3 Chaos

We discussed the Henon map in last section. However, different from the section 1, Logistic map has been wildly used in application problems, we talked less about why we are interested in this model. So, in this section, we will mainly introduce the motivation.

3.1 Lorenz system, Henon map and Poincare section

D I S C U S S I O N* 3.1 *Why are we intersted in Henon map?
First of all, it is necessary to introduce a continuous model. The Lorenz system is a system of ordinary differential equations which notable for having chaotic solutions for certain parameter values and initial conditions. In particular, the Lorenz attractor is a set of chaotic solutions of the Lorenz system.

P r o b l e m 3.1 *Lorenz model*
Lorenz model is a system of three ordinary differential equations now known as the Lorenz equations:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

where σ, ρ, β are parameters.

It is continuous problem, however, based on the knowledge in numerical analysis, we can discrete the continuous to discrete problem in several ways.[1]

We can reconstruct the Lorenz equation, or a normal continuous dynamical system as

$$\frac{dX_i}{dt} = F_i(X_1, X_2, \dots X_m), i = 1, 2, \dots m$$

which is a m-dim dynamical system and t is single independence variable. To simplify this problem, we choose a initial time t_0 and time increment Δt, then let

$$X_{i,n} = X_i(t_0 + n\Delta t)$$

we have several ways to aproximate the equations.

SOLUTION* 3.1 *[i] Auxiliary approximations $X_{i,n+1} = X_{i,n} + F_i(P_n)\Delta t$
[ii] Centered difference procedure $X_{i,n+1} = X_{i,n-1} + 2F_i(P_n)\Delta t$
[iii] Double-approximation procedure $X_{i,n+1} = X_{i,n} + \frac{1}{2}(F_i(P_n) + F_i(P_{n+1}))\Delta t$

[1]EDWARD N LORENZ'S 1963 PAPER, "DETERMINISTIC NONPERIODIC FLOW", IN JOURNAL OF THE ATMOSPHERIC SCIENCES, VOL 20, PAGES 130–141

Even solve this group of function directly is difficult, it is not difficult to find the numerical solution.

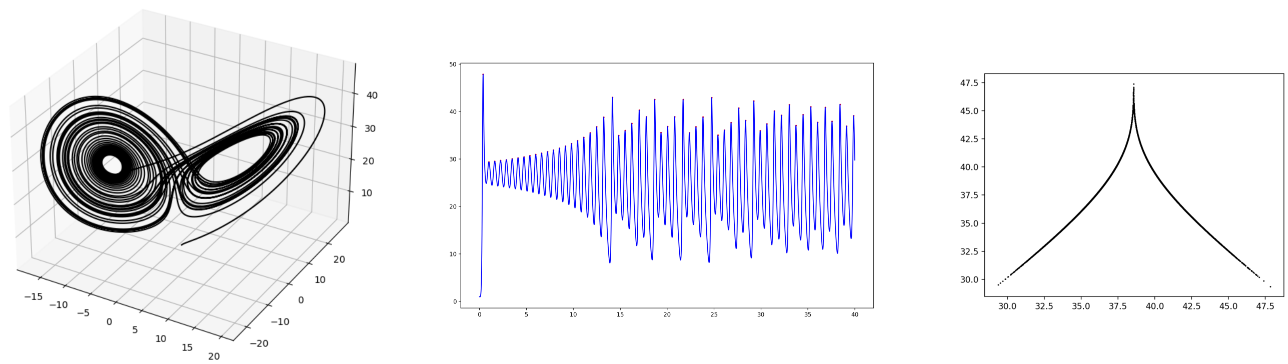

Figure 18: Lorenz system, $z-t$ map and Lorenz map

No we consider the problem in one dimension. In the second part of Fig. 18, we plot the z-t figure of Lorenz model.

Ok, we found that it is still difficult to discribe the $z-t$ figure. However, after the discussion of the logistic map $g(x)=4x(1-x)$ as well as chaotic orbit, we know in most situation, we just care about the boundary of interval of a map. On the other hand, we found that the $z-t$ figure of Lorenz system is familiar with sine function, it is shaking during the time iteration. So if we just consider the maxinum (or the mininum) of this $z-t$ map, we can analysis the problem easier.

Definition 3.1 ***Lorenz map***
The function $z_{n+1}=f(z_n)$ satisfied the last of Fig. 18 is called the Lorenz map. The map can be discribed in following steps.
[i] *Find the $z-t$ function in Lorenz model.*
[ii] *The map $\{z_n\}$ is a point set which is the maxima of $z-t$ function. $z_{n+1}=f(z_n)$ where z_n is a maxima point of $z-t$ function and z_{n+1} is next maxima point of the function with growing of t.*

** The graph of Lorenz map is not actually a curve. It does have some trickness because it is not a well-defined function. However trickness is so small and there is so much to be gained by treating the graph as a curve, that we will simplt make this approximation keeping in mind that the sunsequence analysis is plausible.*

As Lorenz map have no formula to discribe, it is very difficult to research that. However, in Lorenz's paper, he gave a correspondence to analysis the map, called tent map, which we has been introduced in the section 1

$$x_{n+1}=\begin{cases} 2x_n & x_n<1/2 \\ \text{Undefined} & x_n=1/2 \\ 2-2x_n & x_n>1/2 \end{cases}$$

At least this is a piecewise continuous funcion with one discontinuous point $x=1/2$. So we found the property of this map is not good enough to analysis. We hope the function is continuous in all domain. And we found if we try to remove this discontinuous point, then the f' will satisfy $f'^{+}(1/2)=f'^{-}(1/2)=1$. We found it is similar to the Logistic model and it seems we can discuss the property of Logistic map rather than Lorenz map. And we will explain why we can discuss the Logistic map instead of Lorenz map.

On the other hand, we found the Henon map also familiar with Logistic map in Fig. 17. The only different between Henon map and Logistic map is Henon map is fat and Logistic is thin, or a line. However if we change our parameter, like $b \to 0$, then we found

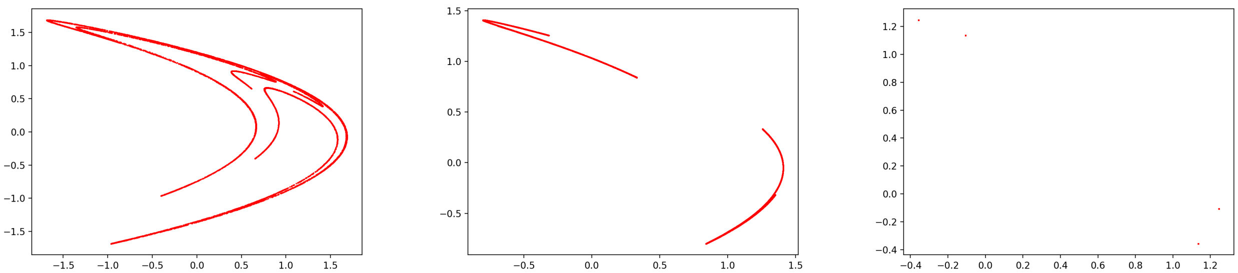

Figure 19: Attractor of Henon map in different $b(0.4, 0.2, 0.05)$

*So it **seems** we can analysis the Henon map instead of Lorenz map. But we still need more proof and in this section, we will try to solve these problems.*

3.2 Lyapunov exponents and Conjugacy

Definition 3.2 ***Asymptotically periodic***
Consider map $f \in C^1(R^1)$. An orbit $\{x_1, x_2, \dots x_n, \dots\}$ is called asymptotically periodic if it convergence to a periodic orbit as $n \to \infty$, that means, $\exists\{y_1, y_2, \dots y_k, y_1, y_2, \dots y_k m, \dots\}$ is a periodic orbit s.t.

$$\lim_{n\to\infty} |x_n - y_n| = 0$$

*Also, wecalled these map as **eventually periodic** beacause their orbit is eventually lands on a periodic orbit.*

For instance, the Lorenz map is a eventually periodic map. Because we found in the begining of the iteration, the map shaking in a wild interation (Just as 1-15 iterates in Fig. 18, second image) and after this period, the map become stable and it is convergent to the map in the 3rd image of Fig. 18, which we called that Lorenz map.

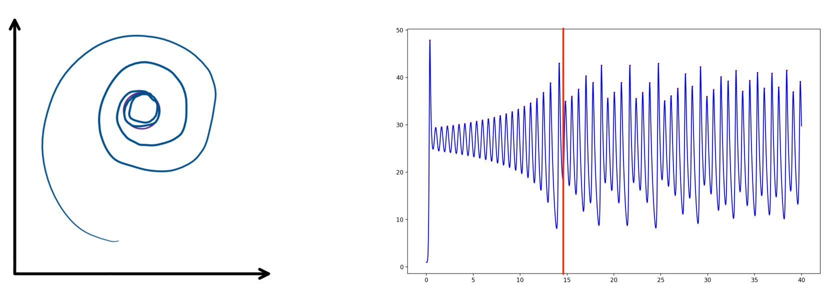

Figure 20: Asymptotically periodic, intro and example in Lorenz map

There are several other maps with this property, for example, in section 1, we introduced the Logistic map $G(x) = 4x(1-x)$, with the initial condition $x_0 = 1/2$, we found after 2 iterates, is coincides with the fixed point $x = 0$.

Now we try to find a method to judge a map is asymptotically periodic to another periodic map. In the section 1, we introduced the stability test for periodic orbits (Theo. 1.3), we called the limitation of value in Theo. 1.2 as **Lyapunov number**.

Definition 3.3 ***Lyapunov number and Lyapunov exponent***
*Consider map $f \in C^1(R^1)$. Define **Lyapunov number** $L(x_1)$ as*

$$L(x_1) = \lim_{n\to\infty} (\prod_{i=1}^{n} |f'(x_i)|)^{1/n}$$

and based on the logarithm function, we can define the Lyapunov exponent as

$$h_f(x_1) = h(x_1) = \lim_{n\to\infty} \frac{1}{n} \left[\sum_{i=1}^{n} \ln(f'(x_i)) \right]$$

Notice that h exists if and only if L exists and is nonzero, also $\ln L = h$.

D I S C U S S I O N* 3.2** ***Why are we interested in Lyapunov exponent?
We know that chaos means if two initial values have small different, then after long time iteration, the system showed us very different property with these two initial valus.(Just like Fig. 11, the error from computer results very different value, one is periodic and the other is chaos.)
Consider a self map f and a orbit $x_0, x_1, \ldots x_n, \ldots$, where $x_{i+1} = f(x_i) = f^i(x_0)(\forall i \in \mathcal{N})$. Now we focus on the point $x_0 + \delta_0$ where δ_0 is small almost near the x_0 point. For other $\delta_i, i \in \mathcal{N}^+$, we define

$$\delta_i = f(x_{i-1} + \delta_{i-1}) - x_i = f^i(x_0 + \delta_0) - f^i(x_0)$$

which discribed the distant between different initial value in i-iterate.
*Based on the definition of **exponentially stable**, we assume*

$$|\delta_i| = \exp(\lambda i)|\delta_0|$$

then, for every $i \in \mathcal{N}$ which is positive, the monotony of $\exp(\lambda i)|\delta_0|$ is decided by λ. If $\lambda > 1$, the sequence $\{\delta_i\}$ is disconvergence and if $\lambda <\in (0, 1)$ the sequence is convergence.

On the other hand, to find property of λ is difficult beacause we cannot discribe the propety of λ directly. However, during the iteration, we will find every $x_1, x_2, \ldots x_n, \ldots$ as well as $\delta_1, \delta_2, \ldots, \delta_n, \ldots$, and we know that

$$\lambda = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right|$$

Based on our thought, we need $\delta_0 \to 0$ and $n \to \infty$ so

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln \left| \lim_{\delta_0 \to 0} \left(\frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right) \right| = \lim_{n\to\infty} \frac{1}{n} \ln |(f^n)'(x_0)| = \lim_{n\to\infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n} f'(x_i) \right|$$

which is the definition of Lyapunov number.

Based on this Lyapunov exponent, we have this theorem.

Theorem 3.1 *Consider map $f \in C^1(R^1)$. If orbits $\{x_1, x_2, \ldots\}$ of f satisfies $f'(x_i) \neq 0 \forall i \in N$ and it is asymptotically periodic to the periodic orbit $y_1, y_2, \ldots$, then two orbit have indentical Lyapunov exponents, assuming both exist.*

***PROOF* 3.1** *[**i**] If we consider a sequence $\{s_i\}$ s.t. $\lim_{i\to\infty 0} s_i = s$, then*

$$\forall \varepsilon > 0, \exists N_1 \in \mathcal{N} \ s.t. \ \forall n > N_1, |s_n - s| < \varepsilon$$

Now we consider the average of $\{s_i\}$, we found for this ε,

$$\lim_{N\to\infty} \frac{1}{N} \left| \sum_{i=1}^{N} s_i - s \right| = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} |s_i - s| = \lim_{N\to\infty} \frac{1}{N} \left[\sum_{i=1}^{N_1-1} |s_i - s| + \sum_{i=N_1}^{N} |s_i - s| \right]$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N_1-1} |s_i - s| + \lim_{N\to\infty} \frac{1}{N} \sum_{i=N_1}^{N} |s_i - s| \leq 0 + \lim_{N\to\infty} \frac{1}{N - N_1} \sum_{i=N_1}^{N} |s_i - s| < \frac{N0N_1}{N - N_1} \varepsilon = \varepsilon$$

So we have this conclusion

$$\forall \varepsilon > 0, \exists N_1 \in \mathcal{N}, \ s.t. \ \forall n > N_1, \lim_{N\to\infty} \frac{1}{N} \left| \sum_{i=1}^{N} s_i - s \right| < \varepsilon \Rightarrow \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} s_i = s$$

*[**ii**] Let y_1 is the fixed point (that means, x_i asymptotically periodic to a periodic-1 orbit), then $\lim_{n\to\infty} x_n = y_1$. As $f \in C^1(R^1)$, then f' is exists and f' Riemann integrable (and of course, Lebesgue integrable), so we can exchange the order of integral(or differential of f) and limitation, then we have*

$$\lim_{n\to\infty} f'(x_n) = f'(\lim_{n\to\infty} x_n) = f'(y_1)$$

On the other hand, as $\ln|x|$ is a continuous, monotony function for $x \in R^+$, then

$$\lim_{n\to\infty} \ln|f'(x_n)| = \ln\left|\lim_{n\to\infty} f'(x_n)\right| = \ln|f'(y_1)| \Rightarrow h(x_1) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln|f'(x_i)| = \ln|f'(y_1)| = h(y_1)$$

*[**iii**] Now we assume $k > 1, k \in \mathcal{N}$, obviously, y_1 is fixed point of f^k, and*

$$h_{f^k}(x_1) = \ln|(f^k)'(y_1)| = h_{f^k}(y_1)$$

Now we will prove $h_{f^k}(x_1) = \frac{1}{k} h_f(x_1)$. Based on the definition, we know

$$h_{f^k}(x_1) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln\left|(f^k)'(x_i)\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln\left| \frac{1}{k} \prod_{j=i}^{i+k} f'(x_i) \right| = \frac{1}{k} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=i}^{i+k} \ln|f'(x_i)| = \frac{1}{k} h_f(x_1)$$

And we proved the theorem. ■

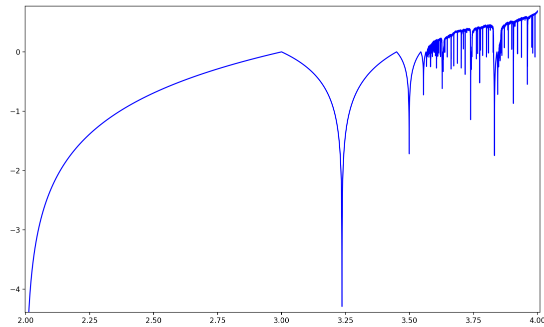

Figure 21: Lyapunov exponent of logistic model in different parameter

Obviously, the chaotic orbit satisfied Def. 1.9 will have no asymptotically periodic and we have this theorem.

Definition 3.4 *Consider map $f \in C^1(R^1)$, the orbit is **chaotic** if it satisfied both*
[i] *$\{x_1, x_2, \ldots\}$ is no asymptotically periodic, and*
[ii] *the Lyapunov exponent $h(x_1)$ is **greater** than zero.*

Now we will discuss the mod map and the tent map.

The mod map has been introduced in section 1 (Fig. 12) and we will mainly discuss the $f(x) = 2x(mod 1)$ in this discussion. A simple way to analysis the stability of map is itinerary.

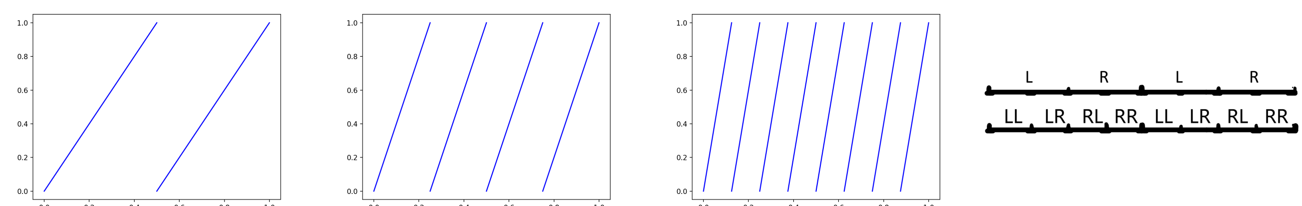

Figure 22: Iterates and itinerary of mod-2 map

[i] Binary and itinerary of mod map

We will focus on the initial value 0.2 of mod-2 map firstly. That is simple to find the itinerary with the code we used in section 1.

Table 6: Logistic4 itinerary with different initial value

Val	$1-8$	$9-16$	$17-24$	$25-32$	$\ldots$
0.2	**LLRRLLRR**	**LLRRLLRR**	**LLRRLLRR**	**LLRRLLRR**	$\ldots$

We found that this point is a periodic-4 point. And now, to analysis the problem simple, we will import a new method based on the binary number. In this problem, we have

$$\frac{1}{5} = 0.\overline{0011}, \quad f(\frac{1}{5}) = 0.011\overline{0011}, \quad f^2(\frac{1}{5}) = 0.11\overline{0011}, \quad f^3(\frac{1}{5}) = 0.1\overline{0011}, \quad f^4(\frac{1}{5}) = \overline{0011},$$

and obviously 1/5 point is a periodic-4 orbit. So in summary, we can use binary value to find the period of every point in this map.

Now we consider the Lyapunov exponent of mod map. Obviously, the differential of map is equal to 2 except the discontinuous point 1/2, so for ever initial value different from the 1/2, we have

$$h_{mod-2}(x_1) = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln\left(f'(x_i)\right)\right] = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln(2)\right] = \lim_{n\to\infty}\left(\frac{1}{n} n\ln(2)\right) = \ln(2) > 0$$

So the conclusion is the orbit of every initial value will result chaotic.

[ii] Mod-Sum model and asymptotically periodic

Now we change the mod model a little bit.

$$f(x) = (x+q)(mod 1) \text{ where } q \text{ is a constant.}$$

We will mainly consider the $q \in [0,1]$, if not, $\exists p \in [0,1]$ and $p = q(mod1) \wedge \forall x \in [0,1] f_q(x) = (x+q)(mod1) = f_p(x) = (x+p)(mod1)$.

That is simple to found the discontinuous point of $f_q(x)$ as $1-q$, futhermore, the differential of model is equal to 1 except this discontinuous point. So we have the Lyapunov exponent as

$$h_{f_q}(x_1) = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln\left(f'(x_i)\right)\right] = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln(1)\right] = 0$$

And

[iii] Tent map Now we consider the tent map.

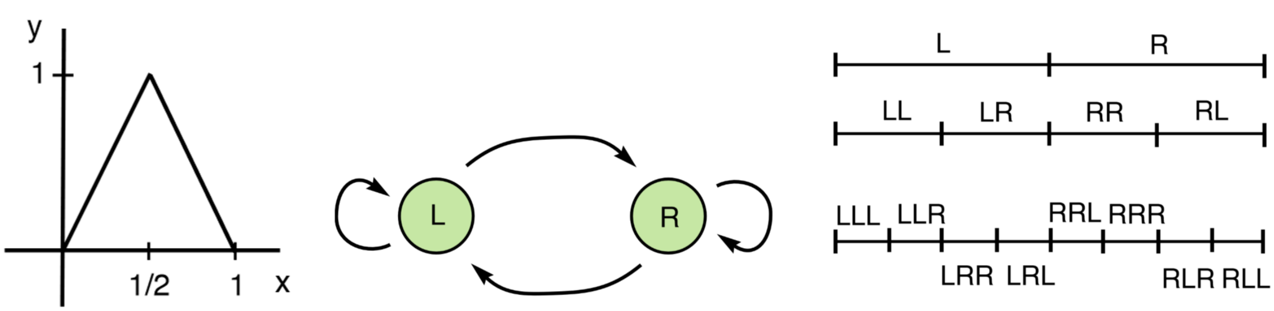

Figure 23: Tent map

We found that tent map have similar property of mod-2 map, that is because of the left side of the tent map is equal to the mod-2 map, and the right side of the tent map is symmetric of mod-2 map. So we can found that the size of interval in itinerary of both mod-2 map and tent map are 2^k and the only different between the mod-2 map and tent map is tent map will exchange the order of the **L** and **R** during iteration.

However, if we consider the topic of asymptotically periodic, we can found this conclusion.

Theorem 3.2 *The tent map T has infinitely many chaotic orbits.*

PROOF 3.2 *[**i**]Consider the set of $[0,1]\backslash Q$ which is combined with all irrational number in $[0,1]$. If $x \in [0,1]\backslash Q$, then*
1. *If $x < 1/2$, then $x_{new} = 2x \in [0,1]\backslash Q$*
2. *If $x > 1/2$, then $x_{new} = 2 - 2x \in [0,1]\backslash Q$*
Now we consider a orbit with irrational begining, obviously, every value of this orbit is based on irrational value. And we know that there are infinity element in $[0,1]\backslash Q$, so we have infinity irrational based orbits.

On the other hand, we know ever orbits avoid have

$$\forall x_1 \in [0,1], x_1 \neq \frac{1}{2}, h_f(x_1) = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln|f'(x_i)|\right] = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} \ln(2)\right] = \ln(2) > 0$$

*As $1/2$ is a rational value, so every orbit in [**i**] will avoid this point. So all of these orbits are chaotic. And in summary we found infinity many chaotic orbits in tent map T.* ■

[iv] Conjugacy and Logistic map

First of all, we will list the main property in both logistic-4 map and tent map.

Table 7: Logistic-4 map VS tent map

Property	Logistic-4 map($G(x)$)	Tent map($T(x)$)
Critical point	$1/2 \to 1 \to 0$	
Fixed point	$0, 3/4$	$0, 2/3$
Period-2 orbit	$\{(5-\sqrt{5})/8, (5+\sqrt{5})/8\}$ (Single)	$\{0.4, 0.8\}$ (Single)
	Lies in the same relation, one in $[0, 1/2]$ and the other in $[1/2, 1]$ where $1/2$ is critical point	

We will check the derivative value for periodic-k orbit in both map.

Periodic-1
[1] **Tent map** $T'(2/3) = -2$
[2] **Logistic map** $G'(3/4) = -2$
CONCLUSION: $T'(2/3) = G'(3/4)$
Periodic-2
[1] **Tent map** $T'(0.4)T'(0.8) = -4$
[2] **Logistic map** $T'((5-\sqrt{5})/8)T'((5+\sqrt{5})/8) = -4$
CONCLUSION: $T'(0.4)T'(0.8) = T'((5-\sqrt{5})/8)T'((5+\sqrt{5})/8) = -4 = -2^2$
...

We found some intersted result, but we cannot confirm how long these property still established. To solve, or to prove the similiarity of tent map and logistic map, we will import the definition of conjugate.

Definition 3.5 *The mape f and g are **conjugate** if they are related by a continuous one-to-one change of coordinates, that is, if $C \circ f = g \circ C$.*

P r o b l e m 3.2 *Proof: Map T and G are conjugate, and the C is $C(x) = (1 - \cos(\pi x))/2$.*

PROOF* 3.3** ***[i] $x \in [0, 1/2]$ $G(C(x)) = 4(C(x))(1 - C(x)) = 1 - \cos^2(\pi x) = \sin^2(\pi x)$
$C(T(x)) = 1 - \cos(2\pi x)/2 = \sin^2(\pi x) = G(C(x))$
[ii] $x \in [1/2, 1]$ $G(C(x)) = 4(C(x))(1 - C(x)) = 1 - \cos^2(\pi x) = \sin^2(\pi x)$
$C(T(x)) = 1 - \cos(\pi(2 - 2x)/2 = 1 - \cos(2\pi x)/2 = \sin^2(\pi x) = G(C(x))$
In summary we proved the problem. ■

Familiar with matrix exponent, if we consider the G^n map, then

$$G^n = G \circ G \circ G \circ G \circ \ldots \circ G = CTC^{-1}CTC^{-1}CTC^{-1}CTC^{-1} \ldots CTC^{-1} = CT^nC^{-1} \Rightarrow G^nC = CT^n$$

Theorem 3.3 *If x is periodic-k point of f, then $C(x)$ is a periodic-k point of g, where f, g are conjugate with C and $Cf = gC$.*

***PROOF* 3.4** *With the conclusion above, we have*

$$f^k = C^{-1}g^kC, \text{ then } \forall x \text{ is a periodic-k point }, f^k(x) = C^{-1}\{g^k[C(x)]\} = x \Rightarrow g^k[C(x)] = C(x) ■$$

Now we focus on the partical of map, because we want to know if the conclusion like $T'(0.4)T'(0.8) = T'((5 - \sqrt{5})/8)T'((5 + \sqrt{5})/8) = -4$ is established on high dimension or not.

Theorem 3.4 *If f and g are conjugate with C and $Cf = gC$ and x is periodic-k point of f (that means $C(x)$ is a periodic-k point of g), then*

$$(g^k)'[C(x)] = (f^k)'(x)$$

***PROOF* 3.5** *[1] The chain rule says that*

$$C[f(x)] = g[C(x)] \Rightarrow C'[f(x)]f'(x) = g'[C(x)]C'(x)$$

If x is periodic-1 (fixed) point, then $f(x) = x \wedge g(C(x)) = C(x)$, so we have

$$C'(x)f'(x) = g'[C(x)]C'(x) \Rightarrow f'(x) = g'[C(x)]$$

[2] Now if x is periodic-2 point, then $f^2(x) = x, g^2[C(x)] = C(x) \wedge C[f^2(x)] = g^2[C(x)]$ and

$$\frac{dC[f^2(x)]}{dx} = C'[f^2(x)]\frac{df[f(x)]}{dx} = C'[x]\frac{df[f(x)]}{dx} = \frac{dg\{g[C(x)]\}}{dx} = [(g^2)'(C(x))]C'(x)$$

So In summary we have

$$(f^2)'(x) = (g^2)'(C(x))$$

[k] *Now we consider arbitrary $k \in \mathcal{N}$, familiar periodic-2, for a periodic-k point x, we have $f^k(x) = x, gk[C(x)] = C(x) \wedge C[f^k(x)] = g^k[C(x)]$ and*

$$\frac{d\{C[f^k(x)]\}}{dx} = C'[f^k(x)](f^k)'(x) = C'(x)(f^k)'(x) = \frac{d\{(g^k)[C(x)]\}}{dx} = (g^k)'[C(x)]C'(x)$$

In summary we proved the conclusion that $\forall k \in \mathcal{N}, (f^k)'(x) = (g^k)'(C(x))$ ■

Now we can simply found this conclusion:

***CONCLUSION* 3.1** *If f and g are conjugate with C and $Cf = gC$ and $\{x_1, x_2, \dots x_k\}$ is periodic-k orbit of f (that means $\{C(x_1), C(x_2), \dots C(x_k)\}$ is a periodic-k orbit of g), then*

$$\prod_{i=1}^{k} f'(x_i) = \prod_{i=1}^{k} g'(C(x_i))$$

***PROOF* 3.6** *Based on the chain rule, we still have*

$$(f^k)'(x_1) = [f^{k-1}(f)]'(x_1) = (f^{k-1})'(f(x_1))f'(x_1) = (f^{k-1})'(x_2)f'(x_1) = \dots = \prod_{i=1}^{k} f'(x_i)$$

also this conclusion is established in $(g^k)'(C(x))$ ■

Based on the conjugacy, we can prove these series of conclusions

***CONCLUSION* 3.2** *All periodic points of logistic map G are source.*

***PROOF* 3.7** *Firstly, we consider the tent map T, and obviously the point $0, 1/2$ and 1 will now be the periodic points because they result asymptotically periodic. And for ever other point of tent map, we know that $|T'(x)| \equiv 2$. On the other hand, we know that exists C s.t. $GC(x) = CT(x)$, so*

$$\prod_{i=1}^{k} |G'(x_i)| = \prod_{i=1}^{k} |T'(C(x_i))| = 2^k > 1 \blacksquare$$

(Also, this conclusion told us that $\prod_{i=1}^{k} |G'(x_i)| = 2^k$ will always be established, which we has beed found in periodic-1 and periodic-2 discussion.)

***CONCLUSION* 3.3** *Consider a itinerary figure of logistic map G, the length of every k-iterates subinterval is lower than $\frac{\pi}{2^{k+1}}$.*

***PROOF* 3.8** *Consider a subinterval of itinerary after k-iterates $[x_p, x_q]$ where $\forall x \in [x_p, x_q], T^k(x)$ have symbol $\boldsymbol{L} \vee T^k(x) \equiv \boldsymbol{R}$ and $\forall \varepsilon > 0 \forall x \in (N_\varepsilon(x_p) \bigcup N_\varepsilon(x_q)) \backslash [x_p, x_q], T^k(x)$ have different symbol of $x \in [x_p, x_q]$. Obviously, we know that $\mu([x_p, x_q]) = 2^{-k}$, where μ is lebesgue measure (or the length of the interval), then*

$$\mu_G([x_p, x_q]) = C(x_q) - C(x_p) = \int_{x_p}^{x_q} C'(x)dx = \int_{x_p}^{x_q} \frac{\pi}{2}\sin(\pi x)d \leq \frac{\pi}{2}\int_{x_p}^{x_q} dx = \frac{\pi}{2}(x_q - x_p) = \frac{\pi}{2^{k+1}} \blacksquare$$

Finally, we will use the tools of Lyapunov exponent as well as conjugacy to proof the following theorem.

Theorem 3.5 *The logistic map G has chaotic orbit.*

Lemma 3.1 *If f and g are conjugate with C and $Cf = gC$ $\{x_1, x_2, \ldots x_k, \ldots\}and\{C(x_1), C(x_2), \ldots C(x_k), \ldots\}$ are orbit of f and g, if C satisfied $\lim_{k\to\infty} \frac{1}{k} C'(x_k) = 0$ then we have*

$$h_f(x_1) = h_g(C(x_1))$$

***PROOF* 3.9** *Consider $C[f(x_i)] = g[C(x_i)]$, then the chain rule told us that*

$$\{C[f(x_i)]\}' = C'[f(x_i)]f'(x_i) = C'(x_{i+1})f'(x_i) = \{g[C(x_i)]\}' = g'[C(x_i)]C'(x_i)$$

$$\Rightarrow f'(x_i) = g'[C(x_i)]\frac{C'(x_i)}{C'(x_{i+1})} \Rightarrow \prod_{i=1}^{k} f'(x_i) = \prod_{i=1}^{k} g'(C(x_i)) \left(\frac{C'x_1}{C'(x_{k+1})} \right)$$

(This formula obey the conclusion above, if we pay attention to the condition, we know that once x_1 is a periodic-k point, then $x_{k+1} = x_1$ and we have $\prod_{i=1}^{k} f'(x_i) = \prod_{i=1}^{k} g'(C(x_i))$)
Moreover, based on the definition of Lyapunov exponent, there is

$$\ln \left| \prod_{i=1}^{k} f'(x_i) \right| = \sum_{i=1}^{k} \ln |f'(x_i)| = \ln \left| \prod_{i=1}^{k} g'[C(x_i)] \right| + \ln |C'(x_1)| - \ln |C'(x_{k+1})|$$

Note that $C'(x_1)$ is constant so $\lim_{k\to\infty} \frac{1}{k} (\ln |C'(x_1)|) = 0$. And based on the condition of theorem, we have $\lim_{k\to\infty} \frac{1}{k} \ln |C'(x_{k+1})| = 0$
So finally

$$h_f(x_1) = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \ln |f'(x_i)|$$

$$= \lim_{k\to\infty} \frac{1}{k} \ln \left| \prod_{i=1}^{k} g'[C(x_i)] \right| + \lim_{k\to\infty} \frac{1}{k} (\ln |C'(x_1)| - \ln |C'(x_{k+1})|)$$

$$= \lim_{k\to\infty} \frac{1}{k} \ln \left| \prod_{i=1}^{k} g'[C(x_i)] \right| = h_g(C(x_1))$$

In summary the theorem is established. ■

Lemma 3.2 *The periodic point of T map is countable, that is, if P_T is the periodic point of T then $P_T \sim \mathcal{N}$, or exists an one-to-one map $R, \forall x \in P_T, \exists$ only one $y \in \mathcal{N} s.t. R(x) = y$ vice versa.*

***PROOF* 3.10** *We can write a binary tree to analysis the periodic-k point of tent map.*

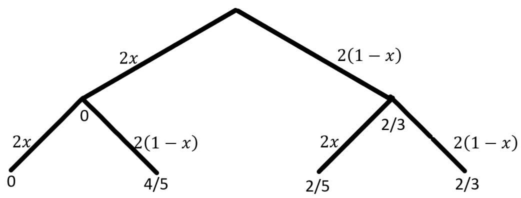

Figure 24: Binary tree of periodic-k point in tent map

We found that in every layer-k, the total of solution is less than 2^k and it is obviously that if we have infinity countable set in total and every set is infinity countable set, then the union of these sets is still countable (which has been introduced in Hilbert's paradox of the Grand Hotel).

On the other hand, in this problem, we have another simple way to build an one-to-one from periodic points P_T to $\mathcal{N}$. In the solusion binary tree above, we can code the solution from line to line. That means, in the first line, we code 0 to 1 and $2/3$ to 2. Then, in the next line, if we coded the solution, we jump over that node. After this iteration, we will build a one-to-one.

Moreover, actually we can proof that $P_T \subset [0,1] \bigcap Q$. It is simple to proof that the solution set $P_T^{(k)}$ of $T^k(x) = x (\forall k \in \mathcal{N})$ is the subset of $P^{(k)}$ and $x \in P^{(k)}$ satisfied $p + q \cdot 2^k x = x$, where $p \in Z$ and $q = \pm 1$, so $x = \frac{p}{q \cdot 2^k - 1} \in Q \bigcap [0,1]$ and we have

$$P_T^{(k)} \subset P^{(k)} \subset Q \bigcap [0,1]$$

So we have

$$P_T = \bigcup_{k=1}^{\infty} P_T^{(k)} \subseteqq \bigcup_{k=1}^{\infty} P^{(k)} \subseteqq Q \bigcap [0,1] \subset Q$$

As Q is countable set, then every subset of Q is countable, so P_T is countable. ■

***PROOF* 3.11** *Now we try to proof that G has chaotic orbit. As T and G are conjugacy, so based on one-to-one C, we know that P_G is countable. On the other hand, we proved that every periodic points of G are source, annd therefore no orbits besides periodic orbits - and eventually periodic orbits, another countable set - can be asymptotically periodic. Then any orbits whose responding symbol sequence is not eventually periodic, and which never contains the sequence **LL**, has Lyapunov exponent $\ln 2$ and is chaotic.*■

In the last of this subsection, we will introduce some other definition.

Definition 3.6 *If $A \subset B$ called A is **dense** in B if*

$$\forall x \in B, \forall \varepsilon > 0, N_\varepsilon(x) \bigcap A \neq \varnothing$$

Also, imitate the linear space, topological space, we can define the **symbol space** to discribe the itinerary of a map.

Definition 3.7 ***Symbol space***
The set S of all infinity itineraries of a map is called the ***symbol space*** *for the map. The* ***shift map*** *s is defined on the symbol space S as follows*

$$s(S_0 S_1 S_2 \dots) = S_1 S_2 S_3 \dots$$

This shift mape chops the leftmost symbol, which is the analogue on the itineraries of iterating the map on the point.

3.3 Fixed point theorem

Theorem 3.6 ***Fixed point theorem***
Let $f \in C(R^n), I = [a, b]$ s.t. $I \subset f(I)$, Then f has a fixed point in I
Moreover, if $I_1, I_2, \dots I_k$ are all closed intervals s.t. $\forall i = 1, 2, \dots, k-1, I_{i+1} \subset f(I_i) \wedge I_1 \subset f(I_n)$, then f^n has a fixed point in I_1, or f has a periodic-k point in I_1.

PROOF **3.12** *Proof: Theo.3.6*

$$I \subset f(I) \Rightarrow \forall x \in I, x \in f(I) \Rightarrow a, b \in f(I) \Rightarrow \exists x_1, x_2 \in [a, b] \text{ s.t. } f(x_1) = a, f(x_2) = b$$

$$(0 = f(x_1) - a \leq f(x_1) - x_1) \wedge (0 = f(x_2) - b \geq f(x_2) - x_2)$$

So we found a point x_1 s.t. $f(x) - x \geq 0$ and a point x_2 s.t. $f(x) - x \leq 0$, as the function f is continuous, based on the Intermediate Value Theorem, $\exists x_3$ s.t. $f(x_3) - x_3 = 0$ ■

Definition 3.8 ***Partition***
The collection of subintervals that are pairwise disjoint except at the endpoints whose union is I is the ***partition*** *of I.*

CONCLUSION **3.4** ***Covering rule for transition graphs***
[i] *An arrow is drawn from set A to set B in a transition graphs if and only if $B \subset f(A)$*
[ii] *Moreover, assume that $\{S_1, S_2, \dots S_n\}$ is a partition and the transition graph of f allows a sequence of symbols that returns to the same symbols s.t. $S_1 S_2 \dots S_k S_1$, then $S_1 \subset f^k(S_1)$*
[iii] *Generally, if $S_1 S_2 \dots S_k S_1$ is a path in the transition graph of a map f, then the subinterval denoted by $S_1 S_2 \dots S_k S_1$ contains a fixed point of f^k.*

3.4 Basins of attraction

Definition 3.9 ***Basin of attraction***
*Let f be a map on R^n and p be an attracting fixed point or periodic point of f, then the **basin of attraction** of p, or just **basin** of p is the set of point s.t.*

$$\lim_{k\to\infty} \left| f^k(x) - f^k(p) \right| = 0$$

Theorem 3.7 *Let f is a continuous map on R^1, then*
[i] *if $f(b) = b \wedge (\forall x \in [a, b), x < f(x) < b)$, then $a \to b, f^k(a) \to b$*
[ii] *if $f(b) = b \wedge (\forall x \in [b, c), b < f(x) < x)$, then $c \to b, f^k(c) \to b$*

***PROOF* 3.13** *We just need to proof **[i]** because it is simple to proof **[ii]** in the same method. Let $x_0 = a, x_{i+1} = f(x_i)\forall i \in \mathcal{N}^*$, obviously, $a \le x < f(x) < b$, thus all $x_i \in [a, b) \wedge a = x_0 < x_1 < x_2 < \ldots < x_\infty < b$ which is strictly increasing and bounded above by b. Since increasing bounded sequence must convergence, $\exists x_*$ s.t. $x_i \to x_*$ and*

$$x_* = \lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} f(x_i) = f(x_*)$$

and x_* is a fixed point ■

Definition 3.10 ***Schwarzian derivative, negative Schwarzian***
*Let $f \in C^\infty(R^1)$, then the **Schwarzian derivative** of f is*

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

If $\forall x$ s.t. $f'(x) \neq 0, S(f)(x) < 0$, then called the map has negative Schwarzian.

Theorem 3.8 *If f, g are negative Schwarzian, then fg has negative Schwarzian. Moreover, if f has negative Schwarzian, then f^k has negative Schwarzian.*

Theorem 3.9 *If map f on R^1 has negative Schwarzian, and p is a fixed point or periodic point of f, then either*
[i] *p has an infinity basin; or*
[ii] *there is a critical point of f in the basin of p; or*
[iii] *p is a source.*

***PROOF* 3.14** *Proof: Theo. 3.8*
If f, g are negative Schwarzian, then

$$\forall x \in D \bigcap \{x|f'(x) \neq 0\} \bigcup \{x|g'(x) \neq 0\} \subset R^1, (S(f)(x) < 0) \wedge (S(g)(x) < 0)$$

On the other hand, we have

$$S(f\circ g)(x) = \frac{(f\circ g)'''}{(f\circ g)'} - \frac{3}{2}\left(\frac{(f\circ g)''}{(f\circ g)'}\right)^2$$

and

$$(f\circ g)' = g'\cdot f'(g) \qquad (f\circ g)'' = g''\cdot f'(g) + (g')^2\cdot f''(g)$$
$$(f\circ g)''' = g'''\cdot f'(g) + g'g''\cdot f'(g) + 2g'g''f''\cdot(g) + (g')^3 f'''\cdot(g)$$

so we have

$$\frac{(f\circ g)'''}{(f\circ g)'} = \frac{g'''\cdot f'(g) + g'g''\cdot f'(g) + 2g'g''f''\cdot(g) + (g')^3 f'''\cdot(g)}{g'\cdot f'(g)} = \frac{g'''}{g'} + \frac{g''}{g'} + 2\frac{g''\cdot f''(g)}{f'(g)} + \frac{(g')^2\cdot f'''(g)}{f'(g)}$$

$$\left(\frac{(f\circ g)''}{(f\circ g)'}\right)^2 = \left(\frac{g''\cdot f'(g) + (g')^2\cdot f''(g)}{g'\cdot f'(g)}\right)^2 = \left(\frac{g''}{g'} + \frac{g'\cdot f''(g)}{f'}\right)^2 = \left(\frac{g''}{g'}\right)^2 + 2\frac{g''\cdot f''(g)}{f'(g)} + \left(\frac{g'\cdot f''(g)}{f'(g)}\right)^2$$

and finally

$$S(f\circ g)(x) = \frac{g'''}{g'} + \frac{g''}{g'} + 2\frac{g''\cdot f''(g)}{f'(g)} + \frac{(g')^2\cdot f'''(g)}{f'(g)} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 - 3\frac{g''\cdot f''}{f'(g)} - \frac{3}{2}\left(\frac{g'\cdot f''}{f'}\right)^2$$
$$= S(g)(x) + (g')^2 S(f)(x) + g''\left(\frac{1}{g'} - \frac{f''(g)}{f'(g)}\right) = (Sf)(g(x))\cdot(g'(x))^2. < 0 \quad \blacksquare$$

E x a m p l e 3.1 *[1]* $f(x) = ax$ *on* R^1 *and* $|a| < 1$, *zero is a fixed point sink whose basin is the entire real line.*
*[**1.1**] A lienar map on* R^n *whose matrix representation has distinct eigenvalue that are less than one in magnitude, then the origin is a fixed sink whose basin is* R^n.
[2] $f(x) = 4/\pi \arg\tan(x)$ *on* R^1 *has 3 fixed point* ± 1 *and* 0 *where* ± 1 *are sink and* 0 *is source. The basin of* 1 *is positive and the basin of* -1 *is negative.*
*[**3.1**]* $g(x) = ax(1-x), a \in (0,1)$, *the fixed point is* 0 *all* R^1 *is basin of this point.*
** We can proved that* $((a-1)/a, 1]$ *lies in the basin of* $x = 0$ *with Theo. 3.7. From graphcal representation orbits, it is clear that in addition, the interval* $[1, 1/a]$, $(-\infty, (a-1)/a)$ *and* $(1/a, \infty)$ *are basin of* 0.
*[**3.2**]* $g(x) = ax(1-x), a \in (1,2)$, *the sink fixed point is* $(a-1)/a$, *the basin is* $(0,1)$.
[4] $f(r,\theta) = (r^2, \theta - \sin\theta), r \in R^+, \theta \in [0, 2\pi)$ *which used the polar coordinates. The fixed points are* $(0,0), (1,0), (1,\pi)$, *the attractors are* $(0,0)$ *and* (∞,∞). *The basin of* $(0,0)$ *is the points inside the unit circle and the points outside the unit circle are basin of* (∞,∞).
[5] $g(x) = ax(1-x)$ *has Schwarzian derivative formed*

$$S(g)(x) = -\frac{3}{2}\left(\frac{-2a}{a-2ax}\right)^2$$

and therefore has negative Schwarzian.

***CONCLUSION* 3.5** *The logistic map* $g(x) = ax(1-x)(a \in [1,4])$ *has at most one periodic sink.*

***PROOF* 3.15** $g'(x) = a - 2ax, g''(x) = -2a, g'''(x) = 0$ *and*

$$S(g)(x) = -\frac{3}{2}\left(\frac{-2a}{1-2ax}\right)^2 < 0$$

*[**i**] Consider a point* $p \in [0,1]$, *as every point in* $R\backslash[0,1]$ *will tent to* $-\infty$ *so no point in* $[0,1]$ *have infinity basin.*
*[**ii**] Since the only critical point of* g *is* $1/2$, *there can be at most one attracting periodic orbit.* $\blacksquare$

3.5 Density function and Ulam-von Neumann transformations

We found the relationship between the tent map, the logistic map(l-4), and the mod map. Now we try to explain the Henon map and the tent map.[2] [3]

Consider a Henon map formed

$$H_{(a,b)}(x,y) = (1 - ax^2 + by, y)$$

when $a = 2$ and $b \to 0$, the map is

$$H_{(2,0)}(x,y) = (1 - 2x^2, y) \Rightarrow q(x) = 1 - 2x^2$$

which is a 1-dim nonlinear map. Obviously, there is exists a one-to-one map from tent map to the map follows

$$\tau(x) = \begin{cases} 2x+1 & x \in [-1,0] \\ -2x+1 & x \in [0,1] \end{cases} \Rightarrow \tau(x) = 1 - 2|x|, x \in [-1,1]$$

So the problem is how to find a one-to-one between the $\tau(x)$ and $q(x)$. Even in the introduction of conjugacy in the last section, we just given the map between the tent map and logistic map and never discuss how to find these kind of map. Here, we will used the definition of probability density function (PDF, or invariable density in dynamic system) to analysis this problem.

Definition 3.11 ***Density Function***
Consider a map $f \in C^1([0,1])$ and a group of initial stats $x_1^{(0)}, x_2^{(0)}, \ldots, x_N^{(0)}$, then define the $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(m)}$ sequence with

$$x_i^{(p)} = f(x_i^{(p-1)}) = f^p(x_i^{(0)}), i = 1, 2, \ldots N$$

And a ***density function*** *of a $x_i^{(p)}$ sequence will satisfy, for every interval $\Delta_0 \subset [0,1]$*

$$\int_{\Delta_0} f_0(u)du \simeq \frac{1}{N}\sum_{k=1}^{N} 1_{\Delta_0}(x_k^{(0)})$$

where $1_\Delta(x)$ is ***indicator function*** *s.t.*

$$1_\Delta(x) = \begin{cases} 1 & x \in \Delta \\ 0 & x \notin \Delta \end{cases}$$

In most situation, we cannot find the density function directly with the formula of map. We can only statistic the value of map and hope we can find a distribution to evaluate it. However, in logistic-4 map $G(x)$, we have a certain distribution.

$$\rho_q = \frac{1}{\pi\sqrt{1-x^2}}$$

On the other hand, based on the bifurcation diagram, we found that in ever parameter in $[1,4]$, there are some continuous line in the image. We can guess that in a normal situation, the

[2] Reference: Jiang, Y., 1995. On Ulam von Neumann transformations.
[3] Reference: Lasota, A. and Mackey, M., n.d. Chaos, Fractals, and Noise.

density function is a mixture model based on the ρ_q just like Gaussian Mixture Model(GMM) in probability.

$$\rho_{(g_a)}(x) = \sum_{i=1}^{K} \frac{p_i}{\pi\sqrt{1-(x-q_i)^2}}, \text{ where } \sum_{i=1}^{K} p_i = 1$$

Moreover, as we know the statistic data $x_i^{(p)}$ sequence, it is easy to get the numerical solution of parameter in the model above. For instance, with the machine learning method like Expectation–maximization(EM) algorithm we can find a mixture model.

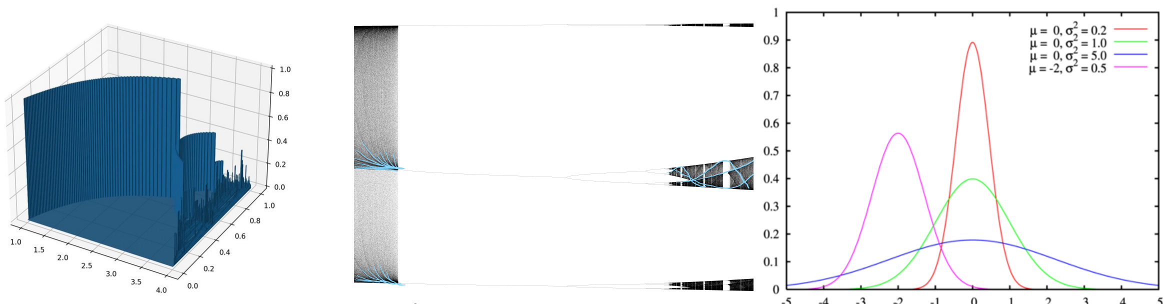

Figure 25: Histogram of logistic map, continuous line in bifurcation diagram

We found the density function of $q(x)$ is $\rho_q = \frac{1}{\pi\sqrt{1-x^2}}$, and the reflection between q and τ will satisfy

$$dy = \frac{2dx}{\pi\sqrt{1-x^2}} \Rightarrow y = h(x) = \frac{2}{\pi}\arg\sin(x) \text{ and } h^{-1}(x) = \sin\left(\frac{\pi x}{2}\right)$$

And we have

$$\bar{q}(x) = h\left\{q\left[h^{-1}(x)\right]\right\} = h\left\{1 - 2\sin^2\left(\frac{\pi x}{2}\right)\right\} = \frac{2}{\pi}\arg\sin\left[1 - 2\sin^2\left(\frac{\pi x}{2}\right)\right] = \frac{2}{\pi}\arg\sin\left[\cos\left(\pi x\right)\right]$$

$$\Rightarrow \sin\left(\frac{\pi\bar{q}}{2}\right) = \cos\left(\pi x\right) \Rightarrow \cos\left(\frac{\pi}{2} - \frac{\pi\bar{q}}{2}\right) = \cos\left(\pi x\right) \Rightarrow 1 - \bar{q} = 2|x| \Rightarrow \bar{q} = 1 - 2|x| = \tau(x)\blacksquare$$

Based on this density function, we can find the one-to-one between most non-linear map with certain condition and τ map, or tent map. Obviously, both τ map and tent map have certain Lyapunov exponent, and this makes the analysis of chaostic in non-linear system become easier.

Definition 3.12 ***Ulam-von Neumann transformation***
A map f is a Ulam-von Neumann transformation if it satisfied
[i] *f is a piecewise C^1 self mapping of $[-1,1]$ with a unique power law singular point 0;*
[ii] *$f|_{[-1,0]}$ is C^1 and increasing, and $f|_{[0,1]}$ is C^1 and decreasing;*
[iii] *$f(0) = 1, f(-1) = f(1) = 1$*
[iv] *$f|_{[-1,0]}$ and $f|_{[0,1]}$ are $C^{1+\alpha}$ for some $\alpha \in (0,1]$ and the restrictions of $r_f(x) = f'(x)/|x|^{\gamma-1}$ to $[-1,0)$ and to $(0,1]$ are $\beta-$ Bolder continuous fo some $\beta \in (0,1]$*
[v] *the sequence $\{\eta_n\}_{n=0}^{\infty}$ of neasted partitions by f decreases exponentially.*

Theorem 3.10 *Any two Ulam-von neumann transformations f and g are topologically conjugate.*

Theorem 3.11 *The Lorenz map is a Ulam-von Neumann transformation.*

Typically, we can discribe the Ulam-von Neumann transformations with properties follows: It is a either linear or non-linear curve like a mountain from point $(-1,-1)$ increase to $(0,1)$ and finally decrease to $(1,-1)$, the only singular point is 0 which may discontinuous. Also, there are some properies in differential of the function.

It is difficult to proof these theorem strickly beacause we need tools of mainfold as well as real analysis. So here we just introduce these definition and conclusions.

Figure 26: CHAOS