4 Fractal

4.1 General tent map, Cantor set and self-similar attractor

In this section, we will mainly focus on the general tent map

$$T_a(x) = \begin{cases} ax & x \leq 1/2 \\ a(1-x) & x \geq 1/2 \end{cases}$$

There are several properties of this map

Property 4.1 ***General tent map - fixed point***
$a \in (0,1)$: *single fixed point* 0*, all initial conditions are attracted to 0;*
$a \in (1, \infty)$: *both* 0 *and* $\frac{1}{1+a}$ *are fixed point.*

In last section, we know that if $a = 2$, then $T_2(x)$ is mapped onto itself and we have

Property 4.2 ***Point leave the interval***
$a \in (0,2]$: *points stay within I;*
$a \in (2, \infty)$: *a.e. points eventually leave the interval and never return, where a.e. means almost everywhere, that means, without a measure zero subset, all set will satisfied this property.*

We will try to proof the second property.

***PROOF* 4.1** *[i] Obviously, based on the definition of tent map, if* $x < 0$*, then* $f(x) < 0$*, if we let* $x_1 = f(x) < 0$*, then* $f(x_1) < 0$ *etc. So we have this conclusion: if* $f^p(x) < 0$*, then* $\forall q > p, f^q(x) < 0$ *where* $p, q \in \mathcal{N}$*.*
Moreover, if $x > 1$*, then* $f(x) < 0$ *and* $\forall n \in \mathcal{N}^+, f^n(x) < 0$*.*

Define the set L s.t.

$$L = \{x \in [0,1] | f^n(x) < 0\}$$

where $n \in \mathcal{N}$ *is a certain value.*
[ii] *If* $a \in (0,2)$*, then* $f([0,1]) = [0, a/2] \subset [0,1]$*. On the other hand, as* $\forall x \in [0,1], f(x) \geq 0$*, so* $L = \varnothing$*.*
[iii] *We will proof that if* $a > 2$*, then* $\mu(L) = 1$ *where* μ *is (lebesgue) measure. Firstly, we consider the interval* $L_1 = (1/a, (a-1)/a)$ *s.t.* $\forall x \in L_1, f(x) > 0 \Rightarrow f^2(x) < 0$*.*
Now we consider the subset $L_0 = (0, 1/a)$*, we found that* $f((0,1/a)) = (0,1)$*, so we can apart the interval* $(0, 1/a)$ *again, where*

$$L_{00} = (0, 1/a^2), L_{01} = (1/a^2, (a-1)/a^2), L_{02} = ((a-1)/a^2, 1/a)$$

We found that $\forall x \in L_{01}, f^2(x) > 1 \Rightarrow f^3(x) < 0$
If we consider the subset $L_2 = ((a-1)/1, 1)$*, we found that is symmetric of* $(0, 1/a)$*, that means we can also apart* L_2 *as* L_{20}, L_{21}, L_{22}*, where* L_{21} *have same property with* L_{01}*. We found the structure of this set is familiar with* ***Cantor set*** *especially when* $a = 3$*.*

Definition 4.1 ***Cantor Set***
Let set G s.t.

$$G = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

then called $C = [0,1] \backslash G = [0,1] \bigcap G^c$ as Cantor set.

Based on the knowledge in real analysis, we know that

Property 4.3 $\mu(C) = 0, \mu(G) = 1$ *where μ is measure of set.*

*The proof of this property is simple because we know that for a certain interval (a,b), the measure $\mu((a,b)) = b - a$ which also be called as **Borel measure**. And we can just calculate the measure of G with limitation.*

Now we come back to the problem above

If $a = 3$, then $L = G$ and we proved the problem. If $a > 2, a \neq 3$, we can found a one-to-one map from $T_a(x)$ to $T_3(x)$ and the attractor will express same property as $a = 3$. So we finally proved that $\forall a > 2$, $x \in [0,1]$ a.e. s.t.$\exists N, \forall n > N, f^n(x) < 0$ ■

Now we come back to the title of this section, so that means "fractal"? We know that for every subset of Cantor set, or attractor of $T_a (a > 2)$ map, these subset showed us same property of the original set and we called this **self-similar** set as **fractal**.

Here are some example of fractal set.

Definition 4.2 ***Iterated function system***.
Consider a group of map on R^m s.t. $f = \{f_1, f_2, \ldots f_r\}$ and for every maps, exists a positive number $p_1, p_2, \ldots p_r$ s.t. $\sum_{i=1}^{r} p_i = 1$ (probabilities). Then we called this group of f_i is iterated function system.

E x a m p l e 4.1 *(A simple iterated function system)*
0 *Roll a point x_0 randomly in $[0,1]$*
[i] *flip a coin,*
[i-1] *if coin comes up heads, then move the point x_{i-1} to $x_i = \frac{1}{3} x_{i-1}$*
[i-2] *if coin comes up tails, then move the point x_{i-1} to $x_i = \frac{1}{3}(2 + x_{i-1})$*

We can simulate this example with code, if we statistic the point, or find the density figure of map, we found that .

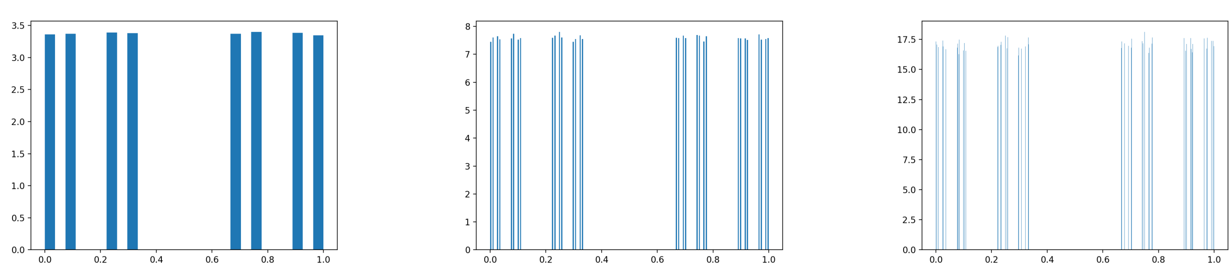


Figure 27: Simulation of Simple Ierated Function System

which showed us the property of Cantor set, and we can guess that the Cantor set is the attractor of the probabilitic constructions.

Here is another example of self-similar map.

E x a m p l e 4.2 ***Sierpinski carpet*** .

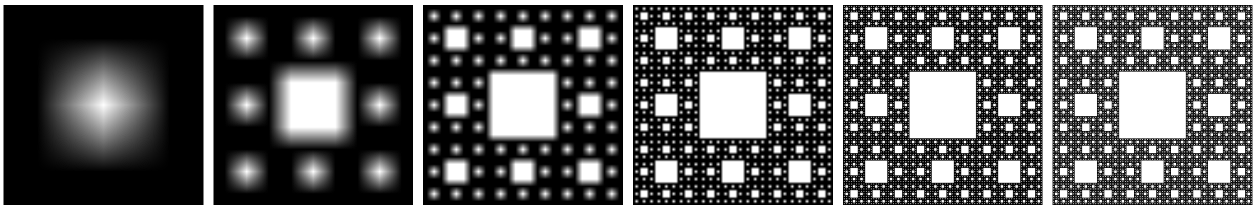


Figure 28: Sierpinski carpet

Of couse, Henon's map also have this fractal property.

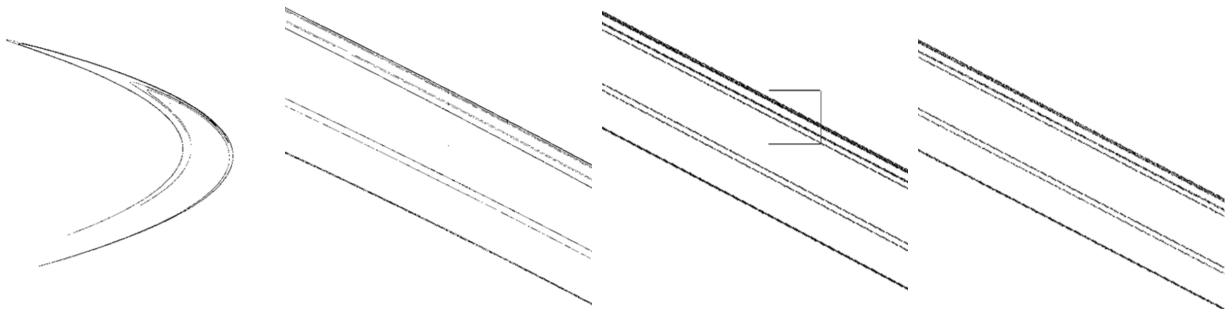


Figure 29: Fractal in Henon's map

Moreover, we can also discuss this fractal property in basin.

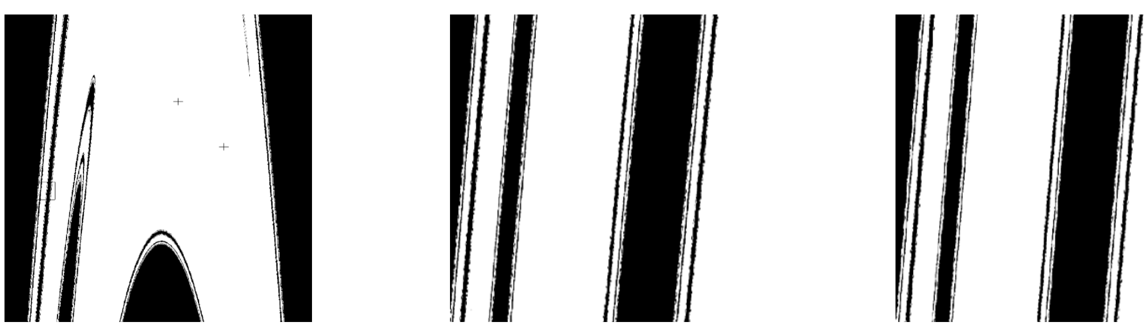


Figure 30: Fractal in basin of Henon's map

E x a m p l e 4.3 ***Julia set, Mandelbrot set***

Now we consider a map in complex values

$$P_c(z) = z^2 + c$$

where z, c *are complex number s.t.* $\exists x, y, c_x, c_y \in \mathcal{R}, z = x + yi, c = c_x + c_y i$ *and* $i = \sqrt{-1}$. *Based on the calculation rules of complex number, we know*

$$P_c(z) = (x + yi)^1 + (c_x + c_y i) = x^2 + 2xyi - y^2 + c_x + c_y i = (x^2 - y^2 + c_x) + i(2xy + c_y)$$

so we have

$$f(x,y) = (Re[P_c(z)], Im[P_c(z)]) = (x^2 - y^2 + c_x) + i(2xy + c_y) \Rightarrow \begin{cases} x_{n+1} = x_n^2 - y_n^2 + c_x \\ y_{n+1} = 2x_n y_n + c_y \end{cases}$$

where $n \in \mathcal{N}, x_n, y_n, c_x, c_y \in \mathcal{R}$ are c_x, c_y is constant.

We know that if we consider a map formed $f(x,y) = x^2 + y^2$, then the unit circle is important, for ever point inside the unit circle, the are all sink to zero point and for every point outside the unit circle, they will go to infinity after enough iteration.

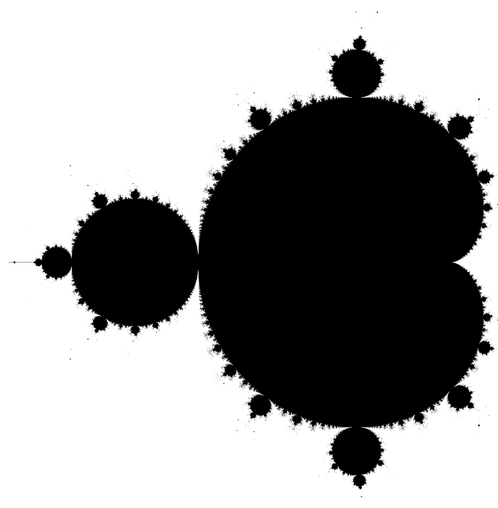


Figure 31: Mandelbrot set

We called the Black area in the image above as Mandelbrot set, more mathematically, the **Mandelbrot set** is

$$M = \{c : 0 \text{ is not in the basin of infinity for the map } P_c(z) = z^2 + c\}$$

We can analysis the convergence and disconvergence of this complex map too. For a certain c, now we consider the convergence and disconvergence for every point in the space. We called this set as Julia set.

Definition 4.3 ***Julia set***

Consider a map $f : \mathcal{R}^n \to \mathcal{R}^n$

$$J(f) = \{x | x \in R^n, \forall \varepsilon > 0, \exists x_1, x_2 \in N(x,\varepsilon) s.t. \left(\lim_{n\to\infty} f^n(x_1) < \infty \wedge \lim_{n\to\infty} f^n(x_2) = \infty\right)\}$$

which is the boundary points between convergence area and disconvergence area.

Figure 32: Julia set

(parameter $c_1 = c_2 = -0.17 + 0.78i, c_3 = 0.38 + 0.32i, c_4 = 0.32 + 0.043i$)

Table 8: Interval

img	x	y	img	x	y
1	$[-1.5, 1.5]$	$[-1.5, 1.5]$	2	$[-0.19, 0.01]$	$[0.89, 1.09]$
3	$[-1.3, 1.3]$	$[-1.3, 1.3]$	4	$[-1.3, 1.3]$	$[-1.3, 1.3]$

4.2 Fractal dimension

Finally, we try to analysis the calculation of these fractal. In analysis we know that **measure** is a describe of area, also we define the Lebesgue outer measure in follow

Definition 4.4 ***Eculidean rectangle Neighbourhood***
1. Opened box $I = \{x = (x_1, x_2, \dots, x_n) | a_i < x_i < b_i, i \in N\}$, where a_i, b_i are constant;
2. Closed box $I = \{x = (x_1, x_2, \dots, x_n) | a_i \leq x_i \leq b_i, i \in N\}$, where a_i, b_i are constant;
Called the $b_i - a_i$ is side length of a box. If $|b_1 - a_1| = |b_2 - a_2| = \dots = |b_n - a_n|$, then called this box as cube.

Definition 4.5 ***Lebesuge outer measure***

$$\mu^* E = \inf \left\{ \sum_{i=1}^{\infty} |I_i| : E \subset \bigcup_{i=1}^{\infty} I_i, I_i \textit{ is an opened box} \right\}$$

Also, we can define the outer measure with ε (Familiar with Cauchy's limitation)

$$\forall \varepsilon > 0, \exists \{I_i\} \textit{ open cover s.t.} E \subset \bigcup_{i=1}^{\infty} I_i \wedge \mu^* E \leq \sum_{i=1}^{\infty} |I_i| + \varepsilon$$

In this subsection, we mainly focus on the area based on the cube.

Definition 4.6 $C(1/\varepsilon)$ ***cube***
Now we consider a special type of cover.
***[i]** Firstly, for a interval $I_1 = [a_1, b_1]$, let the side length of boxes is ε, then we have $N = int(C/\varepsilon) + 1$ subintervals to cover the original interval. (e.g. $[a_1 + (p-1)C/\varepsilon, a_1 + pC/\varepsilon], p = 1, 2, \dots N$)*
***[ii]** Now we consider a surface in R^n space, let the projection on x_i axis is $I_i = [a_i, b_i]$, and $C = \max\{\mu(I_i)\}, i = 1, 2, \dots, n$, then we have a group of cube total $N(\varepsilon) = (C\varepsilon)^n$ and*

$$d = \frac{\ln N(\varepsilon) - \ln C}{\ln(1/\varepsilon)}$$

Definition 4.7 $C(1/\varepsilon)$ ***cube***
A bounded set $s \subset R^n$ has box-counting dimension

$$bd(S) = \lim_{\varepsilon \to \infty} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)}$$

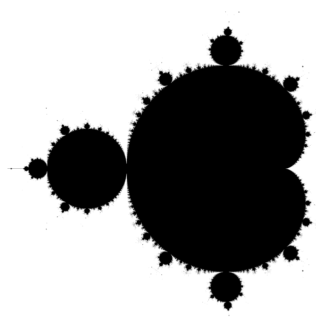
when the limit exists.

***CONCLUSION* 4.1** *Based on the Bolzano–Weierstrass theorem, we know that if we have a sequence $\{b_n\}$ s.t. $\lim_{n\to\infty} b_n = 0$, if b_n is ε then $\lim_{n\to\infty} \frac{\ln b_{n+1}}{\ln b_n} = 1$*

***CONCLUSION* 4.2** *We still consider the sequence above, then*

$$\frac{N(b_n)}{4} \leq N(b_n) \leq 4N(b_{n+1})$$

and the proof is simple with figure follows



Theorem 4.1 *If $\{b_n\}$ is monotony, or assume $b_1 > b_2 > \ldots > b_n > \ldots > b_\infty = 0$ If*

$$\lim_{n\to\infty} \left(\frac{\ln b_{n+1}}{\ln b_n} \right) = 1 \wedge \lim_{n->\infty} \left(\frac{\ln N(b_n)}{\ln(1/b_n)} \right) = d =$$

then $\lim_{\varepsilon\to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = d$ and therefore the box-counting dimension is d.

Theorem 4.2 *If $S \subset R^n$ is bounded and $bd(S) = d < n$, then $\mu(S) = 0$*

Another way to calculate the measure of a area is based on the statistic,

Definition 4.8 ***Correlation dimension*** *Let $S = \{v_0, v_1, \ldots\}$ be an orbit of the map $f : \mathcal{R}^n \to \mathcal{R}^n$, then for ever $r > 0$, define $C(r)$ s.t.*

$$C(r) = \lim_{N\to\infty} \frac{|\{(p,q)|p,q \in S, |p-q| < r\}|}{|\{(p,q)|p,q \in S\}|}$$

Moreover, if $C(r) \approx r^d$, then

$$d \approx cd(S) = \lim_{r\to\infty} \frac{\ln C(r)}{\ln(r)}$$

And here we found another way to calculation the area of attractor.