5 Chaos in high dimension map

5.1 Lyapunov Spectrum

In the section 3, we mainly discussed the judgement of chaos in 1 dimension. Here we focus on the Lyapunov exponent in high dimension. Obviously, in high dimension, the δ_n function which we used in Discussion 3.2 will be a m dimension vector δ_n and we mainly focus on the length(or norm) of this vector.

$$||\delta_t|| = \exp(\lambda t)||\delta_0||$$

$$t \to \infty, ||\delta_t|| = \exp(\lambda t)||\delta_0|| \Rightarrow \lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{||\delta_t||}{||\delta_0||} = \lim_{t\to\infty} \ln \left[\frac{||\delta_t||}{||\delta_0||}\right]^{\frac{1}{t}}$$

However, in high dimension problem, it is different from 1-dim because during the iteration, some direct of vector increase and the others decrease. Fig. 33 showed us an example of this evolution.

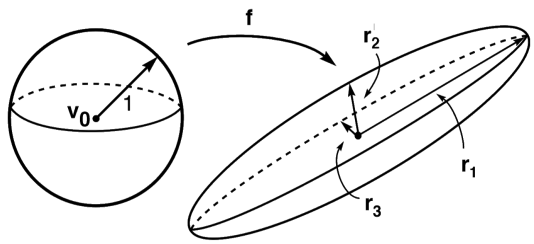

Figure 33: Evolution of unit vector in 3d

To solve this problem, we should consider the evolution of every direct in the space.

Let $\omega_0^{(1)}, \omega_0^{(2)}, \ldots, \omega_0^{(m)}$ is a group of orthogonal basis in $\mathcal{R}^m$ space which satisfied $\forall i, j = 1, 2, \ldots, m, i \neq j$, the inner production $< \omega_0^{(i)}, \omega_0^{(j)} >= 0$, then for every direction, we have a λ value based on the formula above, that means

$$\forall i = 1, 2, \ldots m, \lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln \frac{||\omega_t^{(i)}||}{||\omega_0^{(i)}||}$$

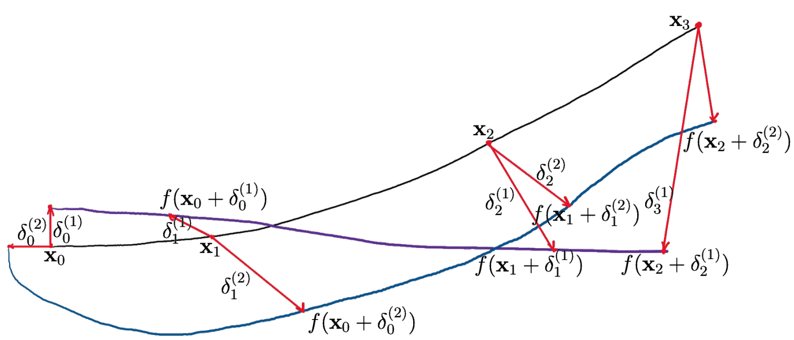

Figure 34: Analysis in two dimension problem

Basically, a simple basis in $\mathcal{R}^m$ space is $\omega_0^{(1)} = (1, 0, \ldots, 0)^T, \omega_0^{(2)} = (0, 1, \ldots, 0)^T, \ldots, \omega_0^{(m)} = (0, 0, \ldots, 1)^T$ and

$$\Omega_0 = (\omega_0^{(1)}, \omega_0^{(2)}, \ldots, \omega_0^{(m)}) = I$$

In this situation, $||\omega_0^{(i)}|| = 1 (i = 1, 2, \ldots, m)$, let

$$\Lambda = (\lambda_1, \lambda_2, \ldots \lambda_m) = \lim_{t \to \infty} \frac{1}{t} \left(\ln ||\omega_t^{(1)}||, \ln ||\omega_t^{(2)}||, \ldots, \ln ||\omega_t^{(m)}|| \right)$$

Let $p_1, p_2, \ldots p_m$ s.t. $\{p_1, p_2, \ldots, p_m\} = \{1, 2, \ldots, m\} \wedge \forall i, j = 1, 2, \ldots m, i \neq j, p_i \neq p_j$ and

$$\ln ||\omega_t^{(i_1)}|| \geq \ln ||\omega_t^{(i_2)}|| \geq \ldots \geq \ln ||\omega_t^{(i_m)}||$$

In the discrete time processing, we know that $t = 0, 1, 2, \ldots$ so $r_t^{(k)} = r_n^{(k)}$ where $n \in \mathcal{N}$.

Let $r_n^{(k)} = \ln ||\omega_n^{(i_1)}||$ be the length of the kth longest orthogonal axis after n time iterate for an initial point $\omega_0^{(i_1)}$. Obviously, these $r_n^{(k)}$ sequence (with k, not n) measured the expansion of initial vectors, so we can define the Lyapunov exponent in follows.

Definition 5.1 ***Lyapunov number, Lyapunov exponent in high dimension problem***
*Let $f \in C^\infty(\mathcal{R}^m)$, $J_n = Df^n(\mathbf{x}_0)$, $r_n^{(k)}$ be the length of the kth logenst orthogonal axis which defined by the explanation above. Then the kth **Lyapunov number** of $\mathbf{x}_0$ is defined by*

$$L_k = \lim_{n \to \infty} \left(r_n^{(k)} \right)^{\frac{1}{n}}, \text{ and the } \textbf{Lyapunov exponent } h_k = \ln L_k$$

Obviously,

$$(L_1 \geq L_2 \geq \ldots \geq L_m) \wedge (h_1 \geq h_2 \geq \ldots \geq h_m)$$

For every single $r_n^{(k)}$ we know that is familiar with λ in 1-dim Lyapunov exponent, so we have The orbit is **chaotic** if it satisfied both
[i] $\{x_1, x_2, \ldots\}$ is no asymptotically periodic, and
[ii] the Lyapunov exponent h_k is **greater** than zero.

As $h_1 \geq h_2 \geq \ldots \geq h_m$, so if $h_1 < 0$, then every Lyapunov exponent is less than zero, that means, we can simplify the definition and only care about h_1

Definition 5.2 ***Orbit chaotic in high dimension***
let f be a map of $\mathcal{R}^m$, and $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_1, \ldots$ be a bounded orbit of f, if
[i] *orbit is not asymptotically periodic; and*
[ii] *$\forall i = 1, 2, \ldots m, h_m \neq 0$ and $h_1 > 0$ then the orbit is **chaotic**.*

Now we consider how to calculate this Lyapunov exponent in normal problem.

[i] f is linear map

If f is a linear map, then $\exists P$ s.t. $f(\mathbf{x}) = P\mathbf{x}$ and

$$\forall i = 1, 2, \ldots, m, \omega_n^{(i)} = f(\mathbf{x}_{n-1} + \omega_{n-1}^{(i)}) - f(\mathbf{x}_{n-1}) = P(\mathbf{x}_{n-1} + \omega_{n-1}^{(i)}) - P\mathbf{x}_{n-1} = P\omega_{n-1}^{(i)} = P^n \omega_0^{(i)}$$

So we have

$$\Lambda = \lim_{n \to \infty} \frac{1}{n} \left(\ln ||\omega_n^{(1)}||, \ln ||\omega_n^{(2)}||, \ldots, \ln ||\omega_n^{(m)}|| \right) = \lim_{n \to \infty} \frac{1}{n} \ln \left(||P^n \omega_0^{(1)}||, ||P^n \omega_0^{(2)}||, \ldots, ||P^n \omega_0^{(m)}|| \right)$$

Define the vector norm function $\xi_p(X)$, where $X = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ s.t.

$$\xi_p(X) = (||\mathbf{x}_1||_p, ||\mathbf{x}_2||_p, \dots, ||\mathbf{x}_n||_p)$$

is the $p-$norm of every vector in the matrix. Then we have

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left(\xi(P^n\Omega_0)\right)$$

We know that Ω_0 is a normal orthogonal basis, and we said the most useful and simple orthogonal basis is I, so here we let $\Omega_0 = I$ then

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left(\xi(P^n)\right)$$

And now, the problem is how to calculate this P^n, we know that if the eigenvector and eigenvalue of P is V and E, then $P^n = V^{-1}E^nV$, that means, if there is a eigenvalue of P absolute greater than 1, then some element in P^n will satisfy $n \to \infty, p_{i,j} \to \infty$. So now we will try to put the $1/n$ into the ξ function.

We know that in the begining of the discussion, Λ satisfied

$$\begin{aligned}\Lambda = \lim_{n\to\infty} \frac{1}{n} \Bigg[&\ln\left(\frac{||\omega_n^{(1)}||}{||\omega_{n-1}^{(1)}||}\frac{||\omega_{n-1}^{(1)}||}{||\omega_{n-2}^{(1)}||}\cdots\frac{||\omega_1^{(1)}||}{||\omega_0^{(1)}||}\right), \ln\left(\frac{||\omega_n^{(2)}||}{||\omega_{n-1}^{(2)}||}\frac{||\omega_{n-1}^{(2)}||}{||\omega_{n-2}^{(2)}||}\cdots\frac{||\omega_1^{(2)}||}{||\omega_0^{(2)}||}\right), \dots \\ &\ln\left(\frac{||\omega_n^{(m)}||}{||\omega_{n-1}^{(m)}||}\frac{||\omega_{n-1}^{(m)}||}{||\omega_{n-2}^{(m)}||}\cdots\frac{||\omega_1^{(m)}||}{||\omega_0^{(m)}||}\right), \Bigg] \\ = \lim_{n\to\infty} \frac{1}{n} &\left[\sum_{i=1}^{n} \ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \sum_{i=1}^{n} \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \dots \sum_{i=1}^{n} \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right), \right. \\ = \lim_{n\to\infty} \frac{1}{n} &\left\{\sum_{i=1}^{n}\left[\ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \dots, \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right)\right]\right\}\end{aligned}$$

Notice for every $\omega_{i-1}^{(j)}, j = 1, 2, \dots, m$ we can find a normal orthogonal basis of $\mathcal{R}^m$ space based on the **Gram Schmidt Processing**

Algorithm 1 Gram-Schmidt process in orthogonal decomposition

INPUT: A n-dimension Euclidean space, a basis $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ of the space.

procedure GRAM-SCHMIDT PROCESS
  for $i = 1, i \le n, i++$ **do**
    $\beta_i = -\sum_{j=1}^{i-1} \frac{<\beta_j,\alpha_i>}{<\beta_j,\beta_j>}\beta_j + \alpha_i$
    $\beta_i = \text{normalization}(\beta_i) = \frac{\beta_i}{||\beta_i||}$
  end for
  return Normal orthogonal basis $\{\beta\}$
end procedure

Let the basis of $\omega_{i-1}^{(j)}$ is $B_{i-1} = (\beta_{i-1}^{(1)}, \beta_{i-1}^{(2)}, \dots \beta_{i-1}^{(m)})$, and $Q_{i-1} = (q_{i-1}^{(1)}, q_{i-1}^{(2)}), \dots, q_{i-1}^{(m)}$ s.t.

$$\forall j = 1, 2, \dots, m, \omega_{i-1}^{(j)} = B_{i-1} q_{i-1}^{(j)} \wedge ||q_{i-1}^{(j)}|| = 1$$

$$\left[\ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \dots, \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right)\right]$$

$$= \left[\ln\left(\frac{||PB_{i-1}q_{i-1}^{(1)}||}{||B_{i-1}q_{i-1}^{(1)}||}\right), \ln\left(\frac{||PB_{i-1}q_{i-1}^{(2)}||}{||B_{i-1}q_{i-1}^{(2)}||}\right), \dots, \ln\left(\frac{||PB_{i-1}q_{i-1}^{(m)}||}{||B_{i-1}q_{i-1}^{(m)}||}\right)\right] = \xi(P) \qquad (*)$$

And finally, we have

$$\Lambda = \lim_{n \to \infty} \frac{1}{n} \left\{ \sum_{i=1}^{n} [\xi(P)] \right\} = \xi(P)$$

[TODO] WRONG PROOF?

[ii] f is non-linear map

Now we focus on the problem that f is not a linear map. Typically, we will use Jacobian matrix to linearize the problem and for a sequence x_i we have

$$\mathbf{x}_{i+1} = f(\mathbf{x}_i) = P_i \mathbf{x}_i$$

where P_i is Jacobian matrix near the point x_i, so we just need to change the P matrix to P_i matrix in non-linear problems. We found in formula $(*)$ we have

$$\left[\ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \dots, \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right)\right]$$

$$= \left[\ln\left(\frac{||P_{i-1}B_{i-1}q_{i-1}^{(1)}||}{||B_{i-1}q_{i-1}^{(1)}||}\right), \ln\left(\frac{||P_{i-1}B_{i-1}q_{i-1}^{(2)}||}{||B_{i-1}q_{i-1}^{(2)}||}\right), \dots, \ln\left(\frac{||P_{i-1}B_{i-1}q_{i-1}^{(m)}||}{||B_{i-1}q_{i-1}^{(m)}||}\right)\right] = \xi(P_{i-1}) \qquad (*)$$

then, we have

$$\Lambda = \lim_{n \to \infty} \frac{1}{n} \left\{ \sum_{i=1}^{n} [\xi(P_{i-1})] \right\}$$

Finally, we can summary this processing in algorithm follows

Algorithm 2 Calculation of Lyapunov spectrum

INPUT: Values of system: $x_0, x_1, \dots x_N$;

procedure GramSchmidt(matrix); ▷ Return the Gram-Schmidt orthogonal matrix.
end procedure

procedure Jacobian(value); ▷ Return the Jacobian matrix at input value
end procedure

procedure VecNorm(matrix); ▷ Return the norm of every vector in the matrix
end procedure

procedure LyaSpec
    float $P = I$;
    list LyaSpec;
    for $i = 1, i \leq N, i++$ **do**
        P = Jacobian$(x_i) \cdot P$
        LyaSpec = LyaSpec + (ln VecNorm(P));
        P = VecNorm(P)
    end for
    LyaSpec = LyaSpec$/N$
    return LyaSpec
end procedure

Now we will apply the method with several example.

Example 5.1 ***Lyapunov spectrum in Henon's map***
Firstly we consider the Henon's map

$$\begin{cases} x = 1 - ax^2 + by \\ y = x \end{cases} \quad \textit{and the Jacobian matrix is } J(x,y) = \begin{bmatrix} -2ax & b \\ 1 & 0 \end{bmatrix}$$

Based on the Algo. 2, we can calculate the Lyapunov spectrum numerically.

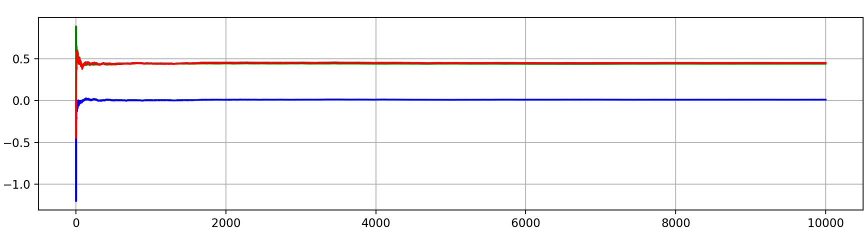

Figure 35: Lyapunov spectrum in Henon's map($a = 2, b = 0.3$)

$$\textit{Lyapunov spectrum} = (0.44004674, 0.01101559), \textit{sum} = 0.4510623256923707$$

We can check the conclusion with the volumn of the map.

Theorem 5.1 ***Volumn of the map***
Consider a map $f = (f_1, f_2, \ldots, f_m)$ in $\mathcal{R}^m$ the volumn of the map can be defined in follows

$$V = \sum_{i=1}^{m} \frac{\partial \dot{f}_m}{\partial x_m}$$

and the sum of the Lyapunov spectrum is equal to the volumn V.

Now we consider a continuous system s.t. $\dot{x} = f(x), x(0) = x_0$ where x_0 is a constant vector at initial time t_0. With Ronge-kutta method, we can find a group of value to simulate the system. So we can change the system to a map formed

$$x_{n+1} = g(x_n, t_n), x_0 = x(0), t_{n+1} = t_n + \Delta t$$

where g is based on the Ronge-kutta method.
Here we don't care about the formula g, we just consider for a certain n, if we still find a Jacobian matrix J_n, then $x_{n+1} = J_n x_n$.
On the other hand, we know that

$$\dot{x} = \frac{x(t_0 + (n+1)\Delta t) - x(t_0 + n\Delta t)}{\Delta t} = \frac{x_{n+1} - x_n}{\Delta t} = f(x_n)$$

In a certain model, we know the formula of f as well as parameter Δt. Let $\bar{J}(x)$ is Jacobian matrix of $f(x)$, then

$$x_{n+1} - x_n = \Delta t \bar{J}(x_n) x_n = (J_n - I_n) x_n \Rightarrow J_n = \Delta t \bar{J}(x_n) + I_n$$

E x a m p l e 5.2 ***Lyapunov spectrum in Lorenz system***
Now we consider the Lorenz system:

$$\begin{cases} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{cases} \quad \textit{and the Jacobian matrix of } f \textit{ is } \bar{J}(x, y, z) = \begin{bmatrix} -\sigma & \sigma & 0 \\ (\rho - z) & -1 & -x \\ y & x & -\beta \end{bmatrix}$$

So the Jacobian matrix of the discrete maps is

$$J(x, y, z) = \Delta t \bar{J}(x, y, z) + I = \begin{bmatrix} 1 - \sigma\Delta t & \sigma\Delta t & 0 \\ (\rho - z)\Delta t & 1 - \Delta t & -x\Delta t \\ y\Delta t & x\Delta t & 1 - \beta\Delta t \end{bmatrix}$$

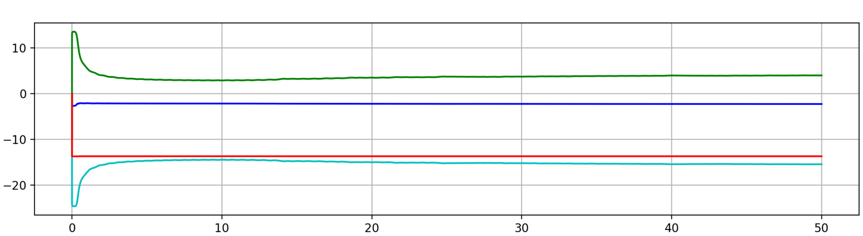

Figure 36: Lyapunov spectrum in Lorenz system($(\sigma, \rho, \beta, \Delta t) = (28, 10, 8/3, 0.0001)$)

$$Lyapunov\ spectrum = (4.00544221, -2.24674614, -15.42821896), sum = -13.669522889292935$$

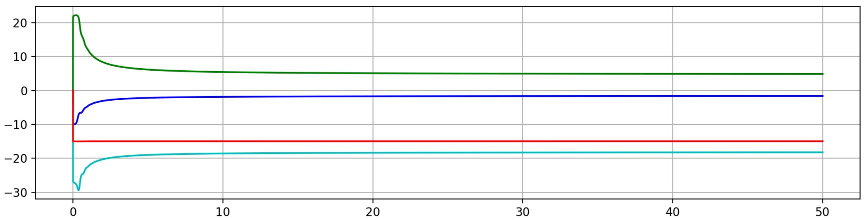

Figure 37: Lyapunov spectrum in Lorenz system($(\sigma, \rho, \beta, \Delta t) = (45.92, 4, 10, 0.0001)$)

$$Lyapunov\ spectrum = (4.8857278, -1.61501712, -18.23236088), sum = -14.961650200700317$$

E x a m p l e 5.3 ***Lyapunov spectrum in Rossler system***

Now we consider the Rossler system:

$$\begin{cases} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{cases} \quad \text{and the Jacobian matrix of } f \text{ is } \bar{J}(x, y, z) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{bmatrix}$$

So the Jacobian matrix of the discrete maps is

$$J(x, y, z) = \Delta t \bar{J}(x, y, z) + I = \begin{bmatrix} 1 & -\Delta t & -\Delta t \\ \Delta t & a\Delta t + 1 & 0 \\ z\Delta t & 0 & (x - c)\Delta t + 1 \end{bmatrix}$$

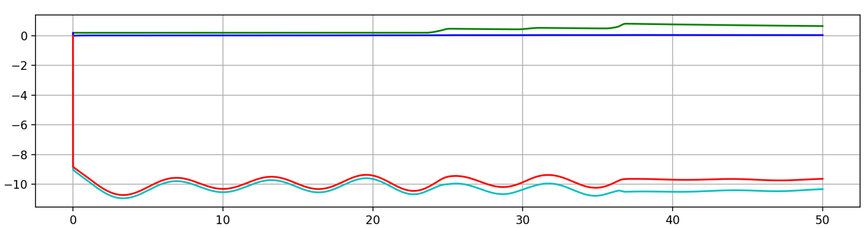

Figure 38: Lyapunov spectrum in Rossler system($(a, b, c, \Delta t) = (0.2, 0.3, 9, 0.001)$)

$$Lyapunov\ spectrum = (0.65174245, 0.04364673, -10.32448076), sum = -9.629091584454333$$

E x a m p l e 5.4 ***Lyapunov spectrum in Duffing system***

Now we consider the Duffing equation:

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t)$$

Let $y = \dot{x}$, then

$$\dot{y} = \ddot{x} = \gamma \cos(\omega t) - \delta\dot{x} - \alpha x - \beta x^3 = \gamma \cos(\omega t) - \delta y - \alpha x - \beta x^3$$

So we have

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= \gamma\cos(\omega t) - \alpha x - \beta x^3 - \delta y \end{cases} \quad \textit{and the Jacobian matrix of } f \textit{ is } \bar{J}(x,y) = \begin{bmatrix} 0 & 1 \\ -\alpha - 3\beta x^2 & \delta \end{bmatrix}$$

So the Jacobian matrix of the discrete maps is

$$J(x,y,z) = \Delta t \bar{J}(x,y) + I = \begin{bmatrix} 1 & \Delta t \\ (-\alpha - 3\beta x^2)\Delta t & \delta\Delta t + 1 \end{bmatrix}$$

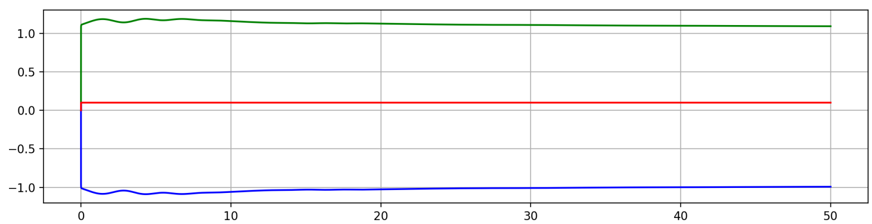

Figure 39: Lyapunov spectrum in Duffing equation($(\alpha, \beta, \gamma, \delta, \omega, \Delta t) = (1, 0.04, 1, 0.1, \pi/2, 0.001)$)

$$\textit{Lyapunov spectrum} = (1.0913024, -0.99141093), \textit{sum} = 0.09989146956531447$$