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Conference Paper · August 2012

DOI: 10.2514/6.2012-4422

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AIAA/AAS Astrodynamics Specialist Conference
13 - 16 August 2012, Minneapolis, Minnesota

AIAA 2012-4422

A Survey Of Attitude Error Representations

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Several attitude error representations are developed for describing the tracking orientation error kinematics. Compact forms of attitude error equation are derived for each case. The attitude error is initially defined as the quaternion (rotation) error between the current and the estimated orientation. The nonlinear kinematic models are valid for arbitrarily large relative rotations and rotation rates. These modes have been developed for supporting the development of nonlinear spacecraft maneuver formulations. All of the kinematic formulations assume that a reference state has been defined. These results are expected to be broadly useful for generalizing extended Kalman filtering formulations. The benefits of paper are discussed.

I. Introduction

The diversity of attitude representations has been one of the important issues to be considered for science and engineering applications. Various attitude representations are available for use.[5,6,8,9,17] Selecting the appropriate representation is highly linked with the kind of the problem being considered. Euler parameters (quaternion) are a frequently used set of attitude variables, consisting of four components; non-singular, redundant coordinates to describe a rigid body orientation.[15] Classical Rodrigues Parameters (CRP's) is another attitude representation that consists of three components; after eliminating the redundant component. The transformation from classical Rodridues Parameters to Quaternion, and vice-versa, is easily performed, similarly for CRP's. The more recently developed Modified Rodrigues Parameters representation is a vector-based, three-component, attitude description. More about these representations and other representations can be found in many references.[5,6,9,12,15,17]For applications requiring large and rapid rotational motions there exists a need for developing attitude error kinematic models that exactly describe arbitrary large rotational motions. The main contribution of this work is to develop exact large motion error kinematic and dynamic representations using quaternions, CRP's and MRP's.

Several attitude parameterizations are compared by solving a nonlinear spacecraft tracking problem. Several authors have considered feedback control strategies. Markley and Coppola[12] have considered different attitude error representations for estimating the state of a maneuvering spacecraft. They have clarified the relationship between the four-component quaternion representation of attitude and the Multiplicative Extended Kalman Filter. Crassidis, Vadali, Markley, and Coppola[4] investigated a variable-structure control strategy for maneuvering vehicles. In their work, they used a feedback linearizing technique and added an additional term to the spacecraft maneuvers to deal with model uncertainties. The addition of the simple term in the control law always provides an optimal response. Ahmed, Coppola, and Bernstein[1] extended previous work to consider adaptive asymptotic tracking during maneuvers while estimating inertia properties. They used a Lyapunov argument to generate an unconditionally robust control law with respect to their assumed parametric uncertainty. Bani Younes, Turner, Majji, and Junkins[18,19] considered generalized optimal control

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formulations that handle nonlinear system dynamics and developing control gain sensitivities to handle plant model uncertainties during maneuvers. Sharma and Tiwari[16] introduced Modified Rodrigues Parameters (MRP's) for parameterization of the orientation error. They defined the attitude error as an additive quantity. Their work is extended by retaining a rigorous nonlinear MRP's-based error equation. Schaub, Junkins, and Robinett[14] have developed a new penalty function for optimal control formulation of spacecraft attitude control problems. This function returns the same scalar penalty for a given physical attitude regardless of the attitude coordinate choice.This definition of the performance index will be considered in future papers.

Error equations are challenging to define for quaternion variables because of the implied coupling effects between the quaternion components.Junkins[7] has discussed the link between designing a good controller and the choice of coordinate to represent the attitude kinematics. He linearized the attitude error equations by defining the departure motion as an additive error from a nominal trajectory. Normally, the position error is described by the distance between the two vectors, which represent the current position state and a prescribed reference position state. However, the error in orientation can not be simply represented in this way.[13] This work presents attitude variable representations for: Quaternion, CRP's and MRP's that account for the coupled nonlinear error kinematics. No simplifications are introduced in description of the vehicle attitude motion. The resulting expressions are very compact, accurate, and computationally efficient. A nonlinear optimal tracking feedback problem is formulated using each set of attitude variables. The attitude error is described using both the reference trajectory and the current trajectory for all three forms.

This paper starts by formulating the spacecraft attitude control (II) and generating an optimal open-loop trajectory (II.B), which will be considered as the nominal (reference) motion. Since, the paper presents various attitude error representations, the authors find the tracking problem (III) a good example to utilize the attitude error kinematics developments. First, the reference motion equation defines the angular velocity error (III.A). Second, various spacecraft attitude error kinematics (III.B) are derived from the quaternion (rotational) error principle. the resulted (nonlinear) differential equations motivated the authors to develop nonlinear optimal feedback control (IV) to drive the terminal state value to zero. The nonlinear feedback gain is presented in (IV.A) and the simulation results (for quaternion, CRP's, and MRP's) are presented in (IV.B). The tracking optimal feedback control results for other attitude error representations will be presented in future paper with a generalized performance index.[14]

II. SPACECRAFT ATTITUDE CONTROL

The goal of this section is to develop a nominal solution for the spacecraft to follow as it maneuvers from its initial conditions to the desired final conditions.[18] The necessary conditions for the nominal solution are defined by an optimal open-loop solution for the rigid body trajectories. A rigid body model is assumed for the spacecraft, leading to equations of motion of the form

$$I\,\dot{\boldsymbol{\omega}} = -[\,\boldsymbol{\omega}\times]\,I\,\boldsymbol{\omega} + \boldsymbol{u} \tag{1}$$

where $\boldsymbol{\omega}$ is the spacecraft angular velocity, I is the spacecraft moments of inertia (see table 1) and $\boldsymbol{u}$ is the external torque (controller) acting on the system. Rotational singularities are avoided for the open-loop

Table 1. Principal components of Inertia ($\mathrm{kg}\cdot\mathrm{m}^2$)

I_1	86.215
I_2	85.070
I_3	113.565

solution by introducing a quaternion for the attitude kinematics variables,[12].[17] The kinematics of quaternion is given by

$$\dot{\boldsymbol{q}} = \frac{d}{dt}\begin{Bmatrix}\boldsymbol{q}_v\\ q_4\end{Bmatrix} = \frac{1}{2}\begin{bmatrix}-[\,\boldsymbol{\omega}\times] & \boldsymbol{\omega}\\ -\boldsymbol{\omega}^{\mathrm{T}} & 0\end{bmatrix}\begin{Bmatrix}\boldsymbol{q}_v\\ q_4\end{Bmatrix} \tag{2}$$

where $\boldsymbol{q}_v = \boldsymbol{e} \sin\left(\frac{\phi}{2}\right)$, $q_4 = \cos\left(\frac{\phi}{2}\right)$, ϕ denotes the principal angle, $\boldsymbol{e}$ denotes the unit vector, and $[\,\boldsymbol{e}\times] = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$ is the cross product matrix operator. These nonlinear equations are used for developing rigorous error kinematic and dynamic models that are suitable for optimal control solution strategies. The quaternion variables are useful for large motions, but the norm constraint complicates the feedback control approach.

A. Classical/Modified Rodrigues Parameter Kinematic Formulation

The quaternion equations of Eq. (2) is useful for defining the open-loop reference solution, but the feedback law is plagued by having to deal with the norm constraint. This problem is overcome by shifting to an alternative attitude representation. This problem is remedied, motivated by the developments of Sharma and Tiwari,[16] by introducing Modified Rodrigues Parameters (MRP's) for parametrization of the orientation error. Both Modified Rodrigues Parameters (CRP's) and MRP's have the desired property of having zero values for the terminal boundary conditions. The work of Sharma and Tiwari is generalized in this paper by retaining a rigorous nonlinear CRP- and MRP-based error equation. As a result, our feedback state variables are nonlinear in both the kinematic and dynamic variables, which motivate our exploration of 2^{nd} order tensor control feedback terms for suppressing the nonlinear behavior of the system. The singularity-free quaternion-based open-loop solution is transformed in terms of Modified Rodrigues Parameters for generating the reference kinematic state for the feedback control problem formulation.

B. Optimal Control Problem: Open-Loop Solution

The open-loop reference trajectory is obtained by minimizing the following optimal control performance index:[10]

$$J = \frac{1}{2}\int_{t_0}^{t_f} (\boldsymbol{x}^{\mathrm{T}} Q \boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}} R \boldsymbol{u})\, dt. \tag{3}$$

subject to the nonlinear state equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + B\,\boldsymbol{u} \tag{4}$$

where $\boldsymbol{f}(\boldsymbol{x})$ is a vector function containing the nonlinear terms. Let the state variables be the spacecraft angular velocity and the quaternion parameters $\boldsymbol{q}$; yielding, the seven-element state vector $\boldsymbol{x} = \{\boldsymbol{\omega}^{\mathrm{T}},\, \boldsymbol{q}^{\mathrm{T}}\}^{\mathrm{T}}$. The Hamiltonian H for this system of equations is[2,3,10]

$$H = \frac{1}{2}(\boldsymbol{x}^{\mathrm{T}} Q \boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}} R \boldsymbol{u}) + \boldsymbol{\lambda}^{\mathrm{T}} \dot{\boldsymbol{x}} \tag{5}$$

Invoking the standard necessary condition for optimality (i.e., $\dot{\boldsymbol{\lambda}} = \frac{\partial H}{\partial \boldsymbol{x}}$), after some algebraic manipulation, results in the following Co-state differential equations:

$$\begin{aligned}
\dot{\lambda}_1 &= -q_{1j}x_j - \left[\lambda_2 K_2 x_3 + \lambda_3 K_3 x_2 + \frac{1}{2}(\lambda_4 x_7 + \lambda_5 x_6 - \lambda_6 x_5 + \lambda_7 x_4)\right] \\
\dot{\lambda}_2 &= -q_{2j}x_j - \left[\lambda_1 K_1 x_3 + \lambda_3 K_3 x_1 - \frac{1}{2}(\lambda_4 x_6 - \lambda_5 x_7 - \lambda_6 x_4 - \lambda_7 x_5)\right] \\
\dot{\lambda}_3 &= -q_{3j}x_j - \left[\lambda_1 K_1 x_2 + \lambda_2 K_2 x_1 + \frac{1}{2}(\lambda_4 x_5 - \lambda_5 x_4 + \lambda_6 x_4 + \lambda_7 x_6)\right] \\
\dot{\lambda}_4 &= -q_{4j}x_j + \frac{1}{2}(\lambda_5 x_3 - \lambda_6 x_2 - \lambda_7 x_1) \\
\dot{\lambda}_5 &= -q_{5j}x_j - \frac{1}{2}(\lambda_4 x_3 - \lambda_6 x_1 + \lambda_7 x_2) \\
\dot{\lambda}_6 &= -q_{6j}x_j + \frac{1}{2}(\lambda_4 x_2 - \lambda_5 x_1 - \lambda_7 x_3) \\
\dot{\lambda}_7 &= -q_{7j}x_j - \frac{1}{2}(\lambda_4 x_1 + \lambda_5 x_2 + \lambda_6 x_3)
\end{aligned} \tag{6}$$

where $K_1 = (I_2 - I_3)/I_1$, $K_2 = (I_3 - I_1)/I_2$, and $K_3 = (I_1 - I_2)/I_3$. The optimal control is given by the first-order necessary conditions for an extremum, $H_u = 0$leading to $\boldsymbol{u} = -R^{-1}B^{\mathrm{T}}\boldsymbol{\lambda}$.

A fixed-time and fixed final-state open-loop optimal control solution is obtained for maneuvering nonlinear spacecraft. The initial and final state variable conditions for this example are given in Table 2. For simplicity the weighting matrix Q and R are chosen to be identity matrices. However, one can sweep those penalties to obtain different solutions sets. The numerical solution is obtained using a shooting method (MATLAB `fsolve`) to solve the boundary value problem (BVP) in the Co-state equation.[11]

time (s)	quaternion $\boldsymbol{q}$	angular velocity $\boldsymbol{\omega}$ (rad/s)
0	$\frac{\sqrt{2}}{2}\{1,\ 0,\ 0,\ 1\}^{\mathrm{T}}$	$\{0.1,\ 0.2,\ 0.3]^{\mathrm{T}}$
50	$\{0,\ 0,\ 0,\ 1\}^{\mathrm{T}}$	$\{0,\ 0,\ 0\}^{\mathrm{T}}$

Table 2. Initial/Boundary Conditions.

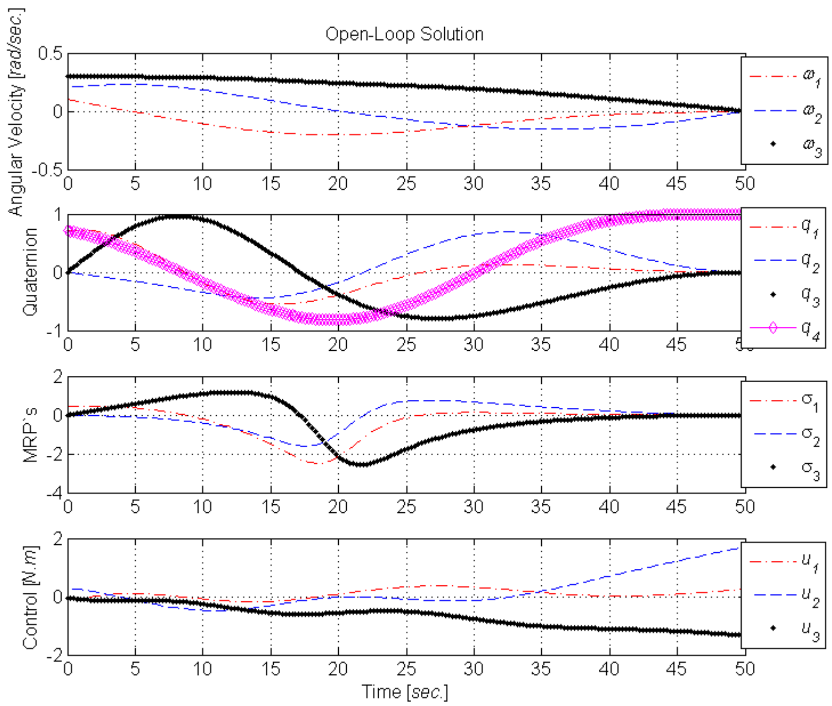

Figure 1. Open-Loop Solution

Figure 1 presents the open-loop solution for the nonlinear optimal control problem. Solution histories are provided for of the spacecraft angular velocities, quaternion, MRP's and the open-loop control. The full model derivation is given in the following section. One can easily see that the MRP's have the very desirable property that all of its components are identically zero at the final time, whereas the quaternion final time values clearly do not vanish. The open-loop optimal solution achieves the initial and final boundary conditions. The quaternion norm constraint is preserved throughout the maneuver. The quaternion and angular velocity solutions define the nominal trajectories for the maneuver.

III. Tracking Problem

The goal is to force the system response to follow a pre-defined solution path, where the maneuver time is fixed.[10] An exact nonlinear error kinematics and dynamics model is developed for spacecraft tracking maneuvers. The new nonlinear error dynamics models are shown to lead to 2^{nd} and 3^{rd} order models. Three feedback control approaches are considered where the spacecraft attitude motion is described by quaternion, CRP's, MRP's, Euler angles, principal axis and principal angle, direction cosine matrix, and Cayley-Klein.

A. Reference motion equation

Nonlinear tracking system dynamics models are developed in terms of the angular velocity error $\delta\boldsymbol{\omega}$. The desired motion is defined in terms of the open-loop reference angular velocity[19] $\hat{\boldsymbol{\omega}}$,

$$I\,\dot{\hat{\boldsymbol{\omega}}} = -[\hat{\boldsymbol{\omega}}\times]\,I\,\hat{\boldsymbol{\omega}} + \boldsymbol{\tau} \qquad (\text{assume } \boldsymbol{\tau} = \mathbf{0}) \tag{7}$$

which is used for the open-loop solution. A nonlinear tracking error dynamics equation is derived by defining the angular velocity error:[19]

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \hat{\boldsymbol{\omega}} \tag{8}$$

and

$$\boxed{\begin{aligned} \delta\dot{\boldsymbol{\omega}} = \quad & -I^{-1}\{[\hat{\boldsymbol{\omega}}\times]I - [(I\hat{\boldsymbol{\omega}})\times]\}\delta\boldsymbol{\omega} - I^{-1}[(\delta\boldsymbol{\omega})\times]I\delta\boldsymbol{\omega} + \\ & +I^{-1}\boldsymbol{u} - \dot{\hat{\boldsymbol{\omega}}} - I^{-1}[\hat{\boldsymbol{\omega}}\times]I\hat{\boldsymbol{\omega}} \end{aligned}} \tag{9}$$

B. Spacecraft Attitude Error

The attitude error is originally represented in underlying quaternion error kinematics equations. Then different attitude error equations are developed using kinematic identities to formulate the nonlinear kinematics model. These developments are numerically verified to the machine error.

1. *Quaternion Error Kinematics*

The kinematic solution for the reference trajectory quaternion defines the desired rotational motion for the spacecraft. The quaternion rotational error is defined as[18, 19]

$$\delta\boldsymbol{q} = \boldsymbol{q} \otimes \hat{\boldsymbol{q}}^{-1} \tag{10}$$

where $\hat{\boldsymbol{q}}^{-1}$ is quaternion inverse of the reference quaternion rotation and $\otimes$ represents the quaternion product. Note that the error, $\delta\boldsymbol{q}$, is a quaternion, that is, a unit-vector. The quaternion rotational error rate is represented as,

$$\delta\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \otimes \hat{\boldsymbol{q}}^{-1} + \boldsymbol{q} \otimes \dot{\hat{\boldsymbol{q}}}^{-1} \tag{11}$$

The quaternion kinematics evolve in time according to the differential equation

$$\dot{\boldsymbol{q}} = \frac{1}{2}\begin{Bmatrix}\boldsymbol{\omega} \\ 0\end{Bmatrix} \otimes \boldsymbol{q} = \frac{1}{2}\begin{bmatrix}-[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^{\mathrm{T}} & 0\end{bmatrix}\boldsymbol{q} = \frac{1}{2}\Omega(\boldsymbol{\omega})\boldsymbol{q} \tag{12}$$

performing the derivative of the identity $\hat{\boldsymbol{q}} \otimes \hat{\boldsymbol{q}}^{-1} = \{0,0,0,1\}^{\mathrm{T}}$, that is, $\dot{\hat{\boldsymbol{q}}} \otimes \hat{\boldsymbol{q}}^{-1} + \hat{\boldsymbol{q}} \otimes \dot{\hat{\boldsymbol{q}}}^{-1} = \mathbf{0}$, we obtain that the inverse quaternion evolves as

$$\dot{\hat{\boldsymbol{q}}}^{-1} = -\frac{1}{2}\hat{\boldsymbol{q}}^{-1} \otimes \begin{Bmatrix}\hat{\boldsymbol{\omega}} \\ 0\end{Bmatrix} = -\frac{1}{2}\Gamma(\hat{\boldsymbol{\omega}})\hat{\boldsymbol{q}}^{-1} \tag{13}$$

Substituting Eq. (12) and Eq. (13) that into Eq. (11), yields

$$\delta\dot{\boldsymbol{q}} = \frac{1}{2}\Omega(\boldsymbol{\omega})\delta\boldsymbol{q} - \frac{1}{2}\boldsymbol{q} \otimes \hat{\boldsymbol{q}}^{-1} \otimes \begin{Bmatrix}\hat{\boldsymbol{\omega}} \\ 0\end{Bmatrix} = \frac{1}{2}\Omega(\boldsymbol{\omega})\delta\boldsymbol{q} - \frac{1}{2}\delta\boldsymbol{q} \otimes \begin{Bmatrix}\hat{\boldsymbol{\omega}} \\ 0\end{Bmatrix} \tag{14}$$

and we can write

$$\delta\boldsymbol{q} \otimes \begin{Bmatrix}\hat{\boldsymbol{\omega}} \\ 0\end{Bmatrix} = \begin{bmatrix}[\hat{\boldsymbol{\omega}}\times] & \hat{\boldsymbol{\omega}} \\ -\hat{\boldsymbol{\omega}}^{\mathrm{T}} & 0\end{bmatrix}\delta\boldsymbol{q} = \Gamma(\hat{\boldsymbol{\omega}})\delta\boldsymbol{q} \tag{15}$$

where $\Gamma(\hat{\boldsymbol{\omega}})$ is the reference angular velocity matrix. The quaternion error rate equation becomes

$$\delta\dot{\boldsymbol{q}} = \frac{1}{2}[\Omega(\boldsymbol{\omega}) - \Gamma(\hat{\boldsymbol{\omega}})]\delta\boldsymbol{q} \tag{16}$$

by substituting the angular velocity error Eq. (8) into Eq. (16), one obtains the bilinear differential equation for the tracking error kinematics

$$\delta\dot{\boldsymbol{q}} = \frac{1}{2}[\Omega(\delta\boldsymbol{\omega} + \hat{\boldsymbol{\omega}}) - \Gamma(\hat{\boldsymbol{\omega}})]\delta\boldsymbol{q} = \frac{1}{2}[\Omega(\delta\boldsymbol{\omega}) - \bar{\Gamma}(\hat{\boldsymbol{\omega}})]\delta\boldsymbol{q} \tag{17}$$

where $\bar{\Gamma}$ is a matrix defined as

$$\bar{\Gamma}(\hat{\boldsymbol{\omega}}) = \Omega(\hat{\boldsymbol{\omega}}) - \Gamma(\hat{\boldsymbol{\omega}}) = \begin{bmatrix} -2[\boldsymbol{\omega}\times] & \boldsymbol{0} \\ \boldsymbol{0}^{\mathrm{T}} & 0 \end{bmatrix} \tag{18}$$

The quaternion error is a four-dimensional vector, defined as

$$\delta\boldsymbol{q} = \begin{Bmatrix} \delta\boldsymbol{q}_v \\ \delta q_4 \end{Bmatrix} \tag{19}$$

with $\delta\boldsymbol{q}_v = \{\delta q_1,\ \delta q_2,\ \delta q_3\}^{\mathrm{T}} = \boldsymbol{e}\,\sin\left(\frac{\phi}{2}\right)$ and $\delta q_4 = \cos\left(\frac{\phi}{2}\right)$, and where $\boldsymbol{e}$ is the principal (Euler) axis and ϕ is the principal (Euler) angle. Eq. (17) can be rewritten in the following compact form

$$\delta\dot{\boldsymbol{q}} = \frac{1}{2}\begin{bmatrix} -([\delta\boldsymbol{\omega}\times] + 2[\hat{\boldsymbol{\omega}}\times]) & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^{\mathrm{T}} & 0 \end{bmatrix}\begin{Bmatrix} \delta\boldsymbol{q}_v \\ \delta q_4 \end{Bmatrix} \tag{20}$$

This can be split into the scalar and vector part of the quaternion as follows

$$\boxed{\begin{cases} \delta\dot{\boldsymbol{q}}_v &= \dfrac{1}{2}\left\{-\left([\delta\boldsymbol{\omega}\times] + 2[\hat{\boldsymbol{\omega}}\times]\right)\delta\boldsymbol{q}_v + \delta q_4\delta\boldsymbol{\omega}\right\} \\ \delta\dot{q}_4 &= -\dfrac{1}{2}\delta\boldsymbol{\omega}^{\mathrm{T}}\delta\boldsymbol{q}_v \end{cases}} \tag{21}$$

2. Classical Rodrigues Parameter Error Kinematics

Classical Rodrigues Parameters (CRP) are a minimum attitude parametrization. CRP vector is defined in terms of quaternion parameters as,[15]

$$\boldsymbol{\rho} = \frac{\boldsymbol{q}_v}{q_4} = \boldsymbol{e}\,\tan\left(\frac{\phi}{2}\right) \tag{22}$$

where the inverse transformation is given by ($\rho^2 = \boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\rho}$)

$$q_4 = \frac{1}{\sqrt{1+\rho^2}} \qquad \text{and} \qquad \boldsymbol{q}_v = \frac{\boldsymbol{\rho}}{\sqrt{1+\rho^2}}. \tag{23}$$

Note that the attitude error given in Eq. (21) is represented in quaternion. Since we have used full non-linear model in quaternion, with no approximation, the quaternion unit constraint is always preserved. This implies that the quaternion (attitude) error still represents a finite orientation that can be mapped to any other attitude representations. Here, we will map the quaternion (attitude) error to Classical Rodrigues Parameters (CRP's) using Eq. (22). Thus, CRP error vector is expressed as

$$\delta\boldsymbol{\rho} = \frac{\delta\boldsymbol{q}_v}{\delta q_4} \tag{24}$$

and the inverse mapping for quaternion variables follow as ($\delta\rho^2 = \delta\boldsymbol{\rho}^{\mathrm{T}}\delta\boldsymbol{\rho}$)

$$\delta q_4 = \frac{1}{\sqrt{1+\delta\rho^2}} \qquad \text{and} \qquad \delta\boldsymbol{q}_v = \frac{\delta\boldsymbol{\rho}}{\sqrt{1+\delta\rho^2}}. \tag{25}$$

A governing differential equation for the CRP's error follows on taking the time derivative of Eq. (24)

$$\delta\dot{\boldsymbol{\rho}} = \frac{\delta\dot{\boldsymbol{q}}_v}{\delta q_4} - \frac{\delta\dot{q}_4\delta\boldsymbol{q}_v}{(\delta q_4)^2} \tag{26}$$

substituting Eq. (21) and Eq. (25) into Eq. (26), yielding

$$\delta\dot{\boldsymbol{\rho}} = \frac{\dfrac{-([\delta\boldsymbol{\omega}\times] + 2[\hat{\boldsymbol{\omega}}\times])\delta\boldsymbol{\rho}}{\sqrt{1+\delta\rho^2}} + \dfrac{\delta\boldsymbol{\omega}}{\sqrt{1+\delta\rho^2}}}{2\dfrac{1}{\sqrt{1+\delta\rho^2}}} + \frac{\dfrac{\delta\boldsymbol{\omega}^{\mathrm{T}}\delta\boldsymbol{\rho}}{\sqrt{1+\delta\rho^2}}\dfrac{\delta\boldsymbol{\rho}}{\sqrt{1+\delta\rho^2}}}{2\dfrac{1}{1+\delta\rho^2}}$$

The equation can be simplified to the compact (nonlinear, third-order) form

$$\boxed{\delta\dot{\boldsymbol{\rho}} = \frac{1}{2}\left[-\left([\delta\boldsymbol{\omega}\times] + 2[\hat{\boldsymbol{\omega}}\times]\right)\delta\boldsymbol{\rho} + \delta\boldsymbol{\omega}\right] + \frac{1}{2}\left(\delta\boldsymbol{\omega}^{\mathrm{T}}\delta\boldsymbol{\rho}\right)\delta\boldsymbol{\rho}} \tag{27}$$

Equation (27) provides the desired nonlinear kinematic differential equations for the vehicle rotational motion.

3. Modified Rodrigues Parameter Error Kinematics

Modified Rodrigues Parameters (MRP's) are an elegant addition to the family of attitude parameters. MRP's vector is defined in terms of the quaternion parameters as the transformation,[15]

$$\boldsymbol{\sigma} = \frac{\boldsymbol{q}_v}{1+q_4} = \boldsymbol{e}\tan\left(\frac{\phi}{4}\right) \tag{28}$$

the inverse transformation is given by

$$q_4 = \frac{1-\sigma^2}{1+\sigma^2} \qquad \text{and} \qquad \boldsymbol{q}_v = \frac{2\boldsymbol{\sigma}}{1+\sigma^2}. \tag{29}$$

Similarly, since the attitude error in Eq. (21) is still quaternion, we can perform the mapping into Modified Rodrigues Parameters (MRP's) using Eq. (28). Thus, MRP's error vector is expressed as

$$\delta\boldsymbol{\sigma} = \frac{\delta\boldsymbol{q}_v}{1+\delta q_4} \tag{30}$$

and the inverse mapping for quaternion variables follow as,

$$\delta q_4 = \frac{1-\delta\sigma^2}{1+\delta\sigma^2} \qquad \text{and} \qquad \delta\boldsymbol{q}_v = \frac{2\delta\boldsymbol{\sigma}}{1+\delta\sigma^2} \tag{31}$$

A governing differential equation for the MRP's error follows on taking the time derivative of Eq. (30)

$$\delta\dot{\boldsymbol{\sigma}} = \frac{\delta\dot{\boldsymbol{q}}_v}{1+\delta q_4} - \frac{\delta\dot{q}_4\delta\boldsymbol{q}_v}{(1+\delta q_4)^2} \tag{32}$$

substituting Eq. (21) and Eq. (31) into Eq. (32), yields

$$\delta\dot{\boldsymbol{\sigma}} = \frac{\dfrac{1}{2}\left[\dfrac{-([\delta\boldsymbol{\omega}\times] + 2[\hat{\boldsymbol{\omega}}\times])2\delta\boldsymbol{\sigma}}{1+\delta\sigma^2} + \dfrac{1-\delta\sigma^2}{1+\delta\sigma^2}\delta\boldsymbol{\omega}\right]}{1+\dfrac{1-\delta\sigma^2}{1+\delta\sigma^2}} + \frac{\dfrac{2\delta\boldsymbol{\omega}^{\mathrm{T}}\delta\boldsymbol{\sigma}}{1+\delta\sigma^2}\dfrac{2\delta\boldsymbol{\sigma}}{1+\delta\sigma^2}}{2\left(1+\dfrac{1-\delta\sigma^2}{1+\delta\sigma^2}\right)^2}$$

This equation can be clearly simplified to the compact (nonlinear, third-order) form

$$\boxed{\delta\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[-2\left([\delta\boldsymbol{\omega}\times] + 2[\hat{\boldsymbol{\omega}}\times]\right)\delta\boldsymbol{\sigma} + (1-\delta\sigma^2)\delta\boldsymbol{\omega}\right] + \frac{1}{2}\left(\delta\boldsymbol{\omega}^{\mathrm{T}}\delta\boldsymbol{\sigma}\right)\delta\boldsymbol{\sigma}} \tag{33}$$

Equation (33) provides the desired nonlinear kinematic differential equations for the vehicle rotational motion.

4. Euler Angles Kinematics

The three angles $(\delta\theta_1, \delta\theta_2, \delta\theta_3)$ giving the three rotation matrices are called Euler angles. There are several conventions for Euler angles, depending on the axes about which the rotations are carried out. In general, the rotation follows an arbitrary sequence (roll-pitch-yaw, yaw-roll-yaw,...). In the following derivation, we assume the rotation is performed following (roll-pitch-yaw or 123) sequence and (yaw-roll-yaw or 313) sequence. We start from the mapping equations from quaternion to Euler angles.[9] For the (3-1-3) set, the transformation is given by

$$\begin{bmatrix} \delta \boldsymbol{q}_v \\ \delta q_4 \end{bmatrix} = \begin{bmatrix} \sin(\frac{\delta\theta_2}{2})\cos(\frac{\delta\theta_1-\delta\theta_3}{2}) \\ \sin(\frac{\delta\theta_2}{2})\sin(\frac{\delta\theta_1-\delta\theta_3}{2}) \\ \cos(\frac{\delta\theta_2}{2})\sin(\frac{\delta\theta_1+\delta\theta_3}{2}) \\ \cos(\frac{\delta\theta_2}{2})\cos(\frac{\delta\theta_1+\delta\theta_3}{2}) \end{bmatrix}_{313} = \boldsymbol{\Theta}_{313}(\delta\theta_1, \delta\theta_2, \delta\theta_3) \tag{34}$$

by differentiating Eq. (34)

$$\begin{bmatrix} \delta \dot{\boldsymbol{q}}_v \\ \delta \dot{q}4 \end{bmatrix} = H_{313}(\delta\theta_1, \delta\theta_2, \delta\theta_3) \begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix}_{313} \tag{35}$$

where $H_{313}(\delta\theta_1, \delta\theta_2, \delta\theta_3)$ is a (4×3) matrix. Thus, Euler angles rates can be written in the following least square solution

$$\begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix}_{313} = (H_{313}^T H_{313})^{-1} H_{313}^T \begin{bmatrix} \delta \dot{\boldsymbol{q}}_v \\ \delta \dot{q}4 \end{bmatrix} \tag{36}$$

substituting Eq. (20) and making use of Eq. (34)

$$\boxed{\begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix}_{313} = \frac{1}{2}(H_{313}^T H_{313})^{-1} H_{313}^T \begin{bmatrix} -([\delta\boldsymbol{\omega}\times] + 2[\hat{\boldsymbol{\omega}}\times]) & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^{\mathrm{T}} & 0 \end{bmatrix} \boldsymbol{\Theta}_{313}(\delta\theta_1, \delta\theta_2, \delta\theta_3)} \tag{37}$$

Similarly, for the (1-2-3) set, the transformation is given by

$$\begin{bmatrix} \delta \boldsymbol{q}_v \\ \delta q_4 \end{bmatrix} = \begin{bmatrix} \sin(\frac{\delta\theta_1}{2})\cos(\frac{\delta\theta_2}{2})\cos(\frac{\delta\theta_3}{2}) + \cos(\frac{\delta\theta_1}{2})\sin(\frac{\delta\theta_2}{2})\sin(\frac{\delta\theta_3}{2}) \\ \cos(\frac{\delta\theta_1}{2})\sin(\frac{\delta\theta_2}{2})\cos(\frac{\delta\theta_3}{2}) - \sin(\frac{\delta\theta_1}{2})\cos(\frac{\delta\theta_2}{2})\sin(\frac{\delta\theta_3}{2}) \\ \cos(\frac{\delta\theta_1}{2})\cos(\frac{\delta\theta_2}{2})\sin(\frac{\delta\theta_3}{2}) + \sin(\frac{\delta\theta_1}{2})\sin(\frac{\delta\theta_2}{2})\cos(\frac{\delta\theta_3}{2}) \\ \cos(\frac{\delta\theta_1}{2})\cos(\frac{\delta\theta_2}{2})\cos(\frac{\delta\theta_3}{2}) - \sin(\frac{\delta\theta_1}{2})\sin(\frac{\delta\theta_2}{2})\sin(\frac{\delta\theta_3}{2}) \end{bmatrix}_{123} = \boldsymbol{\Theta}_{123}(\delta\theta_1, \delta\theta_2, \delta\theta_3) \tag{38}$$

by differentiating Eq. (38)

$$\begin{bmatrix} \delta \dot{\boldsymbol{q}}_v \\ \delta \dot{q}4 \end{bmatrix} = H_{123}(\delta\theta_1, \delta\theta_2, \delta\theta_3) \begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix}_{123} \tag{39}$$

where $H_{123}(\delta\theta_1, \delta\theta_2, \delta\theta_3)$ is a (4×3) matrix. Thus, Euler angles rates can be written in the following least square solution

$$\begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix}_{123} = (H_{123}^T H_{123})^{-1} H_{123}^T \begin{bmatrix} \delta \dot{\boldsymbol{q}}_v \\ \delta \dot{q}4 \end{bmatrix} \tag{40}$$

substituting Eq. (20) and making use of Eq. (38)

$$\boxed{\begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \\ \delta\dot{\theta}_3 \end{bmatrix}_{123} = \frac{1}{2}(H_{123}^T H_{123})^{-1} H_{123}^T \begin{bmatrix} -([\delta\boldsymbol{\omega}\times] + 2[\hat{\boldsymbol{\omega}}\times]) & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^{\mathrm{T}} & 0 \end{bmatrix} \boldsymbol{\Theta}_{123}(\delta\theta_1, \delta\theta_2, \delta\theta_3)} \tag{41}$$

Other rotation sequences can also be easily derived in the same manner.

5. *Principal Axis and Angle Kinematics*

A simple rotation can be defined to be about an axis ($\boldsymbol{e}$) by an angle (ϕ). To derive the kinematics of the principal axis/angle for attitude error, we start from the definition of the quaternion

$$\delta q_4 = \cos(\delta\phi/2) \qquad \text{and} \qquad \delta \boldsymbol{q}_v = \delta \boldsymbol{e} \sin(\delta\phi/2) \tag{42}$$

taking the time derivative and solving for $\delta\dot{\phi}$ and ($\delta\dot{\boldsymbol{e}}$)

$$\delta\dot{\phi} = -\frac{2\ \delta\dot{q}_4}{\sin(\delta\phi/2)} \qquad \text{and} \qquad \delta\dot{\boldsymbol{e}} = \frac{\delta\dot{\boldsymbol{q}}_v - \frac{1}{2}\delta\boldsymbol{e}\delta\dot{\phi}\cos(\delta\phi/2)}{\sin(\delta\phi/2)} \tag{43}$$

substituting Eq. (21) and making use of $\delta\boldsymbol{q}_v = \delta\hat{\mathbf{e}}\ \sin(\delta\phi/2)$ (where $\delta\hat{\mathbf{e}} = \frac{\delta\boldsymbol{e}}{\sqrt{\delta\boldsymbol{e}^T\delta\boldsymbol{e}}}$)

$$\boxed{\delta\dot{\phi} = \delta\boldsymbol{\omega}^T\delta\hat{\mathbf{e}}} \tag{44}$$

and

$$\boxed{\delta\dot{\boldsymbol{e}} = -\frac{1}{2}\Big(([\delta\boldsymbol{\omega}\times] + 2[\hat{\boldsymbol{\omega}}\times])\,\delta\hat{\mathbf{e}} + \big(\left(\delta\boldsymbol{\omega}^T\delta\hat{\mathbf{e}}\right)\delta\hat{\mathbf{e}} - \delta\boldsymbol{\omega}\big)\cot(\delta\phi/2)\Big)} \tag{45}$$

6. *Direction Cosine Matrix Kinematics*

The direction cosine matrix error can be written as

$$\delta C = C\,\hat{C}^{\mathrm{T}} \tag{46}$$

performing the derivative we obtain

$$\delta\dot{C} = \dot{C}\,\hat{C}^{\mathrm{T}} + C\,\dot{\hat{C}}^{\mathrm{T}} \tag{47}$$

The DCM kinematics evolves in time according to the differential equation

$$\dot{C} = -[\boldsymbol{\omega}\times]\,C \tag{48}$$

performing the derivative of the identity matrix $\hat{C}\,\hat{C}^{\mathrm{T}} = I_{3\times3}$, that is, $\dot{\hat{C}}\,\hat{C}^{\mathrm{T}} + \hat{C}\,\dot{\hat{C}}^{\mathrm{T}} = 0_{3\times3}$, we obtain that the inverse DCM evolves as

$$\dot{\hat{C}}^{\mathrm{T}} = -\hat{C}^{\mathrm{T}}\,\dot{\hat{C}}\,\hat{C}^{\mathrm{T}} \tag{49}$$

substituting in Eq. (47) we obtain

$$\begin{aligned}
\delta\dot{C} &= \dot{C}\,\hat{C}^{\mathrm{T}} - C\,\hat{C}^{\mathrm{T}}\,\dot{\hat{C}}\,\hat{C}^{\mathrm{T}} = -[\boldsymbol{\omega}\times]\,C\,\hat{C}^{\mathrm{T}} - \delta C\,\dot{\hat{C}}\,\hat{C}^{\mathrm{T}} = \\
&= -[\boldsymbol{\omega}\times]\,\delta C - \delta C\,\dot{\hat{C}}\,\hat{C}^{\mathrm{T}} = -[\boldsymbol{\omega}\times]\,\delta C + \delta C\,[\hat{\boldsymbol{\omega}}\times]\,\hat{C}\,\hat{C}^{\mathrm{T}} = \\
&= -[(\delta\boldsymbol{\omega} + \hat{\boldsymbol{\omega}})\times]\delta C + \delta C[\hat{\boldsymbol{\omega}}\times]
\end{aligned}$$

and, finally, we obtain

$$\boxed{\delta\dot{C} = -[\delta\boldsymbol{\omega}\times]\delta C - [\hat{\boldsymbol{\omega}}\times]\delta C + \delta C[\hat{\boldsymbol{\omega}}\times]} \tag{50}$$

7. *Cayley-Klein Parameters*

Cayley-Klein parameters are derived from quaternion

$$\delta K = \delta q_4 I + i\,(\delta q_1\sigma_1 + \delta q_2\sigma_2 + \delta q_3\sigma_3) \tag{51}$$

$$\delta K = \begin{bmatrix} \delta q_4 + i\delta q_3 & \delta q_2 + i\delta q_1 \\ -\delta q_2 + i\delta q_1 & \delta q_4 - i\delta q_3 \end{bmatrix} \tag{52}$$

where $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, and $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The principal angle can be computed from

$$\delta\phi = 2\arccos\left(\frac{1}{2}tr\,(\delta K)\right) \tag{53}$$

by rewriting Eq. (52) in a vector form

$$vec(\delta K) = \begin{bmatrix} \delta K_{1,1} \\ \delta K_{2,1} \\ \delta K_{1,2} \\ \delta K_{2,2} \end{bmatrix} = \begin{bmatrix} \delta q_4 + i\delta q_3 \\ -\delta q_2 + i\delta q_1 \\ \delta q_2 + i\delta q_1 \\ \delta q_4 - i\delta q_3 \end{bmatrix} \tag{54}$$

note Eq. (54) can also be written (in another linear form)

$$vec(\delta K) = \begin{bmatrix} \delta K_{1,1} \\ \delta K_{2,1} \\ \delta K_{1,2} \\ \delta K_{2,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & i & 1 \\ i & -1 & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & -i & 1 \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{q}_v \\ \delta q_4 \end{bmatrix} = \Psi_0 \delta \boldsymbol{q} \tag{55}$$

where $\Psi_0 = \begin{bmatrix} 0 & 0 & i & 1 \\ i & -1 & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & -i & 1 \end{bmatrix}$ is a constant and an invertible matrix. By time differentiating Eq. (55) yields

$$vec(\delta \dot{K}) = \Psi_0 \delta \dot{\boldsymbol{q}} \tag{56}$$

substituting Eq. (20) into Eq. (56)

$$vec(\delta \dot{K}) = \frac{1}{2}\Psi_0 \begin{bmatrix} -([\delta\boldsymbol{\omega}\times] + 2[\hat{\boldsymbol{\omega}}\times]) & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^{\mathrm{T}} & 0 \end{bmatrix} \begin{Bmatrix} \delta \boldsymbol{q}_v \\ \delta q_4 \end{Bmatrix} \tag{57}$$

using Eq. (55) to solve for $\delta\boldsymbol{q}$ and substituting it in Eq. (57)

$$\boxed{vec(\delta \dot{K}) = \frac{1}{2}\Psi_0 \begin{bmatrix} -([\delta\boldsymbol{\omega}\times] + 2[\hat{\boldsymbol{\omega}}\times]) & \delta\boldsymbol{\omega} \\ -\delta\boldsymbol{\omega}^{\mathrm{T}} & 0 \end{bmatrix} \Psi_0^{-1} vec(\delta K)} \tag{58}$$

where the $vec(\delta K)$ has to satisfy the constraint equation $\delta q^T \delta q = 1 = vec(\delta K)^T \Psi_0^{-T} \Psi_0^{-1} vec(\delta K)$ or $vec(\delta K)^T vec(\delta K) = 2$

IV. Optimal Tracking Control Formulation

An error tracking dynamics state vector is defined by combining error dynamics and attitude equations,

$$\dot{x}_i = a_{ij_1} x_{j_1} + c_{ijkl} x_j x_k x_l + t_{imn} x_m x_n + d_i + b_{ij_2} u_{j_2} \tag{59}$$

where the state variables are the spacecraft angular velocity error and the spacecraft attitude error. Note that the state coefficients depend on the attitude error representation; for instance $c_{ijkl} = 0$ when attitude error is described using the quaternion error.

A finite-time optimal control problem is designed by minimizing the following performance index:

$$\begin{aligned} J =& \frac{1}{2}\Phi(t_f, \boldsymbol{x}(t_f)) + \frac{1}{2}\int_{t_0}^{t_f} \{\boldsymbol{x}^{\mathrm{T}} Q \boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}} R \boldsymbol{u}\}\, dt = \\ =& \frac{1}{2}\Phi(t_f, \boldsymbol{x}(t_f)) + \frac{1}{2}\int_{t_0}^{t_f} \{q_{j_3 j_4} x_{j_3} x_{j_4} + r_{j_5 j_6} u_{j_5} u_{j_6}\}\, dt \end{aligned} \tag{60}$$

subject to the state equation given in Eq. (59), where $\Phi(t_f, \boldsymbol{x}(t_f))$ is a soft terminal constraint that is a function of the final time t_f and the final state $\boldsymbol{x}_f$. The Hamiltonian for this system of equations is

$$H = \frac{1}{2}(q_{j_3 j_4} x_{j_3} x_{j_4} + r_{j_5 j_6} u_{j_5} u_{j_6}) + \lambda_{i_1}(a_{i_1 j_1} x_{j_1} + c_{i_1 jkl} x_j x_k x_l + t_{i_1 mn} x_m x_n + d_{i_1} + b_{i_1 j_2} u_{j_2})$$

The necessary condition for optimality yields: (1) the Co-state equations $\dot{\lambda} = -\dfrac{\partial H}{\partial \boldsymbol{x}}$, and (2) the optimal control $\dfrac{\partial H}{\partial \boldsymbol{u}} = 0$. The Co-state differential follows as:

$$\begin{aligned}\dot{\lambda}_i =& -\frac{\partial H}{\partial \boldsymbol{x}} = -\frac{1}{2}(q_{j_3j_4}x_{j_3}\delta_{ij_4} + q_{j_3j_4}x_{j_4}\delta_{ij_3}) + \\ &- \lambda_{i_1}[a_{i_1j_1}\delta_{ij_1} + c_{i_1jkl}(x_jx_k\delta_{il} + x_jx_l\delta_{ik} + x_kx_l\delta_{ij}) + t_{i_1mn}(x_m\delta_{in} + x_n\delta_{im})]\end{aligned}$$

or

$$\dot{\lambda}_i = -q_{ij_4}x_{j_4}\lambda_{i_1}(a_{i_1i} + c_{i_1jki}x_jx_k + c_{i_1jil}x_jx_l + c_{i_1ikl}x_kx_l + t_{i_1mi}x_m + t_{i_1in}x_n) \tag{61}$$

and the control follows as

$$\frac{\partial H}{\partial \boldsymbol{u}} = 0 = \frac{1}{2}\{r_{j_5j_6}u_{j_5}\delta_{i_3j_6} + r_{j_5j_6}u_{j_6}\delta_{i_3j_5}\} + \lambda_{i_2}b_{i_2j_7}\delta_{i_3j_7}$$

or

$$u_{j_6} = -r^{-1}_{i_3j_6}b_{i_2i_3}\lambda_{i_2} \tag{62}$$

where $r^{-1}_{i_3j_6}$ denotes the i_3j_6-th element in the inverse of the matrix R (with elements defined by $r_{i_3j_6}$).

A. Nonlinear Feedback Gain

Since the tracking state equation turns to be highly nonlinear, the choice of the feedback control becomes non trivial. By observing the state equation, we can break it in three categories: 1) Linear term, 2) nonlinear (polynomial) term, and 3) the disturbance term. To solve the feedback problem, an appropriate Co-state structure is assumed, which leads to tensor-based gain equations that define the time history for the feedback control solution. Three different forms of the feedback control are assumed,[2],[3],[18]:[19]

- **A classical linear feedback gain**: which is governed by a matrix Riccati equation.
- **A disturbance rejection term**: Since the tracking state equation is perturbed with a disturbance, resulting from the reference open-loop torque, a disturbance rejection tracking term is introduced that provides explicit links to the reference motion torque.
- **A quadratic term**: The quadratic feedback gain strategy seeks to suppress second- and third-order error terms in the tracking error dynamics equations.

Because the gain equations lead to 3^{rd} order tensor equations, it is more natural to develop the governing equations by invoking the use of Eistein's summation notation, leading to a Co-state equations of the form

$$\lambda_{i_4} = s_{i_4j_9j_{10}}x_{j_9}x_{j_{10}} + k_{i_4j_8}x_{j_8} + p_{i_4} \tag{63}$$

where $s_{i_4j_9j_{10}}$, $k_{i_4j_8}$ and p_{i_4} are the control gains sought. By substituting Eq. (63) into the Co-state equation Eq. (61) and Eq. (62) and carrying out the ensuing algebra, we are led to the following equations

$$\begin{aligned}\dot{\lambda}_i =& -q_{ij_4}x_{j_4} - k_{i_1j_8}a_{i_1i}x_{j_8} - k_{i_1j_8}c_{i_1jki}x_jx_kx_{j_8} - k_{i_1j_8}c_{i_1jil}x_jx_lx_{j_8} \\ &- k_{i_1j_8}c_{i_1ikl}x_kx_lx_{j_8} - k_{i_1j_8}t_{i_1mi}x_mx_{j_8} - k_{i_1j_8}t_{i_1in}x_nx_{j_8} \\ &- s_{i_1j_9j_{10}}a_{i_1i}x_{j_9}x_{j_{10}} - s_{i_1j_9j_{10}}c_{i_1jki}x_jx_kx_{j_9}x_{j_{10}} \\ &- s_{i_1j_9j_{10}}c_{i_1jil}x_jx_lx_{j_9}x_{j_{10}} - s_{i_1j_9j_{10}}c_{i_1ikl}x_kx_lx_{j_9}x_{j_{10}} \\ &- s_{i_1j_9j_{10}}t_{i_1mi}x_mx_{j_9}x_{j_{10}} - s_{i_1j_9j_{10}}t_{i_1in}x_nx_{j_9}x_{j_{10}} - p_{i_1}a_{i_1i} \\ &- p_{i_1}c_{i_1jki}x_jx_k - p_{i_1}c_{i_1ikl}x_kx_l - p_{i_1}t_{i_1mi}x_m - p_{i_1}t_{i_1in}x_n\end{aligned} \tag{64}$$

and

$$\begin{aligned}u_{j_6} =& -r^{-1}_{i_3j_6}b_{i_2i_3}\{s_{i_2j_9j_{10}}x_{j_9}x_{j_{10}} + k_{i_2j_8}x_{j_8} + p_{i_2}\} = \\ =& -r^{-1}_{i_3j_6}b_{i_2i_3}s_{i_2j_9j_{10}}x_{j_9}x_{j_{10}} - r^{-1}_{i_3j_6}b_{i_2i_3}k_{i_2j_8}x_{j_8} - r^{-1}_{i_3j_6}b_{i_2i_3}p_{i_2} \\ &\text{or} \\ =& -r^{-1}_{i_5j_6}b_{i_6i_5}s_{i_6j_{31}j_{32}}x_{j_{31}}x_{j_{32}} - r^{-1}_{i_5j_6}b_{i_6i_5}k_{i_6j_{30}}x_{j_{30}} - r^{-1}_{i_5j_6}b_{i_6i_5}p_{i_6}\end{aligned} \tag{65}$$

by time-differentiating Eq. (63) and plugging the state equation Eq. (59) in it, the Co-state equation becomes

$$
\begin{aligned}
\dot{\lambda}_{i_4} = & \dot{k}_{i_4 j_8} x_{j_8} + k_{i_4 j_8} a_{j_8 j_1} x_{j_1} + k_{i_4 j_8} c_{j_8 jkl} x_j x_k x_l + k_{i_4 j_8} t_{j_8 mn} x_m x_n \\
& + k_{i_4 j_8} d_{j_8} + k_{i_4 j_8} b_{j_8 j_2} u_{j_2} + \dot{s}_{i_4 j_9 j_{10}} x_{j_9} x_{j_{10}} + s_{i_4 j_9 j_{10}} a_{j_9 j_1} x_{j_{10}} x_{j_1} \\
& + c_{j_9 jkl} s_{i_4 j_9 j_{10}} x_{j_{10}} x_j x_k x_l + t_{j_9 mn} s_{i_4 j_9 j_{10}} x_{j_{10}} x_m x_n + s_{i_4 j_9 j_{10}} d_{j_9} x_{j_{10}} \\
& + s_{i_4 j_9 j_{10}} b_{j_9 j_2} u_{j_2} x_{j_{10}} + s_{i_4 j_9 j_{10}} a_{j_{10} j_1} x_{j_9} x_{j_1} + c_{j_{10} jkl} s_{i_4 j_9 j_{10}} x_{j_9} x_j x_k x_l \\
& + t_{j_{10} mn} s_{i_4 j_9 j_{10}} x_{j_9} x_m x_n + s_{i_4 j_9 j_{10}} d_{j_{10}} x_{j_9} + s_{i_4 j_9 j_{10}} b_{j_{10} j_2} u_{j_2} x_{j_9} + \dot{p}_{i_4}
\end{aligned}
\tag{66}
$$

also substituting Eq. (64) and Eq. (65) into the Co-state equation Eq. (66) and simplifying,

$$
\begin{aligned}
& q_{i_4 j_4} x_{j_4} + k_{i_1 j_8} a_{i_1 i_4} x_{j_8} + k_{i_1 j_8} c_{i_1 jk i_4} x_j x_k x_{j_8} + k_{i_1 j_8} c_{i_1 j i_4 l} x_j x_l x_{j_8} + k_{i_1 j_8} c_{i_1 i_4 kl} x_k x_l x_{j_8} \\
& + k_{i_1 j_8} t_{i_1 m i_4} x_m x_{j_8} + k_{i_1 j_8} t_{i_1 i_4 n} x_n x_{j_8} + s_{i_1 j_9 j_{10}} a_{i_1 i_4} x_{j_9} x_{j_{10}} + s_{i_1 j_9 j_{10}} c_{i_1 jk i_4} x_j x_k x_{j_9} x_{j_{10}} \\
& + s_{i_1 j_9 j_{10}} c_{i_1 j i_4 l} x_j x_l x_{j_9} x_{j_{10}} + s_{i_1 j_9 j_{10}} c_{i_1 i_4 kl} x_k x_l x_{j_9} x_{j_{10}} + s_{i_1 j_9 j_{10}} t_{i_1 m i_4} x_m x_{j_9} x_{j_{10}} \\
& - s_{i_1 j_9 j_{10}} t_{i_1 i_4 n} x_n x_{j_9} x_{j_{10}} + p_{i_1} a_{i_1 i_4} + p_{i_1} c_{i_1 jk i_4} x_j x_k + p_{i_1} c_{i_1 i_4 kl} x_k x_l + p_{i_1} t_{i_1 m i_4} x_m + p_{i_1} t_{i_1 i_4 n} x_n \\
& + \dot{k}_{i_4 j_8} x_{j_8} + k_{i_4 j_8} a_{j_8 j_1} x_{j_1} + k_{i_4 j_8} c_{j_8 jkl} x_j x_k x_l + k_{i_4 j_8} t_{j_8 mn} x_m x_n + k_{i_4 j_8} d_{j_8} \\
& - k_{i_4 j_8} b_{j_8 j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} s_{i_6 j_{31} j_{32}} x_{j_{31}} x_{j_{32}} - k_{i_4 j_8} b_{j_8 j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} k_{i_6 j_{30}} x_{j_{30}} - k_{i_4 j_8} b_{j_8 j_2} r^{-1}_{i_5 j_6} b_{i_6 i_5} p_{i_6} \\
& + \dot{s}_{i_4 j_9 j_{10}} x_{j_9} x_{j_{10}} + s_{i_4 j_9 j_{10}} a_{j_9 j_1} x_{j_{10}} x_{j_1} + c_{j_9 jkl} s_{i_4 j_9 j_{10}} x_{j_{10}} x_j x_k x_l \\
& + t_{j_9 mn} s_{i_4 j_9 j_{10}} x_{j_{10}} x_m x_n + s_{i_4 j_9 j_{10}} d_{j_9} x_{j_{10}} - s_{i_4 j_9 j_{10}} b_{j_9 j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} s_{i_6 j_{31} j_{32}} x_{j_{31}} x_{j_{32}} x_{j_{10}} \\
& - s_{i_4 j_9 j_{10}} b_{j_9 j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} k_{i_6 j_{30}} x_{j_{30}} x_{j_{10}} - s_{i_4 j_9 j_{10}} b_{j_9 j_2} r^{-1}_{i_5 j_6} b_{i_6 i_5} p_{i_6} x_{j_{10}} + s_{i_4 j_9 j_{10}} a_{j_{10} j_1} x_{j_9} x_{j_1} \\
& + c_{j_{10} jkl} s_{i_4 j_9 j_{10}} x_{j_9} x_j x_k x_l + t_{j_{10} mn} s_{i_4 j_9 j_{10}} x_{j_9} x_m x_n + s_{i_4 j_9 j_{10}} d_{j_{10}} x_{j_9} \\
& - s_{i_4 j_9 j_{10}} b_{j_{10} j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} s_{i_6 j_{31} j_{32}} x_{j_{31}} x_{j_{32}} x_{j_9} \\
& - s_{i_4 j_9 j_{10}} b_{j_{10} j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} k_{i_6 j_{30}} x_{j_{30}} x_{j_9} - s_{i_4 j_9 j_{10}} b_{j_{10} j_2} r^{-1}_{i_5 j_6} b_{i_6 i_5} p_{i_6} x_{j_9} + \dot{p}_{i_4} = 0
\end{aligned}
\tag{67}
$$

note that in Eq. (67), there is ONLY one FREE index, which is (i_4), and the others are dummy indices. Therefore, the dummy indices can be renamed to collect the same power terms. Thus Eq. (67) can be written as

$$
\begin{aligned}
& (k_{i_4 j_8} d_{j_8} + \dot{p}_{i_4} - k_{i_4 j_8} b_{j_8 j_2} r^{-1}_{i_5 j_6} b_{i_6 i_5} p_{i_6} + p_{i_1} a_{i_1 i_4}) \\
& + (k_{i_1 j_{40}} a_{i_1 i_4} + \dot{k}_{i_4 j_{40}} - k_{i_4 j_8} b_{j_8 j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} k_{i_6 j_{40}} + k_{i_4 j_8} a_{j_8 j_{40}} + s_{i_4 j_{40} j_{10}} d_{j_{10}} + s_{i_4 j_9 j_{40}} d_{j_9} \\
& - s_{i_4 j_9 j_{40}} b_{j_9 j_2} r^{-1}_{i_5 j_6} b_{i_6 i_5} p_{i_6} - s_{i_4 j_{40} j_{10}} b_{j_{10} j_2} r^{-1}_{i_5 j_6} b_{i_6 i_5} p_{i_6} + q_{i_4 j_{40}} + p_{i_1} t_{i_1 j_{40} i_4} + p_{i_1} t_{i_1 i_4 j_{40}}) x_{j_{40}} \\
& + (k_{i_1 j_{42}} t_{i_1 j_{41} i_4} + k_{i_1 j_{42}} t_{i_1 i_4 j_{41}} + s_{i_1 j_{41} j_{42}} a_{i_1 i_4} + + k_{i_4 j_8} t_{j_8 j_{41} j_{42}} + s_{i_4 j_9 j_{41}} a_{i_9 i_{42}} \\
& - k_{i_4 j_8} b_{j_8 j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} s_{i_6 j_{41} j_{42}} + \dot{s}_{i_4 j_{41} j_{42}} - s_{i_4 j_9 j_{42}} b_{j_9 j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} k_{i_6 j_{41}} + s_{i_4 j_{41} j_{10}} a_{j_{10} j_{42}} \\
& - s_{i_4 j_{42} j_{10}} b_{j_{10} j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} k_{i_6 j_{41}} + p_{i_1} c_{i_1 j_{41} j_{42} i_4} + p_{i_1} c_{i_1 j_{41} i_4 j_{42}} + p_{i_1} c_{i_1 i_4 j_{41} j_{42}}) x_{j_{41}} x_{j_{42}} \\
& + (k_{i_1 j_{45}} c_{i_1 j_{43} j_{44} i_4} + k_{i_1 j_{45}} c_{i_1 j_{43} i_4 j_{44}} + k_{i_1 j_{45}} c_{i_1 i_4 j_{43} j_{44}} + s_{i_1 j_{44} j_{45}} t_{i_1 j_{43} i_4} \\
& + s_{i_1 j_{44} j_{45}} t_{i_1 i_4 j_{43}} + k_{i_4 j_8} c_{j_8 j_{43} j_{44} j_{45}} - s_{i_4 j_{45} j_{10}} b_{j_{10} j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} s_{i_6 j_{43} j_{44}} \\
& + t_{j_9 j_{44} j_{45}} s_{i_4 j_9 j_{43}} - s_{i_4 j_9 j_{45}} b_{j_9 j_2} r^{-1}_{i_5 j_2} b_{i_6 i_5} s_{i_6 j_{43} j_{44}} + t_{j_{10} j_{44} j_{45}} s_{i_4 j_{43} j_{10}}) x_{j_{43}} x_{j_{44}} x_{j_{45}} \\
& + (s_{i_1 j_{48} j_{49}} c_{i_1 j_{46} j_{47} i_4} + s_{i_1 j_{48} j_{49}} c_{i_1 j_{46} i_4 j_{47}} + s_{i_1 j_{48} j_{49}} c_{i_1 i_4 j_{46} j_{47}} \\
& + c_{j_9 j_{47} j_{48} j_{49}} s_{i_4 j_9 j_{46}} + c_{j_{10} j_{47} j_{48} j_{49}} s_{i_4 j_{46} j_{10}}) x_{j_{46}} x_{j_{47}} x_{j_{48}} x_{j_{49}} = 0
\end{aligned}
$$

therefore, the gain differential equation can be written as

$$
\begin{aligned}
\dot{p}_i = & - k_{ij} d_j + k_{ij} b_{jk} r^{-1}_{lk} b_{ml} p_m - p_n a_{ni} \\
\dot{k}_{ij} = & - k_{kj} a_{ki} + k_{il} b_{lm} r^{-1}_{nm} b_{on} k_{oj} - k_{il} a_{lj} - s_{ijp} d_p - s_{iqj} d_q \\
& + s_{iqj} b_{qm} r^{-1}_{nm} b_{on} p_o + s_{ijp} b_{pm} r^{-1}_{nm} b_{on} p_o - q_{ij} - p_r t_{rji} - p_r t_{rij} \\
\dot{s}_{ijk} = & - k_{lk} t_{lji} - k_{lk} t_{lij} - s_{ljk} a_{li} - k_{im} t_{mjk} - s_{inj} a_{nk} \\
& + k_{im} b_{mo} r^{-1}_{po} b_{qp} s_{qjk} + s_{ink} b_{no} r^{-1}_{po} b_{qp} k_{qj} - s_{ijr} a_{rk} \\
& + s_{ikr} b_{ro} r^{-1}_{po} b_{qp} k_{qj} - p_s c_{sjki} - p_s c_{sjik} - p_s c_{sijk}
\end{aligned}
$$

The gain differential equations are integrated backward in time with the following boundary conditions: $p_i(t_f) = 0$, $k_{ij}(t_f) = K_f$, and $s_{ijk}(t_f) = 0$.

B. Simulation Results

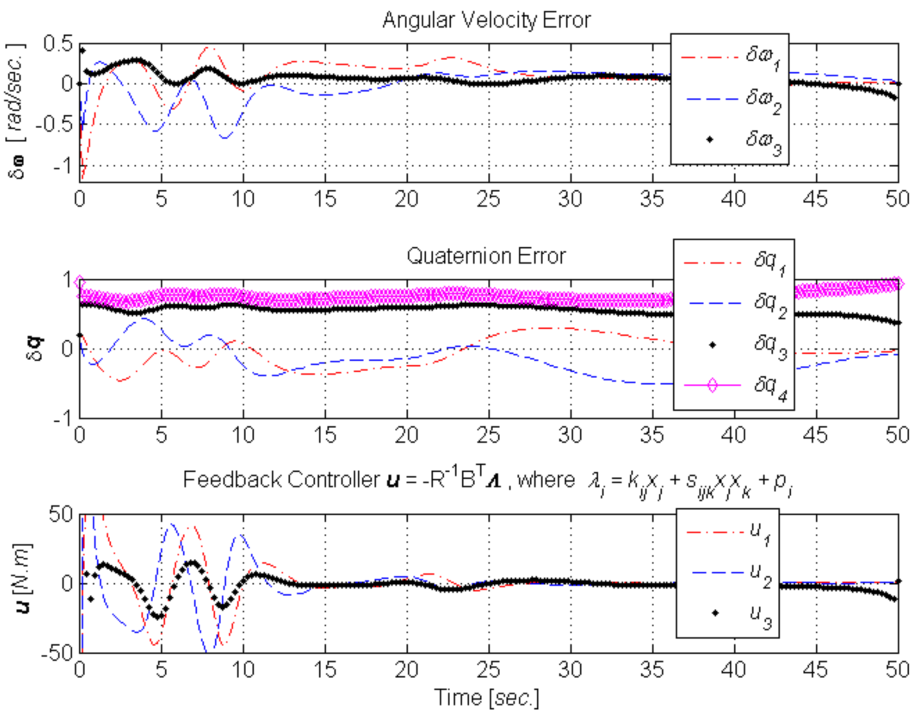

Figure 2. Quaternion Error Representation: Feedback solution

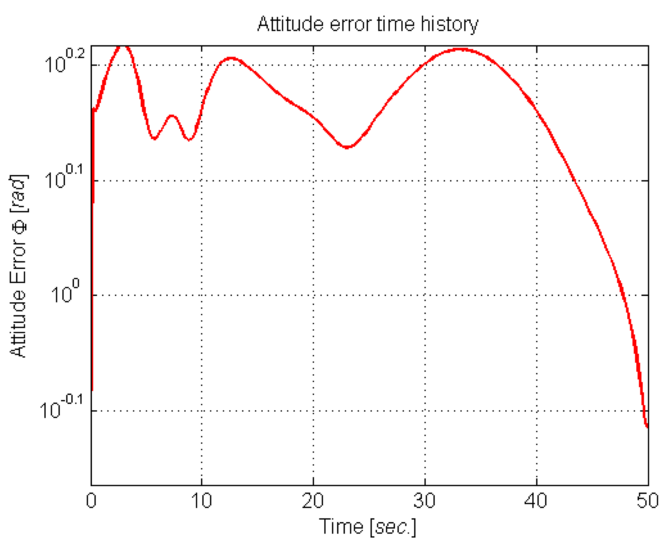

Figure 3. Quaternion Error Representation: Attitude error principal angle

The comparison between the attitude error representation is also summarized in table 3.

Figure 2, 4, and 5 present the feedback solution for the nonlinear optimal control problem. Solution histories are provided for of the spacecraft angular velocities, quaternion, CRP's and MRP's. One can easily see that both CRP's and MRP's have the very desirable property that all of its error components

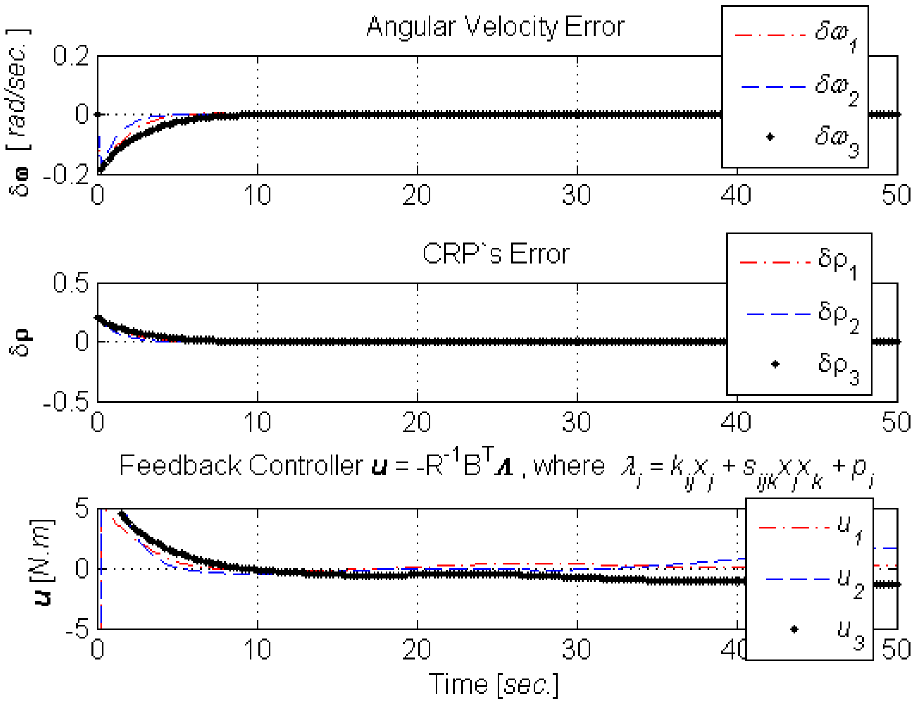

Figure 4. CRP's Error Representation: Feedback solution

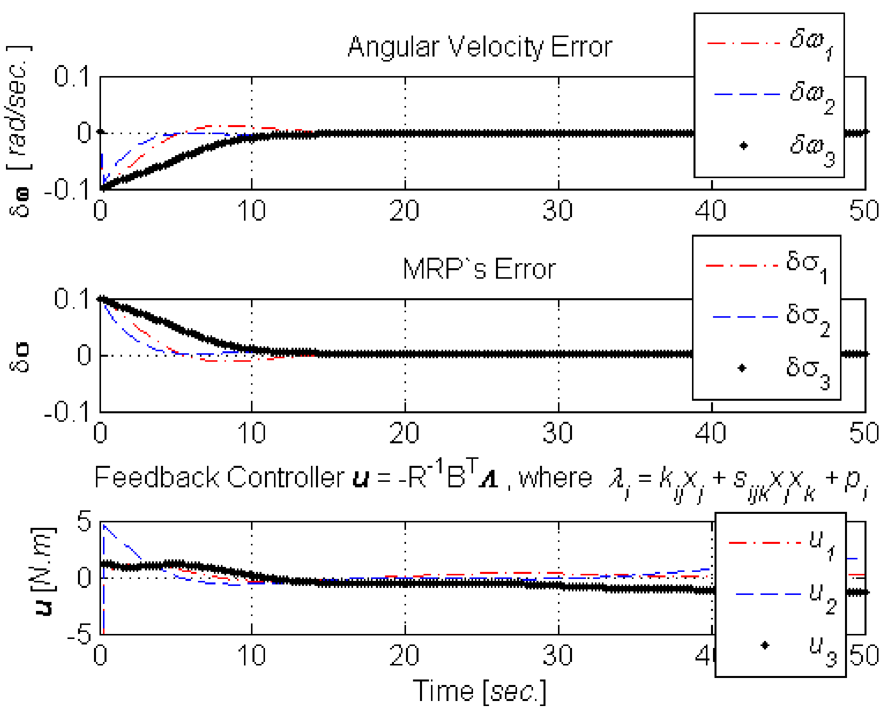

Figure 5. MRP's Error Representation: Feedback solution

are identically zero at the final time, whereas the quaternion-error final time values clearly do not vanish. The quaternion norm constraint is preserved throughout the maneuver. Figure 3 presents the attitude error angle obtained in the quaternion error case. Due to previously mentioned, the attitude error angle has a bad

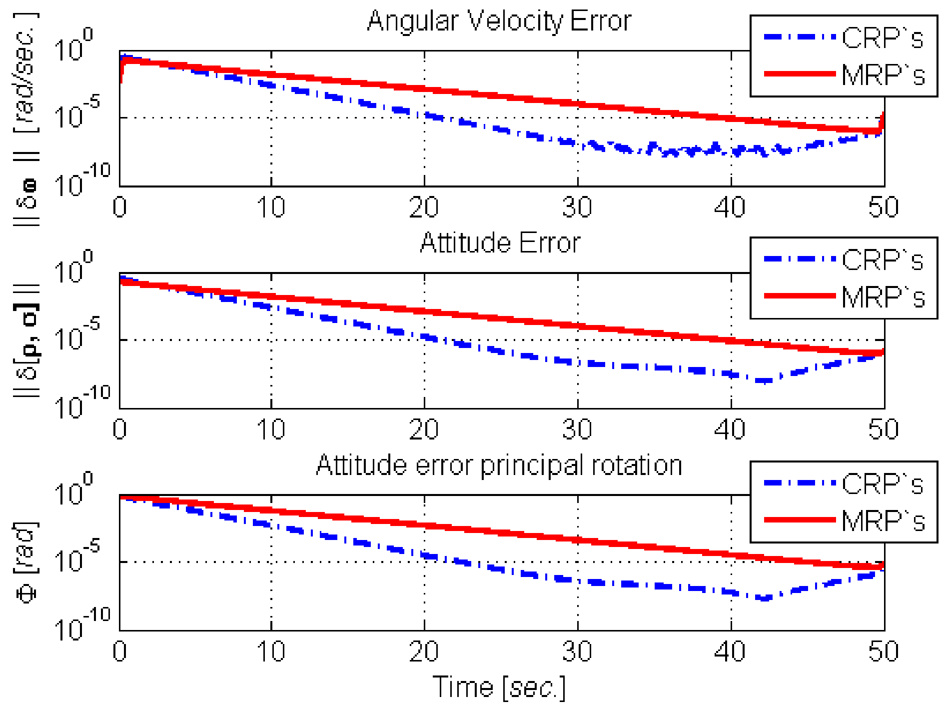

Figure 6. CRP's and MRP's Error Representation: Attitude trajectory

Table 3. Attitude Error Representation Comparisons

	Quaternion	MRP's	CRP's
Speed [*sec.*]	62.2875	25.2622	23.1926
Convergence Error	1.9540×10^{-5}	2.1159×10^{-5}	2.1155×10^{-5}
Performance Index	$3.2153 \times 10^{+4}$	59.5791	115.5169
Control Input $\| \boldsymbol{u} \|_2$	$3.8910 \times 10^{+4}$	553.4828	$1.1357 \times 10^{+3}$
Principal Angle $@t_f$	0.7697	6.4575×10^{-6}	3.4725×10^{-6}

convergence at the final time compared to the attitude error angle in CRP's and MRP's case, see Figure 6, which summaries the comparison between the norm error of CRP's and MRP's.

We mention the evident truth that minimization of quadratic penalties of tracking errors in each of the coordinate choices, while convenient, is not physically consistent across the above examples. This is because a quadratic penalty on the MRP's error is clearly not "the same" physically as a quadratic penalty on the classical Rodrigues parameters. Reference (14) introduces a universal" way to measure attitude error. We anticipate extending the current paper to utilize this universal attitude error measure expressed through approximate transformations as a positive function of each of the above coordinate choices.

V. Conclusions

A compact attitude motion error is derived for various attitude error representations. No simplifications are introduced in the derivation. These equations are very efficient to simulate. Fixed final time, fixed final state nonlinear optimal tracking control formulations have been considered where the state equation consists of the nonlinear state variables with disturbance rejection. A reference trajectory is defined by solving optimal open-loop spacecraft maneuvers control. The full nonlinear error dynamics for kinematics and the equation of motion are retained, yielding a tensor-based series solution for the Co-State as a function of error dynamics. Classical Rodrigues Parameters (CRP's), Modified Rodrigues Parameters (MRP's) and quaternion are used for describing the tracking orientation error dynamics. The resulting expressions are very compact, accurate, and computationally efficient. Both CRP's and MRP's produced a well-converged and inexpensive solution, compared to quaternion error. The unsuccessful convergence in the quaternion error solution is believed be related to the implicit requirement for maintaining the norm of the quaternion error variable, while the error feedback control is designed for achieving a final state of zero magnitude, not explicitly aimed at maintaining a zero norm constraint error. This issue will be resolved before we submit this paper for journal publication.

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