Generalizations of the Riemann Zeta Function, their Numerical Approximations, and their Applications to Physics

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Generalizations of the Riemann zeta function are explored. Numerical methods are used to find values of functions examined here, and explore the validity of relations found. Numerical methods are found to be necessary, since the described functions involve infinite sums that don't simplify well algebraically. Furthermore the physical applications of the topics discussed are explored. In particular the extended zeta functions are used to explore the possibility of observable negative energy states in quantum statistical mechanics. It's found numerically that Bose-Einstein systems can only access negative energy states when the dimensionality of space, D, is of the form $D = 2(n+1) = \{4, 6, 8, 10, ...\}, n \in \mathbb{Z}^+$.

I. INTRODUCTION

The Riemann zeta function [1] is a well known function in mathematics given by

$$\zeta(s) = \sum_{a=1}^{\infty} \frac{1}{a^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + ... \tag{1}$$

for $s \in \mathbb{C}$. Alternatively via an integral transform (specifically the Mellin transform) it can be written as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \tag{2}$$

where $\Gamma(s)$ is the gamma function defined by

$$\Gamma(s) = (s-1)! = \int_0^\infty x^{s-1} e^{-x} dx \tag{3}$$

Some additional properties of the Riemann zeta function are

$$\lim_{s\to 1} (s-1)\zeta(s) = 1 \tag{4}$$

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!} \tag{5}$$

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \tag{6}$$

$$\zeta(s) = 2^s \pi^{s-1} \ \sin\left(\frac{\pi s}{2}\right) \ \Gamma(1-s) \ \zeta(1-s) \tag{7}$$

Where $n \in \mathbb{N}$ and B_{2n} and B_{n+1} are Bernoulli numbers. This function is of great interest in pure mathematics, and plays roles in many different branches of mathematics and physics. The function also reveals many relations interesting in themselves such as the values of the sums $\sum_{a=1}^{\infty}(1) = 1+1+1+1+...$ and $\sum_{a=1}^{\infty}(a) = 1+2+3+4+...$ which can be found via the Riemann zeta function:

$$\zeta(0) = \sum_{a=1}^{\infty}(1) = 1+1+1+1+... = -\frac{1}{2} \tag{8}$$

$$\zeta(-1) = \sum_{a=1}^{\infty}(a) = 1+2+3+4+... = -\frac{1}{12} \tag{9}$$

These alone are astounding results, saying that the sum of an infinite amount of 1s is $-\frac{1}{2}$ and the sum of all natural numbers is $-\frac{1}{12}$. These results sharply contradict basic mathematical intuition, but it's important to note that basic intuition can't necessarily be extended to sums of infinite amounts of numbers. These results, along with other sums obtained in the same way, can be, and have been, applied to physics from quantum field theory [4, 5] to string theory [6]. To see how these are used in the less theoretical field of cosmology see [7].

Due to the applicability of the Riemann zeta function to physics it can be useful to examine the function closer and study generalizations of it. Even if generalizations have no immediate applications to physics today, they would be of mathematical interest, and could eventually become a tool to do physical calculations. The most immediate generalization of note is the Hurwitz zeta function [2]:

$$\zeta(s,q) = \sum_{a=0}^{\infty} \frac{1}{(a+q)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-qx}}{1 - e^{-x}} dx \tag{10}$$

for $s, q \in \mathbb{C}$. The Hurwitz zeta function is distinguished from the Riemann zeta unction simply by a comma between the s and q input variables. Note that when $q = 1$ the Hurwitz zeta function reduces to the Riemann zeta function. Additional properties of the Hurwitz zeta function are

$$\lim_{q\to\infty} \zeta(s,q) = 0 \tag{11}$$

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$$\zeta(1-s,\frac{m}{n})=\frac{2\Gamma(s)}{(2\pi n)^s}\sum_{k=1}^{n}\cos\left(\frac{\pi s}{2}-\frac{2\pi km}{n}\right)\zeta(s,\frac{k}{n}) \tag{12}$$

II. GENERALIZED ZETA FUNCTION

For our purpose we want as general a function as possible, so that we can gain further insight into the properties of the Hurwitz and Riemann zeta functions and adapt the function to be more computationally useful in physics. To construct a more general function that the Hurwitz zeta function we can first extend the beginning of the sum to some value a_0 and change the end of the sum to a_f. To further generalize the function we can add a term bi to the a term and sum over b as we do a. The summation bounds for b are similarly b_0 and b_f. In effect this extends the a term in equations (1) and (10) from a sum of natural numbers to a sum of complex numbers with $\mathrm{Re}(a+bi)\in\mathbb{N}$, and $\mathrm{Im}(a+bi)\in\mathbb{N}$. Defining b as a separate summation index allows (10) to be obtained directly by letting $a_0=-\infty$, $a_f=\infty$, and $b_0=b_f=0$. Now we have

$$\beta(s,q,a_0,a_f,b_0,b_f)=\sum_{a=a_0}^{a_f}\sum_{b=b_0}^{b_f}\frac{1}{(a+bi+q)^s} \tag{13}$$

For $a,b,a_0,a_f,b_0,b_f\in\mathbb{N}$ and $s,q\in\mathbb{C}$. This sum will have singularities at $a_k+b_ji=-q$ $(j,k\in\{0,1,...,f\})$ but these singularities can be avoided by choosing particular a_0,a_f,b_0,b_f, by explicitly excluding the terms at which singularities occur, or by choosing q such that $\mathrm{Re}(q)\notin\mathbb{N}$ and/or $\mathrm{Im}(q)\notin\mathbb{N}$. Note that if we exclude the b terms, i.e. let $b_0=b_f=0$ then

$$\begin{aligned}\beta(s,q,a_0,a_f)&=\sum_{a=a_0}^{a_f}\frac{1}{(a+q)^s}\\&=\zeta(s,q+a_0)-\zeta(s,q+a_f+1)\end{aligned} \tag{14}$$

Now, letting $\beta\equiv\beta(s,q,a_0,a_f,b_0,b_f)$ equation (13) can be written as

$$\beta=\sum_{a=a_0}^{a_f}(\sum_{b=b_0}^{b_f}\frac{1}{(a+bi+q)^s})=\sum_{b=b_0}^{b_f}(\sum_{a=a_0}^{a_f}\frac{1}{(a+bi+q)^s}) \tag{15}$$

Where the parentheses are used to denote the order of summation in each term, i.e. in the first term the summation over the a terms is done before the b summation begins. Let

$$\beta_{ab}=\sum_{a=a_0}^{a_f}(\sum_{b=b_0}^{b_f}\frac{1}{(a+bi+q)^s}) \tag{16}$$

$$\beta_{ba}=\sum_{b=b_0}^{b_f}(\sum_{a=a_0}^{a_f}\frac{1}{(a+bi+q)^s}) \tag{17}$$

Now expand the interior sums

$$\beta_{ab}=\sum_{a=a_0}^{a_f}[\frac{1}{(a+b_0i+q)^s}+...+\frac{1}{(a+b_fi+q)^s}] \tag{18}$$

$$\beta_{ba}=\sum_{b=b_0}^{b_f}[\frac{1}{(a_0+bi+q)^s}+...+\frac{1}{(a_f+bi+q)^s}] \tag{19}$$

Notice that the b_j terms are constants in β_{ab} as are the a_k terms in β_{ba}. Since $q\in\mathbb{C}$ by definition we can absorb the a_k's and b_j's into the q terms as follows

$$\begin{aligned}\beta_{ab}&=\sum_{a=a_0}^{a_f}[\frac{1}{(a+(q+b_0i))^s}+...+\frac{1}{(a+(q+b_fi))^s}]\\&=\sum_{a=a_0}^{a_f}(\sum_{b=b_0}^{b_f}\frac{1}{(a+(q+bi))^s})\end{aligned} \tag{20}$$

$$\begin{aligned}\beta_{ba}&=\sum_{b=b_0}^{b_f}[\frac{1}{(bi+(q+a_0))^s}+...+\frac{1}{(bi+(q+a_f))^s}]\\&=\sum_{b=b_0}^{b_f}(\sum_{a=a_0}^{a_f}\frac{1}{(bi+(q+a))^s})\end{aligned} \tag{21}$$

This is done so that we can now apply the relation found in equation (14) to get

$$\begin{aligned}\beta_{ab}=\sum_{a=a_0}^{a_f}&[\zeta(s,(q+b_0i)+a)\\&-\zeta(s,(q+b_fi)+a+1)]\end{aligned} \tag{22}$$

$$\begin{aligned}\beta_{ba}=\sum_{b=b_0}^{b_f}&[\zeta(s,(q+a_0)+bi)\\&-\zeta(s,(q+a_f)+bi+1)]\end{aligned} \tag{23}$$

And by definition $\beta=\beta_{ab}=\beta_{ba}$ so

$$\begin{aligned}&\beta(s,q,a_0,a_f,b_0,b_f)\\&=\sum_{a=a_0}^{a_f}[\zeta(s,(q+b_0i)+a)-\zeta(s,(q+b_fi)+a+1)]\\&=\sum_{b=b_0}^{b_f}[\zeta(s,(q+a_0)+bi)-\zeta(s,(q+a_f)+bi+1)]\end{aligned} \tag{24}$$

These are rather surprising relations, as they relate the possibly finite sum defined in equation (13) to the infinite sum Hurwitz zeta function. These equations can also be used to find other interesting relations. For instance letting $b_0=b_f=0$ in equation (24) gives

$$\sum_{a=a_0}^{a_f}[\zeta(s,q+a)-\zeta(s,q+a+1)] = \zeta(s,q+a_0)-\zeta(s,q+a_f+1) \tag{25}$$

Which, when combined with (14), gives

$$\sum_{a=a_0}^{a_f}[\zeta(s,q+a)-\zeta(s,q+a+1)] = \sum_{a=a_0}^{a_f}\frac{1}{(a+q)^s} = (-1)^{-s}\zeta(s,1-q)+\zeta(s,q) \tag{26}$$

Implying that

$$\zeta(s,q+a)-\zeta(s,q+a+1) = \frac{1}{(a+q)^s} = (a+q)^{-s} \tag{27}$$

Note that a and q can be condensed into a single complex number. Equation (27) is a useful relation. It holds when the Hurwitz zeta function converges, i.e. for $\mathrm{Re}(s) > 1$, and $\mathrm{Re}(q) + \mathrm{Min}\{a, a+1\} > 0$ (for $\mathrm{Re}(q) + \mathrm{Min}\{a, a+1\} < 0$ the relation still holds for even integer values of s and is negated for odd integer values of s). This relation allows us to put equations of the form $(a+q)^{-s}$ in terms of the $\mathbb{C}$-analytic Hurwitz zeta function, which itself can be related to many other expressions [2, 3].

Also note the identities

$$\sum_{a=a_0}^{a_f}\frac{1}{(a+q)^s} = \zeta(s,b_0+q)-\zeta(s,b_f+q+1) \tag{28}$$

$$\sum_{b=b_0}^{b_f}\frac{1}{(bi+q)^s} = i^{-s}[\zeta(s,b_0-iq)-\zeta(s,b_f-iq+1)] \tag{29}$$

A. Zeta-Beta Function

Define a zeta function corresponding to $\beta(s,q,a_0,a_f,b_0,b_f)$ as

$$\begin{aligned}\zeta_\beta(s,q) &= \lim_{a_0,b_0\to-\infty}\lim_{a_f,b_f\to\infty}\beta(s,q,a_0,a_f,b_0,b_f)\\ &= \lim_{a_0,b_0\to-\infty}\lim_{a_f,b_f\to\infty}\sum_{a=a_0}^{a_f}\sum_{b=b_0}^{b_f}\frac{1}{(a+bi+q)^s}\\ &= \sum_{a=-\infty}^{\infty}\sum_{b=-\infty}^{\infty}\frac{1}{(a+bi+q)^s}\end{aligned} \tag{30}$$

Which, from equation (24), becomes

$$\begin{aligned}\zeta_{\beta a} &= \lim_{a_0,b_0\to-\infty}\lim_{a_f,b_f\to\infty}\sum_{a=a_0}^{a_f}[\zeta(s,(q+b_0 i)+a)\\ &\qquad -\zeta(s,(q+b_f i)+a+1)]\\ \zeta_{\beta b} &= \lim_{a_0,b_0\to-\infty}\lim_{a_f,b_f\to\infty}\sum_{b=b_0}^{b_f}[\zeta(s,(q+a_0)+bi)\\ &\qquad -\zeta(s,(q+a_f)+bi+1)]\end{aligned} \tag{31}$$

Where $\zeta_{\beta a} = \zeta_{\beta b} \equiv \zeta_\beta(s,q)$, the second subscript denoting the index of summation not yet done. For brevity let $Lims \equiv \lim_{a_0,b_0\to-\infty}\lim_{a_f,b_f\to\infty}$. Express (31) as

$$\begin{aligned}\zeta_{\beta a} &= Lims\sum_{a=a_0}^{a_f}\sum_{n=0}^{\infty}[\frac{1}{(n+q+b_0 i+a)^s}\\ &\qquad -\frac{1}{(n+q+b_f i+a+1)^s}]\\ \zeta_{\beta b} &= Lims\sum_{b=b_0}^{b_f}\sum_{n=0}^{\infty}[\frac{1}{(n+q+a_0+bi)^s}\\ &\qquad -\frac{1}{(n+q+a_f+1+bi)^s}]\end{aligned} \tag{32}$$

$$\begin{aligned}\zeta_{\beta a} &= Lims\sum_{a=a_0}^{a_f}\sum_{n=0}^{\infty}[\frac{1}{((n+a+\mathrm{Re}(q))+(b_0+\mathrm{Im}(q))i)^s}\\ &\qquad -\frac{1}{((n+a+\mathrm{Re}(q)+1)+(b_f+\mathrm{Im}(q))i)^s}]\\ \zeta_{\beta b} &= Lims\sum_{b=b_0}^{b_f}\sum_{n=0}^{\infty}[\frac{1}{((n+a_0+\mathrm{Re}(q))+(b+\mathrm{Im}(q)))^s}\\ &\qquad -\frac{1}{(n+a_f+\mathrm{Re}(q)+1)+(b+\mathrm{Im}(q))i)^s}]\end{aligned} \tag{33}$$

Giving

$$\begin{aligned}\zeta_{\beta a} &= Lims\sum_{a=a_0}^{a_f}\sum_{n=0}^{\infty}[(\frac{(n+a+\mathrm{Re}(q))-(b_0+\mathrm{Im}(q))i}{(n+a+\mathrm{Re}(q))^2+(b_0+\mathrm{Im}(q))^2})^s\\ &\qquad -(\frac{(n+a+\mathrm{Re}(q)+1)-(b_f+\mathrm{Im}(q))i}{(n+a+\mathrm{Re}(q)+1)^2+(b_f+\mathrm{Im}(q))^2})^s]\\ \zeta_{\beta b} &= Lims\sum_{b=b_0}^{b_f}\sum_{n=0}^{\infty}[(\frac{(n+a_0+\mathrm{Re}(q))-(b+\mathrm{Im}(q))i}{(n+a_0+\mathrm{Re}(q))^2+(b+\mathrm{Im}(q))^2})^s\\ &\qquad -(\frac{(n+a_f+\mathrm{Re}(q)+1)-(b+\mathrm{Im}(q))i}{(n+a_f+\mathrm{Re}(q)+1)^2+(b+\mathrm{Im}(q))^2})^s]\end{aligned} \tag{34}$$

Notice that these sums can't always be resolved analytically, since some terms will become indeterminate and others will go to ∞. It is therefore necessary to resolve them numerically. $\zeta_\beta(s,q)$ in particular is analyzed numerically in the accompanying file GenZeta.cpp. Notice that GenZeta.cpp can be resolved numerically. For example, for $s=2$ and $q=2.5$ GenZeta(2,2.5) = 4.82649, which is a rather nice value considering the nature of the function.

III. APPLICATIONS TO PHYSICS

The applications of the Riemann zeta function in physics are discussed rigorously in [4]. Here we'll look at applications of these generalized functions to physics.

Notice how in equation (2) the denominator of the integral takes the same form as the denominator in the mean particle number equation of Bose-Einstein Statistics. This, in part, allows the zeta function to be applied to Bose-Einstein Statistics. The most obvious extension of the Zeta function would therefore be to Fermi-Dirac statistics. In [8] the authors derive the equation

$$\frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x+1}dx = (1-2^{1-s})\zeta(s) \tag{35}$$

When combined with the identity $\zeta(s,1/2)-\zeta(s,3/2)=2^{-s}$ derived from equation (27) this becomes

$$\frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x+1}dx = [1-2(\zeta(s,1/2)-\zeta(s,3/2))]\zeta(s) \tag{36}$$

These show that the Fermi-Dirac distribution can also be realted to zeta functions.

Mathematically speaking, equation (27) is very applicable to physics, seeing as it offers more ways to manipulate functions. Its region of applicability $\mathrm{Re}(s)>1$, $\mathrm{Re}(q)+\mathrm{Min}\{a,a+1\}>0$, would apply to the large majority of physics problems, since in physics variables can very often be assumed to be positive.

A. Negative Energy States

In situations where the weak energy condition can't be assumed, i.e. where negative energy states are allowed, we need a method for summing the negative energy states along with the positive ones. The file RealZeta.cpp allows us to do the sum $\zeta(s,q)=\sum_{a=-\infty}^{\infty}(a+q)^{-s}$. Using equation (52) from ref [4] we know that the number of particles in a Bose-Einstein system, N, is proportional to $\zeta(D/2)$ where D is the number of spatial dimensions. Adding the negative energy states we have:

$$N \propto \zeta(D/2) + \sum_{a=-\infty}^{-1}(a)^{-D/2} \tag{37}$$

Or, letting $\zeta_{Neg}(s,q)=\sum_{a=-\infty}^{-1}(a+q)^{-s}$, and letting q be the energy of the ground state, this becomes

$$N \propto \zeta(D/2,q)+\zeta_{Neg}(D/2,q) \tag{38}$$

We can't have $D=0$ or $D=2$, since the Zeta functions diverges for $s=$ and $s=1$. Assume a zero-point energy of $q=(1/2)$ for each state. For $D=3$ we get $RealZeta(3/2,1/2)=-nan$, meaning that the output is complex or indeterminate, effectively meaning that we can't have infinite negative energy states for Bose-Einstein statistics, at least not in 3-D space. For $D=4$ we get $N=RealZeta(2,1/2)=9.8696$, so it is possible to have such an arrangement in 4-D space! So a Bose-Einstein system must be in at least 4-D space to access negative energy states.

This tells us that the fact that we don't have observable negative energy states in our universe is no accident, but a result of the dimensionality of space. More generally this tells us that any negative energy state accessing Bose-Einstein system must have an even dimensionality greater than 3, or $D=2(n+1)=\{4,6,8,10,...\}, n\in\mathbb{Z}^+$. For $D=4,8,12,16,...$, $\zeta(D/2,1/2)=\zeta_{Neg}(D/2,1/2)$ so $RealZeta(D/2,1/2)=2\zeta(D/2,1/2)$ and the availability of negative energy states simply doubles the number of particles in a quantum state (not the same as an energy state here), as would be expected. For $D=6,10,14,18,...$, $\zeta(D/2,1/2)=-\zeta_{Neg}(D/2,1/2)$ so $RealZeta(D/2,1/2)\approx 0$. This means that bosons aren't allowed in spaces of these dimensions. The fact that $|\zeta(D/2,1/2)|=|\zeta_{Neg}(D/2,1/2)|$ is itself surprising given that the positive q term takes away from the magnitude of the denominators of ζ_{Neg}.

If Bose-Einstein systems that should have some predetermined N_0 were observed as having $N\approx 2N_0$, or $N\approx 0N_0$, it could be indicative of the system accessing additional spatial dimensions, or at least acting as such.

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