[Note] Chaos: An Introduction to Dynamical Systems

January 28, 2021

Problem in discrete-time system

1 One-Dimension Maps

Definition 1.1 ***n-order differentiable function, Smooth function, Map***
Consider an open set E and $n \in \mathcal{N}$, called

$$C^n(E) = \{f \in C(E) | \forall \alpha \ s.t. \ |\alpha| \leq n, D^\alpha f \in C(E)\}$$

is n-order differentiable function set of E, where $C(E)$ is continuous function on E.
If f on domian E have infinity-order derivative, or $f \in C^\infty(E)$, then called f ***smooth function****.*
If the function f have same domain and range, then called f is a ***map****.*

The fucntion in this book will be a smooth function if we not emphasize.

Definition 1.2 ***Orbit, initial value, fixed point***
Consider a map $f : X \to X$, x is a point in X then
Called ***orbit*** *of x is a set of point*

$$Orbit(X) = \{x, f(x), f^2(x), \ldots f^n(x), \ldots\}$$

, where $f^n(x) = f(f(\ldots f(x))) = (f \circ f \circ f \circ \ldots \circ f)(x)$.
The starting point of x for a orbit called the ***initial value****.*
If the point p s.t. $f(p) = p$, then called p as fixed point.

OK, and now we consider two dynamical systems, with a input x, the system will always return to $f(x) = 2x$ and $g(x) = 2x(1-x)$. And then the output will become the input value and etc. During this looping, it is simple to find the orbit of a certain initial value.

Table 1: Comparison of exponential growth and logistic growth

f	init	1	2	3	4	5	6	7	8	9
$f(x)$	0.01	0.02	0.04	0.08	0.16	0.32	0.64	1.28	2.56	5.12
$f(x)$	0.01	0.0198	0.0388	0.0746	0.138	0.268	0.362	0.462	0.497	0.499
$g(x)$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$g(x)$	0.8	0.32	0.435	0.492	0.499	0.5	0.5	0.5	0.5	0.5
$g(x)$	1.2	-0.48	-1.42	-6.87	-108.4	-23716.9	-1125030476	-inf	-inf	-inf

We found that in the model of f, the result is growth as exponential function and we called that exponential growth. Also, when intial value $x \in [0, 1]$, with iteration, the result have limition of 0.5 and we called these model as logistic growth.

In this section, we will mainly focus on these kind of dynamical system, obviously, the iteration processing of model are discrete, we also called these dynamical system models as **maps**.

1.1 Cobweb plot, stability

To analysis a maps, the basic method is based on cobweb plot. Fig **??** showed a method to analysis a dynamical system with a certain iteration principle.

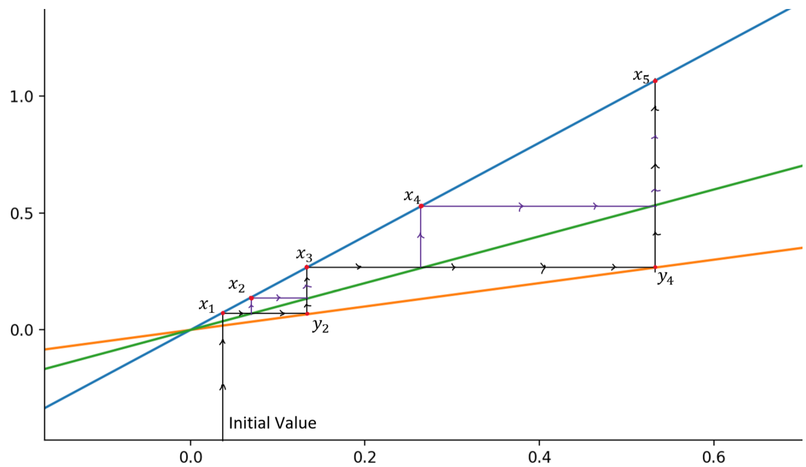


Figure 1: An example of cobweb plot and basic principle

In every iteration, the independent and dependent variable exchanged their location and we can found a group of $\{x_1, y_2, x_3, y_4, \ldots\}$ as orbit of initial value. Or, with the symmetric line $y = x$, it is simple to symmetric all black line to purple line and we can build a cobweb plot with origin image and $y = x$ to find a group of $\{x_1, x_2, \ldots\}$ as orbit from the initial value.

Before we discuss the different of fixed point, it is necessary to review some basic definitions.

Definition 1.3 ε ***Neighbourhood***
In a metric space X, an ε neighbourhood $N_\varepsilon(p)$ of point p is defined

$$N_\varepsilon(p) = \{x \in X | d(x, p) < \varepsilon\}$$

where $d(x, p)$ is the distance bewteen point p and x. Also, in a R^1 space, the ε Neighbourhood is give by

$$N_\varepsilon(p) = \{x \in R || x - p| < \varepsilon\}$$

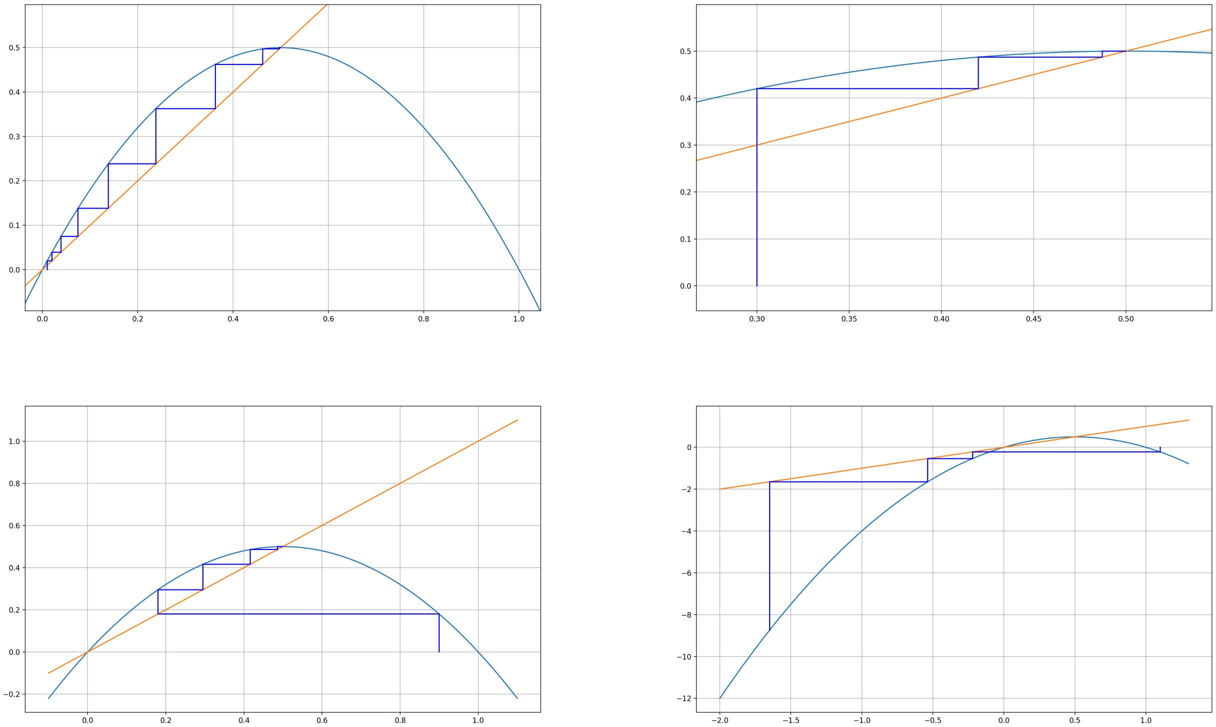


Figure 2: Cobweb plot in different initial value

It is simple to find that for all initial value $x \in (0,1)$, with iteration, the output have limitation in 0.5. On the other hand, to solve the equation $x = 2x(1-x)$ we found $x_1 = 0, x_2 = 1, x_3 = 0.5$ as three fixed point. So we have two kinds of fixed point, the one is limitation point and the other is not.

Definition 1.4 ***Sink, Source (Attracting and Repelling Fixed Point)***
Consider a map $f : R \to R$ and point p s.t. $f(p) = p$, then
*If for evert points sufficiently to p are attracted to p, then called p as **sink**, or **attracting fixed point**. Or*

$$\text{For an } \varepsilon > 0, \forall x \in N_\varepsilon(p), \lim_{k\to\infty} f^k(x) = p \text{ then called } p \text{ as } \textbf{sink}.$$

*If for every points sufficiently to p are repelled to p, then called p as **source**, or **repelling fixed point**.*

$$\text{For an } \varepsilon > 0, \forall x \in N_\varepsilon(p), x \neq p, \lim_{k\to\infty} f^k(x) \notin N_\varepsilon(x) \text{ then called } p \text{ as } \textbf{source}.$$

Theorem 1.1 *Let f is a map on R, assume p is a fixed point of f, then*
[i] If $|f'(p)| < 1$, then p is a sink;
[ii] If $|f'(p)| > 1$, then p is a source.

PROOF* 1.1** ***[i] *Based on definition of derivative, we have*

$$\lim_{x \to p} \frac{|f(x) - f(p)|}{|x - p|} = |f'(p)|$$

Now. let $a \in (\min(|f'(p)|, 1), \max(|f'(p)|, 1))$ (e.g. $a = \frac{1}{2}(1 + |f'(p)|)$), then

$$\forall a \in (\min(|f'(p)|, 1), \max(|f'(p)|, 1)), \exists \varepsilon_0 > 0 \ s.t. \ \forall \varepsilon \in (0, \varepsilon_0], \forall x \in N_\varepsilon(p), \frac{|f(x) - f(p)|}{|x - p|} < a$$

That means, $f(x)$ is closer to p than x (or distant bewteen curve $y = f(x)$ and $y = x$), but at least a factor of a and we have the conclusion

$$\forall x \in N_\varepsilon(p), f(x) \in N_\varepsilon(p)$$

During the iteration processing it is simple to find that all orbit $\{f(x), f^2(x), \ldots f^n(x), \ldots\} \subset N_\varepsilon(p)$, so now we can consider another conclusion in follow.

[ii] *We try to prove the inequality $\forall x \in N_\varepsilon(p), |f^k(x) - p|$*
[ii-1] *Obvious, if $k = 1$, then $|f(x) - p| = |f(x) - f(p)| < a|x - p|$ (p is fixed point so $f(p) = p$)*
[ii-2] *If $k = 2$, Based on the conclusion in **[i]**, $x_1 = f(x) \in N_\varepsilon(p)$ and $|f(x_1) - p| < a|x_1 - p| < a^2|x - p|$*
...
[ii-k+1] *(Assume the inequality is established in k), then*

$$|f^{k+1}(x) - p| < a|f^k(x) - p| < a \cdot a^k|x - p| = a^{k+1}|x - p|$$

In summary, for all $k \in N$, the inequality is established.

[iii-1] *Now we consider the equality condition, if $|f'(p)| < 1$ then $a < 1$ and*

$$\lim_{k \to \infty} |f^k(x) - p| < |x - p| \lim_{k \to \infty} a^k = 0 \ \text{(Because } a \in (0, 1)\text{)}$$

So we have the conclusion,

$$\forall x \in N_\varepsilon(p), \lim_{k \to \infty} f^k(x) = p$$

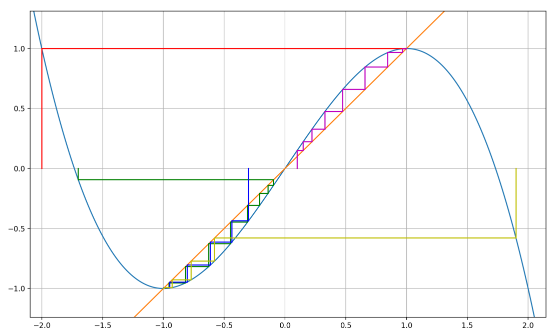
[iii-2] *Also, if $|f'(p)| > 1$ then $a^k \to \infty$, that means, with the iteration, the maps will eventually outside the condition, or the domain interval.* ■

* We will discuss what happened while $f'(p) = 1$ laterly.

** Obviously, this theorem expressed a kind of convergence, as the speed of the convergence is based on the a in exponent function, we called this convergence as **Exponential Convergence**.

Now we consider another map as example.

E x a m p l e 1.1 *Solved the fixed point of $\varphi(x) = (3x - x^2)/2$, find every sink and source point with Theo.* **??**



***SOLUTION* 1.1** *It is simple to find the fixed point with $x = (3x - x^3)/2$ and $x_1 = 1, x_2 = 0, x_3 = -1$. Based on the image, we can found that 1 and -1 are sink and 0 is source. On the other hand*

$$\varphi'(x) = \frac{3}{2}(1 - x^2), \varphi'(-1) = 0 < 1, \varphi'(0) = \frac{3}{2} > 1, \varphi'(1) = 0 < 1$$

and we proved the conclusion we found on figure before. ■

Another way to confirm a point is sink or source is based on the formula identity and algebra. For instance, we consider the distance bewteen $g(x) = 2x(1-x)$ and fixed point $1/2$, then

$$|g(x) - 1/2| = |2x(1-x) - 1/2| = 2|x - 1/2||x - 1/2|$$

and $\forall x \in (0,1), |x - 1/2| < 1 \Rightarrow |g(x) - 1/2| < 1$, that means the distance bewteen $g(x)$ and p is decreasing during time iteration and we can confirm that $1/2$ is a sink point rather than source point.

Next, we will focus on a logistic model with different parameter.

1.2 Periodic points, family of logistic maps

E x a m p l e 1.2 *Find the fixed point of $g(x) = 3.3x(1-x), x \in [0,1]$.*

***SOLUTION* 1.2** *It is simple to find the fixed point with $x = 3.3x(1-x)$ and $x_1 = 0, x_2 = 23/33, x_3 = 1$. Obviously, both 0 and 1 are source. And*

$$g'(x) = 3.3 - 6.6x, |g'(23/33)| = 1.3 > 1$$

So all these three fixed point are source, and it is simple to find the conclusion with cobweb plot. ■

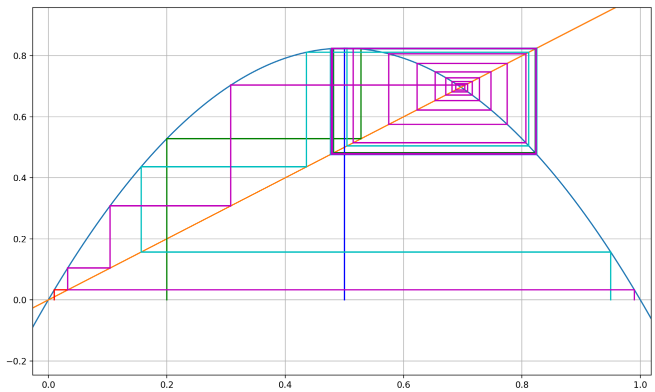


Figure 3: An example of periodic point

Hold on a second, something strange! Even we cannot find a sink fixed point, all of initial value are sank into a group of points!

Definition 1.5 ***Period-k point, Period-k orbit***
*Let f be a map on R, and p is a point in domain, if $f^k(p) = p$, and k is the smallest such positive integer, then called p as **periodic point of period** k, or **period-k point**;*
*Called orbit with initial point p as **periodic orbit of period k**, or **period-k orbit**;*

Definition 1.6 ***Sink and Source in Period point***
Let f be a map and p is a period-k point
If p is a sink, then called this period-k orbit as periodic sink;
If p is a source, then called this period-k orbit as periodic source.

Obviously, based on the chain rule, we have $(fg)'(x) = f'(g(x))g'(x)$, let $f = g, x = p_1$, then

$$g^2(p_1) = g'(g(p_1))g'(p_1) = g'(p_2)g'(p_1)$$

Summary this formula, we have

Theorem 1.2 *For every map f and period-k orbit $\{p_1, p_2, \ldots p_k\}$,*

$$(f^k)'(p_1) = (f^k)'(p_2) = \ldots = (f^k)'(p_k) = \prod_{i=1}^{k} f'(p_i)$$

***PROOF* 1.2** ***Theo.* ??**

$$(f^k)(p_1) = (f(f^{k-1}))'(p_1) = f'(f^{k-1}(p_1))(f^{k-1})'(p_1) = \ldots = \prod_{i=1}^{k} f'(p_i) = (f^k)(p_i)(\forall i = 1, 2, \ldots, k)\blacksquare$$

Same as Theo. **??**, we have stability test for periodic orbits.

Theorem 1.3 ***Stability test for periodic orbits***
Let f is a map and period-k orbit $\{p_1, p_2, \dots p_k\}$,
If $|\prod_{i=1}^{k} f'(p_i)| < 1$ then called this periodic orbit is a sink;
If $|\prod_{i=1}^{k} f'(p_i)| > 1$ then called this periodic orbit is a source;

PROOF* 1.3** ***Theo. ??
Consider a new map $g(x) = f^k(x)$, where f be a map and p is a period-k point, then p is a fixed point of g. Based on Theo. **??**, *$|g(p)| < 1$ if p is sink and $|g(p)| > 1$ if p is a source. On the other hand, $g(p) = f^k(p) = \prod_{i=1}^{k} f'(p_i)$*■

Now we consider another problem.

E x a m p l e 1.3 *Find the fixed point or periodic orbit of $g_{3.5}(x) = 3.5x(1-x), g_{3.86}(x) = 3.86x(1-x), x \in [0,1]$.*

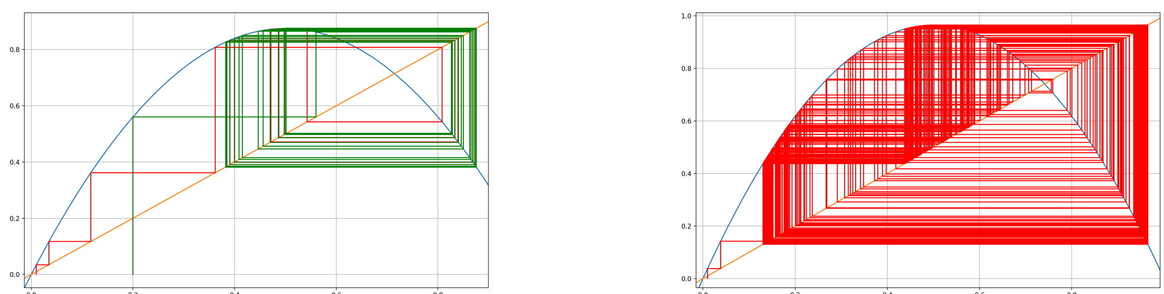


Figure 4: Logistic maps in $a = 3.5$ and $a = 3.86$

We found in $a = 3.5$, even the periodic orbit is difficult to find, the iteration still have a boundary. If we consider every $a \in [1,4]$, we can plot a figure between parameter a and orbits x, and this **bifurcation diagram** was made by following repearting:
[i] Choose a value a, starting with $a = 1$.
[ii] Choose a value $x \in [0,1]$ randomly.
[iii] Calculate the orbit of x under $g_a(x)$ in a certain iteration times t_max.
[iv] Ignore the first t_0 iterates and plot the orbit.

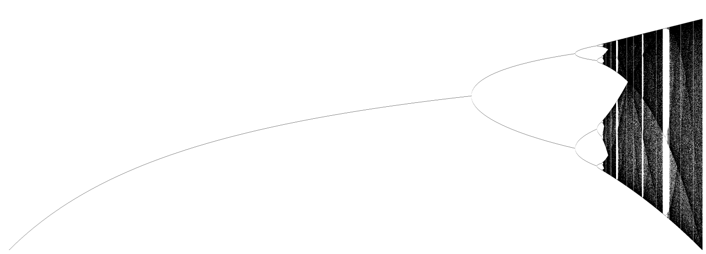


Figure 5: Logistic model stability interval ($a \in [1,4]$)

***D I S C U S S I O N* 1.1** *Now we will discuss the family of logistic maps with Fig.* **??**
[i] Periodic-3 window
We found periodic-1 orbits (or point) and periodic-2 orbits, based on the image above, it seem we also have periodic-3 orbits. And now we focus on the interval of parameter a rather than domain of function, we found there is a interval of a inside the $[3.83, 3.86]$ and we called these kind of interval as "periodic window". For instance, next figure showed the periodic-3 window of a.

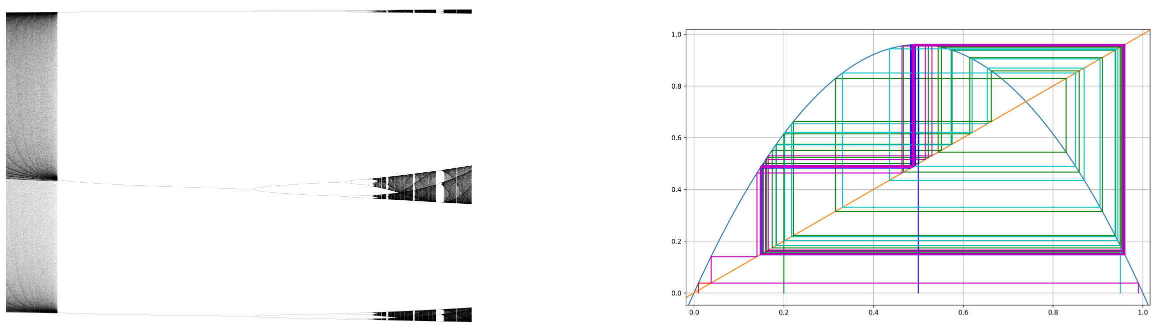


Figure 6: Periodic-3 window and cobweb plot in $a = 3.84$

That's fine, let's check the result by cobweb plot. Ok, hold on a second, something wrong! So we still need more analysis.

Obviously, every periodic-3 orbit of g is a fixed point of g^3, so we can also analysis g^3 map.

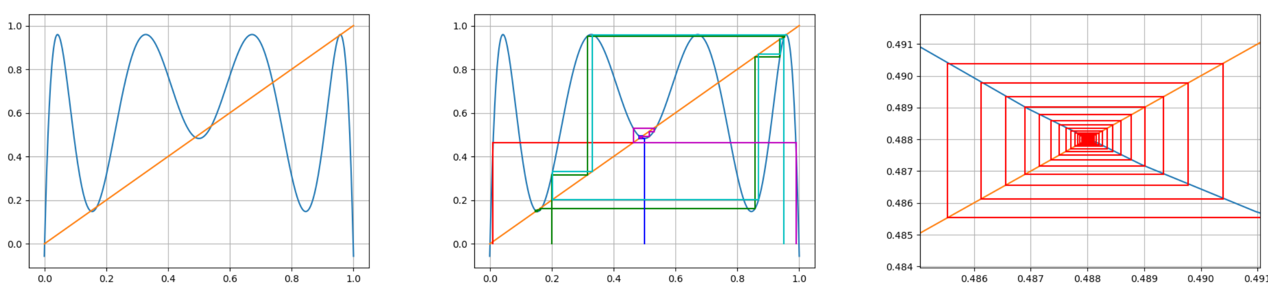


Figure 7: g^3 map and g^3 cobweb plot figure

We found different from periodic-2 orbit, the periodic-3 orbit is nearby(rather than equal) the point and it seems we have periodic-3 orbit. Actually, we will explain all periodic-3 will implies a characteristic we called "chaos".

[ii] The Logistic Map $G(x) = 4x(1-x)$
Now we consider another logistic map where $a \equiv 4$.

Firstly, why we are interested in $g_4(x)$, consider a quadratic function

$$g_a(x) = ax(1-x) = a(-x^2 + x - 1 + 1) = -a(x - \frac{1}{2})^2 + \frac{a}{4}$$

this function have maximum at point $x = 1/2$ and the maximum is $a/4$. As we have the Theo. **??**, *if we consider the sink point set, it is necessary to satisfy $|g_a(x)| < 1$, or$\frac{a}{}4 < 1 \Rightarrow a < 4$. So at the point $a = 4$, this set is empty and this is a critical state. For every $a_{new} = a - \varepsilon(\varepsilon \to 0)$, we have the interval of sink. So at this point, some special property has been result and that is why we interested in this map.*

We can still find the fixed point of $g_4(x)$ to solve $g_4(x) = x$, and we have $x_{11} = 0, x_{21} = 3/4$. If we consider periodic-k orbit, for instance, we consider periodic-2 orbit, then we have solve the function $g(g(x)) = x$ as

$$g(g(x)) = 4(4x(1-x))[1 - 4x(1-x)] = x \Rightarrow (4x^2 - 4x + 1)(x-1)x + \frac{x}{16} = 0$$

$$\Rightarrow (4x - 3)(16x^2 - 20x + 5)x = 0 \Rightarrow x_{21} = 0, x_{22} = \frac{3}{4}, x_{23,24} = \frac{5 \pm \sqrt{5}}{8}$$

Also, it is easy to check the periodic-k orbit in the figure.

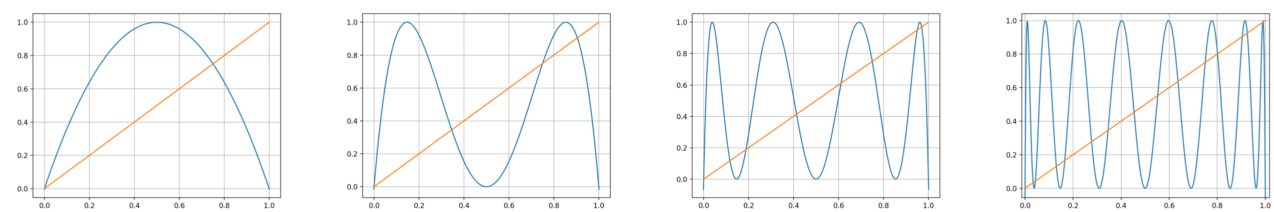


Figure 8: g_4^1, g_4^2, g_4^3 and g_4^4 figure

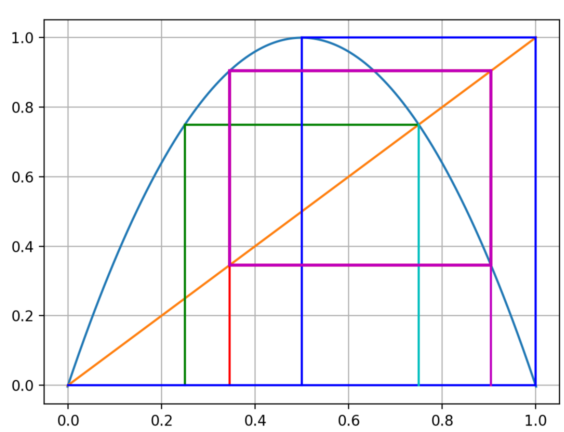


Figure 9: $g_4(x)$ cobweb plot(periodic-1,2 orbits)

We found a conclusion here

***CONCLUSION* 1.1** *For every periodic-k, the model g_4^k have $2^k - 1$ saddle-node bifurcation and 2^k fixed point. And these 2^k points include every fixed point for model $g_4^i, i = 1, 2, \ldots, k-1 \wedge k \equiv 0(mod\ i)$*

The number of orbits of the map for each period can be tabulated in the map's periodic table.

Table 2: The periodic table for the logistic4 map

Period k	1	2	3	4	5	6	7
Number of fixed points of g_4^k	2	4	8	16	32	64	128
Orbits of Period k	2	1	2	3	4	5	6
Fixed points due to lower orbits	0	2	2	4	2	4	2
1	/						
2	*	/					
3	*		/				
4	-	*		/			
5	*				/		
6	-	*	*			/	
7	*						/

(: Greatest common divisor group, -: g^k fixed)*

1.3 Chaos

We still focus on $g_4(x)$ map, we try to check the g_4^2 fixed point $\frac{5-\sqrt{5}}{8}$.

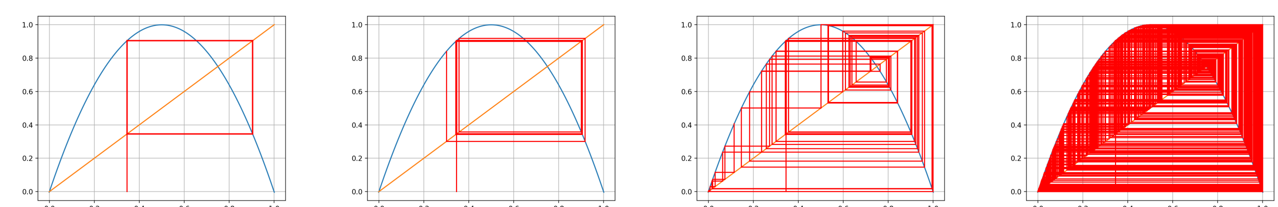


Figure 10: g_4^1, g_4^2, g_4^3 and g_4^4 figure

It seems something wrong. Because we proved that $\frac{5\pm\sqrt{5}}{8}$ is a periodic orbit during the iteration, but once we growth the iteration times, the results filled all the interval.

So what happened? We try to put all of our data into a same image, and we have

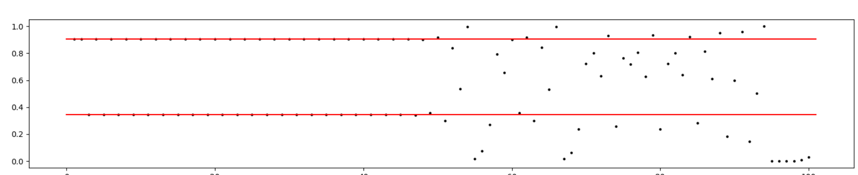


Figure 11: Iteration and "periodic-2 orbit" value

Obviously, in about first 40 times iteration, it was worked for a while, but with the iteration increasing, the error also increaed rapidly. Ok, ok, let's check the data for more details.

Table 3: Logistic4 periodic-2 orbit iteration

1-4	0.3454915028125262	0.9045084971874737	0.3454915028125262	0.9045084971874735
5-8	0.34549150281252694	0.9045084971874745	0.3454915028125237	0.9045084971874705
9-12	0.34549150281253665	0.9045084971874865	0.3454915028124849	0.9045084971874225
13-16	0.34549150281269186	0.9045084971876783	0.3454915028118641	0.9045084971866552
17-20	0.3454915028151752	0.9045084971907479	0.34549150280193086	0.9045084971743771
21-24	0.3454915028549078	0.9045084972398602	0.3454915026430002	0.904508496977928
25-27	0.3454915034906304	0.9045084980256565	0.34549150010010987	...

We noticed that during the iteration, the values of periodic-2 orbit are actually changed very small. Then we realized that is beacause of $\frac{5-\sqrt{5}}{8} \neq 0.3454915028125262$ and this is just a value near the periodic point.(And the computer can only calculate this estimation value rather than real value.) Even this two value are almost nearby, it still have a little difference, and this difference become larger and larger during the iteration.

That is important beacause we found even two value are almost equal, after iterate, this tiny, tiny difference will become a catastrophe and eventually two orbits move apart.

Definition 1.7 ***Sensitive dependence on initial conditions, Sensitive point***
Let f is a map on R, x_0 in domain.
If there is a nonzero distance d s.t. some points arbitrary near x_0 are eventually mapped at least d units from the corresponding image of x_0, then we called x_0 has ***sensitive dependence on initial conditions****;*
If for this $x_0, \exists \varepsilon > 0$ s.t. $\forall x \in N^o_\varepsilon(x_0) = N_\varepsilon(x_0) \backslash \{x_0\}, \exists K$ s.t. $\forall k > K, ||f^k(x) - f^k(x_0)|| \geq \varepsilon$, then called this point is ***sensitive point****.*

Definition 1.8 ***Eventually periodic***
Let f is a map on R, x_0 in domain. If for some positive integer $N, \forall n > N, f^{n+p}(x) = f^n(x)$, then we called x ***eventually periodic*** *with period p, where p is the smallest such positive integer.*

Now we consider another model to explain this definition in another way.

E x a m p l e 1.4 *Consider a map $f(x) = 3x(mod\ 1)$. (e.g. $f(4.33) = 0.33, f(-1.98) = 0.02.$)*

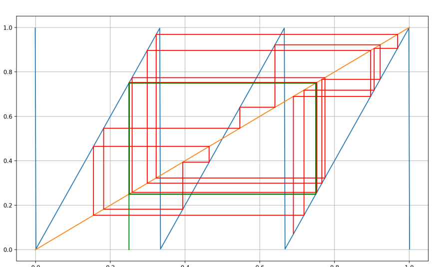


Figure 12: 3x mod 1 cobweb plot(initial value: 0.25(green), 0.2501(red))

Basically, we have

Theorem 1.4 *For any map f, the source has sensitive dependence on initial conditions.*

***PROOF* 1.4** *For a certain ε, as p is a source, then $\forall x \in N_\varepsilon^o(p), \lim_{k\to\infty} f^k(x) \notin N_d^\varepsilon(p) \Rightarrow d(p,x) > \varepsilon$* ■

Is there any way to investigate this sensitive dependence? Yes, and here we will introduce a method called **itinerary** of an orbit.

SOLUTION* 1.3 *Itinerary
*We still consider the g_4 model. Assign the symbol **L** to the left subinterval $[0, 1/2]$ and **R** to the right subinterval $[1/2, 1]$. Then, for every initival condition x_0, we can list the itinerary with **L** and **R**.*
*For instance, the initial point $x_0 = 1/3$ have the itinerary **LRLRLRRLLRR***

Table 4: Logistic4 1/3 itinerary

0-2	0.3333333333333333 (**L**)	0.8888888888888889 (**R**)	0.39506172839506154 (**L**)
3-5	0.9559518366102727 (**R**)	0.16843169076687667(**L**)	0.5602498252491516 (**R**)
6-8	0.9854798342297868 (**R**)	0.05723732222487492 (**L**)	0.21584484467760304(**L**)
9-10	0.6770233908148179 (**R**)	0.874650876417697 (**R**)	...

And we can list all itinerary with different initial value.■

Table 5: Logistic4 itinerary with different initial value

Val	$1-10$	$11-20$	$21-30$	$31-40$	...
0.01	**LLLRRLLLLR**	**RRRLRLRLRR**	**LLRRRLLRRR**	**RLRRLRLRLR**	...
0.25	**LRRRRRRRRR**	**RRRRRRRRRR**	**RRRRRRRRRR**	**RRRRRRRRRR**	...
1/3	**LRLRLRRLLR**	**RLRLLRRRRL**	**LLLRRRRLRL**	**RRRRRRRLRR**	...
0.5	**RRLLLLLLLL**	**LLLLLLLLLL**	**LLLLLLLLLL**	**LLLLLLLLLL**	...
1	**RLLLLLLLLL**	**LLLLLLLLLL**	**LLLLLLLLLL**	**LLLLLLLLLL**	...

Notice that there are some conclusions.

***CONCLUSION* 1.2** *For every periodic-k point, the itinerary of orbit will repeats **L** or **R** infinitely.*

***CONCLUSION* 1.3** *For every k iterate, the itinerary have 2^k choice and the sum of their lengths is 1(or the length of the interval).*

Also, we have a conclusion not very obvious.

***CONCLUSION* 1.4** *Each 2^k itinerary is shorter than $\pi/2^{k+1}$.*

We will prove this conclusion in later sections.

We can also analysis the problem with **transition graph**.

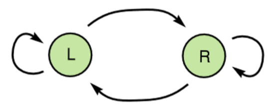


Figure 13: Transition graph

Finally, we focus on the title of this subsection "chaos", after these analysis, it is simple to summary the definition of chaos.

Definition 1.9 ***Chaos***
A chaotic orbit is a bounded, non-periodic orbit that displays sensitive dependence. Chaotic orbits seoarate exponentially fast from their neighbors as the map iterated.

Theorem 1.5 *The existence of periodic-3 orbit alone implies the existence of a large set of sensitive points, or chaotic orbit.*

We will prove this problem in appendix.

2 Two-Dimension and High-Dimension Maps

In this section, we will mainly discusse a new type of model, called Henon map which formed

$$f(x, y) = (a - x^2 + by, x)$$

2.1 Lorenz system, Henon map and Poincare section

A simple way to analysis the problem is analysis all point in the surface if they are convergence or divergence. In figures following, point in black represent initial conditions whose orbits diverge to infinity and the points in white represent initial values whose orbits converge to the period-2 orbit.

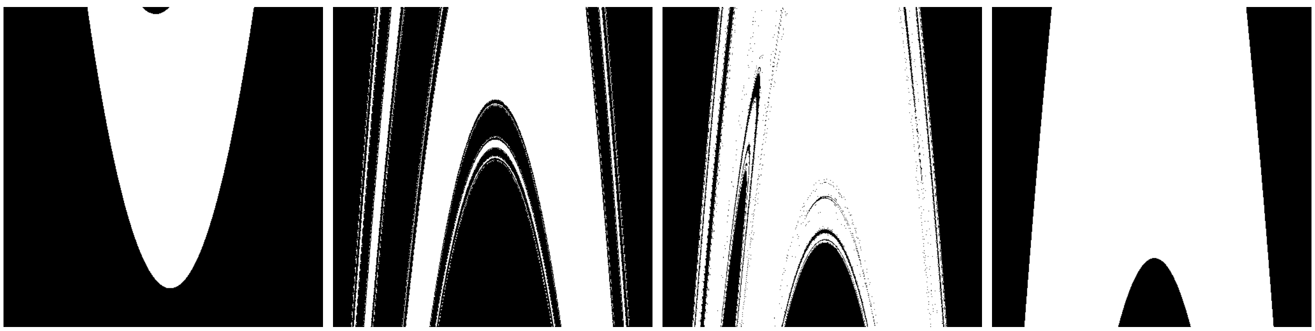


Figure 14: Initial condition square

(Parameter group $(a, b) = (0, 0.4), (2, -0.3), (1.4, -0.3), (1.28, -0.3))$

D I S C U S S I O N* 2.1 *Lorenz model, Henon map and Poincare section
The Lorenz system is a system of ordinary differential equations which notable for having chaotic solutions for certain parameter values and initial conditions. In particular, the Lorenz attractor is a set of chaotic solutions of the Lorenz system.

P r o b l e m 2.1 *Lorenz model*
Lorenz model is a system of three ordinary differential equations now known as the Lorenz equations:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

where σ, ρ, β *are parameters.*

Even solve this group of function directly is difficult, it is not difficult to find the numerical solution.

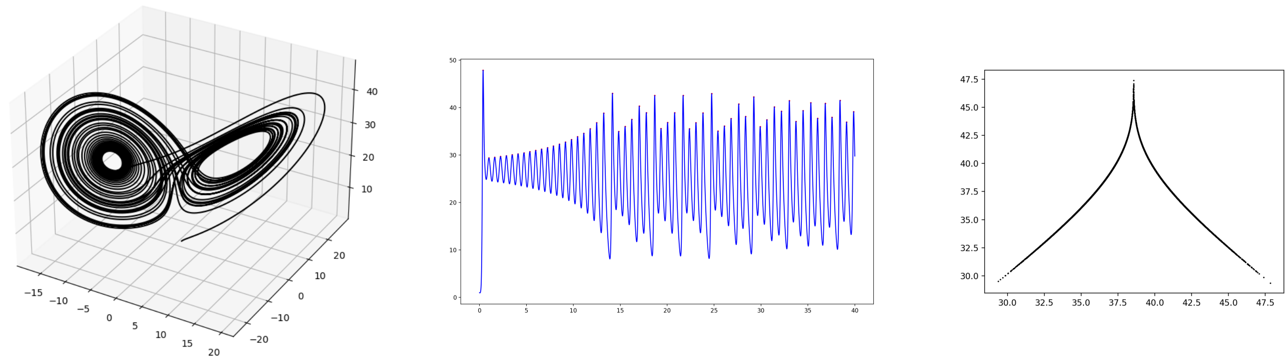


Figure 15: Lorenz system, $z-t$ map and Lorenz map

To analysis this problem, we can consider the problem in one dimension. In the second part of Fig. **??**, *we plot the z-t figure of Lorenz model.*

Ok, we found that it is still difficult to discribe the $z-t$ figure. However, after the discussion of the logistic map $g(x) = 4x(1-x)$ as well as chaotic orbit, we know in most situation, we just care about the boundary of interval of the map. On the other hand, we found that the $z-t$ figure of Lorenz system is familiar with sine function, it is shaking during the time iteration. So if we just consider the maxinum (or the mininum) of this $z-t$ map, we can analysis the problem easier.

Definition 2.1 ***Lorenz map***
The function $z_{n+1} = f(z_n)$ satisfied the last of Fig. **??** *is called the Lorenz map. The map can be discribed in following steps.*
[i] *Find the $z-t$ function in Lorenz model.*
[ii] *The map $\{z_n\}$ is a point set which is the maxima of $z-t$ function. $z_{n+1} = f(z_n)$ where z_n is a maxima point of $z-t$ function and z_{n+1} is next maxima point of the function with growing of t.*

** The graph of Lorenz map is not actually a curve. It does have some trickness because it is not a well-defined function. However trickness is so small and there is so much to be gained by treating the graph as a curve, that we will simplt make this approximation keeping in mind that the sunsequence analysis is plausible.*[1]

Definition 2.2 ***Poincare map and Poincare section***

SOLUTION* 2.1** ***Simply Lorenz model to Henon map with Poincare section
[i] Transfrom the continuous problem to discrete problem
Based on the definition of differential, $\frac{dx}{dt} = \frac{x(t+\Delta t)-x(t)}{\Delta t}$. In continuous time problem, let $\Delta t \to 0$. However, in discrete, let $\Delta t = 1$ (unit). Let $t = k$ then we have

$$x_{k+1} - x_k = \frac{x(k+1) - x(k)}{1} = \frac{dx(k)}{dt} \Rightarrow x_{k+1} = x_k + \frac{dx(k)}{dt}$$

use this method in y and z, finally we can transfrom the origin Lorenz problem into

$$x_{k+1} = x_k + \sigma(y_k - x_k) = (1-\sigma)x_k + \sigma y_k$$

[1]Reference: Nonlinear Dynamics and Chaos With Applications to Physics, Biology, Chemistry, and Engineering, Steven H. Strogatz, Section 9.4, ISBN-13: 978-0813349107

$$y_{k+1} = y_k + x_k(\rho - z_k) - y_k = \rho x_k - x_k z_k$$
$$z_{k+1} = z_k + x_k y_k - \beta z_k = (1-\beta) z_k - x_k y_k$$

[ii] Use Poincare section to reduce dimension *Let C is a constant which satisfied*

$$z_1 = z_2 = \ldots = z_k = z_{k+1} = \ldots = z_\infty = C$$

then

$$z_{k+1} = z_k + x_k y_k - \beta z_k = (1-\beta) z_k - x_k y_k \Rightarrow x_k y_k = C - (1-\beta)C = \beta C$$

and we have

$y_{k+1} = y_k + x_k(\rho - z_k) - y_k = \rho x_k - x_k z_k = (\rho - \beta C) x_k,$ *let* $b = (\rho - \beta C)$ *and* $x_k^1 = b x_k,$ *then* $, y_{k+1} = x_k^1$ *and*

$$x_{k+1}$$

2.2 Analysis of Henon map

Now we focus on Henon map. Familiar with 1 dim map, it is necessary to define the sink and source as well as saddle.

Definition 2.3 ***Neighborhood***
Consider a R^n space, called every point $x = (x_1, x_2, \dots x_n)$ is a vector of R^n space,
Define the ***Euclidean Length*** $|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$, *which is equal to norm;*
And define the distance between two point $d(x, y) = |x - y|$;
Also, the ***ε-neighborhood*** *is*

$$\forall \varepsilon > 0, \textit{the } \varepsilon\textit{-neighborhood of point } p, N_\varepsilon(p) \textit{ is } \{x \in R^n \,||x - p| < \varepsilon\}, \textit{ also define } N_\varepsilon^o(p) = N_\varepsilon(p) \backslash \{p\}$$

Definition 2.4 ***Sink and Source in High-dimension Map***
Let f is a map on R^n, p is a vector on R^n which is the fixed point and $f(p) = p$ then
If there is an $\varepsilon > 0$ s.t. $\forall x \in N_\varepsilon(p), \lim_{k\to\infty} f^k(x) = p$, then p is a sink or attracting fixed point.
If $\forall x \in N_\varepsilon^o(p), \exists K s.t. \forall k > K, f^k(x) \notin N_\varepsilon(p)$, then called the point p as source.

We will explain these definitions with an example

E x a m p l e 2.1 *Analysis the sink point, source point and saddle of Henon map with parameter $a = 0, b = 0.4$*

SOLUTION 2.2 *Obviously, if we consider the funcion $f(x, y) = (-x^2 + 0.4y, x) = (x, y)$, then*

$$-0.2x^2 + 0.4x = x \Rightarrow x_1 = 0, x_2 = -0.6$$

So the fixed points are $(0, 0)$ and $(-0.6, -0.6)$. And now we have a new problem: how to confirm a fixed point is sink or source. Even the definition of sink and source are given above, we still need theory like Theo. **??**. *But here, we can analysis the problem with simulator.*

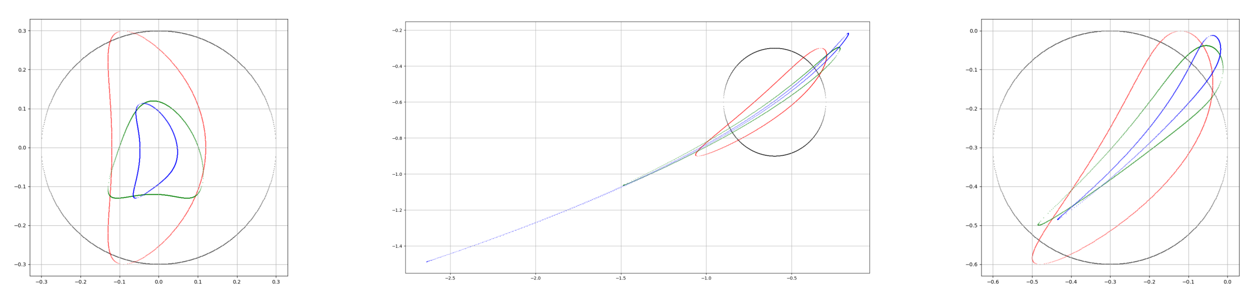


Figure 16: Sink, source and saddle in Henon map with $a = 0, b = -0.4$

(Order of color: Black(neighborhood), Red (Iter = 1), Green (Iter = 2), Blue (Iter = 3))

To solve the problem we faced in e.g.**??**, we will discuss the simple form of the high dimension maps.

Definition 2.5 ***High dimension linear map***
*A map $A : R^m \to R^m$ is **linear** if $\forall a, b \in R, \forall x, y \in R^m$, $f(ax+by) = af(x)+bf(y)$. Equivalently, a lienar map $f(x)$ can be represented as multiplication by an $m \times m$ matrix.*

Now we consider a system s.t. $f(x) = Ax$, if λ is eigenvalue and $\mathbf{v}$ is eigenvector of A, based on the definition fo eigenvalue and eigenvector, we have Let A have eigenvalue λ, based on the definition of eigenvalue, we have

$$A\mathbf{x} = \lambda\mathbf{v}$$

Then, for the initial point $\mathbf{v}$, we have

$$A(\mathbf{v}) = A\mathbf{v} = \lambda\mathbf{v}$$

let $\mathbf{v}_0 = \mathbf{v}, \mathbf{v}^n = A^n(\mathbf{v})$, then

$$\mathbf{v}_1 = A\mathbf{v}_0 = \lambda\mathbf{v}_0, \mathbf{v}_2 = A\mathbf{v}_1 = \lambda^2\mathbf{v}_1 \ldots \mathbf{v}_n = A\mathbf{v}_{n-1} = \lambda^n\mathbf{v}_0$$

Futhermore, if we consider a system in random initial value $\mathbf{x}_0$, still define $\mathbf{x}_n = f(\mathbf{x}_{n-1})$, then

$$\mathbf{x}_n = f(\mathbf{x}_{n-1}) = A\mathbf{x}_{n-1} = Af(\mathbf{x}_{n-2}) = \ldots = A^n\mathbf{x}_0$$

To analysis this problem, firstly we will review some theorems in algebra.

D I S C U S S I O N* 2.2** ***Eigenvalue, eigenvector and Jordan normal form
** We will consider a square matrix $A_{m\times m}$ s.t. $rank(A) = m$ in following discussion.*

[i] If** A **have** m **different Eigenvalue
Based on the discussion above, we know that it is the first step to analysis the A^n to discribe all the linear system. Obviously, if A is a diagonal matrix, then the exponent of the matrix is easy and simple.

Theorem 2.1 *Let A is a diagonal matrix s.t. $A = diag(a_1, a_2, \ldots a_m)$, then $A^n = diag(a_1^n, a_2^n, \ldots a_m^n)$.*

Furthermore, if matrix A have m different eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $\mathbf{v}_i$ is the eigenvector of λ_i. Let

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_m), V = (\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m)$$

then we can easily prove that

$$A = V^{-1}\Lambda V$$

And the calculation of A^n is simple.

$$A^n = V^{-1}\Lambda^n V = V^{-1}diag(\lambda_1^n, \lambda_2^n, \ldots, \lambda_m^n)V$$

Now we back to consider the linear system, if $f(\mathbf{x}) = A\mathbf{x}$ and A have m different eigenvalue, then we know that

$$\mathbf{x}_n = A^n\mathbf{x}_0 = V^{-1}diag(\lambda_1^n, \lambda_2^n, \ldots, \lambda_m^n)V\mathbf{x}_0$$

Based on the analysis in the section 1, we still want to analysis the convergence and divergence for ever system.

$$\lim_{n\to\infty} x_n = V^{-1}diag(\lim_{n\to\infty} \lambda_1^n, \lim_{n\to\infty} \lambda_2^n, \ldots, \lim_{n\to\infty} \lambda_m^n)V\mathbf{x}_0$$

Obviously, with the knowledge of sequence, if $|\lambda_i| \in [0, 1)$, then $\lim_{n\to\infty} \lambda_i^n = 0$ and the sequence is convergence. Also, if $|\lambda_i| \in (1, +\infty)$, then $\lim_{n\to\infty} \lambda_i^n = \infty$ and the sequence is divergence. So we have this conclusion.

Theorem 2.2 ***Sink, source and saddle in linear system***
Consider a linear system $f(\mathbf{x}) = A\mathbf{x}$, where A is a square matrix in m dimension. If the eigenvalue of A are $\lambda_1, \lambda_2, \dots \lambda_m$ and
[i] $\forall i \in 1, 2, \dots, m, |\lambda_i| < 1$*, then the origin point is sink.*
[ii] $\forall i \in 1, 2, \dots, m, |\lambda_i| > 1$*, then the origin point is source.*
[iii] $\{i||\lambda_i| < 1\} \neq \varnothing \wedge \{j||\lambda_j| > 1\} \neq \varnothing$*, that means, if at least one eigenvalue are absolute smaller than one and at least one is upper than one, then the origin point is saddle.*

[i] If A ***have at least two equal eigenvalue***
We can transfrom the matrix A with Jordan normal form rather than eigenvalue diagonal matrix. Consider the matrix $A_{m\times m}$ and the eigenvalue $\lambda_1, \lambda_2, \dots, \lambda_k$ are $r_1, r_2, \dots r_k$ multiple root of function $|\lambda I - A| = 0$, which satisfied the definition of eigenvalue, and $k < m$, $\sum_{i=1}^{k} r_i = m$, $I = diag(1, 1, 1, \dots, 1)$.
Then for every r_i multiple eigenvalue λ_i, $\exists \mathbf{v}_{i1}, \mathbf{v}_{i2}, \dots \mathbf{v}_{ir_i}$ s.t.

$$|\lambda I - A|\mathbf{v}_{i1} = 0, |\lambda I - A|\mathbf{v}_{ij+1} = \mathbf{v}_{ij} (j = 1, 2, \dots r_i - 1)$$

*We can still structure the V matrix same as V in **[i]**, and we can also represent the diagonal eigenvalue matrix Λ to the **Jordan normal form matrix** J which satisfied*

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \dots & \\ & & & J_k \end{bmatrix} = diag(J_1, J_2, \dots J_k), \text{ where } J_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \dots & \dots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}$$

*is r_i dimension square matrix called **Jordan block***

Based on the calculation of block matrix we found that

$$A^n = V^{-1} J^n V = V^{-1} diag(J_1^n, J_2^n, \dots, J_k^n) V$$

*So familiar with the discussion in **[i]**, now it is necessary to discuss the J_i^n. On the other hand, we know that for ever Jordan block, we have*

$$J_i^n = \begin{bmatrix} \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \binom{n}{2}\lambda_i^{n-2} & \dots & \binom{n}{r_i}\lambda_i^{n-r_i} \\ & \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \dots & \binom{n}{r_{i-1}}\lambda_i^{n-r_i+1} \\ & & \dots & \dots & \dots \\ & & & \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} \\ & & & & \lambda_i^n \end{bmatrix}$$

*Obviously, for ever element on the diagonal, the Theo. **??** still established.*

As for the non-linear problem, a wildly used method is **Jacobian matrix**

Definition 2.6 ***Jacobian matrix***
*Let $\mathbf{f} = (f_1, f_2, \ldots f_m)$ be a map on R^m and $\mathbf{p} \in R^m$ is a point on R^m space. The **Jacobian matrix** of $\mathbf{f}$ at $\mathbf{p}$ is the matrix*

$$D\mathbf{f}(\mathbf{p}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{p}) & \frac{\partial f_1}{\partial x_2}(\mathbf{p}) & \ldots \frac{\partial f_1}{\partial x_m}(\mathbf{p}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{p}) & \frac{\partial f_2}{\partial x_2}(\mathbf{p}) & \ldots \frac{\partial f_2}{\partial x_m}(\mathbf{p}) \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{p}) & \frac{\partial f_m}{\partial x_2}(\mathbf{p}) & \ldots \frac{\partial f_m}{\partial x_m}(\mathbf{p}) \end{bmatrix}$$

Jacobian matrix is a linearization estimation of a non-linear system that we can assume the derivative of the system near the point $\mathbf{p}$ is $D\mathbf{f}(\mathbf{p})$. That means, instead of origin non-linear system, we can analysis the estimated system $\mathbf{f}_1(\mathbf{x}) = D\mathbf{f}(\mathbf{p})\mathbf{x}$ where $x \in N(\mathbf{p}, \varepsilon)$ and ε is a certain constant. Based on the Theo. **??**, it is easy to improve the following conclusion.

Theorem 2.3 ***Sink, source and saddle in non-linear system***
Consider a non-linear system $\mathbf{f}(\mathbf{x})$ and a fixed point $\mathbf{p} \in R^m$ s.t. $\mathbf{f}(\mathbf{p}) = \mathbf{p}$. If the Jacobian matrix of $\mathbf{f}$ at $\mathbf{p}$ is $D\mathbf{f}(\mathbf{p})$, and Λ are eigenvalue set of matrix $D\mathbf{f}(\mathbf{p})$
[i] *$\forall\lambda_i \in \Lambda|\lambda_i| < 1$, then the $\mathbf{p}$ is a sink point.*
[ii] *$\forall\lambda_i \in \Lambda|\lambda_i| > 1$, then the $\mathbf{p}$ is a source point.*
[iii] *$\{\lambda_i \in \Lambda||\lambda_i| < 1\} \neq \varnothing \wedge \{\lambda_i \in \Lambda||\lambda_i| > 1\} \neq \varnothing$, that means, if at least one eigenvalue are absolute smaller than one and at least one is upper than one, then the $\mathbf{p}$ is a saddle point.*

Finally, we can analysis the property of fixed point in e.g. **??**.

***SOLUTION* 2.3** *We can consider the Henon map directly*

$$f(x, y) = (a - x^2 + by, x) \Rightarrow Df(x, y) = \begin{bmatrix} -2x & b \\ 1 & 0 \end{bmatrix}$$

Let λ are eigenvalue, then

$$|\lambda I - Df(x, y)| = 0 \Rightarrow \begin{vmatrix} -2x - \lambda & b \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 2x\lambda - b = 0 \Rightarrow \lambda_{12} = -x \pm \sqrt{x^2 + b}$$

When $(a, b) = (0, 0.4)$

$$\text{If } (x, y) = (0, 0), \text{ then } |\lambda_{12}| = |-x \pm \sqrt{x^2 + b}| = |\sqrt{0.4}| < 1$$

If $(x, y) = (0.6, 0.6)$, then $|\lambda_{12}| = |-x \pm \sqrt{x^2 + b}| = |-0.6 \pm \sqrt{0.76}| \Rightarrow |\lambda_1| = 1.472 > 1, |\lambda_2| = 0.272 < 1$
Finally, we proved that $(0, 0)$ is sink and $(-0.6, -0.6)$ is saddle just as what we found on simulation.

We found this conclusion based on the discussion above.

***CONCLUSION* 2.1** *The value of Jacobian matrix of a Henon map is just relevant to variable x and parameter b. That means, if we reduce the dimension of parameter and fixed b as b_0, then the property of fixed point will be determined only with variable x.*

If we consider the fixed point of system with arbitrary paremeter grou (a, b), we found that the fixed point will satisfied

$$x^2 + (1-b)x - a = 0 \Rightarrow x = \frac{1}{2}(b-1) \pm \sqrt{(b-1)^2 + 4a}$$

and the fixed point is $(x, y) = (\frac{1}{2}(b-1) \pm \sqrt{(b-1)^2 + 4a}, \frac{1}{2}(b-1) \pm \sqrt{(b-1)^2 + 4a})$, so we have the Jacobian matrix at this fixed point as

$Df(x, y) = \begin{bmatrix} (b-1) \pm \sqrt{(b-1)^2 + 4a} & b \\ 1 & 0 \end{bmatrix}$, and the eigenvalue λ_i satisfied $\begin{vmatrix} (b-1) \pm \sqrt{(b-1)^2 + 4a} - \lambda \\ 1 \end{vmatrix}$

2.3 Stability and matrix periodic