7 Chaos in attractor

7.1 Attractor and chaotic attractor

Definition 7.1 *Let f be a map and x_0 be an initial condition. The **forward limit set** of the orbit $\{f^n(X_0)\}$ is*

$$\omega(x_0) = \{x|\forall\varepsilon > 0, \forall N > 0, \exists n > N s.t. |f^n(x_0) - x| < \varepsilon\}$$

Definition 7.2 ***Attractor***
Attractor is a forward limit set which attracts a set of initial values that has nonzero measure (nonzero length, area, or volume, depending on whether the dimension of the map's domain is one, two, or higher).
This set of initial conditions is called the basin of attraction (or just basin), of the attractor.

Definition 7.3 ***Chaos set***
Let $\{f^n(x)\}$ be a chaotic orbit.
*If $x_0 \in \omega(x_0)$, then $\omega(x_0)$ us called a **chaotic set**.*

CONCLUSION* 7.1** ***[i] *A chaotic set is the forward limit set of a chaotic orbit which itself is contained in its forward limit set.*
[ii] *A chaotic attractor is a chaotic set that is also an attractor.*
[iii] *A sink is an attractor, since it attracts at least a small neighborhood of initial values.*

E x a m p l e 7.1 ***Ikeda map***
Ikeda map map was proposed first by Kensuke Ikeda as a model of light going around across a nonlinear optical resonator. Is is a discrete-time dynamical system given by the complex map.

$$z_{n+1} = A + Bz_n e^{i(|z_n|^2+C)}$$

Based on the Euler formula, we can transform this complex map to real map

$$\begin{cases} \dot{x} = R + C_2(x\cos\tau - y\sin\tau) \\ \dot{y} = C_2(x\sin\tau + y\cos\tau) \end{cases} \quad \tau = C_1 - \frac{C_3}{1+x^2+y^2}$$

To analysis the problem with Lyapunov exponent, we need to calculate the partical differential of the function. Let $\xi = (1+x^2+y^2)^{-1}$, then

$$\frac{\partial\sin\tau}{\partial *} = \frac{\partial\tau}{\partial *}\cos\tau \qquad \frac{\partial\cos\tau}{\partial *} = -\frac{\partial\tau}{\partial *}\sin\tau \qquad (* = x, y) \text{ and}$$

$$\frac{\partial\tau}{\partial x} = \frac{2C_3x}{(1+x^2+y^2)^2} = 2C_3x\xi^2 \qquad \frac{\partial\tau}{\partial y} = \frac{2C_3y}{(1+x^2+y^2)^2} = 2C_3y\xi^2$$

Now we consider $\partial\dot{*}/\partial-$, where $* = x, y$ and $- = x, y$

$$\frac{\partial\dot{x}}{\partial x} = C_2\left(\frac{\partial(x\cos\tau)}{\partial x} - y\frac{\partial\sin\tau}{\partial x}\right) = C_2\left(cos\tau - \frac{\partial\tau}{\partial x}x\sin\tau - \frac{\partial\tau}{\partial x}y\cos\tau\right)$$

$$\frac{\partial \dot{x}}{\partial y} = C_2\left(x\frac{\partial \cos\tau}{\partial y} - \frac{\partial(y\sin\tau)}{\partial y}\right) = C_2\left(-\sin\tau - \frac{\partial\tau}{\partial y}x\sin\tau - \frac{\partial\tau}{\partial y}y\cos\tau\right)$$
$$\frac{\partial \dot{y}}{\partial x} = C_2\left(\frac{\partial(x\sin\tau)}{\partial x} + y\frac{\partial \cos\tau}{\partial x}\right) = C_2\left(sin\tau + \frac{\partial\tau}{\partial x}x\cos\tau - \frac{\partial\tau}{\partial x}y\sin\tau\right)$$
$$\frac{\partial \dot{y}}{\partial y} = C_2\left(x\frac{\partial \sin\tau}{\partial y} + \frac{\partial(y\cos\tau)}{\partial y}\right) = C_2\left(\cos\tau + \frac{\partial\tau}{\partial y}x\cos\tau - \frac{\partial\tau}{\partial y}y\sin\tau\right)$$

Summarize the formula we wrote above in matrix, then

$$J = C_2\cdot\begin{bmatrix}\cos\tau & -\sin\tau\\ \sin\tau & \cos\tau\end{bmatrix}\cdot\left(I+\begin{bmatrix}-y\\ x\end{bmatrix}\cdot\begin{bmatrix}\frac{\partial\tau}{\partial x} & \frac{\partial\tau}{\partial y}\end{bmatrix}\right)$$
$$= C_2\cdot\begin{bmatrix}\cos(\tau) & -\sin(\tau)\\ \sin(\tau) & \cos(\tau)\end{bmatrix}\cdot\left(I+\begin{bmatrix}-y\\ x\end{bmatrix}\cdot\begin{bmatrix}2C_3x\xi^2 & 2C_3y\xi^2\end{bmatrix}\right)$$
$$= \alpha C_2\cdot\begin{bmatrix}\cos(\tau) & -\sin(\tau)\\ \sin(\tau) & \cos(\tau)\end{bmatrix}\cdot\begin{bmatrix}1/\alpha - xy & -y^2\\ x^2 & 1/\alpha + xy\end{bmatrix}$$

Finally, we have

$$J = \alpha C_2\cdot\begin{bmatrix}\cos(\tau) & -\sin(\tau)\\ \sin(\tau) & \cos(\tau)\end{bmatrix}\cdot\begin{bmatrix}1/\alpha - xy & -y^2\\ x^2 & 1/\alpha + xy\end{bmatrix}, \alpha = 2C_3\xi^2, \tau = C_1 - C_3\xi, \xi = \frac{1}{(1+x^2+y^2)}$$

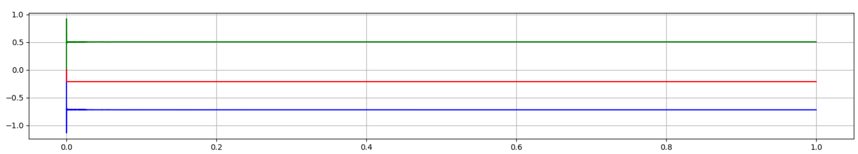

Figure 52: Lyapunov spectrum of Ikeda map

Lyapunov spectrum = (+, -)
$(Ikeda(R = 1, C_1 = 0.4, C_2 = 0.9, C_3 = 6) = (0.51, -0.72))$
So obviously, Ikeda map is a chaotic system.

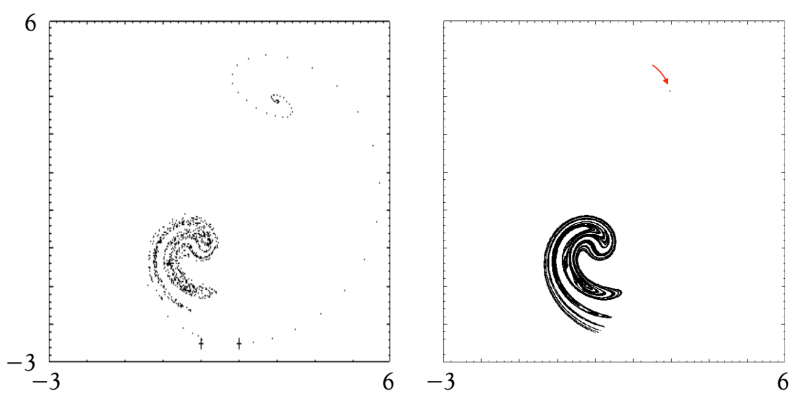

Figure 53: Forward map of Ikeda map

Now we consider two different initial value(+ point in Fig.53, (1)), we ignore first 1,000,000 result and plot 1,000,001 - 2,000,000 in the Fig.53, (2). Obviously, we can assume these points are forward limit set, or the attractor of the system,which has one chaotic set and a sink point.

We can found Conclusion 7.1(i), (ii) and (iii) are established in this Ikeda map easily.

Now we discuss what will be included in attractor set.

CONCLUSION* 7.2** ***Attractor set include:
*[**i**] Fixed point sink*
*[**ii**] Periodic-k points*

Or in summary, a attractor should be a set satisfied:
*[**iii**] Attractor should be **irreducible** in the sense that it includes only what is necessary.*
*[**iv**] The attractor must have the property that a point chosen at random should have a **greater-than-zero probability of converging** to the set.*
*[**vi**] Chaos introduces a new twist. Chaotic orbits can be attracting.*

7.2 n-dim Itinerary

In itinerary we introduced before, we just used two subinterval(**L** and **R** in Tent map(1-dim) and baker map(high-dim))

Definition **7.4** ***Piecewise expanding map***
Consider a interval $I = [p_0, p_k]$ and $\{p_0, p_1, \ldots, p_k\}$ is a partition of I s.t.

$$p_0 < p_1 < p_2 < \ldots < p_k, \text{ and } I_i = [p_{i-1}, p_i], I = \bigcup_{i=1}^{k} I_i$$

*Let $f : I \to I$ be a map s.t. $|f'(x)| \geq \alpha > 1$ except possibly at the p_i point (that means, f may have dicountinuous point). We will call such a map a **piecewise expanding map** with **stretching factor** α.*
*We say that $\{p_0, p_1, \ldots, p_k\}$ is a **stretching partition** for the piecewise expanding map f if, for each i, $f(I_i)$ is exactly the union of some of the intervals $I_1, I_2, \ldots I_k$.*

and we have these conclusion:

***CONCLUSION* 7.3** *[**i**] A stretching partition satisfies the covering rule, which allows the construction of transition graphs for the partition intervals I_i.*
*[**ii**] It is also the one-dimensional analogue of the concept of Markov partition.*

Now we consider the itinerary of a system, we can use $I_1, I_2, \ldots, I_k$ instead of **L** and **R** to describe the symbol of a orbit.

E x a m p l e 7.2 ***W-map***
Consider a 1-dim map f s.t.

$$f(x) = \begin{cases} 2|(x - 1/4)| & x \in [0, 1/2] \\ 2|(x - 3/4)| & x \in [1/2, 1] \end{cases}$$

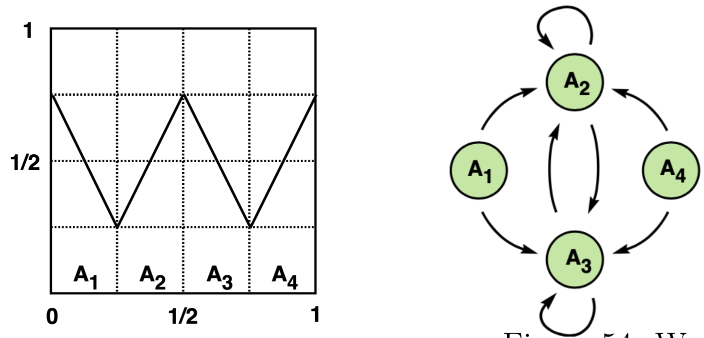

Figure 54: W map

Let $p_0, p_1, p_2, p_3, p_4 = 0, 1/4, 2/4, 3/4, 1$*,* $A_1 = [0, 1/4], A_2 = [1/4, 2/4], A_3 = [2/4, 3/4], A_4 = [3/4, 1]$*, then we can easily found the transition graph of the map.*

$$f(A_1) = f(A_2) = f(A_3) = f(A_4) = A_2 \bigcup A_3 \Rightarrow f\left(\bigcup_{i=1}^{4} A_i\right) = A_2 \bigcup A_3$$

Now we consider the itinerary of the map.
** If* $S_1S_2\ldots$ *is itinerary, then* $S_1S_2 = A_1A_2$ *is an allowable symbol sequence, but* A_2A_1 *is not. In general, we have*

CONCLUSION 7.4 *If* B, C *are two subintervals for a stretching partition,* C *allowed to follow* B *in the symbol sequence of a poitn of the interval* I *if and only if* $f(B) = C\bigcup(\textit{other subintervals})$

*** As* $|f'| \equiv= 2 = \alpha$ *and for every interval* $L = \mu([0,1]) = 1 \Rightarrow \mu(f(I)) = \mu([0,1])/\alpha = 1/2$*. On the other hand, for ever interval* $[0,1] = \bigcup_{i=1}^{4} A_i, f(\bigcup_{i=1}^{4} A_i) = A_2 \bigcup A_3 \Rightarrow \mu(f(I)) = \mu(A_2 \bigcup A_3) = 1/2$*. Generally, we have*

CONCLUSION 7.5 *For an allowable sequence* $B_1B_2\ldots B_n$ *of n symbols, there is a subinterval of length at most* $\frac{L}{\alpha^n}$ *which we call an* ***order n subinterval***

Theorem 7.1 *Let* f *be a continuous piecewise expanding map on an interval* I *of length* L *with stretching factor* α*, and let* $p_0 < p_1 < \ldots < p_k$ *be a stretching partition for* f*.*
[i] *Each allowable finite symbol sequence* $S_1S_2\ldots S_n$ *corresponds to a subinterval of length at most* L/α^{n-1}
[ii] *Each allowable infinite symbol sequence* $S_1S_2\ldots S_n$ *corresponds to a single point* x *of* I *such that* $\forall i \geq 1, f^i(x) \in A_{i+1}$*, and if the symbol sequence is not periodic or eventually periodic, then* x *generates a chaotic orbit.*
[iii] *If, in addition, each pair of symbols* B *and* C *(possibly* $B = C$*) can be connected by allowable finite symbol sequences B...C and C...B, then* f *has a dense chaotic orbit on* I*, and* I *is a chaotic attractor.*

7.3 Measure with fractal dimension

In this section, we mainly discuss the topic of measure. Typically, the measure means Lebesgue measure in real analysis and Borel measure in most problem, however, we know that the box counting dimension can also explained with measure, so that means, we can improve the general measure to fractal dimension.

Definition 7.5 ***$\sigma-$Algebra***
Consider a set X, called $\Sigma \subset P(X)$ is a $\sigma-$algebra of X, if
[i] *(Include Maximum set) $X \in \Sigma$*
[ii] *(Include the Complement set)$A \in \Sigma \Rightarrow A^c \in \Sigma$*
[iii] *(Countable union) $\forall A_1, A_2, \ldots, A_n, \ldots \in \Sigma, \bigcup_{i=1}^{\infty} A_i \in \Sigma$*

Definition 7.6 ***Measure***
Consider a pair (X, A), where X is a set and A is $\sigma-$ algebra on X, if exists a function μ on X s.t.
[i] *$\forall X_0 \subset X, \mu(X_0) \geq 0$, and*
[ii] *$\forall X_1, X_2, \ldots, X_n, \ldots$ s.t. $\forall i, j \in \mathcal{N}, C_i \bigcap X_j = \varnothing, \bigcup_{i=1}^{\infty} \mu(X_i) = \sum_{i=1}^{\infty} \mu(X_i)$*
*then we called the function μ is a **measure** of (X, A). Moreover, if*
[iii] *$\mu(X) = 1$*
*then we called this measure as **probability measure**. And finally, if*
[iv] *$\forall X_0 \subset X, X_0$ is a closed set , $\mu(f^{-1}(S)) = \mu(S)$*
*then this measure is a **f-invariant measure***

Definition 7.7 ***Measure space***
*Consider a set X and $P(X)$ is $\sigma-$algebra on A, μ is the measure on (X, A), then called (X, A, μ) as a **measure space**.*

Definition 7.8 ***Measurable space/Borel space***
*Consider a set X and $P(X)$ is $\sigma-$algebra on A, if exists a function μ satisfed the definition of measure on A, then called (X, A) as **measurable space***

Example 7.3 ***Ikeda measure***
Consider a Ikeda map f and we record location of points with following processing
[i] *Choose an initial point at random to start an orbit.*
[ii] *At each iteration, we record whether the new orbit point fell into the box or not.*
[iii] *We keep this up for a long time, and when we stop we divide the number that landed in the box by the total number of iterates.*
[iv] *The result would be a number between 0 and 1 that we could call the "Ikeda measure" of the box.*

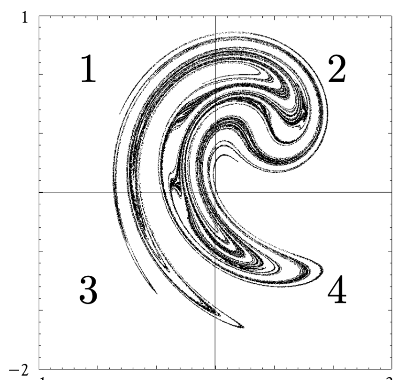

Figure 55: 100,000 points on the Ikeda attractor and Ikeda measure

In the figure above, the Ikeda measure for every box is $0.29798, 0.35857, 0.17342, 0.17003$*, obviouly, this Ikeda measure satisfed the definition of measure, probability measure as well as f-invariant measure*

To have a good measure, we need to require that almost every initial value produces an orbit that in the limit measures every set identically. That is, if we ignore a set of initial values that is a measure zero set, then the limit of the proportion of points that fall into each set is **independent of initial value**. A measure with this property will be called a natural measure. And next, we will define this natural measure more strickly.

Definition 7.9 ***Nature measure***
Consider a map f on R^m space and $S \subset R^m$, define function

$$F(x_0, S) = \lim_{n\to\infty} \frac{1}{n} |\{f^i(x_0) \in S, i = 1, 2, \ldots, n\}|$$

which is the total of the element include in set S during n iterations and $|\{*\}|$ is the total of the element of a set.
Then, a ***nature measure generated by map*** *f, or* ***f-measure*** *function μ is*

$$\mu_f(S) = \lim_{\varepsilon\to 0} F(x_0, N(\varepsilon, S))$$

where $N(\varepsilon, S)$ is the neighbourhood of the set S, or

$$N(\varepsilon, S) = \{x | d(x, S) \le \varepsilon\}$$

Definition 7.10 *A measure is* ***atomic*** *if all of the measure is contained in a finite or countably infinite set of points. In general, a map for which almost every orbit is attracted to a fixed-point sink will, by the same reasoning, have an atomic natural measure located at the sink.*

D I S C U S S I O N **7.1** ***Why measure?***
[i] *We must know that the natural measure of a map is not atomic if there is a chaotic attractor.*
[ii] *The existence of a natural measure allows us to calculate quantities that are sampled and averaged over the basin of attraction and have these quantities be well-defined, for instance, Lyapunov exponent will be a important quantity in our discussion.*
[*] *For an orbit of a one-dimensional map f, the Lyapunov exponent is $\ln|f'|$ averaged over the entire orbit. In order to know that the average really tells us something about the attractor (in this case, that orbits on the attractor separate exponentially), we must know that we will obtain the same average no matter which orbit we choose. We must be guaranteed that an orbit chosen at random spends the same portion of its iterates in a given region as any other such orbit would. That is precisely what a natural measure guarantees.*

7.4 Invariable measure in 1-dim map

Definition 7.11 ***Piecewise smooth***
*Consider a map f on $[0,1]$, if $\exists$ **a finite set** A s.t.*

$$f \in C^2([0,1]\backslash A)$$

*otherwise, $\forall x \in [0,1]\backslash A$, f, f', f'' are continuous and bounded, then called f is **piecewise smooth** on $[0,1]$ interval.*
*Futhermore, if $\forall x \in [0,1]\backslash A, \exists \alpha > 1 s.t. |f'(x)| > \alpha$, then f is **piecewise expanding** on $[0,1]$.*

Theorem 7.2 ***Invariant measure***
For ever map f piecewise smooth and piecewise expanding on $[0,1]$, $\exists$ a function μ is an invariant measure, that means

$$\exists \text{ a constant } c \text{ s.t. } \forall [a,b] \subset [0,1], \mu([a,b]) \le c|a-b|$$

D I S C U S S I O N* 7.2** ***Invariant and probablility
Now we consider $S, A_1, A_2, \ldots, A_n$ s.t.

$$\forall i = 1,2,\ldots n, A_i \subset S \text{ and } A_1, A_2, \ldots A_n \text{ s.t. } \left(\forall i,j = 1,2,\ldots,n, i \neq j, A_i \bigcap A_j = 0\right) \wedge \left(\bigcup_{i=1}^{n} A_i = S\right)$$

[i] *If a function p s.t.*

$$\mu(S) = \int_S p(x)dx$$

then we called $p(x)$ as probability density function (PDF). In discrete condition, $p(x)$ s.t.

$$p(x) = \begin{cases} p_1 & x \in A_1 \\ p_2 & x \in A_2 \\ \cdots & \\ p_n & x \in A_n \end{cases}$$

where $p_i \in [0,1] \wedge \sum_{i=1}^{n} p_i \mu_{lebesgue}(A_i) = 1$

[ii] *Now we consider the transition graph of a map between every subset, for a subset A_i and set $f(A_i)$, define transformation probability*

$$p_{i,j} = p(A_i, A_j) = \frac{|\{x | x \in f(A_i) \wedge x \in A_j\}|}{|f(A_i)|}$$

*we combined every transformation probability as **Markov matrix** s.t.*

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

Obviously, $\forall i = 1,2,\ldots n \sum_{j=1}^{n} p_{ij} = 1$.

E x a m p l e 7.4 ***Markov matrix of W-map***
Consider a W-map, the Markov matrix of this map is

$$P_T = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

Now we consider a Markov chain based on a Markov matrix, if we have a uniform probability distribution vector $X_0 = (1/n, 1/n, \dots 1/n)$*, then we can structure a Markov chain* $X_i, i = 1, 2, \dots, n, \dots, \infty$ *with* P *matrix s.t.*

$$X_{i+1} = X_i P$$

Obviously, we have

$$X_\infty = X_0 P^\infty = X_0 \lim_{n\to\infty} P^n$$

On the other hand, if we consider the X_∞ *for a dynamical system, we can easily found that* $X_{\infty i} = 0$ *is a nature measure of original system* f*. So based on the Markov chain, we have another method, based on the matrix, to consider the f-measure as well as nature measure of every box.*