Contents

Chaos: An Introduction to Dynamical Systems

April 16, 2021

Problem in discrete-time system

1 One-Dimension Maps

Definition 1.1 ***n-order differentiable function, Smooth function, Map***
Consider an open set E and $n \in \mathcal{N}$, called

$$C^n(E) = \{f \in C(E) | \forall \alpha \ s.t. \ |\alpha| \leq n, D^\alpha f \in C(E)\}$$

is n-order differentiable function set of E, where $C(E)$ is continuous function on E.
If f on domian E have infinity-order derivative, or $f \in C^\infty(E)$, then called f ***smooth function****.*
If the function f have same domain and range, then called f is a ***map****.*

The fucntion in this book will be a smooth function if we not emphasize.

Definition 1.2 ***Orbit, initial value, fixed point***
Consider a map $f : X \to X$, x is a point in X then
Called ***orbit*** *of x is a set of point*

$$Orbit(X) = \{x, f(x), f^2(x), \ldots f^n(x), \ldots\}$$

, where $f^n(x) = f(f(\ldots f(x))) = (f \circ f \circ f \circ \ldots \circ f)(x)$.
The starting point of x for a orbit called the ***initial value****.*
If the point p s.t. $f(p) = p$, then called p as fixed point.

OK, and now we consider two dynamical systems, with a input x, the system will always return to $f(x) = 2x$ and $g(x) = 2x(1 - x)$. And then the output will become the input value and etc. During this looping, it is simple to find the orbit of a certain initial value.

Table 1: Comparison of exponential growth and logistic growth

f	init	1	2	3	4	5	6	7	8	9
$f(x)$	0.01	0.02	0.04	0.08	0.16	0.32	0.64	1.28	2.56	5.12
$f(x)$	0.01	0.0198	0.0388	0.0746	0.138	0.268	0.362	0.462	0.497	0.499
$g(x)$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$g(x)$	0.8	0.32	0.435	0.492	0.499	0.5	0.5	0.5	0.5	0.5
$g(x)$	1.2	-0.48	-1.42	-6.87	-108.4	-23716.9	-1125030476	-inf	-inf	-inf

We found that in the model of f, the result is growth as exponential function and we called that exponential growth. Also, when intial value $x \in [0, 1]$, with iteration, the result have limition of 0.5 and we called these model as logistic growth.

In this section, we will mainly focus on these kind of dynamical system, obviously, the iteration processing of model are discrete, we also called these dynamical system models as **maps**.

1.1 Cobweb plot, stability

To analysis a maps, the basic method is based on cobweb plot. Fig 1 showed a method to analysis a dynamical system with a certain iteration principle.

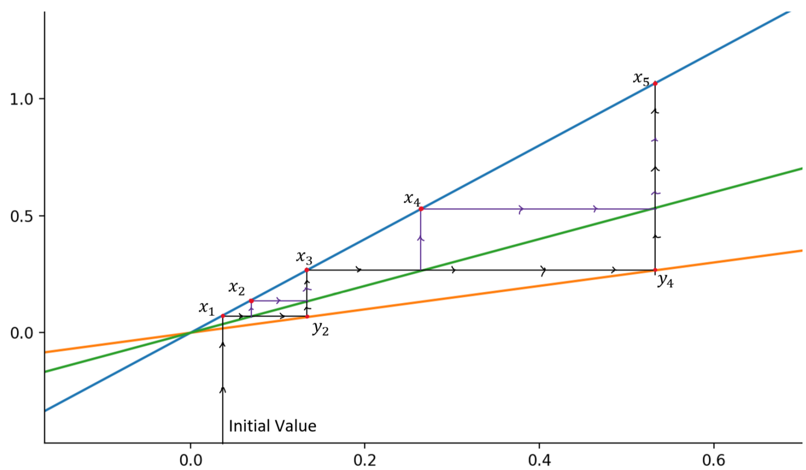


Figure 1: An example of cobweb plot and basic principle

In every iteration, the independent and dependent variable exchanged their location and we can found a group of $\{x_1, y_2, x_3, y_4, \ldots\}$ as orbit of initial value. Or, with the symmetric line $y = x$, it is simple to symmetric all black line to purple line and we can build a cobweb plot with origin image and $y = x$ to find a group of $\{x_1, x_2, \ldots\}$ as orbit from the initial value.

Before we discuss the different of fixed point, it is necessary to review some basic definitions.

Definition 1.3 ε ***Neighbourhood***

In a metric space X, an ε neighbourhood $N_\varepsilon(p)$ of point p is defined

$$N_\varepsilon(p) = \{x \in X | d(x, p) < \varepsilon\}$$

where $d(x, p)$ is the distance bewteen point p and x. Also, in a R^1 space, the ε Neighbourhood is give by

$$N_\varepsilon(p) = \{x \in R || x - p| < \varepsilon\}$$

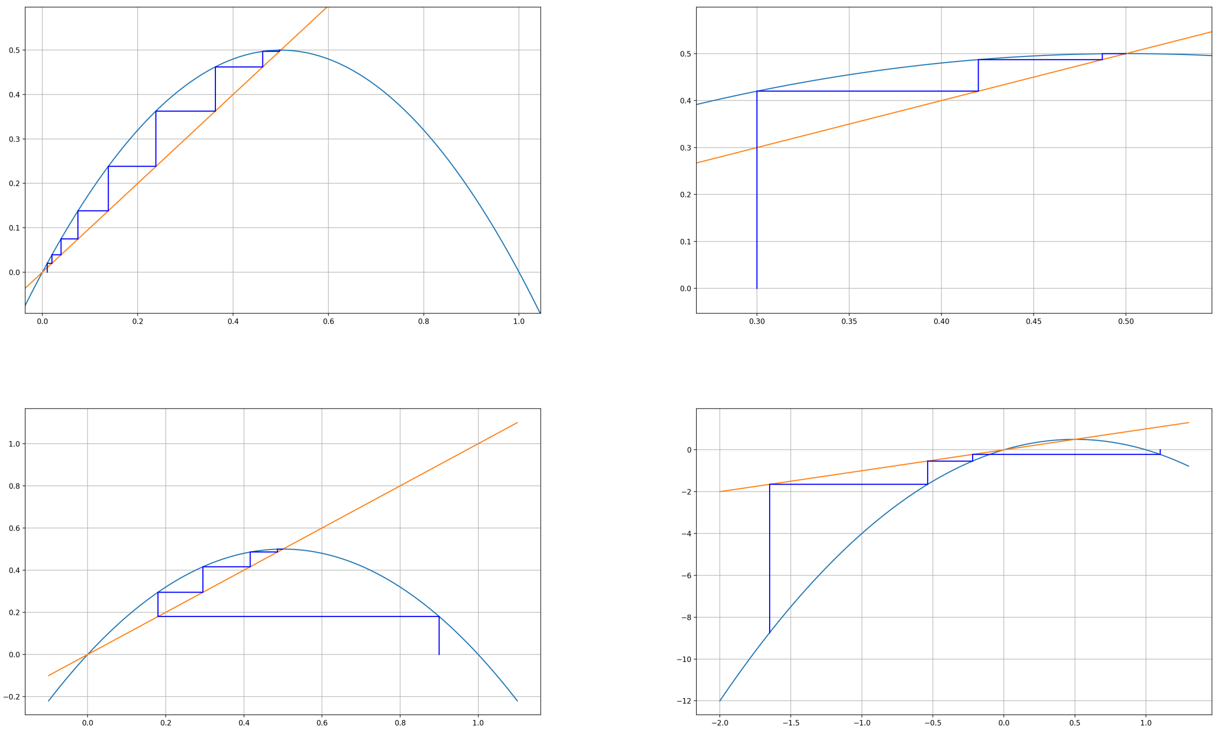


Figure 2: Cobweb plot in different initial value

It is simple to find that for all initial value $x \in (0, 1)$, with iteration, the output have limitation in 0.5. On the other hand, to solve the equation $x = 2x(1-x)$ we found $x_1 = 0, x_2 = 1, x_3 = 0.5$ as three fixed point. So we have two kinds of fixed point, the one is limitation point and the other is not.

Definition 1.4 ***Sink, Source (Attracting and Repelling Fixed Point)***
Consider a map $f : R \to R$ and point p s.t. $f(p) = p$, then
*If for evert points sufficiently to p are attracted to p, then called p as **sink**, or **attracting fixed point**. Or*

$$\text{For an } \varepsilon > 0, \forall x \in N_\varepsilon(p), \lim_{k\to\infty} f^k(x) = p \text{ then called } p \text{ as } \textbf{sink}.$$

*If for every points sufficiently to p are repelled to p, then called p as **source**, or **repelling fixed point**.*

$$\text{For an } \varepsilon > 0, \forall x \in N_\varepsilon(p), x \neq p, \lim_{k\to\infty} f^k(x) \notin N_\varepsilon(x) \text{ then called } p \text{ as } \textbf{source}.$$

Theorem 1.1 *Let f is a map on R, assume p is a fixed point of f, then*
[i] If $|f'(p)| < 1$, then p is a sink;
[ii] If $|f'(p)| > 1$, then p is a source.

PROOF* 1.1** ***[i] *Based on definition of derivative, we have*

$$\lim_{x \to p} \frac{|f(x) - f(p)|}{|x - p|} = |f'(p)|$$

Now. let $a \in (\min(|f'(p)|, 1), \max(|f'(p)|, 1))$ (e.g. $a = \frac{1}{2}(1 + |f'(p)|)$), then

$$\forall a \in (\min(|f'(p)|, 1), \max(|f'(p)|, 1)), \exists \varepsilon_0 > 0 \ s.t. \ \forall \varepsilon \in (0, \varepsilon_0], \forall x \in N_\varepsilon(p), \frac{|f(x) - f(p)|}{|x - p|} < a$$

That means, $f(x)$ is closer to p than x (or distant bewteen curve $y = f(x)$ and $y = x$), but at least a factor of a and we have the conclusion

$$\forall x \in N_\varepsilon(p), f(x) \in N_\varepsilon(p)$$

During the iteration processing it is simple to find that all orbit $\{f(x), f^2(x), \ldots f^n(x), \ldots\} \subset N_\varepsilon(p)$, so now we can consider another conclusion in follow.

[ii] *We try to prove the inequality $\forall x \in N_\varepsilon(p), |f^k(x) - p|$*
[ii-1] *Obvious, if $k = 1$, then $|f(x) - p| = |f(x) - f(p)| < a|x - p|$ (p is fixed point so $f(p) = p$)*
[ii-2] *If $k = 2$, Based on the conclusion in **[i]**, $x_1 = f(x) \in N_\varepsilon(p)$ and $|f(x_1) - p| < a|x_1 - p| < a^2|x - p|$*
...
[ii-k+1] *(Assume the inequality is established in k), then*

$$|f^{k+1}(x) - p| < a|f^k(x) - p| < a \cdot a^k|x - p| = a^{k+1}|x - p|$$

In summary, for all $k \in N$, the inequality is established.

[iii-1] *Now we consider the equality condition, if $|f'(p)| < 1$ then $a < 1$ and*

$$\lim_{k \to \infty} |f^k(x) - p| < |x - p| \lim_{k \to \infty} a^k = 0 \ \textit{(Because } a \in (0, 1)\textit{)}$$

So we have the conclusion,

$$\forall x \in N_\varepsilon(p), \lim_{k \to \infty} f^k(x) = p$$

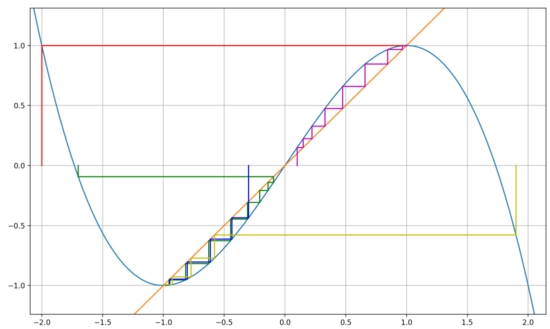
[iii-2] *Also, if $|f'(p)| > 1$ then $a^k \to \infty$, that means, with the iteration, the maps will eventually outside the condition, or the domain interval.* ■

* We will discuss what happened while $f'(p) = 1$ laterly.

** Obviously, this theorem expressed a kind of convergence, as the speed of the convergence is based on the a in exponent function, we called this convergence as **Exponential Convergence**.

Now we consider another map as example.

E x a m p l e 1.1 *Solved the fixed point of $\varphi(x) = (3x - x^2)/2$, find every sink and source point with Theo. 1.1*



***SOLUTION* 1.1** *It is simple to find the fixed point with $x = (3x - x^3)/2$ and $x_1 = 1, x_2 = 0, x_3 = -1$. Based on the image, we can found that 1 and -1 are sink and 0 is source. On the other hand*

$$\varphi'(x) = \frac{3}{2}(1 - x^2), \varphi'(-1) = 0 < 1, \varphi'(0) = \frac{3}{2} > 1, \varphi'(1) = 0 < 1$$

and we proved the conclusion we found on figure before. ■

Another way to confirm a point is sink or source is based on the formula identity and algebra. For instance, we consider the distance bewteen $g(x) = 2x(1 - x)$ and fixed point $1/2$, then

$$|g(x) - 1/2| = |2x(1 - x) - 1/2| = 2|x - 1/2||x - 1/2|$$

and $\forall x \in (0, 1), |x - 1/2| < 1 \Rightarrow |g(x) - 1/2| < 1$, that means the distance bewteen $g(x)$ and p is decreasing during time iteration and we can confirm that $1/2$ is a sink point rather than source point.

Next, we will focus on a logistic model with different parameter.

1.2 Periodic points, family of logistic maps

E x a m p l e 1.2 *Find the fixed point of $g(x) = 3.3x(1 - x), x \in [0, 1]$.*

***SOLUTION* 1.2** *It is simple to find the fixed point with $x = 3.3x(1 - x)$ and $x_1 = 0, x_2 = 23/33, x_3 = 1$. Obviously, both 0 and 1 are source. And*

$$g'(x) = 3.3 - 6.6x, |g'(23/33)| = 1.3 > 1$$

So all these three fixed point are source, and it is simple to find the conclusion with cobweb plot. ■

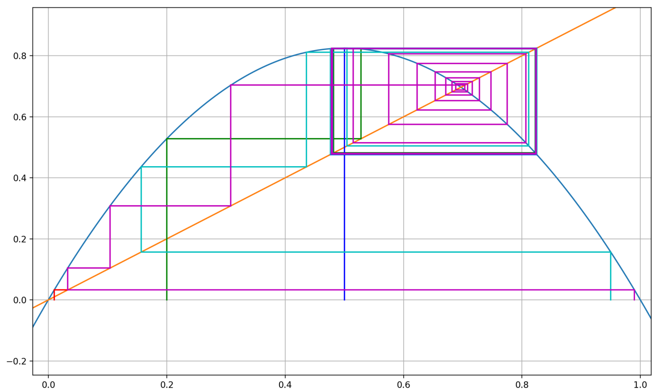


Figure 3: An example of periodic point

Hold on a second, something strange! Even we cannot find a sink fixed point, all of initial value are sank into a group of points!

Definition 1.5 ***Period-k point, Period-k orbit***
Let f be a map on R, and p is a point in domain, if $f^k(p) = p$, and k is the smallest such positive integer, then called p as ***periodic point of period*** *k, or* ***period-k point****;*
Called orbit with initial point p as ***periodic orbit of period k****, or* ***period-k orbit****;*

Definition 1.6 ***Sink and Source in Period point***
Let f be a map and p is a period-k point
If p is a sink, then called this period-k orbit as periodic sink;
If p is a source, then called this period-k orbit as periodic source.

Obviously, based on the chain rule, we have $(fg)'(x) = f'(g(x))g'(x)$, let $f = g, x = p_1$, then

$$g^2(p_1) = g'(g(p_1))g'(p_1) = g'(p_2)g'(p_1)$$

Summary this formula, we have

Theorem 1.2 *For every map f and period-k orbit $\{p_1, p_2, \ldots p_k\}$,*

$$(f^k)'(p_1) = (f^k)'(p_2) = \ldots = (f^k)'(p_k) = \prod_{i=1}^{k} f'(p_i)$$

PROOF* 1.2** ***Theo. 1.2

$$(f^k)(p_1) = (f(f^{k-1}))'(p_1) = f'(f^{k-1}(p_1))(f^{k-1})'(p_1) = \ldots = \prod_{i=1}^{k} f'(p_i) = (f^k)(p_i)(\forall i = 1, 2, \ldots, k)\blacksquare$$

Same as Theo. 1.1, we have stability test for periodic orbits.

Theorem 1.3 ***Stability test for periodic orbits***
Let f is a map and period-k orbit $\{p_1, p_2, \dots p_k\}$,
If $|\prod_{i=1}^{k} f'(p_i)| < 1$ then called this periodic orbit is a sink;
If $|\prod_{i=1}^{k} f'(p_i)| > 1$ then called this periodic orbit is a source;

PROOF* 1.3** ***Theo. 1.1
Consider a new map $g(x) = f^k(x)$, where f be a map and p is a period-k point, then p is a fixed point of g. Based on Theo. 1.1, $|g(p)| < 1$ if p is sink and $|g(p)| > 1$ if p is a source. On the other hand, $g(p) = f^k(p) = \prod_{i=1}^{k} f'(p_i)$ ■

Now we consider another problem.

E x a m p l e 1.3 *Find the fixed point or periodic orbit of $g_{3.5}(x) = 3.5x(1-x), g_{3.86}(x) = 3.86x(1-x), x \in [0,1]$.*

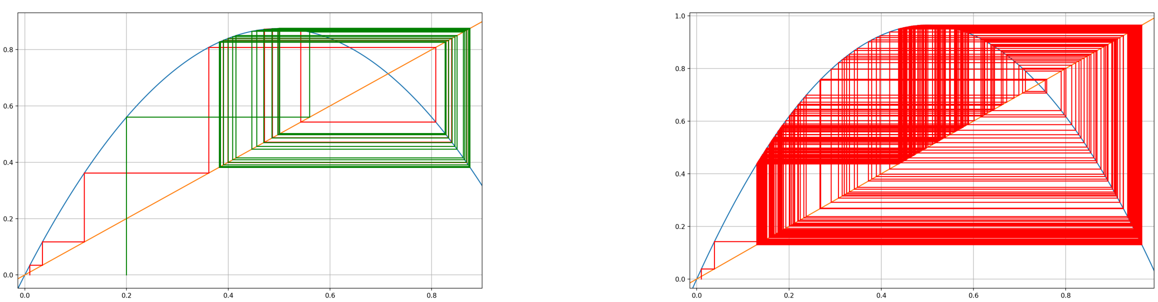


Figure 4: Logistic maps in $a = 3.5$ and $a = 3.86$

We found in $a = 3.5$, even the periodic orbit is difficult to find, the iteration still have a boundary. If we consider every $a \in [1,4]$, we can plot a figure between parameter a and orbits x, and this **bifurcation diagram** was made by following repearting:
[i] Choose a value a, starting with $a = 1$.
[ii] Choose a value $x \in [0,1]$ randomly.
[iii] Calculate the orbit of x under $g_a(x)$ in a certain iteration times t_max.
[iv] Ignore the first t_0 iterates and plot the orbit.

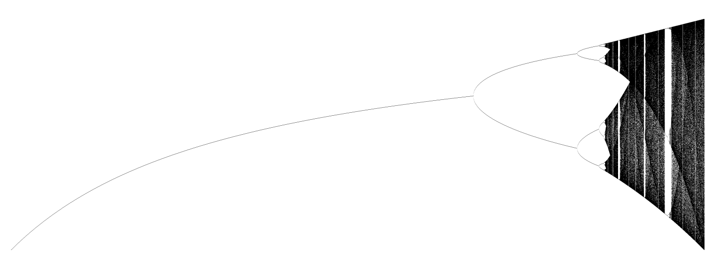


Figure 5: Logistic model stability interval ($a \in [1,4]$)

D I S C U S S I O N **1.1** *Now we will discuss the family of logistic maps with Fig.* **??**
[i] Periodic-3 window
We found periodic-1 orbits (or point) and periodic-2 orbits, based on the image above, it seem we also have periodic-3 orbits. And now we focus on the interval of parameter a rather than domain of function, we found there is a interval of a inside the $[3.83, 3.86]$ and we called these kind of interval as "periodic window". For instance, next figure showed the periodic-3 window of a.

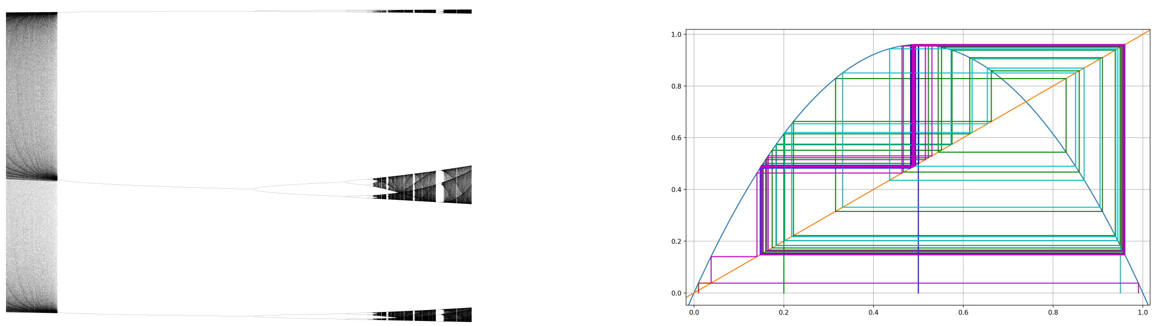


Figure 6: Periodic-3 window and cobweb plot in $a = 3.84$

That's fine, let's check the result by cobweb plot. Ok, hold on a second, something wrong! So we still need more analysis.

Obviously, every periodic-3 orbit of g is a fixed point of g^3, so we can also analysis g^3 map.

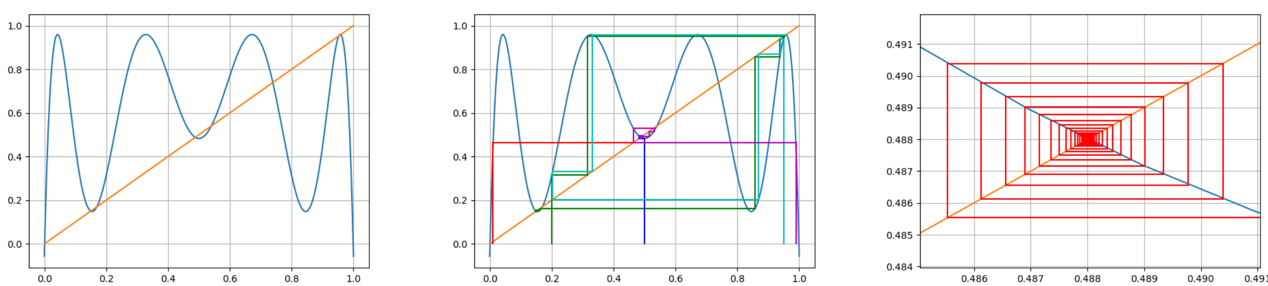


Figure 7: g^3 map and g^3 cobweb plot figure

We found different from periodic-2 orbit, the periodic-3 orbit is nearby(rather than equal) the point and it seems we have periodic-3 orbit. Actually, we will explain all periodic-3 will implies a characteristic we called "chaos".

[ii] The Logistic Map $G(x) = 4x(1-x)$
Now we consider another logistic map where $a \equiv 4$.

Firstly, why we are interested in $g_4(x)$, consider a quadratic function

$$g_a(x) = ax(1-x) = a(-x^2 + x - 1 + 1) = -a(x - \frac{1}{2})^2 + \frac{a}{4}$$

this function have maximum at point $x = 1/2$ and the maximum is $a/4$. As we have the Theo. 1.1, if we consider the sink point set, it is necessary to satisfy $|g_a(x)| < 1$, or$\frac{a}{4} < 1 \Rightarrow a < 4$. So at the point $a = 4$, this set is empty and this is a critical state. For every $a_{new} = a - \varepsilon(\varepsilon \to 0)$, we have the interval of sink. So at this point, some special property has been result and that is why we interested in this map.

We can still find the fixed point of $g_4(x)$ to solve $g_4(x) = x$, and we have $x_{11} = 0, x_{21} = 3/4$. If we consider periodic-k orbit, for instance, we consider periodic-2 orbit, then we have solve the function $g(g(x)) = x$ as

$$g(g(x)) = 4(4x(1-x))[1 - 4x(1-x)] = x \Rightarrow (4x^2 - 4x + 1)(x-1)x + \frac{x}{16} = 0$$

$$\Rightarrow (4x-3)(16x^2 - 20x + 5)x = 0 \Rightarrow x_{21} = 0, x_{22} = \frac{3}{4}, x_{23,24} = \frac{5 \pm \sqrt{5}}{8}$$

Also, it is easy to check the periodic-k orbit in the figure.

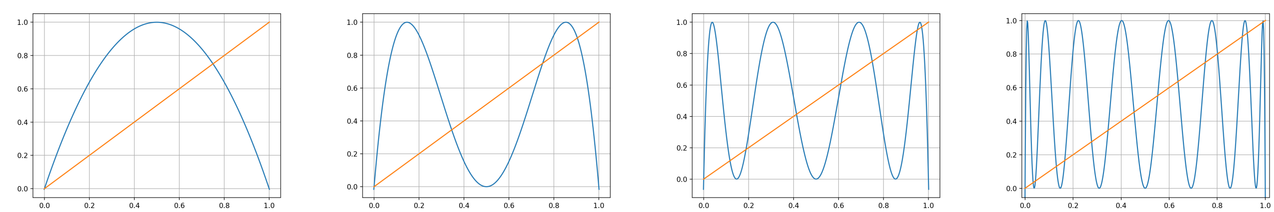


Figure 8: g_4^1, g_4^2, g_4^3 and g_4^4 figure

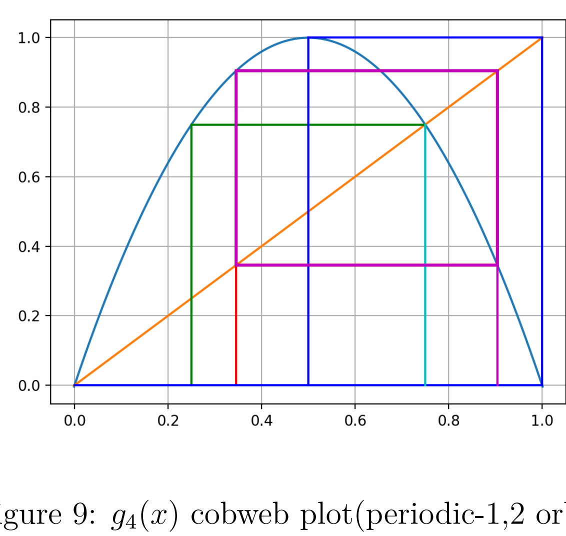


Figure 9: $g_4(x)$ cobweb plot(periodic-1,2 orbits)

We found a conclusion here

***CONCLUSION* 1.1** *For every periodic-k, the model g_4^k have $2^k - 1$ saddle-node bifurcation and 2^k fixed point. And these 2^k points include every fixed point for model $g_4^i, i = 1, 2, \dots, k-1 \wedge k \equiv 0(mod\ i)$*

The number of orbits of the map for each period can be tabulated in the map's periodic table.

Table 2: The periodic table for the logistic4 map

Period k	1	2	3	4	5	6	7
Number of fixed points of g_4^k	2	4	8	16	32	64	128
Orbits of Period k	2	1	2	3	4	5	6
Fixed points due to lower orbits	0	2	2	4	2	4	2
1	/						
2	*	/					
3	*		/				
4	-	*		/			
5	*				/		
6	-	*	*			/	
7	*						/

(: Greatest common divisor group, -: g^k fixed)*

1.3 Chaos

We still focus on $g_4(x)$ map, we try to check the g_4^2 fixed point $\frac{5-\sqrt{5}}{8}$.

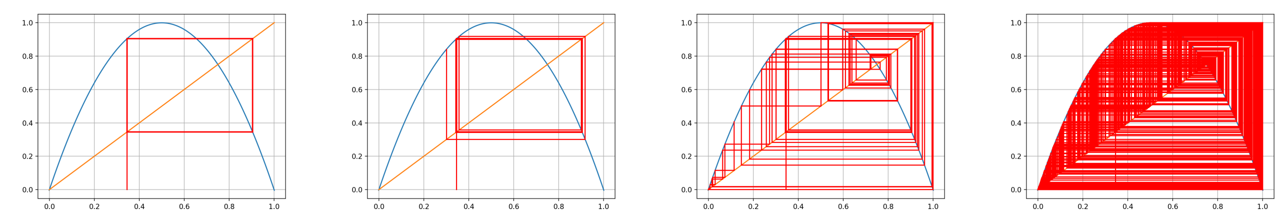


Figure 10: g_4^1, g_4^2, g_4^3 and g_4^4 figure

It seems something wrong. Because we proved that $\frac{5\pm\sqrt{5}}{8}$ is a periodic orbit during the iteration, but once we growth the iteration times, the results filled all the interval.

So what happened? We try to put all of our data into a same image, and we have

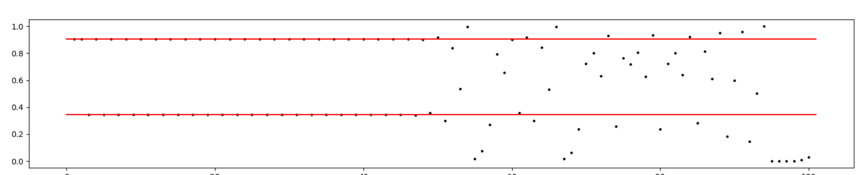


Figure 11: Iteration and "periodic-2 orbit" value

Obviously, in about first 40 times iteration, it was worked for a while, but with the iteration increasing, the error also increaed rapidly. Ok, ok, let's check the data for more details.

Table 3: Logistic4 periodic-2 orbit iteration

1-4	0.3454915028125262	0.9045084971874737	0.3454915028125262	0.9045084971874735
5-8	0.34549150281252694	0.9045084971874745	0.3454915028125237	0.9045084971874705
9-12	0.34549150281253665	0.9045084971874865	0.3454915028124849	0.9045084971874225
13-16	0.34549150281269186	0.9045084971876783	0.3454915028118641	0.9045084971866552
17-20	0.3454915028151752	0.9045084971907479	0.34549150280193086	0.9045084971743771
21-24	0.3454915028549078	0.9045084972398602	0.3454915026430002	0.904508496977928
25-27	0.3454915034906304	0.9045084980256565	0.34549150010010987	...

We noticed that during the iteration, the values of periodic-2 orbit are actually changed very small. Then we realized that is beacause of $\frac{5-\sqrt{5}}{8} \neq 0.3454915028125262$ and this is just a value near the periodic point.(And the computer can only calculate this estimation value rather than real value.) Even this two value are almost nearby, it still have a little difference, and this difference become larger and larger during the iteration.

That is important beacause we found even two value are almost equal, after iterate, this tiny, tiny difference will become a catastrophe and eventually two orbits move apart.

Definition 1.7 ***Sensitive dependence on initial conditions, Sensitive point***
Let f is a map on R, x_0 in domain.
If there is a nonzero distance d s.t. some points arbitrary near x_0 are eventually mapped at least d units from the corresponding image of x_0, then we called x_0 has ***sensitive dependence on initial conditions****;*
If for this $x_0, \exists \varepsilon > 0$ s.t. $\forall x \in N_\varepsilon^o(x_0) = N_\varepsilon(x_0) \backslash \{x_0\}, \exists K$ s.t. $\forall k > K, ||f^k(x) - f^k(x_0)|| \geq \varepsilon$, then called this point is ***sensitive point****.*

Definition 1.8 ***Eventually periodic***
Let f is a map on R, x_0 in domain. If for some positive integer $N, \forall n > N, f^{n+p}(x) = f^n(x)$, then we called x ***eventually periodic*** *with period p, where p is the smallest such positive integer.*

Now we consider another model to explain this definition in another way.

Example 1.4 *Consider a map $f(x) = 3x(mod\ 1)$. (e.g. $f(4.33) = 0.33, f(-1.98) = 0.02$.)*

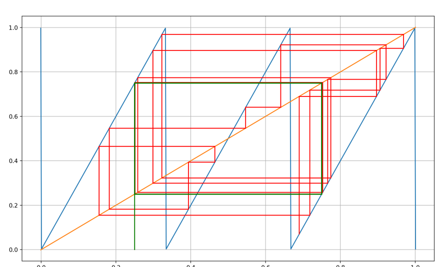


Figure 12: 3x mod 1 cobweb plot(initial value: 0.25(green), 0.2501(red))

Basically, we have

Theorem 1.4 *For any map f, the source has sensitive dependence on initial conditions.*

***PROOF* 1.4** *For a certain ε, as p is a source, then $\forall x \in N^o_\varepsilon(p), \lim_{k\to\infty} f^k(x) \notin N^\varepsilon_d(p) \Rightarrow d(p,x) > \varepsilon$* ■

Is there any way to investigate this sensitive dependence? Yes, and here we will introduce a method called **itinerary** of an orbit.

SOLUTION* 1.3** ***Itinerary
*We still consider the g_4 model. Assign the symbol **L** to the left subinterval $[0, 1/2]$ and **R** to the right subinterval $[1/2, 1]$. Then, for every initival condition x_0, we can list the itinerary with **L** and **R**.*
*For instance, the initial point $x_0 = 1/3$ have the itinerary **LRLRLRRLLRR***

Table 4: Logistic4 1/3 itinerary

0-2	0.333333333333333 (**L**)	0.888888888888889 (**R**)	0.39506172839506154 (**L**)
3-5	0.9559518366102727 (**R**)	0.16843169076687667(**L**)	0.5602498252491516 (**R**)
6-8	0.9854798342297868 (**R**)	0.05723732222487492 (**L**)	0.21584484467760304(**L**)
9-10	0.6770233908148179 (**R**)	0.874650876417697 (**R**)	...

And we can list all itinerary with different initial value.■

Table 5: Logistic4 itinerary with different initial value

Val	$1-10$	$11-20$	$21-30$	$31-40$	...
0.01	**LLLRRLLLLR**	**RRRLRLRLRR**	**LLRRRLLRRR**	**RLRRLRLRLR**	...
0.25	**LRRRRRRRRR**	**RRRRRRRRRR**	**RRRRRRRRRR**	**RRRRRRRRRR**	...
1/3	**LRLRLRRLLR**	**RLRLLRRRRL**	**LLLRRRRLRL**	**RRRRRRRLRR**	...
0.5	**RRLLLLLLLL**	**LLLLLLLLLL**	**LLLLLLLLLL**	**LLLLLLLLLL**	...
1	**RLLLLLLLLL**	**LLLLLLLLLL**	**LLLLLLLLLL**	**LLLLLLLLLL**	...

Notice that there are some conclusions.

***CONCLUSION* 1.2** *For every periodic-k point, the itinerary of orbit will repeats **L** or **R** infinitely.*

***CONCLUSION* 1.3** *For every k iterate, the itinerary have 2^k choice and the sum of their lengths is 1(or the length of the interval).*

Also, we have a conclusion not very obvious.

***CONCLUSION* 1.4** *Each 2^k itinerary is shorter than $\pi/2^{k+1}$.*

We will prove this conclusion in later sections.

We can also analysis the problem with **transition graph**.

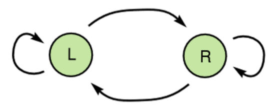


Figure 13: Transition graph

Finally, we focus on the title of this subsection "chaos", after these analysis, it is simple to summary the definition of chaos.

Definition 1.9 ***Chaos***
A chaotic orbit is a bounded, non-periodic orbit that displays sensitive dependence. Chaotic orbits seoarate exponentially fast from their neighbors as the map iterated.

Theorem 1.5 *The existence of periodic-3 orbit alone implies the existence of a large set of sensitive points, or chaotic orbit.*

We will prove this problem in appendix.

2 Two-Dimension and High-Dimension Maps

In this section, we will mainly discusse a new type of model, called Henon map which formed

$$f(x, y) = (a - x^2 + by, x)$$

A simple way to analysis this problem is analysis all point in the surface if they are convergence or divergence. In figures following, point in black represent initial conditions whose orbits diverge to infinity and the points in white represent initial values whose orbits converge to the period-2 orbit.

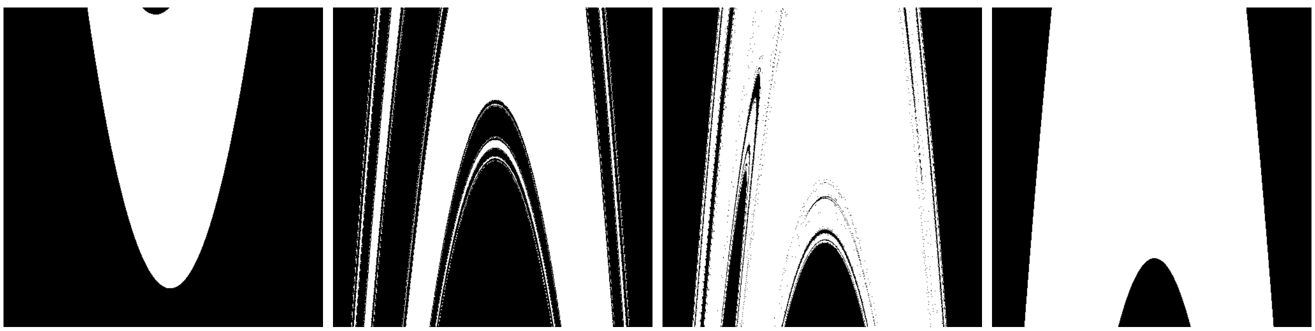


Figure 14: Initial condition square

(Parameter group $(a, b) = (0, 0.4), (2, -0.3), (1.4, -0.3), (1.28, -0.3)$)

2.1 Analysis of Henon map

Now we focus on Henon map. Familiar with 1 dim map, it is necessary to define the sink and source as well as saddle.

Definition 2.1 ***Neighborhood***
Consider a R^n space, called every point $x = (x_1, x_2, \ldots x_n)$ is a vector of R^n space,
Define the ***Euclidean Length*** $|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$*, which is equal to norm;*
And define the distance between two point $d(x, y) = |x - y|$;
Also, the ***ε-neighborhood*** *is*

$\forall \varepsilon > 0,$ *the ε-neighborhood of point $p, N_\varepsilon(p)$ is $\{x \in R^n ||x - p| < \varepsilon\}$, also define $N_\varepsilon^o(p) = N_\varepsilon(p) \backslash \{p\}$*

Definition 2.2 ***Sink and Source in High-dimension Map***
Let f is a map on R^n, p is a vector on R^n which is the fixed point and $f(p) = p$ then
If there is an $\varepsilon > 0$ s.t. $\forall x \in N_\varepsilon(p), \lim_{k\to\infty} f^k(x) = p$, then p is a sink or attracting fixed point.
If $\forall x \in N_\varepsilon^o(p), \exists K s.t. \forall k > K, f^k(x) \notin N_\varepsilon(p)$, then called the point p as source.

We will explain these definitions with an example

E x a m p l e 2.1 *Analysis the sink point, source point and saddle of Henon map with parameter $a = 0, b = 0.4$*

***SOLUTION* 2.1** *Obviously, if we consider the funcion $f(x,y) = (-x^2+0.4y, x) = (x,y)$, then*

$$-0.2x^2 + 0.4x = x \Rightarrow x_1 = 0, x_2 = -0.6$$

So the fixed points are $(0,0)$ and $(-0.6,-0.6)$. And now we have a new problem: how to confirm a fixed point is sink or source. Even the definition of sink and source are given above, we still need theory like Theo. 1.1. But here, we can analysis the problem with simulator.■

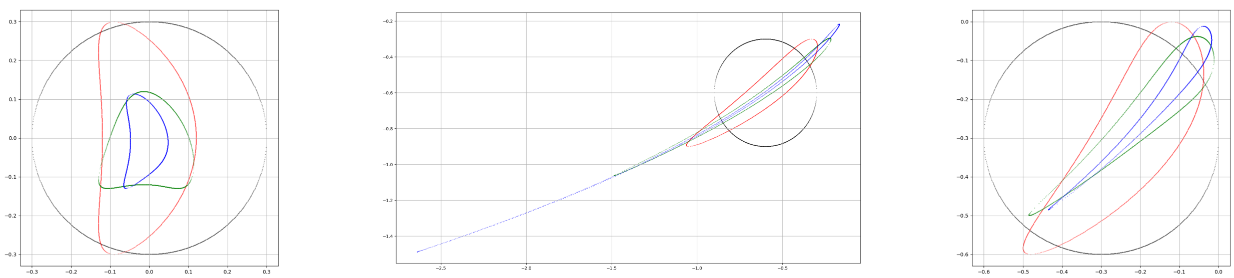


Figure 15: Sink, source and saddle in Henon map with $a = 0, b = -0.4$

(Order of color: Black(neighborhood), Red (Iter = 1), Green (Iter = 2), Blue (Iter = 3))

To solve the problem we faced in e.g.2.1, we will discuss the simple form of the high dimension maps.

Definition 2.3 ***High dimension linear map***
*A map $A: R^m \to R^m$ is **linear** if $\forall a,b \in R, \forall x,y \in R^m$, $f(ax+by) = af(x)+bf(y)$. Equivalently, a lienar map $f(x)$ can be represented as multiplication by an $m \times m$ matrix.*

Now we consider a system s.t. $f(x) = Ax$, if λ is eigenvalue and $\mathbf{v}$ is eigenvector of A, based on the definition fo eigenvalue and eigenvector, we have Let A have eigenvalue λ, based on the definition of eigenvalue, we have

$$A\mathbf{x} = \lambda\mathbf{v}$$

Then, for the initial point $\mathbf{v}$, we have

$$A(\mathbf{v}) = A\mathbf{v} = \lambda\mathbf{v}$$

let $\mathbf{v}_0 = \mathbf{v}, \mathbf{v}^n = A^n(\mathbf{v})$, then

$$\mathbf{v}_1 = A\mathbf{v}_0 = \lambda\mathbf{v}_0, \mathbf{v}_2 = A\mathbf{v}_1 = \lambda^2\mathbf{v}_1 \ldots \mathbf{v}_n = A\mathbf{v}_{n-1} = \lambda^n\mathbf{v}_0$$

Futhermore, if we consider a system in random initial value $\mathbf{x}_0$, still define $\mathbf{x}_n = f(\mathbf{x}_{n-1})$, then

$$\mathbf{x}_n = f(\mathbf{x}_{n-1}) = A\mathbf{x}_{n-1} = Af(\mathbf{x}_{n-2}) = \ldots = A^n\mathbf{x}_0$$

To analysis this problem, firstly we will review some theorems in algebra.

D I S C U S S I O N* 2.1 *Eigenvalue, eigenvector and Jordan normal form

** We will consider a square matrix $A_{m\times m}$ s.t. $rank(A) = m$ in following discussion.*

[i] If** A **have** m **different Eigenvalue

Based on the discussion above, we know that it is the first step to analysis the A^n to discribe all the linear system. Obviously, if A is a diagonal matrix, then the exponent of the matrix is easy and simple.

Theorem 2.1 *Let A is a diagonal matrix s.t. $A = diag(a_1, a_2, \ldots a_m)$, then $A^n = diag(a_1^n, a_2^n, \ldots a_m^n)$.*

Furthermore, if matrix A have m different eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $\mathbf{v}_i$ is the eigenvector of λ_i. Let

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_m), V = (\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m)$$

then we can easily prove that

$$A = V^{-1}\Lambda V$$

And the calculation of A^n is simple.

$$A^n = V^{-1}\Lambda^n V = V^{-1}diag(\lambda_1^n, \lambda_2^n, \ldots, \lambda_m^n)V$$

Now we back to consider the linear system, if $f(\mathbf{x}) = A\mathbf{x}$ and A have m different eigenvalue, then we know that

$$\mathbf{x}_n = A^n\mathbf{x}_0 = V^{-1}diag(\lambda_1^n, \lambda_2^n, \ldots, \lambda_m^n)V\mathbf{x}_0$$

Based on the analysis in the section 1, we still want to analysis the convergence and divergence for ever system.

$$\lim_{n\to\infty} x_n = V^{-1}diag(\lim_{n\to\infty}\lambda_1^n, \lim_{n\to\infty}\lambda_2^n, \ldots, \lim_{n\to\infty}\lambda_m^n)V\mathbf{x}_0$$

Obviously, with the knowledge of sequence, if $|\lambda_i| \in [0, 1)$, then $\lim_{n\to\infty}\lambda_i^n = 0$ and the sequence is convergence. Also, if $|\lambda_i| \in (1, +\infty)$, then $\lim_{n\to\infty}\lambda_i^n = \infty$ and the sequence is divergence. So we have this conclusion.

Theorem 2.2 ***Sink, source and saddle in linear system***

Consider a linear system $f(\mathbf{x}) = A\mathbf{x}$, where A is a square matrix in m dimension. If the eigenvalue of A are $\lambda_1, \lambda_2, \ldots \lambda_m$ and

***[i]** $\forall i \in 1, 2, \ldots, m, |\lambda_i| < 1$, then the origin point is sink.*

***[ii]** $\forall i \in 1, 2, \ldots, m, |\lambda_i| > 1$, then the origin point is source.*

***[iii]** $\{i||\lambda_i| < 1\} \neq \varnothing \wedge \{j||\lambda_j| > 1\} \neq \varnothing$, that means, if at least one eigenvalue are absolute smaller than one and at least one is upper than one, then the origin point is saddle.*

[ii] If** A **have at least two equal eigenvalue

We can transfrom the matrix A with Jordan normal form rather than eigenvalue diagonal matrix. Consider the matrix $A_{m\times m}$ and the eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_k$ are $r_1, r_2, \ldots r_k$ multiple root of function $|\lambda I - A| = 0$, which satisfied the definition of eigenvalue, and $k < m$, $\sum_{i=1}^k r_i = m$, $I = diag(1, 1, 1, \ldots, 1)$.

Then for every r_i multiple eigenvalue λ_i, $\exists \mathbf{v}_{i1}, \mathbf{v}_{i2}, \ldots \mathbf{v}_{ir_i}$ s.t.

$$|\lambda I - A|\mathbf{v}_{i1} = 0, |\lambda I - A|\mathbf{v}_{ij+1} = \mathbf{v}_{ij}(j = 1, 2, \ldots r_i - 1)$$

*We can still structure the V matrix same as V in **[i]**, and we can also represent the diagonal eigenvalue matrix Λ to the **Jordan normal form matrix** J which satisfied*

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \dots & \\ & & & J_k \end{bmatrix} = diag(J_1, J_2, \dots J_k),\ where\ J_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \dots & \dots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}$$

*is r_i dimension square matrix called **Jordan block***

Based on the calculation of block matrix we found that

$$A^n = V^{-1}J^nV = V^{-1}diag(J_1^n, J_2^n, \dots, J_k^n)V$$

*So familiar with the discussion in **[i]**, now it is necessary to discuss the J_i^n. On the other hand, we know that for ever Jordan block, we have*

$$J_i^n = \begin{bmatrix} \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \binom{n}{2}\lambda_i^{n-2} & \dots & \binom{n}{r_i}\lambda_i^{n-r_i} \\ & \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \dots & \binom{n}{r_{i-1}}\lambda_i^{n-r_i+1} \\ & & \dots & \dots & \dots \\ & & & \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} \\ & & & & \lambda_i^n \end{bmatrix}$$

Obviously, for ever element on the diagonal, the Theo. 2.2 still established. To proved that, we will prove the following theorem firstly.

Theorem 2.3 *Let J_i is a Jordan block with eigenvalue λ_i.*
[i] *If $|\lambda_i| < 1$, then $\lim_{n\to\infty} J_i^n = 0$*
[ii] *If $|\lambda_i| > 1$, then $\lim_{n\to\infty} J_i^n = \infty$*

***PROOF* 2.1** *Consider a element $\binom{n}{k}\lambda_i^{n-k}$ of J_i, then*

$$\lim_{n\to\infty} \binom{n}{k}\lambda_i^{n-k} = \lim_{n\to\infty} \left(\frac{n(n-1)\dots(n-k)}{1\cdot 2\cdot \dots \cdot k} \lambda_i^{n-k} \right)$$

As the $\frac{n(n-1)\dots(n-k)}{1\cdot 2\cdot\dots\cdot k}$ is a polynomial of n in k dimension, so $\exists a_1, a_2, \dots a_k \in R$ s.t.

$$\lim_{n\to\infty} \binom{n}{k}\lambda_i^{n-k} = \lim_{n\to\infty} \left(\sum_{p=1}^{k} a_p n^p \right) \lambda_{n-k} = \sum_{p=1}^{k} \lim_{n\to\infty} (a_p n^p \lambda_i^{n-k})$$

Finally, we found, if $|\lambda_i| > 1$, then $\lim_{n\to\infty}\binom{n}{k}\lambda_i^{n-k} = \infty$ and if $|\lambda_i| < 1$, then $\lim_{n\to\infty}\binom{n}{k}\lambda_i^{n-k} = 0$ and the Theo. 2.3 is established. ■

As for the non-linear problem, a wildly used method is **Jacobian matrix**

Definition 2.4 ***Jacobian matrix***
*Let $\mathbf{f} = (f_1, f_2, \dots f_m)$ be a map on R^m and $\mathbf{p} \in R^m$ is a point on R^m space. The **Jacobian matrix** of $\mathbf{f}$ at $\mathbf{p}$ is the matrix*

$$D\mathbf{f}(\mathbf{p}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{p}) & \frac{\partial f_1}{\partial x_2}(\mathbf{p}) & \dots \frac{\partial f_1}{\partial x_m}(\mathbf{p}) & \\ \frac{\partial f_2}{\partial x_1}(\mathbf{p}) & \frac{\partial f_2}{\partial x_2}(\mathbf{p}) & \dots \frac{\partial f_2}{\partial x_m}(\mathbf{p}) & \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{p}) & \frac{\partial f_m}{\partial x_2}(\mathbf{p}) & \dots \frac{\partial f_m}{\partial x_m}(\mathbf{p}) & \end{bmatrix}$$

Jacobian matrix is a linearization estimation of a non-linear system that we can assume the derivative of the system near the point $\mathbf{p}$ is $D\mathbf{f}(\mathbf{p})$. That means, instead of origin non-linear system, we can analysis the estimated system $\mathbf{f}_1(\mathbf{x}) = D\mathbf{f}(\mathbf{p})\mathbf{x}$ where $x \in N(\mathbf{p}, \varepsilon)$ and ε is a certain constant. Based on the Theo. 2.2, it is easy to improve the following conclusion.

Theorem 2.4 ***Sink, source and saddle in non-linear system***
Consider a non-linear system $\mathbf{f}(\mathbf{x})$ and a fixed point $\mathbf{p} \in R^m$ s.t. $\mathbf{f}(\mathbf{p}) = \mathbf{p}$. If the Jacobian matrix of $\mathbf{f}$ at $\mathbf{p}$ is $D\mathbf{f}(\mathbf{p})$, and Λ are eigenvalue set of matrix $D\mathbf{f}(\mathbf{p})$
[i] *$\forall \lambda_i \in \Lambda |\lambda_i| < 1$, then the $\mathbf{p}$ is a sink point.*
[ii] *$\forall \lambda_i \in \Lambda |\lambda_i| > 1$, then the $\mathbf{p}$ is a source point.*
[iii] *$\{\lambda_i \in \Lambda || \lambda_i| < 1\} \neq \varnothing \wedge \{\lambda_i \in \Lambda || \lambda_i| > 1\} \neq \varnothing$, that means, if at least one eigenvalue are absolute smaller than one and at least one is upper than one, then the $\mathbf{p}$ is a saddle point.*

Finally, we can analysis the property of fixed point in e.g. 2.1.

SOLUTION **2.2** *We can consider the Henon map directly*

$$f(x, y) = (a - x^2 + by, x) \Rightarrow Df(x, y) = \begin{bmatrix} -2x & b \\ 1 & 0 \end{bmatrix}$$

Let λ are eigenvalue, then

$$|\lambda I - Df(x, y)| = 0 \Rightarrow \begin{vmatrix} -2x - \lambda & b \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 2x\lambda - b = 0 \Rightarrow \lambda_{12} = -x \pm \sqrt{x^2 + b}$$

When $(a, b) = (0, 0.4)$

$$\text{If } (x, y) = (0, 0), \text{ then } |\lambda_{12}| = |-x \pm \sqrt{x^2 + b}| = |\sqrt{0.4}| < 1$$

If $(x, y) = (0.6, 0.6)$, then $|\lambda_{12}| = |-x \pm \sqrt{x^2 + b}| = |-0.6 \pm \sqrt{0.76}| \Rightarrow |\lambda_1| = 1.472 > 1, |\lambda_2| = 0.272 < 1$
Finally, we proved that $(0, 0)$ is sink and $(-0.6, -0.6)$ is saddle just as what we found on simulation. ■

2.2 Stability and matrix periodic

We found this conclusion based on the discussion above.

CONCLUSION **2.1** *The value of Jacobian matrix of a Henon map is just relevant to variable x and parameter b. That means, if we reduce the dimension of parameter and fixed b as b_0, then the property of fixed point will be determined only with variable x.*

As $y_{n+1} = x_n$, so we can just analysis the bifurcation of $a - x_\infty$ with random initial point.

If we consider the fixed point of system with arbitrary paremeter grou (a, b), we found that the fixed point will satisfied

$$x^2 + (1-b)x - a = 0 \Rightarrow x = \frac{1}{2}(b-1) \pm \sqrt{(b-1)^2 + 4a} \tag{1}$$

and the fixed point is $(x, y) = (\frac{1}{2}(b-1) \pm \sqrt{(b-1)^2 + 4a}, \frac{1}{2}(b-1) \pm \sqrt{(b-1)^2 + 4a})$, so we have the Jacobian matrix at this fixed point as

$$Df(x, y) = \begin{bmatrix} (b-1) \pm \sqrt{(b-1)^2 + 4a} & b \\ 1 & 0 \end{bmatrix}$$, and the eigenvalue λ_i satisfied

$$\lambda^2 - [(b-1) \pm \sqrt{(b-1)^2 + 4a}]\lambda - b = 0 \tag{2}$$

Then we can found the property of sink and source in every fixed point easily.

Now we focus on periodic-k orbit. Firstly, we still plot the bifurcation diagram of Henon map.

Figure 16: Bifurcation diagram for Henon map ($b = 0.4$)

That is simple to analysis the influence of parameter. In following plots, $b \equiv 0.4$ and $a = 0.9, 0.988, 1.0, 1.0293, 1.045, 1.2$. We found $a = 0.9$ is a periodic-4 sink, $a = 0.988$ is a periodic-16 sink, $a = 1.0$ is a four-piece attractor, $a = 1.0293$ is a periodic-10 sink, $a = 1.045$ is two-piece attractor and the points of an orbit alternate between the pieces. Finally $a = 1.2$ two pieces have merged to form one-piece attractor.

Definition 2.5 ***Attractor*** *An attractor is a set of numerical values toward which a system tends to evolve, for a wide variety of starting conditions of the system.*
Futuermore, in discrete time, we called the orbit of a system as periodic-k orbit. However in chaotic orbit, the solution set is a continuous (or uncountable) set. And we called this orbit as attractor.

We will discuss the relationship between matrix and periodic-k orbit. But before that, it is necessary to introduce some new definition.

Definition 2.6 *A map* $\mathbf{f}$ *on* R^m *is* ***one-to-one*** *if and only if* $\mathbf{f}(\mathbf{v}_1)\mathbf{f}(\mathbf{v}_2) \Leftrightarrow \mathbf{v}_1 = \mathbf{v}_2$

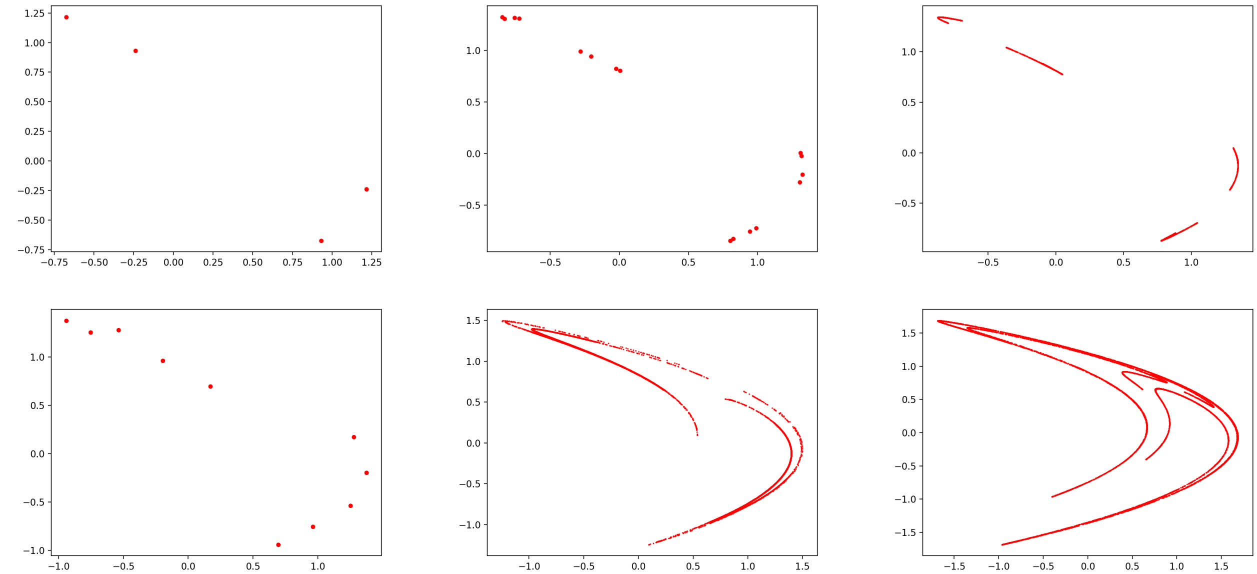


Figure 17: Attractor of Henon map in different parameter

Definition 2.7 ***Inverse map***
Consider a one-to-one map $\mathbf{f}$ *on* R^m. *The inverse map* $\mathbf{f}^{-1}$ *is automatically exists and satisfied* $\forall \mathbf{v} \in D \subset R^m, \mathbf{f}(\mathbf{f}^{-1})(\mathbf{v}) = \mathbf{f}^{-1}(\mathbf{f})(\mathbf{v}) = \mathbf{v}$, *where* D *is domain of map.*

For instance, a one-to-one map $f(x) = 2x$ have an inverse map $f^{-1} = x/2$. Obviously, for every linear map $\mathbf{f}(\mathbf{x}) = A\mathbf{x}, \exists f^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$.

Theorem 2.5 *For every* R^m *linear map* $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ *and* A *s.t.* $rank(A) = m$, *the inverse map* f^{-1} *always be existed.*

PROOF 2.2 *[i] If* A *have* m *different eigenvalue, then* $A = V\Lambda V$ *where* $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_m), V = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ *where* λ_i *is eigenvalue and* $\mathbf{v}_1$ *is eigenvector. Then*

$$A^{-1} = \left(V^{-1}\Lambda V\right)^{-1} = V\Lambda^{-1}V^{-1} = V diag\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_m}\right) V^{-1}$$

[ii] If A *have* $p < m$ *different eigenvalue, based on the Jordan normal form, we still have* J, V *s.t.* $A = V^{-1}JV$ *and* $J = diag(J_1, J_2, \ldots J_k)$ *where* J_i *is Jordan block based on the eigenvalue* λ_i. *And now the problem is prove that for evert Jordan block, the inverse block always be existed. Obviously,* $J_i = \lambda_i I + N$ *where*

$$N = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

And it is simple to found that $N^m = \mathbf{0}_{m\times m}$. *Based on the Taylor expansion, we have*

$$J_i^{-1} = \lambda_i^{-1}(I + \lambda_i^{-1}N + \lambda_i^{-2}N^2 - \ldots + (-\lambda_i)^{-n+1}N^{n-1})$$

Although the inverse of Jordan block is not a Jordan block of $1/\lambda_i$, *it is still exists and we proved the theorem.* ■

3 Chaos

We discussed the Henon map in last section. However, different from the section 1, Logistic map has been wildly used in application problems, we talked less about why we are interested in this model. So, in this section, we will mainly introduce the motivation.

3.1 Lorenz system, Henon map and Poincare section

D I S C U S S I O N* 3.1 *Why are we intersted in Henon map?
First of all, it is necessary to introduce a continuous model. The Lorenz system is a system of ordinary differential equations which notable for having chaotic solutions for certain parameter values and initial conditions. In particular, the Lorenz attractor is a set of chaotic solutions of the Lorenz system.

P r o b l e m 3.1 *Lorenz model*
Lorenz model is a system of three ordinary differential equations now known as the Lorenz equations:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

where σ, ρ, β are parameters.

It is continuous problem, however, based on the knowledge in numerical analysis, we can discrete the continuous to discrete problem in several ways.[1]

We can reconstruct the Lorenz equation, or a normal continuous dynamical system as

$$\frac{dX_i}{dt} = F_i(X_1, X_2, \dots X_m), i = 1, 2, \dots m$$

which is a m-dim dynamical system and t is single independence variable. To simplify this problem, we choose a initial time t_0 and time increment Δt, then let

$$X_{i,n} = X_i(t_0 + n\Delta t)$$

we have several ways to aproximate the equations.

SOLUTION* 3.1 *[i] Auxiliary approximations $X_{i,n+1} = X_{i,n} + F_i(P_n)\Delta t$
[ii] Centered difference procedure $X_{i,n+1} = X_{i,n-1} + 2F_i(P_n)\Delta t$
[iii] Double-approximation procedure $X_{i,n+1} = X_{i,n} + \frac{1}{2}(F_i(P_n) + F_i(P_{n+1}))\Delta t$

[1] EDWARD N LORENZ'S 1963 PAPER, "DETERMINISTIC NONPERIODIC FLOW", IN JOURNAL OF THE ATMOSPHERIC SCIENCES, VOL 20, PAGES 130–141

Even solve this group of function directly is difficult, it is not difficult to find the numerical solution.

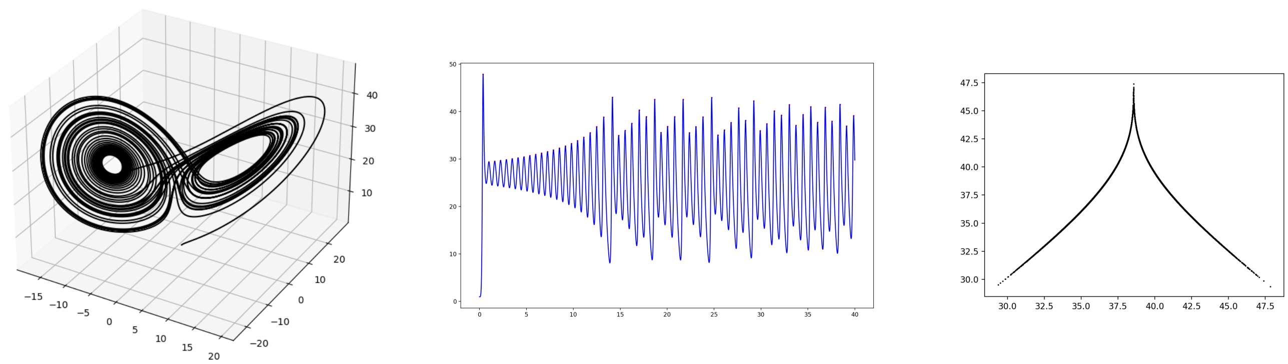


Figure 18: Lorenz system, $z - t$ map and Lorenz map

No we consider the problem in one dimension. In the second part of Fig. 18, we plot the z-t figure of Lorenz model.

Ok, we found that it is still difficult to discribe the $z - t$ figure. However, after the discussion of the logistic map $g(x) = 4x(1-x)$ as well as chaotic orbit, we know in most situation, we just care about the boundary of interval of a map. On the other hand, we found that the $z - t$ figure of Lorenz system is familiar with sine function, it is shaking during the time iteration. So if we just consider the maxinum (or the mininum) of this $z - t$ map, we can analysis the problem easier.

Definition 3.1 ***Lorenz map***
The function $z_{n+1} = f(z_n)$ satisfied the last of Fig. 18 is called the Lorenz map. The map can be discribed in following steps.
[i] Find the $z - t$ function in Lorenz model.
[ii] The map $\{z_n\}$ is a point set which is the maxima of $z - t$ function. $z_{n+1} = f(z_n)$ where z_n is a maxima point of $z - t$ function and z_{n+1} is next maxima point of the function with growing of t.

* The graph of Lorenz map is not actually a curve. It does have some trickness because it is not a well-defined function. However trickness is so small and there is so much to be gained by treating the graph as a curve, that we will simplt make this approximation keeping in mind that the sunsequence analysis is plausible.

As Lorenz map have no formula to discribe, it is very difficult to research that. However, in Lorenz's paper, he gave a correspondence to analysis the map, called tent map, which we has been introduced in the section 1

$$x_{n+1} = \begin{cases} 2x_n & x_n < 1/2 \\ \textit{Undefined} & x_n = 1/2 \\ 2 - 2x_n & x_n > 1/2 \end{cases}$$

At least this is a piecewise continuous funcion with one discontinuous point $x = 1/2$. So we found the property of this map is not good enough to analysis. We hope the function is continuous in all domain. And we found if we try to remove this discontinuous point, then the f' will satisfy $f'^{+}(1/2) = f'^{-}(1/2) = 1$. We found it is similar to the Logistic model and it seems we can discuss the property of Logistic map rather than Lorenz map. And we will explain why we can discuss the Logistic map instead of Lorenz map.

On the other hand, we found the Henon map also familiar with Logistic map in Fig. 17. The only different between Henon map and Logistic map is Henon map is fat and Logistic is thin, or a line. However if we change our parameter, like $b \to 0$, then we found

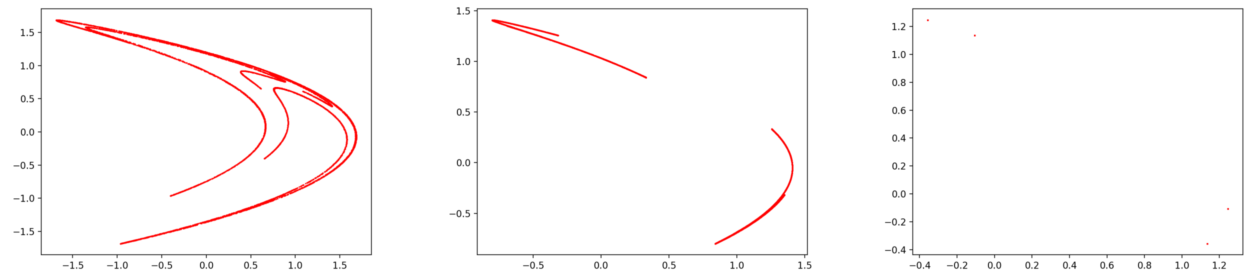


Figure 19: Attractor of Henon map in different b(0.4, 0.2, 0.05)

So it ***seems*** *we can analysis the Henon map instead of Lorenz map. But we still need more proof and in this section, we will try to solve these problems.*

3.2 Lyapunov exponents and Conjugacy

Definition 3.2 ***Asymptotically periodic***

Consider map $f \in C^1(R^1)$. An orbit $\{x_1, x_2, \dots x_n, \dots\}$ is called asymptotically periodic if it convergence to a periodic orbit as $n \to \infty$, that means, $\exists \{y_1, y_2, \dots y_k, y_1, y_2, \dots y_k m, \dots\}$ is a periodic orbit s.t.

$$\lim_{n\to\infty} |x_n - y_n| = 0$$

Also, wecalled these map as ***eventually periodic*** *beacause their orbit is eventually lands on a periodic orbit.*

For instance, the Lorenz map is a eventually periodic map. Because we found in the begining of the iteration, the map shaking in a wild interation (Just as 1-15 iterates in Fig. 18, second image) and after this period, the map become stable and it is convergent to the map in the 3rd image of Fig. 18, which we called that Lorenz map.

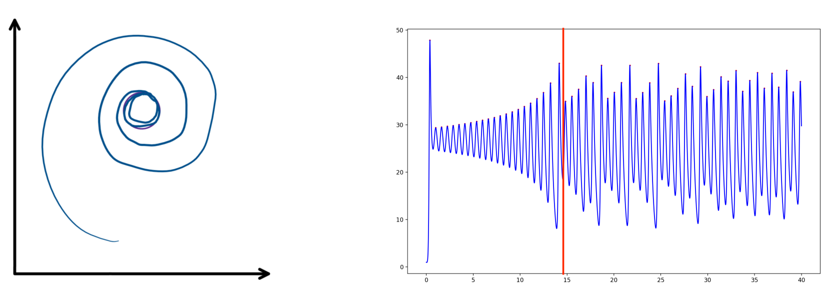


Figure 20: Asymptotically periodic, intro and example in Lorenz map

There are several other maps with this property, for example, in section 1, we introduced the Logistic map $G(x) = 4x(1-x)$, with the initial condition $x_0 = 1/2$, we found after 2 iterates, is coincides with the fixed point $x = 0$.

Now we try to find a method to judge a map is asymptotically periodic to another periodic map. In the section 1, we introduced the stability test for periodic orbits (Theo. 1.3), we called the limitation of value in Theo. 1.2 as **Lyapunov number**.

Definition 3.3 ***Lyapunov number and Lyapunov exponent***
*Consider map $f \in C^1(R^1)$. Define **Lyapunov number** $L(x_1)$ as*

$$L(x_1) = \lim_{n\to\infty} (\prod_{i=1}^{n} |f'(x_i)|)^{1/n}$$

and based on the logarithm function, we can define the Lyapunov exponent as

$$h_f(x_1) = h(x_1) = \lim_{n\to\infty} \frac{1}{n} \left[\sum_{i=1}^{n} \ln(f'(x_i)) \right]$$

Notice that h exists if and only if L exists and is nonzero, also $\ln L = h$.

D I S C U S S I O N* 3.2** ***Why are we interested in Lyapunov exponent?
We know that chaos means if two initial values have small different, then after long time iteration, the system showed us very different property with these two initial valus.(Just like Fig. 11, the error from computer results very different value, one is periodic and the other is chaos.)
Consider a self map f and a orbit $x_0, x_1, \ldots x_n, \ldots$, where $x_{i+1} = f(x_i) = f^i(x_0) (\forall i \in \mathcal{N})$. Now we focus on the point $x_0 + \delta_0$ where δ_0 is small almost near the x_0 point. For other $\delta_i, i \in \mathcal{N}^+$, we define

$$\delta_i = f(x_{i-1} + \delta_{i-1}) - x_i = f^i(x_0 + \delta_0) - f^i(x_0)$$

which discribed the distant between different initial value in i-iterate.
*Based on the definition of **exponentially stable**, we assume*

$$|\delta_i| = \exp(\lambda i)|\delta_0|$$

then, for every $i \in \mathcal{N}$ which is positive, the monotony of $\exp(\lambda i)|\delta_0|$ is decided by λ. If $\lambda > 1$, the sequence $\{\delta_i\}$ is disconvergence and if $\lambda <\in (0, 1)$ the sequence is convergence.

On the other hand, to find property of λ is difficult beacause we cannot discribe the proprety of λ directly. However, during the iteration, we will find every $x_1, x_2, \ldots x_n, \ldots$ as well as $\delta_1, \delta_2, \ldots, \delta_n, \ldots$, and we know that

$$\lambda = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right|$$

Based on our thought, we need $\delta_0 \to 0$ and $n \to \infty$ so

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln \left| \lim_{\delta_0 \to 0} \left(\frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right) \right| = \lim_{n\to\infty} \frac{1}{n} \ln |(f^n)'(x_0)| = \lim_{n\to\infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n} f'(x_i) \right|$$

which is the definition of Lyapunov number.

Based on this Lyapunov exponent, we have this theorem.

Theorem 3.1 *Consider map $f \in C^1(R^1)$. If orbits $\{x_1, x_2, \ldots\}$ of f satisfies $f'(x_i) \neq 0 \forall i \in N$ and it is asymptotically periodic to the periodic orbit $y_1, y_2, \ldots$, then two orbit have indentical Lyapunov exponents, assuming both exist.*

PROOF 3.1 *[i] If we consider a sequence $\{s_i\}$ s.t. $\lim_{i\to\infty 0} s_i = s$, then*

$$\forall \varepsilon > 0, \exists N_1 \in \mathcal{N} \ s.t. \ \forall n > N_1, |s_n - s| < \varepsilon$$

Now we consider the average of $\{s_i\}$, we found for this ε,

$$\lim_{N\to\infty} \frac{1}{N} \left| \sum_{i=1}^{N} s_i - s \right| = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} |s_i - s| = \lim_{N\to\infty} \frac{1}{N} \left[\sum_{i=1}^{N_1-1} |s_i - s| + \sum_{i=N_1}^{N} |s_i - s| \right]$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N_1-1} |s_i - s| + \lim_{N\to\infty} \frac{1}{N} \sum_{i=N_1}^{N} |s_i - s| \leq 0 + \lim_{N\to\infty} \frac{1}{N - N_1} \sum_{i=N_1}^{N} |s_i - s| < \frac{N0N_1}{N - N_1} \varepsilon = \varepsilon$$

So we have this conclusion

$$\forall \varepsilon > 0, \exists N_1 \in \mathcal{N}, \ s.t. \ \forall n > N_1, \lim_{N\to\infty} \frac{1}{N} \left| \sum_{i=1}^{N} s_i - s \right| < \varepsilon \Rightarrow \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} s_i = s$$

[ii] Let y_1 is the fixed point (that means, x_i asymptotically periodic to a periodic-1 orbit), then $\lim_{n\to\infty} x_n = y_1$. As $f \in C^1(R^1)$, then f' is exists and f' Riemann integrable (and of course, Lebesgue integrable), so we can exchange the order of integral(or differential of f) and limitation, then we have

$$\lim_{n\to\infty} f'(x_n) = f'(\lim_{n\to\infty} x_n) = f'(y_1)$$

On the other hand, as $\ln|x|$ is a continuous, monotony function for $x \in R^+$, then

$$\lim_{n\to\infty} \ln|f'(x_n)| = \ln\left|\lim_{n\to\infty} f'(x_n)\right| = \ln|f'(y_1)| \Rightarrow h(x_1) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln|f'(x_i)| = \ln|f'(y_1)| = h(y_1)$$

[iii] Now we assume $k > 1, k \in \mathcal{N}$, obviously, y_1 is fixed point of f^k, and

$$h_{f^k}(x_1) = \ln|(f^k)'(y_1)| = h_{f^k}(y_1)$$

Now we will prove $h_{f^k}(x_1) = \frac{1}{k} h_f(x_1)$. Based on the definition, we know

$$h_{f^k}(x_1) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln\left|(f^k)'(x_i)\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln\left| \frac{1}{k} \prod_{j=i}^{i+k} f'(x_i) \right| = \frac{1}{k} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=i}^{i+k} \ln|f'(x_i)| = \frac{1}{k} h_f(x_1)$$

And we proved the theorem. ■

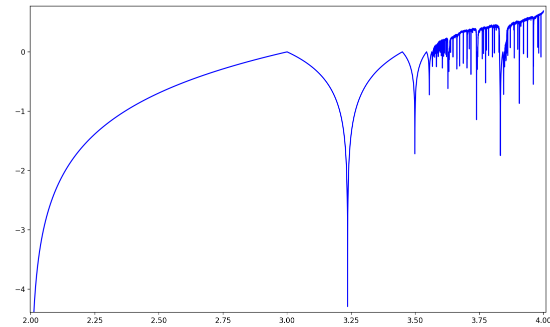


Figure 21: Lyapunov exponent of logistic model in different parameter

Obviously, the chaotic orbit satisfied Def. 1.9 will have no asymptotically periodic and we have this theorem.

Definition 3.4 *Consider map $f \in C^1(R^1)$, the orbit is* ***chaotic*** *if it satisfied both*
[i] *$\{x_1, x_2, \ldots\}$ is no asymptotically periodic, and*
[ii] *the Lyapunov exponent $h(x_1)$ is* ***greater*** *than zero.*

Now we will discuss the mod map and the tent map.

The mod map has been introduced in section 1 (Fig. 12) and we will mainly discuss the $f(x) = 2x(mod 1)$ in this discussion. A simple way to analysis the stability of map is itinerary.

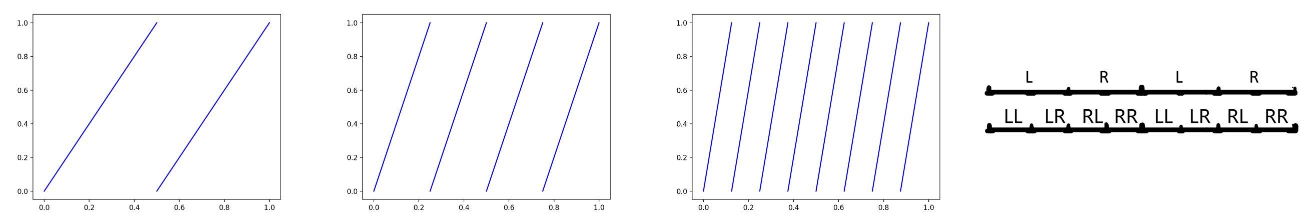


Figure 22: Iterates and itinerary of mod-2 map

[i] Binary and itinerary of mod map

We will focus on the initial value 0.2 of mod-2 map firstly. That is simple to find the itinerary with the code we used in section 1.

Table 6: Logistic4 itinerary with different initial value

Val	$1-8$	$9-16$	$17-24$	$25-32$	$\ldots$
0.2	**LLRRLLRR**	**LLRRLLRR**	**LLRRLLRR**	**LLRRLLRR**	$\ldots$

We found that this point is a periodic-4 point. And now, to analysis the problem simple, we will import a new method based on the binary number. In this problem, we have

$$\frac{1}{5} = 0.\overline{0011}, \quad f(\frac{1}{5}) = 0.011\overline{0011}, \quad f^2(\frac{1}{5}) = 0.11\overline{0011}, \quad f^3(\frac{1}{5}) = 0.1\overline{0011}, \quad f^4(\frac{1}{5}) = \overline{0011},$$

and obviously 1/5 point is a periodic-4 orbit. So in summary, we can use binary value to find the period of every point in this map.

Now we consider the Lyapunov exponent of mod map. Obviously, the differential of map is equal to 2 except the discontinuous point 1/2, so for ever initial value different from the 1/2, we have

$$h_{mod-2}(x_1) = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln\left(f'(x_i)\right)\right] = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln(2)\right] = \lim_{n\to\infty} \left(\frac{1}{n} n \ln(2)\right) = \ln(2) > 0$$

So the conclusion is the orbit of every initial value will result chaotic.

[ii] Mod-Sum model and asymptotically periodic

Now we change the mod model a little bit.

$$f(x) = (x+q)(mod 1) \text{ where } q \text{ is a constant.}$$

We will mainly consider the $q \in [0,1]$, if not, $\exists p \in [0,1]$ and $p = q(mod1) \wedge \forall x \in [0,1] f_q(x) = (x+q)(mod1) = f_p(x) = (x+p)(mod1)$.

That is simple to found the discontinuous point of $f_q(x)$ as $1-q$, futhermore, the differential of model is equal to 1 except this discontinuous point. So we have the Lyapunov exponent as

$$h_{f_q}(x_1) = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln\left(f'(x_i)\right)\right] = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln(1)\right] = 0$$

And

[iii] Tent map Now we consider the tent map.

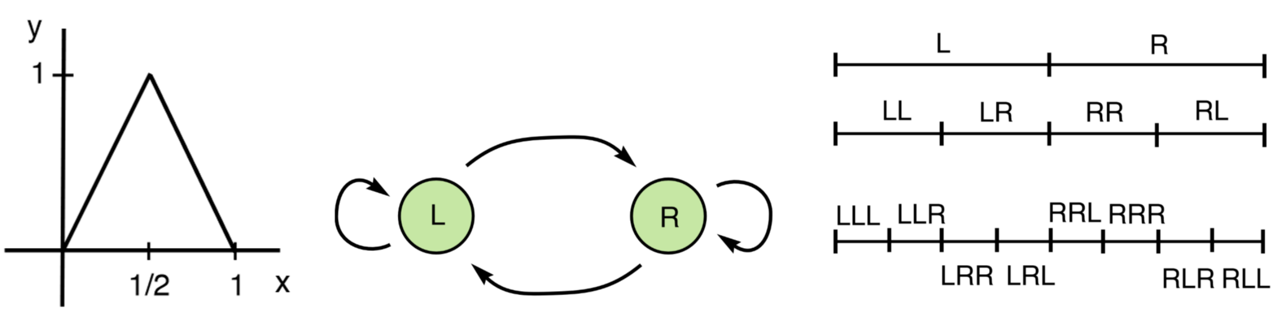


Figure 23: Tent map

We found that tent map have similar property of mod-2 map, that is because of the left side of the tent map is equal to the mod-2 map, and the right side of the tent map is symmetric of mod-2 map. So we can found that the size of interval in itinerary of both mod-2 map and tent map are 2^k and the only different between the mod-2 map and tent map is tent map will exchange the order of the **L** and **R** during iteration.

However, if we consider the topic of asymptotically periodic, we can found this conclusion.

Theorem 3.2 *The tent map T has infinitely many chaotic orbits.*

PROOF 3.2 *[**i**]Consider the set of $[0,1]\backslash Q$ which is combined with all irrational number in $[0,1]$. If $x \in [0,1]\backslash Q$, then*
1. *If $x < 1/2$, then $x_{new} = 2x \in [0,1]\backslash Q$*
2. *If $x > 1/2$, then $x_{new} = 2 - 2x \in [0,1]\backslash Q$*
Now we consider a orbit with irrational begining, obviously, every value of this orbit is based on irrational value. And we know that there are infinity element in $[0,1]\backslash Q$, so we have infinity irrational based orbits.

On the other hand, we know ever orbits avoid have

$$\forall x_1 \in [0,1], x_1 \neq \frac{1}{2}, h_f(x_1) = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln|f'(x_i)|\right] = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln(2)\right] = \ln(2) > 0$$

*As $1/2$ is a rational value, so every orbit in [**i**] will avoid this point. So all of these orbits are chaotic. And in summary we found infinity many chaotic orbits in tent map T.* ■

[iv] Conjugacy and Logistic map

First of all, we will list the main property in both logistic-4 map and tent map.

Table 7: Logistic-4 map VS tent map

Property	Logistic-4 map($G(x)$)	Tent map($T(x)$)
Critical point	$1/2 \to 1 \to 0$	
Fixed point	$0, 3/4$	$0, 2/3$
Period-2 orbit	$\{(5-\sqrt{5})/8, (5+\sqrt{5})/8\}$ (Single)	$\{0.4, 0.8\}$ (Single)
	Lies in the same relation, one in $[0,1/2]$ and the other in $[1/2,1]$ where $1/2$ is critical point	

We will check the derivative value for periodic-k orbit in both map.

Periodic-1
[1] **Tent map** $T'(2/3) = -2$
[2] **Logistic map** $G'(3/4) = -2$
CONCLUSION: $T'(2/3) = G'(3/4)$
Periodic-2
[1] **Tent map** $T'(0.4)T'(0.8) = -4$
[2] **Logistic map** $T'((5-\sqrt{5})/8)T'((5+\sqrt{5})/8) = -4$
CONCLUSION: $T'(0.4)T'(0.8) = T'((5-\sqrt{5})/8)T'((5+\sqrt{5})/8) = -4 = -2^2$
...

We found some intersted result, but we cannot confirm how long these property still established. To solve, or to prove the similiarity of tent map and logistic map, we will import the definition of conjugate.

Definition 3.5 *The mape f and g are **conjugate** if they are related by a continuous one-to-one change of coordinates, that is, if $C \circ f = g \circ C$.*

P r o b l e m 3.2 *Proof: Map T and G are conjugate, and the C is $C(x) = (1 - \cos(\pi x))/2$.*

***PROOF* 3.3** *[i]* $x \in [0, 1/2]$ $G(C(x)) = 4(C(x))(1 - C(x)) = 1 - \cos^2(\pi x) = \sin^2(\pi x)$
$C(T(x)) = 1 - \cos(2\pi x)/2 = \sin^2(\pi x) = G(C(x))$
[ii] $x \in [1/2, 1]$ $G(C(x)) = 4(C(x))(1 - C(x)) = 1 - \cos^2(\pi x) = \sin^2(\pi x)$
$C(T(x)) = 1 - \cos(\pi(2 - 2x)/2 = 1 - \cos(2\pi x)/2 = \sin^2(\pi x) = G(C(x))$
In summary we proved the problem. ■

Familiar with matrix exponent, if we consider the G^n map, then

$$G^n = G \circ G \circ G \circ G \circ \ldots \circ G = CTC^{-1}CTC^{-1}CTC^{-1}CTC^{-1} \ldots CTC^{-1} = CT^nC^{-1} \Rightarrow G^nC = CT^n$$

Theorem 3.3 *If x is periodic-k point of f, then $C(x)$ is a periodic-k point of g, where f, g are conjugate with C and $Cf = gC$.*

***PROOF* 3.4** *With the conclusion above, we have*

$$f^k = C^{-1}g^kC, \text{ then } \forall x \text{ is a periodic-k point }, f^k(x) = C^{-1}\{g^k[C(x)]\} = x \Rightarrow g^k[C(x)] = C(x) ■$$

Now we focus on the partical of map, because we want to know if the conclusion like $T'(0.4)T'(0.8) = T'((5 - \sqrt{5})/8)T'((5 + \sqrt{5})/8) = -4$ is established on high dimension or not.

Theorem 3.4 *If f and g are conjugate with C and $Cf = gC$ and x is periodic-k point of f (that means $C(x)$ is a periodic-k point of g), then*

$$(g^k)'[C(x)] = (f^k)'(x)$$

***PROOF* 3.5** *[1] The chain rule says that*

$$C[f(x)] = g[C(x)] \Rightarrow C'[f(x)]f'(x) = g'[C(x)]C'(x)$$

If x is periodic-1 (fixed) point, then $f(x) = x \wedge g(C(x)) = C(x)$, so we have

$$C'(x)f'(x) = g'[C(x)]C'(x) \Rightarrow f'(x) = g'[C(x)]$$

[2] Now if x is periodic-2 point, then $f^2(x) = x, g^2[C(x)] = C(x) \wedge C[f^2(x)] = g^2[C(x)]$ and

$$\frac{dC[f^2(x)]}{dx} = C'[f^2(x)]\frac{df[f(x)]}{dx} = C'[x]\frac{df[f(x)]}{dx} = \frac{dg\{g[C(x)]\}}{dx} = [(g^2)'(C(x))]C'(x)$$

So In summary we have

$$(f^2)'(x) = (g^2)'(C(x))$$

[k] Now we consider arbitrary $k \in \mathcal{N}$, familiar periodic-2, for a periodic-k point x, we have $f^k(x) = x, gk[C(x)] = C(x) \wedge C[f^k(x)] = g^k[C(x)]$ and

$$\frac{d\{C[f^k(x)]\}}{dx} = C'[f^k(x)](f^k)'(x) = C'(x)(f^k)'(x) = \frac{d\{(g^k)[C(x)]\}}{dx} = (g^k)'[C(x)]C'(x)$$

In summary we proved the conclusion that $\forall k \in \mathcal{N}, (f^k)'(x) = (g^k)'(C(x))$ ■

Now we can simply found this conclusion:

***CONCLUSION* 3.1** *If f and g are conjugate with C and $Cf = gC$ and $\{x_1, x_2, \ldots x_k\}$ is periodic-k orbit of f (that means $\{C(x_1), C(x_2), \ldots C(x_k)\}$ is a periodic-k orbit of g), then*

$$\prod_{i=1}^{k} f'(x_i) = \prod_{i=1}^{k} g'(C(x_i))$$

***PROOF* 3.6** *Based on the chain rule, we still have*

$$(f^k)'(x_1) = [f^{k-1}(f)]'(x_1) = (f^{k-1})'(f(x_1))f'(x_1) = (f^{k-1})'(x_2)f'(x_1) = \ldots = \prod_{i=1}^{k} f'(x_i)$$

also this conclusion is established in $(g^k)'(C(x))$ ■

Based on the conjugacy, we can prove these series of conclusions

***CONCLUSION* 3.2** *All periodic points of logistic map G are source.*

***PROOF* 3.7** *Firstly, we consider the tent map T, and obviously the point $0, 1/2$ and 1 will now be the periodic points because they result asymptotically periodic. And for ever other point of tent map, we know that $|T'(x)| \equiv 2$. On the other hand, we know that exists C s.t. $GC(x) = CT(x)$, so*

$$\prod_{i=1}^{k} |G'(x_i)| = \prod_{i=1}^{k} |T'(C(x_i))| = 2^k > 1 \blacksquare$$

(Also, this conclusion told us that $\prod_{i=1}^{k} |G'(x_i)| = 2^k$ will always be established, which we has beed found in periodic-1 and periodic-2 discussion.)

***CONCLUSION* 3.3** *Consider a itinerary figure of logistic map G, the length of every k-iterates subinterval is lower than $\frac{\pi}{2^{k+1}}$.*

***PROOF* 3.8** *Consider a subinterval of itinerary after k-iterates $[x_p, x_q]$ where $\forall x \in [x_p, x_q], T^k(x)$ have symbol $\boldsymbol{L} \vee T^k(x) \equiv \boldsymbol{R}$ and $\forall \varepsilon > 0 \forall x \in (N_\varepsilon(x_p) \bigcup N_\varepsilon(x_q)) \backslash [x_p, x_q], T^k(x)$ have different symbol of $x \in [x_p, x_q]$. Obviously, we know that $\mu([x_p, x_q]) = 2^{-k}$, where μ is lebesgue measure (or the length of the interval), then*

$$\mu_G([x_p, x_q]) = C(x_q) - C(x_p) = \int_{x_p}^{x_q} C'(x)dx = \int_{x_p}^{x_q} \frac{\pi}{2}\sin(\pi x)d \leq \frac{\pi}{2}\int_{x_p}^{x_q} dx = \frac{\pi}{2}(x_q - x_p) = \frac{\pi}{2^{k+1}} \blacksquare$$

Finally, we will use the tools of Lyapunov exponent as well as conjugacy to proof the following theorem.

Theorem 3.5 *The logistic map G has chaotic orbit.*

Lemma 3.1 *If f and g are conjugate with C and $Cf = gC$ $\{x_1, x_2, \dots x_k, \dots\}and\{C(x_1), C(x_2), \dots C(x_k), \dots\}$ are orbit of f and g, if C satisfied $\lim_{k\to\infty} \frac{1}{k} C'(x_k) = 0$ then we have*

$$h_f(x_1) = h_g(C(x_1))$$

***PROOF* 3.9** *Consider $C[f(x_i)] = g[C(x_i)]$, then the chain rule told us that*

$$\{C[f(x_i)]\}' = C'[f(x_i)]f'(x_i) = C'(x_{i+1})f'(x_i) = \{g[C(x_i)]\}' = g'[C(x_i)]C'(x_i)$$

$$\Rightarrow f'(x_i) = g'[C(x_i)]\frac{C'(x_i)}{C'(x_{i+1})} \Rightarrow \prod_{i=1}^{k} f'(x_i) = \prod_{i=1}^{k} g'(C(x_i)) \left(\frac{C'x_1}{C'(x_{k+1})}\right)$$

(This formula obey the conclusion above, if we pay attention to the condition, we know that once x_1 is a periodic-k point, then $x_{k+1} = x_1$ and we have $\prod_{i=1}^{k} f'(x_i) = \prod_{i=1}^{k} g'(C(x_i))$)
Moreover, based on the definition of Lyapunov exponent, there is

$$\ln\left|\prod_{i=1}^{k} f'(x_i)\right| = \sum_{i=1}^{k} \ln|f'(x_i)| = \ln\left|\prod_{i=1}^{k} g'[C(x_i)]\right| + \ln|C'(x_1)| - \ln|C'(x_{k+1})|$$

Note that $C'(x_1)$ is constant so $\lim_{k\to\infty} \frac{1}{k}(\ln|C'(x_1)|) = 0$. And based on the condition of theorem, we have $\lim_{k\to\infty} \frac{1}{k} \ln|C'(x_{k+1})| = 0$
So finally

$$h_f(x_1) = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \ln|f'(x_i)|$$

$$= \lim_{k\to\infty} \frac{1}{k} \ln\left|\prod_{i=1}^{k} g'[C(x_i)]\right| + \lim_{k\to\infty} \frac{1}{k}(\ln|C'(x_1)| - \ln|C'(x_{k+1})|)$$

$$= \lim_{k\to\infty} \frac{1}{k} \ln\left|\prod_{i=1}^{k} g'[C(x_i)]\right| = h_g(C(x_1))$$

In summary the theorem is established. ■

Lemma 3.2 *The periodic point of T map is countable, that is, if P_T is the periodic point of T then $P_T \sim \mathcal{N}$, or exists an one-to-one map $R, \forall x \in P_T, \exists$ only one $y \in \mathcal{N} s.t. R(x) = y$ vice versa.*

***PROOF* 3.10** *We can write a binary tree to analysis the periodic-k point of tent map.*

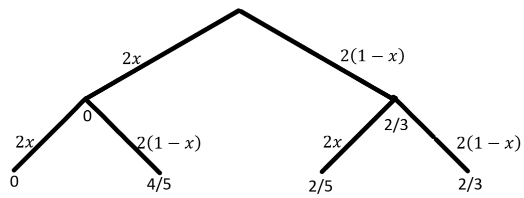


Figure 24: Binary tree of periodic-k point in tent map

We found that in every layer-k, the total of solution is less than 2^k and it is obviously that if we have infinity countable set in total and every set is infinity countable set, then the union of these sets is still countable (which has been introduced in Hilbert's paradox of the Grand Hotel).

On the other hand, in this problem, we have another simple way to build an one-to-one from periodic points P_T to $\mathcal{N}$. In the solusion binary tree above, we can code the solution from line to line. That means, in the first line, we code 0 to 1 and $2/3$ to 2. Then, in the next line, if we coded the solution, we jump over that node. After this iteration, we will build a one-to-one.

Moreover, actually we can proof that $P_T \subset [0,1] \bigcap Q$. It is simple to proof that the solution set $P_T^{(k)}$ of $T^k(x) = x (\forall k \in \mathcal{N})$ is the subset of $P^{(k)}$ and $x \in P^{(k)}$ satisfied $p + q \cdot 2^k x = x$, where $p \in Z$ and $q = \pm 1$, so $x = \frac{p}{q \cdot 2^k - 1} \in Q \bigcap [0,1]$ and we have

$$P_T^{(k)} \subset P^{(k)} \subset Q \bigcap [0,1]$$

So we have

$$P_T = \bigcup_{k=1}^{\infty} P_T^{(k)} \subseteqq \bigcup_{k=1}^{\infty} P^{(k)} \subseteqq Q \bigcap [0,1] \subset Q$$

As Q is countable set, then every subset of Q is countable, so P_T is countable. ■

***PROOF* 3.11** *Now we try to proof that G has chaotic orbit. As T and G are conjugacy, so based on one-to-one C, we know that P_G is countable. On the other hand, we proved that every periodic points of G are source, annd therefore no orbits besides periodic orbits - and eventually periodic orbits, another countable set - can be asymptotically periodic. Then any orbits whose responding symbol sequence is not eventually periodic, and which never contains the sequence **LL**, has Lyapunov exponent $\ln 2$ and is chaotic.*■

In the last of this subsection, we will introduce some other definition.

Definition 3.6 *If $A \subset B$ called A is **dense** in B if*

$$\forall x \in B, \forall \varepsilon > 0, N_\varepsilon(x) \bigcap A \neq \varnothing$$

Also, imitate the linear space, topological space, we can define the **symbol space** to discribe the itinerary of a map.

Definition 3.7 ***Symbol space***
*The set S of all infinity itineraries of a map is called the **symbol space** for the map. The **shift map** s is defined on the symbol space S as follows*

$$s(S_0S_1S_2\dots) = S_1S_2S_3\dots$$

This shift mape chops the leftmost symbol, which is the analogue on the itineraries of iterating the map on the point.

3.3 Fixed point theorem

Theorem 3.6 ***Fixed point theorem***
Let $f \in C(R^n), I = [a,b]$ s.t. $I \subset f(I)$, Then f has a fixed point in I
Moreover, if $I_1, I_2, \dots I_k$ are all closed intervals s.t. $\forall i = 1, 2, \dots, k-1, I_{i+1} \subset f(I_i) \wedge I_1 \subset f(I_n)$, then f^n has a fixed point in I_1, or f has a periodic-k point in I_1.

***PROOF* 3.12** *Proof: Theo.3.6*

$$I \subset f(I) \Rightarrow \forall x \in I, x \in f(I) \Rightarrow a, b \in f(I) \Rightarrow \exists x_1, x_2 \in [a,b] \text{ s.t. } f(x_1) = a, f(x_2) = b$$

$$(0 = f(x_1) - a \leq f(x_1) - x_1) \wedge (0 = f(x_2) - b \geq f(x_2) - x_2)$$

So we found a point x_1 s.t. $f(x) - x \geq 0$ and a point x_2 s.t. $f(x) - x \leq 0$, as the function f is continuous, based on the Intermediate Value Theorem, $\exists x_3$ s.t. $f(x_3) - x_3 = 0$ ■

Definition 3.8 ***Partition***
*The collection of subintervals that are pairwise disjoint except at the endpoints whose union is I is the **partition** of I.*

CONCLUSION* 3.4** ***Covering rule for transition graphs
[i] *An arrow is drawn from set A to set B in a transition graphs if and only if $B \subset f(A)$*
[ii] *Moreover, assume that $\{S_1, S_2, \dots S_n\}$ is a partition and the transition graph of f allows a sequence of symbols that returns to the same symbols s.t. $S_1S_2\dots S_kS_1$, then $S_1 \subset f^k(S_1)$*
[iii] *Generally, if $S_1S_2\dots S_kS_1$ is a path in the transition graph of a map f, then the subinterval denoted by $S_1S_2\dots S_kS_1$ contains a fixed point of f^k.*

3.4 Basins of attraction

Definition 3.9 ***Basin of attraction***
*Let f be a map on R^n and p be an attracting fixed point or periodic point of f, then the **basin of attraction** of p, or just **basin** of p is the set of point s.t.*

$$\lim_{k\to\infty} \left|f^k(x) - f^k(p)\right| = 0$$

Theorem 3.7 *Let f is a continuous map on R^1, then*
[i] *if $f(b) = b \wedge (\forall x \in [a, b), x < f(x) < b)$, then $a \to b, f^k(a) \to b$*
[ii] *if $f(b) = b \wedge (\forall x \in [b, c), b < f(x) < x)$, then $c \to b, f^k(c) \to b$*

***PROOF* 3.13** *We just need to proof **[i]** because it is simple to proof **[ii]** in the same method. Let $x_0 = a, x_{i+1} = f(x_i)\forall i \in \mathcal{N}^*$, obviously, $a \le x < f(x) < b$, thus all $x_i \in [a, b) \wedge a = x_0 < x_1 < x_2 < \ldots < x_\infty < b$ which is strictly increasing and bounded above by b. Since increasing bounded sequence must convergence, $\exists x_*$ s.t. $x_i \to x_*$ and*

$$x_* = \lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} f(x_i) = f(x_*)$$

and x_* is a fixed point ■

Definition 3.10 ***Schwarzian derivative, negative Schwarzian***
*Let $f \in C^\infty(R^1)$, then the **Schwarzian derivative** of f is*

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

If $\forall x$ s.t. $f'(x) \neq 0, S(f)(x) < 0$, then called the map has negative Schwarzian.

Theorem 3.8 *If f, g are negative Schwarzian, then fg has negative Schwarzian. Moreover, if f has negative Schwarzian, then f^k has negative Schwarzian.*

Theorem 3.9 *If map f on R^1 has negative Schwarzian, and p is a fixed point or periodic point of f, then either*
[i] *p has an infinity basin; or*
[ii] *there is a critical point of f in the basin of p; or*
[iii] *p is a source.*

***PROOF* 3.14** *Proof: Theo. 3.8*
If f, g are negative Schwarzian, then

$$\forall x \in D \bigcap \{x|f'(x) \neq 0\} \bigcup \{x|g'(x) \neq 0\} \subset R^1, (S(f)(x) < 0) \wedge (S(g)(x) < 0)$$

On the other hand, we have

$$S(f \circ g)(x) = \frac{(f \circ g)'''}{(f \circ g)'} - \frac{3}{2}\left(\frac{(f \circ g)''}{(f \circ g)'}\right)^2$$

and

$$(f\circ g)' = g'\cdot f'(g) \qquad (f\circ g)'' = g''\cdot f'(g) + (g')^2\cdot f''(g)$$
$$(f\circ g)''' = g'''\cdot f'(g) + g'g''\cdot f'(g) + 2g'g''f''\cdot(g) + (g')^3 f'''\cdot(g)$$

so we have

$$\frac{(f\circ g)'''}{(f\circ g)'} = \frac{g'''\cdot f'(g) + g'g''\cdot f'(g) + 2g'g''f''\cdot(g) + (g')^3 f'''\cdot(g)}{g'\cdot f'(g)} = \frac{g'''}{g'}+\frac{g''}{g'}+2\frac{g''\cdot f''(g)}{f'(g)}+\frac{(g')^2\cdot f'''(g)}{f'(g)}$$

$$\left(\frac{(f\circ g)''}{(f\circ g)'}\right)^2 = \left(\frac{g''\cdot f'(g) + (g')^2\cdot f''(g)}{g'\cdot f'(g)}\right)^2 = \left(\frac{g''}{g'}+\frac{g'\cdot f''(g)}{f'}\right)^2 = \left(\frac{g''}{g'}\right)^2 + 2\frac{g''\cdot f''(g)}{f'(g)} + \left(\frac{g'\cdot f''(g)}{f'(g)}\right)^2$$

and finally

$$S(f\circ g)(x) = \frac{g'''}{g'}+\frac{g''}{g'}+2\frac{g''\cdot f''(g)}{f'(g)}+\frac{(g')^2\cdot f'''(g)}{f'(g)} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 - 3\frac{g''\cdot f''}{f'(g)} - \frac{3}{2}\left(\frac{g'\cdot f''}{f'}\right)^2$$
$$= S(g)(x) + (g')^2 S(f)(x) + g''\left(\frac{1}{g'} - \frac{f''(g)}{f'(g)}\right) = (Sf)(g(x))\cdot(g'(x))^2 . < 0 \qquad \blacksquare$$

E x a m p l e 3.1 *[1]* $f(x) = ax$ *on* R^1 *and* $|a| < 1$*, zero is a fixed point sink whose basin is the entire real line.*
*[**1.1**] A lienar map on* R^n *whose matrix representation has distinct eigenvalue that are less than one in magnitude, then the origin is a fixed sink whose basin is* R^n*.*
[2] $f(x) = 4/\pi \arg\tan(x)$ *on* R^1 *has 3 fixed point* ± 1 *and* 0 *where* ± 1 *are sink and* 0 *is source. The basin of* 1 *is positive and the basin of* -1 *is negative.*
*[**3.1**]* $g(x) = ax(1-x), a \in (0,1)$*, the fixed point is* 0 *all* R^1 *is basin of this point.*
** We can proved that* $((a-1)/a, 1]$ *lies in the basin of* $x = 0$ *with Theo. 3.7. From graphcal representation orbits, it is clear that in addition, the interval* $[1, 1/a]$*,* $(-\infty, (a-1)/a)$ *and* $(1/a, \infty)$ *are basin of* 0*.*
*[**3.2**]* $g(x) = ax(1-x), a \in (1,2)$*, the sink fixed point is* $(a-1)/a$*, the basin is* $(0,1)$*.*
[4] $f(r,\theta) = (r^2, \theta - \sin\theta), r \in R^+, \theta \in [0, 2\pi)$ *which used the polar coordinates. The fixed points are* $(0,0), (1,0), (1,\pi)$*, the attractors are* $(0,0)$ *and* (∞,∞)*. The basin of* $(0,0)$ *is the points inside the unit circle and the points outside the unit circle are basin of* (∞,∞)*.*
[5] $g(x) = ax(1-x)$ *has Schwarzian derivative formed*

$$S(g)(x) = -\frac{3}{2}\left(\frac{-2a}{a-2ax}\right)^2$$

and therefore has negative Schwarzian.

***CONCLUSION* 3.5** *The logistic map* $g(x) = ax(1-x)(a \in [1,4])$ *has at most one periodic sink.*

***PROOF* 3.15** $g'(x) = a - 2ax, g''(x) = -2a, g'''(x) = 0$ *and*

$$S(g)(x) = -\frac{3}{2}\left(\frac{-2a}{1-2ax}\right)^2 < 0$$

*[**i**] Consider a point* $p \in [0,1]$*, as every point in* $R\backslash[0,1]$*will tent to* $-\infty$ *so no point in* $[0,1]$ *have infinity basin.*
*[**ii**] Since the only critical point of* g *is* $1/2$*, there can be at most one attracting periodic orbit.* $\blacksquare$

3.5 Density function and Ulam-von Neumann transformations

We found the relationship between the tent map, the logistic map(l-4), and the mod map. Now we try to explain the Henon map and the tent map.[2] [3]

Consider a Henon map formed

$$H_{(a,b)}(x,y) = (1 - ax^2 + by, y)$$

when $a = 2$ and $b \to 0$, the map is

$$H_{(2,0)}(x,y) = (1 - 2x^2, y) \Rightarrow q(x) = 1 - 2x^2$$

which is a 1-dim nonlinear map. Obviously, there is exists a one-to-one map from tent map to the map follows

$$\tau(x) = \begin{cases} 2x+1 & x \in [-1,0] \\ -2x+1 & x \in [0,1] \end{cases} \Rightarrow \tau(x) = 1 - 2|x|, x \in [-1,1]$$

So the problem is how to find a one-to-one between the $\tau(x)$ and $q(x)$. Even in the introduction of conjugacy in the last section, we just given the map between the tent map and logistic map and never discuss how to find these kind of map. Here, we will used the definition of probability density function (PDF, or invariable density in dynamic system) to analysis this problem.

Definition 3.11 ***Density Function***
Consider a map $f \in C^1([0,1])$ and a group of initial stats $x_1^{(0)}, x_2^{(0)}, \ldots, x_N^{(0)}$, then define the $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(m)}$ sequence with

$$x_i^{(p)} = f(x_i^{(p-1)}) = f^p(x_i^{(0)}), i = 1, 2, \ldots N$$

And a ***density function*** *of a $x_i^{(p)}$ sequence will satisfy, for every interval $\Delta_0 \subset [0,1]$*

$$\int_{\Delta_0} f_0(u)du \simeq \frac{1}{N} \sum_{k=1}^{N} 1_{\Delta_0}(x_k^{(0)})$$

where $1_\Delta(x)$ is ***indicator function*** *s.t.*

$$1_\Delta(x) = \begin{cases} 1 & x \in \Delta \\ 0 & x \notin \Delta \end{cases}$$

In most situation, we cannot find the density function directly with the formula of map. We can only statistic the value of map and hope we can find a distribution to evaluate it. However, in logistic-4 map $G(x)$, we have a certain distribution.

$$\rho_q = \frac{1}{\pi\sqrt{1-x^2}}$$

On the other hand, based on the bifurcation diagram, we found that in ever parameter in $[1,4]$, there are some continuous line in the image. We can guess that in a normal situation, the

[2] Reference: Jiang, Y., 1995. On Ulam von Neumann transformations.

[3] Reference: Lasota, A. and Mackey, M., n.d. Chaos, Fractals, and Noise.

density function is a mixture model based on the ρ_q just like Gaussian Mixture Model(GMM) in probability.

$$\rho_{(g_a)}(x) = \sum_{i=1}^{K} \frac{p_i}{\pi\sqrt{1-(x-q_i)^2}}, \text{ where } \sum_{i=1}^{K} p_i = 1$$

Moreover, as we know the statistic data $x_i^{(p)}$ sequence, it is easy to get the numerical solution of parameter in the model above. For instance, with the machine learning method like Expectation–maximization(EM) algorithm we can find a mixture model.

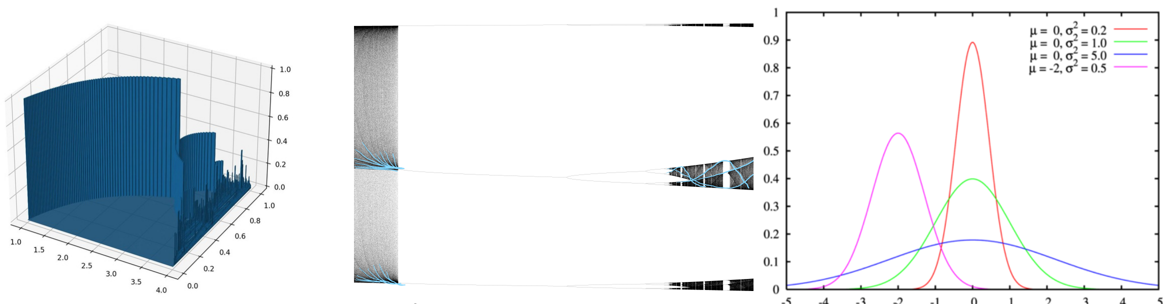


Figure 25: Histogram of logistic map, continuous line in bifurcation diagram

We found the density function of $q(x)$ is $\rho_q = \frac{1}{\pi\sqrt{1-x^2}}$, and the reflection between q and τ will satisfy

$$dy = \frac{2dx}{\pi\sqrt{1-x^2}} \Rightarrow y = h(x) = \frac{2}{\pi}\arg\sin(x) \text{ and } h^{-1}(x) = \sin\left(\frac{\pi x}{2}\right)$$

And we have

$$\bar{q}(x) = h\left\{q\left[h^{-1}(x)\right]\right\} = h\left\{1-2\sin^2\left(\frac{\pi x}{2}\right)\right\} = \frac{2}{\pi}\arg\sin\left[1-2\sin^2\left(\frac{\pi x}{2}\right)\right] = \frac{2}{\pi}\arg\sin\left[\cos\left(\pi x\right)\right]$$

$$\Rightarrow \sin\left(\frac{\pi\bar{q}}{2}\right) = \cos\left(\pi x\right) \Rightarrow \cos\left(\frac{\pi}{2}-\frac{\pi\bar{q}}{2}\right) = \cos\left(\pi x\right) \Rightarrow 1-\bar{q} = 2|x| \Rightarrow \bar{q} = 1-2|x| = \tau(x)\blacksquare$$

Based on this density function, we can find the one-to-one between most non-linear map with certain condition and τ map, or tent map. Obviously, both τ map and tent map have certain Lyapunov exponent, and this makes the analysis of chaostic in non-linear system become easier.

Definition 3.12 ***Ulam-von Neumann transformation***
A map f is a Ulam-von Neumann transformation if it satisfied
[i] *f is a piecewise C^1 self mapping of $[-1,1]$ with a unique power law singular point 0;*
[ii] *$f|_{[-1,0]}$ is C^1 and increasing, and $f|_{[0,1]}$ is C^1 and decreasing;*
[iii] *$f(0) = 1, f(-1) = f(1) = 1$*
[iv] *$f|_{[-1,0]}$ and $f|_{[0,1]}$ are $C^{1+\alpha}$ for some $\alpha \in (0,1]$ and the restrictions of $r_f(x) = f'(x)/|x|^{\gamma-1}$ to $[-1,0)$ and to $(0,1]$ are $\beta-$ Bolder continuous fo some $\beta \in (0,1]$*
[v] *the sequence $\{\eta_n\}_{n=0}^{\infty}$ of neasted partitions by f decreases exponentially.*

Theorem 3.10 *Any two Ulam-von neumann transformations f and g are topologically conjugate.*

Theorem 3.11 *The Lorenz map is a Ulam-von Neumann transformation.*

Typically, we can discribe the Ulam-von Neumann transformations with properties follows: It is a either linear or non-linear curve like a mountain from point $(-1,-1)$ increase to $(0,1)$ and finally decrease to $(1,-1)$, the only singular point is 0 which may discontinuous. Also, there are some properies in differential of the function.

It is difficult to proof these theorem strickly beacause we need tools of mainfold as well as real analysis. So here we just introduce these definition and conclusions.

Figure 26: CHAOS

4 Fractal

4.1 General tent map, Cantor set and self-similar attractor

In this section, we will mainly focus on the general tent map

$$T_a(x) = \begin{cases} ax & x \leq 1/2 \\ a(1-x) & x \geq 1/2 \end{cases}$$

There are several properties of this map

Property 4.1 ***General tent map - fixed point***
$a \in (0,1)$*: single fixed point* 0*, all initial conditions are attracted to 0;*
$a \in (1,\infty)$*: both* 0 *and* $\frac{1}{1+a}$ *are fixed point.*

In last section, we know that if $a = 2$, then $T_2(x)$ is mapped onto itself and we have

Property 4.2 ***Point leave the interval***
$a \in (0,2]$*: points stay within I;*
$a \in (2,\infty)$*: a.e. points eventually leave the interval and never return, where a.e. means almost everywhere, that means, without a measure zero subset, all set will satisfied this property.*

We will try to proof the second property.

PROOF* 4.1** ***[i] *Obviously, based on the definition of tent map, if* $x < 0$*, then* $f(x) < 0$*, if we let* $x_1 = f(x) < 0$*, then* $f(x_1) < 0$ *etc. So we have this conclusion: if* $f^p(x) < 0$*, then* $\forall q > p, f^q(x) < 0$ *where* $p, q \in \mathcal{N}$*.*
Moreover, if $x > 1$*, then* $f(x) < 0$ *and* $\forall n \in \mathcal{N}^+, f^n(x) < 0$*.*

Define the set L *s.t.*

$$L = \{x \in [0,1] | f^n(x) < 0\}$$

where $n \in \mathcal{N}$ *is a certain value.*
[ii] *If* $a \in (0,2)$*, then* $f([0,1]) = [0, a/2] \subset [0,1]$*. On the other hand, as* $\forall x \in [0,1], f(x) \geq 0$*, so* $L = \varnothing$*.*
[iii] *We will proof that if* $a > 2$*, then* $\mu(L) = 1$ *where* μ *is (lebesgue) measure. Firstly, we consider the interval* $L_1 = (1/a, (a-1)/a)$ *s.t.* $\forall x \in L_1, f(x) > 0 \Rightarrow f^2(x) < 0$*.*
Now we consider the subset $L_0 = (0, 1/a)$*, we found that* $f((0,1/a)) = (0,1)$*, so we can apart the interval* $(0,1/a)$ *again, where*

$$L_{00} = (0, 1/a^2), L_{01} = (1/a^2, (a-1)/a^2), L_{02} = ((a-1)/a^2, 1/a)$$

We found that $\forall x \in L_{01}, f^2(x) > 1 \Rightarrow f^3(x) < 0$
If we consider the subset $L_2 = ((a-1)/1, 1)$*, we found that is symmetric of* $(0, 1/a)$*, that means we can also apart* L_2 *as* L_{20}, L_{21}, L_{22}*, where* L_{21} *have same property with* L_{01}*. We found the structure of this set is familiar with* ***Cantor set*** *especially when* $a = 3$*.*

Definition 4.1 ***Cantor Set***
Let set G s.t.

$$G = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

then called $C = [0,1] \backslash G = [0,1] \bigcap G^c$ as Cantor set.

Based on the knowledge in real analysis, we know that

Property 4.3 $\mu(C) = 0, \mu(G) = 1$ *where μ is measure of set.*

The proof of this property is simple because we know that for a certain interval (a,b), the measure $\mu\left((a,b)\right) = b - a$ which also be called as ***Borel measure****. And we can just calculate the measure of G with limitation.*

Now we come back to the problem above
If $a = 3$, then $L = G$ and we proved the problem. If $a > 2, a \neq 3$, we can found a one-to-one map from $T_a(x)$ to $T_3(x)$ and the attractor will express same property as $a = 3$. So we finally proved that $\forall a > 2$, $x \in [0,1]$ a.e. s.t$\exists N, \forall n > N, f^n(x) < 0$ $\blacksquare$

Now we come back to the title of this section, so that means "fractal"? We know that for every subset of Cantor set, or attractor of $T_a(a > 2)$ map, these subset showed us same property of the original set and we called this **self-similar** set as **fractal**.

Here are some example of fractal set.

Definition 4.2 ***Iterated function system****.*
Consider a group of map on R^m s.t. $f = \{f_1, f_2, \dots f_r\}$ and for every maps, exists a positive number $p_1, p_2, \dots p_r$ s.t. $\sum_{i=1}^{r} p_i = 1$ (probabilities). Then we called this group of f_i is iterated function system.

Example 4.1 *(A simple iterated function system)*
0 *Roll a point x_0 randomly in $[0,1]$*
[i] *flip a coin,*
[i-1] *if coin comes up heads, then move the point x_{i-1} to $x_i = \frac{1}{3}x_{i-1}$*
[i-2] *if coin comes up tails, then move the point x_{i-1} to $x_i = \frac{1}{3}(2 + x_{i-1})$*

We can simulate this example with code, if we statistic the point, or find the density figure of map, we found that .

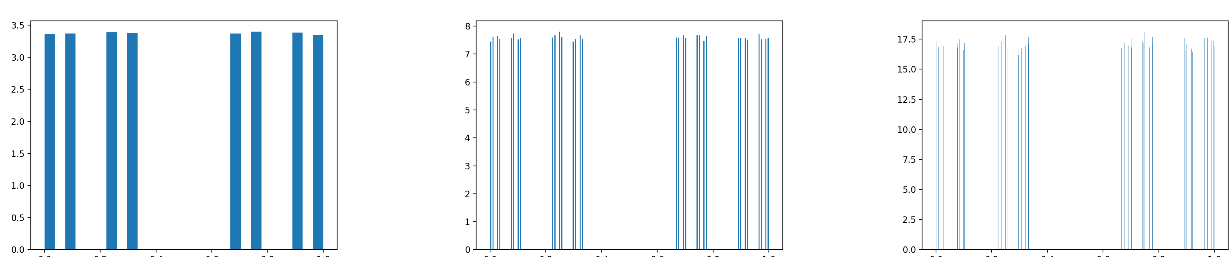


Figure 27: Simulation of Simple Ierated Function System

which showed us the property of Cantor set, and we can guess that the Cantor set is the attractor of the probabilitic constructions.

Here is another example of self-similar map.

E x a m p l e 4.2 ***Sierpinski carpet*** .

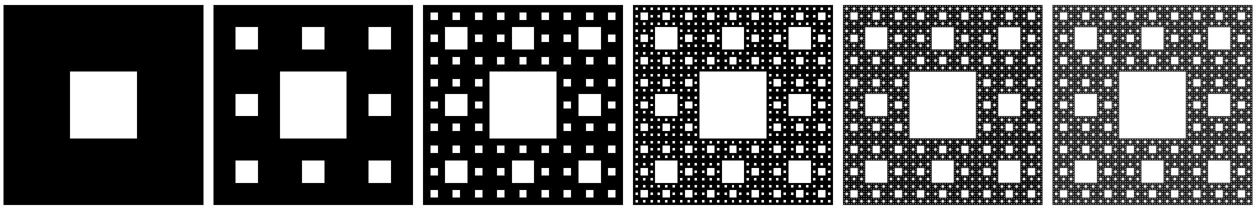


Figure 28: Sierpinski carpet

Of couse, Henon's map also have this fractal property.

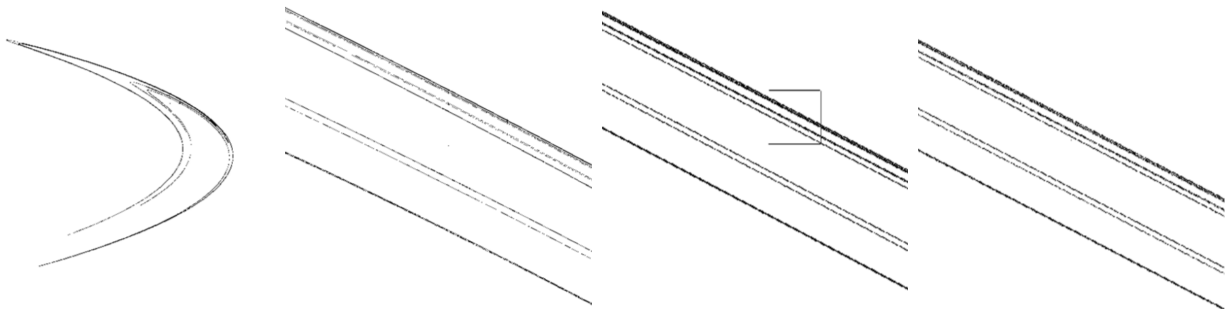


Figure 29: Fractal in Henon's map

Moreover, we can also discuss this fractal property in basin.

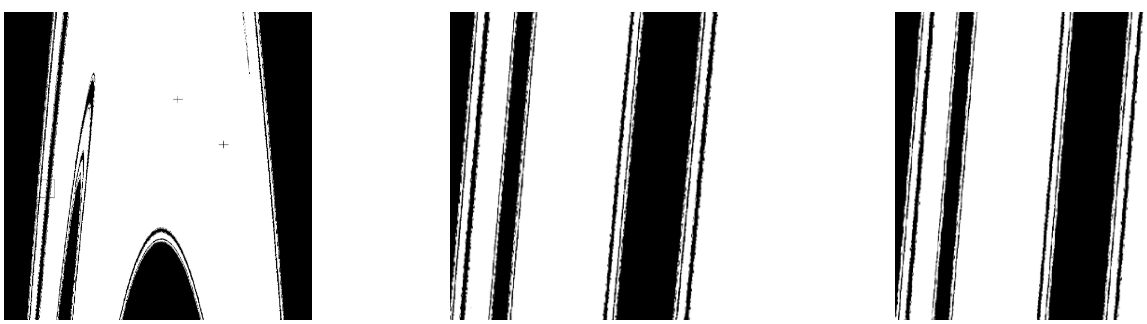


Figure 30: Fractal in basin of Henon's map

E x a m p l e 4.3 ***Julia set, Mandelbrot set***
Now we consider a map in complex values

$$P_c(z) = z^2 + c$$

where z, c *are complex number s.t.* $\exists x, y, c_x, c_y \in \mathcal{R}, z = x + yi, c = c_x + c_y i$ *and* $i = \sqrt{-1}$. *Based on the calculation rules of complex number, we know*

$$P_c(z) = (x + yi)^1 + (c_x + c_y i) = x^2 + 2xyi - y^2 + c_x + c_y i = (x^2 - y^2 + c_x) + i(2xy + c_y)$$

so we have

$$f(x,y) = (Re[P_c(z)], Im[P_c(z)]) = (x^2 - y^2 + c_x) + i(2xy + c_y) \Rightarrow \begin{cases} x_{n+1} = x_n^2 - y_n^2 + c_x \\ y_{n+1} = 2x_n y_n + c_y \end{cases}$$

where $n \in \mathcal{N}, x_n, y_n, c_x, c_y \in \mathcal{R}$ *are* c_x, c_y *is constant.*

We know that if we consider a map formed $f(x,y) = x^2 + y^2$*, then the unit circle is important, for ever point inside the unit circle, the are all sink to zero point and for every point outside the unit circle, they will go to infinity after enough iteration.*

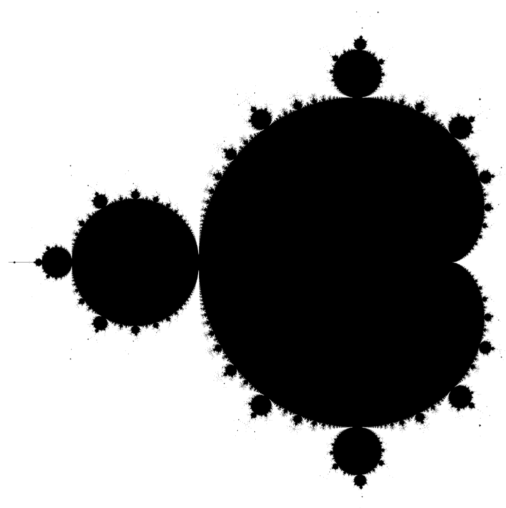


Figure 31: Mandelbrot set

We called the Black area in the image above as Mandelbrot set, more mathematically, the **Mandelbrot set** is

$$M = \{c : 0 \text{ is not in the basin of infinity for the map } P_c(z) = z^2 + c\}$$

We can analysis the convergence and disconvergence of this complex map too. For a certain c, now we consider the convergence and disconvergence for every point in the space. We called this set as Julia set.

Definition 4.3 ***Julia set***

Consider a map $f : \mathcal{R}^n \to \mathcal{R}^n$

$$J(f) = \{x | x \in R^n, \forall \varepsilon > 0, \exists x_1, x_2 \in N(x,\varepsilon) s.t. \left(\lim_{n\to\infty} f^n(x_1) < \infty \wedge \lim_{n\to\infty} f^n(x_2) = \infty \right)\}$$

which is the boundary points between convergence area and disconvergence area.

Figure 32: Julia set

(parameter $c_1 = c_2 = -0.17 + 0.78i, c_3 = 0.38 + 0.32i, c_4 = 0.32 + 0.043i$)

Table 8: Interval

img	x	y	img	x	y
1	$[-1.5, 1.5]$	$[-1.5, 1.5]$	2	$[-0.19, 0.01]$	$[0.89, 1.09]$
3	$[-1.3, 1.3]$	$[-1.3, 1.3]$	4	$[-1.3, 1.3]$	$[-1.3, 1.3]$

5 Chaos in high dimension map

5.1 Lyapunov Spectrum

In the section 3, we mainly discussed the judgement of chaos in 1 dimension. Here we focus on the Lyapunov exponent in high dimension. Obviously, in high dimension, the δ_n function which we used in Discussion 3.2 will be a m dimension vector δ_n and we mainly focus on the length(or norm) of this vector.

$$||\delta_t|| = \exp(\lambda t)||\delta_0||$$

$$t \to \infty, ||\delta_t|| = \exp(\lambda t)||\delta_0|| \Rightarrow \lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{||\delta_t||}{||\delta_0||} = \lim_{t\to\infty} \ln \left[\frac{||\delta_t||}{||\delta_0||}\right]^{\frac{1}{t}}$$

However, in high dimension problem, it is different from 1-dim because during the iteration, some direct of vector increase and the others decrease. Fig. 33 showed us an example of this evolution.

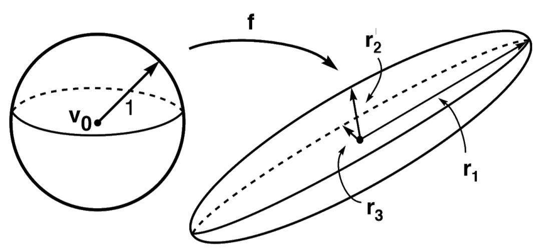


Figure 33: Evolution of unit vector in 3d

To solve this problem, we should consider the evolution of every direct in the space.

Let $\omega_0^{(1)}, \omega_0^{(2)}, \ldots, \omega_0^{(m)}$ is a group of orthogonal basis in $\mathcal{R}^m$ space which satisfied $\forall i, j = 1, 2, \ldots, m, i \neq j$, the inner production $< \omega_0^{(i)}, \omega_0^{(j)} >= 0$, then for every direction, we have a λ value based on the formula above, that means

$$\forall i = 1, 2, \ldots m, \lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln \frac{||\omega_t^{(i)}||}{||\omega_0^{(i)}||}$$

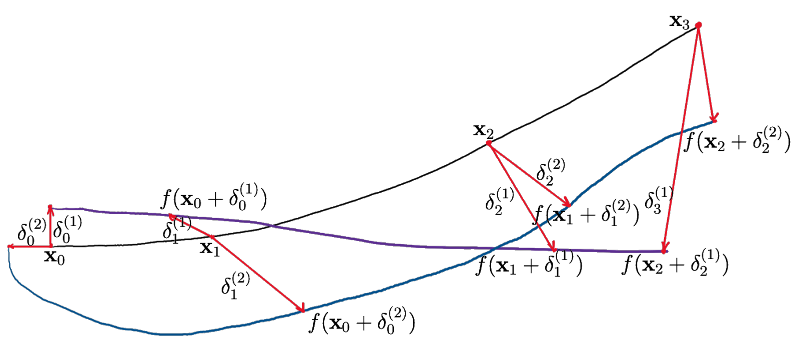


Figure 34: Analysis in two dimension problem

Basically, a simple basis in $\mathcal{R}^m$ space is $\omega_0^{(1)} = (1, 0, \dots, 0)^T, \omega_0^{(2)} = (0, 1, \dots, 0)^T, \dots, \omega_0^{(m)} = (0, 0, \dots, 1)^T$ and

$$\Omega_0 = (\omega_0^{(1)}, \omega_0^{(2)}, \dots, \omega_0^{(m)}) = I$$

In this situation, $||\omega_0^{(i)}|| = 1 (i = 1, 2, \dots, m)$, let

$$\Lambda = (\lambda_1, \lambda_2, \dots \lambda_m) = \lim_{t \to \infty} \frac{1}{t} \left(\ln ||\omega_t^{(1)}||, \ln ||\omega_t^{(2)}||, \dots, \ln ||\omega_t^{(m)}|| \right)$$

Let $p_1, p_2, \dots p_m$ s.t. $\{p_1, p_2, \dots, p_m\} = \{1, 2, \dots, m\} \wedge \forall i, j = 1, 2, \dots m, i \neq j, p_i \neq p_j$ and

$$\ln ||\omega_t^{(i_1)}|| \geq \ln ||\omega_t^{(i_2)}|| \geq \dots \geq \ln ||\omega_t^{(i_m)}||$$

In the discrete time processing, we know that $t = 0, 1, 2, \dots$ so $r_t^{(k)} = r_n^{(k)}$ where $n \in \mathcal{N}$.

Let $r_n^{(k)} = \ln ||\omega_n^{(i_1)}||$ be the length of the kth longest orthogonal axis after n time iterate for an initial point $\omega_0^{(i_1)}$. Obviously, these $r_n^{(k)}$ sequence (with k, not n) measured the expansion of initial vectors, so we can define the Lyapunov exponent in follows.

Definition 5.1 ***Lyapunov number, Lyapunov exponent in high dimension problem***
Let $f \in C^\infty(\mathcal{R}^m)$, $J_n = Df^n(\mathbf{x}_0)$, $r_n^{(k)}$ be the length of the kth logenst orthogonal axis which defined by the explanation above. Then the kth ***Lyapunov number*** *of $\mathbf{x}_0$ is defined by*

$$L_k = \lim_{n \to \infty} \left(r_n^{(k)} \right)^{\frac{1}{n}}, \text{ and the } \textbf{Lyapunov exponent } h_k = \ln L_k$$

Obviously,

$$(L_1 \geq L_2 \geq \dots \geq L_m) \wedge (h_1 \geq h_2 \geq \dots \geq h_m)$$

For every single $r_n^{(k)}$ we know that is familiar with λ in 1-dim Lyapunov exponent, so we have The orbit is **chaotic** if it satisfied both
[i] $\{x_1, x_2, \dots\}$ is no asymptotically periodic, and
[ii] the Lyapunov exponent h_k is **greater** than zero.

As $h_1 \geq h_2 \geq \dots \geq h_m$, so if $h_1 < 0$, then every Lyapunov exponent is less than zero, that means, we can simplify the definition and only care about h_1

Definition 5.2 ***Orbit chaotic in high dimension***
let f be a map of $\mathcal{R}^m$, and $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_1, \dots$ be a bounded orbit of f, if
[i] *orbit is not asymptotically periodic; and*
[ii] *$\forall i = 1, 2, \dots m, h_m \neq 0$ and $h_1 > 0$ then the orbit is* ***chaotic****.*

Now we consider how to calculate this Lyapunov exponent in normal problem.

[i] f is linear map

If f is a linear map, then $\exists P$ s.t. $f(\mathbf{x}) = P\mathbf{x}$ and

$$\forall i = 1, 2, \dots, m, \omega_n^{(i)} = f(\mathbf{x}_{n-1} + \omega_{n-1}^{(i)}) - f(\mathbf{x}_{n-1}) = P(\mathbf{x}_{n-1} + \omega_{n-1}^{(i)}) - P\mathbf{x}_{n-1} = P\omega_{n-1}^{(i)} = P^n \omega_0^{(i)}$$

So we have

$$\Lambda = \lim_{n \to \infty} \frac{1}{n} \left(\ln ||\omega_n^{(1)}||, \ln ||\omega_n^{(2)}||, \dots, \ln ||\omega_n^{(m)}|| \right) = \lim_{n \to \infty} \frac{1}{n} \ln \left(||P^n \omega_0^{(1)}||, ||P^n \omega_0^{(2)}||, \dots, ||P^n \omega_0^{(m)}|| \right)$$

Define the vector norm function $\xi_p(X)$, where $X = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ s.t.

$$\xi_p(X) = (||\mathbf{x}_1||_p, ||\mathbf{x}_2||_p, \ldots, ||\mathbf{x}_n||_p)$$

is the $p-$norm of every vector in the matrix. Then we have

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left(\xi(P^n\Omega_0)\right)$$

We know that Ω_0 is a normal orthogonal basis, and we said the most useful and simple orthogonal basis is I, so here we let $\Omega_0 = I$ then

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left(\xi(P^n)\right)$$

And now, the problem is how to calculate this P^n, we know that if the eigenvector and eigenvalue of P is V and E, then $P^n = V^{-1}E^nV$, that means, if there is a eigenvalue of P absolute greater than 1, then some element in P^n will satisfy $n \to \infty, p_{i,j} \to \infty$. So now we will try to put the $1/n$ into the ξ function.

We know that in the begining of the discussion, Λ satisfied

$$\begin{aligned}\Lambda = \lim_{n\to\infty} \frac{1}{n} \Bigg[\ln\left(\frac{||\omega_n^{(1)}||}{||\omega_{n-1}^{(1)}||}\frac{||\omega_{n-1}^{(1)}||}{||\omega_{n-2}^{(1)}||}\cdots\frac{||\omega_1^{(1)}||}{||\omega_0^{(1)}||}\right), \ln\left(\frac{||\omega_n^{(2)}||}{||\omega_{n-1}^{(2)}||}\frac{||\omega_{n-1}^{(2)}||}{||\omega_{n-2}^{(2)}||}\cdots\frac{||\omega_1^{(2)}||}{||\omega_0^{(2)}||}\right), \ldots \\ \ln\left(\frac{||\omega_n^{(m)}||}{||\omega_{n-1}^{(m)}||}\frac{||\omega_{n-1}^{(m)}||}{||\omega_{n-2}^{(m)}||}\cdots\frac{||\omega_1^{(m)}||}{||\omega_0^{(m)}||}\right), \Bigg] \\ = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \sum_{i=1}^{n} \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \ldots \sum_{i=1}^{n} \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right), \right. \\ = \lim_{n\to\infty} \frac{1}{n}\left\{\sum_{i=1}^{n}\left[\ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \ldots, \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right)\right]\right\}\end{aligned}$$

Notice for every $\omega_{i-1}^{(j)}, j = 1, 2, \ldots, m$ we can find a normal orthogonal basis of $\mathcal{R}^m$ space based on the **Gram Schmidt Processing**

Algorithm 1 Gram-Schmidt process in orthogonal decomposition

INPUT: A n-dimension Euclidean space, a basis $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ of the space.

procedure GRAM-SCHMIDT PROCESS
  for $i = 1, i \leq n, i++$ **do**
    $\beta_i = -\sum_{j=1}^{i-1} \frac{<\beta_j, \alpha_i>}{<\beta_j, \beta_j>}\beta_j + \alpha_i$
    $\beta_i = \text{normalization}(\beta_i) = \frac{\beta_i}{||\beta_i||}$
  end for
  return Normal orthogonal basis $\{\beta\}$
end procedure

Let the basis of $\omega_{i-1}^{(j)}$ is $B_{i-1} = (\beta_{i-1}^{(1)}, \beta_{i-1}^{(2)}, \dots \beta_{i-1}^{(m)})$, and $Q_{i-1} = (q_{i-1}^{(1)}, q_{i-1}^{(2)}), \dots, q_{i-1}^{(m)}$ s.t.

$$\forall j = 1, 2, \dots, m, \omega_{i-1}^{(j)} = B_{i-1} q_{i-1}^{(j)} \wedge ||q_{i-1}^{(j)}|| = 1$$

$$\left[\ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \dots, \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right)\right]$$

$$= \left[\ln\left(\frac{||PB_{i-1}q_{i-1}^{(1)}||}{||B_{i-1}q_{i-1}^{(1)}||}\right), \ln\left(\frac{||PB_{i-1}q_{i-1}^{(2)}||}{||B_{i-1}q_{i-1}^{(2)}||}\right), \dots, \ln\left(\frac{||PB_{i-1}q_{i-1}^{(m)}||}{||B_{i-1}q_{i-1}^{(m)}||}\right)\right] = \xi(P) \qquad (*)$$

And finally, we have

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \left\{\sum_{i=1}^{n} [\xi(P)]\right\} = \xi(P)$$

[ii] f is non-linear map

Now we focus on the problem that f is not a linear map. Typically, we will use Jacobian matrix to linearize the problem and for a sequence x_i we have

$$\mathbf{x}_{i+1} = f(\mathbf{x}_i) = P_i \mathbf{x}_i$$

where P_i is Jacobian matrix near the point x_i, so we just need to change the P matrix to P_i matrix in non-linear problems. We found in formula $(*)$ we have

$$\left[\ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \dots, \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right)\right]$$

$$= \left[\ln\left(\frac{||P_{i-1}B_{i-1}q_{i-1}^{(1)}||}{||B_{i-1}q_{i-1}^{(1)}||}\right), \ln\left(\frac{||P_{i-1}B_{i-1}q_{i-1}^{(2)}||}{||B_{i-1}q_{i-1}^{(2)}||}\right), \dots, \ln\left(\frac{||P_{i-1}B_{i-1}q_{i-1}^{(m)}||}{||B_{i-1}q_{i-1}^{(m)}||}\right)\right] = \xi(P_{i-1}) \qquad (*)$$

then, we have

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \left\{\sum_{i=1}^{n} [\xi(P_{i-1})]\right\}$$

Finally, we can summary this processing in algorithm follows[4]

[4] Reference: Malik Hassanaly, Venkat Raman, Numerical convergence of the Lyapunov spectrum computed using low Mach number solvers, Journal of Computational Physics, Volume 386, 2019, Pages 467-485, ISSN 0021-9991.

Algorithm 2 Benettin's algorithm (Jacobian differential)

INPUT: Jacobian matrix for every point: $J_0, J_1, \ldots J_N$;

procedure GRAM SCHMIDT(matrix); ▷ Return the Gram-Schmidt orthogonal matrix.
end procedure

procedure NORMALIZATION(matrix); ▷ Normalization every vector in the matrix
end procedure

procedure VECTORNORM(matrix); ▷ Return the norm of every vector in the matrix
end procedure

procedure LYAPUNOV SPECTRUM
  float $P = I$;
  list Spectrum;
  for $i = 1, i \leq N, i++$ **do**
    $P = J_i \cdot P$
    $P = \text{Gram Schmidt}(P)$
    $\text{Spectrum} = \text{Spectrum} + (\ln \text{VectorNorm}(P))$;
    $\text{P} = \text{Normalization}(P)$
  end for
  $\textit{Spectrum} = \text{Spectrum}/N$
  return Spectrum
end procedure

Now we will apply the method with several example.

E x a m p l e 5.1 ***Lyapunov spectrum in Henon's map***
Firstly we consider the Henon's map

$$\begin{cases} x = 1 - ax^2 + by \\ y = x \end{cases} \quad \textit{and the Jacobian matrix is } J(x, y) = \begin{bmatrix} -2ax & b \\ 1 & 0 \end{bmatrix}$$

Based on the Algo. 2, we can calculate the Lyapunov spectrum numerically.

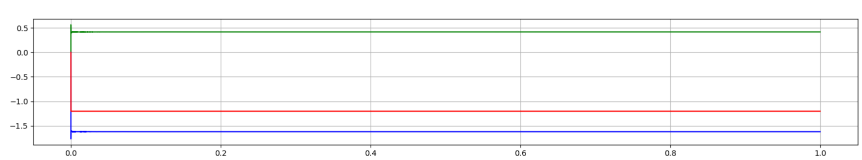


Figure 35: Lyapunov spectrum in Henon's map($a = 1.4, b = 0.3$)

Table 9: Result of Lyapunov spectrum in differen problem ($\Delta t = 0.0001$)

Model	λ_1	λ_2	λ_3	λ_4	$\sum \lambda_i$	Parameter
Logistic map	0.4961				0.4961(+)	3.9
Henon's map	0.4204	-1.6130			-1.1665(+, -)	$1.4, 0.3$
Duffing map	1.0660	-0.9661			0.0998(+, -)	$1, 0.04, 1, 0.1, \pi/2$
Lorenz model	0.9170	0.0031	-14.590		-13.666(+, 0, -)	$28, 10, 8/3$
Rossler system	0.0705	0.0020	-5.3882		-5.3157(+, 0, -)	$0.2, 0.2, 5.7$
Extend Rossler	3.4475	3.5305	2.7098	10.7696	20.4574(+, +, 0, -)	/

Now we consider a continuous system s.t. $\dot{x} = f(x), x(0) = x_0$ where x_0 is a constant vector at initial time t_0. With Ronge-kutta method, we can find a group of value to simulate the system. So we can change the system to a map formed

$$x_{n+1} = g(x_n, t_n), x_0 = x(0), t_{n+1} = t_n + \Delta t$$

where g is based on the Ronge-kutta method.
Here we don't care about the formula g, we just consider for a certain n, if we still find a Jacobian matrix J_n, then $x_{n+1} = J_n x_n$.
On the other hand, we know that

$$\dot{x} = \frac{x(t_0 + (n+1)\Delta t) - x(t_0 + n\Delta t)}{\Delta t} = \frac{x_{n+1} - x_n}{\Delta t} = f(x_n)$$

In a certain model, we know the formula of f as well as parameter Δt. Let $\bar{J}(x)$ is Jacobian matrix of $f(x)$, then

$$x_{n+1} - x_n = \Delta t \bar{J}(x_n) x_n = (J_n - I_n) x_n \Rightarrow J_n = \Delta t \bar{J}(x_n) + I_n$$

E x a m p l e 5.2 ***Lyapunov spectrum in Lorenz system***
Now we consider the Lorenz system:

$$\begin{cases} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{cases} \quad \text{and the Jacobian matrix of } f \text{ is } \bar{J}(x, y, z) = \begin{bmatrix} -\sigma & \sigma & 0 \\ (\rho - z) & -1 & -x \\ y & x & -\beta \end{bmatrix}$$

So the Jacobian matrix of the discrete maps is

$$J(x, y, z) = \Delta t \bar{J}(x, y, z) + I = \begin{bmatrix} 1 - \sigma\Delta t & \sigma\Delta t & 0 \\ (\rho - z)\Delta t & 1 - \Delta t & -x\Delta t \\ y\Delta t & x\Delta t & 1 - \beta\Delta t \end{bmatrix}$$

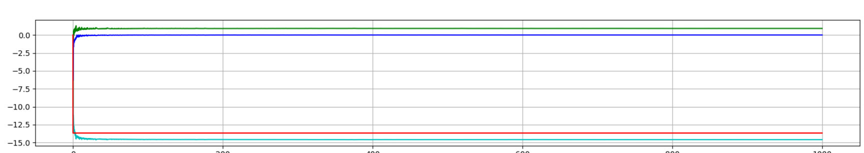


Figure 36: Lyapunov spectrum in Lorenz system(($\sigma, \rho, \beta, \Delta t$) = ($28, 10, 8/3, 0.0001$))

E x a m p l e 5.3 ***Lyapunov spectrum in Rossler system***
Now we consider the Rossler system:

$$\begin{cases} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x-c) \end{cases} \quad \text{and the Jacobian matrix of } f \text{ is } \bar{J}(x,y,z) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{bmatrix}$$

So the Jacobian matrix of the discrete maps is

$$J(x,y,z) = \Delta t \bar{J}(x,y,z) + I = \begin{bmatrix} 1 & -\Delta t & -\Delta t \\ \Delta t & a\Delta t + 1 & 0 \\ z\Delta t & 0 & (x-c)\Delta t + 1 \end{bmatrix}$$

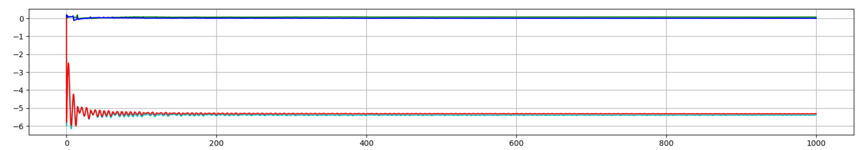


Figure 37: Lyapunov spectrum in Rossler system($(a, b, c, \Delta t) = (0.2, 0.3, 9, 0.001)$)

E x a m p l e 5.4 ***Lyapunov spectrum in Duffing system***
Now we consider the Duffing equation:

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t)$$

Let $y = \dot{x}$*, then*

$$\dot{y} = \ddot{x} = \gamma\cos(\omega t) - \delta\dot{x} - \alpha x - \beta x^3 = \gamma\cos(\omega t) - \delta y - \alpha x - \beta x^3$$

So we have

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= \gamma\cos(\omega t) - \alpha x - \beta x^3 - \delta y \end{cases} \quad \text{and the Jacobian matrix of } f \text{ is } \bar{J}(x,y) = \begin{bmatrix} 0 & 1 \\ -\alpha - 3\beta x^2 & \delta \end{bmatrix}$$

So the Jacobian matrix of the discrete maps is

$$J(x,y,z) = \Delta t \bar{J}(x,y) + I = \begin{bmatrix} 1 & \Delta t \\ (-\alpha - 3\beta x^2)\Delta t & \delta\Delta t + 1 \end{bmatrix}$$

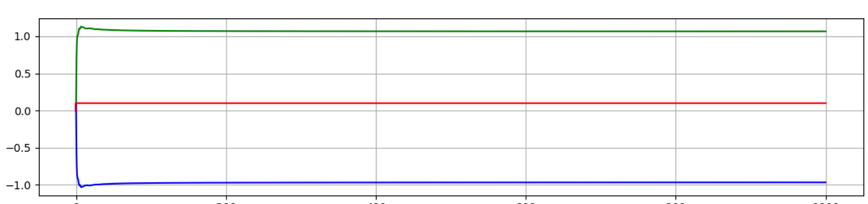


Figure 38: Lyapunov spectrum in Duffing equation($(\alpha, \beta, \gamma, \delta, \omega, \Delta t) = (1, 0.04, 1, 0.1, \pi/2, 0.001)$)

E x a m p l e 5.5 ***Lyapunov spectrum in Extend Rossler system***
In the last of this part, we consider Extend Rossler system:

$$\begin{cases} \dot{x} &= -y - z \\ \dot{y} &= x + 0.25y + w \\ \dot{z} &= 3 + xz \\ \dot{w} &= -0.5z + 0.05w \end{cases} \quad \textit{and the Jacobian matrix of } f \textit{ is } \bar{J}(x,y,z,w) = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ z & 0 & x & 0 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix}$$

So the Jacobian matrix of the discrete maps is

$$J(x,y,z,w) = \Delta t \bar{J}(x,y,z,w) + I = \begin{bmatrix} 1 & -\Delta t & -\Delta t & 0 \\ \Delta t & 1 + 0.25\Delta t & 0 & \Delta t \\ z\Delta t & 0 & 1 + x\Delta t & 0 \\ 0 & 0 & -0.5\Delta t & 1 + 0.05\Delta t \end{bmatrix}$$

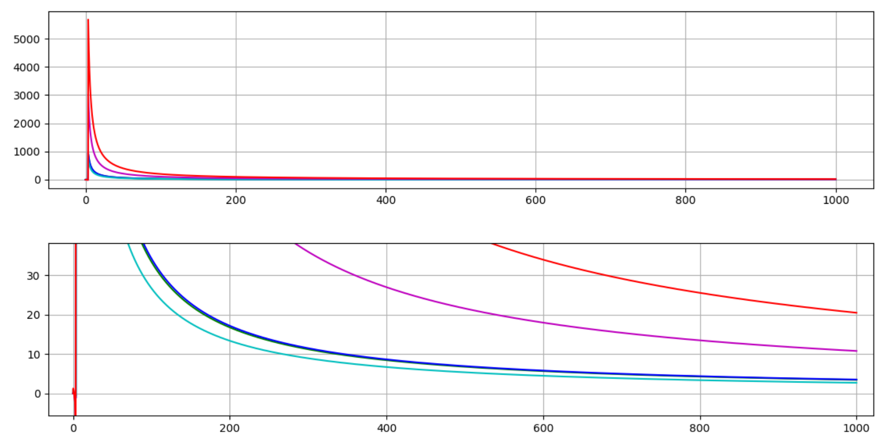


Figure 39: Lyapunov spectrum in this 4-dim equation

5.2 Fixed-point theorem in high dimension

We introduced the fixed point theorem in section, we found that if an inital interval I_0 s.t.

$$I_{n+1} = f(I_n) \wedge I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$$

then, based on the Nested interval theorem, exist at least one point x_0 s.t. $\forall i \in \mathcal{N}, x_0 \in I_i$ which is fixed point.

However, this condition is too strict, if we consider another group of set, for instance, with the relations in the figure follow.

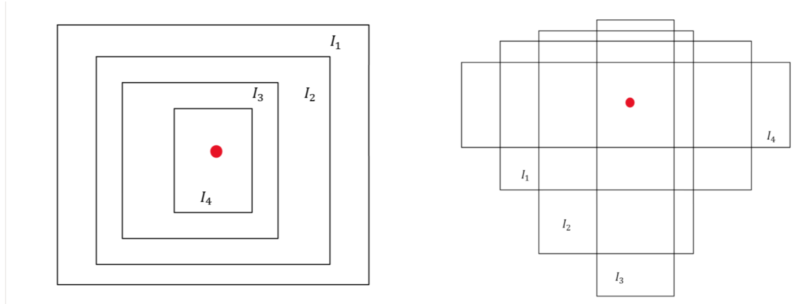


Figure 40: Colorado corollary

Then we can found a fixed point in the center of every interval. And we called this kind of map as **Colorado corollary**

Now the problem is how to describe this property that every $f(I_n)$ covered some part of the I_n. If $\mathbf{x}$ is a boundary point on I_n, then $f(\mathbf{x})$ also be a boundary point of $f(I_n)$. If we consider this kind of vector $V(\mathbf{x}) = \frac{f(\mathbf{x})-\mathbf{x}}{||f(\mathbf{x})-\mathbf{x}||}$ which is the direction of these boundary vector, then we can easily found that the vectors will travel through a cumulative $k2\pi$ where $k \in \mathcal{N}$

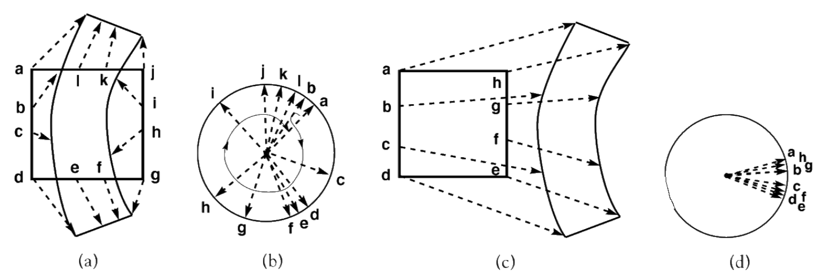


Figure 41: Proof of fixed point theorem

Theorem 5.1 ***Index theory*** *Let f be a continuous map on R^2, S is a ractangular region and ∂S is the boundary of S. Now consider an point $\mathbf{x} = \mathbf{x}_0$ on ∂S, then we have the vector $V(\mathbf{x}) = \frac{f(\mathbf{x})-\mathbf{x}}{||f(\mathbf{x})-\mathbf{x}||}$. Move the point with the boundary continuous and sign every vector $V(\mathbf{x})$ until the point back to the $\mathbf{x}_0$ and sum the total rotation of these vector sequence, if it is nonzero, then f has a fixed point in S.*

***PROOF* 5.1** *[1] If center c of rectangle is fixed point, then there is nothing to do.*
[2] If center c is not fixed point, we can shrink the rectangle down from its original size tothe point c.
[2-1] Obviously, the vector $V(\mathbf{x})$ defined along the ∂S and change continuously, and they must continue to make at least one full trun. So finally, some point $\mathbf{x}$ results $V(\mathbf{x}) = 0$ and $\mathbf{x}$ is fixed point because the definition of $V(x)$
[3] Now we consider the situation that the image move completely away from it. Obviously, there is no fixed point if the image move away. And in this condition, we can simply find net rorate of $V()$ is zero. ■

CONCLUSION* 5.1** *The fixed point theorem is a property of the topology of the map alone, and doesn't depend on starting with a prefect rectangle, as long as region has* ***no holes

Now we back to consider the Colorado corollary. In some situation the region not only expand from the origin interval, but also fold from the original region. For instance in the image below, S_T, S_B fold in the subimage 3. We found this kind of map have no influence of the conclusion we introduced before.

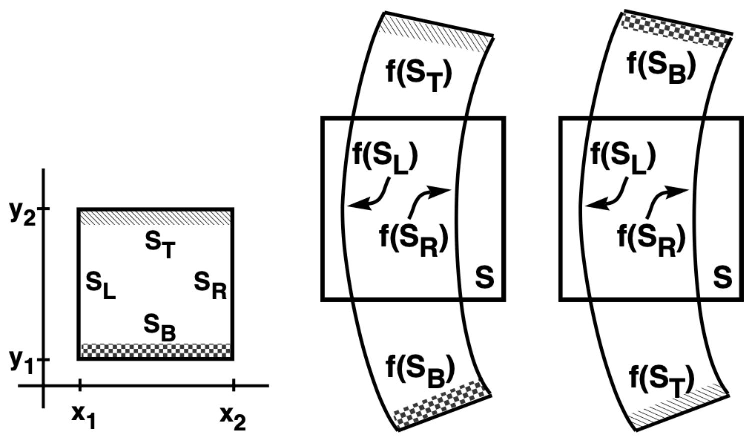


Figure 42: Colorado corollary after map

CONCLUSION* 5.2** ***The Colorado Corollary
Let $f \in C(\mathcal{R}^2)$ is a map, S be a rectangle in $\mathbf{R}^2$, with vertical sides s_L, s_R and horizontal side s_T, s_B. Assume that $f(s_L), f(s_R)$ are surrounded by s_L, s_R and $f(s_T), f(s_B)$ are surrounded by s_T, s_B, then f has a fixed point in S

Finally we consider a kind of map which is "lying across" from the original region.

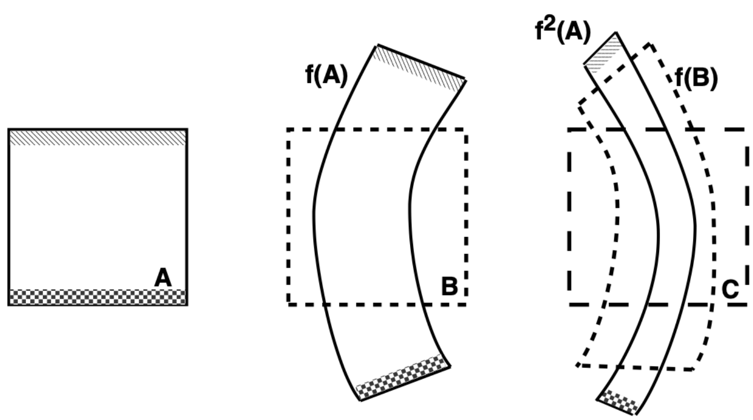


Figure 43: Transitivity of "lying across"

Obviously, that is a special condition of fixed point theorem and we have

***CONCLUSION* 5.3** *Let f be a map and $S_1, S_2, \dots S_k$ be rectangular sets in $\mathcal{R}^2$ s.t. $f(S_i)$ lies across the S_{i+1} and S_1, then f^k has a fixed point in S_1*

In the last of this section, we will dicsuss the definition of Markov partitions

Definition 5.3 ***Markov Partitions***
Assume S is a rectangle set and $S_1, S_2, \ldots, S_r$ s.t.

$$\forall i, j = 1, 2, \ldots, r, i \neq j, (S_i \subset S) \wedge \left(S_i \bigcap S_j = \varnothing\right) \wedge (\partial S_i \text{ parallel to the coordinate axes })$$

and $f(S_i)$ lying across S_i, that means, in one direction of axis, f stretches the rectangles and in the other directions the f contract the rectangles.
Then, the stretching directions are mapped to stretching directions, and shrinking directions to shrinking directions. Then we called this $S_1, S_2, \ldots, S_r$ as ***Markov partition*** *of S for f.*

Based on the Markov partition, we can define the itinerary in high dimension problems and find the fixed point easier. Here we will discuss two example, one is **skinny baker map**, and the other is Henon's map, or the **Horseshoe map**.

5.3 Example: Skinny baker map

The skinny baker map is

$$B(x_1, x_2) = \begin{cases} \begin{pmatrix} a & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & x_2 \in [0, 1/2] \\ \begin{pmatrix} a & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b \\ -1 \end{pmatrix} & x_2 \in (1/2, 1] \end{cases}$$

If we consider the itinerary of the map, let $x_1 < 1/2$ as **L** and $x_1 > 1/2$ as **R**$(\forall x_1, x_2 \in [0, 1],$ point $(x_1, x_2))$then

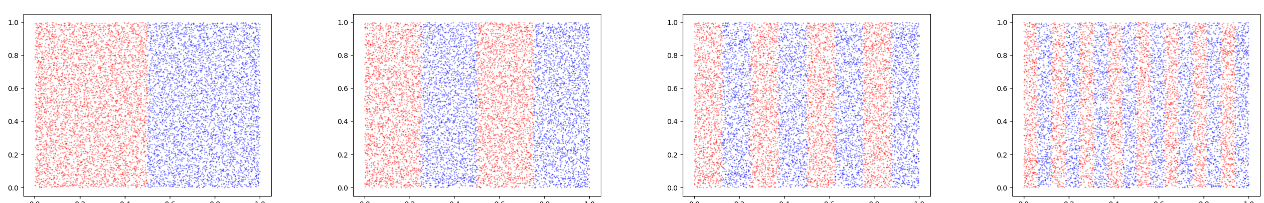


Figure 44: Baker map$(a = 0.5, b = 0.5)$

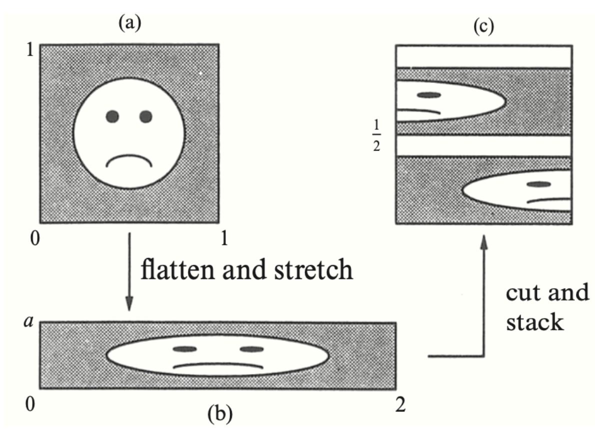


Figure 45: Baker map iteration

We found that the map is similar as baker. However, if we change the parameter, the map will show a little different from baker.

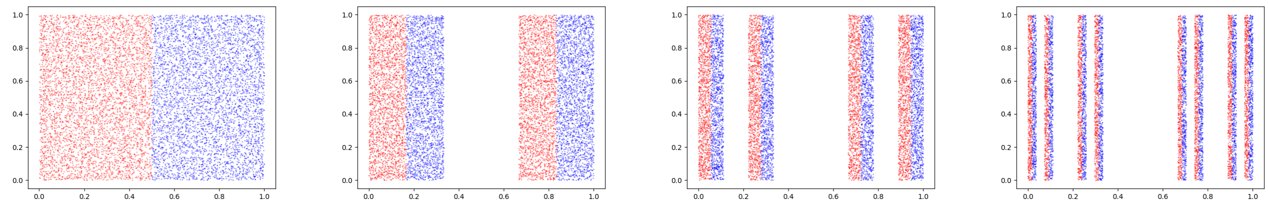


Figure 46: Baker map($a = 1/3, b = 2/3$)

And we now discuss the change of b.

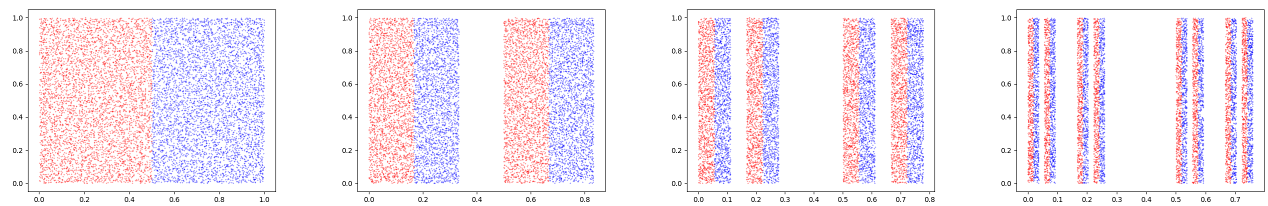


Figure 47: Baker map($a = 1/3, b = 0.5$)

Note that we will consider $a = 1/3, b = 2/3$ in the discussion below. Obviously the Lyapunov exponent are $-\ln(3), \ln(2)$, and now will will try to proof the map will never asymptotically periodic with itineraries.

We can easily found that the itineraries are "bi-infinite", that means, they are defined for $-\infty < i < \infty$ and for every point (x_1, x_2) it is the result after infinity iteration and the begining of next infinity iteration.

$$\text{itineraries: } \ldots S_{-2}S_{-1}S_0S_1S_3$$

Obviously if we consider the problem in $x_1 \leq 0.5$ and $x_1 > 0.5$, then this two subset combined Markov partition.

To analysis this problem, first, we consider a special map.

Definition 5.4 ***Shift map***
Consider a map s, if

$$s(\ldots S_{-2}S_{-1}S_0S_1S_3\ldots) = \ldots S_{-2}S_{-1}S_0S_1S_3\ldots$$

then we called this map as shift map.

So the orbit is asymptotically periodic if and only if the itinerary is eventually periodic toward right. Any itinerary that is not periodic toward right is not asymptotically periodic. So finally we have

***CONCLUSION* 5.4** *The skinny baker map has chaotic orbits.*

5.4 Example: Horseshoe map

Before we consider the horseshoe map, we consider the Henon's map firstly and explain why are we interested in horseshoe map.

Consider a rectangle region same as we discussed in the fixed point theorem above,

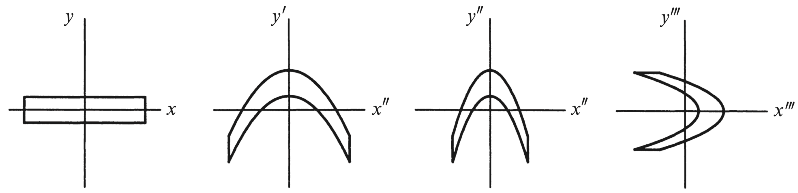


Figure 48: Henon's map, transform to horseshoe map

$$T_1(x,y) = \begin{cases} x^{(1)} = x \\ y^{(1)} = 1 + y - ax^2 \end{cases} \qquad T_2(x,y) = \begin{cases} x^{(2)} = bx \\ y^{(2)} = y \end{cases} \qquad T_3(x,y) = \begin{cases} x^{(3)} = y \\ y^{(3)} = x \end{cases}$$

Obviously, we found that the Henons map $H(x,y) = T_3(T_2(T_1(x,y)))$ which transformed a rectangle region to a Horseshoe region. We called this kind of map as Horseshoe map. (Fig. 49, (1))

The **horseshoe map** h is a continuous one-to-one map on R^2 s.t.
[i] Map the square $W = ABCD$ to the overlapping horseshoe image, $h(A) = A^*, h(B) = B^*, h(C) = C^*, h(D) = D^*$.(Fig. 49, (3))
[ii] In the W, the map uniformly contracts distances horizontally and expands distances vertically.
[iii] Points are stretched vertically by a factor 4 and squeezed horizontally by 4.
[iv] Outside W we only retriction we make on h hs that it be contiuous and one-to-one.dd

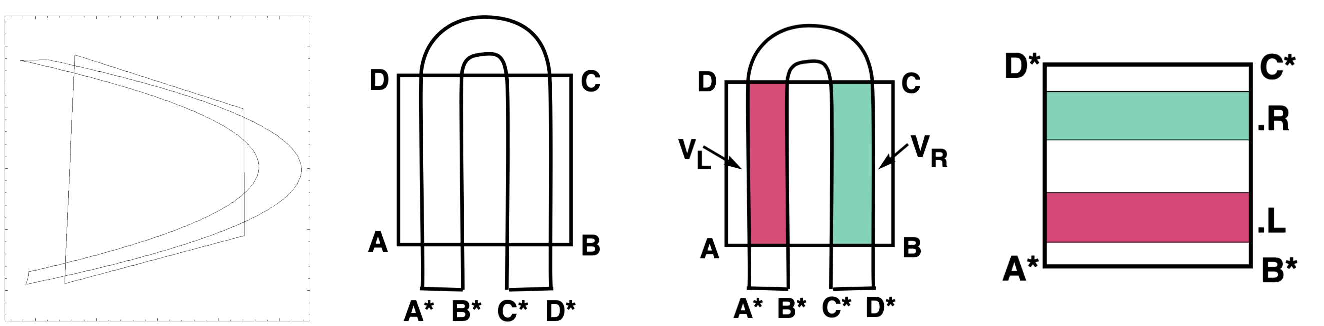


Figure 49: Horseshoe map

Here we consider the points set H remains every point of W with forward and backware iterates. For every point $x \in H$, it must be in either left leg V_L or right leg V_R, define the itinerary
[i] if $h^i(x)$ lies in V_L, then set $S_i = L$
[ii] if $h^i(x)$ lies in V_R, then set $S_i = R$
where S_i is itineraries, $i \in Z$ and the Itinerary is $\ldots S_{-2}S_{-1}S_0S_1S_2\ldots$

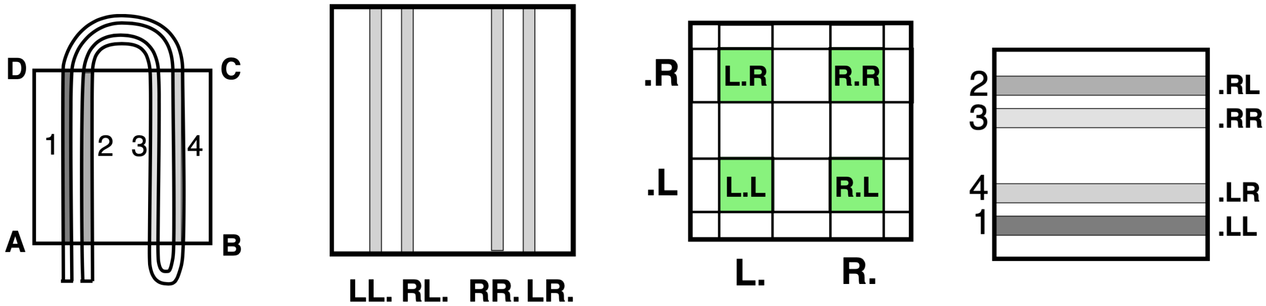


Figure 50: Horseshoe map - 2

We know that itinerary is a tool to consider a map is asymptotically periodic or not, in horseshoe map, we can easily found that the Lyapunov exponent is $\ln 3$ and $-\ln 4$, and same as baker map, the horseshoe will never asymptotically periodic. So finally we have the conclusion below.

***CONCLUSION* 5.5** *The horseshoe map has chaotic orbits.*

Fractal dimension

In real analysis, if we consider the size of a subspace, we basically use **measure** to describe that.

Definition 5.5 ***Eculidean rectangle Neighbourhood***
1. Opened box $I = \{x = (x_1, x_2, \dots, x_n) | a_i < x_i < b_i, i \in N\}$, where a_i, b_i are constant;
2. Closed box $I = \{x = (x_1, x_2, \dots, x_n) | a_i \le x_i \le b_i, i \in N\}$, where a_i, b_i are constant;
Called the $b_i - a_i$ is side length of a box. If $|b_1 - a_1| = |b_2 - a_2| = \dots = |b_n - a_n|$, then called this box as cube.

Definition 5.6 ***Lebesuge outer measure***

$$\mu^* E = \inf \left\{ \sum_{i=1}^{\infty} |I_i| : E \subset \bigcup_{i=1}^{\infty} I_i, I_i \text{ is an opened box} \right\}$$

Also, we can define the outer measure with ε (Familiar with Cauchy's limitation)

$$\forall \varepsilon > 0, \exists \{I_i\} \text{ open cover s.t.} E \subset \bigcup_{i=1}^{\infty} I_i \wedge \mu^* E \le \sum_{i=1}^{\infty} |I_i| + \varepsilon$$

Now, we consider a cube with side length 1, and neigibourhood cube with side length ε, the total of the neigibourhood include in the unix cube as $V = N(\varepsilon)$, the dimension of subspace as k, then

$$V = N(\varepsilon) = \left(\frac{1}{\varepsilon}\right)^k \Rightarrow k = \log_{1/\varepsilon} N(\varepsilon) = -\frac{\ln N(\varepsilon)}{\ln(\varepsilon)}$$

In normal problem, $k \in \mathcal{N}^+$ called **Lebesgue covering dimension** or **topological dimension**.

Definition 5.7 ***Lebesgue covering dimension***
*A topological space X is said to have the Lebesgue covering dimension $d < \infty$ if d is the smallest **non-negative integer** with the property that each open cover of X has a refinement in which no point of X is included in more than $d + 1$ elements.*

Definition 5.8 $C(1/\varepsilon)$ ***cube***
Now we consider a special type of cover.
[i] *Firstly, for a interval $I_1 = [a_1, b_1]$, let the side length of boxes is ε, then we have $N = int(C/\varepsilon) + 1$ subintervals to cover the original interval. (e.g. $[a_1 + (p-1)C/\varepsilon, a_1 + pC/\varepsilon], p = 1, 2, \dots N$)*
[ii] *Now we consider a surface in R^n space, let the projection on x_i axis is $I_i = [a_i, b_i]$, and $C = \max\{\mu(I_i)\}, i = 1, 2, \dots, n$, then we have a group of cube total $N(\varepsilon) = (C\varepsilon)^n$ and*

$$d = \frac{\ln N(\varepsilon) - \ln C}{\ln(1/\varepsilon)}$$

Definition 5.9 $C(1/\varepsilon)$ ***cube***
A bounded set $s \subset R^n$ has box-counting dimension

$$bd(S) = \lim_{\varepsilon \to \infty} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)}$$

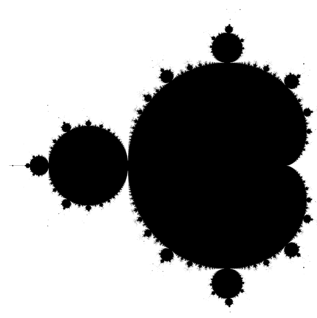
when the limit exists.

***CONCLUSION* 5.6** *Based on the Bolzano–Weierstrass theorem, we know that if we have a sequence* $\{b_n\}$ *s.t.* $\lim_{n\to\infty} b_n = 0$*, if* b_n *is* ε *then* $\lim_{n\to\infty} \frac{\ln b_{n+1}}{\ln b_n} = 1$

***CONCLUSION* 5.7** *We still consider the sequence above, then*

$$\frac{N(b_n)}{4} \le N(b_n) \le 4N(b_{n+1})$$

and the proof is simple with figure follows



Theorem 5.2 *If* $\{b_n\}$ *is monotony, or assume* $b_1 > b_2 > \ldots > b_n > \ldots > b_\infty = 0$ *If*

$$\lim_{n\to\infty}\left(\frac{\ln b_{n+1}}{\ln b_n}\right) = 1 \wedge \lim_{n->\infty}\left(\frac{\ln N(b_n)}{\ln(1/b_n)}\right) = d =$$

then $\lim_{\varepsilon\to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = d$ *and therefore the box-counting dimension is* d*.*

Theorem 5.3 *If* $S \subset R^n$ *is bounded and* $bd(S) = d < n$*, then* $\mu(S) = 0$

Another way to calculate the measure of a area is based on the statistic,

Definition 5.10 ***Correlation dimension*** *Let* $S = \{v_0, v_1, \ldots\}$ *be an orbit of the map* $f : \mathcal{R}^n \to \mathcal{R}^n$*, then for ever* $r > 0$*, define* $C(r)$ *s.t.*

$$C(r) = \lim_{N\to\infty} \frac{|\{(p,q)|p,q \in S, |p-q| < r\}|}{|\{(p,q)|p,q \in S\}|}$$

Moreover, if $C(r) \approx r^d$*, then*

$$d \approx cd(S) = \lim_{r\to\infty} \frac{\ln C(r)}{\ln(r)}$$

And here we found another way to calculation the area of attractor.

In the last section, we introduced the Lyapunov spectrum. However, if we want to analysis the whole system, the Lyapunov exponent is not a good choice because it described the property in every different direction.

So to describe the property of a set, or a space, the basic element is the definition of open set. Now we consider a box neigibourhood rather than the circle one, signed as I_0 with certain side length ω_0 and the volumn $V_0 = \omega_0^m$. Then let the set $I_1 = f(I_0), I_2 = f(I_1) \dots$ and we now try to consider the volumn of these sets. If the Lyapunov spectrum of system is $(h_1, h_2, \dots, h_m)$, then

$$||\omega_n^{(j)}|| = \exp(nh_j)||\omega_0^{(j)}||$$

and the volume of box neigibourhood is

$$V_i = \prod_{j=1}^{m} ||\omega_i^{(j)}|| = \prod_{j=1}^{m} \exp(ih_j)||\omega_0^{(j)}||$$

$$V_n = \prod_{j=0}^{m} \exp(nh_j)||\omega_0^{(j)}||$$

But that is not the all conclusion, as some of Lyapunov exponent is less than 0, the side length in that direction will decrease and decrease and finally equal to 0. And that is unnecessary to consider such direction.

And now, the problem is "Which dimension is enough to include the $f^{\infty}(I_0)$"

We know that if the dimension is 3 as normal world, then the volumn V and side length d have relationship as

$$V = d^3 \Rightarrow 3 = \log_d V = \frac{\ln V}{\ln d}$$

So if we assume the dimension of space is $k \leq m$, then

$$k = \frac{\ln V_\infty}{\ln d_0} \qquad (*)$$

[i] In 2-dim problem Now we consider a 2-dim problem.

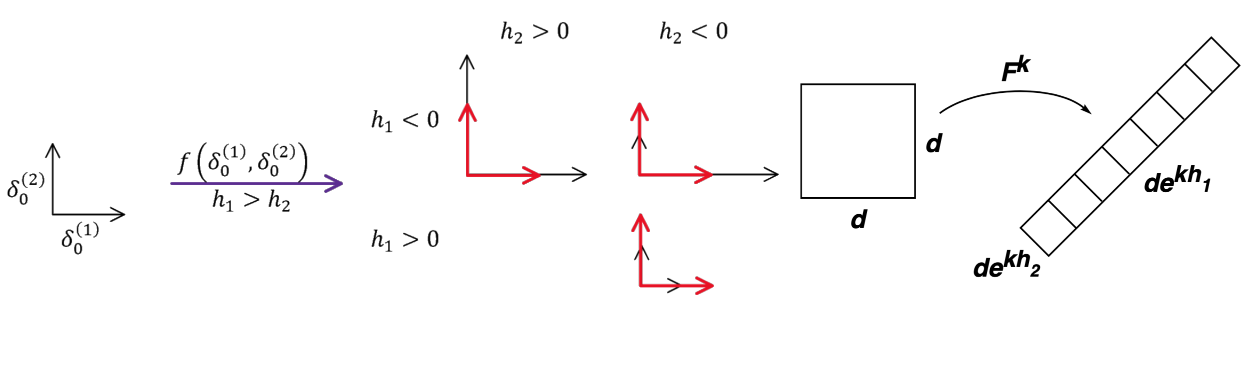


Figure 51: Volumn and vector change with Lyapunov exponent

Same as the figure above, if $h_1 > h_2 > 0$, then, obviously, we need a 2-dim neigibourhood to cover all the system after iteration. On the other hand, if $0 > h_1 > h_2$, then we just need a point to cover it. So the main problem is how can we cover the situation $h_1 > 0 > h_2$.

$$V_n = \omega_0^{(1)} \exp(nh_1)\omega_0^{(2)} \exp(nh_2) = d^2 \left(\exp(h_1 + h_2)\right)^n$$

So if $h_1 + h_2 < 0$, the volumn will still decrese to 0, that means the dimension of V_∞ is 0. And now we consider the condition $h_1 > 0 > h_2 \wedge h_1 + h_2 > 0$

Definition 5.11 ***Lyapunov dimension***
Let f be a map on $\mathcal{R}^m$, the Lyapunov exponent of an orbit is $h_1 \geq h_2 \geq \ldots \geq h_m$, let p

$$p = \arg\max_p \left(\sum_{i=1}^{p} h_i \geq 0 \right) \text{ then, the Lyapunov dimension is } D_L = \begin{cases} 0 & \text{if } p \text{ is not exists} \\ p + \frac{1}{|h_{p+1}|} \sum_{i=1}^{p} h_i & \text{if } p < m \\ m & \text{if } p = m \end{cases}$$

D I S C U S S I O N* 5.1** ***What can Lyapunov dimension describe? *We just introduced the definition of Lyapunov dimension above, and here, we will explain why we are interested in Lyapunov dimension.*

So why are we interested in these fractal dimension, like Hausforff dimensionD_H, Box counting dimension D_B, Correlation Dimension D_C as well as Lyapunov dimension D_{Lya}, typically we have this conclusion

***CONCLUSION* 5.8** *Consider a dynamic system which have the fractal dimension D_H, D_B, D_C, D_{Lya} where D_H is Hausforff dimension, D_B is Box counting dimension, D_C is Correlation Dimension and D_{Lya} is Lyapunov dimension D_{Lya}, then*

$$D_H \leq D_B \leq D_C \leq D_{Lya}$$

On the other hand, we found calculation of Hausforff dimension is difficult or impossible, so if we want to consider the fractal dimension of a system, at least, the box counting dimension is easier to calculate, and more easier to calculate the Correlation Dimension as well as Lyapunov dimension. On the other hand, if the system satisfied some condition, then $D_H = D_B = D_C = D_{Lya}$ which means we can calculate the Lyapunov dimension instead of Hausforff dimension or box counting dimension which is really difficult to calculate.

6 Chaos in attractor

6.1 Attractor and chaotic attractor

Definition 6.1 *Let f be a map and x_0 be an initial condition. The* ***forward limit set*** *of the orbit $\{f^n(X_0)\}$ is*

$$\omega(x_0) = \{x | \forall \varepsilon > 0, \forall N > 0, \exists n > N s.t. |f^n(x_0) - x| < \varepsilon\}$$

Definition 6.2 ***Attractor***
Attractor is a forward limit set which attracts a set of initial values that has nonzero measure (nonzero length, area, or volume, depending on whether the dimension of the map's domain is one, two, or higher).
This set of initial conditions is called the basin of attraction (or just basin), of the attractor.

Definition 6.3 ***Chaos set***
Let $\{f^n(x)\}$ be a chaotic orbit.
If $x_0 \in \omega(x_0)$, then $\omega(x_0)$ us called a ***chaotic set****.*

CONCLUSION **6.1** ***[i]*** *A chaotic set is the forward limit set of a chaotic orbit which itself is contained in its forward limit set.*
[ii] *A chaotic attractor is a chaotic set that is also an attractor.*
[iii] *A sink is an attractor, since it attracts at least a small neighborhood of initial values.*

E x a m p l e 6.1 ***Ikeda map***
Ikeda map map was proposed first by Kensuke Ikeda as a model of light going around across a nonlinear optical resonator. Is is a discrete-time dynamical system given by the complex map.

$$z_{n+1} = A + B z_n e^{i(|z_n|^2 + C)}$$

Based on the Euler formula, we can transform this complex map to real map

$$\begin{cases} \dot{x} = R + C_2(x\cos\tau - y\sin\tau) \\ \dot{y} = C_2(x\sin\tau + y\cos\tau) \end{cases} \quad \tau = C_1 - \frac{C_3}{1 + x^2 + y^2}$$

To analysis the problem with Lyapunov exponent, we need to calculate the partical differential of the function. Let $\xi = (1 + x^2 + y^2)^{-1}$, then

$$\frac{\partial \sin\tau}{\partial *} = \frac{\partial \tau}{\partial *}\cos\tau \qquad \frac{\partial \cos\tau}{\partial *} = -\frac{\partial \tau}{\partial *}\sin\tau \qquad (* = x, y) \ \textit{and}$$

$$\frac{\partial \tau}{\partial x} = \frac{2C_3 x}{(1 + x^2 + y^2)^2} = 2C_3 x\xi^2 \qquad \frac{\partial \tau}{\partial y} = \frac{2C_3 y}{(1 + x^2 + y^2)^2} = 2C_3 y\xi^2$$

Now we consider $\partial\dot{*}/\partial-$, where $* = x, y$ and $- = x, y$

$$\frac{\partial \dot{x}}{\partial x} = C_2\left(\frac{\partial(x\cos\tau)}{\partial x} - y\frac{\partial \sin\tau}{\partial x}\right) = C_2\left(cos\tau - \frac{\partial\tau}{\partial x}x\sin\tau - \frac{\partial\tau}{\partial x}y\cos\tau\right)$$

$$\frac{\partial \dot{x}}{\partial y} = C_2\left(x\frac{\partial \cos\tau}{\partial y} - \frac{\partial(y\sin\tau)}{\partial y}\right) = C_2\left(-\sin\tau - \frac{\partial\tau}{\partial y}x\sin\tau - \frac{\partial\tau}{\partial y}y\cos\tau\right)$$

$$\frac{\partial \dot{y}}{\partial x} = C_2\left(\frac{\partial(x\sin\tau)}{\partial x} + y\frac{\partial\cos\tau}{\partial x}\right) = C_2\left(sin\tau + \frac{\partial\tau}{\partial x}x\cos\tau - \frac{\partial\tau}{\partial x}y\sin\tau\right)$$

$$\frac{\partial \dot{y}}{\partial y} = C_2\left(x\frac{\partial\sin\tau}{\partial y} + \frac{\partial(y\cos\tau)}{\partial y}\right) = C_2\left(\cos\tau + \frac{\partial\tau}{\partial y}x\cos\tau - \frac{\partial\tau}{\partial y}y\sin\tau\right)$$

Summarize the formula we wrote above in matrix, then

$$J = C_2 \cdot \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix} \cdot \left(I + \begin{bmatrix} -y \\ x \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial\tau}{\partial x} & \frac{\partial\tau}{\partial y} \end{bmatrix}\right)$$

$$= C_2 \cdot \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix} \cdot \left(I + \begin{bmatrix} -y \\ x \end{bmatrix} \cdot \begin{bmatrix} 2C_3x\xi^2 & 2C_3y\xi^2 \end{bmatrix}\right)$$

$$= \alpha C_2 \cdot \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix} \cdot \begin{bmatrix} 1/\alpha - xy & -y^2 \\ x^2 & 1/\alpha + xy \end{bmatrix}$$

Finally, we have

$$J = \alpha C_2 \cdot \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix} \cdot \begin{bmatrix} 1/\alpha - xy & -y^2 \\ x^2 & 1/\alpha + xy \end{bmatrix}, \alpha = 2C_3\xi^2, \tau = C_1 - C_3\xi, \xi = \frac{1}{(1+x^2+y^2)}$$

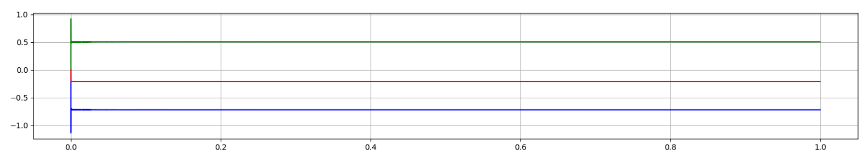


Figure 52: Lyapunov spectrum of Ikeda map

Lyapunov spectrum = (+, -)
$(Ikeda(R = 1, C_1 = 0.4, C_2 = 0.9, C_3 = 6) = (0.51, -0.72))$
So obviously, Ikeda map is a chaotic system.

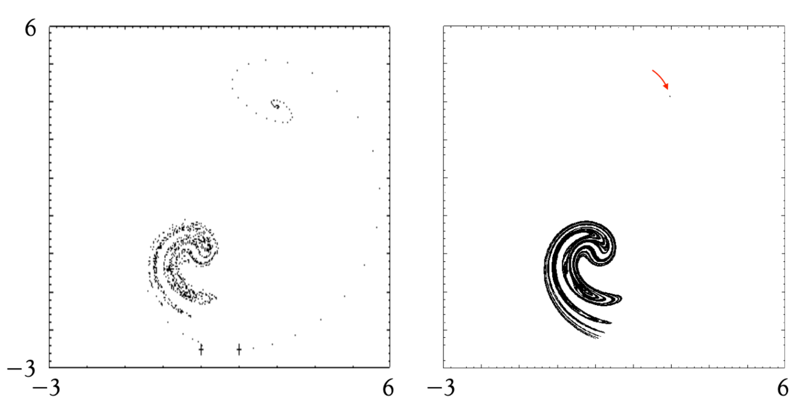


Figure 53: Forward map of Ikeda map

Now we consider two different initial value(+ point in Fig.53, (1)), we ignore first 1,000,000 result and plot 1,000,001 - 2,000,000 in the Fig.53, (2). Obviously, we can assume these points are forward limit set, or the attractor of the system,which has one chaotic set and a sink point.

We can found Conclusion 6.1(i), (ii) and (iii) are established in this Ikeda map easily.

Now we discuss what will be included in attractor set.

CONCLUSION* 6.2** ***Attractor set include:
[i] *Fixed point sink*
[ii] *Periodic-k points*

Or in summary, a attractor should be a set satisfied:
[iii] *Attractor should be* ***irreducible*** *in the sense that it includes only what is necessary.*
[iv] *The attractor must have the property that a point chosen at random should have a* ***greater-than-zero probability of converging*** *to the set.*
[vi] *Chaos introduces a new twist. Chaotic orbits can be attracting.*

6.2 n-dim Itinerary

In itinerary we introduced before, we just used two subinterval(**L** and **R** in Tent map(1-dim) and baker map(high-dim))

Definition 6.4 ***Piecewise expanding map***
Consider a interval $I = [p_0, p_k]$ and $\{p_0, p_1, \ldots, p_k\}$ is a partition of I s.t.

$$p_0 < p_1 < p_2 < \ldots < p_k, \text{ and } I_i = [p_{i-1}, p_i], I = \bigcup_{i=1}^{k} I_i$$

Let $f : I \to I$ be a map s.t. $|f'(x)| \geq \alpha > 1$ except possibly at the p_i point (that means, f may have dicountinuous point). We will call such a map a ***piecewise expanding map*** *with* ***stretching factor*** *α.*
We say that $\{p_0, p_1, \ldots, p_k\}$ is a ***stretching partition*** *for the piecewise expanding map f if, for each i, $f(I_i)$ is exactly the union of some of the intervals $I_1, I_2, \ldots I_k$.*

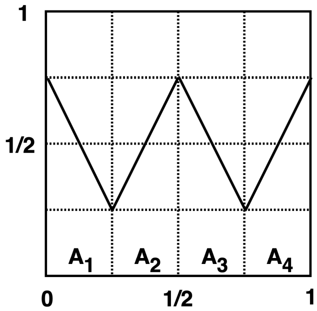
and we have these conclusion:

CONCLUSION* 6.3** ***[i] *A stretching partition satisfies the covering rule, which allows the construction of transition graphs for the partition intervals I_i.*
[ii] *It is also the one-dimensional analogue of the concept of Markov partition.*

Now we consider the itinerary of a system, we can use $I_1, I_2, \ldots, I_k$ instead of **L** and **R** to describe the symbol of a orbit.

E x a m p l e 6.2 ***W-map***
Consider a 1-dim map f s.t.

$$f(x) = \begin{cases} 2|(x - 1/4)| & x \in [0, 1/2] \\ 2|(x - 3/4)| & x \in [1/2, 1] \end{cases}$$



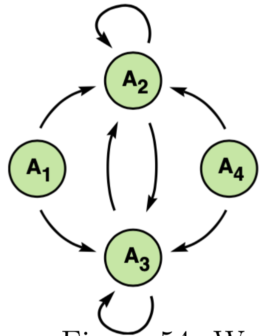


Figure 54: W map

Let $p_0, p_1, p_2, p_3, p_4 = 0, 1/4, 2/4, 3/4, 1$, $A_1 = [0, 1/4], A_2 = [1/4, 2/4], A_3 = [2/4, 3/4], A_4 = [3/4, 1]$, then we can easily found the transition graph of the map.

$$f(A_1) = f(A_2) = f(A_3) = f(A_4) = A_2 \bigcup A_3 \Rightarrow f\left(\bigcup_{i=1}^{4} A_i\right) = A_2 \bigcup A_3$$

Now we consider the itinerary of the map.
** If $S_1 S_2 \ldots$ is itinerary, then $S_1 S_2 = A_1 A_2$ is an allowable symbol sequence, but $A_2 A_1$ is not. In general, we have*

***CONCLUSION* 6.4** *If B, C are two subintervals for a stretching partition, C allowed to follow B in the symbol sequence of a poitn of the interval I if and only if $f(B) = C \bigcup (other\ subintervals)$*

*** As $|f'| \equiv= 2 = \alpha$ and for every interval $L = \mu([0,1]) = 1 \Rightarrow \mu(f(I)) = \mu([0,1])/\alpha = 1/2$. On the other hand, for ever interval $[0,1] = \bigcup_{i=1}^{4} A_i, f(\bigcup_{i=1}^{4} A_i) = A_2 \bigcup A_3 \Rightarrow \mu(f(I)) = \mu(A_2 \bigcup A_3) = 1/2$. Generally, we have*

CONCLUSION* 6.5** *For an allowable sequence $B_1 B_2 \ldots B_n$ of n symbols, there is a subinterval of length at most $\frac{L}{\alpha^n}$ which we call an **order n subinterval

Theorem 6.1 *Let f be a continuous piecewise expanding map on an interval I of length L with stretching factor α, and let $p_0 < p_1 < \ldots < p_k$ be a stretching partition for f.*
[i] *Each allowable finite symbol sequence $S_1 S_2 \ldots S_n$ corresponds to a subinterval of length at most L/α^{n-1}*
[ii] *Each allowable infinite symbol sequence $S_1 S_2 \ldots S_n$ corresponds to a single point x of I such that $\forall i \geq 1, f^i(x) \in A_{i+1}$, and if the symbol sequence is not periodic or eventually periodic, then x generates a chaotic orbit.*
[iii] *If, in addition, each pair of symbols B and C (possibly $B = C$) can be connected by allowable finite symbol sequences $B \ldots C$ and $C \ldots B$, then f has a dense chaotic orbit on I, and I is a chaotic attractor.*

6.3 Measure with fractal dimension

In this section, we mainly discuss the topic of measure. Typically, the measure means Lebesgue measure in real analysis and Borel measure in most problem, however, we know that the box counting dimension can also explained with measure, so that means, we can improve the general measure to fractal dimension.

Definition 6.5 ***σ−Algebra***
Consider a set X, called $\Sigma \subset P(X)$ is a $\sigma-$algebra of X, if
[i] *(Include Maximum set)* $X \in \Sigma$
[ii] *(Include the Complement set)* $A \in \Sigma \Rightarrow A^c \in \Sigma$
[iii] *(Countable union)* $\forall A_1, A_2, \ldots, A_n, \ldots \in \Sigma, \bigcup_{i=1}^{\infty} A_i \in \Sigma$

Definition 6.6 ***Measure***
Consider a pair (X, A), where X is a set and A is $\sigma-$ algebra on X, if exists a function μ on X s.t.
[i] $\forall X_0 \subset X, \mu(X_0) \geq 0$, *and*
[ii] $\forall X_1, X_2, \ldots, X_n, \ldots$ *s.t.* $\forall i, j \in \mathcal{N}, C_i \bigcap X_j = \varnothing, \bigcup_{i=1}^{\infty} \mu(X_i) = \sum_{i=1}^{\infty} \mu(X_i)$
then we called the function μ is a ***measure*** *of (X, A). Moreover, if*
[iii] $\mu(X) = 1$
then we called this measure as ***probability measure****. And finally, if*
[iv] $\forall X_0 \subset X, X_0$ *is a closed set* , $\mu(f^{-1}(S)) = \mu(S)$
then this measure is a ***f-invariant measure***

Definition 6.7 ***Measure space***
Consider a set X and $P(X)$ is $\sigma-$algebra on A, μ is the measure on (X, A), then called (X, A, μ) as a ***measure space****.*

Definition 6.8 ***Measurable space/Borel space***
Consider a set X and $P(X)$ is $\sigma-$algebra on A, if exists a function μ satisfed the definition of measure on A, then called (X, A) as ***measurable space***

Example 6.3 ***Ikeda measure***
Consider a Ikeda map f and we record location of points with following processing
[i] *Choose an initial point at random to start an orbit.*
[ii] *At each iteration, we record whether the new orbit point fell into the box or not.*
[iii] *We keep this up for a long time, and when we stop we divide the number that landed in the box by the total number of iterates.*
[iv] *The result would be a number between 0 and 1 that we could call the "Ikeda measure" of the box.*

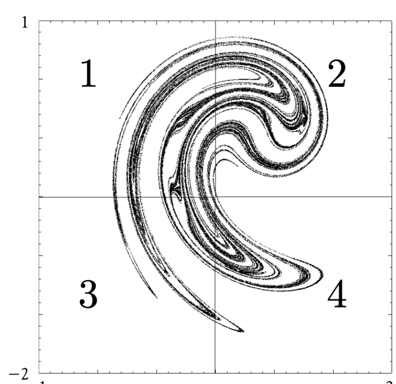


Figure 55: 100,000 points on the Ikeda attractor and Ikeda measure

In the figure above, the Ikeda measure for every box is $0.29798, 0.35857, 0.17342, 0.17003$*, obviouly, this Ikeda measure satisfed the definition of measure, probability measure as well as f-invariant measure*

To have a good measure, we need to require that almost every initial value produces an orbit that in the limit measures every set identically. That is, if we ignore a set of initial values that is a measure zero set, then the limit of the proportion of points that fall into each set is **independent of initial value**. A measure with this property will be called a natural measure. And next, we will define this natural measure more strickly.

Definition 6.9 ***Nature measure***
Consider a map f on R^m space and $S \subset R^m$, define function

$$F(x_0, S) = \lim_{n\to\infty} \frac{1}{n} |\{f^i(x_0) \in S, i = 1, 2, \ldots, n\}|$$

which is the total of the element include in set S during n iterations and $|\{*\}|$ is the total of the element of a set.
Then, a ***nature measure generated by map*** *f, or* ***f-measure*** *function μ is*

$$\mu_f(S) = \lim_{\varepsilon\to 0} F(x_0, N(\varepsilon, S))$$

where $N(\varepsilon, S)$ is the neighbourhood of the set S, or

$$N(\varepsilon, S) = \{x | d(x, S) \le \varepsilon\}$$

Definition 6.10 *A measure is* ***atomic*** *if all of the measure is contained in a finite or countably infinite set of points. In general, a map for which almost every orbit is attracted to a fixed-point sink will, by the same reasoning, have an atomic natural measure located at the sink.*

D I S C U S S I O N* 6.1** ***Why measure?
[i] *We must know that the natural measure of a map is not atomic if there is a chaotic attractor.*
[ii] *The existence of a natural measure allows us to calculate quantities that are sampled and averaged over the basin of attraction and have these quantities be well-defined, for instance, Lyapunov exponent will be a important quantity in our discussion.*
[*] For an orbit of a one-dimensional map f, the Lyapunov exponent is $\ln|f'|$ averaged over the entire orbit. In order to know that the average really tells us something about the attractor (in this case, that orbits on the attractor separate exponentially), we must know that we will obtain the same average no matter which orbit we choose. We must be guaranteed that an orbit chosen at random spends the same portion of its iterates in a given region as any other such orbit would. That is precisely what a natural measure guarantees.

6.4 Invariable measure in 1-dim map

Definition 6.11 ***Piecewise smooth***
Consider a map f on $[0,1]$, if $\exists$ ***a finite set*** *A s.t.*

$$f \in C^2([0,1]\backslash A)$$

otherwise, $\forall x \in [0,1]\backslash A$, f, f', f'' are continuous and bounded, then called f is ***piecewise smooth*** *on $[0,1]$ interval.*
Futhermore, if $\forall x \in [0,1]\backslash A, \exists \alpha > 1 s.t. |f'(x)| > \alpha$, then f is ***piecewise expanding*** *on $[0,1]$.*

Theorem 6.2 ***Invariant measure***
For ever map f piecewise smooth and piecewise expanding on $[0,1]$, $\exists$ a function μ is an invariant measure, that means

$$\exists \text{ a constant } c \text{ s.t. } \forall [a,b] \subset [0,1], \mu([a,b]) \leq c|a-b|$$

D I S C U S S I O N **6.2** ***Invariant and probablility***
Now we consider $S, A_1, A_2, \ldots, A_n$ s.t.

$$\forall i = 1,2,\ldots n, A_i \subset S \text{ and } A_1, A_2, \ldots A_n \text{ s.t. } \left(\forall i,j = 1,2,\ldots,n, i \neq j, A_i \bigcap A_j = 0\right) \wedge \left(\bigcup_{i=1}^{n} A_i = S\right)$$

[i] *If a function p s.t.*

$$\mu(S) = \int_S p(x)dx$$

then we called $p(x)$ as probability density function (PDF). In discrete condition, $p(x)$ s.t.

$$p(x) = \begin{cases} p_1 & x \in A_1 \\ p_2 & x \in A_2 \\ \ldots \\ p_n & x \in A_n \end{cases}$$

where $p_i \in [0,1] \wedge \sum_{i=1}^{n} p_i \mu_{lebesgue}(A_i) = 1$

[ii] *Now we consider the transition graph of a map between every subset, for a subset A_i and set $f(A_i)$, define transformation probability*

$$p_{i,j} = p(A_i, A_j) = \frac{|\{x | x \in f(A_i) \wedge x \in A_j\}|}{|f(A_i)|}$$

we combined every transformation probability as ***Markov matrix*** *s.t.*

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

Obviously, $\forall i = 1,2,\ldots n \sum_{j=1}^{n} p_{ij} = 1$.

E x a m p l e 6.4 ***Markov matrix of W-map***
Consider a W-map, the Markov matrix of this map is

$$P_T = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

Now we consider a Markov chain based on a Markov matrix, if we have a uniform probability distribution vector $X_0 = (1/n, 1/n, \ldots 1/n)$, *then we can structure a Markov chain* $X_i, i = 1, 2, \ldots, n, \ldots, \infty$ *with* P *matrix s.t.*

$$X_{i+1} = X_i P$$

Obviously, we have

$$X_\infty = X_0 P^\infty = X_0 \lim_{n\to\infty} P^n$$

On the other hand, if we consider the X_∞ *for a dynamical system, we can easily found that* $X_{\infty i} = 0$ *is a nature measure of original system* f. *So based on the Markov chain, we have another method, based on the matrix, to consider the f-measure as well as nature measure of every box.*

Problem in continuous time system

7 Differential equations

7.1 Basic Definition

Definition 7.1 ***Terms***
Autonomous: *The time variable t does not explicitly appear, for instance*

$$\text{Autonomous: } \dot{x} = ax, \text{ Nonautonomous:} \ddot{x} = -c\dot{x} - \sin x + \rho \sin x$$

Initial Value *The number $x_0 = x(0)$ is called the **initial value** of the function x.*
Flow *The **flow** F of an autonomous differential equation is the function of time t and initial value x_0 which represents the set of solutions. Thus $F(t, x_0)$ is the value at time t of the solution with initial value x_0 . We will often use the slightly different notation $F_t(x_0)$ to mean the same thing.*
Equilibrium *A constant solution of the autonomous differential equation $\dot{x} = f(x)$ is called an **equilibrium** of the equation.*
Sink *An equilibrium solution is called **attracting** or a **sink** if the trajectories of nearby initial conditions converge to it.*
Source *It is called **repelling** or a **source** if the solutions through nearby initial conditions diverge from it.*

* *I. One dimension system.*

Example 7.1 ***Logistic differential equation***
Consider a differential equation $\dot{x} = ax(1 - x)$ where $a > 0$ is a constant.

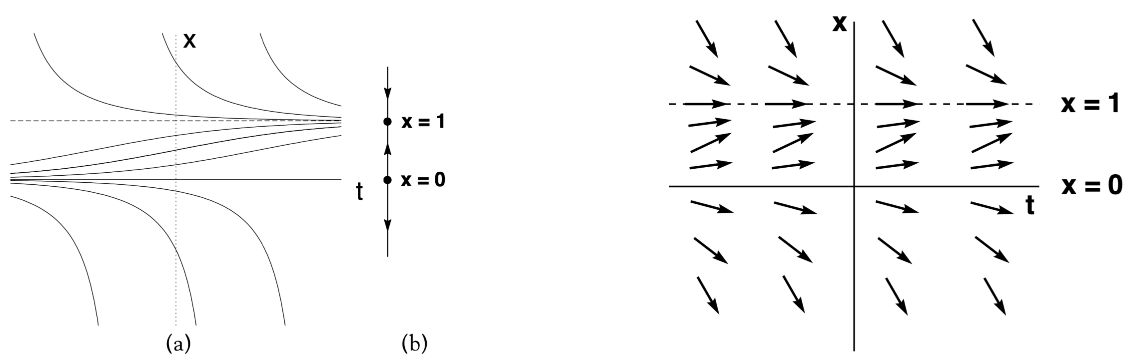


Figure 56: $x_0 - t$ plot of logistic DE

[i] Existence *Each point in the (t, x)-plane has a solution passing through it. The solution has slope given by the differential equation at that point.*
[ii] Uniqueness *Only one solution passes through any particular (t, x).*
[iii] Continuous dependence *Solutions through nearby initial conditions remain close over short time intervals. In other words, the flow $F(t, x_0)$ is a continuous function of x_0 as well as t.*
*Using the first concept, we can draw a **slope field** in the (t, x)-plane by evaluating $\dot{x} = ax(1 - x)$ at several points and putting a short line segment with the evaluated slope at each point*

** II. High dimension linear system.*

Definition 7.2 ***Lyapunov Stable*** *An equilibrium point* $\mathbf{v}$ *is called* ***stable or Lyapunov stable*** *if every initial point* $\mathbf{v_0}$ *that is chosen very close to* $\mathbf{v}$ *has the property that the solution* $F(t, \mathbf{v_0})$ *stays close to* v *for* $t \geq 0$.
More formally, for any neighborhood N *of* $\mathbf{v}$ *there exists a neighborhood* N_1 *of* $\mathbf{v}$*, contained in* N*, such that for each initial point* $\mathbf{v_0}$ *in* N_1*, the solution* $F(t, \mathbf{v_0})$ *is in* N *for all* $t \geq 0$.
An equilibrium is called ***asymptotically stable*** *if it is both stable and attracting.*
An equilibrium is called ***unstable*** *if it is not stable.*
Finally, an equilibrium is ***globally asymptotically stable*** *if it is asymptotically stable and all initial values converge to the equilibrium.*

Theorem 7.1 ***Stability of Origin*** *Let* A *be an* $n \times n$ *matrix, and consider the equation* $\dot{\mathbf{v}} = A\mathbf{v}$*. If the real parts of all eigenvalues of* A *are negative, then the equilibrium* $\mathbf{v} = 0$ *is globally asymptotically stable. If* A *has* n *distinct eigenvalues and if the real parts of all eigenvalues of* A *are nonpositive, then* $\mathbf{v} = 0$ *is stable.*

III. High dimension nonlinear system.

CONCLUSION **7.1** *For all high dimension ODE, it can be transformed into a first-order system.*

So if we consider a high dimension nonlinear equation, that is same as consider a 1-dim ODE system.

Theorem 7.2 ***Existence and Uniqueness***
Consider the first-order differential equation $\dot{\mathbf{v}} = f(\mathbf{v})$ *where both* f *and its first partial derivatives with respect to* v *are continuous on an open set* U*. Then for any real number* t_0 *and real vector* v_0*, there is an open interval containing* t_0*, on which there exists a solution satisfying the initial condition* $v(t_0) = v_0$*, and this solution is unique.*

Definition 7.3 ***Lipschitz Constant***
Let U *be an open set in* $\mathcal{R}^n$*. A function* f *on* $\mathcal{R}^n$ *is said to be Lipschitz on* U *if there exists a constant* L *s.t.*

$$\forall v, w \in U, ||f(v) - f(w)|| \leq L||v - w||$$

The constant L *is called a Lipschitz constant for f.*

Theorem 7.3 ***Continuous dependence on initial conditions***
Let f *be defined on the open set* U *in* $\mathcal{R}^n$*, and assume that* f *has Lipschitz constant* L *in the variables* v *on* U*. Let* $v(t)$ *and* $w(t)$ *be solutions of* $\dot{\mathbf{v}} = f(\mathbf{v})$*, and let* $[t_0, t_1]$ *be a subset of the domains of both solutions. Then*

$$\forall t \in [t_0, t_1], ||v(t) - w(t)|| \leq ||v(t_0) - w(t_0)|| \exp(L(t - t_0))$$

Next two definition introduced the sink, source and saddle in continuous problem.

Definition 7.4 *An equilibrium* v_0 *of* $\dot{v} = f(v)$ *is called hyperbolic if none of the eigenvalues of* $Df(v_0$ *has real part 0.*

Definition 7.5 *Let* v_0 *be an equilibrium of* $\dot{v} = f(v)$*. If the real part of each eigenvalue of* $Df(v_0)$ *is strictly negative, then* v_0 *is asymptotically stable. If the real part of at least one eigenvalue is strictly positive, then* v_0 *is unstable.*

7.2 Energy Function, Lyapunov Function

Definition 7.6 ***Level curve***
Given a real number c and a function $E : \mathcal{R}^2 \to \mathcal{R}$, the set $E_c = \{(x, y)|E(x, y) = c\}$ is called a ***level curve*** *of the function E.*

Definition 7.7 ***Lyapunov Function***
Let v_0 be an equilibrium of $\dot{\mathbf{v}} = f(\mathbf{v})$. A function $E : \mathcal{R}^n \to \mathcal{R}$ is called a Lyapunov function for v_0 if for some neighborhood $N(v_0)$, the following conditions are satisfied:
[i] $\forall v \in N(v_0)\backslash\{v_0\}, E(v_0) = 0 \wedge E(v) > 0$
[ii] $\forall v \in N(V_0), \dot{E}(v) \leq 0$

If the stronger inequality

$$\textbf{\textit{[ii']}} \forall v \in N(V_0), \dot{E}(v) < 0$$

holds, then E is called a strict Lyapunov function.

Theorem 7.4 *Let v_0 be an equilibrium of $\dot{\mathbf{v}} = f(\mathbf{v})$. If there exists a Lyapunov function for v, then v is stable. If there exists a strict Lyapunov function for v, then v is asymptotically stable.*

Definition 7.8 ***Basin of attraction***
Let v_0 be an asymptotically stable equilibrium of $\dot{\mathbf{v}} = f(\mathbf{v})$. Then the basin of attraction of v_0, denoted $B(v_0)$, is the set of initial conditions v_{init} s.t.

$$\lim_{t\to\infty} F(t, v_{init}) = v_0$$

Definition 7.9 *A set $U \subset \mathcal{R}^n$ is called a forward invariant set for $\dot{\mathbf{v}} = f(\mathbf{v})$ if for each $v_0 \in U$, the forward orbit $\{F(t, v_0) : t \geq 0\}$ is contained in U. A forward invariant set that is bounded is called a* ***trapping region****. We also require that a trapping region be an n-dimensional set.*

CONCLUSION* 7.2** ***Barbashin-LaSalle
Let E be a Lyapunov function for v_0 on the neighborhood $N(v_0)$. Let $Q = \{v \in N : \dot{E}(v) = 0\}$. Assume that N is forward invariant. If the only forward-invariant set contained completely in Q is v_0, then v_0 is asymptotically stable. Furthermore, N is contained in the basin of v_0; that is, $\forall v \in N, \lim_{t\to\infty} F(t, v) = v_0$

We will proof this conclusion in next section.

8 Periodic Orbits and Limit Sets

In this section, we mainly consider the fixed point, periodic-k orbit in continuous problem.

I. Fixed point, Periodic-k points $\rightarrow$ Limit set

Definition 8.1 ***Limit set***
A point $z \in \mathcal{R}^n$ is in the ω-limit set $\omega(v_0)$(or α-limit set $\alpha(v_0)$) of the solution curve $F(t, v_0)$ if there is a sequence of points increasingly(decreasingly) far out along the orbit which converges to z, or

$$z \text{ is in } \omega(v_0) \text{ if } \exists\{t_n\} \text{ s.t. } \forall i < j, t_i < t_j, t_\infty = \infty, \text{ have } \lim_{n\to\infty} F(t_n, v_0) = z$$

$$z \text{ is in } \alpha(v_0) \text{ if } \exists\{t_n\} \text{ s.t. } \forall i < j, t_i > t_j, t_\infty = -\infty, \text{ have } \lim_{n\to\infty} F(t_n, v_0) = z$$

Theorem 8.1 *All solutions of the scalar differential equation $\dot{x} = f(x)$ are either monotonic increasing or monotonic decreasing as a function of t. For $x_0 \in R$, if the orbit $F(t, x0), t \geq 0$, is bounded, then $\omega(x_0)$ consists solely of an equilibrium.*

II. Asymptotically Periodic $\rightarrow$ Limit cycle

Definition 8.2 ***Limit Cycle***
If there exists a $T > 0$ s.t. $\forall t > 0, F(t + T, v_0) = F(t, v_0)$ and v_0 is not an equilibrium, then the solution $F(t, v_0)$ is called a periodic orbit, or a limit cycle, and the smallest value T is called the period of the orbit.
If the limit cycle have same begining and endding, then called this curve is closed.
A simple closed curve is one that does not cross itself.

Theorem 8.2 ***Poincare-Bendixson theorem***
Let f be a smooth vector field of the plane, for which the equilibria of $\dot{v} = f(v)$ are isolated. If the forward orbit $F(t, v_0), t \geq 0$ is bounded, then either
[i] *$\omega(v_0)$ is an equilibrium, or*
[ii] *$\omega(v_0)$ is a periodic orbit(cycle), or*
[iii] *$\forall u \in \omega(v_0)$, the limit set $\alpha(u)$ and $\omega(u)$ are equilibria.*

If we explain the Poincare-Bendixson theorem in map, then it is for all bounded attractor, the forward orbit is fixed point([i]) or periodic-k orbit ([ii]), or asymptotically periodic orbit ([iii])

We mainly introduced the definition and theorem, and the proof will be put in the end of this section.

Property 8.1 ***Property of ω-limit sets***
Existence: *If the orbit $F(t, v_0)$ is bounded for all $t \geq 0$, then $\omega(v_0)$ is non-empty.*
Closure: *$\omega(v_0)$ is closed, that means $\omega(v_0)$ contains all of its limit points.*
Invariance: *$\omega(v_0)$ is invariant under the flow, that is, if $u \in \omega(v_0)$, then $\forall t \geq 0, F(t, u) \in \omega(v_0)$.*
Connectedness: *If $\{F(t, v_0)\}$ is a bounded set, then $\omega(v_0)$ is connected.*
Transitivity: *If $z \in \omega(u) \wedge u \in \omega(v_0)$, then $z \in \omega(v_0)$*

We introduced the Barbashin-LaSalle theorem in last section, and here, we will try to proof that.

Barbashin-LaSalle

Let E be a Lyapunov function for v_0 on the neighborhood $N(v_0)$. Let $Q = \{v \in N : \dot{E}(v) = 0\}$. Assume that N is forward invariant. If the only forward-invariant set contained completely in Q is v_0, then v_0 is asymptotically stable. Furthermore, N is contained in the basin of v_0; that is, $\forall v \in N, \lim_{t\to\infty} F(t,v) = v_0$

***PROOF* 8.1** *Theo. 8.1*

Lemma 8.1 *Let $W_c = \{v \in R^n : E(v) \le c\}$ and $v_0 \in W_c$, for some $c > 0$. Then there is a number $d, 0 \le d \le c$ s.t. $\forall v \in \omega(v_0), E(v) = d$*

***CONCLUSION* 8.1** $\forall v \in \omega(V_0), \dot{E}(v) = 0$

■

***PROOF* 8.2** *Theo. 8.1* ■

***PROOF* 8.3** *Theo. 7.2* ■

***PROOF* 8.4** *Theo. 8.2* ■

9 Chaos Systems

9.1 Lorenz System

9.2 Lorenz Map

9.3 Rossler System

9.4 Other Systems

9.4.1 Chua's Circuit

9.4.2 Forced Oscillators

9.5 Lyapunov Exponents in Flow