Problem formulation and control strategy

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1 Problem description

Given an inertial system in the form of a CubeSat with uniform mass distribution, impose a control strategy that can achieve quaternion positioning and speed control with respect to a given reference.

1.1 Speed Control

The speed control implies that when a given reference is received, the system should:

- Accelerate
- Maintain constant angular velocity
- Decelerate
- Drive the angular velocity system state to "0"

For the control method, an error feedback must be implemented.

1.2 Quaternion control

Quaternion control is also based on a error (quaternion) feedback method that:

- Takes an **Euler angle** reference
- Updates the system **Directional Cosine Matrix**
- Converts angular data into quaternions

Inputs into the system **coupled dynamics** :

- Input quaternion positioning error into the nonlinear controller
- output control signal "?" (motor voltage?-this means implementing motor dynamics into the system plant)
- **Motor actuation is done inside the plant ?**
- Output system state : angular velocity
- Output system state : relative quaternion positioning?

The feedback error:

- Communicates at all times with **an absolute positioning sensor(IMU)**
  - IMU outputs Euler angles
  - Turn angle information into quaternions for the feedback loop

2 System dynamics

The system dynamics are characterized as such:

$$\dot{\omega}_{ob}^b = \hat{f}_{inert} + \hat{f}_\tau + \hat{f}_g + \hat{f}_{add}$$

$$\dot{\omega}_s = \bar{f}_{inert} + \bar{f}_\tau + \bar{f}_g$$

$$\dot{\eta} = -\frac{1}{2}\epsilon^T \omega_{ob}^b$$

$$\dot{\epsilon} = \frac{1}{2}\left[\eta\mathbf{1} + \epsilon^\times\right]\omega_{ob}^b$$

Where the component functions are:

$$\hat{f}_{inert} = \mathbf{J}^{-1}\left[-\left(\omega_{ob}^b - \omega_0 c_2\right)^\times \left(\mathbf{I}\left[\omega_{ob}^b - \omega_0 c_2\right] + A i_s \omega_s\right)\right]$$

$$\bar{f}_{inert} = -A^T\mathbf{J}^{-1}\left[-\left(\omega_{ob}^b - \omega_0 c_2\right)^\times \left(\mathbf{I}\left[\omega_{ob}^b - \omega_0 c_2\right] + A i_s \omega_s\right)\right]$$

$$\hat{f}_\tau = -\mathbf{J}^{-1}A\tau_a$$

$$\bar{f}_\tau = \left[A^T\mathbf{J}^{-1}A + i_s^{-1}\right]\tau_a$$

$$\hat{f}_g = \mathbf{J}^{-1}\left[3\omega_0^2\left(c_3\right)^\times \mathbf{I}c_3\right]$$

$$\bar{f}_g = -A^T\mathbf{J}^{-1}\left[3\omega_0^2\left(c_3\right)^\times \mathbf{I}c_3\right]$$

$$\hat{f}_{add} = \omega_0\dot{c}_2$$

I/O States		
Input	Output	Feedback Loop
1	$\dot{\omega}_{ob}^b$	$\dot{\omega}_{ob}^b$
τ_a	$\dot{\omega}_s$	$\dot{\omega}_s$
1	$\dot{\eta}$	q_o^b
1	$\dot{\epsilon}$	
1	R_o^b	R_o^b

3 Nonlinear control

As the linear control method has failed, in this current form, (The Jacobian matrix of the coupled dynamics is not full rank and therefore not all states are controllable), a solution may be to develop a method of nonlinear control. Nonlinear control is inherently more precise but also more computationally expensive. Hence this fact, through the nonlinear method we shall reach an informed conclusion about the system and later on revert back to a linear form (PWA approximation).

3.1 Global stability

A Lyapunov function candidate, should be chosen, this should be an energy-like function describing the correlation between the kinetic and potential energy of the system, all the while containing the essential system states that will later make possible the partial derivation along the system trajectories

Topland Morten, describes a LFC suited for reaction wheel control where **the reaction wheel angular velocity is omitted from the state space and used as an external signal** Hypothesis: this is done so that the Lyapunov function is highly convergent.

$$
\tau_c = -k_{\epsilon,1}\tilde{\epsilon} - \mathbf{C}\tilde{\omega} \tag{4.72a}
$$

$$
\begin{aligned}
\mathbf{A}\tau_a = k_{\epsilon,2}\tilde{\epsilon} + \mathbf{D}\tilde{\omega} + \omega_0(\mathbf{c}_2)^{\times}\mathbf{A}\mathbf{I}_s\left(\mathbf{A}^T\tilde{\omega} + \omega_s\right) \\
- \omega_0^2(\mathbf{c}_2)^{\times}\mathbf{I}\mathbf{c}_2 + 3\omega_0^2(\mathbf{c}_3)^{\times}\mathbf{I}\mathbf{c}_3
\end{aligned} \tag{4.72b}
$$

where $\mathbf{C}$ and $\mathbf{D}$ are constant matrices, and $k_{\epsilon,i}$ $(i = 1, 2)$ is a constant. The control law for τ_a cancels the nonlinearities in $\dot{V}$. After inserting (4.72) into (4.71), we get:

$$
\dot{V} = (k_3 - k_{\epsilon,1} - k_{\epsilon,2})\,\tilde{\omega}^T\tilde{\epsilon} - \tilde{\omega}^T\,(\mathbf{C} + \mathbf{D})\,\tilde{\omega} \tag{4.73}
$$

We choose $k_3 = k_{\epsilon,1} + k_{\epsilon,2}$, thus:

$$
\dot{V} = -\tilde{\omega}^T\,(\mathbf{C} + \mathbf{D})\,\tilde{\omega} \tag{4.74}
$$

If $(\mathbf{C} + \mathbf{D}) > 0$ then $\dot{V} \le 0$. To ensure this, we choose $\mathbf{C} = k_{\omega,1}\mathbf{1}$ and $\mathbf{D} = k_{\omega,2}\mathbf{1}$ where $k_{\omega,i}$ $(i = 1, 2)$ is a constant and $(k_{\omega,1} + k_{\omega,2}) > 0$. Thus $\tilde{\omega} \to 0 \Rightarrow \dot{\tilde{\omega}} \to 0$. We will now apply corollary 3.2. When $\dot{\tilde{\omega}} = \tilde{\omega} = 0$, (4.18) becomes $k_0\tilde{\epsilon} = 0 \Rightarrow \tilde{\epsilon} \to 0$. Thus corollary 3.2 states that the system is globally asymptotically stable.

Figure 1: Topland Lyapunov control law for reaction wheels

B. Spacecraft Attitude Error

The attitude error is originally represented in underlying quaternion error kinematics equations. Then different attitude error equations are developed using kinematic identities to formulate the nonlinear kinematics model. These developments are numerically verified to the machine error.

1. *Quaternion Error Kinematics*

The kinematic solution for the reference trajectory quaternion defines the desired rotational motion for the spacecraft. The quaternion rotational error is defined as[18,19]

$$
\delta\boldsymbol{q} = \boldsymbol{q} \otimes \hat{\boldsymbol{q}}^{-1} \tag{10}
$$

where $\hat{\boldsymbol{q}}^{-1}$ is quaternion inverse of the reference quaternion rotation and $\otimes$ represents the quaternion product. Note that the error, $\delta\boldsymbol{q}$, is a quaternion, that is, a unit-vectbr. The quaternion rotational error rate is represented as,

$$
\delta\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \otimes \hat{\boldsymbol{q}}^{-1} + \boldsymbol{q} \otimes \dot{\hat{\boldsymbol{q}}}^{-1} \tag{11}
$$

The quaternion kinematics evolve in time according to the differential equation

$$
\dot{\boldsymbol{q}} = \frac{1}{2}\begin{Bmatrix}\boldsymbol{\omega} \\ 0\end{Bmatrix} \otimes \boldsymbol{q} = \frac{1}{2}\begin{bmatrix}-[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^{\mathrm{T}} & 0\end{bmatrix}\boldsymbol{q} = \frac{1}{2}\Omega(\boldsymbol{\omega})\boldsymbol{q} \tag{12}
$$

Figure 2: Possible quaternion error solution...

3.1.1 Closing words

Even though a lyapunov controller has been proposed, it should be tested before or after the simulation. Nevertheless feedback control cannot be achieved until quaternion positioning error problem, stated in the last figure, is not assessed.

Moreover it should also be brought into discussion the possible need for a form of **interpolation** that will ensure a smooth transitioning between states.