2 Exercise

Computing the Parameter Update

Your task is to perform one update step of the weights and biases using the given data and multiclass logistic regression. The provided data includes a feature vector $\mathbf{x}$, initial weights matrix $\mathbf{W}$, bias vector $\mathbf{b}$, and the true class labels in one-hot encoded format y.

$$X = [1, 3, 0]$$

$$W = \begin{bmatrix} 0.3 & 0.1 & -2 \\ -0.6 & -0.5 & 2 \\ -1 & -0.5 & 0.1 \end{bmatrix}$$

$$b = [0.1,\ 0.1,\ 0.1]$$

$$y = [0, 1, 0]$$

① Begin by computing the linear combinations z for each class:

$$z = W^T x + b$$

$$z_1 = W_1^T x + b_1 = \begin{bmatrix} 0.3 \\ 0.1 \\ -2 \end{bmatrix} [1, 3, 0] + 0.1 = (0.3 \times 1) + (0.1 \times 3) + (-2 \times 0) + 0.1$$

$$= 0.3 + 0.3 + 0.1$$

$$= 0.7$$

$$z_2 = W_2^T x + b_2 = \begin{bmatrix} -0.6 \\ -0.5 \\ 2 \end{bmatrix} [1, 3, 0] + 0.1 = (-0.6 \times 1) + (-0.5 \times 3) + (2 \times 0) + 0.1$$

$$= -0.6 - 1.5 + 0.1$$

$$= -2$$

$$z_3 = W_3^T x + b_3 = \begin{bmatrix} -1 \\ -0.5 \\ 0.1 \end{bmatrix} [1, 3, 0] + 0.1 = (-1 \times 1) + (-0.5 \times 3) + (0.1 \times 0) + 0.1$$

$$= -1 - 1.5 + 0.1$$

$$= -2.4$$

② Apply the softmax function to get the predicted probabilities $\hat{y}$:

$$\hat{y} = \text{softmax}(z)$$

$$\hat{y}_1 = \text{softmax}(z_1) = \frac{\exp(z_1)}{\sum_{j=1}^{3} \exp(z_j)} = \frac{\exp(0.7)}{\exp(0.7) + \exp(-2) + \exp(-2.4)} = \frac{2.01}{2.01 + 0.14 + 0.09}$$

$$= \frac{2.01}{2.24} \cong 0.9$$

$$\hat{y}_2 = \text{softmax}(z_2) = \frac{\exp(z_2)}{\sum_{j=1}^{3} \exp(z_j)} = \frac{\exp(-2)}{\exp(0.7) + \exp(-2) + \exp(-2.4)} = \frac{0.14}{2.01 + 0.14 + 0.09}$$

$$= \frac{0.14}{2.24} \cong 0.06$$

$$\hat{y}_3 = \text{softmax}(z_3) = \frac{\exp(z_3)}{\sum_{j=1}^{3} \exp(z_j)} = \frac{\exp(-2.4)}{\exp(0.7) + \exp(-2) + \exp(-2.4)} = \frac{0.09}{2.01 + 0.14 + 0.09}$$

$$= \frac{0.09}{2.24} \cong 0.04$$

$$\Rightarrow \hat{y} = [0.9 \ , \ 0.06 \ , \ 0.04]$$

③ Compute the gradient of the loss with respect to z using the cross-entropy loss and the true labels y:

$$\nabla_z L = \hat{y} - y$$

$$\nabla_{z_i} L = [0.9\ ,\ 0.06\ ,\ 0.04] - [0\ ,\ 1\ ,\ 0]$$

$$\nabla_{z_i} L = [0.9\ ,\ -0.94\ ,\ 0.04]$$

(4) Now, compute the gradients with respect to the weights W and biases b:

$$\nabla_W L = \nabla_{z_i} L\, X^T$$

$$\nabla_b L = \nabla_{z_i} L$$

$$\nabla_{W_1} L = \nabla_{z_1} L\, X^T = 0.9 \times \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 2.7 \\ 0 \end{bmatrix}$$

$$\nabla_{W_2} L = \nabla_{z_2} L \cdot X^T = -0.94 \times \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.94 \\ -2.82 \\ 0 \end{bmatrix} \Rightarrow \nabla_W L = \begin{bmatrix} 0.9 & -0.94 & 0.04 \\ 2.7 & -2.82 & 0.12 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\nabla_{W_3} L = \nabla_{z_3} L \cdot X^T = 0.04 \times \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.04 \\ 0.12 \\ 0 \end{bmatrix}$$

$$\nabla_b L = [0.9\ ,\ -0.94\ ,\ 0.04]$$

(5) Finally, update the weights and biases using a learning rate η:

$$W \leftarrow W - \eta \nabla_W L$$

$$b \leftarrow b - \eta \nabla_b L$$

Let's consider $\eta = 0.1$

$$W = \begin{bmatrix} 0.3 & 0.1 & -2 \\ -0.6 & -0.5 & 2 \\ -1 & -0.5 & 0.1 \end{bmatrix} - 0.1 \begin{bmatrix} 0.9 & -0.94 & 0.04 \\ 2.7 & -2.82 & 0.12 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3 & 0.1 & -2 \\ -0.6 & -0.5 & 2 \\ -1 & -0.5 & 0.1 \end{bmatrix} - \begin{bmatrix} 0.09 & -0.094 & 0.004 \\ 0.27 & -0.282 & 0.012 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.21 & 0.194 & -2.004 \\ -0.87 & -0.218 & 1.988 \\ -1 & -0.5 & 0.1 \end{bmatrix}$$

$$b = [0.1\,,\ 0.1\,,\ 0.1] - 0.1\,[0.9\,,\ -0.94\,,\ 0.04]$$

$$= [0.1,\ 0.1,\ 0.1] - [0.09,\ -0.094,\ 0.004]$$

$$= [0.01,\ 0.19,\ 0.09]$$