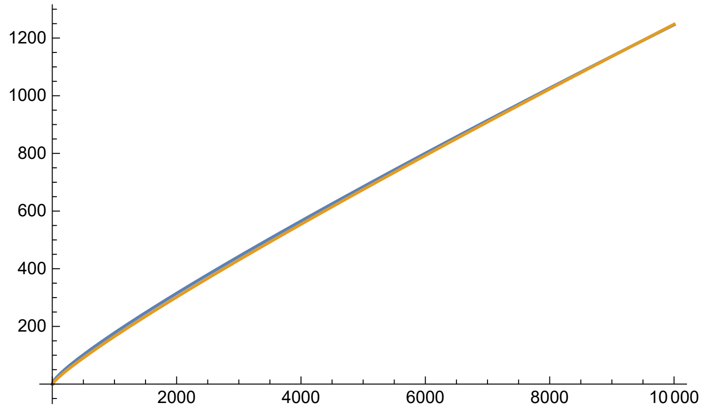
```
(*^ Small r appx*)
p = 0.94;
κ0 = 0.23;
σ0 = 0.16;(* GeV^2 *)
σ = 0.19;(* GeV^2 *)
c = 0.9; (* GeV^2 *)
(* Appx numerical values *)
```

```
Plot[{V[r], LogIntegral[r]}, {r, 0, 100}]
```



$$\text{PotIntegrand}[\nu\_, \lambda\_] := \frac{1}{\nu^2}\left(-1 + \frac{e^{\frac{\lambda \nu^2}{2}}}{\sqrt{1 - e^{\lambda(1-\nu^2)}\,\nu^4}}\right);$$

```
Plot[{LogIntegral[r],
  r / (Log[r] - Log[(Log[r])] + Log[Log[(Log[r])]] - Log[(Log[Log[(Log[r])]])])}, {r,
  0, 10000}]
```

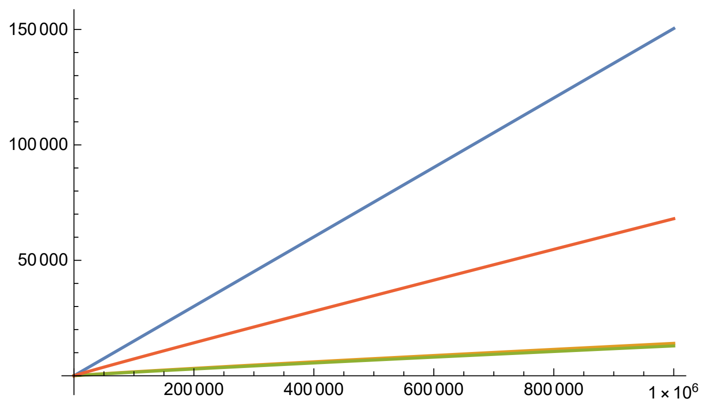



```
(*Ansatz =(p*κ0) LogIntegral[r];(* Ansatz for V[r] dependence on r *)*)
```

```
Log[(Log[10000000000])] // N
```

3.13662

$$\texttt{Plot}\Big[\Big\{V[r],\ (p*\sigma)\ \texttt{LogIntegral}[r],\ (p*\sigma)\left(\frac{r}{\texttt{Log}[r]}\right),\ (p*\sigma)\ \frac{r}{\texttt{Log}[(\texttt{Log}[r])]}\Big\},\ \{r,\ 0,\ 1000000\}\Big]$$




```
PotIntegrand[ν, λ]
PotIntegrand[LogIntegral[r], λ] // FullSimplify
```

$$\frac{-1+\frac{e^{\frac{\lambda\,\nu^2}{2}}}{\sqrt{1-e^{\lambda\,(1-\nu^2)}\,\nu^4}}}{\nu^2}$$

$$\frac{-1+\frac{e^{\frac{1}{2}\lambda\,\texttt{LogIntegral}[r]^2}}{\sqrt{1-e^{\lambda-\lambda\,\texttt{LogIntegral}[r]^2}\,\texttt{LogIntegral}[r]^4}}}{\texttt{LogIntegral}[r]^2}$$

```
Integrate[PotIntegrand[LogIntegral[r], 0], {r, 0, 1}]
```

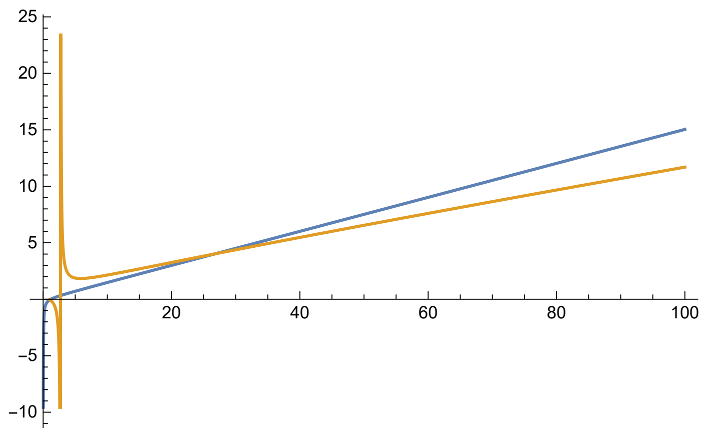
$$\int_0^1 \frac{-1+\frac{1}{\sqrt{1-\text{LogIntegral}[r]^4}}}{\text{LogIntegral}[r]^2}\,dr$$

$$-1+\frac{1}{\sqrt{1-\text{LogIntegral}[r]^4}} = \text{LogIntegral}[r]^2$$

```
D[LogIntegral[r], {r, 1}]
```

$$\frac{1}{\text{Log}[r]}$$

$$\text{Plot}\left[\left\{V[r],\ (p*\sigma)\ \frac{r}{\text{Log}[(\text{Log}[r])]}\right\},\ \{r, 0, 100\}\right]$$



$$\text{Integrate}\left[\frac{1}{\text{Log}[t]},\ \{t, 0, x\}\right]$$

ConditionalExpression[LogIntegral[x], Re[x] ≤ 1 || x ∉ Reals]

$$\text{Integrate}\left[\frac{1}{\text{Log}[\text{Log}[t]]},\ \{t, 0, x\}\right]$$

$$\int_0^x \frac{1}{\text{Log}[\text{Log}[t]]}\,dt$$

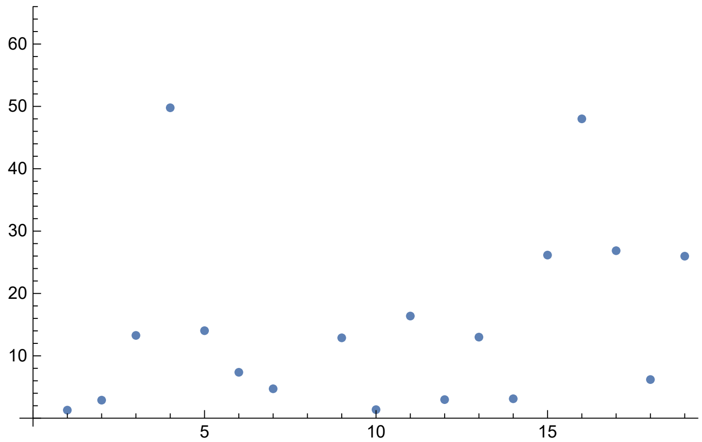
$$\text{Integrate}\left[\frac{1}{(\text{Log}[t]\,\hat{}\,2)},\ \{t, 0, x\}\right]$$

$$\text{ConditionalExpression}\left[-\frac{x}{\text{Log}[x]}+\text{LogIntegral}[x],\ \text{Re}[x] \le 1\ ||\ x \notin \text{Reals}\right]$$

$$\text{list1} = \text{Table}\left[\text{Abs}\left[\text{NIntegrate}\left[\frac{1}{\text{Log}[\text{Log}[t]]},\ \{t, 0, x\}\right]\right],\ \{x, 2, 20\}\right]$$

{1.2929, 2.89164, 13.2742, 49.7807, 14.0332, 7.35991, 4.71906, 1692.32, 12.8875, 1.37097,
16.3802, 2.98056, 13.0007, 3.11831, 26.1565, 48.0036, 26.8611, 6.19622, 25.9789}

```
ListPlot[list1]
```




```
Table[NIntegrate[1/LogIntegral[t], {t, 0, x}], {x, 2, 10}]
```

$\{-1.12413\times10^{10}, -1.12412\times10^{10}, -1.12411\times10^{10}, -1.12411\times10^{10}, -1.12411\times10^{10}, -1.1241\times10^{10}, -1.1241\times10^{10}, -1.1241\times10^{10}, -1.1241\times10^{10}\}$

```
Assuming[x ∈ Reals, Integrate[1/Log[t], {t, 0, x}]]
```

ConditionalExpression[LogIntegral[x], 0 ≤ x < 1]

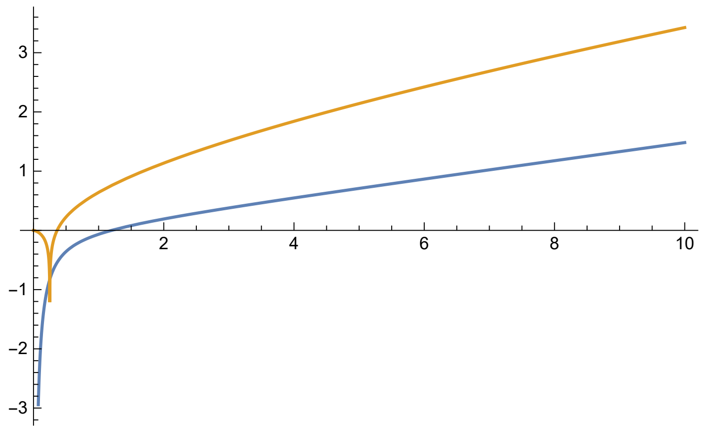
```
nnr = 100;
V[nnr]
(p * σ) nnr/Log[(Log[nnr])] // N
(nnr/Log[(Log[nnr])]) / V[nnr] // N
```

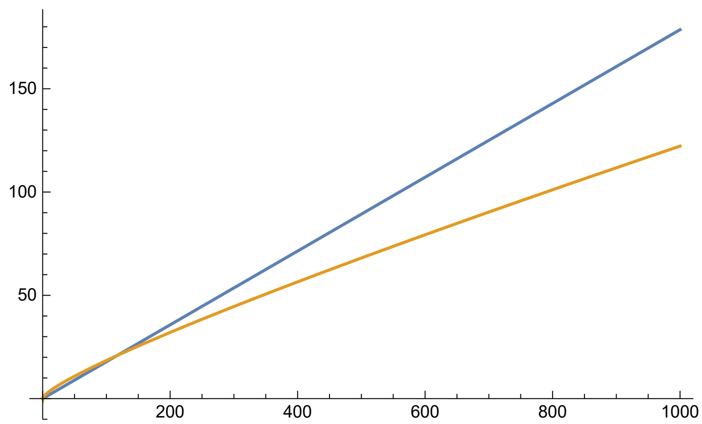
15.0378

11.6948

4.35436

```
Ansatz = (p * κ0) (LogIntegral[4 r]); (* Ansatz for V[r] dependence on r *)
Plot[{V[r], Ansatz}, {r, 0, 10}] (*Small r appx*)
Plot[{(p * σ) r, Ansatz}, {r, 0, 1000}] (*Large r appx*)
```





$$\texttt{PotIntegrand}[\nu\_,\ \lambda\_] := \frac{1}{\nu^2}\left(-1 + \frac{e^{\frac{\lambda \nu^2}{2}}}{\sqrt{1 - e^{\lambda\,(1-\nu^2)}\ \nu^4}}\right)$$

```
PotIntegrand[ν, λ]
PotIntegrand[LogIntegral[r], λ] // FullSimplify
```

$$\frac{-1 + \frac{e^{\frac{\lambda \nu^2}{2}}}{\sqrt{1 - e^{\lambda\,(1-\nu^2)}\ \nu^4}}}{\nu^2}$$

$$\frac{-1 + \frac{e^{\frac{1}{2}\lambda\,\texttt{LogIntegral}[r]^2}}{\sqrt{1 - e^{\lambda - \lambda\,\texttt{LogIntegral}[r]^2}\ \texttt{LogIntegral}[r]^4}}}{\texttt{LogIntegral}[r]^2}$$

```
Integrate[PotIntegrand[LogIntegral[r], 0], {r, 0, 1}]
```

$$\int_0^1 \frac{-1 + \frac{1}{\sqrt{1 - \texttt{LogIntegral}[r]^4}}}{\texttt{LogIntegral}[r]^2}\, dr$$

```
D[LogIntegral[r], {r, 1}]
```

$$\frac{1}{\text{Log}[r]}$$

```
PotIntegrand[ν, λ]
```

$$\frac{-1+\frac{e^{\frac{\lambda \nu^2}{2}}}{\sqrt{1-e^{\lambda\left(1-\nu^2\right)}\nu^4}}}{\nu^2}$$

```
Table[NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}], {ν, 1, 20}]
```

```
{NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}], NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}],
 NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}], NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}],
 NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}], NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}],
 NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}], NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}],
 NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}], NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}],
 NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}], NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}],
 NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}], NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}],
 NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}], NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}],
 NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}], NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}],
 NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}], NIntegrate[PotIntegrand[ν, λ], {ν, 0, 1}]}
```

```
e^9 // N
e^(3^2) // N
e^3 (3^2) // N
```

8103.08

8103.08

180.77

```
Series[e^(LogIntegral[r]^2), {r, 1, 3}, Assumptions → r > 1]
Series[LogIntegral[r]^2, {r, 1, 1}, Assumptions → r > 1]
```

$e^{\text{LogIntegral}[r]^2}$

$(\text{EulerGamma} + \text{Log}[-1+r])^2 + (\text{EulerGamma} + \text{Log}[-1+r])\,(r-1) + O[r-1]^2$

```
Normal[(EulerGamma + Log[-1 + r])^2 + (EulerGamma + Log[-1 + r]) (r - 1) + O[r - 1]^2] //
 FullSimplify
```

$(\text{EulerGamma} + \text{Log}[-1+r])\,(-1 + \text{EulerGamma} + r + \text{Log}[-1+r])$

```
Series[Exp[ν], {ν, 1, 1}, Assumptions → ν > 1] /.
   ν → (EulerGamma + Log[-1 + r]) (-1 + EulerGamma + r + Log[-1 + r]) // Normal
```

$e + e\,((\text{EulerGamma} + \text{Log}[-1+r])\,(-1 + \text{EulerGamma} + r + \text{Log}[-1+r]) - 1) +$
$O[(\text{EulerGamma} + \text{Log}[-1+r])\,(-1 + \text{EulerGamma} + r + \text{Log}[-1+r]) - 1]^2$

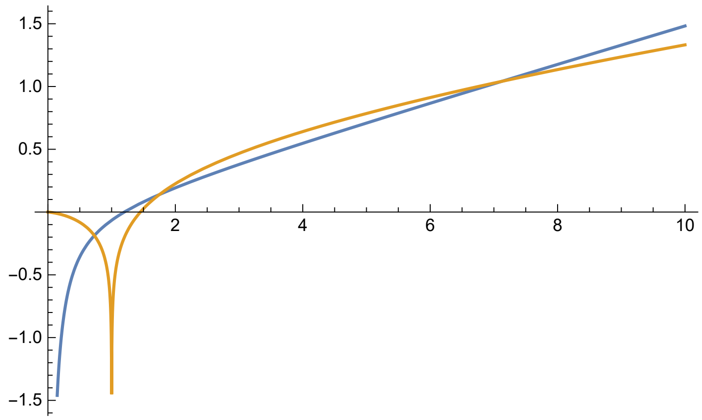
```
p * κ0
p * σ0
p * σ
```

0.2162

0.1504

0.1786

```
Plot[{V[r], .216 LogIntegral[r]}, {r, 0, 10}]
```




```
Series[LogIntegral[r], {r, 1, 1}, Assumptions → r > 1]
```

$$(\text{EulerGamma} + \text{Log}[-1 + r]) + \frac{r - 1}{2} + O[r - 1]^2$$

```
Series[0.94 LogIntegral[r], {r, 1, 1}, Assumptions → r > 1]
```

$$(0.542583 + 0.94\,\text{Log}[-1 + r]) + 0.47\,(r - 1) + O[r - 1]^2$$

Electric potential energy between two protons in Lorentz-Heaviside units:

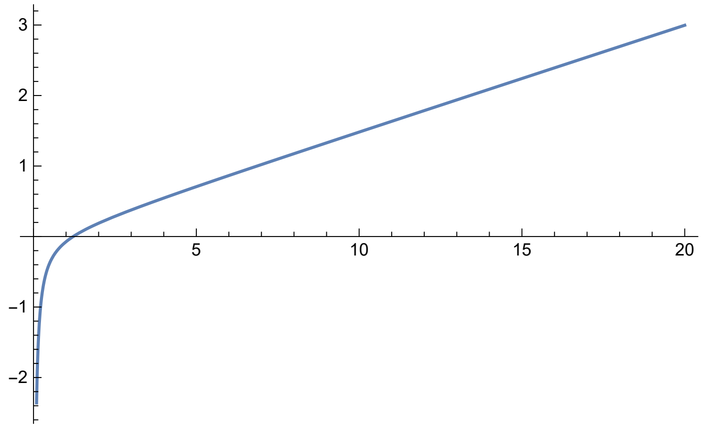
$\alpha = \frac{1}{137}$

Abs[e] = $\sqrt{4\pi\alpha}$

F = $\frac{1}{(4\pi)}\frac{e\char`^2}{r\char`^2}$

$$\text{ElecPot}[r\_] := \frac{(1/137)}{r}$$

```
Plot[{V[r] - ElecPot[r]}, {r, 0, 20}]
```



AdS/QCD

Integrals

```
h[ν_, λ_] := ((1 - ((ν^4) Exp[λ (1 - (ν^2))]))^(-1/2))
```

radius = $2\sqrt{\frac{\lambda}{c}}$ Integrate[(ν^2) Exp[$\frac{\lambda}{2}$ (1 - (ν^2))] h[ν, λ], {ν, 0, 1}]

$$2\sqrt{\frac{\lambda}{c}}\int_0^1 \frac{e^{\frac{1}{2}\lambda\left(1-\nu^2\right)}\,\nu^2}{\sqrt{1-e^{\lambda\left(1-\nu^2\right)}\,\nu^4}}\,d\nu$$

$$\frac{e^{\frac{1}{2}\lambda\left(1-\nu^2\right)}\,\nu^2}{\sqrt{1-e^{\lambda\left(1-\nu^2\right)}\,\nu^4}} = \frac{1}{\sqrt{\frac{e^{-\lambda\left(1-\nu^2\right)}}{\nu^4}-1}}$$

let $t = \frac{1}{\sqrt{\frac{e^{-\lambda\left(1-\nu^2\right)}}{\nu^4}-1}}$

so that

$$\text{radius} = 2\sqrt{\frac{\lambda}{c}}\int_0^1 t\,d\nu$$

Integrate[(ν^(-2)) ((Exp[$\frac{\lambda}{2}$ ((ν^2))] h[ν, λ]) - 1), {ν, 0, 1}]

$$\int_0^1 \frac{-1+\frac{e^{\frac{\lambda\nu^2}{2}}}{\sqrt{1-e^{\lambda\left(1-\nu^2\right)}\,\nu^4}}}{\nu^2}\,d\nu$$

let $t = \frac{1}{\sqrt{\frac{e^{-\lambda\left(1-\nu^2\right)}}{\nu^4}-1}}$

$$\text{PotentialIntegrand} = \frac{1}{\nu^2}\left(-1 + \frac{e^{\frac{\lambda \nu^2}{2}}}{\sqrt{1 - e^{\lambda (1-\nu^2)}\,\nu^4}}\right) =$$

$$\frac{1}{\nu^2}\left(-1 + \frac{e^{\frac{\lambda \nu^2}{2}}}{\nu^2 \sqrt{\frac{1}{\nu^4} - e^{\lambda (1-\nu^2)}}}\right) = \frac{1}{\nu^2}\left(-1 + \frac{e^{\frac{\lambda \nu^2}{2}}}{\nu^2 e^{\frac{1}{2}\lambda (1-\nu^2)} \sqrt{\frac{e^{-\lambda (1-\nu^2)}}{\nu^4} - 1}}\right)$$

$$= \frac{1}{\nu^2}\left(-1 + \frac{e^{\frac{\lambda \nu^2}{2}}}{\nu^2 e^{\frac{1}{2}\lambda (1-\nu^2)}}\, t\right) = \frac{1}{\nu^2}\left(-1 + \frac{e^{\frac{\lambda \nu^2}{2}}}{\nu^2 e^{\frac{1}{2}\lambda}\; e^{-\frac{1}{2}\lambda \nu^2}}\, t\right) =$$

$$\frac{1}{\nu^2}\left(-1 + \frac{e^{\lambda \nu^2}}{\nu^2 e^{\frac{1}{2}\lambda}}\, t\right) = \frac{1}{\nu^2}\left(-1 + \frac{e^{-\frac{1}{2}\lambda (1-2\nu^2)}}{\nu^2}\, t\right)$$

$$\text{PotentialIntegrand} = -\frac{1}{\nu^2} + \frac{e^{-\frac{1}{2}\lambda (1-2\nu^2)}}{\nu^4}\, t = -\frac{1}{\nu^2} + \frac{e^{-\lambda\left(\frac{1}{2}-\nu^2\right)}}{\nu^4}\, t = -\frac{1}{\nu^2} + \frac{1}{\nu^4}\, t\, e^{-\lambda\left(\frac{1}{2}-\nu^2\right)}$$

$$e^{-\lambda\left(\frac{1}{2}-\nu^2\right)} = e^{\lambda \nu^2}\, e^{-\frac{1}{2}\lambda}$$

$$e^{-\lambda (1-\nu^2)} = e^{\lambda\nu^2}\, e^{-\lambda} = e^{\lambda\nu^2}\, e^{-\frac{1}{2}\lambda}\; e^{-\frac{1}{2}\lambda} = \left(e^{-\lambda\left(\frac{1}{2}-\nu^2\right)}\right) e^{-\frac{1}{2}\lambda}$$

$$t = \frac{1}{\sqrt{\frac{e^{-\lambda (1-\nu^2)}}{\nu^4} - 1}}$$

$$\Rightarrow \frac{e^{-\lambda (1-\nu^2)}}{\nu^4} = \frac{1}{t^2} + 1$$

$$\Rightarrow e^{-\lambda (1-\nu^2)} = \left(\frac{1}{t^2} + 1\right)\nu^4$$

$$\Rightarrow e^{-\lambda (1-\nu^2)} = \left(e^{-\lambda\left(\frac{1}{2}-\nu^2\right)}\right) e^{-\frac{1}{2}\lambda} = \left(\frac{1}{t^2} + 1\right)\nu^4$$

$$\Rightarrow e^{-\lambda\left(\frac{1}{2}-\nu^2\right)} = \left(\frac{1}{t^2} + 1\right)\nu^4\, e^{\frac{1}{2}\lambda}$$

$$\Rightarrow \frac{1}{\nu^4}\, e^{-\lambda\left(\frac{1}{2}-\nu^2\right)} = \left(\frac{1}{t^2} + 1\right) e^{\frac{1}{2}\lambda}$$

we can note that $e^{\lambda \nu^2} =$

$D\left[\frac{1}{\nu^4}\, e^{\lambda \nu^2}, \{\lambda, 2\}\right]$ in case we later want to use parameter differentiation unde integral

Then

PotentialIntegrand $= -\frac{1}{\nu^2} + \frac{1}{\nu^4}\, t\, e^{-\lambda\left(\frac{1}{2}-\nu^2\right)} = -\frac{1}{\nu^2} + \frac{1}{\nu^4}\, t\left(\frac{1}{t^2}+1\right)\nu^4\, e^{\frac{1}{2}\lambda}$

PotentialIntegrand $= -\frac{1}{\nu^2} + \left(\frac{1}{t}+t\right) e^{\frac{1}{2}\lambda}$

Can already see resemblance to first order terms in $r \to 0$ limit

Note that

radius $= 2\sqrt{\frac{\lambda}{c}}\int_0^1 t\, d\nu = A\int_0^1 t\, d\nu$ by defn of t and letting $A = 2\sqrt{\frac{\lambda}{c}}$

$V(r) = \frac{p}{\pi}\sqrt{\frac{\lambda}{c}}\left(\int_0^1 (\text{PotentialIntegrand})\, d\nu - 1\right)$

for PotentialIntegrand $= -\frac{1}{\nu^2} + \left(\frac{1}{t}+t\right) e^{\frac{1}{2}\lambda}$

Then

$$\int_0^1 (\text{PotentialIntegrand})\, d\nu = \int_0^1 \left(-\frac{1}{\nu^2} + \left(\frac{1}{t}+t\right) e^{\frac{1}{2}\lambda}\right) d\nu$$

$$= \int_0^1 \left(-\frac{1}{\nu^2}\right) d\nu + \int_0^1 \left(\frac{1}{t}+t\right) e^{\frac{1}{2}\lambda}\, d\nu$$

$$= \int_0^1 \left(-\frac{1}{\nu^2}\right) d\nu + \int_0^1 \left(\frac{1}{t}+t\right) e^{\frac{1}{2}\lambda}\, d\nu$$

Note that first integral diverges,
but we ' re doing this in nonpertubative regime so drop that term

And for the second integral use the fact that λ is independent of ν and $r = A\int_0^1 t\, d\nu$

Then

$$\int_0^1 \left(\frac{1}{t}+t\right) e^{\frac{1}{2}\lambda}\, d\nu =$$

$$e^{\frac{1}{2}\lambda}\int_0^1 \left(\frac{1}{t}+t\right) d\nu = e^{\frac{1}{2}\lambda}\int_0^1 \left(\frac{1}{t}\right) d\nu + e^{\frac{1}{2}\lambda}\int_0^1 (t)\, d\nu = e^{\frac{1}{2}\lambda}\int_0^1 \left(\frac{1}{t}\right) d\nu + e^{\frac{1}{2}\lambda}\frac{r}{A}$$

$$= e^{\frac{1}{2}\lambda}\int_0^1 \left(\frac{1}{t}\right) d\nu + e^{\frac{1}{2}\lambda}\frac{r}{A}$$

For low t we have approx $\left(\frac{1}{t}+t\right) \approx \frac{1}{\text{Log}[t+1]}$ (See plot below)

Suggests logarithm is pertubative form

So $\int_0^1 \left(\frac{1}{t} + t\right) d\nu \;\approx\; \int_0^1 \frac{1}{\text{Log}[t+1]} d\nu$

$$\text{Log}[t+1] = \text{Log}\left[\frac{1}{\sqrt{\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1}} + 1\right] =$$

$$\text{Log}\left[\frac{1 + \sqrt{\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1}}{\sqrt{\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1}}\right] = \text{Log}\left[1 + \sqrt{\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1}\right] - \frac{1}{2}\,\text{Log}\left[\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1\right]$$

Note $t \to 0$ as $\nu \to 0$ (so low t appx holds for low ν) and $t \to \infty$ as $\nu \to 1$

$$t = \frac{1}{\sqrt{\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1}}$$

$$\int_0^1 \left(\frac{1}{t}\right) d\nu = \int_0^1 \sqrt{\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1}\; d\nu$$

$$\left(\frac{1}{t} + t\right) = \frac{1}{\sqrt{\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1}} + \sqrt{\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1} =$$

$$\frac{\left(1 + \left(\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1\right)\right)}{\sqrt{\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1}} = \frac{\left(\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4}\right)}{\sqrt{\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1}} = \frac{1}{\frac{\nu^4}{e^{-\lambda\,(1-\nu^2)}}\sqrt{\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4} - 1}}$$

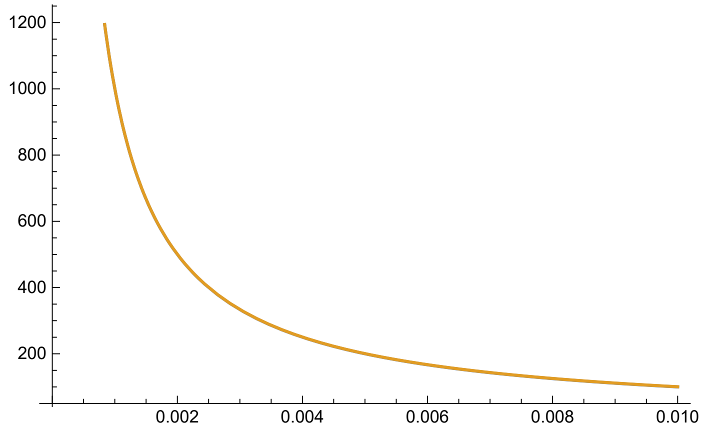
Integrate[Log[$\frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4}$ - 1], {ν, 0, 1}]

$$\int_0^1 \text{Log}\left[-1 + \frac{e^{-\lambda\,(1-\nu^2)}}{\nu^4}\right] d\nu$$

$$\text{Series}\left[\frac{1}{\sqrt{\frac{e^{-\lambda(1-\nu^2)}}{\nu^4}-1}}+\sqrt{\frac{e^{-\lambda(1-\nu^2)}}{\nu^4}-1},\ \{\nu, 0, 2\}\right]$$

$$\frac{\sqrt{e^{-\lambda}}}{\nu^2}+\frac{1}{2}\sqrt{e^{-\lambda}}\,\lambda+\frac{1}{8}\sqrt{e^{-\lambda}}\left(4\,e^{\lambda}+\lambda^2\right)\nu^2+O[\nu]^3$$

$$\text{Plot}\left[\left\{\left(\frac{1}{t}+t\right),\ \frac{1}{\text{Log}[t+1]}\right\},\ \{t, 0, .01\}\right]$$



$$\text{Series}\left[\frac{1}{\text{Log}[(1-x)]},\ \{x, 0, 5\}\right]$$

$$\text{Series}\left[\frac{1}{\text{Log}[(1+x)]},\ \{x, 0, 5\}\right]$$

$$\text{Series}\left[\frac{1}{\text{Log}[(1-x)]},\ \{x, 0, 5\}\right]-\text{Series}\left[\frac{1}{\text{Log}[(1+x)]},\ \{x, 0, 5\}\right]$$

$$-\frac{1}{x}+\frac{1}{2}+\frac{x}{12}+\frac{x^2}{24}+\frac{19\,x^3}{720}+\frac{3\,x^4}{160}+\frac{863\,x^5}{60480}+O[x]^6$$

$$\frac{1}{x}+\frac{1}{2}-\frac{x}{12}+\frac{x^2}{24}-\frac{19\,x^3}{720}+\frac{3\,x^4}{160}-\frac{863\,x^5}{60480}+O[x]^6$$

$$-\frac{2}{x}+\frac{x}{6}+\frac{19\,x^3}{360}+\frac{863\,x^5}{30240}+O[x]^6$$

$$\text{Series}\left[\text{Log}\left[\frac{1}{(1-x)}\right],\ \{x, 0, 5\}\right]$$

$$\text{Sum}\left[\frac{\text{Log}[(x)\,\hat{}\,(n)]}{n},\ \{n, 1, 5\}\right]$$

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+O[x]^6$$

$$\text{Log}[x]+\frac{\text{Log}[x^2]}{2}+\frac{\text{Log}[x^3]}{3}+\frac{\text{Log}[x^4]}{4}+\frac{\text{Log}[x^5]}{5}$$

$$\text{Sum}\left[\frac{\text{Log}[(x)\,\hat{}\,(n)]}{n}, \{n, 1, 5\}\right] =$$

$$\text{Log}[x] + \frac{\text{Log}[x^2]}{2} + \frac{\text{Log}[x^3]}{3} + \frac{\text{Log}[x^4]}{4} + \frac{\text{Log}[x^5]}{5} = \text{Log}[x]\left(1 + \frac{2}{2} + \frac{3}{3} + \frac{4}{4} + \frac{5}{5}\right) = 5\,\text{Log}[x]$$