5 Chaos in high dimension map

5.1 Lyapunov Spectrum

In the section 3, we mainly discussed the judgement of chaos in 1 dimension. Here we focus on the Lyapunov exponent in high dimension. Obviously, in high dimension, the δ_n function which we used in Discussion 3.2 will be a m dimension vector δ_n and we mainly focus on the length(or norm) of this vector.

$$||\delta_t|| = \exp(\lambda t)||\delta_0||$$

$$t \to \infty, ||\delta_t|| = \exp(\lambda t)||\delta_0|| \Rightarrow \lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{||\delta_t||}{||\delta_0||} = \lim_{t\to\infty} \ln \left[\frac{||\delta_t||}{||\delta_0||} \right]^{\frac{1}{t}}$$

However, in high dimension problem, it is different from 1-dim because during the iteration, some direct of vector increase and the others decrease. Fig. 33 showed us an example of this evolution.

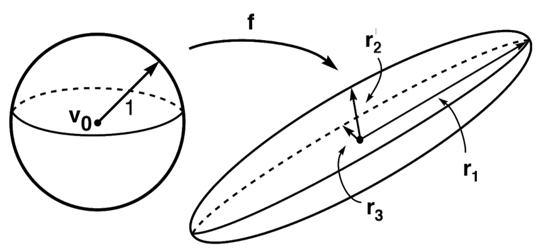


Figure 33: Evolution of unit vector in 3d

To solve this problem, we should consider the evolution of every direct in the space.

Let $\omega_0^{(1)}, \omega_0^{(2)}, \ldots, \omega_0^{(m)}$ is a group of orthogonal basis in $\mathcal{R}^m$ space which satisfied $\forall i, j = 1, 2, \ldots, m, i \neq j$, the inner production $< \omega_0^{(i)}, \omega_0^{(j)} >= 0$, then for every direction, we have a λ value based on the formula above, that means

$$\forall i = 1, 2, \ldots m, \lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln \frac{||\omega_t^{(i)}||}{||\omega_0^{(i)}||}$$

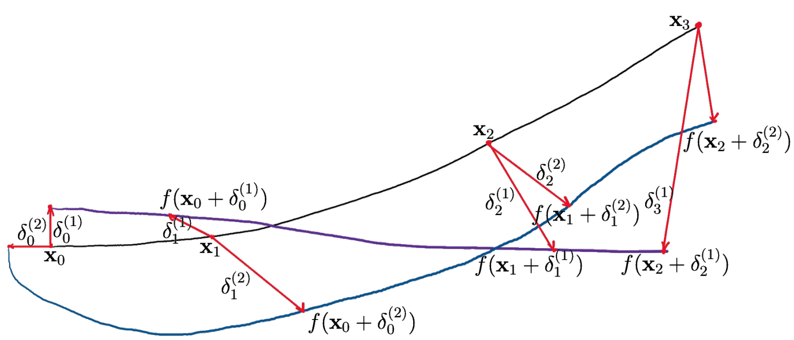


Figure 34: Analysis in two dimension problem

Basically, a simple basis in $\mathcal{R}^m$ space is $\omega_0^{(1)} = (1, 0, \dots, 0)^T, \omega_0^{(2)} = (0, 1, \dots, 0)^T, \dots, \omega_0^{(m)} = (0, 0, \dots, 1)^T$ and

$$\Omega_0 = (\omega_0^{(1)}, \omega_0^{(2)}, \dots, \omega_0^{(m)}) = I$$

In this situation, $||\omega_0^{(i)}|| = 1(i = 1, 2, \dots, m)$, let

$$\Lambda = (\lambda_1, \lambda_2, \dots \lambda_m) = \lim_{t\to\infty} \frac{1}{t} \left(\ln ||\omega_t^{(1)}||, \ln ||\omega_t^{(2)}||, \dots, \ln ||\omega_t^{(m)}|| \right)$$

Let $p_1, p_2, \dots p_m$ s.t. $\{p_1, p_2, \dots, p_m\} = \{1, 2, \dots, m\} \wedge \forall i, j = 1, 2, \dots m, i \neq j, p_i \neq p_j$ and

$$\ln ||\omega_t^{(i_1)}|| \geq \ln ||\omega_t^{(i_2)}|| \geq \dots \geq \ln ||\omega_t^{(i_m)}||$$

In the discrete time processing, we know that $t = 0, 1, 2, \dots$ so $r_t^{(k)} = r_n^{(k)}$ where $n \in \mathcal{N}$.

Let $r_n^{(k)} = \ln ||\omega_n^{(i_1)}||$ be the length of the kth longest orthogonal axis after n time iterate for an initial point $\omega_0^{(i_1)}$. Obviously, these $r_n^{(k)}$ sequence (with k, not n) measured the expansion of initial vectors, so we can define the Lyapunov exponent in follows.

Definition 5.1 ***Lyapunov number, Lyapunov exponent in high dimension problem***
Let $f \in C^\infty(\mathcal{R}^m)$, $J_n = Df^n(\mathbf{x}_0)$, $r_n^{(k)}$ be the length of the kth logenst orthogonal axis which defined by the explanation above. Then the kth ***Lyapunov number*** *of $\mathbf{x}_0$ is defined by*

$$L_k = \lim_{n\to\infty} \left(r_n^{(k)}\right)^{\frac{1}{n}}, \text{ and the } \textbf{Lyapunov exponent } h_k = \ln L_k$$

Obviously,

$$(L_1 \geq L_2 \geq \dots \geq L_m) \wedge (h_1 \geq h_2 \geq \dots \geq h_m)$$

For every single $r_n^{(k)}$ we know that is familiar with λ in 1-dim Lyapunov exponent, so we have The orbit is **chaotic** if it satisfied both
[i] $\{x_1, x_2, \dots\}$ is no asymptotically periodic, and
[ii] the Lyapunov exponent h_k is **greater** than zero.

As $h_1 \geq h_2 \geq \dots \geq h_m$, so if $h_1 < 0$, then every Lyapunov exponent is less than zero, that means, we can simplify the definition and only care about h_1

Definition 5.2 ***Orbit chaotic in high dimension***
let f be a map of $\mathcal{R}^m$, and $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_1, \dots$ be a bounded orbit of f, if
[i] *orbit is not asymptotically periodic; and*
[ii] *$\forall i = 1, 2, \dots m, h_m \neq 0$ and $h_1 > 0$ then the orbit is* ***chaotic****.*

Now we consider how to calculate this Lyapunov exponent in normal problem.

[i] f is linear map

If f is a linear map, then $\exists P$ s.t. $f(\mathbf{x}) = P\mathbf{x}$ and

$$\forall i = 1, 2, \dots, m, \omega_n^{(i)} = f(\mathbf{x}_{n-1} + \omega_{n-1}^{(i)}) - f(\mathbf{x}_{n-1}) = P(\mathbf{x}_{n-1} + \omega_{n-1}^{(i)}) - P\mathbf{x}_{n-1} = P\omega_{n-1}^{(i)} = P^n\omega_0^{(i)}$$

So we have

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \left(\ln ||\omega_n^{(1)}||, \ln ||\omega_n^{(2)}||, \dots, \ln ||\omega_n^{(m)}||\right) = \lim_{n\to\infty} \frac{1}{n} \ln \left(||P^n\omega_0^{(1)}||, ||P^n\omega_0^{(2)}||, \dots, ||P^n\omega_0^{(m)}|| \right)$$

Define the vector norm function $\xi_p(X)$, where $X = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ s.t.

$$\xi_p(X) = (||\mathbf{x}_1||_p, ||\mathbf{x}_2||_p, \ldots, ||\mathbf{x}_n||_p)$$

is the $p-$norm of every vector in the matrix. Then we have

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left(\xi(P^n\Omega_0)\right)$$

We know that Ω_0 is a normal orthogonal basis, and we said the most useful and simple orthogonal basis is I, so here we let $\Omega_0 = I$ then

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left(\xi(P^n)\right)$$

And now, the problem is how to calculate this P^n, we know that if the eigenvector and eigenvalue of P is V and E, then $P^n = V^{-1}E^nV$, that means, if there is a eigenvalue of P absolute greater than 1, then some element in P^n will satisfy $n \to \infty, p_{i,j} \to \infty$. So now we will try to put the $1/n$ into the ξ function.

We know that in the begining of the discussion, Λ satisfied

$$\begin{aligned}
\Lambda = \lim_{n\to\infty} \frac{1}{n} \Bigg[&\ln\left(\frac{||\omega_n^{(1)}||}{||\omega_{n-1}^{(1)}||}\frac{||\omega_{n-1}^{(1)}||}{||\omega_{n-2}^{(1)}||}\cdots\frac{||\omega_1^{(1)}||}{||\omega_0^{(1)}||}\right), \ln\left(\frac{||\omega_n^{(2)}||}{||\omega_{n-1}^{(2)}||}\frac{||\omega_{n-1}^{(2)}||}{||\omega_{n-2}^{(2)}||}\cdots\frac{||\omega_1^{(2)}||}{||\omega_0^{(2)}||}\right), \ldots \\
&\ln\left(\frac{||\omega_n^{(m)}||}{||\omega_{n-1}^{(m)}||}\frac{||\omega_{n-1}^{(m)}||}{||\omega_{n-2}^{(m)}||}\cdots\frac{||\omega_1^{(m)}||}{||\omega_0^{(m)}||}\right), \Bigg] \\
= \lim_{n\to\infty} \frac{1}{n} &\left[\sum_{i=1}^{n} \ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \sum_{i=1}^{n} \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \ldots \sum_{i=1}^{n} \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right), \right. \\
= \lim_{n\to\infty} \frac{1}{n} &\left\{\sum_{i=1}^{n}\left[\ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \ldots, \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right)\right]\right\}
\end{aligned}$$

Notice for every $\omega_{i-1}^{(j)}, j = 1, 2, \ldots, m$ we can find a normal orthogonal basis of $\mathcal{R}^m$ space based on the **Gram Schmidt Processing**

Algorithm 1 Gram-Schmidt process in orthogonal decomposition

INPUT: A n-dimension Euclidean space, a basis $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ of the space.

procedure Gram-Schmidt process
  for $i = 1, i \leq n, i++$ **do**
    $\beta_i = -\sum_{j=1}^{i-1} \frac{<\beta_j,\alpha_i>}{<\beta_j,\beta_j>}\beta_j + \alpha_i$
    $\beta_i = \text{normalization}(\beta_i) = \frac{\beta_i}{||\beta_i||}$
  end for
  return Normal orthogonal basis $\{\beta\}$
end procedure

Let the basis of $\omega_{i-1}^{(j)}$ is $B_{i-1} = (\beta_{i-1}^{(1)}, \beta_{i-1}^{(2)}, \ldots \beta_{i-1}^{(m)})$, and $Q_{i-1} = (q_{i-1}^{(1)}, q_{i-1}^{(2)}), \ldots, q_{i-1}^{(m)}$ s.t.

$$\forall j = 1, 2, \ldots, m, \omega_{i-1}^{(j)} = B_{i-1} q_{i-1}^{(j)} \wedge ||q_{i-1}^{(j)}|| = 1$$

$$\left[\ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \ldots, \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right)\right]$$

$$= \left[\ln\left(\frac{||PB_{i-1}q_{i-1}^{(1)}||}{||B_{i-1}q_{i-1}^{(1)}||}\right), \ln\left(\frac{||PB_{i-1}q_{i-1}^{(2)}||}{||B_{i-1}q_{i-1}^{(2)}||}\right), \ldots, \ln\left(\frac{||PB_{i-1}q_{i-1}^{(m)}||}{||B_{i-1}q_{i-1}^{(m)}||}\right)\right] = \xi(P) \qquad (*)$$

And finally, we have

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \left\{ \sum_{i=1}^{n} [\xi(P)] \right\} = \xi(P)$$

[ii] f is non-linear map

Now we focus on the problem that f is not a linear map. Typically, we will use Jacobian matrix to linearize the problem and for a sequence x_i we have

$$\mathbf{x}_{i+1} = f(\mathbf{x}_i) = P_i \mathbf{x}_i$$

where P_i is Jacobian matrix near the point x_i, so we just need to change the P matrix to P_i matrix in non-linear problems. We found in formula $(*)$ we have

$$\left[\ln\left(\frac{||\omega_i^{(1)}||}{||\omega_{i-1}^{(1)}||}\right), \ln\left(\frac{||\omega_i^{(2)}||}{||\omega_{i-1}^{(2)}||}\right), \ldots, \ln\left(\frac{||\omega_i^{(m)}||}{||\omega_{i-1}^{(m)}||}\right)\right]$$

$$= \left[\ln\left(\frac{||P_{i-1}B_{i-1}q_{i-1}^{(1)}||}{||B_{i-1}q_{i-1}^{(1)}||}\right), \ln\left(\frac{||P_{i-1}B_{i-1}q_{i-1}^{(2)}||}{||B_{i-1}q_{i-1}^{(2)}||}\right), \ldots, \ln\left(\frac{||P_{i-1}B_{i-1}q_{i-1}^{(m)}||}{||B_{i-1}q_{i-1}^{(m)}||}\right)\right] = \xi(P_{i-1}) \qquad (*)$$

then, we have

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \left\{ \sum_{i=1}^{n} [\xi(P_{i-1})] \right\}$$

Finally, we can summary this processing in algorithm follows

Algorithm 2 Calculation of Lyapunov spectrum

INPUT: Values of system: $x_0, x_1, \ldots x_N$;

procedure GRAMSCHMIDT(matrix); ▷ Return the Gram-Schmidt orthogonal matrix.
end procedure

procedure JACOBIAN(value); ▷ Return the Jacobian matrix at input value
end procedure

procedure VECNORM(matrix); ▷ Return the norm of every vector in the matrix
end procedure

```
procedure LyaSpec
    float P = I;
    list LyaSpec;
    for i = 1, i ≤ N, i++ do
        P = Jacobian(x_i) · P
        LyaSpec = LyaSpec + (ln VecNorm(P));
        P = VecNorm(P)
    end for
    LyaSpec = LyaSpec/N
    return LyaSpec
end procedure
```

Now we will apply the method with several example.

E x a m p l e 5.1 ***Lyapunov spectrum in Henon's map***
Firstly we consider the Henon's map

$$\begin{cases} x = 1 - ax^2 + by \\ y = x \end{cases} \quad \textit{and the Jacobian matrix is } J(x, y) = \begin{bmatrix} -2ax & b \\ 1 & 0 \end{bmatrix}$$

Based on the Algo. 2, we can calculate the Lyapunov spectrum numerically. $a = 1.4, b = 0.3$
Lyapunov spectrum $= (0.42040807, -1.61305949)$, *sum* $= -1.1665324333361191$

Now we consider a continuous system s.t. $\dot{x} = f(x), x(0) = x_0$ where x_0 is a constant vector at initial time t_0. With Ronge-kutta method, we can find a group of value to simulate the system. So we can change the system to a map formed

$$x_{n+1} = g(x_n, t_n), x_0 = x(0), t_{n+1} = t_n + \Delta t$$

where g is based on the Ronge-kutta method.

Table 9: Result of Lyapunov spectrum in differen problem ($\Delta t = 0.0001$)

Model	λ_1	λ_2	λ_3	λ_4	$\sum \lambda_i$	Parameter
Logistic map	0.5769				0.5769	$a = 3.95$
Henon's map	0.4204	-1.6130			-1.1665	$(a, b) = 1.4, 0.3$
Duffing map	1.0660	-0.9661			0.0998	$(\alpha, \beta, \gamma, \delta, \omega) = (1, 0.04, 1, 0.1, \pi/2)$
Lorenz model	0.9038	0.0048	14.574		-13.666	$(\sigma, \rho, \beta) = (28, 10, 8/3)$
Lorenz model -2	0.1040	0.0028	-9.7590		-9.6520	$(\sigma, \rho, \beta) = (45.92, 4, 10)$
Rossler system	0.1040	0.0028	-9.7590		-9.6290	$(a, b, c) = (0.2, 0.3, 9)$
4-dim model	3.4475	3.5305	2.7098	10.7696	20.4574	/

Here we don't care about the formula g, we just consider for a certain n, if we still find a Jacobian matrix J_n, then $x_{n+1} = J_n x_n$.
On the other hand, we know that

$$\dot{x} = \frac{x(t_0 + (n+1)\Delta t) - x(t_0 + n\Delta t)}{\Delta t} = \frac{x_{n+1} - x_n}{\Delta t} = f(x_n)$$

In a certain model, we know the formula of f as well as parameter Δt. Let $\bar{J}(x)$ is Jacobian matrix of $f(x)$, then

$$x_{n+1} - x_n = \Delta t \bar{J}(x_n) x_n = (J_n - I_n) x_n \Rightarrow J_n = \Delta t \bar{J}(x_n) + I_n$$

Example 5.2 ***Lyapunov spectrum in Lorenz system***
Now we consider the Lorenz system:

$$\begin{cases} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{cases} \quad \textit{and the Jacobian matrix of } f \textit{ is } \bar{J}(x, y, z) = \begin{bmatrix} -\sigma & \sigma & 0 \\ (\rho - z) & -1 & -x \\ y & x & -\beta \end{bmatrix}$$

So the Jacobian matrix of the discrete maps is

$$J(x, y, z) = \Delta t \bar{J}(x, y, z) + I = \begin{bmatrix} 1 - \sigma\Delta t & \sigma\Delta t & 0 \\ (\rho - z)\Delta t & 1 - \Delta t & -x\Delta t \\ y\Delta t & x\Delta t & 1 - \beta\Delta t \end{bmatrix}$$

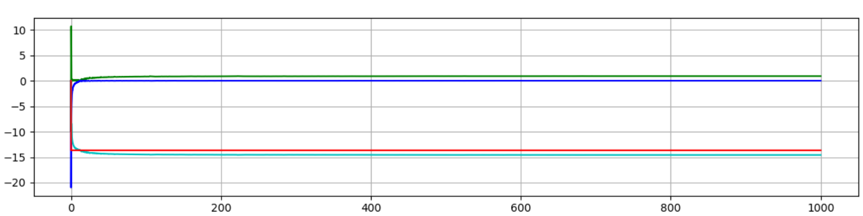


Figure 35: Lyapunov spectrum in Lorenz system($(\sigma, \rho, \beta, \Delta t) = (28, 10, 8/3, 0.0001)$)

$(\sigma, \rho, \beta, \Delta t) = (28, 10, 8/3, 0.0001)$
Lyapunov spectrum $= (0.903833632, 0.00483117315 - 14.5749841)$, *sum* $= -13.666319315391025$
$(\sigma, \rho, \beta, \Delta t) = (45.92, 4, 10, 0.0001)$
Lyapunov spectrum $= (0.104081796, 0.00285865072 - 9.75901813)$, *sum* $= -9.652077679944226$

E x a m p l e 5.3 ***Lyapunov spectrum in Rossler system***
Now we consider the Rossler system:

$$\begin{cases} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x-c) \end{cases} \quad \text{and the Jacobian matrix of } f \text{ is } \bar{J}(x,y,z) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{bmatrix}$$

So the Jacobian matrix of the discrete maps is

$$J(x,y,z) = \Delta t \bar{J}(x,y,z) + I = \begin{bmatrix} 1 & -\Delta t & -\Delta t \\ \Delta t & a\Delta t + 1 & 0 \\ z\Delta t & 0 & (x-c)\Delta t + 1 \end{bmatrix}$$

$(a, b, c, \Delta t) = (0.2, 0.3, 9, 0.001)$
Lyapunov spectrum $= (0.104081796, 0.002.85865072, -9.75901813)$, *sum* $= -9.629091584454333$

E x a m p l e 5.4 ***Lyapunov spectrum in Duffing system***
Now we consider the Duffing equation:

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t)$$

Let $y = \dot{x}$*, then*

$$\dot{y} = \ddot{x} = \gamma\cos(\omega t) - \delta\dot{x} - \alpha x - \beta x^3 = \gamma\cos(\omega t) - \delta y - \alpha x - \beta x^3$$

So we have

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= \gamma\cos(\omega t) - \alpha x - \beta x^3 - \delta y \end{cases} \quad \text{and the Jacobian matrix of } f \text{ is } \bar{J}(x,y) = \begin{bmatrix} 0 & 1 \\ -\alpha - 3\beta x^2 & \delta \end{bmatrix}$$

So the Jacobian matrix of the discrete maps is

$$J(x,y,z) = \Delta t \bar{J}(x,y) + I = \begin{bmatrix} 1 & \Delta t \\ (-\alpha - 3\beta x^2)\Delta t & \delta\Delta t + 1 \end{bmatrix}$$

Lyapunov spectrum $= (1.06601852, -0.96612192)$, *sum* $= 0.09989660859631133$

E x a m p l e 5.5 ***Lyapunov spectrum in a 4-dim system***
In the last of this part, we consider a 4-dim dynamic system:

$$\begin{cases} \dot{x} &= -y - z \\ \dot{y} &= x + 0.25y + w \\ \dot{z} &= 3 + xz \\ \dot{w} &= -0.5z + 0.05w \end{cases} \quad \text{and the Jacobian matrix of } f \text{ is } \bar{J}(x,y,z,w) = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ z & 0 & x & 0 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix}$$

So the Jacobian matrix of the discrete maps is

$$J(x,y,z,w) = \Delta t \bar{J}(x,y,z,w) + I = \begin{bmatrix} 1 & -\Delta t & -\Delta t & 0 \\ \Delta t & 1 + 0.25\Delta t & 0 & \Delta t \\ z\Delta t & 0 & 1 + x\Delta t & 0 \\ 0 & 0 & -0.5\Delta t & 1 + 0.05\Delta t \end{bmatrix}$$

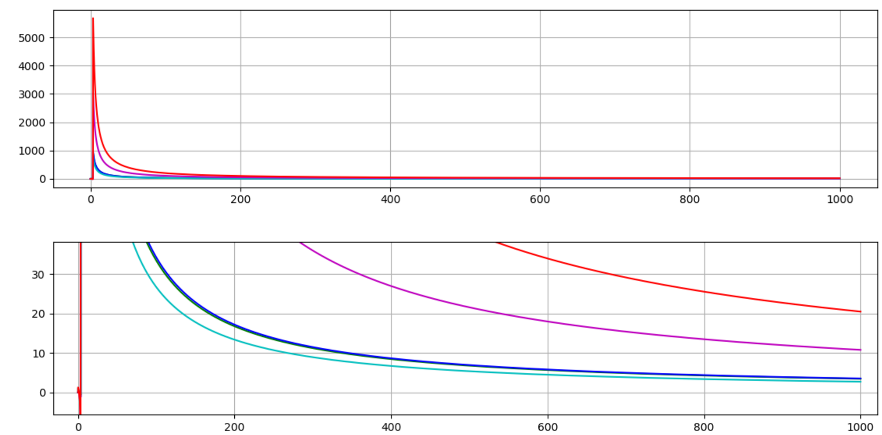


Figure 36: Lyapunov spectrum in this 4-dim equation

$$Lyapunov\ spectrum = (3.44750533, 3.53053891, 2.70980462, 10.76963212), sum = 20.457480973829277$$

5.2 Fixed-point theorem in high dimension

We introduced the fixed point theorem in section, we found that if an inital interval I_0 s.t.

$$I_{n+1} = f(I_n) \wedge I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$$

then, based on the Nested interval theorem, exist at least one point x_0 s.t. $\forall i \in \mathcal{N}, x_0 \in I_i$ which is fixed point.

However, this condition is too strict, if we consider another group of set, for instance, with the relations in the figure follow.

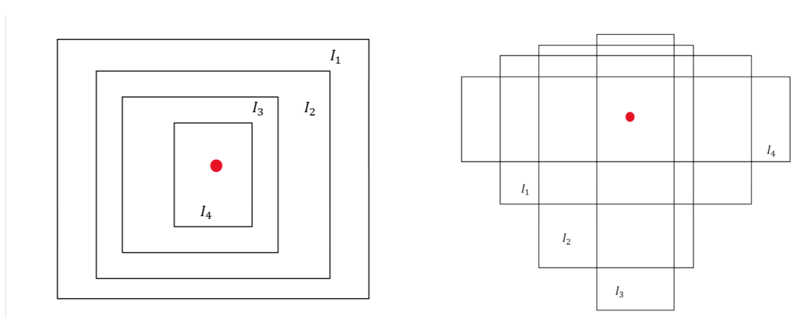


Figure 37: Colorado corollary

Then we can found a fixed point in the center of every interval. And we called this kind of map as **Colorado corollary**

Now the problem is how to describe this property that every $f(I_n)$ covered some part of the I_n. If $\mathbf{x}$ is a boundary point on I_n, then $f(\mathbf{x})$ also be a boundary point of $f(I_n)$. If we consider this kind of vector $V(\mathbf{x}) = \frac{f(\mathbf{x})-\mathbf{x}}{||f(\mathbf{x})-\mathbf{x}||}$ which is the direction of these boundary vector, then we can easily found that the vectors will travel through a cumulative $k2\pi$ where $k \in \mathcal{N}$

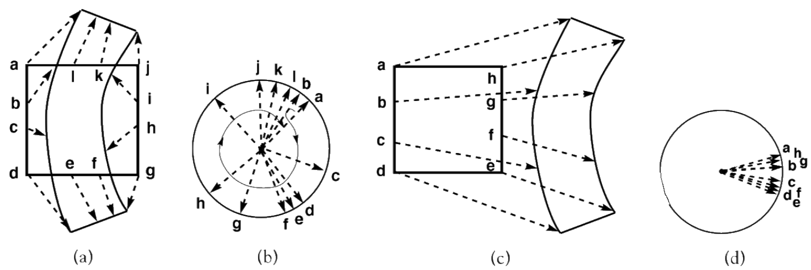


Figure 38: Proof of fixed point theorem

Theorem 5.1 *Let f be a continuous map on R^2, S is a ractangular region and ∂S is the boundary of S. Now consider an point $\mathbf{x} = \mathbf{x}_0$ on ∂S, then we have the vector $V(\mathbf{x}) = \frac{f(\mathbf{x})-\mathbf{x}}{||f(\mathbf{x})-\mathbf{x}||}$. Move the point with the boundary continuous and sign every vector $V(\mathbf{x})$ until the point back to the $\mathbf{x}_0$ and sum the total rotation of these vector sequence, if it is nonzero, then f has a fixed point in S.*

***PROOF* 5.1** *[1] If center c of rectangle is fixed point, then there is nothing to do.*
[2] If center c is not fixed point, we can shrink the rectangle down from its original size tothe point c.
[2-1] Obviously, the vector $V(\mathbf{x})$ defined along the ∂S and change continuously, and they must continue to make at least one full trun. So finally, some point $\mathbf{x}$ results $V(\mathbf{x}) = 0$ and $\mathbf{x}$ is fixed point because the definiton of $V(x)$
[3] Now we consider the situation that the image move completely away from it. Obviously, there is no fixed point if the image move away. And in this condition, we can simply find net rorate of $V()$ is zero.■

CONCLUSION* 5.1** *The fixed point theorem is a property of the topology of the map alone, and doesn't depend on starting with a prefect rectangle, as long as region has **no holes

Now we back to consider the Colorado corollary. In some situation the region not only expand from the origin interval, but also fold from the original region. For instance in the image below, S_T, S_B fold in the subimage 3. We found this kind of map have no influence of the conclusion we introduced before.

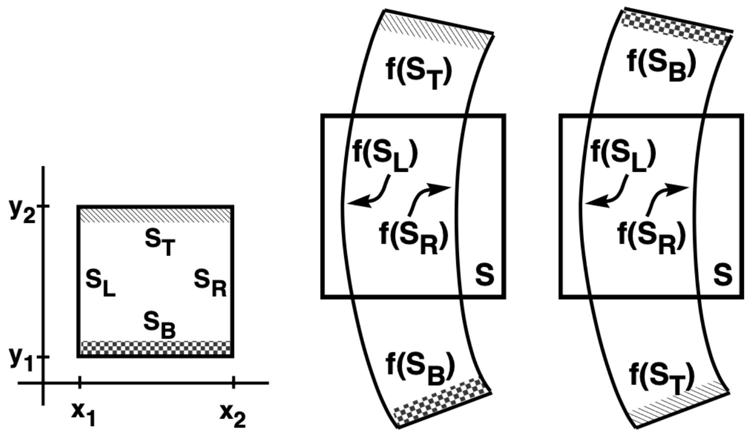


Figure 39: Colorado corollary after map

CONCLUSION* 5.2 *The Colorado Corollary
Let $f \in C(\mathcal{R}^2)$ is a map, S be a rectangle in $\mathbf{R}^2$, with vertical sides s_L, s_R and horizontal side s_T, s_B. Assume that $f(s_L), f(s_R)$ are surrounded by s_L, s_R and $f(s_T), f(s_B)$ are surrounded by s_T, s_B, then f has a fixed point in S

Finally we consider a kind of map which is "lying across" from the original region.

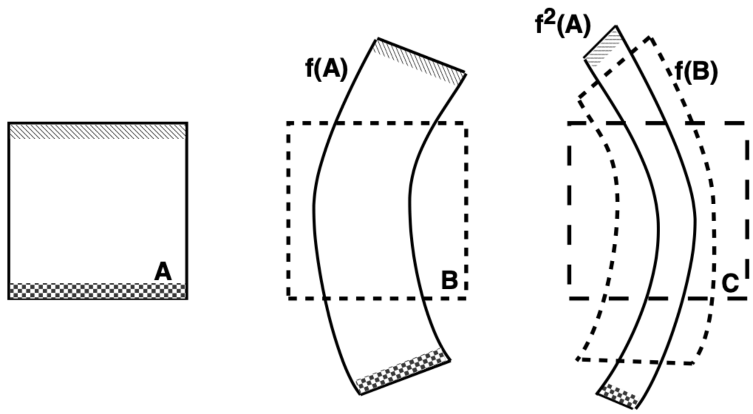


Figure 40: Transitivity of "lying across"

Obviously, that is a special condition of fixed point theorem and we have

***CONCLUSION* 5.3** *Let f be a map and $S_1, S_2, \dots S_k$ be rectangular sets in $\mathcal{R}^2$ s.t. $f(S_i)$ lies across the S_{i+1} and S_1, then f^k has a fixed point in S_1*

In the last of this section, we will dicsuss the definition of Markov partitions

Definition 5.3 ***Markov Partitions***
Assume S is a rectangle set and $S_1, S_2, \dots, S_r$ s.t.

$$\forall i, j = 1, 2, \dots, r, i \neq j, (S_i \subset S) \wedge \left(S_i \bigcap S_j = \varnothing\right) \wedge (\partial S_i \text{ parallel to the coordinate axes })$$

and $f(S_i)$ lying across S_i, that means, in one direction of axis, f stretches the rectangles and in the other directions the f contract the rectangles.
Then, the stretching directions are mapped to stretching directions, and shrinking directions to shrinking directions. Then we called this $S_1, S_2, \dots, S_r$ as ***Markov partition*** *of S for f.*

Based on the Markov partition, we can define the itinerary in high dimension problems and find the fixed point easier. Here we will discuss two example, one is **skinny baker map**, and the other is Henon's map, or the **Horseshoe map**.

5.3 Example: Skinny baker map

The skinny baker map is

$$B(x_1, x_2) = \begin{cases} \begin{pmatrix} a & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & x_2 \in [0, 1/2] \\ \begin{pmatrix} a & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b \\ -1 \end{pmatrix} & x_2 \in (1/2, 1] \end{cases}$$

If we consider the itinerary of the map, let $x_1 < 1/2$ as **L** and $x_1 > 1/2$ as **R**($\forall x_1, x_2 \in [0, 1]$, point (x_1, x_2))then

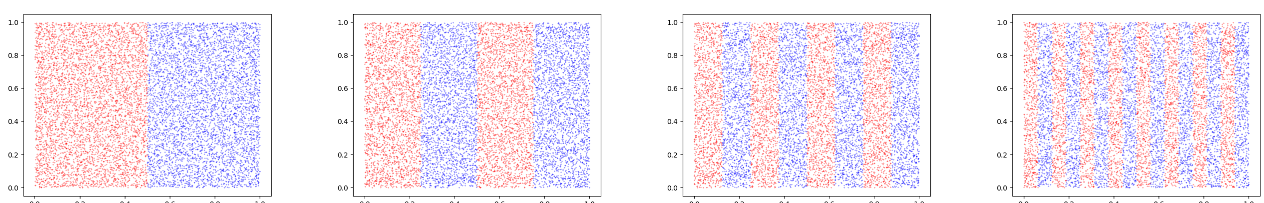


Figure 41: Baker map($a = 0.5, b = 0.5$)

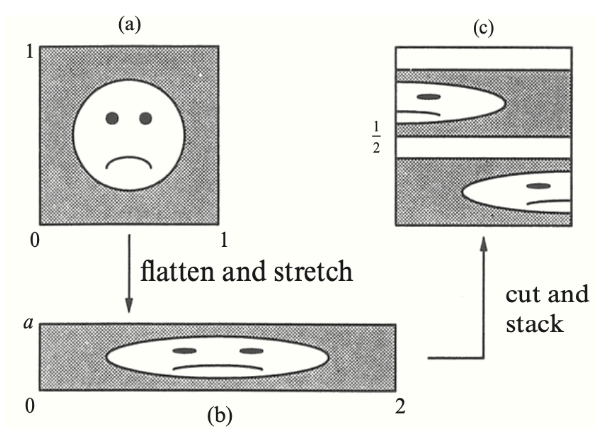


Figure 42: Baker map iteration

We found that the map is similar as baker. However, if we change the parameter, the map will show a little different from baker.

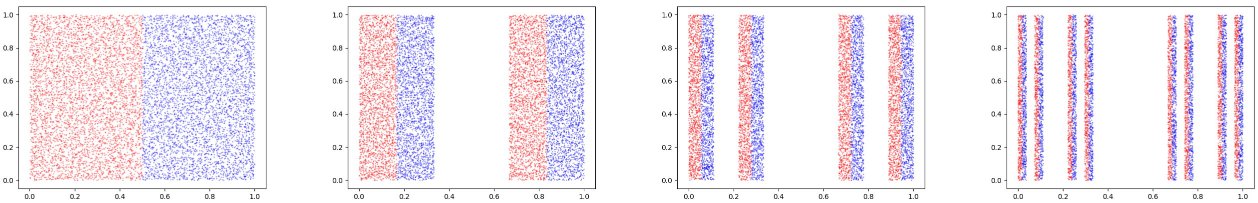


Figure 43: Baker map($a = 1/3, b = 2/3$)

And we now discuss the change of b.

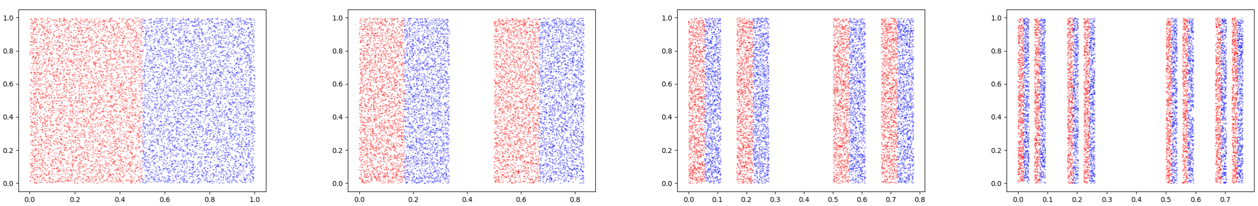


Figure 44: Baker map($a = 1/3, b = 0.5$)

Note that we will consider $a = 1/3, b = 2/3$ in the discussion below. Obviously the Lyapunov exponent are $-\ln(3), \ln(2)$, and now will will try to proof the map will never asymptotically periodic with itineraries.

We can easily found that the itineraries are “bi-infinite”, that means, they are defined for $-\infty < i < \infty$ and for every point (x_1, x_2) it is the result after infinity iteration and the begining of next infinity iteration.

$$\text{itineraries: } \dots S_{-2}S_{-1}S_0S_1S_3$$

Obviously if we consider the problem in $x_1 \leq 0.5$ and $x_1 > 0.5$, then this two subset combined Markov partition.

To analysis this problem, first, we consider a special map.

Definition 5.4 ***Shift map***
Consider a map s, if

$$s(\dots S_{-2}S_{-1}S_0S_1S_3\dots) = \dots S_{-2}S_{-1}S_0S_1S_3\dots$$

then we called this map as shift map.

So the orbit is asymptotically periodic if and only if the itinerary is eventually periodic toward right. Any itinerary that is not periodic toward right is not asymptotically periodic. So finally we have

***CONCLUSION* 5.4** *The skinny baker map has chaotic orbits.*

5.4 Example: Horseshoe map

Before we consider the horseshoe map, we consider the Henon's map firstly and explain why are we interested in horseshoe map.

Consider a rectangle region same as we discussed in the fixed point theorem above,

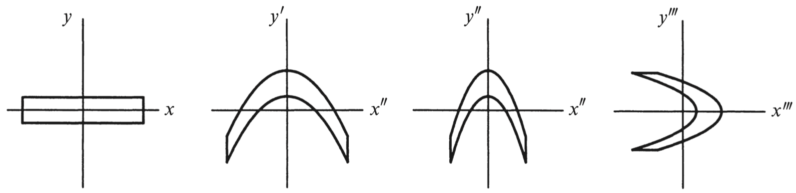


Figure 45: Henon's map, transform to horseshoe map

$$T_1(x,y) = \begin{cases} x^{(1)} = x \\ y^{(1)} = 1 + y - ax^2 \end{cases} \qquad T_2(x,y) = \begin{cases} x^{(2)} = bx \\ y^{(2)} = y \end{cases} \qquad T_3(x,y) = \begin{cases} x^{(3)} = y \\ y^{(3)} = x \end{cases}$$

Obviously, we found that the Henons map $H(x,y) = T_3(T_2(T_1(x,y)))$ which transformed a rectangle region to a Horseshoe region. We called this kind of map as Horseshoe map. (Fig. 46, (1))

The **horseshoe map** h is a continuous one-to-one map on R^2 s.t.

[i] Map the square $W = ABCD$ to the overlapping horseshoe image, $h(A) = A^*, h(B) = B^*, h(C) = C^*, h(D) = D^*$.(Fig. 46, (3))

[ii] In the W, the map uniformly contracts distances horizontally and expands distances vertically.

[iii] Points are stretched vertically by a factor 4 and squeezed horizontally by 4.

[iv] Outside W we only retriction we make on h hs that it be contiuous and one-to-one.dd

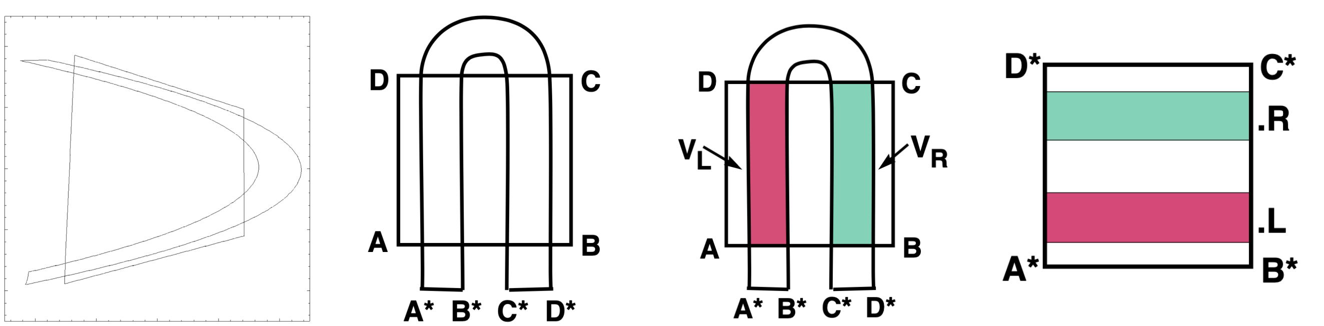


Figure 46: Horseshoe map

Here we consider the points set H remains every point of W with forward and backware iterates. For every point $x \in H$, it must be in either left leg V_L or right leg V_R, define the itinerary

[i] if $h^i(x)$ lies in V_L, then set $S_i = L$

[ii] if $h^i(x)$ lies in V_R, then set $S_i = R$

where S_i is itineraries, $i \in Z$ and the Itinerary is $\ldots S_{-2}S_{-1}S_0S_1S_2\ldots$

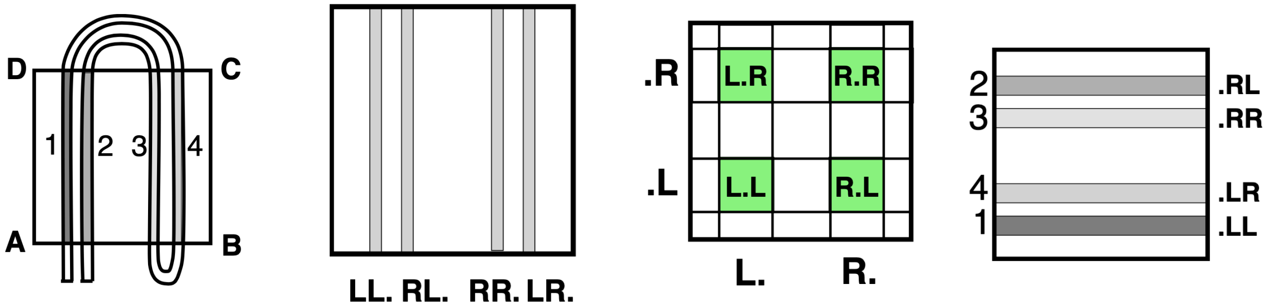


Figure 47: Horseshoe map - 2

We know that itinerary is a tool to consider a map is asymptotically periodic or not, in horseshoe map, we can easily found that the Lyapunov exponent is $\ln 3$ and $-\ln 4$, and same as baker map, the horseshoe will never asymptotically periodic. So finally we have the conclusion below.

***CONCLUSION* 5.5** *The horseshoe map has chaotic orbits.*

6 Fractal dimension

In real analysis, if we consider the size of a subspace, we basically use **measure** to describe that.

Definition 6.1 ***Eculidean rectangle Neighbourhood***
1. Opened box $I = \{x = (x_1, x_2, \ldots, x_n) | a_i < x_i < b_i, i \in N\}$, where a_i, b_i are constant;
2. Closed box $I = \{x = (x_1, x_2, \ldots, x_n) | a_i \leq x_i \leq b_i, i \in N\}$, where a_i, b_i are constant;
Called the $b_i - a_i$ is side length of a box. If $|b_1 - a_1| = |b_2 - a_2| = \ldots = |b_n - a_n|$, then called this box as cube.

Definition 6.2 ***Lebesuge outer measure***

$$\mu^* E = \inf \left\{ \sum_{i=1}^{\infty} |I_i| : E \subset \bigcup_{i=1}^{\infty} I_i, I_i \textit{ is an opened box} \right\}$$

Also, we can define the outer measure with ε (Familiar with Cauchy's limitation)

$$\forall \varepsilon > 0, \exists \{I_i\} \textit{ open cover s.t.} E \subset \bigcup_{i=1}^{\infty} I_i \wedge \mu^* E \leq \sum_{i=1}^{\infty} |I_i| + \varepsilon$$

Now, we consider a cube with side length 1, and neigibourhood cube with side length ε, the total of the neigibourhood include in the unix cube as $V = N(\varepsilon)$, the dimension of subspace as k, then

$$V = N(\varepsilon) = \left(\frac{1}{\varepsilon}\right)^k \Rightarrow k = \log_{1/\varepsilon} N(\varepsilon) = -\frac{\ln N(\varepsilon)}{\ln(\varepsilon)}$$

In normal problem, $k \in \mathcal{N}^+$ called **Lebesgue covering dimension** or **topological dimension**.

Definition 6.3 ***Lebesgue covering dimension***
A topological space X is said to have the Lebesgue covering dimension $d < \infty$ if d is the smallest ***non-negative integer*** *with the property that each open cover of X has a refinement in which no point of X is included in more than $d + 1$ elements.*

Definition 6.4 $C(1/\varepsilon)$ ***cube***
Now we consider a special type of cover.
[i] *Firstly, for a interval $I_1 = [a_1, b_1]$, let the side length of boxes is ε, then we have $N = int(C/\varepsilon) + 1$ subintervals to cover the original interval. (e.g. $[a_1 + (p-1)C/\varepsilon, a_1 + pC/\varepsilon], p = 1, 2, \ldots N$)*
[ii] *Now we consider a surface in R^n space, let the projection on x_i axis is $I_i = [a_i, b_i]$, and $C = \max\{\mu(I_i)\}, i = 1, 2, \ldots, n$, then we have a group of cube total $N(\varepsilon) = (C\varepsilon)^n$ and*

$$d = \frac{\ln N(\varepsilon) - \ln C}{\ln(1/\varepsilon)}$$

Definition 6.5 $C(1/\varepsilon)$ ***cube***
A bounded set $s \subset R^n$ has box-counting dimension

$$bd(S) = \lim_{\varepsilon \to \infty} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)}$$

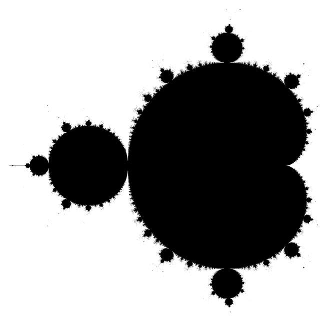
when the limit exists.

***CONCLUSION* 6.1** *Based on the Bolzano–Weierstrass theorem, we know that if we have a sequence* $\{b_n\}$ *s.t.* $\lim_{n\to\infty} b_n = 0$*, if* b_n *is* ε *then* $\lim_{n\to\infty} \frac{\ln b_{n+1}}{\ln b_n} = 1$

***CONCLUSION* 6.2** *We still consider the sequence above, then*

$$\frac{N(b_n)}{4} \leq N(b_n) \leq 4N(b_{n+1})$$

and the proof is simple with figure follows



Theorem 6.1 *If* $\{b_n\}$ *is monotony, or assume* $b_1 > b_2 > \ldots > b_n > \ldots > b_\infty = 0$ *If*

$$\lim_{n\to\infty}\left(\frac{\ln b_{n+1}}{\ln b_n}\right) = 1 \wedge \lim_{n->\infty}\left(\frac{\ln N(b_n)}{\ln(1/b_n)}\right) = d =$$

then $\lim_{\varepsilon\to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = d$ *and therefore the box-counting dimension is* d*.*

Theorem 6.2 *If* $S \subset R^n$ *is bounded and* $bd(S) = d < n$*, then* $\mu(S) = 0$

Another way to calculate the measure of a area is based on the statistic,

Definition 6.6 ***Correlation dimension*** *Let* $S = \{v_0, v_1, \ldots\}$ *be an orbit of the map* $f : \mathcal{R}^n \to \mathcal{R}^n$*, then for ever* $r > 0$*, define* $C(r)$ *s.t.*

$$C(r) = \lim_{N\to\infty} \frac{|\{(p,q)|p,q \in S, |p-q| < r\}|}{|\{(p,q)|p,q \in S\}|}$$

Moreover, if $C(r) \approx r^d$*, then*

$$d \approx cd(S) = \lim_{r\to\infty} \frac{\ln C(r)}{\ln(r)}$$

And here we found another way to calculation the area of attractor.

In the last section, we introduced the Lyapunov spectrum. However, if we want to analysis the whole system, the Lyapunov exponent is not a good choice because it described the property in every different direction.

So to describe the property of a set, or a space, the basic element is the definition of open set. Now we consider a box neigibourhood rather than the circle one, signed as I_0 with certain side length ω_0 and the volumn $V_0 = \omega_0^m$. Then let the set $I_1 = f(I_0), I_2 = f(I_1)\dots$ and we now try to consider the volumn of these sets. If the Lyapunov spectrum of system is $(h_1, h_2, \dots, h_m)$, then

$$||\omega_n^{(j)}|| = \exp(nh_j)||\omega_0^{(j)}||$$

and the volume of box neigibourhood is

$$V_i = \prod_{j=1}^{m} ||\omega_i^{(j)}|| = \prod_{j=1}^{m} \exp(ih_j)||\omega_0^{(j)}||$$

$$V_n = \prod_{j=0}^{m} \exp(nh_j)||\omega_0^{(j)}||$$

But that is not the all conclusion, as some of Lyapunov exponent is less than 0, the side length in that direction will decrease and decrease and finally equal to 0. And that is unnecessary to consider such direction.

And now, the problem is "Which dimension is enough to include the $f^\infty(I_0)$"

We know that if the dimension is 3 as normal world, then the volumn V and side length d have relationship as

$$V = d^3 \Rightarrow 3 = \log_d V = \frac{\ln V}{\ln d}$$

So if we assume the dimension of space is $k \le m$, then

$$k = \frac{\ln V_\infty}{\ln d_0} \qquad (*)$$

[i] In 2-dim problem Now we consider a 2-dim problem.

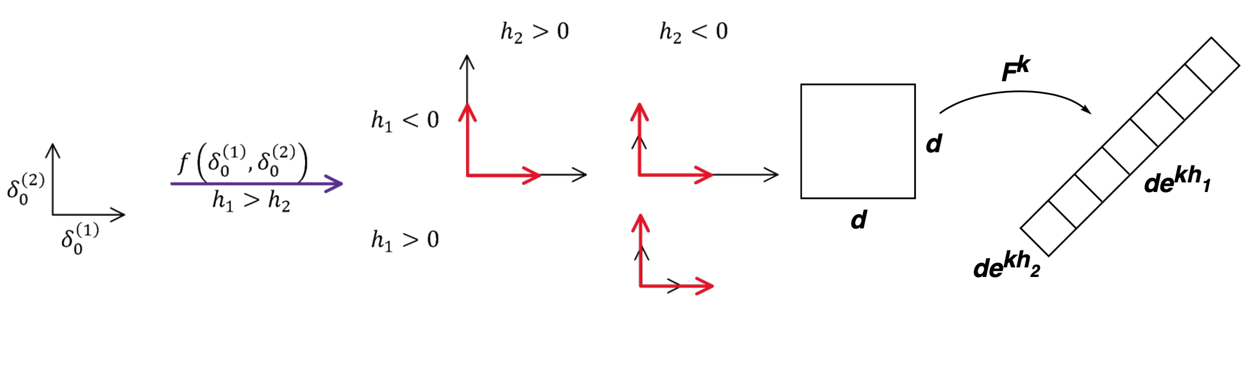


Figure 48: Volumn and vector change with Lyapunov exponent

Same as the figure above, if $h_1 > h_2 > 0$, then, obviously, we need a 2-dim neigibourhood to cover all the system after iteration. On the other hand, if $0 > h_1 > h_2$, then we just need a point to cover it. So the main problem is how can we cover the situation $h_1 > 0 > h_2$.

$$V_n = \omega_0^{(1)} \exp(nh_1) \omega_0^{(2)} \exp(nh_2) = d^2 \left(\exp(h_1 + h_2)\right)^n$$

So if $h_1 + h_2 < 0$, the volumn will still decrese to 0, that means the dimension of V_∞ is 0. And now we consider the condition $h_1 > 0 > h_2 \wedge h_1 + h_2 > 0$

Definition 6.7 ***Lyapunov dimension***
Let f be a map on $\mathcal{R}^m$, the Lyapunov exponent of an orbit is $h_1 \geq h_2 \geq \ldots \geq h_m$, let p

$p = \arg\max_p \left(\sum_{i=1}^{p} h_i \geq 0 \right)$ *then, the Lyapunov dimension is* $D_L = \begin{cases} 0 & \textit{if } p \textit{ is not exists} \\ p + \frac{1}{|h_{p+1}|} \sum_{i=1}^{p} h_i & \textit{if } p < m \\ m & \textit{if } p = m \end{cases}$

D I S C U S S I O N* 6.1** ***What can Lyapunov dimension describe? *We just introduced the definition of Lyapunov dimension above, and here, we will explain why we are interested in Lyapunov dimension.*