6 Fractal dimension

In real analysis, if we consider the size of a subspace, we basically use **measure** to describe that.

Definition 6.1 ***Eculidean rectangle Neighbourhood***
1. Opened box $I = \{x = (x_1, x_2, \ldots, x_n) | a_i < x_i < b_i, i \in N\}$, where a_i, b_i are constant;
2. Closed box $I = \{x = (x_1, x_2, \ldots, x_n) | a_i \leq x_i \leq b_i, i \in N\}$, where a_i, b_i are constant;
Called the $b_i - a_i$ is side length of a box. If $|b_1 - a_1| = |b_2 - a_2| = \ldots = |b_n - a_n|$, then called this box as cube.

Definition 6.2 ***Lebesuge outer measure***

$$\mu^* E = \inf\left\{\sum_{i=1}^{\infty} |I_i| : E \subset \bigcup_{i=1}^{\infty} I_i, I_i \text{ is an opened box}\right\}$$

Also, we can define the outer measure with ε (Familiar with Cauchy's limitation)

$$\forall \varepsilon > 0, \exists \{I_i\} \text{ open cover s.t.} E \subset \bigcup_{i=1}^{\infty} I_i \land \mu^* E \leq \sum_{i=1}^{\infty} |I_i| + \varepsilon$$

Now, we consider a cube with side length 1, and neigibourhood cube with side length ε, the total of the neigibourhood include in the unix cube as $V = N(\varepsilon)$, the dimension of subspace as k, then

$$V = N(\varepsilon) = \left(\frac{1}{\varepsilon}\right)^k \Rightarrow k = \log_{1/\varepsilon} N(\varepsilon) = -\frac{\ln N(\varepsilon)}{\ln(\varepsilon)}$$

In normal problem, $k \in \mathcal{N}^+$ called **Lebesgue covering dimension** or **topological dimension**.

Definition 6.3 ***Lebesgue covering dimension***
A topological space X is said to have the Lebesgue covering dimension $d < \infty$ if d is the smallest ***non-negative integer*** *with the property that each open cover of X has a refinement in which no point of X is included in more than $d + 1$ elements.*

Definition 6.4 $C(1/\varepsilon)$ ***cube***
Now we consider a special type of cover.
[i] *Firstly, for a interval $I_1 = [a_1, b_1]$, let the side length of boxes is ε, then we have $N = int(C/\varepsilon) + 1$ subintervals to cover the original interval. (e.g. $[a_1 + (p-1)C/\varepsilon, a_1 + pC/\varepsilon], p = 1, 2, \ldots N$)*
[ii] *Now we consider a surface in R^n space, let the projection on x_i axis is $I_i = [a_i, b_i]$, and $C = \max\{\mu(I_i)\}, i = 1, 2, \ldots, n$, then we have a group of cube total $N(\varepsilon) = (C\varepsilon)^n$ and*

$$d = \frac{\ln N(\varepsilon) - \ln C}{\ln(1/\varepsilon)}$$

Definition 6.5 $C(1/\varepsilon)$ ***cube***
A bounded set $s \subset R^n$ has box-counting dimension

$$bd(S) = \lim_{\varepsilon \to \infty} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)}$$

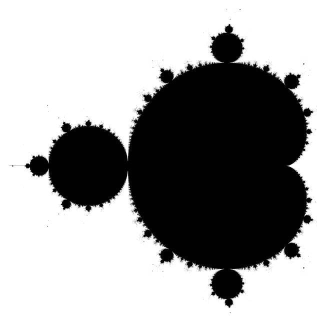
when the limit exists.

***CONCLUSION* 6.1** *Based on the Bolzano–Weierstrass theorem, we know that if we have a sequence* $\{b_n\}$ *s.t.* $\lim_{n\to\infty} b_n = 0$*, if* b_n *is* ε *then* $\lim_{n\to\infty} \frac{\ln b_{n+1}}{\ln b_n} = 1$

***CONCLUSION* 6.2** *We still consider the sequence above, then*

$$\frac{N(b_n)}{4} \le N(b_n) \le 4N(b_{n+1})$$

and the proof is simple with figure follows



Theorem 6.1 *If* $\{b_n\}$ *is monotony, or assume* $b_1 > b_2 > \ldots > b_n > \ldots > b_\infty = 0$ *If*

$$\lim_{n\to\infty}\left(\frac{\ln b_{n+1}}{\ln b_n}\right) = 1 \wedge \lim_{n->\infty}\left(\frac{\ln N(b_n)}{\ln(1/b_n)}\right) = d =$$

then $\lim_{\varepsilon\to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = d$ *and therefore the box-counting dimension is* d*.*

Theorem 6.2 *If* $S \subset R^n$ *is bounded and* $bd(S) = d < n$*, then* $\mu(S) = 0$

Another way to calculate the measure of a area is based on the statistic,

Definition 6.6 ***Correlation dimension*** *Let* $S = \{v_0, v_1, \ldots\}$ *be an orbit of the map* $f : \mathcal{R}^n \to \mathcal{R}^n$*, then for ever* $r > 0$*, define* $C(r)$ *s.t.*

$$C(r) = \lim_{N\to\infty} \frac{|\{(p,q)|p,q \in S, |p-q| < r\}|}{|\{(p,q)|p,q \in S\}|}$$

Moreover, if $C(r) \approx r^d$*, then*

$$d \approx cd(S) = \lim_{r\to\infty} \frac{\ln C(r)}{\ln(r)}$$

And here we found another way to calculation the area of attractor.

In the last section, we introduced the Lyapunov spectrum. However, if we want to analysis the whole system, the Lyapunov exponent is not a good choice because it described the property in every different direction.

So to describe the property of a set, or a space, the basic element is the definition of open set. Now we consider a box neigibourhood rather than the circle one, signed as I_0 with certain side length ω_0 and the volumn $V_0 = \omega_0^m$. Then let the set $I_1 = f(I_0), I_2 = f(I_1)\dots$ and we now try to consider the volumn of these sets. If the Lyapunov spectrum of system is $(h_1, h_2, \dots, h_m)$, then

$$||\omega_n^{(j)}|| = \exp(nh_j)||\omega_0^{(j)}||$$

and the volume of box neigibourhood is

$$V_i = \prod_{j=1}^{m} ||\omega_i^{(j)}|| = \prod_{j=1}^{m} \exp(ih_j)||\omega_0^{(j)}||$$

$$V_n = \prod_{j=0}^{m} \exp(nh_j)||\omega_0^{(j)}||$$

But that is not the all conclusion, as some of Lyapunov exponent is less than 0, the side length in that direction will decrease and decrease and finally equal to 0. And that is unnecessary to consider such direction.

And now, the problem is "Which dimension is enough to include the $f^\infty(I_0)$"

We know that if the dimension is 3 as normal world, then the volumn V and side length d have relationship as

$$V = d^3 \Rightarrow 3 = \log_d V = \frac{\ln V}{\ln d}$$

So if we assume the dimension of space is $k \leq m$, then

$$k = \frac{\ln V_\infty}{\ln d_0} \qquad (*)$$

[i] **In 2-dim problem** Now we consider a 2-dim problem.

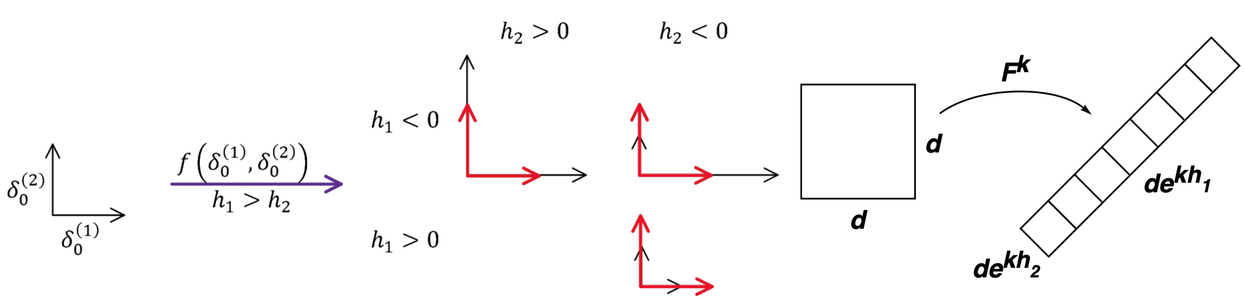


Figure 48: Volumn and vector change with Lyapunov exponent

Same as the figure above, if $h_1 > h_2 > 0$, then, obviously, we need a 2-dim neigibourhood to cover all the system after iteration. On the other hand, if $0 > h_1 > h_2$, then we just need a point to cover it. So the main problem is how can we cover the situation $h_1 > 0 > h_2$.

$$V_n = \omega_0^{(1)} \exp(nh_1)\omega_0^{(2)} \exp(nh_2) = d^2 \left(\exp(h_1 + h_2)\right)^n$$

So if $h_1 + h_2 < 0$, the volumn will still decrese to 0, that means the dimension of V_∞ is 0. And now we consider the condition $h_1 > 0 > h_2 \wedge h_1 + h_2 > 0$

Definition 6.7 ***Lyapunov dimension***
Let f be a map on $\mathcal{R}^m$, the Lyapunov exponent of an orbit is $h_1 \geq h_2 \geq \ldots \geq h_m$, let p

$$p = \arg\max_p \left(\sum_{i=1}^{p} h_i \geq 0\right) \text{ then, the Lyapunov dimension is } D_L = \begin{cases} 0 & \text{if } p \text{ is not exists} \\ p + \frac{1}{|h_{p+1}|} \sum_{i=1}^{p} h_i & \text{if } p < m \\ m & \text{if } p = m \end{cases}$$

D I S C U S S I O N* 6.1** ***What can Lyapunov dimension describe? *We just introduced the definition of Lyapunov dimension above, and here, we will explain why we are interested in Lyapunov dimension.*

So why are we interested in these fractal dimension, like Hausforff dimensionD_H, Box counting dimension D_B, Correlation Dimension D_C as well as Lyapunov dimension D_{Lya}, typically we have this conclusion

***CONCLUSION* 6.3** *Consider a dynamic system which have the fractal dimension D_H, D_B, D_C, D_{Lya} where D_H is Hausforff dimension, D_B is Box counting dimension, D_C is Correlation Dimension and D_{Lya} is Lyapunov dimension D_{Lya}, then*

$$D_H \leq D_B \leq D_C \leq D_{Lya}$$

On the other hand, we found calculation of Hausforff dimension is difficult or impossible, so if we want to consider the fractal dimension of a system, at least, the box counting dimension is easier to calculate, and more easier to calculate the Correlation Dimension as well as Lyapunov dimension. On the other hand, if the system satisfied some condition, then $D_H = D_B = D_C = D_{Lya}$ which means we can calculate the Lyapunov dimension instead of Hausforff dimension or box counting dimension which is really difficult to calculate.