2 Two-Dimension and High-Dimension Maps

In this section, we will mainly discusse a new type of model, called Henon map which formed

$$f(x,y) = (a - x^2 + by, x)$$

A simple way to analysis this problem is analysis all point in the surface if they are convergence or divergence. In figures following, point in black represent initial conditions whose orbits diverge to infinity and the points in white represent initial values whose orbits converge to the period-2 orbit.

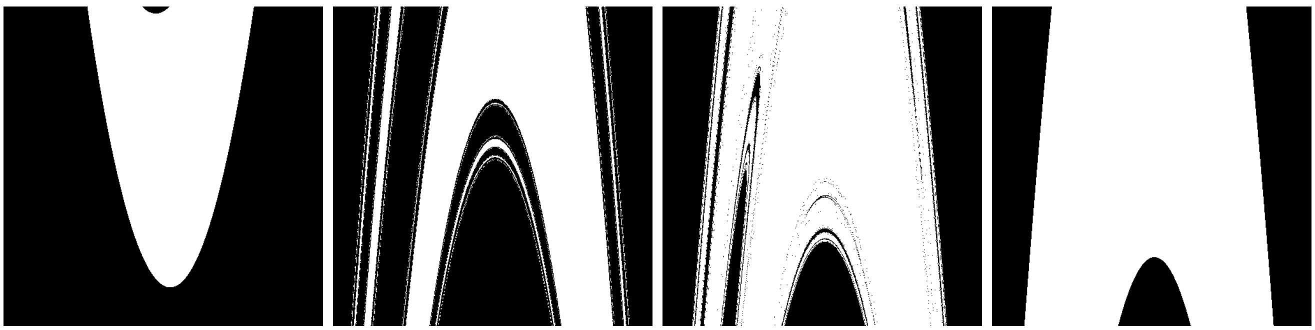

Figure 14: Initial condition square

(Parameter group $(a, b) = (0, 0.4), (2, -0.3), (1.4, -0.3), (1.28, -0.3)$)

2.1 Analysis of Henon map

Now we focus on Henon map. Familiar with 1 dim map, it is necessary to define the sink and source as well as saddle.

Definition 2.1 ***Neighborhood***
Consider a R^n space, called every point $x = (x_1, x_2, \ldots x_n)$ is a vector of R^n space,
Define the ***Euclidean Length*** $|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$*, which is equal to norm;*
And define the distance between two point $d(x, y) = |x - y|$;
Also, the ***ε-neighborhood*** *is*

$\forall \varepsilon > 0,$ *the ε-neighborhood of point $p, N_\varepsilon(p)$ is $\{x \in R^n \mid |x - p| < \varepsilon\}$, also define $N_\varepsilon^o(p) = N_\varepsilon(p) \backslash \{p\}$*

Definition 2.2 ***Sink and Source in High-dimension Map***
Let f is a map on R^n, p is a vector on R^n which is the fixed point and $f(p) = p$ then
If there is an $\varepsilon > 0$ s.t. $\forall x \in N_\varepsilon(p), \lim_{k\to\infty} f^k(x) = p$, then p is a sink or attracting fixed point.
If $\forall x \in N_\varepsilon^o(p), \exists K s.t. \forall k > K, f^k(x) \notin N_\varepsilon(p)$, then called the point p as source.

We will explain these definitions with an example

E x a m p l e 2.1 *Analysis the sink point, source point and saddle of Henon map with parameter $a = 0, b = 0.4$*

***SOLUTION* 2.1** *Obviously, if we consider the funcion* $f(x,y) = (-x^2+0.4y, x) = (x,y)$*, then*

$$-0.2x^2 + 0.4x = x \Rightarrow x_1 = 0, x_2 = -0.6$$

So the fixed points are $(0,0)$ *and* $(-0.6,-0.6)$*. And now we have a new problem: how to confirm a fixed point is sink or source. Even the definition of sink and source are given above, we still need theory like Theo. 1.1. But here, we can analysis the problem with simulator.*■

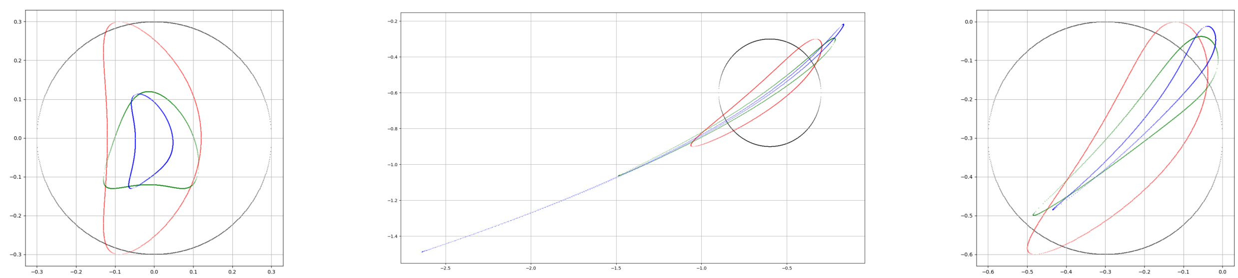

Figure 15: Sink, source and saddle in Henon map with $a = 0, b = -0.4$

(Order of color: Black(neighborhood), Red (Iter = 1), Green (Iter = 2), Blue (Iter = 3))

To solve the problem we faced in e.g.2.1, we will discuss the simple form of the high dimension maps.

Definition 2.3 ***High dimension linear map***
A map $A: R^m \to R^m$ *is* ***linear*** *if* $\forall a, b \in R, \forall x, y \in R^m$, $f(ax+by) = af(x)+bf(y)$*. Equivalently, a lienar map* $f(x)$ *can be represented as multiplication by an* $m \times m$ *matrix.*

Now we consider a system s.t. $f(x) = Ax$, if λ is eigenvalue and $\mathbf{v}$ is eigenvector of A, based on the definition fo eigenvalue and eigenvector, we have Let A have eigenvalue λ, based on the definition of eigenvalue, we have

$$A\mathbf{x} = \lambda\mathbf{v}$$

Then, for the initial point $\mathbf{v}$, we have

$$A(\mathbf{v}) = A\mathbf{v} = \lambda\mathbf{v}$$

let $\mathbf{v}_0 = \mathbf{v}, \mathbf{v}^n = A^n(\mathbf{v})$, then

$$\mathbf{v}_1 = A\mathbf{v}_0 = \lambda\mathbf{v}_0, \mathbf{v}_2 = A\mathbf{v}_1 = \lambda^2\mathbf{v}_1 \ldots \mathbf{v}_n = A\mathbf{v}_{n-1} = \lambda^n\mathbf{v}_0$$

Futhermore, if we consider a system in random initial value $\mathbf{x}_0$, still define $\mathbf{x}_n = f(\mathbf{x}_{n-1})$, then

$$\mathbf{x}_n = f(\mathbf{x}_{n-1}) = A\mathbf{x}_{n-1} = Af(\mathbf{x}_{n-2}) = \ldots = A^n\mathbf{x}_0$$

To analysis this problem, firstly we will review some theorems in algebra.

D I S C U S S I O N* 2.1 *Eigenvalue, eigenvector and Jordan normal form

** We will consider a square matrix $A_{m\times m}$ s.t. $rank(A) = m$ in following discussion.*

[i] If A have m different Eigenvalue

Based on the discussion above, we know that it is the first step to analysis the A^n to discribe all the linear system. Obviously, if A is a diagonal matrix, then the exponent of the matrix is easy and simple.

Theorem 2.1 *Let A is a diagonal matrix s.t. $A = diag(a_1, a_2, \ldots a_m)$, then $A^n = diag(a_1^n, a_2^n, \ldots a_m^n)$.*

Furthermore, if matrix A have m different eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $\mathbf{v}_i$ is the eigenvector of λ_i. Let

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_m), V = (\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m)$$

then we can easily prove that

$$A = V^{-1}\Lambda V$$

And the calculation of A^n is simple.

$$A^n = V^{-1}\Lambda^n V = V^{-1}diag(\lambda_1^n, \lambda_2^n, \ldots, \lambda_m^n)V$$

Now we back to consider the linear system, if $f(\mathbf{x}) = A\mathbf{x}$ and A have m different eigenvalue, then we know that

$$\mathbf{x}_n = A^n\mathbf{x}_0 = V^{-1}diag(\lambda_1^n, \lambda_2^n, \ldots, \lambda_m^n)V\mathbf{x}_0$$

Based on the analysis in the section 1, we still want to analysis the convergence and divergence for ever system.

$$\lim_{n\to\infty} x_n = V^{-1}diag(\lim_{n\to\infty}\lambda_1^n, \lim_{n\to\infty}\lambda_2^n, \ldots, \lim_{n\to\infty}\lambda_m^n)V\mathbf{x}_0$$

Obviously, with the knowledge of sequence, if $|\lambda_i| \in [0, 1)$, then $\lim_{n\to\infty}\lambda_i^n = 0$ and the sequence is convergence. Also, if $|\lambda_i| \in (1, +\infty)$, then $\lim_{n\to\infty}\lambda_i^n = \infty$ and the sequence is divergence. So we have this conclusion.

Theorem 2.2 ***Sink, source and saddle in linear system***

Consider a linear system $f(\mathbf{x}) = A\mathbf{x}$, where A is a square matrix in m dimension. If the eigenvalue of A are $\lambda_1, \lambda_2, \ldots \lambda_m$ and

[i] *$\forall i \in 1, 2, \ldots, m, |\lambda_i| < 1$, then the origin point is sink.*

[ii] *$\forall i \in 1, 2, \ldots, m, |\lambda_i| > 1$, then the origin point is source.*

[iii] *$\{i||\lambda_i| < 1\} \neq \varnothing \wedge \{j||\lambda_j| > 1\} \neq \varnothing$, that means, if at least one eigenvalue are absolute smaller than one and at least one is upper than one, then the origin point is saddle.*

[ii] If A have at least two equal eigenvalue

We can transfrom the matrix A with Jordan normal form rather than eigenvalue diagonal matrix. Consider the matrix $A_{m\times m}$ and the eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_k$ are $r_1, r_2, \ldots r_k$ multiple root of function $|\lambda I - A| = 0$, which satisfied the definition of eigenvalue, and $k < m$, $\sum_{i=1}^{k} r_i = m$, $I = diag(1, 1, 1, \ldots, 1)$.

Then for every r_i multiple eigenvalue λ_i, $\exists \mathbf{v}_{i1}, \mathbf{v}_{i2}, \ldots \mathbf{v}_{ir_i}$ s.t.

$$|\lambda I - A|\mathbf{v}_{i1} = 0, |\lambda I - A|\mathbf{v}_{ij+1} = \mathbf{v}_{ij}(j = 1, 2, \ldots r_i - 1)$$

*We can still structure the V matrix same as V in **[i]**, and we can also represent the diagonal eigenvalue matrix Λ to the **Jordan normal form matrix** J which satisfied*

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \dots & \\ & & & J_k \end{bmatrix} = diag(J_1, J_2, \dots J_k),\ where\ J_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \dots & \dots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}$$

*is r_i dimension square matrix called **Jordan block***

Based on the calculation of block matrix we found that

$$A^n = V^{-1} J^n V = V^{-1} diag(J_1^n, J_2^n, \dots, J_k^n) V$$

*So familiar with the discussion in **[i]**, now it is necessary to discuss the J_i^n. On the other hand, we know that for ever Jordan block, we have*

$$J_i^n = \begin{bmatrix} \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \binom{n}{2}\lambda_i^{n-2} & \dots & \binom{n}{r_i}\lambda_i^{n-r_i} \\ & \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \dots & \binom{n}{r_{i-1}}\lambda_i^{n-r_i+1} \\ & & \dots & \dots & \dots \\ & & & \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} \\ & & & & \lambda_i^n \end{bmatrix}$$

Obviously, for ever element on the diagonal, the Theo. 2.2 still established. To proved that, we will prove the following theorem firstly.

Theorem 2.3 *Let J_i is a Jordan block with eigenvalue λ_i.*
[i] *If $|\lambda_i| < 1$, then $\lim_{n\to\infty} J_i^n = 0$*
[ii] *If $|\lambda_i| > 1$, then $\lim_{n\to\infty} J_i^n = \infty$*

***PROOF* 2.1** *Consider a element $\binom{n}{k}\lambda_i^{n-k}$ of J_i, then*

$$\lim_{n\to\infty} \binom{n}{k}\lambda_i^{n-k} = \lim_{n\to\infty} \left(\frac{n(n-1)\dots(n-k)}{1\cdot 2\cdot \dots \cdot k} \lambda_i^{n-k} \right)$$

As the $\frac{n(n-1)\dots(n-k)}{1\cdot 2\cdot\dots\cdot k}$ is a polynomial of n in k dimension, so $\exists a_1, a_2, \dots a_k \in R$ s.t.

$$\lim_{n\to\infty} \binom{n}{k}\lambda_i^{n-k} = \lim_{n\to\infty} \left(\sum_{p=1}^{k} a_p n^p \right) \lambda_{n-k} = \sum_{p=1}^{k} \lim_{n\to\infty} (a_p n^p \lambda_i^{n-k})$$

Finally, we found, if $|\lambda_i| > 1$, then $\lim_{n\to\infty} \binom{n}{k}\lambda_i^{n-k} = \infty$ and if $|\lambda_i| < 1$, then $\lim_{n\to\infty} \binom{n}{k}\lambda_i^{n-k} = 0$ and the Theo. 2.3 is established. ■

As for the non-linear problem, a wildly used method is **Jacobian matrix**

Definition 2.4 ***Jacobian matrix***
*Let $\mathbf{f} = (f_1, f_2, \dots f_m)$ be a map on R^m and $\mathbf{p} \in R^m$ is a point on R^m space. The **Jacobian matrix** of $\mathbf{f}$ at $\mathbf{p}$ is the matrix*

$$D\mathbf{f}(\mathbf{p}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{p}) & \frac{\partial f_1}{\partial x_2}(\mathbf{p}) & \dots \frac{\partial f_1}{\partial x_m}(\mathbf{p}) & \\ \frac{\partial f_2}{\partial x_1}(\mathbf{p}) & \frac{\partial f_2}{\partial x_2}(\mathbf{p}) & \dots \frac{\partial f_2}{\partial x_m}(\mathbf{p}) & \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{p}) & \frac{\partial f_m}{\partial x_2}(\mathbf{p}) & \dots \frac{\partial f_m}{\partial x_m}(\mathbf{p}) & \end{bmatrix}$$

Jacobian matrix is a linearization estimation of a non-linear system that we can assume the derivative of the system near the point $\mathbf{p}$ is $D\mathbf{f}(\mathbf{p})$. That means, instead of origin non-linear system, we can analysis the estimated system $\mathbf{f}_1(\mathbf{x}) = D\mathbf{f}(\mathbf{p})\mathbf{x}$ where $x \in N(\mathbf{p}, \varepsilon)$ and ε is a certain constant. Based on the Theo. 2.2, it is easy to improve the following conclusion.

Theorem 2.4 ***Sink, source and saddle in non-linear system***
Consider a non-linear system $\mathbf{f}(\mathbf{x})$ and a fixed point $\mathbf{p} \in R^m$ s.t. $\mathbf{f}(\mathbf{p}) = \mathbf{p}$. If the Jacobian matrix of $\mathbf{f}$ at $\mathbf{p}$ is $D\mathbf{f}(\mathbf{p})$, and Λ are eigenvalue set of matrix $D\mathbf{f}(\mathbf{p})$
[i] *$\forall \lambda_i \in \Lambda |\lambda_i| < 1$, then the $\mathbf{p}$ is a sink point.*
[ii] *$\forall \lambda_i \in \Lambda |\lambda_i| > 1$, then the $\mathbf{p}$ is a source point.*
[iii] *$\{\lambda_i \in \Lambda | |\lambda_i| < 1\} \neq \varnothing \wedge \{\lambda_i \in \Lambda | |\lambda_i| > 1\} \neq \varnothing$, that means, if at least one eigenvalue are absolute smaller than one and at least one is upper than one, then the $\mathbf{p}$ is a saddle point.*

Finally, we can analysis the property of fixed point in e.g. 2.1.

***SOLUTION* 2.2** *We can consider the Henon map directly*

$$f(x, y) = (a - x^2 + by, x) \Rightarrow Df(x, y) = \begin{bmatrix} -2x & b \\ 1 & 0 \end{bmatrix}$$

Let λ are eigenvalue, then

$$|\lambda I - Df(x, y)| = 0 \Rightarrow \begin{vmatrix} -2x - \lambda & b \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 2x\lambda - b = 0 \Rightarrow \lambda_{12} = -x \pm \sqrt{x^2 + b}$$

When $(a, b) = (0, 0.4)$

$$\text{If } (x, y) = (0, 0), \text{ then } |\lambda_{12}| = |-x \pm \sqrt{x^2 + b}| = |\sqrt{0.4}| < 1$$

If $(x, y) = (0.6, 0.6)$, then $|\lambda_{12}| = |-x \pm \sqrt{x^2 + b}| = |-0.6 \pm \sqrt{0.76}| \Rightarrow |\lambda_1| = 1.472 > 1, |\lambda_2| = 0.272 < 1$
Finally, we proved that $(0, 0)$ is sink and $(-0.6, -0.6)$ is saddle just as what we found on simulation. ■

2.2 Stability and matrix periodic

We found this conclusion based on the discussion above.

***CONCLUSION* 2.1** *The value of Jacobian matrix of a Henon map is just relevant to variable x and parameter b. That means, if we reduce the dimension of parameter and fixed b as b_0, then the property of fixed point will be determined only with variable x.*

As $y_{n+1} = x_n$, so we can just analysis the bifurcation of $a - x_\infty$ with random initial point.

If we consider the fixed point of system with arbitrary paremeter grou (a, b), we found that the fixed point will satisfied

$$x^2 + (1-b)x - a = 0 \Rightarrow x = \frac{1}{2}(b-1) \pm \sqrt{(b-1)^2 + 4a} \tag{1}$$

and the fixed point is $(x, y) = (\frac{1}{2}(b-1) \pm \sqrt{(b-1)^2 + 4a}, \frac{1}{2}(b-1) \pm \sqrt{(b-1)^2 + 4a})$, so we have the Jacobian matrix at this fixed point as

$$Df(x, y) = \begin{bmatrix} (b-1) \pm \sqrt{(b-1)^2 + 4a} & b \\ 1 & 0 \end{bmatrix}, \text{ and the eigenvalue } \lambda_i \text{ satisfied}$$

$$\lambda^2 - [(b-1) \pm \sqrt{(b-1)^2 + 4a}]\lambda - b = 0 \tag{2}$$

Then we can found the property of sink and source in every fixed point easily.

Now we focus on periodic-k orbit. Firstly, we still plot the bifurcation diagram of Henon map.

Figure 16: Bifurcation diagram for Henon map ($b = 0.4$)

That is simple to analysis the influence of parameter. In following plots, $b \equiv 0.4$ and $a = 0.9, 0.988, 1.0, 1.0293, 1.045, 1.2$. We found $a = 0.9$ is a periodic-4 sink, $a = 0.988$ is a periodic-16 sink, $a = 1.0$ is a four-piece attractor, $a = 1.0293$ is a periodic-10 sink, $a = 1.045$ is two-piece attractor and the points of an orbit alternate between the pieces. Finally $a = 1.2$ two pieces have merged to form one-piece attractor.

Definition 2.5 ***Attractor*** *An attractor is a set of numerical values toward which a system tends to evolve, for a wide variety of starting conditions of the system.*
Futuermore, in discrete time, we called the orbit of a system as periodic-k orbit. However in chaotic orbit, the solution set is a continuous (or uncountable) set. And we called this orbit as attractor.

We will discuss the relationship between matrix and periodic-k orbit. But before that, it is necessary to introduce some new definition.

Definition 2.6 *A map* $\mathbf{f}$ *on* R^m *is* ***one-to-one*** *if and only if* $\mathbf{f}(\mathbf{v}_1)\mathbf{f}(\mathbf{v}_2) \Leftrightarrow \mathbf{v}_1 = \mathbf{v}_2$

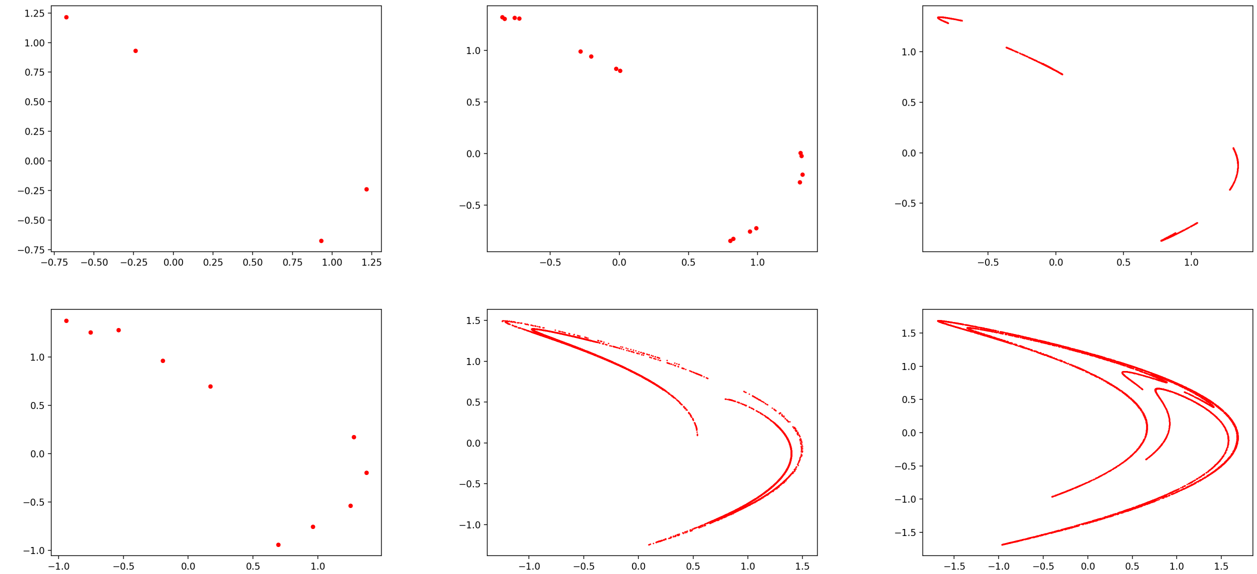

Figure 17: Attractor of Henon map in different parameter

Definition 2.7 ***Inverse map***
Consider a one-to-one map $\mathbf{f}$ *on* R^m. *The inverse map* $\mathbf{f}^{-1}$ *is automatically exists and satisfied* $\forall \mathbf{v} \in D \subset R^m, \mathbf{f}(\mathbf{f}^{-1})(\mathbf{v}) = \mathbf{f}^{-1}(\mathbf{f})(\mathbf{v}) = \mathbf{v}$, *where* D *is domain of map.*

For instance, a one-to-one map $f(x) = 2x$ have an inverse map $f^{-1} = x/2$. Obviously, for every linear map $\mathbf{f}(\mathbf{x}) = A\mathbf{x}, \exists f^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$.

Theorem 2.5 *For every* R^m *linear map* $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ *and* A *s.t.* $rank(A) = m$, *the inverse map* f^{-1} *always be existed.*

PROOF 2.2 ***[i]*** *If* A *have* m *different eigenvalue, then* $A = V\Lambda V$ *where* $\Lambda = diag(\lambda_1, \lambda_2, \dots, \lambda_m), V = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$ *where* λ_i *is eigenvalue and* $\mathbf{v}_1$ *is eigenvector. Then*

$$A^{-1} = \left(V^{-1}\Lambda V\right)^{-1} = V\Lambda^{-1}V^{-1} = V diag\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_m}\right)V^{-1}$$

[ii] *If* A *have* $p < m$ *different eigenvalue, based on the Jordan normal form, we still have* J, V *s.t.* $A = V^{-1}JV$ *and* $J = diag(J_1, J_2, \dots J_k)$ *where* J_i *is Jordan block based on the eigenvalue* λ_i. *And now the problem is prove that for evert Jordan block, the inverse block always be existed.*
Obviously, $J_i = \lambda_i I + N$ *where*

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

And it is simple to found that $N^m = \mathbf{0}_{m\times m}$. *Based on the Taylor expansion, we have*

$$J_i^{-1} = \lambda_i^{-1}(I + \lambda_i^{-1}N + \lambda_i^{-2}N^2 - \dots + (-\lambda_i)^{-n+1}N^{n-1})$$

Although the inverse of Jordan block is not a Jordan block of $1/\lambda_i$, *it is still exists and we proved the theorem.* ■