If $n=0, Fib(0)=0$
If $n=1, Fib(1)=1$
For $n=0, \frac{\phi^0-\psi^0}{\sqrt{5}}=0$

For $n=1, \frac{\phi^1-\psi^1}{\sqrt{5}}=\frac{\frac{1+\sqrt{5}}{2}-\frac{1+\sqrt{5}}{2}}{\sqrt{5}}=1$

Base cases hold

Assume true for $n=k, n=k-1$
For $n=k+1$,

$$Fib(k+1)=\frac{\phi^k-\psi^k+\phi^{k-1}-\psi^{k-1}}{\sqrt{5}}$$

$$Fib(k+1)=\frac{\phi^{k-1}(\phi+1)-\psi^{k-1}(\psi+1)}{\sqrt{5}}$$

The properties of ϕ and ψ *are such that*

$$\phi+1=\phi^2 \text{ and } \psi+1=\psi^2$$

Therefore

$$Fib(k+1)=\frac{\phi^{k-1}\phi^2-\psi^{k-1}\psi^2}{\sqrt{5}}=\frac{\phi^{k+1}-\psi^{k+1}}{\sqrt{5}}$$

By induction, the proof holds for all n

Next we must prove that $Fib(n)$ *is the closest integer* to $\frac{\phi^n}{\sqrt{5}}$

This is equivalent to showing that $\left|\frac{\psi^n}{\sqrt{5}}\right|<\frac{1}{2}$

For the base case $n=0$,

$$\left|\frac{\psi^0}{\sqrt{5}}\right|=\frac{1}{\sqrt{5}}<\frac{1}{2}$$

Assuming $\left|\frac{\psi^k}{\sqrt{5}}\right|<\frac{1}{2}$,

Note that

$$\left|\frac{1-\sqrt{5}}{2}\right|<1$$

$$\psi^{k+1}=\psi^k\frac{1-\sqrt{5}}{2}$$

Therefore

$$\left|\frac{\psi^{k+1}}{\sqrt{5}}\right|<\left|\frac{\psi^k}{\sqrt{5}}\right|$$

By induction $\left|\frac{\psi^n}{\sqrt{5}}\right|<\frac{1}{2}$ *for all* n