Problem in continuous time system

7 Differential equations

7.1 Basic Definition

Definition 7.1 ***Terms***
Autonomous: *The time variable t does not explicitly appear, for instance*

$$\text{Autonomous: } \dot{x} = ax, \text{ Nonautonomous:} \ddot{x} = -c\dot{x} - \sin x + \rho \sin x$$

Initial Value *The number* $x_0 = x(0)$ *is called the* ***initial value*** *of the function* x.
Flow *The* ***flow*** F *of an autonomous differential equation is the function of time* t *and initial value* x_0 *which represents the set of solutions. Thus* $F(t, x_0)$ *is the value at time* t *of the solution with initial value* x_0 *. We will often use the slightly different notation* $F_t(x_0)$ *to mean the same thing.*
Equilibrium *A constant solution of the autonomous differential equation* $\dot{x} = f(x)$ *is called an* ***equilibrium*** *of the equation.*
Sink *An equilibrium solution is called* ***attracting*** *or a* ***sink*** *if the trajectories of nearby initial conditions converge to it.*
Source *It is called* ***repelling*** *or a* ***source*** *if the solutions through nearby initial conditions diverge from it.*

* *I. One dimension system.*

Example 7.1 ***Logistic differential equation***
Consider a differential equation $\dot{x} = ax(1 - x)$ *where* $a > 0$ *is a constant.*

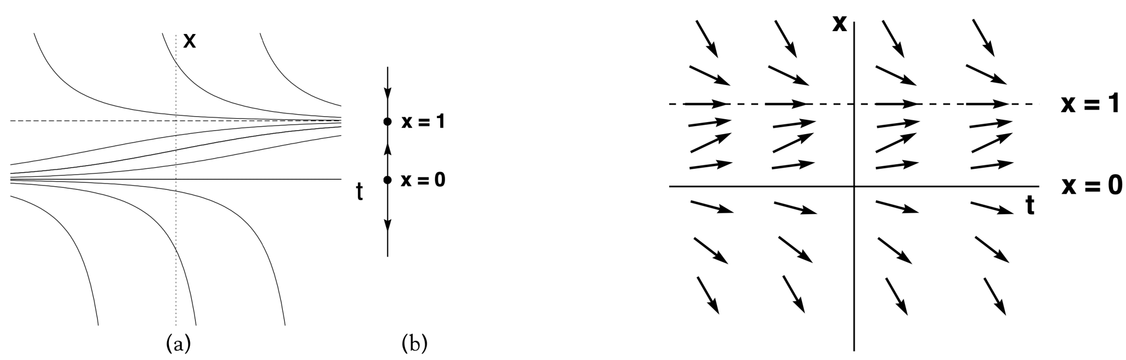

Figure 56: $x_0 - t$ plot of logistic DE

[i] ***Existence*** *Each point in the* (t, x)*-plane has a solution passing through it. The solution has slope given by the differential equation at that point.*
[ii] ***Uniqueness*** *Only one solution passes through any particular* (t, x).
[iii] ***Continuous dependence*** *Solutions through nearby initial conditions remain close over short time intervals. In other words, the flow* $F(t, x_0)$ *is a continuous function of* x_0 *as well as* t.
Using the first concept, we can draw a ***slope field*** *in the* (t, x)*-plane by evaluating* $\dot{x} = ax(1 - x)$ *at several points and putting a short line segment with the evaluated slope at each point*

* II. High dimension linear system.

Definition 7.2 ***Lyapunov Stable*** *An equilibrium point* $\mathbf{v}$ *is called* ***stable or Lyapunov stable*** *if every initial point* $\mathbf{v_0}$ *that is chosen very close to* $\mathbf{v}$ *has the property that the solution* $F(t, \mathbf{v_0})$ *stays close to* v *for* $t \geq 0$.
More formally, for any neighborhood N *of* $\mathbf{v}$ *there exists a neighborhood* N_1 *of* $\mathbf{v}$*, contained in* N*, such that for each initial point* $\mathbf{v_0}$ *in* N_1*, the solution* $F(t, \mathbf{v_0})$ *is in* N *for all* $t \geq 0$.
An equilibrium is called ***asymptotically stable*** *if it is both stable and attracting.*
An equilibrium is called ***unstable*** *if it is not stable.*
Finally, an equilibrium is ***globally asymptotically stable*** *if it is asymptotically stable and all initial values converge to the equilibrium.*

Theorem 7.1 ***Stability of Origin*** *Let* A *be an* $n \times n$ *matrix, and consider the equation* $\dot{\mathbf{v}} = A\mathbf{v}$*. If the real parts of all eigenvalues of* A *are negative, then the equilibrium* $\mathbf{v} = 0$ *is globally asymptotically stable. If* A *has* n *distinct eigenvalues and if the real parts of all eigenvalues of* A *are nonpositive, then* $\mathbf{v} = 0$ *is stable.*

III. High dimension nonlinear system.

CONCLUSION **7.1** *For all high dimension ODE, it can be transformed into a first-order system.*

So if we consider a high dimension nonlinear equation, that is same as consider a 1-dim ODE system.

Theorem 7.2 ***Existence and Uniqueness***
Consider the first-order differential equation $\dot{\mathbf{v}} = f(\mathbf{v})$ *where both* f *and its first partial derivatives with respect to* v *are continuous on an open set* U*. Then for any real number* t_0 *and real vector* v_0*, there is an open interval containing* t_0*, on which there exists a solution satisfying the initial condition* $v(t_0) = v_0$*, and this solution is unique.*

Definition 7.3 ***Lipschitz Constant***
Let U *be an open set in* $\mathcal{R}^n$*. A function* f *on* $\mathcal{R}^n$ *is said to be Lipschitz on* U *if there exists a constant* L *s.t.*

$$\forall v, w \in U, ||f(v) - f(w)|| \leq L||v - w||$$

The constant L *is called a Lipschitz constant for f.*

Theorem 7.3 ***Continuous dependence on initial conditions***
Let f *be defined on the open set* U *in* $\mathcal{R}^n$*, and assume that* f *has Lipschitz constant* L *in the variables* v *on* U*. Let* $v(t)$ *and* $w(t)$ *be solutions of* $\dot{\mathbf{v}} = f(\mathbf{v})$*, and let* $[t_0, t_1]$ *be a subset of the domains of both solutions. Then*

$$\forall t \in [t_0, t_1], ||v(t) - w(t)|| \leq ||v(t_0) - w(t_0)|| \exp(L(t - t_0))$$

Next two definition introduced the sink, source and saddle in continuous problem.

Definition 7.4 *An equilibrium* v_0 *of* $\dot{v} = f(v)$ *is called hyperbolic if none of the eigenvalues of* $Df(v_0$ *has real part 0.*

Definition 7.5 *Let* v_0 *be an equilibrium of* $\dot{v} = f(v)$*. If the real part of each eigenvalue of* $Df(v_0)$ *is strictly negative, then* v_0 *is asymptotically stable. If the real part of at least one eigenvalue is strictly positive, then* v_0 *is unstable.*

7.2 Energy Function, Lyapunov Function

Definition 7.6 ***Level curve***
Given a real number c and a function $E : \mathcal{R}^2 \to \mathcal{R}$, the set $E_c = \{(x, y)|E(x, y) = c\}$ is called a ***level curve*** *of the function E.*

Definition 7.7 ***Lyapunov Function***
Let v_0 be an equilibrium of $\dot{\mathbf{v}} = f(\mathbf{v})$. A function $E : \mathcal{R}^n \to \mathcal{R}$ is called a Lyapunov function for v_0 if for some neighborhood $N(v_0)$, the following conditions are satisfied:
[i] $\forall v \in N(v_0)\backslash\{v_0\}, E(v_0) = 0 \wedge E(v) > 0$
[ii] $\forall v \in N(V_0), \dot{E}(v) \leq 0$

If the stronger inequality

$$\textbf{\textit{[ii']}} \forall v \in N(V_0), \dot{E}(v) < 0$$

holds, then E is called a strict Lyapunov function.

Theorem 7.4 *Let v_0 be an equilibrium of $\dot{\mathbf{v}} = f(\mathbf{v})$. If there exists a Lyapunov function for v, then v is stable. If there exists a strict Lyapunov function for v, then v is asymptotically stable.*

Definition 7.8 ***Basin of attraction***
Let v_0 be an asymptotically stable equilibrium of $\dot{\mathbf{v}} = f(\mathbf{v})$. Then the basin of attraction of v_0, denoted $B(v_0)$, is the set of initial conditions v_{init} s.t.

$$\lim_{t\to\infty} F(t, v_{init}) = v_0$$

Definition 7.9 *A set $U \subset \mathcal{R}^n$ is called a forward invariant set for $\dot{\mathbf{v}} = f(\mathbf{v})$ if for each $v_0 \in U$, the forward orbit $\{F(t, v_0) : t \geq 0\}$ is contained in U. A forward invariant set that is bounded is called a* ***trapping region****. We also require that a trapping region be an n-dimensional set.*

CONCLUSION 7.2 ***Barbashin-LaSalle***
Let E be a Lyapunov function for v_0 on the neighborhood $N(v_0)$. Let $Q = \{v \in N : \dot{E}(v) = 0\}$. Assume that N is forward invariant. If the only forward-invariant set contained completely in Q is v_0, then v_0 is asymptotically stable. Furthermore, N is contained in the basin of v_0; that is, $\forall v \in N, \lim_{t\to\infty} F(t, v) = v_0$