Topological Fixed Point Theory and Its Applications

Homotopy Methods in Topological Fixed and Periodic Points Theory

Jerzy Jezierski and
Wacław Marzantowicz

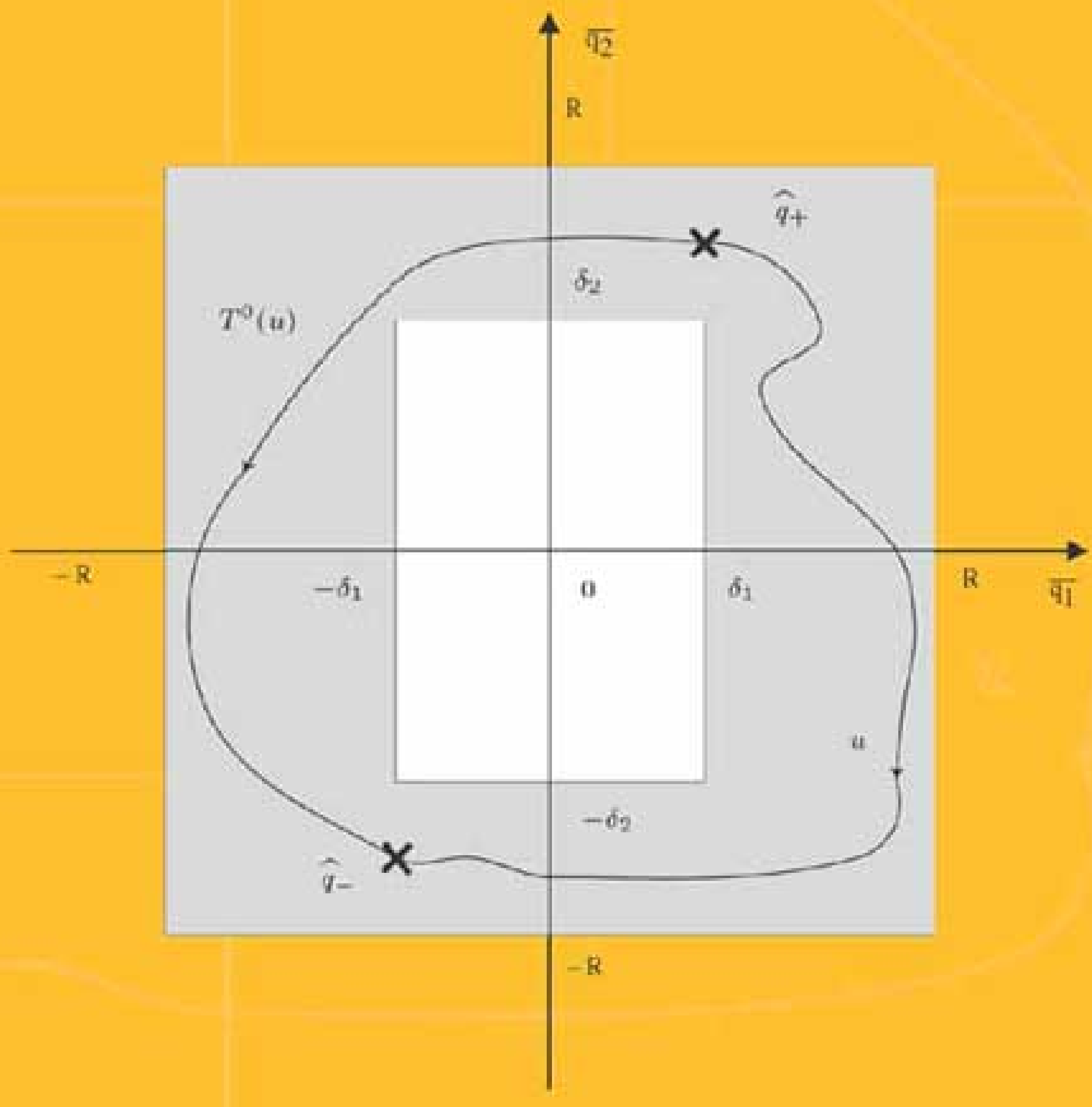

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TABLE OF CONTENTS

PREFACE

The notion of a fixed point plays a crucial role in numerous branches of mathematics and its applications. Information about the existence of such points is often the crucial argument in solving a problem. In particular, topological methods of fixed point theory have been an increasing focus of interest over the last century.

These topological methods of fixed point theory are divided, roughly speaking, into two types. The first type includes such as the Banach Contraction Principle where the assumptions on the space can be very mild but a small change of the map can remove the fixed point. The second type, on the other hand, such as the Brouwer and Lefschetz Fixed Point Theorems, give the existence of a fixed point not only for a given map but also for any its deformations.

This book is an exposition of a part of the topological fixed and periodic point theory, of this second type, based on the notions of Lefschetz and Nielsen numbers. Since both notions are homotopy invariants, the deformation is used as an essential method, and the assertions of theorems typically state the existence of fixed or periodic points for every map of the whole homotopy class, we refer to them as homotopy methods of the topological fixed and periodic point theory.

The first purpose of the book was to give a systematic exposition of classical fixed point theory based on the Lefschetz–Hopf theorem and Nielsen theory, that would be accessible to students and mathematicians of varying background. The second aim of the book was to create an ordered presentation, suitable for research mathematicians in a broad number of fields and applications, of the most recent results on the homotopy theory of fixed and periodic points that is now spread throughout the literature of the last decade.

Originally we planned to include more relations of this theory to other branches of mathematics, but in order to preserve a more concise character of the text, we finally reduced this aspect to one chapter. The starting point and inspiration for the book were exquisite expositions of the topological fixed point theory of R. Brown [Br2], J. Dugundji and A. Granas [DuGr], and B. Jiang [Ji4]. Despite the fact that our presentation is different and self-contained, we highly recommend these books for their background material as well as for many applications that we felt would be repetitive in our book.

The book is directed to graduate students and mathematicians active in related areas with interests in the topological methods of periodic points theory. Prerequisites to the book are standard linear algebra, analysis and primary topology. Although our exposition starts from basic notions and definitions the reader is assumed to have some knowledge and skill in algebraic topology. Also some experience in algebra and number theory would make the reading easier.

The book is organized as follows. There are seven chapters, the first of which have an introductory character. Chapter II contains an exposition of the classical Lefschetz–Hopf fixed point theory.

In the Chapter III we give a survey of various methods of finding periodic points which are based on a use of the Lefschetz numbers of all iterations of a given map. It begins with a methodical study of arithmetical, algebraic and asymptotic properties of the mentioned sequences. It is sufficient to derive some theorems on the existence of periodic points in a general situation (e.g. for a C^1-map). However more interesting statements hold if we assume either a nice local analytical or geometrical structure of a map (e.g. holomorphic, transversal) or a special global structure of the space (e.g. low dimension, a form of the cohomology ring). These are discussed in the remaining sections of this chapter.

Chapter IV includes a self-contained presentation of the Nielsen fixed point theory. We give a detailed proof of the Wecken theorem and we present some computational methods especially for the fibre maps.

In Chapter V we study the periodic points and minimal periods by use of the Nielsen theory. We introduce some Nielsen type invariants, estimating the number of periodic points of a given period, or given minimal period respectively. Next we prove Wecken type theorems minimizing the number of periodic points.

In Chapter VI we apply the above methods to describe the set of homotopy minimal periods for a self-map of the compact nilmanifolds and so-called NR-solvmanifold. We also prove some rigid Šarkovski type theorems: the existence of a given small homotopy minimal period implies that the set of homotopy minimal periods of a map consists of all natural numbers.

As a finale, Chapter VII contains topics which are either direct consequences of the material of preceding chapters or are related to by the methods explored previously. We begin with showing that a commutativity of a map of a sphere with a fixed point free homeomorphism generates infinitely many periodic points. In the next section we give information about a proof of the Entropy Conjecture for the topological entropy of a map of a compact nilmanifold and its relations to the set of homotopy minimal periods. Subsequently we include a survey of results on the local fixed point index of iterations of planar maps. Finally, we describe applications of the Nielsen theory to the existence of periodic points of multivalued maps, e.g. solutions of differential inclusions.

Since the Schauder techniques generalize the Lefschetz and Nielsen theory onto noncompact spaces (assuming a kind of compactness of the maps), many of the theorems are also true in the noncompact case. Although such generalizations are important in applications, we decided to focus on the compact case.

The authors would like to express their thanks to Lech Górniewicz for encouraging them to take up the effort of writing this book. It is also a pleasant duty to thank our friends and colleagues, in particular Grzegorz Graff for many discussions concerning the text and pointing out errors. Finally we record to our indebtedness to the staff of the Kluwer Edition at Dordrecht, and in particular to Dr. Elizabeth Moll and Mrs Marlies Vlot, for patience and understanding throughout this long project.

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CHAPTER I

FIXED POINT PROBLEMS

Suppose that we have a set X and a self-map $f\colon X \to X$. The fundamental notions for us are as follows

(1.0.1) DEFINITION. A point $x \in X$ is called a *fixed point* of f if $f(x) = x$. The set of all $\{x \in X : f(x) = x\}$ is called the *fixed point set* of f and denoted by $\mathrm{Fix}\,(f)$. A point $x \in X$ is called a *periodic point* of f if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$. The set of all $\{x \in X : x \text{ is periodic}\}$ is called the *set of periodic points* of f and denoted by $P(f)$.

Note that every fixed point is a periodic point, i.e. $\mathrm{Fix}\,(f) \subset P(f)$.

Surprisingly many mathematical problems lead naturally to a fixed, or periodic, point problem or could be easily reformulated as such a problem. Let us only mention the problem of existence of an optimal solution in game theory (thus of importance economics) or the problem of existence of fixed points of mappings of partially ordered sets in logic (computer science). Periodic points are studied in the dynamical systems (chaos theory). To study all these problems, a fixed point theory was founded and periodic point theory is developed. It consists of various genres determined by used mathematical tools and technics which depend on the structures of spaces and properties of maps in problem.

Now let us make a simple observation.

(1.0.2) *Let $F\colon X \to X$ be a map of a linear space (e.g. $X = \mathbb{R}^n$, or $\mathbb{C}^n$). A point $x \in X$ is a solution of the equation $F(x) = 0$ if and only if it is a fixed point of the map $f = \mathrm{id} - F$.*

Using the observation (1.0.2) we see that solving a system of linear equations $A\vec{x} = \vec{v}$ where A is an $n \times n$ matrix and $\vec{v} \in \mathbb{R}^n$, finding a root of w, where $w(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ is a polynomial with real or complex coefficients, we deal with a fixed point problem. Similarly several other algebraic equations could be posed as the fixed point problems. Since our reformulation is very commonplace, one would hardly expect that it could help to study the above algebraic problems. But, to an amazement, in some classical problems of algebra the topological approach is the most natural and successful. One of such problems is *the Fundamental Theorem of Algebra*, which proof we also derive as an outcome of the theory presented in this book. On the other hand there are many fixed point problems which are posed either on spaces without any natural topological structure or with maps that are not continuous. Such fixed point problems are not objects of discussion by topological fixed point theory.

Summing up, as the *first requirement for the topological fixed point theory* is:

(1.0.3) *Formulations and assumptions of such a theory have to be stated in terms of topological, geometrical properties of a space and its self-maps.*

Our aim is to present a part of the topological fixed and periodic points theory in which the homotopy methods are the dominating tools. In this chapter we would like to present rather intuitively the nature of this part. We will give examples representative for this part and also some which are not covered by this theory.

We introduce next some other notions which also will be fundamental for us. As previously, let $f\colon X \to X$ be a self-map of a set X.

(1.0.4) Definition. If $x \in X$ is a periodic point of f then any $m \in \mathbb{N}$ such that $f^m(x) = x$ is called a *period of* x. The smallest period of x is called the *minimal period* of x with respect to f. The set of all minimal periods of $x \in X$ is called the set *of minimal periods* of f and denoted by $\operatorname{Per}(f)$.

Observe next that the set of periodic points can be decomposed following the periods and split into a disjoint union following the minimal periods.

Put a notation

$$\begin{aligned} P^m(f) &:= \operatorname{Fix}(f^m) = \{x \in X : m \textit{ is a period of } x\}, \\ P_m(f) &:= \{x \in X : m \textit{ is the minimal period of } x\}. \end{aligned} \tag{1.0.5}$$

It is easy to check that

$$P^m(f) = \bigcup_{k|m} P_k(f) \quad \text{and consequently } P(f) = \bigcup_{m=1}^{\infty} P^m(f) = \bigcup_{m=1}^{\infty} P_m(f),$$

where $k|m$ means that k divides m. Notice that the last is the disjoint union of sets. Later we will define the fundamental algebraic invariants of a map f which allow us to study the following notions:

- Lefschetz number $L(f)$ (cf. (2.3.12)), correspondingly Lefschetz numbers $\{L(f^m)\}$ of all iterations and their algebraic combinations, informing about the existence of fixed, respectively periodic points.
- Nielsen number $N(f)$ (cf. (4.1.2)), correspondingly Nielsen periodic numbers $NF_m(f)$ (cf. (5.1.16)), $NP_m(f)$ (cf. (5.1.14)) estimating from below the number of fixed, respectively points of period m and m-periodic points.

Conditions posed on the map

We begin with a group of problems in which the conditions forcing fixed points are posed on the map rather than on the space.

It is usual in the study of mathematics to see the importance of the fixed point theory for the first time when we consider the classical problem of the existence of solutions of a differential equation. Here we show a method of converting an equation into a fixed point problem.

(1.0.6) Let $\phi(u,t)$, be a continuous function $\phi\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ of two real variables. Consider the ordinary differential equation

$$\frac{du}{dt} = \phi(u(t),t),$$

with the Cauchy initial condition $u(t_0) = u_0$, with $t_0, u_0 \in \mathbb{R}$. This equation can be written as the integral equation

$$u(t) = u_0 + \int_{t_0}^{t} \phi(u(s),s)\,ds.$$

The latter is a fixed point problem for the space $X = C[a,b]$, the Banach space of all continuous functions on a closed interval $[a,b]$, with $a = t_0$ and b appropriately chosen. Indeed, if we set $f(u)(t) := u_0 + \int_{t_0}^{t} \phi(u(s),s)\,ds$, then f maps X into X, because an integral of a continuous function is a function with continuous derivative, thus continuous. Moreover, $u(t)$ is a solution of the integral equation if and only if it satisfies $u = f(u)$.

It is a classical method of the differential equation theory to pose on ϕ, and on one end b of the interval if necessary, an appropriate assumption such that the above problem has a fixed point. One can find the proof in many elementary courses of differential equations. However we refer the reader to the book [Br3] where a topological point of view is exposed. The proof of the fixed point is based on the *Banach contraction principle*, which says that

(1.0.7) (Banach Contraction Principle). Suppose that $f\colon X\to X$ is a map of a complete metric space with a metric ρ. If f is a *contraction*, i.e. there exists $\kappa < 1$ such that for every $x,y\in X$, $\rho(f(x),f(y)) \le \kappa\rho(x,y)$, then there exists a unique fixed point x_0 of f.

Using the Darboux theorem it is easy to show that every contraction of the interval I has a unique fixed point (see Exercise (1.0.16)). Moreover it is not difficult to prove the Banach contraction principle (1.0.7) by considering the sequence $x_{i+1} := f(x_i)$, $x_1 = x$ any point of X. Note also that the proof of the Banach contraction principle (1.0.7) gives us in effect the iteration method of finding the solution by its approximation.

We say that a map $f: X \to X$ of a metric space X is the *non-expansive map* if for every $x, y \in X$ we have $\rho(f(x), f(y)) \leq \rho(x, y)$.

The following theorem says that to ensure that a self-map $f: X \to X$ has a fixed point it is enough to assume that it is non-expansive provided X has some geometrical properties (see [DuGr] for a proof and more details).

(1.0.8) THEOREM (F. Browder, D. Goehde, W. Kirk) *Let $f: X \to X$ be a non-expansive map of a closed, bounded and convex subset $X \subset H$ of a Hilbert space H. Then f has a fixed point.*

Note that the fixed point of such a map is not unique in general. For more information about theorems of this kind we refer to [DuGr], [GoE], and also [HandII].

As we already mentioned, to ensure that the map of example (1.0.6) is a contraction we have to put some assumption on the map ϕ. In the abstract form it is called *the Lipschitz map.*

(1.0.9) We say that a map $f: X \to X$ of a metric space X satisfies the Lipschitz condition with the constant $L \geq 0$ if for every $x, y \in X$ we have

$$\rho(f(x), f(y) \leq L\rho(x, y).$$

It is easy to show that if the function ϕ of (1.0.6) satisfies the Lipschitz condition, with respect to the first variable uniformly with respect to the second, then the associated map f is a contraction provided the interval $[a, b]$ is small enough (see [Br2], [HandI] for details). For us it is notable that a contraction is a special case of the Lipschitz map (with $L < 1$). The following we leave to the reader as an exercise.

(1.0.10) EXERCISE. Show that for every continuous $\phi: [0, 1] \to \mathbb{R}$ and every $\varepsilon > 0$ there exists a function $\psi: [0, 1] \to \mathbb{R}$ such that ψ is Lipschitz and $\|\psi - \phi\| < \varepsilon$. (Use the theorem about the uniform approximation of continuous functions by polynomials of the classical calculus).

Show that for every continuous $\phi: [0, 1] \to \mathbb{R}$ (e.g. for a Lipschtizian) and every $\varepsilon > 0$ there exists a function $\psi: [0, 1] \to \mathbb{R}$ such that ψ is not Lipschitzian and $\|\psi - \phi\| < \varepsilon$. (Use the previous part of exercise and approximate uniformly a polynomial by a spline function with edges being of the type as it of the function $g(x) = \sqrt[3]{x}$ at 0).

Conclude from the above that there is a contraction $f: C[a, b] \to C[a, b]$ such that for every $\varepsilon > 0$ there exists a map $h: C[a, b] \to C[a, b]$ which is not a contraction and such that $\|h - f\| < \varepsilon$.

The examples of Exercise (1.0.10) show that the fixed point theorem based on the Banach contraction principle, or for a non-expansive map, does not satisfy a property stated below (1.0.11) which we wish to have for a topological fixed point theory. Consequently the fixed point theorems based on metric assumptions posed on the map (as in the Banach principle) are not of the kind we are going to study.

(1.0.11) A topological fixed and periodic point theory is called *perturbation stable* if, when the assumptions of its theorem are verified for a given map f, then they are satisfied for any of its small perturbation as well.

Moreover, we would like to have the above property not only for small perturbations but for any deformation of a map.

(1.0.12) DEFINITION. Let $f_0, f_1 \colon X \to X$ be continuous maps of a topological space X. We say that f_0 is *homotopic* to f_1 if there exists a continuous map $H \colon X \times I \to X$, $I = [0,1]$ such that $H(x,0) = f_0(x)$, $H(x,1) = f_1(x)$. $H(x,t)$ is called a homotopy between f_0 and f_1.

Note that homotopy defines an equivalence relation in the set of self-maps.

We are in a position to formulate our final requirement for a topological fixed and periodic point theory to be of interest to us.

(1.0.13) In this book we will present a *homotopy fixed and periodic point theory* i.e. a theory such that, if the assumptions of a theorem are verified for a given map f, then they are satisfied for any map homotopic to it.

(1.0.14) REMARK. For many spaces with a nice local structure i.e. for smooth manifolds, polyhedrons or ENRs (see Subsection 2.2.2 for the definition of ENR) every small perturbation of a self-map f is homotopic to f (cf. [Do1], [Sp]). This means that every homotopy fixed point theory is perturbation stable.

Conditions posed on the space

In this book we would like to present the homotopy fixed and periodic point theory originated by Brouwer in the beginning of the twentieth century. Roughly speaking it claims the existence of a fixed (periodic) point for any continuous self-map of a space X provided X has a special geometric property. If we consider a map $f \colon I \to I$ and draw its graph then we quickly realize the following.

(1.0.15) Let X be a set and $f \colon X \to X$ a self-map of it. By Δ we denote the *diagonal* $\{(x,x)\} \subset X \times X$. A point $x_0 \in X$ is a fixed point of f if and only if the graph $\{(x, f(x))\} \subset X \times X$ intersects Δ at x_0, i.e. $(x_0, f(x_0)) = (x_0, x_0)$.

(1.0.16) EXERCISE. It is easy to see that it is impossible to draw a graph of a continuous function $f: I \to I$ which would not intersect the diagonal. Prove that every continuous function $f: I \to I$ has a fixed point. (Take $h(x) = f(x) - x$ and use the *Darboux property.*)

The above property of the interval holds also for the m-cube I^m. This is the statement of the Brouwer theorem (cf. [Br2]) and also will be derived from our general considerations (Corollary (2.4.4)). Any proof of the Brouwer theorem is much more difficult than the proof for $m = 1$ (Exercise (1.0.16)). One can say that every continuous map has a fixed point, because our space is very simple. Indeed, it is true but we are going to present a theory developed by Solomon Lefschetz in the 1920s–30s (cf. [Lef]),which gives a sufficient condition for the existence of a fixed, consequently periodic, point, and is a homotopy fixed point theory (see Sections 2.3 and 2.4 of Chapter II). From the point of view of this theory "simplicity" of the cube means that every self-map of it is homotopic to the constant map. The basic invariant of the Lefschetz theory is the *Lefschetz number* $L(f)$ of a map $f: X \to X$ (cf. Definition (2.3.12) and Theorem (2.3.16)), which is defined in terms of the linear map induced by the rational homology, or equivalently rational cohomology, spaces of X and is a homotopy invariant. The main theorem of this theory says that if $f: X \to X$ is a map of geometrically fine space (cf. Theorem (2.4.1)) then $L(f) \neq 0$ implies $\operatorname{Fix}(f) \neq 0$. It is an extension of the Brouwer theorem, because $L(f) = 1$ for any self-map of the cube. This means that if the geometrical structure of the space X is plain, e.g. is homotopically equivalent to the point like the cube, then the Lefschetz theorem works effectively. On the other hand, then the sequence $\{L(f^m)\}$ of the Lefschetz number of all iterations of any map f, is constant and does not allow us to distinguish different iterations of it, which is important in the study of periodic points of f. But in general the sequence $\{L(f^m)\}_1^\infty$ contains information about the periodic points. Thus as the Lefschetz number is our first main tool in the study of fixed points of a map so the behaviour of sequence $\{L(f^n)\}_1^\infty$ would be our first main tool in the study of the existence of periodic points.

The next example shows that if a given map has a fixed point, this does not yet mean that any map homotopic to it has a fixed point too.

(1.0.17) EXAMPLE. Consider the unit circle $S^1 \subset \mathbb{C}$. Let $f_\theta(z) = e^\theta z$ be a twist by a small angle θ. Then f_θ is a small perturbation of the identity homotopic to it by $H(z,t) = e^{t\theta} z$. Now $\operatorname{Fix}(f_\theta) = \emptyset$ although $\operatorname{Fix}(\mathrm{id}) = S^1$.

Note that the above example shows indirectly that $L(\mathrm{id}\,|_{S^1}) = 0$, otherwise we get a contradiction.

The class of the identity represents the only homotopy class of the circle containing maps without fixed points. Indeed, it is known [Ji4], [Sp] that every map of $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is homotopic to the map $z \mapsto z^r$ for some $r \in \mathbb{Z}$. Coincidentally we have ([Br2], [Ji4], [Sp]),Theorem (2.4.1), Example (2.2.31) $L(f) = 1-r$ for a map of the circle.

Considering further this example we observe that each of the above representatives of every homotopy class has $r-1$ fixed points, because the equation $z^r = z$ has $r-1$ solutions with $|z| = 1$, which are exactly the roots of unity of degree $r-1$. One can ask the following question:

(1.0.18) For a given set $A \subset X$ by $\#A$ we denote the cardinality of A (identifying all infinite cardinalities). Is it possible to give any fine estimate of $\#\mathrm{Fix}\,(f)$ for all f in the homotopy class? In particular is it true that for every $f\colon S^1 \to S^1$ homotopic to z^r we have $\#\mathrm{Fix}\,(f) \geq r-1$? Note that if it is true, then the number $r-1$ is achieved in a given class for the map z^r, thus it is a sharp estimate.

The answer for the above question is positive and the problem for $X = S^1$ is discussed in detail in the first chapter of the Jiang book [Ji4], in the Kiang book [Ki] or in Chapter 4 of this book. This is the leading example for the Nielsen theory. The Nielsen number $N(f)$ of a self-map $f\colon X \to X$ is an integer that has the property $N(f) \leq \#\mathrm{Fix}\,(f)$, is a homotopy invariant, and has the above estimate as the best for majority of spaces X (see [Ji4] and Chapter IV).

This leads to a formulation of the last requirement for a fixed point theory which we desire.

(1.0.19) A topological fixed and periodic point or theory, e.g. a homotopy fixed point theory, is called a *multiplicity fixed point theory* if it provides nontrivial estimates of $\#\mathrm{Fix}\,(f)$, $\#P^n(f)$, or $\#P_n(f)$.

An invariant of multiplicity, homotopy fixed point theory is called *fine* if it gives a sharp estimate of $\#\mathrm{Fix}\,(f)$, $\#P^n(f)$, or $\#P_n(f)$ in a given homotopy class.

As we have already mentioned, the Nielsen theory is an example of a fine multiplicity, homotopy theory. Remark that it is difficult to derive the Nielsen number of a self-map from its geometrical definition. In the Nielsen theory the geometric complexity of the space X is reflected by its fundamental group $\pi_1(X, *)$, (cf. [Sp]). In Chapter V we show that in several cases $N(f)$ can be derived from the homomorphism $f_\#$ induced by f on the fundamental group. In particular, if the fundamental group of X is trivial, then the Nielsen theory reduces to the Lefschetz point theory i.e. $N(f) = 1$ if and only if $L(f) \neq 0$ and $N(f) = 0$ if and only if $L(f) \neq 0$ (cf. Theorem (2.4.3). The following example of Michael Shub ([Sh]) illustrates many interesting phenomena, in particular shows that does not exist yet a reasonable fine multiplicity homotopy fixed point theory if the fundamental group of the space is trivial.

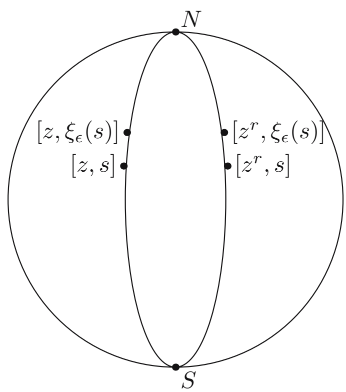
(1.0.20) EXAMPLE. For a given $r \geq 2$, let $h\colon S^1 \to S^1$ be a map of the circle of degree r, e.g. $h(z) := z^r$. Next for a $\varepsilon > 0$, let $\xi_\varepsilon\colon [0,1] \to [0,1]$ be a map satisfying $\xi_\varepsilon(0) = 0$, $\xi_\varepsilon(1) = 1$ and $t < \xi_\varepsilon(t) < t + \varepsilon$ for $0 < t < 1$; for example

$$\xi_\varepsilon(s) := \varepsilon(\sqrt{s} - s) + s$$

for a sufficiently small $\varepsilon > 0$.

Representing the two-dimensional sphere S^2 as the suspension of S^1, i.e. $S^2 = S^1 \times [0,1]/\sim$ where $S^1 \times \{0\} \sim *$, $S^1 \times \{1\} \sim *$, are contracted to $\mathrm{S} = [S^1 \times 0]$ and $\mathrm{N} = [S^1 \times 1]$ South and North Pole respectively, we define a map

$$f_\varepsilon([z,s]) := [(h(z), \xi_\varepsilon(s))].$$



If ε is small then f is a small perturbation of the suspension of z^r i.e. the map $g[z,s]; = [z^r, s]$ is homotopic to g by the homotopy

$$f_t([z,s]) := [z^r, (1-t)\varepsilon(\sqrt{s} - s) + s].$$

Observe that we have $P^n(g) = \mathrm{Fix}\,(g^n) = \{[z,s] : z \in \mathrm{Fix}\,(z^r \circ \ldots \circ z^r)\}$ for every $n > 0$, because g maps only the first coordinate. Also $n \in \mathbb{N}$ is a minimal period for $[z,s]$ with respect to g if and only if it is a minimal period for z^r, which gives $P_n(g) = \{[z,s] : z \in P_n(z^r)\}$. Furthermore, for every $n > 0$ the set $P_n(z^r) \neq \emptyset$, because the equation $z^r \circ \ldots \circ z^r = z^{r^n} = z$ has solutions consisting of the primitive roots of unity of degree $r^n - 1$. Consequently $\mathrm{Per}(g) = \mathbb{N}$.

On the other hand, for every $n \in \mathbb{N}$, $f_\varepsilon^n([z,s]) = [z,s]$ if and only if $s = 0, 1$, because the function $\xi_\varepsilon(s) > s$ for $0 < s < 1$. This shows that

$$\mathrm{Fix}\,(f_\varepsilon) = P^1(f_\varepsilon) = P_1(f_\varepsilon) = P(f_\varepsilon) = [S^1 \times \{0\}] \cup [S^1 \times \{1\}]$$

consists of the “poles” only. Consequently $\mathrm{Per}(f_\varepsilon) = \{1\}$.

The above example shows that for the sets $P^n(f)$, $P_n(f)$ their cardinalities, as well as $\mathrm{Per}(f)$, are not perturbation stable invariants of the periodic point theory. By the positive answer for (1.0.18) for maps of the circle, we see that $P^n(h)$ and $\mathrm{Per}(h)$ are stable notions for a map $h: S^1 \to S^1$, but they lose this property after the suspension to S^2. It is a consequence of the fact that the fundamental group changes from $\mathbb{Z}$ for S^1 into the trivial for S^2, making the Nielsen theory meaningless.

As we said: we are going to present the Lefschetz and Nielsen fixed and periodic points theory of self-maps of a space X having a nice local geometrical structure e.g. is a finite dimensional manifold, a finite dimensional polyhedron etc. This and the compactness of X allows us to define and use the Lefschetz and Nielsen theory to study the fixed and periodic points of f. Then a global assumption on X as contractible (convex) implies the existence of a fixed point for every self-map. If X is not contractible, then the existence (an estimate) of fixed, correspondingly periodic points of f, follows from the non-vanishing of $L(f)$ (a quantity of $N(f)$, respectively), or from a behaviour of the sequence $\{L(f^m)\}$ (a growth of the sequence $\{N(f^m)\}$ respectively). But to define $L(f)$, or $N(f)$ for any continuous map we have to use compactness of X.

On the other hand it was observed by J. Schauder that the correspondent of the Brouwer theorem holds also for a non-compact X if only some compactness assumption is put on the map f. A map $f: X \to E$ of a subset X of a normed linear space is called *completely continuous* if for every bounded subset $\Omega \subset X$ the image $f(\Omega)$ is relatively compact in E, i.e. the closure of $f(\Omega)$ is compact in E (cf. [DuGr]).

(1.0.21) Theorem (Schauder). *Every completely continuous map $f: X \to X$ of a convex subset of a normed space has a fixed point.*

We emphasize that the completely continuous maps appear in a natural way (due to the Arzelo-Ascoli, or Sobolev embedding theorem) in the analysis and consequently the Schauder theorem and its modifications have several important applications. In particular the Schauder theorem yields the existence of solutions of integral (thus differential) equations of the form

$$u(t) = u_0 + \int_{t_0}^{t} \phi(u(s), s)\, ds,$$

where we seek solutions $u(t)$ defined on the whole line $\mathbb{R}^1$. To use the Schauder theorem we need to assume that the continuous function $\phi(u, s)$ is bounded.

It is worthy of note that the Schauder theorem has been developed to a construction of a homotopy, and perturbation stable, fixed point theory for self-maps of bounded subsets of a normed linear space provided the homotopies and perturbations are by completely continuous maps (we refer the reader to [DuGr] and [HandII]). In our terms the latter means that the Schauder theory is not a perturbation stable and homotopy fixed point theory but it is a completely continuous perturbation stable and respectively completely continuous homotopy fixed point theory. It seems to be of interest (and possible at least for a few problems) to extend to this case some parts of the theory of periodic points presented in this book.

CHAPTER II

LEFSCHETZ–HOPF FIXED POINT THEORY

In this chapter we present the classical Lefschetz fixed point theory which culminates in the Lefschetz–Hopf fixed point theorem. This theorem had been well presented in many books ([Br2], [DuGr], [Sp]; however, to make this book more self-contained we include an exposition of it. This theory has two complementing components: geometric (local) the fixed point index theory, and algebraic (global) the Lefschetz number theory.

In Section 2.1 we begin with the geometric part of the Lefschetz–Hopf theory introducing the fixed point index, which theory was developed by Heinz Hopf in [HpI], [HpII]. In order to do this we present the differential definition of the degree based on the Sard theorem. Next we show the homological definition of this notion. It will allow us to define, and give the properties of, the fixed point index of a map.

On the other hand in Section 2.3 we study the algebraic part of the theory directing out our attention to the algebraic derivation of the Lefschetz number. We focus our consideration on those properties that will enable the computation of the Lefschetz numbers of iterations of a map. On the other hand we would like to present the theory in a more general version. To do this we include some information about the trace of a matrix including a definition of the generalized trace for projective modules. The latter allows us to define Lefschetz number for a cohomology theory with any coefficients as well as for the generalized cohomology theory such that cohomology groups are finitely generated over the cohomology of point. Next we show that all the generalized Lefschetz numbers are either equal to the classical Lefschetz number or are remainders of it, modulo an integer m, and consequently are useless for the fixed point theory.

In the last Section 2.4 we give a proof of the Lefschetz–Hopf theorem. We include a new definition based on the simplicial complex structure of the Euclidean space. It gives a proof of the Lefschetz–Hopf formula identifying the Lefschetz number with the sum of local fixed point indices. It indicates also a potential possibility of an effective derivation of the local fixed point.

2.1. Degree of a map

2.1.1. Definition of degree of a map. Let $E = \mathbb{R}^m$ be a Euclidean space, $\mathcal{U} \quad E$ its open subset. A continuous map $f\colon \mathcal{U} \quad E$ is called *d-compact* if $f^{-1}(0)$ is compact. For each d-compact map we will define an integer $\deg(f) \in \mathbb{Z}$ called *the degree of the map*. This integer measures algebraically the number of elements of $f^{-1}(0)$. Since the Euclidean space E is homogeneous, we will not lose generality assuming that $0 \in E$ is the chosen point.

Let us start with some examples illustrating what one should expect under this notion.

(2.1.1) PROPERTIES.

(2.1.1.1) If $f = \mathrm{id}_E$, then the degree equals 1.

(2.1.1.2) If f is a finite covering of multiplicity k, then the degree equals $+k$ or $-k$ since the point 0 is covered k times.

(2.1.1.3) The degree of the map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(t) = t^2 - 1$ is zero since $f^{-1}(0) = \{-1, +1\}$ but the map f is increasing at $+1$ and decreasing at -1. In other words f preserves the orientation of $\mathbb{R}$ at the point $+1$ and reverses it at -1.

As we will see the degree may be characterized as the function subordinating to each d-compact map an integer and satisfying five properties: Localization, Units, Additivity, Multilplicativity and Homotopy Invariance. It will not be so difficult to prove that these properties determine the degree, i.e. that there is at most one function satisfying them. However it is not so easy to prove the existence of such a function. The optimal tool for the construction of the degree is, in our opinion, homology. However we will start by presenting the definition of the degree using differential methods instead of algebraic ones. This will emphasize the geometry of this notion and also, we hope, may encourage the reader preferring analytical methods.

2.1.2. Differential definition of the degree. The main tool which will enable us to define the degree, avoiding the homology, will be the famous Sard Theorem. We start with some definitions. Let E, E' be Euclidean spaces, $\mathcal{U} \quad E$ an open subset and let $f\colon \mathcal{U} \quad E'$ be a smooth, i.e. C^∞, map. The point $x \in \mathcal{U}$ is called *regular* if the differential Df_x has maximal rank, i.e. is an epimorphism. Otherwise the point x will be called *critical.* A point $y \in E'$ will be called *a regular value* if $f^{-1}(0)$ contains only regular points. Otherwise (i.e. if $f^{-1}(0)$ contains a critical point of f) the point y is called a *critical value.* Notice that then each point $y \notin f(\mathcal{U})$ is a regular value.

We base our argument on the following theorem (cf. [Mil]).

(2.1.2) Theorem (Sard Theorem). *Let E, E' be Euclidean spaces* (*possibly of different dimensions*), *let $\mathcal{U} \quad E$ be an open subset and let $f\colon \mathcal{U} \quad E'$ be a smooth map. Then the set of critical values*

$$\{y \in E' : y \text{ is a critical value of } f\} \subset E'$$

has the Lebesgue measure zero. As a consequence the set of regular values

$$\{y \in E' : y \text{ is a regular value of } f\} \subset E'$$

is dense in E'.

Let us recall also another classical result (cf. [Mil]).

(2.1.3) Lemma. *Let E, E' be Euclidean spaces* (*possibly of different dimensions*), *let $\mathcal{U} \quad E$ be an open subset and let $f\colon \mathcal{U} \quad E'$ be a continuous map. Moreover, let $\alpha\colon \mathcal{U} \quad \mathbb{R}$ be a continuous function satisfying $\alpha(x) > 0$. Then there is a smooth map $g\colon \mathcal{U} \quad E'$ satisfying $\|f(x) - g(x)\| < \alpha(x)$ for all $x \in U$. In particular for any number $\varepsilon > 0$ there is a smooth ε-approximation g_ε of the given continuous map f.*

(2.1.4) Lemma. *Let $f\colon \mathcal{U} \quad E$ be a continuous d-compact map. Then:*

(2.1.4.1) *f is d-compactly homotopic to a smooth map g.*
(2.1.4.2) *Moreover, the map g may be so chosen that 0 is its regular value.*
(2.1.4.3) *Any two smooth maps d-compactly homotopic to f* (*as continuous maps*) *are smoothly d-compactly homotopic.*

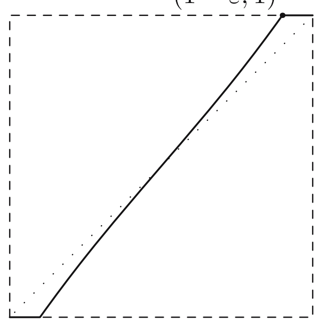
Proof. (2.1.4.1) Since f is d-compact, there is a compact set $K \subset \mathcal{U}$ such that $f^{-1}(0) \subset \operatorname{int} K$. We define the function $\alpha\colon \mathcal{U} \quad K \to \mathbb{R}$ by the formula $\alpha(x) = \|f(x)\|/2$. Then $\alpha(x) > 0$ for all $x \in \mathcal{U} \quad K$. By Tietze's Theorem (cf. [DuGr]) α extends to a function $\alpha\colon E \to \mathbb{R}$ and here we may moreover assume that $\alpha(x) > 0$ for all $x \in E$. Now let $g\colon \mathcal{U} \quad E$ be a smooth α-approximation of f (from Lemma (2.1.3)). We will show that the segment homotopy $H(x,t) = (1-t)f(x) + tg(x)$ between these maps is d-compact. In fact for any $x \notin K$ we have $H(x,t) \neq 0$ since then

$$\begin{aligned}\|H(x,t)\| &= \|f(x) + t(g(x) - f(x))\| \geq \|f(x)\| - \|t(g(x) - f(x))\| \\ &\geq \|f(x)\| - \|\alpha(x)\| \geq \frac{1}{2}\|f(x)\| > 0\end{aligned}$$

which implies $H(x,t) \neq 0$ for $x \notin K$.

(2.1.4.2) It may occur that the point 0 is not a regular value of the smooth map g. Nevertheless by Theorem (2.1.2) the set of regular values is dense, hence we may choose a regular value y_0 lying close to 0. Let ϕ be a local diffeomorphism of E sending y_0 to 0. Then the composition ϕg has 0 as the regular value. On the other hand we may take the deformation ϕ with a compact support, hence the segment homotopy between ϕg and g is d-compact.

(2.1.4.3) Let $\eta\colon [0,1] \to [0,1]$ be a smooth function given by the Figure below (i.e. $\eta(t) = 0$ for $t < \varepsilon$, $\eta(t) = 1$ for $t > 1-\varepsilon$ and η is increasing on $[\varepsilon, 1-\varepsilon]$). Then $H'(x,t) = H(x,\eta(t))$ is also d-compact. Now we may approximate H' inside $\mathcal{U} \quad (\varepsilon/2, 1-\varepsilon/2)$ by a smooth map and extend it by the constant homotopy onto the whole $\mathcal{U} \quad [0,1]$ to a homotopy $H''\colon \mathcal{U} \quad I \to E'$.



It remains to show that H'' may be taken d-compact. Since H' is d-compact, there is a compact set $K \subset \mathcal{U}$ such that $H'(x,t) = 0$ implies $x \in \operatorname{int} K$. Let $\alpha\colon (\mathcal{U} \quad \operatorname{int} K) \times I \to \mathbb{R}$ be given by $\alpha(x,t) = \|H'(x,t)\|$. Since $\alpha(x,t) > 0$ (for $(x,t) \in (\mathcal{U} \quad \operatorname{int} K) \times I$), α admits an extension by Tietze's theorem to a continuous function $\alpha\colon \mathcal{U} \quad I \to \mathbb{R}$ which may be positive: $\alpha(x,t) > 0$ for all $(x,t) \in \mathcal{U} \quad I$. If we demand that H'' satisfies $\|H''(x,t) - H'(x,t)\| < \alpha(x,t)$, then $H''(x,y) \neq 0$ for $x \notin \operatorname{int} K$ hence H'' is d-compact. $\square$

(2.1.5) Lemma. *Let $K \subset E$ be compact and let the homotopy $H\colon K \times I \to E'$ have no zeroes on the boundary. Then the restriction $H_|\colon \operatorname{int} K \times I \to E'$ is d-compact.*

Proof. $H_|^{-1}(0) = H^{-1}(0)$ is compact as the closed subset of the compact space $K \times I$. $\square$

Now we are going to define the degree of a continuous d-compact map $f\colon \mathcal{U} \quad E$ where $\mathcal{U} \quad E$ is an open subset.

First we moreover assume that f is smooth and 0 is its regular value. Then by the Inverse Function Theorem each $x \in f^{-1}(0)$ is an isolated zero. Now $f^{-1}(0)$, both compact and discrete, is also finite. Let Df_x denote its derivative at the point $x \in f^{-1}(0)$. Since $0 \in E$ is a regular value, the linear map $Df_x\colon E \to E$ has the maximal rank i.e. is epi hence iso. We will use a shorthand $\operatorname{sgn}(Df_x) = \operatorname{sgn}(\det(Df_x))$ for the sign of the determinant of the matrix representing the linear map $Df_x\colon E \to E$. Let us notice that this sign does not depend on the choice of the basis since any two matrices A, B representing the same linear map (in different bases) are conjugate $B = PAP^{-1}$ and their determinants are equal. Now we define the degree of the map $f\colon \mathcal{U} \quad E$ as

$$\deg(f) = \sum_{x \in f^{-1}(0)} \operatorname{sgn}(Df_x).$$

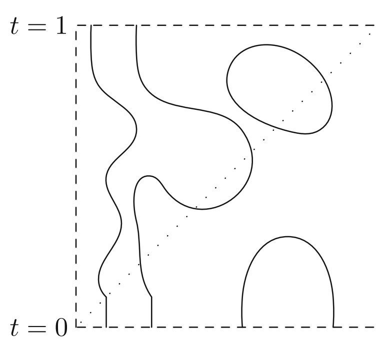
Let $f\colon \mathcal{U} \quad E$ be an arbitrary continuous d-compact map. We find a smooth map g which is d-compactly homotopic to f (see Lemma (2.1.4)) and such that 0 is its regular value. We define

$$\deg(f) = \deg(g).$$

We have to show that this definition is correct. By Lemma (2.1.4) f is d-compactly homotopic to a smooth map g such that 0 is a regular value of g. Moreover, any other such d-compact smooth approximation g' is d-compactly homotopic to g. The correctness of the definition follows from:

(2.1.6) Lemma. *If $g_0, g_1\colon \mathcal{U} \quad E$ are d-compactly homotopic smooth maps having* 0 *as the regular value then* $\deg(g_0) = \deg(g_1)$.

Proof. By (2.1.4.3) there is a smooth d-compact homotopy $H\colon \mathcal{U} \quad I \to E$ between g_0 and g_1. Moreover, by Sard's Theorem, we may assume that H is transverse to the point 0, i.e. $H^{-1}(0)$ is a smooth 1-dimensional manifold and the differential $DH_{(x,t)}\colon E \times I \to E$ is epi for each point $(x,t) \in H^{-1}(0)$. If 0 is not a regular value of H we may, as in the proof of Lemma (2.1.4), find a regular value y_0 close to 0, then we may consider a small isotopy ϕ satisfying $\phi(0) = y_0$ and finally we may replace H with ϕH. After this correction $H^{-1}(0)$ is a compact manifold, hence it splits into a finite number of components (arcs and circles) in $\mathcal{U} \quad I$ see the figure below.



We will show that each of these components gives the same contribution to both $\deg(g_0)$ and $\deg(g_1)$.

If the component is a circle this contribution is zero. Now we consider an arc $\gamma\colon I \to H^{-1}(0)$ going from $\gamma(0) \in \mathcal{U} \quad 0$ to a point $\gamma(1) \in \mathcal{U} \quad 1$. We observe that the restriction of $DH_{(x,t)}$ to the tangent space of $\gamma \subset H^{-1}(0) \subset U \times I$ is zero, hence the restriction of $DH_{(x,t)}$ to the normal space at this point must be epi hence iso. Let $\mathcal{B}_0$ denote the canonical basis of the vector space E regarded as the tangent space to the manifold $\mathcal{U} \quad 0$ at the point $\gamma(0)$. This tangent space coincides with the normal space to $\gamma \subset \mathcal{U} \quad I$. Let $\mathcal{B}_t$ be a basis of the normal space to $H^{-1}(0) \subset \mathcal{U} \quad I$ at the point $\gamma(t)$ obtained by a continuous shift of $\mathcal{B}_0$. Let us notice that the image of $\mathcal{B}_t$ by the map H is still a basis of the vector space E determining the same orientation i.e. its determinant has a constant sign. On the other hand we notice that the basis $\mathcal{B}_1$ gives the standard orientation of the normal space to $H^{-1}(0)$ at the point $\gamma(1)$. But this normal space coincides with the tangent space to $\mathcal{U} \quad 1$ at this point. All the above implies that the tangent map $D(g_0)_{\gamma(0)}$ sends the basis $\mathcal{B}_0$ to the basis giving the same orientation of E as the image of the basis $\mathcal{B}_1$ by $D(g_1)_{\gamma(1)}$. But as we have noticed $\mathcal{B}_0$ and $\mathcal{B}_1$ give the same orientation, hence the contribution to the degree in these points are the same.

If the arc $\gamma \subset H^{-1}(0)$ begins and ends at the same level, say $\mathcal{U} \quad 0$, then we may follow all the above, but then the bases $\mathcal{B}_0$ and $\mathcal{B}_1$ give inverse orientations of E, hence the contribution of these two points to $\deg(g_0)$ is zero. □

(2.1.7) Example. (2.1.7.1) $\deg(\mathrm{id}_{\mathbb{R}^m}) = 1$. In fact $(\mathrm{id}_{\mathbb{R}^m})^{-1}(0) = 0$, hence

$$\deg(\mathrm{id}_{\mathbb{R}^m}) = \operatorname{sgn}(\mathrm{id}) = +1.$$

(2.1.7.2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(t) = at$. If $a \neq 0$, then $f^{-1}(0) = \{0\}$ is compact. Now $\deg(f) = \operatorname{sgn}(Df) = \operatorname{sgn}(a)$. For $a = 0$ the map f is not d-compact, hence the degree is not defined.

(2.1.7.3) Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^k$ for $k \in \mathbb{N}$. We will show that $\deg(f) = k$. Here as above $f^{-1}(0) = 0$ but unfortunately 0 is no longer a regular value if $k \geq 2$. Nevertheless any other point in $\mathbb{C}$ is a regular value. Let us fix such a point $z_0 \neq 0$. Then the formula $h_t(z) = z^k - tz_0$ is a d-compact fixed homotopy from $h_0(z) = z^k$ to $h_1(z) = z^k - z_0$. Now

$$h_1^{-1}(0) = n \text{ roots of the point } z_0 \in \mathbb{C}.$$

It remains to notice that the determinant of the derivative of a holomorphic function $h(z)$ is positive at each point where $h'(z) \neq 0$. In fact let us identify

$z = x + iy = (x, y)$ and $h(x, y) = (u(x, y), v(x, y))$. Then the determinant of the map h equals

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x \cdot v_y - u_y \cdot v_x = u_x^2 + u_y^2$$

where the last equality comes from the Cauchy–Riemann equations. Now $h'(z) \neq 0$ implies that the determinant is positive at this point. Finally we get $\deg(z^k) = k$.

(2.1.8) EXERCISE. Let $f\colon \mathbb{R} \supset \mathcal{U} \quad \mathbb{R}$ be a d-compact map of an open interval of the real line and $f^{-1}(0) \subset \mathcal{U}$. Show that $\deg(f, \mathcal{U}) \in \{-1, 0, 1\}$.

2.1.3. Properties of the degree. We will show the basic properties of the degree.

Let $\mathcal{U} \quad E$ be an open subset of a Euclidean space E and let $f\colon \mathcal{U} \quad E$ be a d-compact map.

(2.1.9) LEMMA (Localization). *Let $i\colon \mathcal{V} \quad$ be the inclusion of an open subset satisfying $f^{-1}(0) \subset \mathcal{V}$ and $f_{|\mathcal{V}}\colon \mathcal{V} \quad E$ denote the restriction of f. Then* $\deg(f_{|\mathcal{V}}) = \deg(f)$.

PROOF. Let us notice that we may choose a d-compact smooth approximation g of f satisfying $g^{-1}(0) \subset \mathcal{V}$. In fact it is enough to choose in the proof of Lemma (2.1.4) the compact set $K \subset \mathcal{V}$. Now the restriction $g_{|\mathcal{V}}$ is also a d-compact approximation of $f_{|\mathcal{V}}$ and, since $g_{|\mathcal{V}}^{-1}(0) = g^{-1}(0)$,

$$\deg(f) = \sum_{x \in g^{-1}(0)} \operatorname{sgn}(Dg_x) = \sum_{x \in g_{|\mathcal{V}}^{-1}(0)} \operatorname{sgn}(Dg_x) = \deg(f_{|\mathcal{V}}). \qquad \square$$

(2.1.10) LEMMA (Units). *Let $i\colon \mathcal{U} \quad E$ be the inclusion. Then*

$$\deg(i) = \begin{cases} 1 & \text{if } 0 \in \mathcal{U}, \\ 0 & \text{if } 0 \notin \mathcal{U}. \end{cases}$$

PROOF. If $0 \in \mathcal{U}$, then $\deg(i) = \operatorname{sgn}(Di_0) = 1$. $\square$

(2.1.11) LEMMA (Additivity). *If $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{U}$ are open subsets such that the restrictions $f_1 = f_{|\mathcal{U}_1}$, $f_2 = f_{|\mathcal{U}_2}$ are d-compact and $\mathcal{U}_1 \cap \mathcal{U}_2$ is disjoint from $f^{-1}(0)$, then*

$$\deg(f) = \deg(f_1) + \deg(f_2).$$

PROOF. The equality is obvious if f smooth and 0 is its regular value, since then

$$\begin{aligned} \deg(f) &= \sum_{x \in f^{-1}(0)} \operatorname{sgn}(Df_x) \\ &= \sum_{x \in f_1^{-1}(0)} \operatorname{sgn}(Df_x) + \sum_{x \in f_2^{-1}(0)} \operatorname{sgn}(Df_x) = \deg(f_1) + \deg(f_2), \end{aligned}$$

since $f^{-1}(0) = f_1^{-1}(0) \cup f_2^{-1}(0)$ is the disjoint sum.

In the general case it will be enough to find a compact d-homotopy $H\colon\mathcal{U} \quad I \to E$ starting from $H(\,\cdot\,,0) = f$ to a smooth map $g = H(\,\cdot\,,1)$ and satisfying $H(x,t) \neq 0$ for $x \notin \mathcal{U}_1 \cap \mathcal{U}_2$ and 0 is a regular value of H. Then the restrictions of H to $\mathcal{U}_1$ and $\mathcal{U}_2$ are also d-compact and

$$\deg(f) = \deg(g) = \deg(g_{|\mathcal{U}_1}) + \deg(g_{|\mathcal{U}_2}) = \deg(f_1) + \deg(f_2)$$

where the middle equality comes from the first part.

It remains to find the homotopy H. Since $f_i^{-1}(0) \subset \mathcal{U}_i$ is compact, we can find a compact set K_i satisfying $f_i^{-1}(0) \subset \operatorname{int} K_i \subset K_i \subset \mathcal{U}_i \setminus (\mathcal{U}_1 \cap \mathcal{U}_2)$ for $i = 1,2$. Let $K = K_1 \cup K_2$. Now we may repeat the proof of Lemma (2.1.4) with the above K. The obtained homotopy has all zeroes inside K which is disjoint from $\mathcal{U}_1 \cap \mathcal{U}_2$. $\square$

(2.1.12) Lemma (Homotopy Invariance). *Let* $\mathcal{U} \quad E$ *be an open subset and let* $F\colon\mathcal{U} \quad I \to E$ *be a d-compact map. Then* $\deg(f_0) = \deg(f_1)$ (*where* $f_t = F(\,\cdot\,,t)$ *for* $0 \leq t \leq 1$).

Proof. Let g_0, g_1 be d-compactly smooth approximations of f_0 and f_1 respectively such that 0 is a regular value of both g_0 and g_1. Then by Lemma (2.1.4) there is a d-compact smooth homotopy between g_0 and g_1. Now

$$\deg(f_0) = \deg(g_0) = \deg(g_1) = \deg(f_1),$$

where the middle equality comes from Lemma (2.1.6). $\square$

(2.1.13) Lemma (Multiplicativity). *If* $f\colon\mathcal{U} \quad E$, $f'\colon\mathcal{U}' \to E'$ *are compactly fixed, then so is* $f \times f'\colon\mathcal{U} \quad ' \to E \times E'$ *and* $\deg(f \times f') = \deg(f) \cdot \deg(f')$.

Proof. Let g, g' be d-compact smooth approximations of f, f' respectively such that 0 is a regular value of both g_0 and g_1. Then $g \times g'$ is a d-compact approximation of $f \times f'$. It remains to show that the equality holds for g and g'. Identifying the linear map with the matrix representing it in the canonical basis we have

$$D(g \times g')_{(x,x')} = \begin{bmatrix} Dg_x & 0 \\ 0 & Dg'_{x'} \end{bmatrix},$$

which implies $\operatorname{sgn}(D(g \times g')_{(x,x')}) = \operatorname{sgn}(D(g_x)) \cdot \operatorname{sgn}(D(g'_{x'}))$. Now

$$\begin{aligned} \deg(g \times g') &= \sum_{(x,x') \in (g \times g')^{-1}(0)} \operatorname{sgn}(D(g \times g')_{(x,x')}) \\ &= \sum_{x \in g^{-1}(0),\ x' \in g'^{-1}(0)} \operatorname{sgn}(Dg_x) \cdot \operatorname{sgn}(Dg'_{x'}) \\ &= \sum_{x \in g^{-1}(0)} \operatorname{sgn}(Dg_x) \cdot \sum_{x' \in g'^{-1}(0)} \operatorname{sgn}(Dg'_{x'}) = \deg(g) \cdot \deg(g'). \quad \square \end{aligned}$$

2.1.4. Orientation of the Euclidean space. Now we will define the degree using homology. The obtained number will be also denoted $\deg(f)$ although it will not be clear that it coincides with the degree defined in the previous section using differential methods. This equality will be proved in Section 2.1.8. We will start with an analysis of the notion of orientation of the Euclidean space. This notion is intuitively evident and we have already used it it the previous section. It can be expressed exactly as a homology element and we are going to do it now.

Let E denote n-dimensional Euclidean space. Usually the orientation of a finitely dimensional vector space is defined as a choice of an ordered basis $e_1, \ldots, e_n$. Another basis $e'_1, \ldots, e'_n$ determines the same orientation if and only if the matrix transforming one basis into the other has positive determinant. Another (equivalent) method is the choice of an ordered sequence of $n+1$ affinely independent points $a_0, \ldots, a_n \in E$: we assume that these points determine the same orientation as the basis $a_1 - a_0, \ldots, a_n - a_0$ of the linear space E.

The following fact we leave as an exercise

(2.1.14) Exercise. Let $e_1, \ldots, e_n$ be a basis and let the affinely independent points $a_0, \ldots, a_n$ have the coordinates: $a_i = \sum_{j=1}^{n} a_{ij} e_j$. Then the orientations determined by the basis $e_1, \ldots, e_n$ and the points $a_0, \ldots, a_n$, considered as a basis, are equal if and only if

$$\det \begin{bmatrix} 1 & \ldots & 1 \\ a_{01} & \ldots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{0n} & \ldots & a_{nn} \end{bmatrix} > 0.$$

Later we will need a definition which could be generalized into manifolds; spaces which are only locally Euclidean. The orientation of a manifold will be defined as a coherent choice of orientations near each point.

(2.1.15) Definition. The *orientation* of a Euclidean n-dimensional vector space E at *a point* $x \in E$ is a choice of a generator of the n-homology group $H_n(E, E \setminus x) \approx \mathbb{Z}$.

This definition is compatible with the traditional one as follows. Let

$$\Delta_n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : t_0 + \cdots + t_n = 1,\ t_i \geq 0\}$$

denote the n-dimensional standard simplex. This simplex is the convex set spanned by the vertices of the standard basis of

$$\mathbb{R}^{n+1} : v_0 = (1, 0, \ldots, 0), v_1 = (0, 1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1).$$

Now any choice of affinely independent points $a_0, \dots, a_n$ determines the affine map $\phi\colon \Delta_n \to \mathbb{R}^n$ given by $\phi(v_i) = a_i$. If $y \in \operatorname{int}(\phi(\Delta_n))$, then the homology class $[\phi]$ is a generator of the homology group $H_n(E, E \setminus \{y\}) = \mathbb{Z}$. Let us denote this generator by z_y.

Let us fix a local orientation z_{y_0} (a generator of $H_n(E, E \setminus y_0)$) near y_0. Let B be a closed ball containing y_0 in its interior. Then for any $y \in \operatorname{int} B$ we have isomorphisms induced by the natural inclusions

$$H_n(E, E \setminus y_0) \xleftarrow{=} H_n(E, E \setminus B) \xrightarrow{=} H_n(E, E \setminus y).$$

Elements corresponding to z_{y_0} by these isomorphisms are denoted z_B, z_y respectively. In general for a compact subset $K \subset E$ we find a closed ball B containing K in its interior and we define the *orientation along* K as the image of z_B (defined above) by the homomorphism $\iota_{KB}\colon H_n(E, E \setminus B) \to H_n(E, E \setminus K)$ induced by the natural inclusion.

(2.1.16) LEMMA. *The definition of the local orientation along a compact subset $K \subset E$ does not depend on the choice of the ball B. It depends only on the choice of the orientation at a point $x \in K$.*

PROOF. Suppose that B' is another ball containing K and x. Let z_K, z'_K denote the orientations obtained by the use of the ball B and B' respectively. Assume at first that $B' \subset B$. Then $z_K = i_{KB}(z_B) = i_{KB'}(i_{B'B} z_B) = i_{KB'} z_{B'} = z'_K$ where i_{KB} denotes the homology homomorphism induced by the inclusion $(E, E \setminus B) \subset (E, E \setminus K)$. In the general case we fix a ball B'' containing both B, B'. Then by the above $z_K = z''_K$ and $z'_K = z''_K$ which implies $z_K = z'_K$. □

Thus fixing the orientation in a point x we determine the orientation along each compact subset of the Euclidean space.

If $\mathcal{U}$ E is an open subset containing the compact set K, then $z_K \in H_m(E, E \setminus K)$, the orientation along K, may be considered as the element of $H_m(\mathcal{U}, \mathcal{U}$ $K) = H_m(E,\, E \setminus K)$, where the last equality is the excision isomorphism.

2.1.5. Homologic definition of the degree. Now we are in a position to present the homologic definition of the degree of a d-compact map.

Let E denote n-dimensional Euclidean space, $\mathcal{U}$ E its open subset. Let us fix an orientation of the space E by fixing a generator $z_0 \in H_n(E, E \setminus 0)$. Let $f\colon U \to E$ be a d-compact map.

Let $z_{f^{-1}(0)} \in H_n(\mathcal{U}, \mathcal{U}$ $f^{-1}(0)) = H_n(E, E \setminus f^{-1}(0))$ denote the orientation along the compact set $f^{-1}(0)$. The map f induces the homomorphism

$$f_*\colon H_n(\mathcal{U}, \mathcal{U} \quad f^{-1}(0)) \to H_n(E, E \setminus 0) = \mathbb{Z}.$$

Now $f_*(z_{f^{-1}(0)}) = d \cdot z_0$ for an integer d. This number will be called the *degree* of the d-compact map f and denoted $\deg(f)$.

Let us notice that $\deg(f)$ does not depend on the choice of the orientation of the space E since the change of the orientation reverses simultaneously signs of both $z_{f^{-1}(0)} \in H_n(\mathcal{U},\mathcal{U} \quad f^{-1}(0))$ and $z_0 \in H_n(E, E \setminus 0)$.

(2.1.17) EXAMPLE. Below we derive the homologic degree for some simple maps.

(2.1.17.1) $\deg(\mathrm{id}_{\mathbb{R}^m}) = 1$. In fact $\mathrm{id}_*(z_0) = 1 \cdot z_0$.

(2.1.17.2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(t) = at$. If $a \neq 0$, then $f^{-1}(0) = \{0\}$ is compact. The homomorphism $f_*\colon H_1(\mathbb{R}, \mathbb{R} \setminus 0) \to H_1(\mathbb{R}, \mathbb{R} \setminus 0)$ is given by $f_*(z) = \mathrm{sgn}(a) \cdot z$ which implies $\deg(f) = \mathrm{sgn}(a)$. For $a = 0$, the map f is not d-compact, hence the degree is not defined.

(2.1.17.3) Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^k$ for $k \in \mathbb{Z}$. We will show that $\deg(f) = k$. Since $f^{-1}(0) = 0$, we have to show that $f_*\colon H_2(\mathbb{C}, \mathbb{C} \setminus 0) \to H_2(\mathbb{C}, \mathbb{C} \setminus 0) = \mathbb{Z}$ is the multiplication by k. We consider the following commutative diagram:

$$\begin{tikzcd}
H_2(\mathbb{C}, \mathbb{C} \setminus 0) \arrow[r, "="] \arrow[d, "f_*"'] & H_1(\mathbb{C} \setminus 0) \arrow[d, "f_*"] & H_1(S^1) \arrow[l, "="'] \arrow[d, "f_*"] \\
H_2(\mathbb{C}, \mathbb{C} \setminus 0) \arrow[r, "i_*"'] & H_1(\mathbb{C} \setminus 0) & H_1(S^1) \arrow[l, "="]
\end{tikzcd}$$

The horizontal arrows are isomorphisms: those on the left-hand side by the long exact sequence of the pair $(\mathbb{C}, \mathbb{C}\setminus 0)$ and those on the right-hand side since $S^1 \subset \mathbb{C}\setminus 0$ is the homotopy equivalence. It remains to notice that the homomorphism induced by $f(z) = z^k$ on $H_1(S^1) = \pi_1(S^1) = \mathbb{Z}$ is the multiplication by k.

2.1.6. Properties of the degree. We will show that the homologic degree also satisfies five basic properties.

Let $\mathcal{U} \quad E$ be an open subset of a Euclidean space E and let $f\colon \mathcal{U} \quad E$ be a d-compact map.

(2.1.18) LEMMA (Localization). *Let $i\colon \mathcal{V} \qquad$ be the inclusion of an open subset satisfying $f^{-1}(0) \subset \mathcal{V}$. Then $\deg(f_{|\mathcal{V}}) = \deg(f)$.*

PROOF. The diagram

$$\begin{tikzcd}
H_n(\mathcal{V}, \mathcal{V} \quad f^{-1}(0)) \arrow[r, "f_*"] \arrow[d, "i_*"'] & H_n(E, E \setminus 0) \arrow[d, "="] \\
H_n(\mathcal{U}, \mathcal{U} \quad f^{-1}(0)) \arrow[r, "f_*"'] & H_n(E, E \setminus 0)
\end{tikzcd}$$

is commutative. $\square$

(2.1.19) Lemma (Units). *Let* $i\colon \mathcal{U} \quad E$ *be the inclusion. Then*

$$\deg(i) = \begin{cases} 1 & \text{if } 0 \in \mathcal{U}, \\ 0 & \text{if } 0 \notin \mathcal{U}. \end{cases}$$

Proof. We notice that the composition

$$H_n(E, E \setminus 0) \xleftarrow{i_*} H_n(\mathcal{U}, \mathcal{U} \quad 0) \xrightarrow{i_*} H_n(E, E \setminus 0)$$

is the identity map (by excision) if $0 \in \mathcal{U}$ and is zero if $0 \notin \mathcal{U}$ since then

$$H_n(\mathcal{U}, \mathcal{U} \quad 0) = 0 \qquad \square.$$

(2.1.20) Lemma (Additivity). *If* $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{U}$ *are open subsets such that the restrictions* $f_{|\mathcal{U}_1}$, $f_{|\mathcal{U}_2}$ *are compactly fixed and* $\mathcal{U}_1 \cap \mathcal{U}_2$ *is disjoint from* $f^{-1}(0)$*, then*

$$\deg(f) = \deg(f_{|\mathcal{U}_1}) + \deg(f_{|\mathcal{U}_2}).$$

Proof. Since $(f_{|\mathcal{U}_1})^{-1}(0)$, $(f_{|\mathcal{U}_2})^{-1}(0)$ are compact and disjoint, there exist open disjoint subsets $\mathcal{U}_i'$ satisfying $(f_{|\mathcal{U}_i})^{-1}(0) \subset \mathcal{U}_i' \subset \mathcal{U}_i$ $(i = 1, 2)$. By the Localization Property $\deg(f_{|\mathcal{U}_i}) = \deg(f_{|\mathcal{U}_i'})$ hence we may assume that the sets $\mathcal{U}_1$ and $\mathcal{U}_2$ are disjoint. Now the lemma follows from the commutativity of the diagram

$$\begin{tikzcd}
H_n(E, E \setminus f^{-1}0) & H_n(\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_1 \cup \mathcal{U}_2 \setminus f^{-1}(0)) \arrow[l, "="'] \arrow[r, "f_*"] & H_n(E, E \setminus 0) \\
H_n(E, E \setminus f^{-1}0) \arrow[u, "="] & H_n(\mathcal{U}_1, \mathcal{U}_1 \setminus f^{-1}(0)) \oplus H_n(\mathcal{U}_2, \mathcal{U}_2 \setminus f^{-1}(0)) \arrow[l, "="] \arrow[u, "i_*"] \arrow[r, "f_{|1*} \oplus f_{|2*}"'] & H_n(E, E \setminus 0) \arrow[u, "="']
\end{tikzcd}$$

$\square$

(2.1.21) Lemma (Homotopy Invariance). *Let* $\mathcal{U} \quad E$ *be an open subset and let* $F\colon \mathcal{U} \quad I \to E$ *be a d-compact map. Then* $\deg(f_0) = \deg(f_1)$*, where* $f_t = F(\,\cdot\,, t)$ *for* $0 \le t \le 1$.

Proof. Since $F^{-1}(0)$ is compact, $p_1(F^{-1}(0)) \subset \mathcal{U}$ is also compact, where $p_1\colon \mathcal{U} \quad I \to \mathcal{U}$ is the projection. Now we have a commutative diagram

$$\begin{tikzcd}
H_n(\mathcal{U}, \mathcal{U} \quad p_1 F^{-1}(0)) \arrow[r] \arrow[d, "="'] & H_n(\mathcal{U}, \mathcal{U} \quad f_0^{-1}(0)) \arrow[r, "f_{0*}"] \arrow[d, "i_{0*}"'] & H_n(E, E \setminus 0) \arrow[d, "="] \\
H_n((\mathcal{U}, \mathcal{U} \quad p_1 F^{-1}(0)) \times I) \arrow[r] & H_n(\mathcal{U} \quad I, \mathcal{U} \quad I \setminus F^{-1}(0)) \arrow[r, "F_*"] & H_n(E, E \setminus 0) \\
H_n(\mathcal{U}, \mathcal{U} \quad p_1 F^{-1}(0)) \arrow[r] \arrow[u, "="] & H_n(\mathcal{U}, \mathcal{U} \quad f_1^{-1}(0)) \arrow[r, "f_{1*}"'] \arrow[u, "i_{i*}"] & H_n(E, E \setminus 0) \arrow[u, "="']
\end{tikzcd}$$

It remains to notice that $i_{0*}(z_{f_0^{-1}(0)}) = i_{1*}(z_{f_1^{-1}(0)}) = j_*(z_{p_1 F^{-1}(0)}) \in H_n(\mathcal{U} \quad I, \mathcal{U} \quad I \setminus F^{-1}(0))$, where $j\colon (\mathcal{U}, \mathcal{U} \quad p_1 F^{-1}(0)) \times I) \to (\mathcal{U} \quad I, \mathcal{U} \quad I \setminus F^{-1}(0))$ is the inclusion and $z_{p_1 F^{-1}(0)} \in H_m(\mathcal{U}, \mathcal{U} \quad p_1 F^{-1}(0)) = H_m((\mathcal{U}, \mathcal{U} \quad p_1 F^{-1}(0)) \times I)$. $\square$

(2.1.22) LEMMA (Multiplicativity). *If* $f\colon\mathcal{U}\quad E$, $f'\colon\mathcal{U}'\to E'$ *are d-compact, then so is* $f\times f'\colon\mathcal{U}\quad '\to E\times E'$ *and* $\deg(f\times f')=\deg(f)\cdot\deg(f')$.

PROOF. Since $(f\times f')^{-1}(0)=f^{-1}(0)\times f'^{-1}(0)$, $f\times f'$ is d-compact. Now the product formula for the degree follows from the commutativity of the diagram

$$\begin{tikzcd}
H_{n+n'}(U\times U', U\times U'\setminus(f\times f')^{-1}(0)) \arrow[r, "(f\times f')_*"] \arrow[d, "="'] & H_{n+n'}(\mathbb{R}^n\times\mathbb{R}^{n'}, \mathbb{R}^n\times\mathbb{R}^{n'}\setminus 0) \arrow[d, "="] \\
H_{n+n'}((U, U\setminus f^{-1}(0))\times(U', U'\setminus f'^{-1}(0))) \arrow[r, "(f\times f')_*"] \arrow[d, "="'] & H_{n+n'}(\mathbb{R}^n\times\mathbb{R}^{n'}, \mathbb{R}^n\times\mathbb{R}^{n'}\setminus 0) \arrow[d, "="] \\
H_n((U, U\setminus f^{-1}(0))\otimes H_{n'}(U', U'\setminus f'^{-1}(0))) \arrow[r, "f_*\otimes f'_*"'] & H_n(\mathbb{R}^n, \mathbb{R}^n\setminus 0)\otimes H_{n'}(\mathbb{R}^{n'}, \mathbb{R}^{n'}\setminus 0)
\end{tikzcd}$$

since the homology element representing the orientation $z_{(f\times f')^{-1}(0)}\in H_{n+n'}(\mathcal{U}$ $\mathcal{U}',\mathcal{U}\quad '\setminus(f\times f')^{-1}(0))$ corresponds to $z_{f^{-1}(0)}\otimes z_{f'^{-1}(0)}\in H_n((\mathcal{U},\mathcal{U}\quad f^{-1}(0))\otimes$ $H_{n'}(\mathcal{U}',\mathcal{U}'\setminus f'^{-1}(0)))$ by the left vertical arrows. □

2.1.7. Degree at a regular point. Now we will show some formulae for the degree. We have two definitions of the degree and we still do not know whether they are equal. However we already know that they both satisfy the Five Basic Properties and only these properties will be used to prove the formulae of this section. The reader may have in mind the definition of the degree he prefers.

(2.1.23) COROLLARY. *A linear map* $f\colon E\to E$ *is d-compact if and only if is an isomorphism. Then*

$$\deg(f)=\begin{cases} +1 & \textit{if } f \textit{ preserves the orientation,}\\ -1 & \textit{if } f \textit{ reverses the orientation.}\end{cases}$$

In other words $\deg(f)=\operatorname{sgn}(\det A)$ *where* A *is a matrix representing* f.

PROOF. Let $f\colon E\to E$ be a linear map. Then $f^{-1}(0)=\operatorname{Ker}(f)$ and the last is compact if and only if f is an isomorphism. We assume that f is an isomorphism. Let us recall that the group $\mathrm{Gl}(\mathbb{R},n)$ has two connected components. If the matrix A representing the map f (in a fixed basis) satisfies $\det A>0$, then there is a path in $\mathrm{Gl}(\mathbb{R},n)$ joining A with the identity matrix. This yields a homotopy from f to the identity map id_E. Now $\deg(f)=\deg(\mathrm{id}_E)=1$. If $\det A<0$, then f can be connected with the map $g(t_1,\dots,t_n)=(t_1,\dots,t_{n-1},-t_n)$. Now by Multiplicity $\deg(f)=-1$. In each case $\deg(f)=\operatorname{sgn}(\det(A))$. □

(2.1.24) Lemma. *Let* $\mathcal{U}$ $\quad E$ *be a compact subset of a Euclidean space and let* $f\colon\mathcal{U}$ $\quad E$ *be a* C^1 *map. Let* $x_0 \in f^{-1}(0)$ *be a regular point, i.e. the derivative map* $Df_{x_0}\colon E \to E$ *is the isomorphism. Then* x_0 *is the isolated zero of* f *and*

$$\deg(f, x_0) = \deg(Df_{x_0}) = \operatorname{sgn}\det(A),$$

where A *is the matrix representing* Df_{x_0}. *In other words the degree at a regular point equals the degree of the derivative map.*

Proof. The regular point x_0 is the isolated zero by the Inverse Function Theorem. It remains to show that $\deg(f, x_0) = \operatorname{sgn}(\det Df_{x_0})$. The right-hand side of the equality in the lemma follows from Corollary (2.1.23). We will show that the segment homotopy between the maps $f(x)$ and $L(x) = Df_{x_0}(x - x_0)$ has no zeroes on the boundary of a sufficiently small ball $\mathrm{K}(x_0, \delta)$. Then the equalities

$$\deg(f, x_0) = \deg(f_{K(x_0,\varepsilon)}, x_0) = \deg(L_{K(x_0,\varepsilon)}, x_0) = \deg(Df_{x_0})$$

prove the lemma.

Let us denote $\varepsilon = \inf\{Df_{x_0}(h) : \|h\| = 1\}$. Since Df_{x_0} is iso, $\varepsilon > 0$ so, for $h \in E$, $\|Df_{x_0}(h)\| \geq \varepsilon\|h\|$. On the other hand from the definition of the derivative we have

$$\lim_{x\to x_0} \frac{f(x) - f(x_0) - Df_{x_0}(x - x_0)}{\|x - x_0\|} = 0.$$

Since $f(x_0) = 0$, there exists a $\delta > 0$ such that

$$\frac{\|f(x) - L(x)\|}{\|x - x_0\|} = \frac{\|f(x) - f(x_0) - Df_{x_0}(x - x_0)\|}{\|x - x_0\|} < \frac{\varepsilon}{2}$$

for $\|x - x_0\| \leq \delta$. Now $\|f(x) - L(x)\| \leq (\varepsilon/2)\|x - x_0\|$ for $\|x - x_0\| \leq \delta$.

We show that the segment homotopy between the maps $f(x)$ and $L(x)$ has no zero on the boundary of the ball $K(x_0, \delta)$. In fact since by the above $\|L(x)\| = \|Df_{x_0}(x - x_0)\| \geq \varepsilon\|x - x_0\| = \varepsilon\delta$ and $\|f(x) - L(x)\| \leq (\varepsilon/2)\delta$ for $x \in \operatorname{bd} K(x_0, \delta)$, zero does not belong to the interval $[f(x), L(x)]$. $\square$

As we have seen $\deg(f) \neq 0$ implies the existence of a point $x \in f^{-1}(0)$. It is easy to find maps with $\deg(f) = 0$ possessing zeroes: take as the example the map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(t) = t^2 - 1$. However the following lemma shows that in some situations $\deg(f) \neq 0$ gives the deformation of f to a map with no zeroes.

(2.1.25) Theorem (Hopf Lemma for the degree). *Let* $B \subset E$ *be a ball in a Euclidean* n*-dimensional space and let* $f\colon \operatorname{cl} B \to E$ *be a map satisfying* $f(x) \neq 0$

for $x \in \operatorname{bd} B$ *and* $\deg(f) = 0$. *Then there is a homotopy* $\{f_t\}$ *such that* $f_0 = f$ *and* $f_1(\operatorname{cl} B) \subset E \setminus 0$. *Moreover, the homotopy may be constant on the boundary.*

PROOF. Let $z \in H_n(\operatorname{cl} B, \operatorname{bd} B) = \mathbb{Z}$ be the generator corresponding to the chosen orientation. Then the homology homomorphism induced by the inclusion $i\colon (\operatorname{cl} B, \operatorname{bd} B) \to (\operatorname{cl} B, \operatorname{cl} B \setminus f^{-1}(0))$ satisfies $i_*(z) = z_{f^{-1}(0)}$. Now $(fi)_*(z) = f_* i_*(z) = f_*(z_{f^{-1}(0)}) = 0 \in H_n(E, E \setminus 0)$. Consider the commutative diagram

$$\begin{tikzcd}
H_n(\operatorname{cl} B, \operatorname{bd} B)) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{n-1}(\operatorname{bd} B) \arrow[d, "f_*"'] & \pi_{n-1}(\operatorname{bd} B) \arrow[l, "="'] \arrow[d, "f_\#"] \\
H_n(E, E \setminus 0) \arrow[r, "\partial"'] & H_{n-1}(E \setminus 0) & \pi_{n-1}(E \setminus 0) \arrow[l, "="]
\end{tikzcd}$$

where the horizontal arrows are isomorphisms: ∂ from the homology long exact sequence, "=" as Hurewicz isomorphism. Now $(fi)_* = 0$ implies $(fi)_\# = 0$. Since moreover the identity on $\operatorname{bd} B = S^{n-1}$ represents a generator of the group $\pi_{n-1}(\operatorname{bd} B) = \mathbb{Z}$, $fi\colon \operatorname{bd} B \to E \setminus 0$ is homotopic to a constant map to a point $y_0 \neq 0$. Let $h_s\colon \operatorname{bd} B \to E \setminus 0$ denote this homotopy. We define the map $H'\colon \operatorname{bd}(B \times I) \to E$ by the formula

$$H'(x,t) = \begin{cases} fi(x) & \text{for } t = 0, \\ y_0 & \text{for } t = 1, \\ h_t(x) & \text{for } x \in \operatorname{bd} B. \end{cases}$$

Since E is contractible, H' admits an extension $H'\colon B \times I \to E$. The map H' is a homotopy satisfying the first part of the Theorem. Now we will correct H' to a homotopy constant on the boundary. Since $0 \notin H'(\operatorname{bd} B \times I)$ and $\operatorname{bd} B \times I$ is compact, there is an open set $\mathcal{U} \quad B$ satisfying $\operatorname{bd} B \subset \mathcal{U}$ and $0 \notin H'((\operatorname{cl} \mathcal{U} \quad B) \times I)$. Let $\eta\colon \operatorname{cl} B \to [0,1]$ be an Urysohn function satisfying $\eta(\operatorname{cl} B \setminus \mathcal{U}) = 1$ and $\eta(\operatorname{bd} B) = 0$. Then $H(x,t) = H'(x, \eta(x)t)$ is a homotopy constant on the boundary. □

(2.1.26) COROLLARY. *Let* x_0 *be the unique zero of the map* $f\colon \mathcal{U} \quad E$ *with* $\deg(f, x_0) = 0$. *Then there is a d-compact homotopy* f_t *from* $f_0 = f$ *to a map* f_1 *with no zeroes. Moreover,* $\{f_t\}$ *may be chosen an* ε*-homotopy with the support in* $\mathcal{U}_0$ *for prescribed number* $\varepsilon > 0$ *and a prescribed neighbourhood* $\mathcal{U}_0 \subset \mathcal{U}$ *of the point* x_0.

PROOF. We may apply the above lemma to the restriction of f to a ball B centered at x_0 and contained in the prescribed neighbourhood $\mathcal{U}_0$. This yields a homotopy constant on the boundary of B and removing zeroes. This homotopy extends by a constant homotopy onto the whole $\mathcal{U}$. If moreover the ball is sufficiently small then the homotopy is smaller than the prescribed ε, because in the proof of Theorem (2.1.25) we used only the homotopy with the support in $\operatorname{cl} B$ and this support was deformed in $\operatorname{conv}(\operatorname{cl} B \cup f(\operatorname{cl} B))$. □

(2.1.27) THEOREM (Fundamental Theorem of Algebra). *Any polynomial*

$$W(z) = a_0 z^n + \cdots + a_{n-1} z + a_n, \quad a_i \in \mathbb{C},\ a_0 \neq 0,\ n > 0,$$

has a complex root.

PROOF. It is enough to show that $\deg(W) = n \neq 0$. Let us denote

$$W(z) = z^n \left(a_0 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n} \right) = z^n (a_0 + V(z)),$$

where $V(z) = a_1/z + \cdots + a_n/z^n$ is a meromorphic function satisfying $\lim_{z \to \infty} V(z) = 0$. Then there exists $R > 0$ such that $|V(z)| \leq |a_0|/2$ for $|z| \geq R$. Let us denote $W_t(z) = z^n(a_0 + tV(z))$. Now

$$|W_t(z)| = |z^n||a_0 + t\,V(z)| \geq |z^n|(|a_0| - |V(z)|) \geq |z^n|\frac{|a_0|}{2} \geq R^n \frac{|a_0|}{2} > 0$$

for $|z| \geq R$. Thus the homotopy $W_t\colon \mathbb{C} \to \mathbb{C}$ is d-compact hence

$$\deg W = \deg W_1 = \deg W_0,$$

where $W_0(z) = a_0 z^n$. Let a_t $(0 \leq t \leq 1)$ be a path joining in $\mathbb{C} \setminus 0$ the points a_0 and 1. Then $H_t(z) = a_t z^n$ is a d-homotopy from W_0 to the map z^n hence $\deg W_0 = n$. □

(2.1.28) EXERCISE. Prove that the degree of a complex polynomial in each root is positive and equals the multiplicity of this root.

Hint. Let z_0 be a root of a polynomial $W(z)$ of multiplicity r. Then $W(z) = V(z)(z - z_0)^r$ where $V(z)$ is a polynomial satisfying $V(z_0) = v_0 \neq 0$. Show that the segment homotopy between $V(z)$ and the constant map v_0 has no zeroes near z_0.

2.1.8. Uniqueness of the degree.

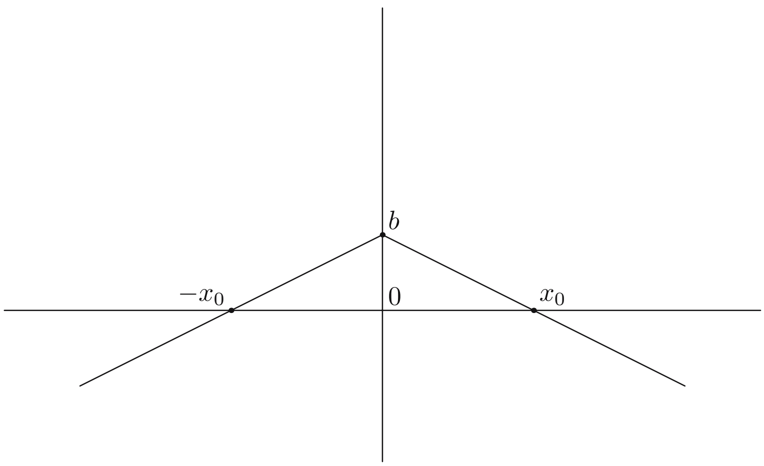
(2.1.29) THEOREM. *There exists a unique function subordinating to each d-compact map $f\colon \mathcal{U} \quad E$, $\mathcal{U}$ an open subset of the Euclidean space E, an integer* $\deg(f, \mathcal{U})$ *which satisfies the properties: Localization, Units, Additivity, Homotopy Invariance, Multiplicativity.*

PROOF. We first show that the above properties determine the degree $\deg(f)$ for special cases.

Case 1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = ax + b$ for $a, b \in \mathbb{R}$, $a \neq 0$. We show that $\deg(f) = \operatorname{sgn}(a)$.

Assume that moreover $a > 0$. Then the concatenation of the homotopies $h_s(x) = ax + (1-s)b$ and $h'_s(x) = ((1-s)a + s)x$ gives a d-fixed homotopy from f to $\mathrm{id}_{\mathbb{R}}$, hence by Homotopy and Units $\deg(f) = \deg(\mathrm{id}_{\mathbb{R}}) = 1$.

Now we assume that $a < 0$ and $b > 0$. Consider the map $g(x) = f(|x|)$ given by the figure below. This map is homotopic by $g_s(x) = \min\{g(x), (1-2s)b\}$ to a map with no zeroes. Thus $0 = \deg(g) = \deg(g, -x_0) + \deg(g, x_0)$. By the first part $\deg(g, -x_0) = 1$ which implies $\deg(g, x_0) = -1$, hence $\deg(f) = \deg(f, x_0) = \deg(g, x_0) = -1$.



If $a < 0$ and $b > 0$, then we consider the function $g(x) = f(-|x|)$.

Case 2. Let $f\colon E \to E$ be a linear map represented by a matrix $A \in \mathrm{Gl}(\mathbb{R}, n)$. Let $\det A > 0$. As in the proof of (2.1.24) we show that f is homotopic, via isomorphisms, to the identity on E, hence $\deg(f) = \deg(\mathrm{id}_E) = 1$. If $\det A < 0$, then f is homotopic to a map $f_1(x,t) = (x, -t) \in E' \times \mathbb{R}$. Now by the Mutliplicativity $\deg(f) = \deg(\mathrm{id}_{E'}) \cdot \deg(-I) = (+1)(-1) = -1$.

Case 3. Let $\mathcal{U}$ $\quad E$ be an open subset of the Euclidean space and let $f\colon \mathcal{U}$ $\quad E$ be a C^1 map. Let $x_0 \in f^{-1}(0)$ be a regular point, i.e. the derivative map $Df_{x_0}\colon E \to E$ is the isomorphism. Then x_0 is the isolated zero of f and

$$\deg(f, x_0) = \deg(Df_{x_0})$$

We follow the proof of Lemma (2.1.24) where only the Five Properties were used.

Case 4. Finally we discuss the general case. Let $f\colon \mathcal{U}$ $\quad E$ be a d-compact map. We may assume that $\mathcal{U}$ is bounded and the map f extends to a (continuous) map on the boundary. If not, we find an open bounded subset $\mathcal{U}'$ satisfying

$f^{-1}(0) \subset \mathcal{U}' \subset \mathrm{cl}\,\mathcal{U}' \subset \mathcal{U}$. By the Localization property $\deg(f,\mathcal{U}) = \deg(f,\mathcal{U}')$ and we may consider $\mathcal{U}'$ instead of $\mathcal{U}$. Let $\delta = \inf\{\|f(x)\| : x \in \mathrm{bd}\,\mathcal{U}$. By the compactness of $\mathrm{cl}\,\mathcal{U}$, $\delta > 0$. By Sard's Theorem there exists a $\delta/2$ homotopy f_t from $f_0 = f_{\mathrm{cl}\,\mathcal{U}'}$ to a map f_1 with $f_1^{-1}(0)$ finite and Df_1 an isomorphism at each point $x \in f_1^{-1}(0)$. Now $\|f_t(x)\| > \delta - \delta/2 > 0$ for all $x \in \mathrm{bd}\,\mathcal{U}$. Hence Homotopy and Additivity Properties imply

$$\deg(f) = \deg(f_1) = \sum_{x_i} \deg(f_1, x_i),$$

where the summation runs over the finite set $f_1^{-1}(0)$. But the values $\deg(f_1, x_i)$ are already determined, since f_1 is a C^1-map. □

(2.1.30) COROLLARY. *The degrees defined in Subsections* 2.1.2 *and* 2.1.5 *are equal.*

Now the proof of any property of the degree can be derived from the Five Properties. However it is often convenient to use any of the above definitions.

We will prove the following modifications of the Homotopy and Units Properties.

(2.1.31) LEMMA (Generalized Homotopy Invariance). *Let $\mathcal{W} \subset E \times I$ be an open subset and let $F\colon \mathcal{W} \to E$ be a d-compact map. Then* $\deg(f_0) = \deg(f_1)$, *where $f_t = F(\,\cdot\,, t)$ for* $0 \le t \le 1$.

PROOF. We may assume that $\mathrm{cl}\,\mathcal{W}$ is compact and F is defined on the boundary of $\mathcal{W}$ since otherwise we may replace $\mathcal{W}$ by a bounded subset $\mathcal{W}' \subset \mathcal{W}$ satisfying $F^{-1}(0) \subset \mathcal{W}' \subset \mathrm{cl}\,\mathcal{W}' \subset E \times I$.

We denote $\mathcal{W}_t = \{x \in E : (x,t) \in \mathcal{W}\}$ and we will show that the map $[0,1] \ni t \to \deg(f_t, \mathcal{W}_t) \in \mathbb{Z}$ is locally constant. Then the connectedness of $[0,1]$ will imply $\deg(f_0) = \deg(f_1)$.

Let us fix a point $t_0 \in [0,1]$. We will find an open subset $\mathcal{U}$ $\quad_{t_0}$ and a number $\varepsilon > 0$ such that $f_t^{-1}(0) \subset \mathcal{U}$ $\quad_t$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. Then by the Localization Property $\deg(f_t) = \deg(f_{t|U})$. But $F_|\colon \mathcal{U}$ $\quad [t_0 - \varepsilon, t_0 + \varepsilon] \to E$ is d-compact, hence by the Homotopy Property, $\deg(f_{t|\mathcal{U}})$ is constant in $[t_0 - \varepsilon, t_0 + \varepsilon]$.

It remains to find ε and $\mathcal{U}$. Since $f_{t_0}^{-1}(0) \subset \mathcal{W}_{t_0}$ is compact, there is an open set $\mathcal{U}$ $\quad E$ satisfying $f_{t_0}^{-1}(0) \subset \mathcal{U}$ $\quad \mathrm{cl}\,\mathcal{U}$ $\quad_{t_0}$. Now $\mathrm{cl}\,\mathcal{U}$ is compact, hence $\mathrm{cl}\,\mathcal{U}$ $\quad [t_0 - \varepsilon', t_0 + \varepsilon'] \subset \mathcal{W}$ for a sufficiently small number $\varepsilon' > 0$. On the other hand $0 \notin F((\mathrm{cl}\,\mathcal{W}_{t_0} \setminus \mathcal{U}) \times t_0)$ and the compactness of $\mathrm{cl}\,W$ imply the existence of $\varepsilon'' > 0$ such that $0 \notin F(\mathcal{W} \cap (E \times [t_0 - \varepsilon'', t_0 + \varepsilon'']) \setminus \mathcal{U}$ $\quad I)$. We put $\varepsilon := \min\{\varepsilon', \varepsilon''\}$. Then for each $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ the set $f_t^{-1}(0)$ is disjoint from $\mathcal{W}_{t_0} \setminus \mathcal{U}$, hence $f_t^{-1}(0) \subset \mathcal{U}$ as required. □

(2.1.32) LEMMA. *Let $\mathcal{U} \quad E$ be an open subset, $x_0 \in E$ and let $s_{x_0}: \mathcal{U} \quad E$ be given by $s_{x_0}(x) = x - x_0$. Then*

$$\deg(s_{x_0}) = \begin{cases} +1 & \textit{if } x_0 \in \mathcal{U}, \\ 0 & \textit{if } x_0 \notin \mathcal{U}. \end{cases}$$

PROOF. We use the differential definition of degree. If $x_0 \notin \mathcal{U}$, then $s_{x_0}^{-1}(0) = \emptyset$, hence $\deg(s_{x_0}) = 0$. If $x_0 \in \mathcal{U}$ then $s_{x_0}^{-1}(0) = \{x_0\}$ and x_0 is a regular value with $D(s_{x_0}) = \mathrm{id}$ hence $\deg(s_{x_0}) = \mathrm{sgn}(D(s_{x_0})_{x_0}) = 1$. □

2.2. Fixed point index

We will use the degree to define the fixed point index which is the algebraic measure of the number of fixed points (cf. Definition (1.0.1)) of a continuous self-map. Let X be a topological space, $\mathcal{U}$ its subset and $f: \mathcal{U} \quad X$ a map. Recall that $\mathrm{Fix}(f) = \{x \in \mathcal{U} : f(x) = x\}$ denotes the *fixed point set* which is a closed set there.

If $X = E$ is a Euclidean space, $\mathrm{Fix}(f) = F^{-1}(0)$ where $F: \mathcal{U} \quad E$ is given by the formula $F(x) = x - f(x)$.

(2.2.1) DEFINITION. We say that a map $f: X \to X$ is *compactly fixed* if its fixed point set is compact.

Let us notice that a map $f: E \to E$ of Euclidean space is compactly fixed if and only if $F(x) = x - f(x)$ is d-compact. Moreover, a point x is a fixed point of f if and only if x is a zero of F.

(2.2.2) DEFINITION. Let $\mathcal{U} \quad E$ be an open subset of a Euclidean space and let $f: \mathcal{U} \quad E$ be a compactly fixed map. We define the *fixed point index* of f as $\mathrm{ind}(f) := \deg(F)$ where $F(x) = x - f(x)$.

(2.2.3) REMARK. $\mathrm{ind}(f) \neq 0$ implies the existence of a fixed point of the map f. In fact $\deg(F) = \mathrm{ind}(f) \neq 0$, hence there is a zero of F which is a fixed point of f.

When the domain of $f: \mathcal{U} \quad E$ is obvious we will write simply $\mathrm{ind}(f)$. However we will often restrict the domain of the given map. Then we will write $\mathrm{ind}(f, \mathcal{U})$ to make precise which domain is the actual one.

(2.2.4) EXAMPLE. The formulae given below follow from the corresponding formulae for the degree. (2.2.4.1) Let $s: \mathcal{U} \quad E$ be a constant map $s(\mathcal{U}) = x_0$. Then

$$\mathrm{ind}(s) = \begin{cases} +1 & \text{if } x_0 \in \mathcal{U}, \\ 0 & \text{if } x_0 \notin \mathcal{U}. \end{cases}$$

In fact the map $F(x) = x - s(x) = x - x_0$ is the inclusion of the open subset and $0 \in F(\mathcal{U})$ if and only if $x_0 \in \mathcal{U}$. Now the formula follows from Lemma (2.1.32).

(2.2.4.2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(t) = at$ for $a \neq 1$. Then

$$\operatorname{ind}(f) = \begin{cases} +1 & \text{for } a < 1, \\ -1 & \text{for } a > 1. \end{cases}$$

(2.2.4.3) Let $f\colon E \to E$ be a linear map represented by the matrix A. Then $\operatorname{Fix}(f) = \operatorname{Ker}(\mathrm{id} - f)$ is compact $\Leftrightarrow$ $\mathrm{id} - f$ is an isomorphism $\Leftrightarrow$ 1 is not an eigenvalue of the linear map f. Now $\operatorname{ind}(f) = \deg(\mathrm{id} - f) = \operatorname{sgn}(\det(I - A))$.

(2.2.4.4) Let $W\colon \mathbb{C} \to \mathbb{C}$ be a polynomial map a degree $n \geq 1$ different than the identity $\mathrm{id}_{\mathbb{C}}$. Then $F(z) = z - W(z)$ is also a polynomial of degree $n > 0$. Since the number of the roots of a (nonconstant) polynomial is finite, $F(z)$ is d-compact and $\operatorname{ind}(f) = \deg(F) = n$.

(2.2.5) Example. Let $f\colon \mathbb{R} \supset \mathcal{U} \quad \mathbb{R}$ be a compactly fixed map of the real line such that $\operatorname{Fix}(f) \subset \mathcal{U}$. Show that $\operatorname{ind}(f, \mathcal{U}) \in \{-1, 0, 1\}$.

Hint. Use Exercise (2.1.8).

2.2.1. Properties of the fixed point index. The properties of the degree imply similar properties of the fixed point index in the Euclidean space. Let $f\colon \mathcal{U} \quad E$ be a compactly fixed map of an open subset to the Euclidean space.

(2.2.6) Lemma (Localization). *Let $i\colon \mathcal{U}' \to \mathcal{U}$ be the inclusion of an open subset satisfying* $\operatorname{Fix}(f) \subset \mathcal{U}'$. *Then $f_{|\mathcal{U}}$ is compactly fixed and* $\operatorname{ind}(f_{|\mathcal{U}'}) = \operatorname{ind}(f)$.

Proof. Recall that $\operatorname{ind}(f) = \deg(\mathrm{id} - f)$ and $\operatorname{ind}(f_{\mathcal{U}'}) = \deg(\mathrm{id} - f)_{\mathcal{U}'}$. Moreover,

$$(\mathrm{id} - f)^{-1}(0) = \operatorname{Fix}(f) = \operatorname{Fix}(f_{\mathcal{U}'}) = (\mathrm{id} - f)^{-1}_{\mathcal{U}'}(0).$$

Now by Lemma (2.1.9) the degrees are equal. □

(2.2.7) Lemma Units. *Let $\rho\colon \mathcal{U} \quad E$ be the constant map $\rho(\mathcal{U}) = x_0$. Then*

$$\operatorname{ind}(\rho) = \begin{cases} +1 & \textit{if } x_0 \in \mathcal{U}, \\ 0 & \textit{if } x_0 \notin \mathcal{U}. \end{cases}$$

Proof. Since $\operatorname{ind}(\rho) = \deg(s_{x_0})$ where $s_{x_0} = \mathrm{id} - \rho$, it remains to apply Lemma (2.1.10) (or (2.1.32)). □

(2.2.8) Lemma (Additivity). *If $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{U}$ are open subsets such that the restrictions $f_{|\mathcal{U}_1}$, $f_{|\mathcal{U}_2}$ are compactly fixed and $\mathcal{U}_1 \cap \mathcal{U}_2$ is disjoint from* $\operatorname{Fix}(f)$, *then*

$$\operatorname{ind}(f) = \operatorname{ind}(f_{|\mathcal{U}_1}) + \operatorname{ind}(f_{|\mathcal{U}_2}),$$

Proof. The map $\mathrm{id} - f$ satisfies the assumptions of Lemma (2.1.11), hence

$$\operatorname{ind}(f) = \deg(\mathrm{id} - f) = \deg(\mathrm{id} - f_{|\mathcal{U}_1}) + \deg(\mathrm{id} - f_{|\mathcal{U}_2}) = \operatorname{ind}(f_{|\mathcal{U}_1}) + \operatorname{ind}(f_{|\mathcal{U}_2}). \; □$$

(2.2.9) LEMMA (Homotopy Invariance). *Let* $\mathcal{W} \subset E \times [0,1]$ *be an open subset and let* $F\colon \mathcal{W} \to E$ *be a compactly-fixed map, i.e. the set*

$$\mathrm{Fix}\,(F) = \{(x,t) \in \mathcal{W} : F(x,t) = x\}$$

is compact. Then $\mathrm{ind}\,(f_0) = \mathrm{ind}\,(f_1)$*, where* $f_t = F(\,\cdot\,,t)$ *for* $0 \le t \le 1$.

PROOF. We may apply Lemma (2.1.12), since $(p_1 - F)^{-1}(0) = \mathrm{Fix}\,(f)$ is compact. Now

$$\mathrm{ind}\,(f_0) = \deg\,(p_1 - f_0) = \deg\,(p_1 - f_1) = \mathrm{ind}\,(f_1). \qquad \square$$

(2.2.10) LEMMA (Multiplicativity). *If* $f\colon \mathcal{U} \quad E$, $f'\colon \mathcal{U}' \to E'$ *are compactly fixed, then so is* $f \times f'\colon \mathcal{U} \quad ' \to E \times E'$ *and* $\mathrm{ind}\,(f \times f') = \mathrm{ind}\,(f) \cdot \mathrm{ind}\,(f')$.

PROOF. Since $\mathrm{Fix}\,(f \times f') = \mathrm{Fix}\,(f) \times \mathrm{Fix}\,(f')$ we may apply Lemma (2.1.13). Now

$$\mathrm{ind}\,(f \times f') = \deg\,(\mathrm{id} \times \mathrm{id'} - f \times f') = \deg\,(\mathrm{id} - f)\deg\,(\mathrm{id'} - f') = \mathrm{ind}\,(f) \cdot \mathrm{ind}\,(f'). \quad \square$$

The fixed point index possesses also a very important property which will enable us to extend its definition to a much larger class of spaces.

(2.2.11) LEMMA (Commutativity Property). *Let* $\mathcal{U} \quad E$, $\mathcal{U}' \subset E'$ *be open subsets of Euclidean spaces and let* $f\colon \mathcal{U} \quad E'$ *and* $g\colon \mathcal{U}' \to E$ *be continuous maps. Then the composites*

$$gf\colon \mathcal{V} = f^{-1}(\mathcal{U}') \to E, \qquad fg\colon \mathcal{V}' = g^{-1}(\mathcal{U}) \to E'$$

have homeomorphic fixed point sets $\mathrm{Fix}\,(gf) = \mathrm{Fix}\,(fg)$. *If moreover these sets are compact, then* $\mathrm{ind}\,(fg) = \mathrm{ind}\,(gf)$.

PROOF. The homeomorphisms are given by the restrictions:

$$f_|\colon \mathrm{Fix}\,(gf) \rightleftarrows \mathrm{Fix}\,(fg) : g_|.$$

In fact if $gf(x) = x$, then $f(x) = f(gf(x)) = fg(f(x))$, hence $f(x) \in \mathrm{Fix}(fg)$. Similarly $g(\mathrm{Fix}(fg)) \subset \mathrm{Fix}(gf)$. It remains to notice that the restrictions $gf_|$ and $fg_|$ are identities on $\mathrm{Fix}\,(gf)$ and $\mathrm{Fix}\,(fg)$, respectively.

We will prove the equality $\mathrm{ind}\,(fg) = \mathrm{ind}\,(gf)$ by showing that both sides equal $\mathrm{ind}\,(\gamma)$, where $\gamma\colon \mathcal{V} \quad ' \to E \times E'$ is given by the formula

$$\gamma(x,y) = (g(y), f(x)).$$

Consider the homotopy $\gamma_t(x,y) = (tgf(x) + (1-t)g(y), f(x))$. Then $\gamma_t(x,y) = (x,y)$ means $y = f(x)$ and $x = tgf(x) + (1-t)g(y)$ which implies $x = tgf(x) + (1-t)gf(x)$, hence $x = gf(x)$. Now $\operatorname{Fix}(\gamma_t) = \{(x, f(x)) : x \in \operatorname{Fix}(gf)\}$. Thus $\operatorname{Fix}(\gamma_t)$ is compact and does not depend on t, hence we get the compactly fixed homotopy. We get $\operatorname{ind}(\gamma_0) = \operatorname{ind}(\gamma_1)$ where $\gamma_1\colon \mathcal{V} \quad {}' \to E \times E'$ is given by $\gamma_1(x,y) = (gf(x), f(x))$. Consider the map $\delta\colon \mathcal{V} \quad E' \to E \times E'$ given by the same formula: $\delta(x,y) = (gf(x), f(x))$. Moreover, $\operatorname{Fix}(\gamma_1) = \operatorname{Fix}(\delta)$, hence by the Localization Property we get get $\operatorname{ind}(\gamma_1) = \operatorname{ind}(\delta)$. Now we consider the homotopy $\delta_t(x,y) = (gf(x), (1-t)f(x))$. Since $\delta_t(x,y) = (x,y)$ means $x = gf(x)$ and $y = (1-t)f(x)$, $\bigcup_{0 \le t \le 1} \operatorname{Fix}(\delta_t) \subset \operatorname{Fix}(gf) \times (I \cdot f(\operatorname{Fix}(gf))$ is also compact which in turn implies $\operatorname{ind}(\delta_0) = \operatorname{ind}(\delta_1)$. But $\delta_1\colon \mathcal{V} \quad E' \to E \times E'$ is given by $\delta_1(x,y) = (gf(x), 0)$, hence the Multiplicativity Property gives $\operatorname{ind}(\delta_1) = \operatorname{ind}(gf)$.

By the symmetry we may also show that $\operatorname{ind}(\gamma) = \operatorname{ind}(fg)$ which ends the proof of the lemma. □

2.2.2. Fixed point index of self-maps of ENRs. Since the question about the existence of a fixed point has sense for self-maps of an arbitrary topological space it is natural to ask whether the notion of the fixed point index can be generalized onto a larger class of spaces.

Let us recall that a topological space X is called a *retract* of another space Y if there exist maps $s\colon X \to Y$, $r\colon Y \to X$ satisfying $rs = \mathrm{id}_X$. Then the map r is called retraction. Let us notice that s is mono and r is epi so we may identify X with a closed subset of Y. A metric space X is called an *Euclidean neighbourhood retract*, or shortly an ENR, if X is a retract of an open subset of a Euclidean space. We write then $X \in \mathrm{ENR}$. It turns out that the class of ENRs is large (cf. [Do1]). In particular it contains all the classes of topological spaces we need for our considerations (cf. [Do1]).

(2.2.12) Theorem. *Every finite polyhedron and every finite* CW*-complex is an* ENR*. Moreover every finitely dimensional separable polyhedron or* CW*-complex is an* ENR*.*

(2.2.13) Theorem. *Every topological manifold is an* ENR*, in particular every smooth manifold is an* ENR*.*

Let X be an ENR, $\mathcal{U} \quad X$ its open subset and $f\colon \mathcal{U} \quad X$ a compactly fixed map. Let us fix an open subset of a Euclidean space $\mathcal{V} \quad E$ with the maps $r\colon \mathcal{V} \quad X$, $s\colon X \to \mathcal{V}$, $rs = \mathrm{id}_X$. We consider the map $sfr_|\colon r^{-1}(\mathcal{U}) \to \mathcal{V}$ given by $sfr_|(x) = sfr(x)$.

(2.2.14) Lemma. *The fixed point sets* $\operatorname{Fix}(sfr_|)$, $\operatorname{Fix}(f)$ *are homeomorphic. The homeomorphisms are given by the restrictions of the maps r and s.*

PROOF. Let $x \in \mathrm{Fix(f)}$. We will show that $s(x) \in \mathrm{Fix}\,(sfr_|)$. First we notice that $s(x) \in r^{-1}(\mathcal{U}) =$ domain of $sfr_|$. In fact $rs(x) = x \in \mathcal{U} =$ domain of f, hence $s(x) \in r^{-1}(\mathcal{U})$. Now $sfr_|(s(x)) = sf(rs(x)) = sf(x) = s(x)$, hence $s(x) \in \mathrm{Fix}\,(sfr_|)$.

Let $y \in \mathrm{Fix}\,(sfr_|)$. We show that $r(y) \in \mathrm{Fix}\,(f)$. Since $y \in r^{-1}(U)$ and $y = sfr(y)$, $r(y) = r(sfr(y)) = fr(y)$, hence $r(y) \in \mathrm{Fix}\,(f)$. □

The above equality of fixed point sets suggests the following extension of the definition of the fixed point index. Under the above notation we define the fixed point index of a compactly fixed map $f\colon \mathcal{U}\ \ X$, $X \in \mathrm{ENR}$, $\mathcal{U}\ \ X$, open subset, as

$$\mathrm{ind}\,(f) = \mathrm{ind}\,(sfr_|).$$

Since f is compactly fixed, by the above lemma $sfr_|\colon r^{-1}(\mathcal{U}) \to E$ is also compactly fixed, hence the index on the right hand side is defined. It remains to check that the definition is correct, i.e. does not depend on the choice of the maps r and s.

(2.2.15) LEMMA (Correctness of the definition). *Let $X \in \mathrm{ENR}$, let $\mathcal{V}$ be its open subset and $f\colon \mathcal{V}\ \ X$ a compactly fixed map. Suppose that $\mathcal{U}\ \ E$, $\mathcal{U}' \subset E'$ are open subsets of Euclidean spaces and let $\mathcal{U} \xrightarrow{r} X \xrightarrow{s} \mathcal{U}$, $\mathcal{U}' \xrightarrow{r'} X \xrightarrow{s'} \mathcal{U}'$ be the maps satisfying $rs = \mathrm{id}_X$, $r's' = \mathrm{id}_X$. Then*

$$\mathrm{ind}\,(sfr_|, r^{-1}(\mathcal{V})) = \mathrm{ind}\,(s'fr'_|, r'^{-1}(\mathcal{V})).$$

PROOF. Consider the maps: $s'fr_|\colon r^{-1}(\mathcal{V}) \to \mathcal{U}' \subset E'$ $sr'_|\colon \mathcal{U}' \to \mathcal{U}\ \ E$. Then $(sr'_|)(s'fr_|) = sfr_|$, $(s'fr_|)(sr'_|) = s'fr'_|$. Since these maps are compactly fixed, the Commutativity Property implies $\mathrm{ind}\,(sfr_|, r^{-1}(\mathcal{V})) = \mathrm{ind}\,(s'fr'_|, r'^{-1}(\mathcal{V}))$. □

Thus to each triple $(\mathcal{V}, f, X)$ (where X is an ENR, $\mathcal{V}\ \ X$ an open subset, $f\colon \mathcal{V}\ \ X$ a compactly fixed map) an integer $\mathrm{ind}\,(f, \mathcal{V})$ is subordinated. It turns out that the properties from the previous section (for the case of an open subset of an Euclidean space) also hold in the general situation.

Let $\mathcal{V}$ be an open subset of an ENR X and let $f\colon \mathcal{V}\ \ X$ be a compactly fixed map. Let $\mathcal{U}\ \ E$ be an open subset of the Euclidean space and let the maps $s\colon X \to \mathcal{U}$, $r\colon \mathcal{U}\ \ X$ satisfy $rs = \mathrm{id}_X$.

(2.2.16) LEMMA (Localization). *Let $i\colon \mathcal{V}' \to \mathcal{V}$ be the inclusion of an open subset satisfying* $\mathrm{Fix}\,(f) \subset \mathcal{V}'$. *Then $f_{|\mathcal{V}'}$ is compactly fixed and* $\mathrm{ind}\,(f_{|\mathcal{V}'}) = \mathrm{ind}\,(f)$.

PROOF. The idea of the proof of this lemma and the next ones is to use the retraction map r and the inclusion s to reduce the problem to the map of an open subset of the Euclidean space and then use the corresponding property from the previous section.

Let us notice that $\operatorname{ind}(f_{|\mathcal{V}}) = \operatorname{ind}(sfr_{|r^{-1}(\mathcal{V})})$ and $\operatorname{ind}(f_{|\mathcal{V}'}) = \operatorname{ind}(sfr_{|r^{-1}(\mathcal{V}')})$. But

$$\operatorname{Fix}(sfr_{|r^{-1}(\mathcal{V})}) = s(\operatorname{Fix}(f)) = s(\operatorname{Fix}(f_{|\mathcal{V}'})) = \operatorname{Fix}(sfr_{|r^{-1}(\mathcal{V}')}),$$

hence we may apply Lemma (2.2.6) to get $\operatorname{ind}(sfr_{|r^{-1}(\mathcal{V})}) = \operatorname{ind}(sfr_{|r^{-1}(\mathcal{V}')})$, which implies the desired equality $\operatorname{ind}(f_{|\mathcal{V}'}) = \operatorname{ind}(f)$. □

(2.2.17) LEMMA (Units). *Let $\rho\colon \mathcal{V} \quad X$ be the constant map $\rho(\mathcal{V}) = x_0$. Then*

$$\operatorname{ind}(\rho) = \begin{cases} +1 & \text{if } x_0 \in \mathcal{V}, \\ 0 & \text{if } x_0 \notin \mathcal{V}. \end{cases}$$

PROOF. We notice that the map $s\rho r$ is also constant $(s\rho r(r^{-1}(\mathcal{V}) = s(x_0))$ and that $x_0 \in \mathcal{V}$ if and only if $s(x_0) \in r^{-1}(\mathcal{V})$. Now the lemma follows from the equality $\operatorname{ind}(\rho) = \operatorname{ind}(s\rho r)$ and Lemma (2.2.7). □

(2.2.18) LEMMA (Additivity). *If $\mathcal{V}_1, \mathcal{V}_2 \subset V$ are open subsets such that the restrictions $f_{|\mathcal{V}_1}$, $f_{|\mathcal{V}_2}$ are compactly fixed and $\mathcal{V}_1 \cap \mathcal{V}_2$ is disjoint from* $\operatorname{Fix}(f)$, *then*

$$\operatorname{ind}(f) = \operatorname{ind}(f_{\mathcal{V}_1}) + \operatorname{ind}(f_{\mathcal{V}_2}).$$

PROOF. We notice that $\operatorname{Fix}(sfr_{|r^{-1}(\mathcal{V}_j)}) = s(\operatorname{Fix}(f_{|\mathcal{V}_j})$ for $j = 1, 2$. Since $s\colon X \to \mathcal{U}$ is mono, $\operatorname{Fix}(sfr_{|r^{-1}(\mathcal{V}_1)}) \cap \operatorname{Fix}(sfr_{|r^{-1}(\mathcal{V}_2)}) = \emptyset$, hence

$$\begin{aligned} \operatorname{ind}(f) &= \operatorname{ind}(sfr; r^{-1}(\mathcal{V})) \\ &= \operatorname{ind}(sfr; r^{-1}(\mathcal{V}_1)) + \operatorname{ind}(sfr; r^{-1}(\mathcal{V}_2)) = \operatorname{ind}(f_{\mathcal{V}_1}) + \operatorname{ind}(f_{\mathcal{V}_2}), \end{aligned}$$

where the middle equality comes from Lemma (2.2.8). □

(2.2.19) LEMMA (Homotopy Invariance). *Let $\mathcal{W} \subset X \times [0,1]$ be an open subset and let $F\colon \mathcal{W} \to X$ be compactly fixed. Then* $\operatorname{ind}(f_0) = \operatorname{ind}(f_1)$.

PROOF. We notice that $\operatorname{Fix}(sF(r \times \mathrm{id}_I)|) = s(\operatorname{Fix}(F))$, hence the homotopy $sF(r \times \mathrm{id}_I)$ is also compactly fixed. Now

$$\operatorname{ind}(f_0) = \operatorname{ind}(sf_0 r_{r^{-1}(\mathcal{V}_0)}) = \operatorname{ind}(sf_1 r_{r^{-1}(\mathcal{V}_1)}) = \operatorname{ind}(f_1). \qquad \square$$

(2.2.20) LEMMA (Multiplicativity). *If $f\colon \mathcal{V} \quad X$, $f'\colon \mathcal{V}' \to X'$ are compactly fixed, then so also is $f \times f'\colon \mathcal{V} \quad X \times X'$ and* $\operatorname{ind}(f \times f') = \operatorname{ind}(f) \cdot \operatorname{ind}(f')$.

PROOF.

$$\begin{aligned} \operatorname{ind}(f \times f') &= \operatorname{ind}((s \times s')(f \times f')(r \times r')_{|(r\times r')_*^{-1}(\mathcal{V}\times\mathcal{V}')}) \\ &= \operatorname{ind}((sfr)_{r^{-1}(\mathcal{V})} \times (s'f'r')_{r'^{-1}(\mathcal{V}')}) \\ &= \operatorname{ind}(sfr)_{r^{-1}(\mathcal{V})} \cdot \operatorname{ind}(s'f'r')_{r'^{-1}(\mathcal{V}')} = \operatorname{ind}(f) \cdot \operatorname{ind}(f'). \end{aligned} \qquad \square$$

(2.2.21) Lemma (Commutativity Property). *Let* $\mathcal{V} \quad X$, $\mathcal{V}' \subset X'$ *be open subsets of compact* ENR*s and let* $f\colon \mathcal{V} \quad X'$ *and* $g\colon \mathcal{V}' \to X$ *be continuous maps. Then the composites*

$$gf\colon f^{-1}(\mathcal{V}') \to X \quad \textit{and} \quad fg\colon g^{-1}(\mathcal{V}) \to X'$$

have homeomorphic fixed point sets $\mathrm{Fix}\,(gf) \quad \mathrm{Fix}(fg)$. *If moreover these sets are compact, then* $\mathrm{ind}\,(fg) = \mathrm{ind}\,(gf)$.

Proof. Let us fix retractions $X \xrightarrow{s} \mathcal{U} \xrightarrow{r} X$, $X' \xrightarrow{s'} \mathcal{U}' \xrightarrow{r'} X'$ from open subsets of Euclidean spaces E, and E', respectively. We consider the maps $sgr'_|\colon r'^{-1}(\mathcal{V}') \to \mathcal{U} \quad E$, $s'fr_|\colon r^{-1}(\mathcal{V}) \to \mathcal{U}' \subset E'$ (from open subsets into the Euclidean spaces). Since

$$\mathrm{Fix}\,((sgr'_|)(s'fr_|)) = \mathrm{Fix}\,(s(gf)r_|)) = s(\mathrm{Fix}\,(gf_|))$$

is compact, by Lemma (2.2.11) we get

$$\mathrm{ind}\,(s(gf)r_|) = \mathrm{ind}\,((sgr'_|)(s'fr_|)) = \mathrm{ind}\,((s'fr_|)(sgr'_|)) = \mathrm{ind}\,(s'(fg)r'_|).$$

It remains to notice that by the definition $\mathrm{ind}\,(gf) = \mathrm{ind}\,(s(gf)r_|)$ and $\mathrm{ind}\,(fg) = \mathrm{ind}\,(s'(f'g')r'_|)$. This implies $\mathrm{ind}\,(fg) = \mathrm{ind}\,(gf)$. $\square$

(2.2.22) Theorem. *There exists a unique function subordinating to each compactly fixed map* $f\colon \mathcal{V} \quad X$, $\mathcal{V}$ *an open subset of an* ENR X, *an integer satifying the properties: Localization, Units, Additivity, Homotopy Invariance, Multiplication and Commutativity.*

Proof. We have just shown that the fixed point index $\mathrm{ind}\,(f)$ is such a function. Now suppose that $\mathrm{ind}'(f)$ also satisfies all these properties. Then the function $\deg'(g) = \mathrm{ind}'(\mathrm{id} - g)$ satisfies the properties of the degree. The latter is defined for all maps $g\colon \mathcal{U} \quad E$, where $\mathcal{U}$ is an open subset of a Euclidean space E, hence $\deg'(g) = \deg\,(g)$. This in turn gives

$$\mathrm{ind}'(f) = \deg'(\mathrm{id} - f) = \deg\,(\mathrm{id} - f) = \mathrm{ind}\,(f)$$

for all compactly fixed maps $f\colon \mathcal{U} \quad E$. Thus the equality $\mathrm{ind}'(f) = \mathrm{ind}\,(f)$ holds for the Euclidean case and it remains to prove that it also holds for each compactly fixed map $f\colon \mathcal{V} \quad X$ of an ENR. But then

$$\mathrm{ind}\,(f) = \mathrm{ind}\,(sfr_|) = \mathrm{ind}'(sfr_|) = \mathrm{ind}'(rsf) = \mathrm{ind}'(f),$$

where the third equality comes from the Commutativity Property of the fixed point index. $\square$

2.2.3. Computations of the fixed point index. The fixed point index can be also generalized onto maps defined on closed subsets provided that $f(x) \neq x$ on the boundary.

(2.2.23) DEFINITION. Let $D \subset E$ be a compact subset and let $f\colon D \to E$ satisfy $f(x) \neq x$ for $x \in \operatorname{bd} D$. Then the restriction $f_|\colon \operatorname{int} D \to E$ is compactly fixed. We define $\operatorname{ind}(f) = \operatorname{ind}(f_|)$.

(2.2.24) LEMMA. *Under the assumptions of the above definition:*

(2.2.24.1) $\operatorname{ind}(f) \neq 0$ *implies the existence of a fixed point of the map* $f\colon D \to E$.
(2.2.24.2) *If f_s is a homotopy $f_s\colon D \to E$ with no fixed points on the boundary, i.e. $f_s(x) \neq x$ for all $x \in \operatorname{bd} D$, then* $\operatorname{ind}(f_0) = \operatorname{ind}(f_1)$.

PROOF. (2.2.24.1) $\operatorname{ind}(f_|) = \operatorname{ind}(f) \neq 0$ implies the existence of a fixed point $x \in \operatorname{Fix}(f_|)$. This is also a fixed point of f.

(2.2.24.2) Since the homotopy $\{f_s\}$ has no fixed point on the boundary, its restriction $f_{s|}\colon \operatorname{int} D \to E$ is compactly fixed hence $\operatorname{ind}(f_{0|}) = \operatorname{ind}(f_{1|})$. It remains to recall that by the definition $\operatorname{ind}(f_i) = \operatorname{ind}(f_{i|})$ for $i = 0, 1$. □

(2.2.25) LEMMA. *Let $B = B(x_0, \varepsilon) \subset E$ be an open ball in a Euclidean space and let $f\colon \operatorname{cl} B \to E$ be a map satisfying $f(\operatorname{bd} B) \subset B$. Then* $\operatorname{ind}(f) = 1$. *In particular f has a fixed point.*

PROOF. Let $f_t(x) = (1-t)f(x) + tx_0$ denote the segment homotopy between $f_0 = f$ and the constant map $f_1(x) \equiv x_0 \in B$. The homotopy has no fixed points on the boundary. In fact if $x \in \operatorname{bd} B$, then $f(x) \in B$, hence x can not lie on the segment $[f(x), x_0)] \subset B$. Thus $\operatorname{ind}(f) = \operatorname{ind}(\mathrm{const}) = +1$. □

(2.2.26) COROLLARY. *Let $\mathcal{U}$ $\quad E$ be an open subset and let $f\colon\mathcal{U}$ $\quad E$ be a contraction with respect to a fixed point $x_0 \in \mathcal{U}$, i.e. $\|f(x) - f(x_0)\| < \|x - x_0\|$ for $0 < \|x - x_0\| \leq \varepsilon$ for an $\varepsilon > 0$. Then x_0 is an isolated fixed point and* $\operatorname{ind}(f; x_0) = +1$.

(2.2.27) LEMMA. *Let $B = B(x_0, \varepsilon) \subset E$ be an open ball in the Euclidean space and let $f\colon \operatorname{cl} B \to E$ be a map satisfying $f(\operatorname{bd} B) \cap \operatorname{cl} B = \emptyset$. Then* $\operatorname{ind}(f) = \deg(x_0 - f(x))$.

PROOF. Since $\operatorname{ind}(f) = \deg(\mathrm{id} - f)$, we consider the segment homotopy: $h_t(x) = x_0 + (1-t)(x - x_0) - f(x)$. Then $h_0(x) = x - f(x)$, $h_1(x) = x_0 - f(x)$. Now we show that $h_t(x) \neq 0$ on the boundary. In fact if $x \in \operatorname{bd}(B)$, then the segment $[x, x_0] \subset \operatorname{cl} B$, hence it does not contain the point $f(x) \notin \operatorname{cl} B$ which gives $h_t(x) = (x_0 + (1-t)(x - x_0)) - f(x) \neq 0$. Now $\operatorname{ind}(f) = \deg(x - f(x)) = \deg(x_0 - f(x))$. □

(2.2.28) Corollary. *Let $\mathcal{U}\quad E$ be an open subset and let $f\colon \mathcal{U}\quad E$ be an expanding map near a fixed point $x_0 \in U$, i.e. $\|f(x) - f(x_0)\| > \|x - x_0\|$ for $0 < \|x - x_0\| \leq \varepsilon$ for an $\varepsilon > 0$. Then x_0 is an isolated fixed point and* $\operatorname{ind}(f; x_0) = \deg(x_0 - f(x))$.

(2.2.29) Lemma (Hopf Lemma for the fixed point index). *Let B be an open ball in the Euclidean space E and let $f\colon \operatorname{cl} B \to E$ be a map with no fixed point on the boundary. Then* $\operatorname{ind}(f) = 0$ *implies a homotopy $f_t\colon \operatorname{cl} B \to E$ from $f_0 = f$ to a fixed point free map. Moreover, the homotopy f_t may be constant on the boundary.*

Proof. Let us notice that $F\colon \operatorname{cl} B \to E$ given by $F(x) = x - f(x)$ is a d-compact map and $\deg(F) = \operatorname{ind}(f) = 0$. Moreover, F has no zeroes on the boundary, hence by Theorem (2.1.25) there is a homotopy $\phi_t\colon \operatorname{cl} B \to E$ from $\phi_0 = F$ to a map ϕ_1 with no zeroes on the boundary and which may be constant on the boundary. Now $\mathrm{id} - \phi_t$ is a homotopy from f to a fixed point free map with no fixed points on the boundary. Finally since the homotopy ϕ_t may be constant on the boundary, so is the homotopy $\mathrm{id} - \phi_t$. □

(2.2.30) Corollary (Hopf). *Let $\mathcal{U}$ be an open subset of a manifold M and let $f\colon \mathcal{U}\quad M$ be a map with an isolated fixed point $x_0 \in \mathcal{U}$ satisfying* $\operatorname{ind}(f; x_0) = 0$. *Then there is a homotopy $f_t\colon \mathcal{U}\quad M$ constant outside a prescribed neighbourhood of x_0 starting from $f_0 = f$, and satisfying*

$$\operatorname{Fix}(f_1) = \operatorname{Fix}(f) \setminus x_0.$$

Moreover, the homotopy may be arbitrarily small.

(2.2.31) Example. Below we derive the fixed point index for some maps.

(2.2.31.1) Let $f_r\colon S^1 \to S^1$ be the map of degree $r \in \mathbb{Z}$. We may assume that $f_r(z) = z^r$. We show that $\operatorname{ind}(f_r) = 1 - r$.

We first notice that $f_0 =$ constant, hence $\operatorname{ind}(f_0) = +1$. Moreover, $f_1(z) = z$, hence by a small rotation f_1 is homotopic to a fixed point free map, hence $\operatorname{ind}(f_1) = 0$.

Let $r \geq 2$. Then $z^r = z$ means $z(z^{r-1} - 1) = 0$, hence z is a root of unity of degree $r - 1$ which yields $r - 1$ fixed points. Now we notice that near each fixed point the map f is expands (r-times) and preserves the orientation. Thus after a local homeomorphism (with a neighbourhood of $0 \in \mathbb{R}$) f corresponds to the map $t \mapsto rt \in \mathbb{R}$. Now $r \geq 2$ implies $\operatorname{ind}(f, z)) = -1$. Finally $\operatorname{ind}(f_r) = (r-1)(-1) = 1 - r$.

Now let $r < 0$. Then $z^r = z$ means $z^{1-r} = 1$, $1 - r > 0$, hence we get $1 - r$ fixed points. The index at each fixed point equals $+1$ since f_r reverses the orientation. We get again $\operatorname{ind}(f_r) = (1-r)(+1) = 1 - r$.

(2.2.31.2) Let us consider again the Shub's example (Example (1.0.20)). It may be easily adapted for any dimension $n \geq 2$. The sphere S^n can be represented as the suspension $S^n = S^{n-1} \times [0,1]/\sim$, where $(x,0) \sim (y,0) = S$, $(x,1) \sim (y,1) = N$ for every $x, y \in S^{n-1}$. For a given map $h\colon S^{n-1} \to S^{n-1}$ we define the suspension of h as a map given by the formula $\Sigma h([x,s]) = [(h(x),s)]$. Let $r = \deg(h)$. Now take a map $\xi_\varepsilon\colon [0,1] \to [0,1]$ satisfying $\xi_\varepsilon(0) = 0$, $\xi_\varepsilon(s) > s$ for $0 < s < 1$. Finally we form the map $h_\varepsilon([x,s]) := [(h(x), \xi_\varepsilon(s))]$. It is obvious that h_ε is homotopic to Σh, thus of degree r. On the other hand h_ε has only two periodic points S and N. Note also that h_ε is a contraction near N and is an expanding map near S. Thus $\operatorname{ind}(\alpha_1 \phi_r, N) = +1$. On the other hand $\operatorname{ind}(\alpha_1 h_\varepsilon, S) = (-1)^n \deg(\alpha_1 h_\varepsilon, S) = (-1)^n r$. Finally

$$\operatorname{ind}(h_\varepsilon) = \operatorname{ind}(\alpha_1 h_\varepsilon) = \operatorname{ind}(\alpha h_\varepsilon, N) + \operatorname{ind}(\alpha h_\varepsilon, S) = 1 + (-1)^n\, r.$$

The above examples demonstrate a method of computing the fixed point index of a self-map of a manifold. The given self-map is deformed to a map with a finite number of fixed points and we may determine the local indices passing to the Euclidean coordinates near each point. At last we add these indices using the Additivity Property.

2.3. The Lefschetz number

In this subsection we give an exposition of the Lefschetz number theory. In view of applications we define it in a very general situation assuming that the reader is familiar with the algebra, e.g. a basic ring theory, as in [La]. Despite the generality the technical difficulties are the same as in the case of the classical definition of the Lefschetz number over the field of rational numbers (cf. [Br2]). We must add that all considerations hold for this case.

2.3.1. Trace of a matrix. Let $\mathcal{R}$ be a commutative ring with unity and $\mathrm{A}\colon \mathbb{M} \to \mathbb{M}$ an endomorphism of a free $\mathcal{R}$-module of dimension d, i.e. $\mathbb{M} \equiv \mathcal{R}^d$. By this isomorphism, i.e. by a choice of a basis, we can identify A with a $d \times d$ matrix A with coefficients in $\mathcal{R}$. The space of all such matrices we denote by $\mathcal{M}_{d\times d}(\mathcal{R})$.

(2.3.1) Definition. Let $A = [a_{ij}] \in \mathcal{M}_{d\times d}(\mathcal{R})$ be a matrix of an endomorphism of $\mathbb{M}$. We assign to A an element $\operatorname{tr} A \in \mathcal{R}$, called the *trace* of A, and defined as

$$\operatorname{tr} A := \sum_{i=1}^{d} a_{ii}.$$

Of the most interest for us will be the following rings: the ring of integral numbers $\mathbb{Z}$, the fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, of rational, real, and complex numbers correspondingly.

We also use the ring $\mathbb{Z}_m$ of remainders modulo $m \in \mathbb{N}$, in particular the field $\mathbb{Z}_p$, p-prime. The following property of the trace follows by direct computation of matrices.

(2.3.2) PROPOSITION. *Let $A, B \in \mathcal{M}_{d\times d}(\mathcal{R})$. Then*

$$\operatorname{tr}(AB) = \operatorname{tr}(BA), \quad \operatorname{tr}\mathrm{I} = d, \quad \operatorname{tr} 0 = 0,$$

where AB means the multiplication of matrices

(2.3.3) EXERCISE. Prove Proposition (2.3.2).

As a direct consequence we get.

(2.3.4) COROLLARY. *The trace is an invariant of the endomorphism $A\colon \mathbb{M} \to \mathbb{M}$. In other words it does not depend on the matrix representing A, i.e. on a choice of the basis in $\mathbb{M}$.*

PROOF. If A is a matrix of an endomorphism in a fixed basis and B a matrix of the same endomorphism in another basis, then $B = SAS^{-1}$, where $S \in \mathcal{M}_{d\times d}(\mathcal{R})$ is an invertible matrix expressing one basis by the other. By this and Proposition (2.3.2), $\operatorname{tr} B = \operatorname{tr}(SAS^{-1}) = \operatorname{tr}((AS^{-1})S) = \operatorname{tr} A$. □

(2.3.5) DEFINITION. For a matrix $A \in \mathcal{M}_{d\times d}(\mathcal{R})$ the characteristic polynomial $\chi_A(\lambda) \in \mathcal{R}[\lambda]$, of degree d, is defined as $\chi_A(\lambda) = \det(\lambda\mathrm{I} - A)$.

Since for every invertible matrix S $\det(\lambda\mathrm{I} - S^{-1}AS) = \det(S^{-1}(\lambda\mathrm{I} - A)S) = \det(\lambda\mathrm{I} - A)$, the characteristic polynomial does not depend on the matrix representation of an endomorphisms, i.e. is an invariant of the endomorphism (cf. [La]).

(2.3.6) DEFINITION. We say that a ring $\mathcal{R}$ is a number ring if $\mathbb{Z} \subset \mathcal{R} \subset \mathbb{C}$; if a number ring is a field we call it a number field. In particular the fields $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ are number fields.

Note that for a free module $\mathbb{M} \equiv \mathcal{R}^d$, $\mathcal{R}$ a number field, any matrix $A \in \mathcal{M}_{d\times d}(\mathcal{R})$ can be considered as a matrix with complex coefficients, since $\mathcal{R} \subset \mathbb{C}$. In the terms of endomorphisms it corresponds to an endomorphism $A\colon \mathbb{C}^d \to \mathbb{C}^d$, called *the complexification* A (see [La]). Using it we can express the trace as a sum of eigenvalues of the matrix if the ring $\mathcal{R}$ is a number ring.

(2.3.7) THEOREM. *If $\mathcal{R}$ is a number ring, then for a matrix $A \in \mathcal{M}_{m\times m}(\mathcal{R})$ we have*

$$\operatorname{tr} A = \sum_{j=1}^{d} \lambda_j,$$

where λ_j are all eigenvalues (counted with multiplicities) of the matrix A considered as a complex matrix.

Proof. Since $\operatorname{tr} A = \sum_{i=1}^{d} a_{ii}$, is the same for the matrix as for its complexification, we can assume that our matrix is complex, i.e. A is a matrix of an endomorphism of $\mathbb{C}^d$, denoted by the same letter. The Jordan theorem (cf. [La]) says that there exists an invertible $d \times d$ matrix S such that the matrix of $B = SAS^{-1}$ is of the form, called *Jordan form*,

$$(2.3.8) \qquad B = \begin{bmatrix} \begin{matrix} \lambda_1 & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_1 & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \lambda_1 \end{matrix} & 0 & 0 \\ & \ddots & \\ 0 & 0 & \begin{matrix} \lambda_r & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_r & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_r & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \lambda_r \end{matrix} \end{bmatrix}$$

where the i-th cell $1 \le i \le r$, $1 \le r \le d$, is a $d_i \times d_i$ matrix and $\sum_1^r d_i = d$. By Corollary (2.3.4)

$$\operatorname{tr} A = \operatorname{tr} B = d_1\lambda_1 + \dots + d_r\lambda_r = \sum_{j=1}^{d} \lambda_j.$$

On the other hand the characteristic polynomials $\chi_A(\lambda) = \chi_B(\lambda)$ are equal. This gives $\chi_A(\lambda) = (\lambda - \lambda_1)^{d_1} \dots (\lambda - \lambda_r)^{d_r}$ by the property of the determinant. This shows that λ_j are all the eigenvalues of A and d_j their multiplicities, and consequently proves the statement. □

As a consequence we get the following.

(2.3.9) Corollary. *If $\mathcal{R}$ is a number ring and $A \in \mathcal{M}_{d\times d}(\mathcal{R})$ a matrix, then for every $n \in \mathbb{N}$*

$$\operatorname{tr} A^n = \sum_{j=1}^{d} \lambda_j^n,$$

where λ_j are all eigenvalues of the matrix A considered as a complex matrix.

Proof. Let $\lambda_1, \dots, \lambda_d$ be all eigenvalues of A, i.e. $\lambda_1, \dots, \lambda_d$ are all the entries on the diagonal of a Jordan matrix $B = SAS^{-1}$. Then $\lambda_1^n, \dots, \lambda_d^n$ are standing on the diagonal of $B^n = SA^nS^{-1}$, hence $\operatorname{tr} A^n = \operatorname{tr} B^n = \sum_{i=1}^{d} \lambda_i^n$ by Theorem (2.3.7). $\square$

(2.3.10) EXERCISE. Let $w(\lambda) \in \mathcal{R}[\lambda]$, $w(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ be a monic polynomial with coefficients in a number ring $\mathcal{R}$, e.g. $w(\lambda) = \chi_A(\lambda)$, the characteristic polynomial of a matrix $A \in \mathcal{M}_{d\times d}(\mathcal{R})$.

Show that for $0 \leq k \leq n-1$ we have

$$a_k = \sum_{j_1<\cdots<j_k} \lambda_{j_1} \ldots \lambda_{j_k}.$$

In particular $a_0 = \det(A)$, $a_{n-1} = \operatorname{tr} A$, for the characteristic polynomial.

Hint. Use the decomposition $w(\lambda) = (\lambda - \lambda_1)^{d_1} \ldots (\lambda - \lambda_r)^{d_r}$ which is possible due to the embedding $\mathcal{R} \subset \mathbb{C}$ and the Fundamental Theorem of Algebra (2.1.27) and the Bezout Lemma.

Now we show a property that will be of use in the next.

(2.3.11) PROPOSITION. *Let $\mathcal{K}$ be a field and $\mathbb{M}$ a finitely generated $\mathcal{K}$-module, thus a vector space over $\mathcal{K}$. Let next $\mathbb{M}' \subset \mathbb{M}$ be a linear subspace and $\mathrm{A}\colon \mathbb{M} \to \mathbb{M}$ an endomorphism such that $\mathrm{A}(\mathbb{M}') \subset \mathbb{M}'$. Then A induces an endomorphism $\widetilde{\mathrm{A}}$ of the quotient space $\widetilde{\mathbb{M}} := \mathbb{M}/\mathbb{M}'$ and we have $\operatorname{tr}\mathrm{A} = \operatorname{tr}\mathrm{A}' + \operatorname{tr}\widetilde{\mathrm{A}}$, where A' denotes $A|_{\mathbb{M}'}$.*

PROOF. Let $d = \dim_{\mathcal{K}} \mathbb{M}$ and $\dim_{\mathcal{K}} \mathbb{M}' = d' \leq d$. Take any basis $e_1', \ldots, e_{d'}'$, of $\mathbb{M}'$ and extend it to a basis of $\mathbb{M}$ by adding vectors $\widetilde{e}_1, \ldots, \widetilde{e}_{d-d'}$. In this basis a matrix of A is of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11}, A_{12}, A_{21}, respectively A_{22}, are $d' \times d'$, $d' \times (d-d')$, $(d-d') \times d'$, and correspondingly $(d-d') \times (d-d')$ matrices. Since $\mathrm{A}(\mathbb{M}') \subset \mathbb{M}'$, $A_{11} = A'$ and $A_{21} = 0$. Furthermore the classes $[\widetilde{e}_i]$ form a basis of $\mathbb{M}/\mathbb{M}'$ and the induced endomorphism $\widetilde{A}\colon \mathbb{M}/\mathbb{M}' \to \mathbb{M}/\mathbb{M}'$ is represented by A_{22} in this basis. Now the statement follows from the definition of trace. $\square$

We are in a position to define the Lefschetz number of a map. It is a topological notion, thus we have to dispose a class of spaces in problem to be sure that all the topological invariants are well defined.

We will assume that our topological space X is a *finite* CW-*complex* of dimension D (see [Sp] for a definition of CW-complex)).

The class of all finite CW-complexes is large enough, it contains finite simplicial complexes (finite polyhedrons) and compact smooth manifolds (with or without boundary) (cf. [Sp]). By its definition, a compact CW-complex is finite and conversely.

For any fixed ring $\mathcal{R}$ and a finite complex X let $H_*(X,\mathcal{R})$, $H^*(X,\mathcal{R})$, respectively be the $\mathcal{R}$-modules of homology, correspondingly cohomology modules of X with coefficients in $\mathcal{R}$ (cf. [Sp]). For a finite CW-complex it is finitely generated $\mathcal{R}$-modules with gradation, i.e. $H_*(X,\mathcal{R}) = \bigoplus_{i=0}^{D} H_i(X,\mathcal{R})$ and all $H_i(X,\mathcal{R})$, $i = 0,\dots,D$, are finitely generated $\mathcal{R}$-modules, and the same for the cohomology. In particular $H_i(X;\mathcal{R}) = 0$, $i > D$. In the case if $\mathcal{R} = \mathcal{K}$ is a field, then $H_*(X;\mathcal{K})$ and $H^*(X;\mathcal{K})$ are vector spaces over $\mathcal{K}$. Moreover, for every continuous map $f\colon X \to Y$ there is uniquely defined $\mathcal{R}$-homomorphism $H_*(f)\colon H_*(X;\mathcal{R}) \to H_*(Y;\mathcal{R})$, and correspondingly $H^*(f)\colon H^*(Y;\mathcal{R}) \to H^*(X;\mathcal{R})$ for cohomology. Furthermore if $f,h\colon X \to Y$ are homotopic maps, then $H_*(f) = H_*(h)$ analogously for cohomology. The latter means that the ordering $X \mapsto H_*(X;\mathcal{R})$, or correspondingly $X \mapsto H^*(X;\mathcal{R})$, is a homotopy functor from the category of finite complexes and continuous maps to the category of finitely generated $\mathcal{R}$-modules with gradation and $\mathcal{R}$-homomorphisms. In particular, if $\mathcal{R} = \mathcal{K}$ is a field, then they are functors to the category of finite dimensional vector spaces with gradation. For details on homology, or cohomology, we refer the reader to the books [Br2], [Sp].

2.3.2. Definition of the Lefschetz number. We are in a position to define the Lefschetz number of a map.

(2.3.12) DEFINITION. Let X be a finite complex and $f\colon X \to X$ a self-map and $H_*(X;\mathcal{K}) = \bigoplus_{i=0}^{D} H_i(X;\mathcal{K})$ homology of X with coefficients in a field $\mathcal{K}$. Let next $H_*(f) = \bigoplus_{i=0}^{D} H_i(f)$ be the induced linear map. The element $L(f;\mathcal{K})$ of $\mathcal{K}$ defined by the formula

$$L(f;\mathcal{K}) := \sum_{i=0}^{D} (-1)^i \operatorname{tr} H_i(f),$$

is called the Lefschetz number of f with respect to the field $\mathcal{K}$, or shortly in the field $\mathcal{K}$. The Lefschetz number in cohomology is defined in the same way.

(2.3.13) REMARK. Note that the Lefschetz numbers in homology and cohomology are equal. Indeed, since $\mathcal{K}$ is a field, we have $H^i(X;\mathcal{K}) = \operatorname{Hom}(H_i(X;\mathcal{K}),\mathcal{K})$ for every $0 \le i \le D$ ([Sp]). From this follows that $H^i(f)\colon H^i(X;\mathcal{K}) \to H^i(X;\mathcal{K})$ is the endomorphism dual to $H_i(f)$ thus given by the transposed matrix. Consequently $\operatorname{tr} H^i(f) = \operatorname{tr} H_i(f)$ for every $0 \le i \le D$, which gives the desired claim.

(2.3.14) REMARK. Observe that the Lefschetz number can be defined as a difference of traces of two matrices of linear maps. Indeed put

$$H_{\mathrm{ev}}(X;\mathcal{K}) := \bigoplus_{i \cong 0 \mod 2} H_i(X;\mathcal{K}), \qquad H_{\mathrm{od}}(X;\mathcal{K}) := \bigoplus_{i \cong 1 \mod 2} H_i(X;\mathcal{K})$$

and analogously for the induced homomorphisms

$$H_{\mathrm{ev}}(f)\colon H_{\mathrm{ev}}(X;\mathcal{K}) \to H_{\mathrm{ev}}(X;\mathcal{K}), \qquad H_{\mathrm{od}}(f)\colon H_{\mathrm{od}}(X;\mathcal{K}) \to H_{\mathrm{od}}(X;\mathcal{K}).$$

Then $L(f;\mathcal{K}) = \operatorname{tr} H_{\mathrm{ev}}(f) - \operatorname{tr} H_{\mathrm{od}}(f)$.

2.3.3. Independence of the Lefschetz number on the field of coefficients. The remaining part of this section contains the proof of Theorem (2.3.16) which says that the Lefschetz number does not depend on the field of coefficients of the homology. First we need to remind a notion of the characteristic of a field ([La]).

(2.3.15) DEFINITION. Let $\mathcal{K}$ be a field. If there exists a natural number m such that $m \cdot 1 = 0$ then the smallest m with this property is called the characteristic of K, and denoted by $\operatorname{char} \mathcal{K}$. If there is not such a number then we say that the characteristic of K is 0. Since a field does not have zero divisors, a finite characteristic must be a prime number.

(2.3.16) THEOREM. *Let X be a finite complex and $\mathcal{K}$ a field and $L(f;\mathcal{K})$ the Lefschetz number with respect to $\mathcal{K}$. Then the Lefschetz numbers derived in homology and cohomology are equal. Moreover, the Lefschetz number $L(f;\mathbb{Q})$ with respect to the field of rational numbers is an integer and we have*

$$L(f;\mathcal{K}) = \begin{cases} L(f;\mathbb{Q}) & \textit{if } \operatorname{char}(\mathcal{K}) = 0, \\ \textit{the reminder} \bmod p \textit{ of } L(f;\mathbb{Q}) & \textit{if } \operatorname{char}(\mathcal{K}) = p. \end{cases}$$

(2.3.17) REMARK. We underlined the field of rational numbers for two reasons: it is the smallest field of characteristic zero and traditionally the Lefschetz number is defined by use of it. □

Before the proof we have to recall some notions of algebraic topology we shall use. It is known that for cohomology with coefficients in a field $\mathcal{K}$ we have

$$H^*(X;\mathcal{K}) = \operatorname{Hom}_{\mathcal{K}}(H_*(X;\mathcal{K});\mathcal{K}),$$

thus it is the space of all linear functionals on $H_*(X;\mathcal{K})$, and the map $H^*(f)$ is induced by $H_*(f)$ in this duality ([Sp]). For a fixed i take a basis $\{\hat{e}^i_j\}$ of $H^i(X;\mathcal{K})$ dual to a given $\{e^i_j\}$, of $H_i(X;\mathcal{K})$, i.e. $\widetilde{e}^i_j(e^i_k) = \delta_{jk}$, where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$. In the basis $\{\widetilde{e}^i_j\}$, the matrix of $H^i(f)$ is transposed to the matrix of $H_i(f)$ in $\{e^i_j\}$, which shows the first part of the statement.

The second part of our thesis is a consequence of a geometric determination of the Lefschetz number. We need to recall the definition of the homology spaces of a finite complex. In this case we can replace the group of all singular simplices of X with the subgroup generated by a finite number of simplical simplices. More precisely, for a finite complex X, a field $\mathcal{K}$ and fixed $0 \le i \le D$, let $C_i(X;\mathcal{K})$, or shortly $C_i(X)$, denote the linear space over $\mathcal{K}$ of i chains, i.e. finite-dimensional vector space with a basis formed by i-dimensional cells, or i-dimensional simplexes if X is a simplicial complex. In other words $C_i(X;\mathcal{K}) = C_i(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{K}$, where $C_i(X;\mathbb{Z})$ is a free $\mathbb{Z}$-module with the same basis. Also there exist, geometrically defined, homomorphisms $d_i\colon C_i(X) \to C_{i-1}(X)$, $d_0 = 0$, called the boundary operators and shortly denoted by d, with the property $d^2 = 0$. Then $Z_i(X) := \operatorname{Ker} d_i$, and, respectively $B_i(X) := \operatorname{Im} d_{i+1}$, called cycles and boundaries correspondingly, are subgroups of $C_i(X)$, and $B_i(X) \subset Z_i(X)$ (cf. [Sp]). By the definition,

$$H_i(X;\mathbb{Z}) := Z_i(X)/B_i(X). \tag{2.3.18}$$

Furthermore, for any continuous map $f\colon X \to Y$ of two finite complexes there exists a linear map $C_i(f)\colon C_i(X;\mathbb{Z}) \to C_i(Y;\mathbb{Z})$, which commutes with d (cf. [Sp]), thus preserves Z_i and B_i. By the definition (cf. [Sp])

$$H_i(f)\colon H_i(X;\mathbb{Z}) = Z_i(X)/B_i(X) \to Z_i(Y)/B_i(Y) = H_i(Y;\mathbb{Z}) \tag{2.3.19}$$

is the quotient map.

Observe that any homomorphism ϕ of a free finitely generated $\mathbb{Z}$-module, i.e. a $\mathbb{Z}$-linear map, defines a $\mathcal{K}$-linear map of the linear space over $\mathcal{K}$ with the same basis. It follows from the fact that $\mathcal{K}$ is a $\mathbb{Z}$-module as an abelian group, i.e. we can multiply elements of $\mathcal{K}$ by integers. Using once more the isomorphism $\mathcal{K}^n \equiv \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathcal{K}$, we can write this homomorphism as $\phi \otimes_{\mathbb{Z}} \mathrm{I}|_{\mathcal{K}}$. By definition, in the given basis ϕ and $\phi \otimes_{\mathbb{Z}} \mathrm{I}|_{\mathcal{K}}$ have the same matrix with integral coefficients.

Applying the above to the homomorphism $C_i(f) \otimes_{\mathbb{Z}} \mathrm{I}|_{\mathcal{K}}$, for every $0 \le i \le D$ we get the commutative diagram

$$\begin{tikzcd}
C_i(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{K} \arrow[r, "C_i(f)\otimes_{\mathbb{Z}}\mathrm{I}|_{\mathcal{K}}"] \arrow[d, "d_i\otimes_{\mathbb{Z}}\mathrm{I}|_{\mathcal{K}}"'] & C_i(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{K} \arrow[d, "d_i\otimes_{\mathbb{Z}}\mathrm{I}|_{\mathcal{K}}"] \\
C_{i-1}(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{K} \arrow[r, "C_{i-1}(f)\otimes_{\mathbb{Z}}\mathrm{I}|_{\mathcal{K}}"'] & C_{i-1}(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{K},
\end{tikzcd} \tag{2.3.20}$$

which follows from the corresponding diagram for $C_i(f)$. Denoting $d_i \otimes_{\mathbb{Z}} \mathrm{I}|_{\mathcal{K}}$ by d_i, shortly d, and $C_i(f) \otimes_{\mathbb{Z}} \mathrm{I}|_{\mathcal{K}}$ by $C_i(f)$, for a given map $f\colon X \to Y$ we have a complex with gradation $C_*(X;\mathcal{K})$ and preserving gradation linear map $C_*(f) := \bigoplus C_i(f)$, which commutes with d. Consequently, $Z_i(X;\mathcal{K}) := d_i^{-1}(0)$, and respectively $B_i(X;\mathcal{K}) := \operatorname{Im} d_{i+1}$, called the cycles and boundaries correspondingly, are linear subspaces of $C_i(X;\mathcal{K})$, and $B_i(X;\mathcal{K}) \subset Z_i(X;\mathcal{K})$. By the definition (cf. [Sp])

(2.3.21) $H_i(X;\mathcal{K}) := Z_i(X;\mathcal{K})/B_i(X;\mathcal{K})$ and, for every continuous map $f\colon X \to Y$, $H_i(f)\colon H_i(X;\mathcal{K}) \to H_i(X;\mathcal{K})$ is the quotient linear map induced by $C_i(f)$.

(2.3.22) Remark. The existence of a homomorphism $C_i(f)$ follows from the homotopy approximation theorem ([Sp]). Consequently the map $C_i(f)$: $C_i(X) \to C_i(X)$ depends on this approximation of f, thus is not unique. However, the induced map $H_i(f)$ on homology does note depend on the homotopy approximation, thus is unique [Sp].

The crucial tool for a proof of Theorem (2.3.16) is the following Hopf lemma.

(2.3.23) Lemma (Hopf Lemma for Trace). *Let $C_*(f) = \bigoplus_{i=0}^D C_i(f)$ be a map of a finite dimensional linear space $C_*(X;\mathcal{K})$ with gradation which commutes with the boundary operator d and preserves gradation $\mathcal{K}$-linear. Let next*

$$H_*(f)\colon H_i(X;\mathcal{K}) \to H_i(X;\mathcal{K})$$

be the map induced by $C_i(f)$ on homology spaces. Then

$$\sum_{i=0}^{D}(-1)^i \mathrm{tr}\, H_i(f) = \sum_{i=0}^{D}(-1)^i \mathrm{tr}\, C_i(f).$$

Proof. By the theorem on decomposition of a homomorphism, we have

$$C_i(X)/Z_i(X) \equiv B_{i-1}(X),$$

since $Z_i(X) = \mathrm{Ker}\, d_i$, and $B_{i-1}(X) = \mathrm{Im}\, d_i$. Consequently,

$$\mathrm{tr}\, C_i(f) = \mathrm{tr}\, C_i(f)|_{Z_i(X)} + \mathrm{tr}\, C_i(f)|_{B_{i-1}(X)},$$

as follows from Proposition (2.3.2). Analogously, by the same proposition

$$\mathrm{tr}\, C_i(f)|_{Z_i(X)} = \mathrm{tr}\, C_i(f)|_{B_i(X)} + \mathrm{tr}\, H_i(f)|_{H_i(X)},$$

because $H_i(X) = Z_i(X)/B_i(X)$. Substituting $\mathrm{tr}\, C_i(f)|_{Z_i(X)}$ from the second equality to the first and taking the alternating sum we get the statement, because the terms $\mathrm{tr}\, C_i(f)|_{B_i(X)}$ are cancelled. $\square$

Proof of Theorem (2.3.16). Let $F\colon X \to X$ be a self-map of a finite complex and $\mathcal{K}$ a field. By the Hopf lemma, Theorem (5.2.18),

$$L(f;\mathcal{K}) = \sum_{i=0}^{D}(-1)^i \mathrm{tr}\, C_i(f), \quad \text{where } C_i(f) = C_i(f) \otimes_{\mathbb{Z}} \mathrm{I}|_{\mathcal{K}}.$$

But $C_i(f)\otimes_{\mathbb{Z}}\mathrm{I}|_{\mathcal{K}}$ is represented by a matrix with integral coefficients by definition. It means that for every i, $\operatorname{tr} C_i(f)$ is an integer but considered as an element of $\mathcal{K}$, and the same for $L(f;\mathcal{K})$. This shows the statement, because for integer $m \in \mathbb{Z}$ we have

$$m = m{\cdot}1 = \begin{cases} m & \text{if } \operatorname{char}(\mathcal{K}) = 0, \\ \text{the reminder of } m \bmod p & \text{if } \operatorname{char}(\mathcal{K}) = p. \end{cases}$$ □

As follows from Theorem (2.3.16), the Lefschetz number with respect to any field of coefficients can be derived from the Lefschetz number computed in the homology with rational coefficients.

(2.3.24) Definition. The Lefschetz number defined by the homology with rational coefficients we call, the universal Lefschetz number, or simply the Lefschetz number, and denote by $L(f)$.

Note that for a finite complex X, $H_*(X;\mathbb{Z})$ is a finitely generated abelian group (cf. [Sp]), thus decomposes into its torsion $\operatorname{Tor}(H_*(X;\mathbb{Z}))$ and non-torsion part $\operatorname{Fr}(H_*(X;\mathbb{Z}))$, i.e. a finite subgroup consisting of all torsion elements and a finitely generated free abelian subgroup respectively. Moreover, every self-homomorphism preserves the torsion part and consequently induces a homomorphism of the quotient group isomorphic to $\operatorname{Fr}(H_*(X;\mathbb{Z}))$.

(2.3.25) Proposition. *Let $f\colon X \to X$ be a self-map of a finite complex X. Then the matrix of $H_*(f)\colon H_*(X;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ has integral coefficients and is equal to a matrix of the induced homomorphism on $H_*(X;\mathbb{Z})/\operatorname{Tor}(H_*(X;\mathbb{Z}))$. Consequently $L(f)$ does not depend on the torsion part of $H_*(X,\mathbb{Z})$.*

Proof. For given $0 \le i \le D$, let $\{e^i_l\}$ be a basis of the free part of $H_i(X;\mathbb{Z})$. For the image of e^i_k by $H_i(f)$ we have

$$H_i(f)(e^i_k) = \sum_l a^i_{kl} e^i_l + g$$

where g is a torsion element and $a_{kl} \in \mathbb{Z}$. Obviously the matrix of induced map $H_i(f)\colon H_i(X;\mathbb{Z})/\operatorname{Tor}(H_i(X;\mathbb{Z}))$ is equal to $\{a^i_{lk}\}$. On the other hand from the universal coefficients formula it follows that $H_*(X;\mathbb{Q}) = H_*(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$ (cf. [Sp]). This shows that the image of $e^k_k \otimes 1$ by $H_i(f)\otimes_{\mathbb{Z}}\mathrm{I}|_{\mathbb{Q}}$ is equal to

$$\sum_l a^i_{kl}(e^i_l \otimes 1),$$

because $g \otimes 1 = 0$ if g is a torsion element. The proof is complete. □

Note that Proposition (2.3.25) shows also that $L(f)$ is an integer. Anyway the proof of Theorem (2.3.16) discovers a geometrical interpretation of the Lefschetz number at least for the cellular or simplical map f.

We would like now to argue briefly that the above universal property of the classical Lefschetz number extends into much more general situations e.g. it does not depend on the definition of the trace we used. There are extensions of the notion of trace into the case of larger class of modules than finitely generated free modules. However in any case (a ring $\mathcal{R}$) for which a Lefschetz number is defined, it can be easily derived from the classical Lefschetz number (Exercise (2.3.32)). This means that such a Lefschetz number gives nothing new to the fixed point theory (as Theorem (2.4.1)).

To introduce a more general definition of a trace we need a notion of the projective module (cf. [La])

(2.3.26) Definition. Let $\mathcal{R}$ be a ring and M an $\mathcal{R}$-module. We say that M is a projective module if it is a direct summand of a free $\mathcal{R}$-module.

We say that M has finite projective resolution of the length K if there exists a family $\{M_k\}_{k=1}^K$ of $\mathcal{R}$-modules such that the sequence

$$0 \longrightarrow M_K \longrightarrow M_{K-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M \longrightarrow 0$$

is exact.

Note that every free module is projective and every projective module has a finite projective resolution – consisting of itself.

(2.3.27) Exercise. A ring is called the principal ideals domain if all its ideals are principal, i.e. generated by one element. Show that every module M over a ring $\mathcal{R}$ which is a principal ideal domain is projective. Use the fact that every module is an image of a free module.

(2.3.28) Example. We recall that $\mathbb{Z}$, the ring of reminders modulo m, $\mathbb{Z} := \mathbb{Z}/m\mathbb{Z}$, and the polynomial rings $\mathbb{C}[x]$, $\mathbb{R}[x]$, $\mathbb{Q}[x]$, $\mathbb{Z}[x]$ are principal ideal domains.

In 1970 A. Thomas gave a construction of a trace for homomorphisms of a class of modules that contains modules having a finite projective resolution ([Th]). Restricted to this category his theorem states the following.

(2.3.29) Theorem. *Suppose that a finitely generated $\mathcal{R}$-module M has a finite projective resolution. Then for every $\mathcal{R}$-endomorphism $\phi\colon M \to M$ there exists well-defined element of $\mathcal{R}$, denoted by $\operatorname{tr}\phi$ and called the trace of ϕ. The assignment $\phi \mapsto \operatorname{tr}\phi$ has the following properties.*

(2.3.29.1) (Exactness) *For every short exact sequence of such $\mathcal{R}$-modules and their endomorphisms*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d, "f_1"'] & M \arrow[r] \arrow[d, "f"'] & M_2 \arrow[r] \arrow[d, "f_2"] & 0 \arrow[d] \\
0 \arrow[r] & M_1 \arrow[r] & M \arrow[r] & M_2 \arrow[r] & 0
\end{tikzcd}$$

we have $\operatorname{tr} f = \operatorname{tr} f_1 + \operatorname{tr} f_2$.

(2.3.29.2) (Commutativity) *For every two endomorphisms* $\phi, \psi \colon M \to M$ *we have* $\operatorname{tr}(\phi \circ \psi) = \operatorname{tr}(\psi \circ \phi)$.

For a proof of this theorem we refer to [Th].

Let $\mathcal{R}$ be a ring such that every finitely generated $\mathcal{R}$-module has finite projective resolution, e.g. a principal ideal domain. Let next $f \colon X \to X$ be a self-map of a finite complex. Since the $\mathcal{R}$-module $C_*(X;\mathcal{R})$ is finitely generated $\mathcal{R}$-module, so also is the quotient module of the homology $H_*(X;\mathcal{R})$. Using the definition of extended trace given by Theorem (2.3.29) we can define the Lefschetz number with respect to $\mathcal{R}$ in the same way as previously:

$$L(f;\mathcal{R}) = \sum_{i=0}^{D} (-1)^i \operatorname{tr} H_i(f). \tag{2.3.30}$$

(2.3.31) Exercise. Show that the Hopf Lemma for Trace (2.3.23) still holds for the above defined in (2.3.30) Lefschetz number. Use property (2.3.29.1) of the trace of Theorem (2.3.29).

Observe that for every ring $\mathcal{R}$ there exists a canonical homomorphism $\varepsilon \colon \mathbb{Z} \to \mathcal{R}$ defined as follows. Let $\mathbf{1}$ be the unit of $\mathcal{R}$. We put $\varepsilon(1) := \mathbf{1}$ and next $\varepsilon(m) = m\mathbf{1}$.

(2.3.32) Exercise. Show that for every ring $\mathcal{R}$ such that every finitely generated $\mathcal{R}$-module has finite projective resolution, and every self-map $f \colon X \to X$ of a finite complex, we have

$$L(f;\mathcal{R}) = \varepsilon(L(f)).$$

Use the Hopf lemma of Exercise (2.3.31) and the next procedure as in the proof of Theorem (2.3.16).

(2.3.33) Exercise. Show that a Lefschetz number defined in cohomology with coefficients in a ring $\mathcal{R}$ such that every finitely generated $\mathcal{R}$-module has finite projective resolution is equal to the Lefschetz number $L(f;\mathcal{R})$ of (2.3.30). Use the Hopf lemma for this Lefschetz number and the fact that $C^*(X;\mathcal{R}) = \operatorname{Hom}(C_*(X);\mathcal{R}),\mathcal{R})$, because $(C_*(X);\mathcal{R})$ is a free module.

Summing up our consideration we can say the following: The classical Lefschetz number $L(f)$ of a self-map of a finite complex, defined by use of the homology with rational coefficients as the difference of traces of two matrices, is an integer and is expressed by the eigenvalues of these matrices. It is possible to define a Lefschetz number using homology or cohomology with coefficients in fields and some rings, but the obtained generalized Lefschetz number is equal to the classical Lefschetz number, or could be easily derived from it. In particular every such Lefschetz number is not zero only if the classical is so.

(2.3.34) Remark. We included the notion of generalized trace to emphasize another thing. In algebraic topology there are studied *generalized* cohomology theories h^* such as K-theory, co-bordisms etc. The abelian groups $h^i(X)$ are different (larger) than $H^i(X;\mathbb{Z})$ then, but are modules over $h^0(\mathrm{pt})$. One can ask whether the Lefschetz number $L_{h^*}(f) \in \mathbf{h}^0(\mathrm{pt})$ gives more information about the fixed points of a map $f\colon X \to X$ provided it is well-defined. In [MarPr] it is shown that for any generalized cohomology theory h^*, for if the Lefschetz number $L_{h^*(\mathrm{pt})}(f)$ is defined then we have $L_{h^*(\mathrm{pt})}(f) = \varepsilon(L(f))$, where $\varepsilon\colon \mathbb{Z} = H^0(\mathrm{pt}) \to h^0(\mathrm{pt})$ is the homomorphism as in Exercise (2.3.32). Such a generalized Lefschetz number gives new fixed point information if we study spaces with actions of a Lie group G and G-equivariant maps (cf. [MarPr]).

2.4. The Lefschetz–Hopf theorem

In this section we prove a fundamental theorem of the fixed point theory which compares the geometric, local fixed point invariant, the fixed point index with the global invariant – the Lefschetz number of a map. Here we present the proof where the emphasis is put rather on geometry than algebra. For an elegant and more general algebraic proof we refer the reader to [Do1].

(2.4.1) Theorem (Lefschetz–Hopf fixed point formula). *Let $f\colon X \to X$ be a self-map of a compact* CW*-complex, i.e. a finite complex. Then* $L(f) = \mathrm{ind}\,(f)$.

Proof. Since both sides of the equality are homotopy invariant, we may deform f to a cellular map i.e. f sends the k-skeleton $X^{(k)}$ to itself $f(X^{(k)}) \subset X^{(k)}$ for all $k = 0, 1, \dots, \dim X$ (cf. [Sp]). Moreover, we may assume that f sends a neighbourhood of each skeleton $X^{(k)}$ into $X^{(k)}$ as follows. For each standard unit ball $D^n \subset \mathbb{R}^n$ we define the map $r\colon D^n \to D^n$ sending a neighbourhood of S^{n-1} into S^{n-1} by the formula

$$r(x) = \begin{cases} 2x & \text{for } |x| \le 1/2, \\ x/|x| & \text{for } |x| \ge 1/2. \end{cases}$$

Now composing each characteristic map $\phi_\sigma\colon D^n \to \sigma \subset X^{(n)}$ with the above retraction $\phi_\sigma \cdot r$ we get the self-map $R\colon X \to X$. The map R is homotopic to the identity hence we may replace f with Rf. Then $f^{-1}(X^{(n)})$ is an open neighbourhood of $X^{(n)}$ for each $n \in \mathbb{N}$. For a given cell σ let $\breve{\sigma}$ denote the interior of the cell σ. Moreover, for each cell $\sigma \in X$, the set $F_\sigma = \mathrm{Fix}\,(f) \cap \breve{\sigma}$ is a closed-open subset of $\mathrm{Fix}\,(f)$. Now the map f sends a neighbourhood of F_σ into $\breve{\sigma}$, hence

$$\mathrm{ind}\,(f; F_\sigma) = \mathrm{ind}\,(f_{|\mathcal{U}_\sigma}; F_\sigma)$$

where $f_{|\mathcal{U}_\sigma}$ denotes the restriction of $f\colon \mathcal{U}_\sigma \to \breve{\sigma}$ where $\mathcal{U}_\sigma = \breve{\sigma} \cap f^{-1}(\breve{\sigma})$.

By the above the fixed point sets split into closed-open subsets $\mathrm{Fix}\,(f) = \sum_\sigma F_\sigma$ (where σ runs all cells in X). Now

$$\mathrm{ind}\,(f) = \sum_\sigma \mathrm{ind}\,(f; F_\sigma) = \sum_\sigma \mathrm{ind}\,(f_{|U_\sigma}) = (*).$$

Now we are going to analyze the Lefschetz number. Let us recall that the homology of the (finite) CW-complex X may be obtained from the chain complex

$$C_n(X) = H_n(X^{(n)}, X^{(n-1)}) = \bigoplus H_n(\sigma, \mathrm{bd}\,\sigma) = \bigoplus_\sigma \mathbb{Z}\sigma,$$

where the summation runs the set of all n-cells σ. Then the generator corresponding to σ is given by the characteristic map

$$\phi_\sigma\colon (D^n, S^{n-1}) \to (\sigma, \mathrm{bd}\,\sigma)(\subset (X^{(n)}, X^{(n-1)})).$$

If $f\colon X \to Y$ is a cellular map, then the induced homomorphism sends the generator into

$$f_*[\phi_\sigma] = \sum_\tau d_\tau \psi_\tau \in C_n(Y) = H_n(Y^{(n)}, Y^{(n-1)}) = \bigoplus_\tau H_n(\tau, \mathrm{bd}\,\tau) = \bigoplus_\tau \mathbb{Z}\tau,$$

where d_τ denotes the degree of the map $\rho_\tau \cdot f \cdot \phi_\sigma\colon D^n/S^{n-1} \to \tau/(\mathrm{bd}\tau)$ induced by the composition

$$D^n \xrightarrow{\phi_\sigma} X^{(n)} \xrightarrow{f} Y^{(n)} \xrightarrow{\rho_\tau} Y^{(n)}/(Y^{(n)} \setminus \breve{\tau}) = \tau/(\mathrm{bd}\,\tau),$$

where $\rho_\tau\colon Y^{(n)} \to \tau/\mathrm{bd}\,\tau$ is the collapsing map, being the identity on τ and sending the rest into the point $\mathrm{bd}\,\tau$.

Now for a cellular map $f\colon X \to X$ we have

$$\begin{aligned}
L(f) &= \sum_{n=0}^{\infty} (-1)^n [\mathrm{trace}\,(f_*\colon H_n(X) \to H_n(X))] \\
&\overset{\text{Hopf lemma}}{=} \sum_{n=0}^{\infty} (-1)^n [\mathrm{trace}\,(f_*\colon H_n(X^{(n)}, X^{(n-1)}) \to H_n(X^{(n)}, X^{(n-1)}))] \\
&= \sum_{n=0}^{\infty} (-1)^n \left[\sum_{\{\sigma; \dim\sigma = n\}} \deg\,(\rho_\sigma \cdot f \cdot \phi_\sigma\colon (D^n, S^{n-1}) \to (\sigma/\breve{\sigma})) \right] \\
&= \sum_\sigma (-1)^{\dim\sigma} [\deg\,(\rho_\sigma \cdot f \cdot \phi_\sigma)\colon (D^n, S^{n-1}) \to (\sigma/\breve{\sigma})] = (**).
\end{aligned}$$

In the last sum σ runs the set of all cells.

The proof of the theorem will be complete once we show that $(*) = (**)$. The last reduces to the equality

$$\operatorname{ind}(f_{|U_\sigma}) = (-1)^{\dim\sigma}\deg(\rho_\sigma f\phi_\sigma)$$

which follows from:

$$\deg(\rho_\sigma f\phi_\sigma\colon D^n/S^{n-1} \to \sigma/\operatorname{bd}\sigma) = \deg(\rho_\sigma f\colon \sigma/\operatorname{bd}\sigma \to \sigma/\operatorname{bd}\sigma).$$

By Lemma (2.4.2) applied to $S^n = \sigma/\operatorname{bd}\sigma$, $h = \rho_\sigma f$, and $\mathcal{U} = \mathcal{U}_\sigma$ the latter is equal to

$$\begin{aligned}(-1)^n\operatorname{ind}(\rho_\sigma f\colon \mathcal{U}_\sigma \to \sigma/\operatorname{bd}\sigma) &= (-1)^n\operatorname{ind}(\mathcal{U}_\sigma \xrightarrow{f} \breve{\sigma} \subset \sigma/\operatorname{bd}\sigma)\\ &= (-1)^n\operatorname{ind}(f\colon \mathcal{U}_\sigma \to X) = (-1)^n\operatorname{ind}(f_{|\mathcal{U}_\sigma}).\end{aligned}$$

It remains to prove Lemma (2.4.2). □

(2.4.2) Lemma. *Let $h\colon S^n \to S^n$ map a neighbourhood of a point x_0 to x_0, i.e. $x_0 \in \operatorname{int}(h^{-1}(x_0))$. Then x_0 is the isolated fixed point and* $\operatorname{ind}(h; S^n \setminus \{x_0\}) =$ $(-1)^n\deg h$.

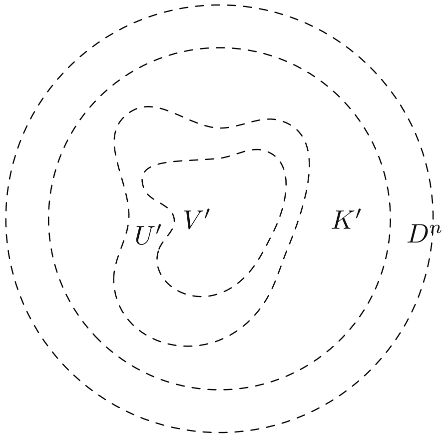
Proof. Let us fix a map $\phi\colon D^n \to S^n$ satisfying $\phi(S^{n-1}) = x_0$ and $\phi\colon \operatorname{int} D^n \to S^n \setminus \{x_0\}$ be a homeomorphism. For any subset $A \subset S^n$ we will denote $A' = \phi^{-1}(A)$. Let us denote $\mathcal{U} = h^{-1}(S^n \setminus \{x_0\})$. Then $\operatorname{cl}\mathcal{U} \quad S^n \setminus \{x_0\}$ implies $\operatorname{cl}\mathcal{U}' \subset \operatorname{int} D^n$. We have the commutative diagram

$$\begin{tikzcd} \mathcal{U}' \arrow[r, "h'"] \arrow[d, "\phi"'] & \operatorname{int} D^n \arrow[d, "\phi"] \\ \mathcal{U} \arrow[r, "h"'] & S^n \setminus \{x_0\} \end{tikzcd}$$

where $h'(x) = \phi^{-1}h\phi(x)$ and vertical arrows are homeomorphisms. Since x_0 is isolated in $\operatorname{Fix}(h)$ and $\mathcal{U} \quad \operatorname{Fix}(h) = \operatorname{Fix}(h) \setminus \{x_0\}$,

$$\operatorname{ind}(h, S^n \setminus \{x_0\}) = \operatorname{ind}(h, \mathcal{U}) = \operatorname{ind}(h') = (*).$$

Since $\operatorname{cl}\mathcal{U}' \subset \operatorname{int} D^n$, there is a ball $K' \subset \operatorname{int} D^n$ containing $\operatorname{cl}\mathcal{U}'$.



Let us denote $\mathcal{V}' = {h'}^{-1}(\operatorname{int} K')$. Then $\operatorname{Fix}(h') \subset \mathcal{V}' \subset \mathcal{U}'$ hence

$$(*) = \operatorname{ind}(h') = \operatorname{ind}(h', \mathcal{V}') = \deg(\mathrm{id} - h') = (**),$$

where the degree is taken with the respect to the point $0 \in \operatorname{int} K' \subset D^n$. Now we notice that $|h'(x)| > |x|$ for $x \in \operatorname{bd} \mathcal{V}'$ (since then $h'(x) \in \operatorname{bd} K'$ and $x \in \operatorname{cl} \mathcal{V}' \subset \operatorname{int} K'$). Now the homotopy $H \colon \operatorname{cl} \mathcal{V}' \times I \to \mathbb{R}^n$ given by the formula

$$H(x,t) = (1-t)x - h'(x)$$

has no zeroes on the boundary, hence

$$(**) = \deg(\mathrm{id} - h') = \deg(-h') = (-1)^n \deg(h').$$

Finally $\operatorname{ind}(h, S^n \setminus \{x_0\}) = (*) = (**) = (-1)^n \deg(h')$. It remains to show that $\deg(h') = \deg(h)$. Since ϕ is a homeomorphism,

$$\deg(h' \colon \mathcal{V}' \to K') = \deg(h \colon \mathcal{V} \quad K).$$

We notice that $h^{-1}(x) \subset \mathcal{V}$ for any $x \in K$ which implies $\deg(h \colon \mathcal{V} \quad K) = \deg(h)$. $\square$

As a direct consequence we get the classical Lefschetz fixed point theorem (see [Lef]).

(2.4.3) Theorem (Lefschetz Fixed Point Theorem). *Let $f \colon X \to X$ be a continuous map of a compact* CW*-complex. If $L(f) \neq 0$ then* $\operatorname{Fix}(f) \neq \emptyset$.

Proof. $L(f) \neq 0 \Rightarrow \operatorname{ind}(f) \neq 0 \Rightarrow \operatorname{Fix}(f) \neq \emptyset$. $\square$

We say that a topological space X is *contractible* if id_X is homotopic to the constant map $x \mapsto *$.

(2.4.4) Corollary (Brouwer Theorem). *Let X be a contractible compact* CW-*complex. Then every self-map $f\colon X \to X$ has a fixed point. In particular if X is a closed, convex and bounded subset of the Euclidean space, then the hypothesis holds.*

Proof. Let $H(x,t)\colon X \times I \to X$ be a homotopy between the identity and the point map. Composing it with f we get $H'(x,t)\colon X \times I \to X$ a homotopy between f and $*$, which gives $L(f) = 1$ for every self-map f. The statement follows from the Lefschetz theorem, because the induced homomorphisms $f_i = 0$ for all positive dimensions i and $f_0 = \mathrm{id}$. □

Note that the simplex, n-dimensional cube, and disc are convex. For any point x_0 of a convex set the mapping $H(x,t) := (1-t)x + tx_0$ defines a homotopy between the identity and point map $X \to x_0$.

For a complete exposition of applications and theorems which are equivalent to the Lefschetz fixed point theorem we refer the reader to the book of Dugundji and Granas [DuGr].

CHAPTER III

PERIODIC POINTS BY THE LEFSCHETZ THEORY

Let $f: X \to X$ be a map of a finite complex and $f^m := f \circ \cdots \circ f: X \to X$ its m-th iteration. The Lefschetz Theorem (2.4.1) ensures that if the Lefschetz number $L(f^m) \neq 0$, then $\mathrm{Fix}\,(f^m) \neq \emptyset$. This leads to a natural question: how much information about the sets of periodic points $P^m(f)$, $P_m(f)$, or about the set $\mathrm{Per}(f) \subset \mathbb{N}$ ((1.0.5) and Definition (1.0.4)) one can get studying the sequence $\{l_m = L(f^m)\}$ of the Lefschetz numbers of iterations of f. The approach is based on a direct generalization of Theorem (2.4.1).

(3.0.1) THEOREM (Lefschetz–Hopf Periodic Point Formula). *Let $f: X \to X$ be a self-map of a compact* CW*-complex, i.e. a finite complex. Then*

$$L(f^m) = \mathrm{ind}\,(f^m) \quad \textit{for every } m \in \mathbb{N}.$$

It is natural to study the behaviour of the sequence $\{L(f^m)\}$ and to expect that this may bring some information about the existence of periodic points. A naive conjecture is that if $|L(f^m)|$ is unbounded, then each consecutive iteration f^m has more and more fixed points, thus m-periodic of points f for m sufficiently large. But this naive claim is false, because the Lefschetz number does not estimate the number of the fixed points. In other words the left-hand side of the equality of Theorem (3.0.1), i.e. the fixed point index of the iteration grows due not to the fact that each consecutive iteration has more fixed points but because the local fixed point index of iteration at a given point is unbounded in general. An illustrative example is already presented, Example (1.0.20) in Chapter I.

(3.0.2) EXAMPLE. Let $f = f_\varepsilon([z,s])$ be the map of S^2 defined in Example (1.0.20). By the definition $L(f^m) = 1 + \deg(f^m) = 1 + r^m$. On the other hand we have $P(f) = P(f_\varepsilon)$ consists of the poles $[S^1 \times \{0\}]$, $[S^1 \times \{1\}]$. Note that near the North Pole $x_1 := [z,1]$ the map f^m is a contraction with respect to x_1, i.e. for every $x \neq x_1$ $|f^m(x) - x_1| < |x - x_1|$ in the local coordinates. By Corollary (2.2.26) we have $\mathrm{ind}\,(f^m; x_1) = 1$ for every $m \in \mathbb{N}$. From the Lefschetz–Hopf Theorem it follows that $\mathrm{ind}\,(f^m; x_0) = L(f^m) - \mathrm{ind}\,(f^m; x_1) = r^m$, which shows that the sequence of the local fixed point indices is unbounded.

(3.0.3) REMARK. It is worth emphasizing that any approximation f_ε can not be differentiable at $x_0 \in S^2$. Indeed, note first that this map is locally near the "South Pole" equivalent to $2z^r|z|^{-1}$, thus not differentiable. Moreover, observe next that every map $f\colon\mathcal{U} \quad \mathbb{C}$ of a neighbourhood $\mathcal{U}$ of 0 which in the polar coordinates is of the form $f(\theta,\rho) = (r\cdot\theta, \xi(\rho))$, $\xi(\rho) > \rho$, for $\rho > 0$, can not be smooth at 0. Indeed, assuming that f is C^1 at 0 and knowing $B_\rho \subset \ f(B_\rho)$, from the changes of variables rule we have $\mu(f(B_\rho)) = \int_{B_\rho} |Df(z)| d\mu(z) \geq \int_{B_\rho} d\mu(z) = \mu(B_\rho)$ for every ball $B_\rho(x_0)$. By the Mean Value Theorem the first integral is equal to $|Df(z_0)| \int_{B_\rho} d\mu(z)$, where $z_0 \in B_\rho$. Passing with $\rho \to 0$ we get $|Df(0)| \geq 1$, which yields that f is a local diffeomorphism at 0 contrary to its form along the angle coordinate.

3.1. Properties of the Lefschetz numbers of iterations

In this section we will discuss first arithmetical and analytical properties of the sequence of Lefschetz numbers of iterations of a given map.

Let X be a finite complex and $f\colon X \to X$, its self-map. To shorten notation put

$$A_{\mathrm{e}} := H_{\mathrm{ev}}(f) := \bigoplus_{i\text{-even}} H_i(f), \qquad A_{\mathrm{o}} := H_{\mathrm{od}}(f) = \bigoplus_{i\text{-odd}} H_i(f),$$

where $H_i(f)\colon H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q})$ is the map induced on the homology spaces. By Remark (2.3.14) the Lefschetz number $L(f)$ of f is equal to $L(f) = L^+(f) - L^-(f) := \operatorname{tr} A_{\mathrm{e}} - \operatorname{tr} A_{\mathrm{o}}$.

3.1.1. Arithmetic properties of the Lefschetz numbers of iterations. Our aim is to study the sequence

$$l_m = L(f^m) = l_m^+ - l_m^- := \operatorname{tr} A_{\mathrm{e}}^m - \operatorname{tr} A_{\mathrm{o}}^m. \tag{3.1.1}$$

First we have to study the sequence of traces of all natural powers of a given integer matrix.

Let $A \in M_{n\times n}(\mathbb{Z})$ be an $n \times n$ matrix with integer elements. Then the characteristic polynomial $\omega(\lambda) = \det(\lambda\mathrm{I} - A)$ of A has integral coefficients and is monic (normed) i.e. its leading coefficient is equal to 1. Let $\sigma(A) \subset \mathbb{C}$ denote the set of all eigenvalues of A, i.e. all roots of $\omega(z)$, called the spectrum of A. The *spectral radius* of A, is defined as $\operatorname{sp}(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}$. By Corollary (2.3.9),

$$\operatorname{tr} A = \sum_{j=1}^{n} \lambda_j \quad \text{and} \quad \operatorname{tr} A^m = \sum_{j=1}^{n} \lambda_j^m,$$

where the sum is taken over all $\lambda_j \in \sigma(A)$, counted with multiplicities.

Let μ be the Möbius function $\mu\colon \mathbb{N} \to \mathbb{Z}$ given as

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \dots p_k,\ p_i \text{ different primes}, \\ 0 & \text{if } p^2 | n,\ p \text{ for a prime } p. \end{cases} \tag{3.1.2}$$

One of the basic facts in elementary number theory *the Möbius inversion formula* (cf. [Ch]): *Let $f, g\colon \mathbb{N} \to \mathbb{C}$ be two functions (called arithmetic functions* [Ch]*).*

$$\textit{If} \quad \forall_{k\in\mathbb{N}}\ f(k) = \sum_{n|k} g(n), \quad \textit{then} \quad \forall_{k\in\mathbb{N}}\ g(k) = \sum_{n|k} \mu(n) f(k/n)$$

and conversely. The sum ranges over all divisors of k.

Let $A \in M_n(\mathbb{Z})$, be a matrix, or $f\colon X \to X$ be a self-map of a space as assumed. By $i_k(A) \in \mathbb{Z}$, or $i_k(f) \in \mathbb{Z}$ (shortly i_k) we denote an arithmetic function given by the formula

$$i_k = \sum_{n|k} \mu(n) l_{k/n}, \tag{3.1.3}$$

where $l_m = \operatorname{tr} A^m$, or $l_m = L(f^m)$ respectively. By the inversion formula, for every $k \in \mathbb{N}$ we have

$$l_k = \sum_{n|k} i_n.$$

The fundamental arithmetic property of the sequence $\{i_k\}$ is given in the following theorem.

(3.1.4) Theorem. *Let i_m be equal to $i_m(A)$ or $i_m(f)$. Then for every $m \in \mathbb{N}$,*

$$i_m \cong 0 \pmod m.$$

We call the congruences of Theorem (3.1.4) the Dold congruences (see Remark (3.1.11)). To prove Theorem (3.1.4) it is sufficient to show it for $i_m(A)$, $A \in M_n(\mathbb{Z})$, and use Definition (2.3.14). We show it using a correspondent of the little Fermat theorem for quadratic matrices with integer terms (cf. [MarPrz]). We recall that the little Fermat theorem for integers says that for a prime number p and every integer a we have $a^p \cong a \mod p$.

(3.1.5) Theorem. *For every $A \in M_n(\mathbb{Z})$, p-prime, $\alpha \in \mathbb{N}$ we have*

$$\operatorname{tr}(A^{p^\alpha}) \cong \operatorname{tr}(A^{p^{\alpha-1}}) \pmod{p^\alpha}$$

We begin with two lemmas.

(3.1.6) LEMMA. *Let $\chi(x_1,\dots,x_n)$ be a polynomial in $\mathbb{Z}[x_1,\dots,x_n]$ and let p be a prime number. Then there exists $v\in\mathbb{Z}[x_1,\dots,x_n]$ such that*

$$\chi(x_1^p,\dots,x_n^p)=(\chi(x_1,\dots,x_n))^p+pv(x_1,\dots,x_n).$$

PROOF. We will show that if the lemma holds for $\chi(x_1,\dots,x_n)$ so does it for $\widetilde{\chi}(x_1,\dots,x_n)=\chi(x_1,\dots,x_n)+x_1^{t_1}\cdots x_n^{t_n}$. By inductive assumption

$$(*)\qquad \begin{aligned}\widetilde{\chi}(x_1^p,\dots,x_n^p)&=\chi(x_1^p,\dots,x_n^p)+x_1^{pt_1}\cdots x_n^{pt_n}\\&=(\chi(x_1,\dots,x_n))^p+pv(x_1,\dots,x_n)+(x_1^{t_1}\cdots x_n^{t_n})^p.\end{aligned}$$

On the other hand

$$(**)\quad \begin{aligned}(\widetilde{\chi}(x_1^p,\dots,x_n^p))^p&=(\chi(x_1,\dots,x_n)+x_1^{t_1}\cdots x_n^{t_n})^p\\&=(\chi(x_1,\dots,x_n))^p+\sum_{i=1}^{p}\binom{p}{i}v_i(x_1,\dots,x_n)+(x_1^{t_1}\cdots x_n^{t_n})^p.\end{aligned}$$

Now $(*)-(**)=pv(x_1,\dots,x_n)-\sum_{i=1}^{p}\binom{p}{i}v_i(x_1,\dots,x_n)$ is divisible by p as follows from the fact that for $0<i<p$, p divides $\binom{p}{i}=p!/(i!(p-i)!)$. □

(3.1.7) LEMMA. *Let $a,b\in\mathbb{Z}$ and $\alpha\in\mathbb{N}$. If $a\cong b \pmod{p^\alpha}$, then*

$$a^p\cong b^p \pmod{p^{\alpha+1}}.$$

PROOF. First observe that $a^{p-1}+a^{p-2}b+\dots+b^{p-1}$ is divisible by p, because each of p components of the sum is congruent to b^{p-1} modulo p by assumption. Now we see that $p^\alpha|a-b$ and the lemma follows from the equality

$$a^p-b^p=(a-b)(a^{p-1}+\dots+b^{p-1}).$$ □

In our next proposition we shall use the Newton theorem on symmetric polynomials. We recall it for convenience of the reader. Let

$$\begin{aligned}&\sigma_1(x_1,\dots,x_n):=x_1+x_1+\dots+x_n,\\&\sigma_2(x_1,\dots,x_n):=x_1x_2+x_1x_3+\dots+x_1x_n+x_2x_3+\dots+x_{n-1}x_n,\end{aligned}$$

and generally for every $1\le i\le n$ let $\sigma_I(x_1,\dots,x_n):=\sum_{j_1<\dots<j_i}x_{j_1}\cdots x_{j_i}$. The polynomials σ_i are symmetric (i.e. they are preserved by any permutation of variables), and called canonical symmetric polynomials. The Newton theorem says that for every symmetric polynomial $\omega\in\mathbb{Z}[x_1,\dots,x_n]$ there exists polynomial $W(x_1,\dots,x_n)\in\mathbb{Z}[x_1,\dots,x_n]$ such that $\omega(x_1,\dots,x_n)=W(\sigma_1,\dots,\sigma_n)$. In particular, the polynomial W_i expressing $\omega_k:=\sum_{j=1}^{n}x_j^k$ as polynomial of $\sigma_1,\dots,\sigma_n$ is called the k-th *Newton polynomial.*

(3.1.8) Proposition. *Suppose that $\lambda_1, \dots, \lambda_n$ are all the roots, counted with multiplicities, of a certain monic polynomial ω from $\mathbb{Z}[x]$. For any symmetric polynomial $\chi(x) \in \mathbb{Z}[x_1, \dots, x_n]$ and $\alpha \in \mathbb{N}$ we have*

$$\chi(\lambda_1^{p^\alpha}, \dots, \lambda_n^{p^\alpha}) \cong \chi(\lambda_1^{p^{\alpha-1}}, \dots, \lambda_n^{p^{\alpha-1}}) \pmod{p^\alpha}.$$

Proof. We show the above by induction over α. From Lemma (3.1.6) there exists $v(x) \in \mathbb{Z}[x_1, \dots, x_n]$ such that

$$\chi(x_1^p, \dots, x_n^p) = \chi(x_1, \dots, x_n)^p + pv(x_1, \dots, x_n).$$

For $\alpha = 1$ the statement is clear. Notice that $v(x_1, \dots, x_n)$ is also symmetric, so its value taken at every $(\lambda_1, \dots, \lambda_n)$ is integral. Indeed, by the Newton theorem $v(x_1, \dots, x_n) = W(\sigma_1, \dots, \sigma_n)$, where w is a polynomial with integral coefficients. On the other hand $\sigma_k(\lambda_1, \dots, \lambda_n) = a_k$, where a_k is the coefficient of polynomial ω (cf. Exercise (2.3.10)), thus an integer. By the same argument and Exercise (3.1.14), for any symmetric polynomial with integral coefficients and every m the value $v(\lambda_1^m, \dots, \lambda_n^m)$ is integral.

Now assume the theorem holds for $\alpha = k$. Then

$$v(\lambda_1^{p^k}, \dots, \lambda_n^{p^k}) \cong v(\lambda_1^{p^{k-1}}, \dots, \lambda_n^{p^{k-1}}) \pmod{p^k}$$

and from Lemma (3.1.7) we have

$$\chi(\lambda_1^{p^k}, \dots, \lambda_n^{p^k})^p \cong \chi(\lambda_1^{p^{k-1}}, \dots, \lambda_n^{p^{k-1}})^p \pmod{p^{k+1}}.$$

Consequently

$$\begin{aligned}\chi(\lambda_1^{p^{k+1}}, \dots, \lambda_n^{p^{k+1}}) &= \chi(\lambda_1^{p^k}, \dots, \lambda_n^{p^k})^p + pv(\lambda_1^{p^k}, \dots, \lambda_n^{p^k}) \cong_{p^{k+1}} \\ &\cong_{p^{k+1}} \chi(\lambda_1^{p^{k-1}}, \dots, \lambda_n^{p^{k-1}})^p + pv(\lambda_1^{p^{k-1}}, \dots, \lambda_n^{p^{k-1}}) \\ &= \chi(\lambda_1^{p^k}, \dots, \lambda_n^{p^k}).\end{aligned}$$

□

Proof of Theorem (3.1.5). Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of A counted with multiplicities, i.e. all the roots of its characteristic polynomial. We put $\chi(x_1, \dots, x_n) = x_1 + \dots + x_n$ in Proposition (3.1.8). Now

$$\operatorname{tr} A^{p^\alpha} = \sum_{i=1}^{n} \lambda_j^{p^\alpha} = \chi(\lambda_1^{p^\alpha}, \dots, \lambda_n^{p^\alpha}) \cong \chi(\lambda_1^{p^{\alpha-1}}, \dots, \lambda_n^{p^{\alpha-1}}) = \operatorname{tr} A^{p^{\alpha-1}},$$

where the middle congruence is $\mod p^\alpha$ as in Proposition (3.1.8). The statement follows by the characterization of trace given in Theorem (2.3.7). □

Proof of Theorem (3.1.4). We use induction with respect to the number of primes dividing m. Assume that for all m', $m' = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $\alpha_i \geq 1$, $r \geq 1$ the statement holds. Let $m = m'p^\alpha$ with m' as above and $(m', p) = 1$. We have

$$i_m(A) = \sum_{k|m} \mu(k) \mathrm{tr}\, A^{m/k} = \sum_{k'|m'} \mu(k') \mathrm{tr}\, A^{p^\alpha m'/k'} - \sum_{k'|m'} \mu(k') \mathrm{tr}\, A^{p^{\alpha-1} m'/k'},$$

since $\mu(pk') = -\mu(k')$ and $\mu(k) = 0$ if $p^2|k$. Applying Theorem (3.1.5) to each matrix $A^{m'/k'}$ we have $i_m(A) \cong 0 \pmod{p^\alpha}$. By the inductive assumption applied to $A^{p^{\alpha-1}}$ and A^{p^α} we also get $i_m(A) \cong 0 \pmod{m'}$, which proves Theorem (3.1.4).$\square$

(3.1.9) Exercise. Show directly, by reducing coefficients modulo p, a weak version of Theorem (3.1.5): For every $A \in M_n(\mathbb{Z})$, p-prime, $\alpha \in \mathbb{N}$ we have

$$\mathrm{tr}\,(A^{p^\alpha}) \cong \mathrm{tr}\,(A^{p^{\alpha-1}}) \pmod{p}.$$

In [Do2] Dold showed that the congruence of Theorem (3.1.4) holds if we take the sequence of fixed point indices $I_m = \mathrm{ind}\,(f^m, \mathcal{U})$ of a map $f\colon \mathcal{U} \quad X$ of an open subset of an ENR X, provided that all these indices are defined. More precisely he proved the following.

(3.1.10) Theorem. *An integral sequence* $\mathcal{I} = \{I_m\}_1^\infty$ *satisfies the congruences*

$$\sum_{k|m} \mu(m/k)\, I_k \cong 0 \quad \mathrm{mod}\ m, \quad m = 1, 2, \dots,$$

if and only if there exist a map $f\colon X \to X$ *of an* ENR X (*not compact in general*) *and an open subset* $\mathcal{U} \quad X$ *such that for* $\mathcal{U}_1 := \mathcal{U}$, $\mathcal{U}_m := f^{-1}(\mathcal{U}_{m-1}) \cap \mathcal{U}$ *the set* $\mathrm{Fix}\,(f^m) \cap \mathcal{U}_m$ *is compact and* $I_m = \mathrm{ind}\,(f^m, \mathcal{U}_m)$.

(3.1.11) Remark. Since a map $f\colon X \to X$ of a finite complex satisfies this assumption with $\mathcal{U} = X$ and from the Lefschetz–Hopf Theorem (3.0.1) we have $I(f^m, \mathcal{U}) = L(f^m)$, the statement of Theorem (3.1.4) follows from the mentioned Dold Theorem (3.1.10). Anyway it is not a direct proof and we must add that Dold uses a geometric argument – an intricate transversality theorem. Also, since the argument of Theorem (3.1.4) is purely algebraic we do not need any assumption about the geometric structure of X as a finite CW-complex or ENR. Consequently its statement holds for every space X with $\dim_{\mathbb{Q}} H_*(X; \mathbb{Q}) < \infty$ provided the homology is well defined.

Theorem (3.1.4) gives a necessary algebraic condition on an integral sequence l_m to be the Lefschetz number of iterations of a map of a finite CW-complex. Next we show that the fact that the matrices $A_{\mathrm{e}}, A_{\mathrm{o}}$ defining the Lefchetz number in Remark (2.3.14) are the matrices of induced homomorphisms in homology does not impose any condition on $l_m = L(f^m)$ in general (cf. [BaBo], [MarPrz]).

(3.1.12) PROPOSITION. *Let X be the bouquets of n_2 circles and n_1 2-spheres. Then for each pair of matrices $A_{\mathrm{e}} \in M_{n_1}(\mathbb{Z})$, $A_{\mathrm{o}} \in M_{n_2}(\mathbb{Z})$ there exists self-map $f\colon X \to X$ satisfying*

$$L(f^m) = \operatorname{tr} A_{\mathrm{e}}^m - \operatorname{tr} A_{\mathrm{o}}^m \quad \textit{for every } m \in \mathbb{N}.$$

PROOF. Let $A_{\mathrm{e}} = \{a_{ij}\}$, $1 \le i, j \le n_1$. Take $X_1 := \bigvee_1^{n_1} S^2$, the bouquet of n_1 copies of 2-dimensional spheres. Let $p_j\colon X_1 \to S^2$ be the map contracting all but S_j^2 spheres of X_1, and ι_i the embedding of S_i^2 into X_1. According to a standard construction, we can define a map $f_1\colon X_1 \to X_1$ such that $\deg p_j f_1 \iota_i = a_{ij}$. We have $H_2(f_1) = A_{\mathrm{e}}$, $H_1(f_1) = 0$, $H_0(f_1) = 1$.

Analogously, we can construct a map $f_2\colon X_2 \to X_2$, $X_2 := \bigvee_1^{n_2} S^1$, such that $H_1(f_2) = A_{\mathrm{o}}$, $H_0(f_2) = 1$. Set $X_0 := S^1$, and $f_0 = \mathrm{id}\colon S^1 \to S^1$. From the Mayer–Vietoris exact sequence it follows that, for $f = f_0 \vee f_1 \vee f_2$, we have $L(f) = \operatorname{tr} A_{\mathrm{e}} - \operatorname{tr} A_{\mathrm{o}}$, which proves the proposition. $\square$

As we have just showed it is equivalent to study the behaviour of the sequences $l_m := \operatorname{tr} A^m$, $A \in M_n(\mathbb{Z})$, and the sequences $l_m := L(f^m)$, $f\colon X \to X$ a map of a finite complex. Another equivalence is left as an exercise.

(3.1.13) EXERCISE. Let $w(z) = z^q + a_{q-1}z^{q-1} + \cdots + a_1 z + a_0$ be a monic polynomial with integral coefficients and $\lambda_1, \ldots, \lambda_q$ all its roots. Show that there exists an integral $q \times q$ matrix A such that $\sigma(A) = \{\lambda_1, \ldots, \lambda_q\}$ and consequently $\operatorname{tr} A^m = \lambda_1^m + \cdots + \lambda_q^m$.

Hint. Consider the $q \times q$ matrix

$$A := \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{q-2} & -a_{q-1} \end{bmatrix}$$

and use Theorem (2.3.7), Corollary (2.3.9).

(3.1.14) EXAMPLE. Let $w(z) = z^q + a_{q-1}z^{q-1} + \cdots + a_1 z + a_0$ be a monic polynomial with integral coefficients, $\lambda_1, \ldots, \lambda_q$ all its roots and A the integral matrix with $\sigma(A) = \{\lambda_1, \ldots, \lambda_q\}$ given by Exercise (3.1.13). Using the fact that for each m, $\sigma(A^m) = \{\lambda_1^m, \ldots, \lambda_q^m\}$, show that $\lambda_1^m, \ldots, \lambda_q^m$ are all roots of a monic polynomial with integral coefficients (the characteristic polynomial of A^m).

It is natural to ask whether the Dold congruences are the only necessary conditions to be a sequence of Lefschetz numbers of iterations. More precisely, can every sequence of integers satisfying Dold congruences be obtained as the Lefschetz numbers of a self-map of a compact space? The answer is negative with respect to an asymptotic behaviour of this sequence that we will discuss in details in the next section.

(3.1.15) EXAMPLE. Take $l_m = \sum_{k|m} k^k$. Then $i_m = m^m$, thus $i_m \cong 0 \pmod m$. On the other hand we have the estimate by spectral radius $|\operatorname{tr} A^m| \leq q \operatorname{sp}(A)^m$, where q is the dimension of A. Consequently the estimation $|L(f^m)| \leq d \operatorname{sp}(A)^m$, where d is the dimension of $H_*(X;\mathbb{Q})$, holds for the Lefschetz numbers of iterations. This shows that the above sequence $\{l_m\}$ can not be obtained as the sequence $\{\operatorname{tr} A^m\}$, or $\{L(f^m)\}$ although it satisfies Dold's congruences.

3.1.2. Analytic properties of the Lefschetz numbers of iterations. Before presenting a complete analytic and algebraic characterization of a sequence l_m, where $l_m = L(f^m)$ is the Lefschetz number, we need some additional notation.

(3.1.16) DEFINITION. Let $L := \{l_m\}_{m=1}^{\infty}$ be a sequence of integers. We associate with L a sequence $i(L) = \{i_m\}_1^{\infty}$ of integers and a sequence of rational numbers $a(L) = \{a_m\}_1^{\infty}$ defined as follows:

$$i_m(L) := \sum_{k|m} \mu(m/k) l_k \quad \text{and} \quad a_m(L) := \frac{1}{m} \sum_{k|m} \mu(m/k) l_k, \tag{3.1.17}$$

where $\mu \colon N \to \{-1, 0, 1\}$ is the Möbius function. The sequence i_m is called the sequence of *Dold multiplicities*, and a_m the sequence of *algebraic multiplicities* of $\{l_m\}$.

Some authors (e.g. [FgLl]) refer to the the Dold multiplicities of the sequence $l_m = L(f^m)$ of the Lefschetz numbers of iteration of a map as the *periodic Lefschetz numbers* i.e. $i_m(f) = i_m(L) := \sum_{k|m} \mu(m/k) L(f^k)$ is called the m-th periodic Lefschetz number of f and denoted by l_m.

(3.1.18) DEFINITION. For the sequence $l_m = L(f^m)$ the set of indices $\mathcal{A} := \{m \in \mathbb{N} : a_m \neq 0\}$, or equivalently $\mathcal{A} := \{m \in \mathbb{N} : i_m \neq 0\}$, is called the set of *algebraic periods* of f.

Note that $i_m = m a_m$, and from the Möbius inverse formula (3.1.3) it follows that

$$l_m = \sum_{k|m} i_k = \sum_{k|m} k a_k. \tag{3.1.19}$$

(3.1.20) DEFINITION. We say that L satisfies the *Dold condition* if $a(L)$ is integral valued, or equivalently if $i_m(L) \cong 0 \bmod(m)$ for every $m \in \mathbb{N}$. The latter is called the Dold congruences.

With a given sequence $L = \{l_m\}$ we can associate the two generating functions

$$(3.1.21) \qquad \zeta(L;z) := \exp\left(\sum_{m=1}^{\infty}(l_m/m)z^m\right), \quad \text{and} \quad S(L;z) := \sum_{m=1}^{\infty} l_m z^{m-1},$$

called the ζ-function, and the S-function of L respectively. By their definition they are formal power series in the variable z with complex coefficients.

There are formal relations between the generating functions

$$(3.1.22) \qquad S(L;z) = \frac{d}{dz}\ln\zeta(L;\,z), \qquad \zeta(L;z) = \prod_{m=1}^{\infty}(1-z^m)^{-a_m}.$$

Indeed, the first equality is a direct consequence of the formal derivation of $\zeta(L;z)$, the second follows from the definition of $S(L;z)$ and and definition (3.1.19) of l_m.

An algebraic global description of the sequence $L = \{l_m\}$ is given in the following theorem of Babienko and Bogatyi ([BaBo]).

We recall that a polynomial with the leading coefficient equal to 1 is called monic. A root of a monic polynomial with integral coefficients is called *an integral algebraic number*. A function is called *rational* if it is the quotient of two polynomials.

(3.1.23) THEOREM. *For an integral sequence $L = \{l_m\}_1^{\infty}$, the following conditions are equivalent:*

(3.1.23.1) *There exists a map $f\colon X \to X$, of a compact* ENR *X, such that $l_m = L(f^m)$ for every $m \in \mathbb{N}$.*

(3.1.23.2) *$\zeta(L;z)$ is a rational function.*

(3.1.23.3) *$S(L;z)$ is a rational function with simple poles and integral residues equal to zero at infinity.*

(3.1.23.4) *For every $m \in \mathbb{N}$ we have $l_m = \sum_{i=1}^{r}\chi_i\lambda_i^m$, where $\lambda_i \in \mathbb{C}$ are algebraic integral numbers and $\chi_i \in \mathbb{Z}$, $i = 1, \dots, r$.*

(3.1.24) REMARK. In the condition (3.1.23.1) of the statement, X could be a finite CW-complex or even a two-dimensional finite CW-complex (cf. Proposition (3.1.12)).

PROOF. We show first the (3.1.23.1) $\Rightarrow$ (3.1.23.4) and (3.1.23.4) $\Rightarrow$ (3.1.23.2).

Let λ_i^+, $1 \le i \le r^+$, be all distinct eigenvalues of A_{e}, each of multiplicity χ_i^+ and λ_j^-, $1 \le j \le r^-$, be all distinct eigenvalues of A_{o}, each of multiplicity χ_i^-. By the definition and Corollary (2.3.9), we have

$$l_m = \sum_{1\le i\le r^+} \chi_i^+ {\lambda_i^+}^m - \sum_{1\le i\le r^-} \chi_i^- {\lambda_i^-}^m, \tag{3.1.25}$$

where $\sum_{1\le i\le r^+} \chi_i^+ = \dim H_{\mathrm{ev}}(X;\mathbb{Q})$, $\sum_{1\le i\le r^-} \chi_i^- = \dim H_{\mathrm{od}}(X;\mathbb{Q})$. This shows the first implication. Using the main property of exp, the above, and the definition of ζ we get

$$\zeta(L;z) = \frac{\prod\limits_{1\le i\le r^+} \left(\exp\left(\sum\limits_{m=1}^{\infty} \frac{(\lambda_i^+ z)^m}{m}\right)\right)^{\chi_i^+}}{\prod\limits_{1\le i\le r^-} \left(\exp\left(\sum\limits_{m=1}^{\infty} \frac{(\lambda_i^- z)^m}{m}\right)\right)^{\chi_i^-}}. \tag{3.1.26}$$

From the formal identity $\ln(1/(1-t)) = \sum_{n=1}^{\infty} t^n/n$ and (3.1.26) it follows that

$$\zeta(L;z) = \frac{\prod\limits_{1\le i\le r^-} (1-\lambda_i^- z)^{\chi_i^-}}{\prod\limits_{1\le i\le r^+} (1-\lambda_i^+ z)^{\chi_i^+}} = \frac{\det(\mathrm{Id} - zH_{\mathrm{od}}(f))}{\det(\mathrm{Id} - zH_{\mathrm{ev}}(f))}, \tag{3.1.27}$$

since the determinant of a matrix is equal to the product of eigenvalues counted with multiplicities and each eigenvalue of $\mathrm{Id} - zA$ is equal to $1 - z\lambda$, where λ is an eigenvalue of A. This shows the second implication with the polynomials given in an effective way.

Now we show that (3.1.23.2) $\Rightarrow$ (3.1.23.3) The rationality of $\zeta(L;z)$ implies that

$$\zeta(L;z) = \frac{u(z)}{v(z)} = \frac{\prod\limits_i (1-\beta_i z)}{\prod\limits_j (1-\gamma_j z)}, \tag{3.1.28}$$

where the numerator and denominator are expanded into linear factors in terms of the roots of the conjugate polynomials $u(1/z)$, $v(1/z)$, because the factor z does not appear, as follows from $\zeta(L;0) = 1$. Combining the sequence $\{\beta_i\}$ and $\{\gamma_j\}$ into one, and taking multiplication into account, we get $\zeta(L;z) = \prod_i (1-\lambda_i z)^{\chi_i}$, where $\chi_i \in \mathbb{Z}$ and $\lambda_i \ne 0$. From the first equality of (3.1.22) it follows that

$$S(L;z) = \frac{d}{dt}\left(\sum_i \chi_i \ln(1-\lambda_i z)\right) = \sum_i \chi_i \frac{-\lambda_i}{1-\lambda_i z} = \sum_i \frac{\chi_i}{z - \lambda_i^{-1}}.$$

This shows that (3.1.23.2) $\Rightarrow$ (3.1.23.3).

Now we show the implication (3.1.23.3) $\Rightarrow$ (3.1.23.4). Suppose that $S(L;z)$ is a rational function as in (3.1.23.3) of Theorem (3.1.23). Then

$$S(L;z) = H(z) + \sum_{i=1}^{r}\sum_{k=1}^{k_i} \frac{A_k^i}{(1-\lambda_i)^k} + \frac{G(z)}{z^s}, \tag{3.1.29}$$

where $H(z)$ is a polynomial and $G(z)$ a polynomial of degree less than s. Since $S(L;\infty) = 0$, we have $H(z) = 0$. Since 0 is a regular point by the definition of $S(L;z)$, we have $G(z) = 0$ and, consequently,

$$S(L;z) = \sum_{i=1}^{r}\sum_{k=1}^{k_i} \frac{A_k^i}{(1-\lambda_i)^k} = \sum_{i=1}^{r} \frac{A_k^i}{(1-\lambda_i)} = \sum_{i=1} \frac{-\lambda_i A_k^i}{(z-\lambda_i)},$$

because the condition that the poles are simple implies that $A_k^i = 0$ for $k > 1$. The integrality of the residues implies that $A_k^i = \chi_i\lambda_i$, where the χ_i are integers. It follows that

$$S(L;z) = \sum_{i=1}^{r} \frac{\chi_i\lambda_i}{1-\lambda_i z} = \sum_{i=1}^{r} \chi_i\lambda_i \sum_{m=0}^{\infty} \lambda_i^m z^m = \sum_{m=0}^{\infty}\left(\sum_{i=1}\chi_i\lambda_i^{m+1}\right) z^m, \tag{3.1.30}$$

and consequently $l_m = \sum_{i=1}^{r} \chi_i \lambda_i^m$.

We are left with the task to show that λ_i are algebraic integers. First note that from the rationality of $S(L;z)$ it follows that $S(L;z) = u(z)/v(z)$, where $u(z), v(z) \in \mathbb{Q}(z)$. Indeed the coefficients of $S(L;z)$ are integers, hence the coefficients of $u(z)$ and $v(z)$, given as the solutions of a sequence of linear equations, belong to the field of quotients of $\mathbb{Z}$, i.e. to $\mathbb{Q}$. Now we will use the following Fatou lemma.

(3.1.31) Lemma. *Let $\sum_{m=0}^{\infty} c_m z^m = u(z)/v(z)$ be a rational function, $c_i \in \mathbb{Z}$. Then the polynomials $u(z)$, $v(z)$ have the form*

$$u(z) = c_0 + \sum_{i=1}^{s} a_i z^i, \qquad v(z) = 1 + \sum_{j=1}^{q} b_j z^j,$$

where the coefficients $\{a_i\}_1^s$ $\{b_j\}_1^q$ are integers.

Proof. As we already noted $\sum_{m=0}^{\infty} c_m z^m = u(z)/v(z)$, where $u(z)$, $v(z)$ have rational coefficients. Multiplying by the least common multiple of all denominators and dividing by a power of z if necessary, we can represent our function as

$$\sum_{m=0}^{\infty} c_m z^m = \frac{a}{b}\cdot\frac{a_0 + \cdots + a_s z^s}{b_0 + \cdots + b_q z^q} = \frac{a}{bb_0}\cdot\frac{a_0 + a_1 z + \cdots + a_s z^s}{1 + b_1/b_0 + \cdots + (b_q/b_0)z^q},$$

where $(a,b)=1$, $(a_0,\ldots,a_s)=1$, $(b_0,\ldots,b_q)=1$. Expanding the denominator of the second fraction in the geometric series we get

$$\sum_{m=0}^{\infty} c_m z^m = \frac{a}{bb_0}(a_0+a_1z+a_2z^2+\cdots+a_sz^s)\sum_{l=0}^{\infty}(-1)^l\left(\frac{b_1}{b_0}z+\cdots+\frac{b_q}{b_0}z^q\right)^l.$$

If a prime $p|b_0$, then p does not divide b_i, for $i\geq 1$ and the right-hand side series would not have integral coefficients then. This shows that $b_0=\pm 1$. Consequently, we can assume that

$$\sum_{m=0}^{\infty} c_m z^m = \frac{a}{b}\cdot\frac{a_0+\cdots+a_sz^s}{1+\cdots+b_q\,z^q}, \tag{3.1.32}$$

Suppose that a prime $p|b$. Since $(a_0,\ldots,a_s)=1$, there exists $1\leq i\leq s$ such that $p|a_0,\ldots,p|a_{i-1}$ but p does not divide a_i. Rewriting (3.1.32) we have

$$a(a_0+\cdots+a_sz^s)=b(1+b_1z+\cdots+b_qz^q)\bigg(\sum_{m=0}^{\infty}c_mz^m\bigg).$$

Comparing the coefficient at z^i on both sides of the above equality we get

$$aa_i=b\bigg(\sum_{k=0}^{i}c_kb_{i-k}\bigg).$$

This shows that $p|a$ or $p|a_i$ which leads to a contradiction. From this it follows that $b=\pm 1$, and the Fatou lemma is proved. $\square$

Now, applying the Fatou lemma to $S(L;z)=\sum_{m=1}^{\infty}(\sum_{i=1}^{r}\chi_i\lambda_i^m)z^{m-1}$ we see that $u(z)=l_1+\sum_{i=1}^{s}a_iz^i$, $v(z)=1+\sum_{j=1}^{q}b_jz^j$, and $\{\lambda_i\}_1^r$ are the roots of the polynomial conjugate to the denominator, that is to $\widetilde{v}(z)=z^q+\sum_{j=1}^{q}b_jz^{q-j}$ as follows from (3.1.30). Consequently, $r=q$ and λ_i are algebraic integers, which shows (3.1.23.3) $\Rightarrow$ (3.1.23.4).

Finally we prove that (3.1.23.4) $\Rightarrow$ (3.1.23.1). We first show that if λ_i, λ_j are algebraically conjugate roots of $\widetilde{v}(z)$, then $\chi_i=\chi_j$. We recall that two algebraic numbers are called *conjugate* if they are roots of the same irreducible polynomial. Let $\Sigma=\mathbb{Q}(\lambda_1,\ldots,\lambda_r)\subset\mathbb{C}$ be the field of the polynomial $\widetilde{v}(z)$ and let σ be an automorphism of Σ over $\mathbb{Q}$, i.e. σ is the identity on $\mathbb{Q}$. (The group of all such automorphisms is called the *Galois group* of Σ). Since $\sigma(w(z))=w(z)$ for every polynomial with rational coefficients, $\{\sigma(\lambda_i)\}_1^r$ is again the set of all roots of $\widetilde{q}(z)$, and consequently $\sigma(\lambda_i)=\lambda_{\sigma(i)}$ where $\{\sigma(i)\}_1^r$, denoted also by σ, is a permutation of indices. Applying σ to the sequence L we get

$$\sigma(l_m)=\sigma\bigg(\sum_{i=1}^{r}\chi_i\lambda_i^m\bigg)=\sum_{i=1}^{r}\chi_i(\sigma(\lambda_i))^m=\sum_{i=1}^{r}\chi_i\lambda_{\sigma(i)}^m=\sum_{i=1}^{r}\chi_{\sigma^{-1}(i)}\lambda_i^m.$$

Since $L = \{l_m\}$ is an integral sequence, $\sigma(l_m) = l_m$, $m = 1, 2, \ldots$, and consequently

$$\sum_{i=1}^{r} \chi_i \lambda_i^m = \sum_{i=1}^{r} \chi_{\sigma^{-1}(i)} \lambda_i^m, \quad m = 1, 2, \ldots$$

Writing the first r of the above equations in matrix form gives

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^r & \lambda_2^r & \ldots & \lambda_r^r \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_r \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^r & \lambda_2^r & \ldots & \lambda_r^r \end{bmatrix} \begin{bmatrix} \chi_{\sigma^{-1}(1)} \\ \chi_{\sigma^{-1}(2)} \\ \vdots \\ \chi_{\sigma^{-1}(r)} \end{bmatrix}.$$

Since λ_i are distinct, the matrices in this equation are nonsingular (the Vandermonde determinant). Thus $\{\chi_i = \chi_{\sigma(i)}\}_{i=1}^r$.

On the other hand, it is known that the Galois group acts transitively on the $\{\lambda_i\}$, i.e. for every conjugate λ_i, λ_j there exists an automorphism σ such that $\sigma(\lambda_i) = \lambda_j$ (cf. [La]). From it follows that $\chi_i = \chi_j$ if λ_i and λ_j are conjugate.

Summing up, we have shown the following. Let $\widetilde{v}(z) = \prod_\alpha \widetilde{v}_\alpha(z)$, $\alpha \in \mathcal{A}$ be the decomposition of $\widetilde{v}(z)$ into irreducible polynomials. Then

$$(3.1.33) \qquad l_m = \sum_{\alpha \in \mathcal{A}} \chi_\alpha \left(\sum_{j=1}^{r_\alpha} \lambda_i^m \right) = \sum_{\alpha \in \mathcal{A}^+} \chi_\alpha^+ \left(\sum_{j=1}^{r_\alpha} \lambda_i^m \right) - \sum_{\alpha \in \mathcal{A}^-} \chi_\alpha^- \left(\sum_{j=1}^{r_\alpha} \lambda_i^m \right),$$

where $\chi_\alpha \in \mathbb{Z}$, $\chi_\alpha^+ \in \mathbb{N}$, $\chi_\alpha^- \in \mathbb{N}$, $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-$ is the decomposition of indices depending on whether χ_α is positive or negative.

Observe that from Exercise (3.1.13) it follows that the right hand side of (3.1.33) is equal to $\operatorname{tr} A_{\mathrm{e}}^m - \operatorname{tr} A_{\mathrm{o}}^m$ for some integral matrices A_{e} and A_{o}. Indeed, for a given $\alpha \in \mathcal{A}$ let A_α be the $q_\alpha \times q_\alpha$ integral matrix given for polynomial $\widetilde{v}_\alpha$ by Exercise (3.1.13). Put

$$A_{\mathrm{e}} := \bigoplus_{\alpha \in \mathcal{A}^+} \chi_\alpha^+ A_\alpha, \qquad A_{\mathrm{o}} := \bigoplus_{\alpha \in \mathcal{A}^-} \chi_\alpha^- A_\alpha,$$

where kA, for $k \in \mathbb{N}$, means $\bigoplus_1^k A$. The additivity of trace (Proposition (2.3.11)) and Exercise (3.1.13) show that

$$(3.1.34) \qquad l_m = \sum_{\alpha \in \mathcal{A}^+} \chi_\alpha^+ \left(\sum_{j=1}^{r_\alpha} \lambda_i^m \right) - \sum_{\alpha \in \mathcal{A}^-} \chi_\alpha^- \left(\sum_{j=1}^{r_\alpha} \lambda_i^m \right) = \operatorname{tr} A_{\mathrm{e}}^m - \operatorname{tr} A_{\mathrm{o}}^m.$$

Now the implication (3.1.23.4) $\Rightarrow$ (3.1.23.1) follows from (3.1.34) and Proposition (3.1.12). This completes the proof of Theorem (3.1.23). $\square$

Theorem (3.1.23) is a nice tool for constructing integral sequences $\{l_m\}$ which satisfy the Dold congruences but are not of the form $\{L(f^m)\}$ (cf. [BaBo, Examples 1 and 2]).

(3.1.35) Example.

(3.1.35.1) Let us take the sequence of algebraic multiplicities (cf. (3.1.17))

$$a_m := \begin{cases} 0 & \text{if } m \neq 2^k, \\ 1 & \text{if } m = 2^m, \end{cases} \qquad k = 1, 2, \ldots$$

Then $l_m = 2^{k+1} - 1$ if $m = 2^k s$. The Dold congruences are satisfied by the construction of $L = \{l_m\}$. One can show that $S(L; z)$ cannot be a rational function then (see [BaBo] for details).

(3.1.35.2) Let us take the sequence

$$l_m := \frac{1}{2}[2^m + (-2)^m] = \begin{cases} 0 & \text{if } m = 2i - 1, \\ 2^m & \text{if } m = 2i, \end{cases} \qquad i = 1, 2, \ldots$$

One can verify that the Dold congruences are satisfied, $S(L; z)$ is a rational function with poles $z = \pm 1/2$, but the residues at these poles are equal to $-1/2$, which contradicts to the condition (3.1.23.3) of Theorem (3.1.23) (see [BaBo] for details).

It is worth pointing that in the above examples the sequence $\{l_m\}$ has an exponential rate of growth so that the estimate used in Example (3.1.15) is not violated.

Remarkably, the purely algebraic stuff used in the proof of this theorem allows us to prove the existence of periodic points. To show it we need the following notions.

Let $A = A_{\mathrm{e}} \oplus A_{\mathrm{o}} \colon E_{\mathrm{e}} \oplus E_{\mathrm{o}} \to E_{\mathrm{e}} \oplus E_{\mathrm{o}}$ be an endomorphism of a vector space with $\mathbb{Z}_2$-gradation. Let $\lambda = \lambda_i \in \sigma(A) = \sigma(A_{\mathrm{o}}) \cup \sigma(A_{\mathrm{e}})$ be an eigenvalue. By E_{o}^{λ}, correspondingly E_{e}^{λ}, we denote the generalized eigenspace of A_{o}, i.e. $\bigcup_{j=1}^{\infty} \operatorname{Ker}(\lambda \mathrm{I} - A_0)^j$, or A_{e}, respectively. We put

$$\text{(3.1.36)} \qquad \chi_\lambda(A), \quad \text{or sometimes shortly,} \quad \chi_i(A) := \dim E_{\mathrm{e}}^{\lambda_i} - \dim E_{\mathrm{o}}^{\lambda_i}.$$

We call an eigenvalue $\lambda \in \sigma(A)$ *essential* if $\chi_\lambda(A) \neq 0$.

(3.1.37) Definition. We define the *essential spectral radius* of $A = A_{\mathrm{o}} \oplus A_{\mathrm{e}}$, as

$$\mathrm{sp}_{\mathrm{e}}(A) := \max\{\lambda \in \sigma(A) \text{ is an essential eigenvalue}\}.$$

We let $\Upsilon(A)$ denote the number of distinct essential eigenvalues λ of A. It is called the *index of periodicity* of A.

(3.1.38) Definition. Let $A = A_{\mathrm{o}} \oplus A_{\mathrm{e}}$ be an endomorphism of a $\mathbb{Z}_2$-graded vector space. Let next $E^0 = E^0_{\mathrm{o}} \oplus E^0_{\mathrm{e}}$ be the generalized kernel of A. Put $\widetilde{E}_{\mathrm{o}} = E_{\mathrm{o}}/E^0_{\mathrm{o}}$, correspondingly $\widetilde{E}_{\mathrm{e}} = E_{\mathrm{e}}/E^0_{\mathrm{e}}$, and next define $\chi_{\mathrm{o}}(A) := \dim \widetilde{E}_{\mathrm{o}}$, and respectively $\chi_{\mathrm{e}}(A) := \dim \widetilde{E}_{\mathrm{e}}$. The number

$$\chi(A) = \chi_{\mathrm{e}}(A) - \chi_{\mathrm{o}}(A)$$

is called the Euler characteristic of A.

Finally we define the last invariant of this endomorphism

$$\mathfrak{h}(A) := \max\Big\{ \sum_{\lambda_i \in \mathcal{A}^+} \chi_i(A), -\sum_{\lambda_i \in \mathcal{A}^-} \chi_i(A) \Big\},$$

where $\mathcal{A}^+$, $\mathcal{A}^-$, are as in the formula (3.1.33), i.e. correspond to the positive and negative χ_i, respectively.

Let $f\colon X \to X$ be a map of a finite complex X. For the induced map $A = H_*(f) = H_{\mathrm{ev}}(f) \oplus H_{\mathrm{od}}(f)$ on the homology spaces we denote $\chi_\lambda(A)$ by $\chi_\lambda(f)$, and call it the λ-Euler characteristic of f. Analogously we define $\chi_i(f)$, $\mathrm{sp}_{\mathrm{es}}(f)$, $\chi(f)$, $\Upsilon(f)$, and $\mathfrak{h}(f)$, respectively. The invariant $\chi(f)$, is called the Euler–Poincaré characteristic or sometimes the Fuller index, and $\Upsilon(f)$, is called the index of periodicity of f. The latter were introduced and used by Bowszyc ([Bo]) to study periodic points. $\mathrm{sp}_{\mathrm{es}}(f)$ and $\mathfrak{h}(A)$ were introduced by by Babenko and Bogatyi (see [BaBo]) .

Remark that if $f\colon X \to X$ is a homeomorphism, then $\chi(f) = \chi(X)$ is the Euler characteristic of X. Indeed, the induced map has no zero eigenvalue then and consequently $\chi(f) = \dim H_{\mathrm{ev}}(X) - \dim H_{\mathrm{od}}(X) = \chi(X)$, because $E = \bigoplus_{\lambda \in \sigma(A)} E^\lambda$ for every endomorphism $A\colon E \to E$. Observe that

$$\chi(f) \neq 0 \Rightarrow \Upsilon(f) \neq 0, \tag{3.1.39}$$

but the converse is not true in general. In the terms of Theorem (3.1.23) we have $\Upsilon(f) = r$, where r is a number of poles of $S(L, z)$ or equivalently the degree of polynomial $v(z)$ in the rational representation $S(L; z) = u(z)/v(z)$. The integer $\chi_i(f)$ is equal to the coefficient χ_i of the expression (3.1.23.4) of Theorem (3.1.23) and the residue of $S(L; z)$ at the pole λ_i. $\chi(f)$ is equal to the sum of all residues of $S(L; z)$, and $\mathfrak{h}(f) = \max\{\sum_{\alpha \in \mathcal{A}^+} \chi_\alpha, -\sum_{\alpha \in \mathcal{A}^-} \chi_\alpha\}$ of the formula (3.1.33), i.e. the maximum of the sum of all positive residues of $S(L; z)$ and the absolute value of the sum of all negative residues of $S(L; z)$.

The analytic characterization of the sequence $\{l_m\}$ leads to theorems on the existence of periodic points that are based on Theorem (3.0.1).

We have the following theorem ([Bo], also [BaBo], [DuGr]).

(3.1.40) THEOREM. *Let $f\colon X \to X$ be a map of a compact* ENR*, e.g. a finite* CW*-complex. If $\Upsilon(f) \neq 0$, then $P^m(f) \neq \emptyset$ for some $m \leq \Upsilon(f)$. In particular it holds if $\chi(f) \neq 0$.*

PROOF. With respect to Theorem (3.0.1) it is enough to show that $l_m = L(f^m) \neq 0$ for some $m \leq \Upsilon(f)$. From the formula (3.1.29) of the proof of implication (3.1.23.3) $\Rightarrow$ (3.1.23.4) it follows that $S(L;z) = u(z)/v(z)$ where $u(z)$, $v(z)$ are polynomials and $\deg(u(z)) < \deg(v(z))$. On the other hand $S(L;z)$ is regular at 0 by the definition. Multiplying power series it is easy to check that if $S(L;z) = \sum_{m=1}^{\infty} l_m z^{m-1}$ is not the zero series, then for at least one $m \leq \deg(u(z))$, the coefficient $l_m \neq 0$. $\square$

(3.1.41) REMARK. For the study of properties of Lefschetz numbers of iterations one can use also the series $\mathcal{L}(L;z) := \chi(f) + \sum_{m=1}^{\infty} l_m z^m = \chi(f) + zS(L;z)$ as it was done in [Bo]. Under our notation we have $\mathcal{L}(L;z) = \sum_{i=1}^{r}\sum_{m=0}^{\infty} \chi_i(f)\lambda_i^m z^m$, which justified the convention $l_0 := \chi(f)$.

The implication $\chi(f) \neq 0 \Rightarrow \operatorname{Fix}(f^m) \neq \emptyset$ has been shown by Fuller for a homeomorphism (cf. [Fu]) and next by Bowszyc [Bo] for the general case.

As a consequence we get the following ([BaBo]).

(3.1.42) COROLLARY. *If $\chi(f) = 0$ and $\Upsilon(f) \geq 1$, then $\Upsilon(f) \geq 2$ and, for some $1 \leq m \leq \Upsilon(f) - 1$, $P^m(f) \neq \emptyset$.*

PROOF. Since $\sum_{i=1}^{\Upsilon(f)} \chi_i(f) = 0$ and there is i_0 such that $\chi_{i_0}(f) \neq 0$, for at least two indices $\chi_i(f) \neq 0$. Note also that the multiplying of series used in Theorem (3.1.40) shows that for the series $\mathcal{L}(L;z) = \sum_{m=0}^{\infty} l_m z^m = \chi(f) + zS(L;z) = \chi(f) + u(z)/v(z)$, the condition $\Upsilon(f) \geq 1$ yields the existence of two indices $0 \leq m_1 < m_2 \leq \Upsilon(f)$ such that $l_{m_1} \neq 0$, $l_{m_2} \neq 0$. On the other hand $l_0 = \chi(f) = 0$, which shows that $0 < m_1 \leq \Upsilon(f) - 1$, and proves the statement. $\square$

As another application of the above method we have the following ([BaBo]).

(3.1.43) COROLLARY. *If $\Upsilon(f) \geq 1$, then $P^m(f) \neq \emptyset$ for some*

$$m \leq \mathfrak{h}(f) = \frac{1}{2}\bigg(\sum_{i=1}^{\Upsilon(f)} |\chi_i(f)| + \chi(f)\bigg).$$

PROOF. The equality of statement follows directly from the definitions of $\mathfrak{h}(f)$ and $\chi(f)$. If for all indices $1 \leq m \leq \mathfrak{h}(f)$ we have $l_m = L(f^m) = 0$, then

$$(3.1.44)\quad 0 = \sum_{\lambda_i \in \mathcal{A}^+} \chi_i(f)\lambda_i^m + \sum_{\lambda_i \in \mathcal{A}^-} \chi_i(f)\lambda_i^m \Rightarrow \sum_{\lambda_i \in \mathcal{A}^+} \chi_i(f)\lambda_i^m = -\sum_{\lambda_i \in \mathcal{A}^-} \chi_i(f)\lambda_i^m$$

for all such m. Put $J_m := \sum_{\lambda_i \in \mathcal{A}^+} \chi_i(f)\lambda_i^m = -\sum_{\lambda_i \in \mathcal{A}^-} \chi_i(f)\lambda_i^m$. Then we get two systems of $\mathfrak{h}(f)$ equations for λ_i. Note that in each of them the number of unknown variables λ_i, counted with the multiplicity, does not exceed $\mathfrak{h}(f)$. Suppose for simplicity that $s := \sum_{\lambda_i \in \mathcal{A}^+} \chi_i(f) \leq \sum_{\lambda_i \in \mathcal{A}^-} \chi_i(f)$. Then repeating each λ_i $\chi_i(f)$ times if $\lambda_i \mathcal{A}^+$ or correspondingly $-\chi_i(f)$ times if $\widetilde{\lambda}_i \in \mathcal{A}^-$, we can write first s equations of our systems as

$$
(3.1.45) \qquad \begin{array}{c}
\lambda_1 + \lambda_2 + \cdots + \lambda_s = J_1 = \widetilde{\lambda}_1 + \widetilde{\lambda}_2 + \cdots + \widetilde{\lambda}_s \\
\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_s^2 = J_2 = \widetilde{\lambda}_1^2 + \widetilde{\lambda}_2^2 + \cdots + \widetilde{\lambda}_s^2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\lambda_1^s + \lambda_2^s + \cdots + \lambda_s^s = J_s = \widetilde{\lambda}_1^s + \widetilde{\lambda}_2^s + \cdots + \widetilde{\lambda}_s^s.
\end{array}
$$

The numbers J_m are called consecutive moments of $\{\lambda_i\}$, or $\{\widetilde{\lambda}_i\}$, respectively. Now we use a classical fact about the moments of s elements of a field of characteristic zero (cf. [Bu, Appendix to Chapter V, Corollary 3]). It states that if $\lambda_1^k + \cdots + \lambda_s^k = \widetilde{\lambda}_1^k + \cdots + \widetilde{\lambda}_s^k$ for all $1 \leq k \leq s$, then there exists a permutation of indices σ such that $\widetilde{\lambda}_i = \lambda_{\sigma(i)}$. Consequently, up to permutation of indices, we can assume that $\lambda_i = \widetilde{\lambda}_i$ for $1 \leq i \leq s$. Since $\mathcal{A}^+ \cap \mathcal{A}^- = \emptyset$ we get a contradiction if at least one of them is nonempty. But supposition $\mathcal{A}^+ = \mathcal{A}^- = \emptyset$ contradicts $\Upsilon(f) \geq 1$. □

Before an estimate of the asymptotic behaviour of the sequence l_m we show a complete characterization of a bounded sequence l_m satisfying the Dold congruences (cf. [BaBo]).

(3.1.46) THEOREM. *Let $L = \{l_m\}$ be a sequence of integers. The following conditions are equivalent:*

(3.1.46.1) *The sequence $a(L)$ of algebraic multiplicities of L is integral, i.e. L satisfies the Dold condition, and $\{l_m\}$ is bounded.*

(3.1.46.2) *$a(L)$ is integral and there exists a natural number k_0 such that $a_k = 0$ for $k \geq k_0$.*

(3.1.46.3) *There exists a natural number l such that $l_m = l_{GCD(m,l)}$ and a_k is integral for all $k|l$.*

(3.1.46.4) *There exists a natural number l such that $l_m = l_{m \bmod l}$ and a_{kp} is an integer whenever $k|l$ and p is prime.*

(3.1.46.5) *There exists a natural number l and integers a_d for all $d|l$ such that*

$$l_m = \sum_{d|l} a_d \sum_{i=1}^{d} (\varepsilon_d)^{im}, \quad m = 1, 2, \dots,$$

where $(\varepsilon_d)^d = 1$ is a primitive root of unity.

(3.1.46.6) *$\zeta(L;z)$ is a rational function with no zeros or poles inside the unit disc.*

(3.1.46.7) *$S(L;z)$ is a rational function which is equal to zero at infinity and regular on the unit disc and which has only simple poles with residues which are integers.*

(3.1.46.8) *There exists a periodic homeomorphism g of a compact* ENR *such that $\{l_m = L(g^m)\}_{m=1}^{\infty}$.*

Proof. (cf. [BaBo]) (3.1.46.1) ⇒ (3.1.46.2) If there exists $|l_m| \le M$ for all m, then

$$|a_k| = \frac{1}{k}\left|\sum_{m|k} \mu(k/m) l_m\right| \le \frac{1}{k}\sum_{m|k}\left|\mu(k/m)\right||l_m| \le \frac{M}{k}\sum_{m|k}|\mu(k/m)| \le \frac{M}{k}\tau(k),$$

where $\tau(k)$ is the number of divisors of k. On the other hand it is known that for every $\delta > 0$, $\tau(n)/n^\delta \to 0$ if $n \to \infty$ ([Ch VI.3 Theorem 5]). This shows that $|a_k| \to 0$, and consequently $a_k = 0$ for large k, because a_k is an integer.

(3.1.46.2) ⇒ (3.1.46.6) This implication follows from the representation (3.1.22) of $\zeta(L;z)$. Note that all zeros and or poles of $\zeta(z)$ are roots of unity.

(3.1.46.6) ⇒ (3.1.46.7) Note that every pole of $S(L;z)$ is the inverse $1/z_0$ of a zero or a pole z_0 of $\zeta(L;z)$, thus also a root of unity. The remaining part of the claim is the condition (3.1.23.3) of Theorem (3.1.23) which we have already shown.

(3.1.46.7) ⇒ (3.1.46.5) First observe that if a series $S(L;z)$ with integer coefficients represents a rational function which is regular inside the unit disc, then all poles of the function are roots of unity. Indeed then $S(L;z) = u(z)/v(z)$, where $u(z)$, $v(z)$ are polynomial with integer coefficients and $v(z)$ has no root inside of the unit disc. By this $\widetilde{v}(z) := z^r v(1/z)$, r the degree of $v(z)$, has all roots of the module less than or equal to 1. From the already used Kronecker theorem (cf. [Nar]) it follows that they are roots of unity and zero. Consequently $v(z)$ has roots λ_i being the roots of unity. This shows that

$$l_m = \sum_{i=1}^{r} \chi_i \lambda_i^m, \quad \text{where } \lambda_i^{d(i)} = 1. \tag{3.1.47}$$

The form (3.1.47) of $\{l_m\}$ and Exercise (3.1.51) give

$$\zeta(L;z) = \prod_{i=1}^{r}(1-\lambda_i z)^{-\chi_i}. \tag{3.1.48}$$

In the proof of implication (3.1.23.4) ⇒ (3.1.23.1) of Theorem (3.1.23) we have shown that if λ_i appears in the formula (3.1.48), then all numbers algebraically conjugated to it appear too. Since λ_i is a root of unity we can write (3.1.48) as

$$\zeta(L;z) = \prod_{d|l} \omega_d(z)^{-\chi(d)}, \tag{3.1.49}$$

where l is the lowest common multiple of all powers $d(i)$ of $\{\lambda_i\}_{i=1}^r$, $\chi(d)$ is an integer equal to 0 if d does not divide l, and $\omega_d(z)$ is the d-th cyclotomic polynomial (cf. [La]), i.e. the polynomial of degree $\phi(d)$ whose roots are the primitive roots of unity of degree d, e.g. $\omega_1(z) = z-1$, $\omega_2(z) = z+1$, $\omega_3(z) = z^2+z+1, \dots$ Now using the identity

$$\omega_d(z) = \prod_{q|d} (1-z^q)^{\mu(d/q)}$$

(cf. Exercise (3.1.52)) we can write $\zeta(z)$ in the form (3.1.22) as the expression

$$\zeta(L;z) = \prod_{q|l} (1-z^q)^{-\sum_{s|q/l} \mu(s)\chi(sd)}. \tag{3.1.50}$$

Comparing (3.1.50) with (3.1.22) we conclude that if $q|l$, then

$$a_q(L) = \sum_{s|q/l} \mu(s)\chi(sd)$$

and if q does not divide l, then $a_q(L) = 0$. In particular $i_q(L)$ is integral. Consequently, we have the following representation for l_m:

$$l_m = \sum_{d|m} d a_d = \sum_{d|l} a_d \sum_{i=1}^{d} \varepsilon_d^{i_k}, \quad m = 1, 2, \dots,$$

where ε_d is a primitive root of unity of degree d.

(3.1.46.5) $\Rightarrow$ (3.1.46.8) (cf. Proposition (3.1.12)). Consider automorphism $\psi_d = \psi_{(1,\dots,d)}$ of the Euclidean space $\mathbb{R}^d$ given by the cyclic permutation of the basis vectors $\{e_i\}$. By its definition $\psi_d^d = \mathrm{I}$ and $\operatorname{tr} \psi_d^k = 0$ if $d \nmid k$. Put

$$K_d := \begin{cases} \bigvee_{i=1}^{d} S_i^2 & \text{if } a_k > 0, \\ \bigvee_{i=1}^{d} S_i^1 & \text{if } a_k > 0, \end{cases}$$

and take $f_d \colon K_d \to K_d$ the homeomorphism of the bouquet K_d given by the cyclic permutation of summands. It is easy to check that $L(f_d^k) = 1 + \operatorname{sgn} a_d \operatorname{tr} \psi_d^k$. Next take $\widetilde{K}_d = \bigvee K_d$ a bouquet of $|a_d|$ copies of K_d with the homeomorphism $F_d := \bigvee f_d$. Then $L(f^k) = 1 + a_d \operatorname{tr} \psi_d^k$. Finally, let $K := S^1 \bigvee_{d|l} (\widetilde{K}_d)$ and $g := \mathrm{id} \vee (\bigvee_{d|l} F_d)$. The map g is the required homeomorphism of the bouquet K.

(3.1.46.8) $\Rightarrow$ (3.1.46.4) Let $g \colon K \to K$ be a periodic homeomorphism of a compact ENR K of period l. Since $g^m = g^{m \bmod l}$, $l_m = l_n$ for any $m = n + kl$, which proves the first part of (3.1.46.4). Since the Dold congruences are satisfied for any map of compact ENR (cf. Theorem (3.1.4), every algebraic multiplicity $a_m(f)$ is integral.

(3.1.46.4) $\Rightarrow$ (3.1.46.3) We show it using an induction with respect to the greatest common divisor $\mathrm{GCD}(m,l)$. Let us denote $M := \max_m |l_m| = \max_{1\le m\le l} |l_m|$. Assume first that $\mathrm{GCD}(m,l)=1$. Then in the arithmetic progression $\{m+k\,l\}_{k=1}^{\infty}$ there exists infinitely many prime numbers by the Dirichlet theorem (see [Ch]). Let us take k_0 such that $p=m+k_0 l>2M$ and is prime. Then $l_p - l_1 = pa_p$ and thus is divisible by p. Since also $|l_p-l_1|\le 2M$, we have $a_p=0$. This means that $l_m = l_{m+k_0 l} = l_p = l_1$.

Now assuming that the thesis holds for $\mathrm{GCD}(m,l)\le l_0$ we show that (3.1.46.3) holds for $\mathrm{GCD}(m,l)=l_0+1$. We have $m+kl=q(m'+kl')$, where $q=(m,l)$ and $(m',l')=1$. Using once more the Dirichlet theorem we find a prime number $p=m'+kl'>2M$ and $p>q$. Now

$$\begin{aligned} qpa_{qp} &= \sum_{d|qp}\mu(d)l_{qp/d} = \sum_{d|qp}\mu(d)l_{qp/d} - \sum_{d|qp}\mu(d)l_{q/d} \\ &= \sum_{d|q}\mu(d)(l_{qp/d}-l_{q/d}) \ = l_{qp}-l_q + \sum_{d|qp}\mu(d)(l_{pq/d}-l_{q/p}). \end{aligned}$$

By the inductive assumption $l_{qp/d}=l_{q/d}$ for $d>1$, and consequently we get $qpa_{pq}=l_{qp}-l_p$. Since $|l_{qp}-l_q|\le 2M$, $a_{qp}=0$, and consequently

$$l_m = l_{m,+k_1 l} = l_{qp} - l_q = l_{\mathrm{mod}(m,l)}.$$

(3.1.46.3) $\Rightarrow$ (3.1.46.1) Obviously the sequence $\{l_m\}_1^{\infty}$ is bounded. Let us take the integral numbers a_m for $l|l$ and put

$$\widetilde{a}_m := \begin{cases} 0 & \text{if } m\nmid l, \\ a_m & \text{if } m|l. \end{cases}$$

The sequence $\{\widetilde{a}_m\}_1^{\infty}$ determines the sequence $\{\widetilde{l}_m\}_1^{\infty}$. By the definition of $\{\widetilde{a}_m\}$, $\widetilde{l}_m = l_m$ for $m|l$. On the other hand we have

$$\widetilde{l}_m = \sum_{d|m} d\widetilde{a}_d = \sum_{d|\mathrm{GCD}(m,l)} da_d = l_{\mathrm{GCD}(m,l)} = l_m,$$

i.e. $\widetilde{L}=\{\widetilde{l}_m\}_1^{\infty}=\{l_m\}_1^{\infty}=L$. By the uniqueness of correspondence between $\{l_m\}_1^{\infty}$ and $\{a_m\}_1^{\infty}$ (the Möbius formula), $\{a_m(L)\}_1^{\infty}=\{a_m(\widetilde{L})\}_1^{\infty}=\{\widetilde{a}_m\}_1^{\infty}$, thus a_m are integral. This completes the proof of Theorem (3.1.46). $\square$

(3.1.51) Exercise. Show that if the sequence $L=\{l_m\}_1^{\infty}$ is of the form $l_m=\sum_{i=1}^r \chi_i\lambda_i^m$, where $\lambda_i^{d(i)}=1$ then its zeta function is $\zeta(L;z)=\prod_{i=1}^r(1-\lambda_i z)^{-\chi_i}$.

(3.1.52) EXERCISE. Show that for the d-th cyclotomic polynomial we have $\omega_d(z) = \prod_{q|d}(1-z^q)^{\mu(d/q)}$.

Hint. See [La, VII §3].

3.1.3. Asymptotic behaviour of Lefschetz numbers of iterations. Now we discuss the asymptotic behaviour of $\{L(f^m)\}$, which is important if $\{L(f^m)\}$ is unbounded. In all this subsection we mainly follow [BaBo]. The next theorem is a number theory result based on another Kronecker theorem.

(3.1.53) THEOREM. *Let $X \to X$ be a map of a space X with the finitely generated real homology spaces, e.g. a map of a compact* ENR. *Then one of the three, mutually disjoint, possibilities holds:*

(3.1.53.1) $L(f^m) = 0$, $m = 1, 2, \ldots$, *which happens if and only if* $\mathrm{sp}_{\mathrm{es}}(f) = 0$.

(3.1.53.2) *The sequence $\{L(f^m)/\mathrm{sp}_{\mathrm{es}}(f)^m\}_1^\infty$ has the same limit points as the periodic sequence $\{\sum_i \alpha_i \varepsilon_i^m\}_{m=1}^\infty$, where $\alpha_i \in \mathbb{Z}$, $\varepsilon_i \in \mathbb{C}$ and $\varepsilon_i^k = 1$ for some $k \in \mathbb{N}$.*

(3.1.53.3) *The set of limit points of the sequence $\{|L(f^m)|/\mathrm{sp}_{\mathrm{es}}(f)^m\}_1^\infty$ contains an interval.*

PROOF. For simplicity, denote $\mathrm{sp}_{\mathrm{es}}(f)$ by ρ_0. Suppose first that $\rho_0 \geq 1$, otherwise we have the case (3.1.53.1) (cf. Remark (3.1.54)). Let $\lambda_1 = \rho_0 \exp(2\pi\imath\theta_1)$, $\lambda_2 = \rho_0 \exp(2\pi\imath\theta_2)$, $\ldots$, $\lambda_r = \rho_0 \exp(2\pi\imath\theta_r)$ be all the essential eigenvalues of $H_*(f)$, i.e. $\chi_j \neq 0$ for $1 \leq j \leq r$.

First observe that $L(f^m)/\rho_0^m$ has the same asymptotic behaviour as the sequence $\sum_{i=1}^r \chi_j \exp(2\pi\imath m\theta_j)$. Indeed

$$L(f^m) = \rho_0^m\left(\sum_{i=1}^r \chi_j \exp(2\pi\imath m\theta_j)\right) + \sum_\lambda \chi_\lambda \lambda^m,$$

where in the second summand all moduli $|\lambda| \leq \rho_1 < \rho_0$. Consequently

$$\left|\frac{|L(f^m)|}{\rho_0^m} - \sum_{i=1}^r \chi_j \exp(2\pi\imath m\theta_j)\right| \leq d\,\frac{\rho_1^m}{\rho_0^m},$$

where d is the dimension of $H_*(X;\mathbb{Q})$, i.e. the size of matrix $H_*(f)$. Now observe that $\rho_1^m/\rho_0^m \xrightarrow{m} 0$ (If $\rho_0 > 1$ it is obvious; if $\rho_0 = 1$ it follows from Remark (3.1.54), because $\rho_1 = 0$ then).

Let us form a continuous function $h : \mathbb{T}^r \to [0,\infty)$ being the composition of the embedding $\overrightarrow{\theta} = (\theta_1, \ldots, \theta_r) \mapsto (\exp(2\pi\imath\theta_1), \ldots, \exp(2\pi\imath\theta_r))$, $\theta_j \in \mathbb{R}/\mathbb{Z}$, and the real-valued function $\widetilde{h} : \mathbb{C}^r \to [0,\infty)$, $\widetilde{h}(z) = |\sum_{i=1}^r \chi_j \exp(2\pi\imath\theta_j)|$. As a matter of fact this function h is C^∞, because the norm in $\mathbb{C}^r$ is C^∞ out of point $\overrightarrow{0}$. For every given subset of indices $\mathcal{S} = \{j_1, \ldots, j_s\} \subset \{1, \ldots, r\}$, we denote by $\mathbb{T}^s_{\mathcal{S}}$

a sub-torus consisted of points $(\theta_1, \ldots, \theta_r)$ with $\theta_j = 0$ if $j \notin \mathcal{S}$. Since $\chi_j \neq 0$ for every $1 \leq j \leq r$ the restriction of h to $\mathbb{T}^s_{\mathcal{S}}$ is also continuous.

Put $a_{\mathcal{S}} := \max_{\overrightarrow{\theta} \in \mathbb{T}^s_{\mathcal{S}}} h(\overrightarrow{\theta})$. Now we show that either $\sum_{i=1}^r \chi_j \exp(2\pi\imath m\theta_j)\}_1^\infty = \{\sum_i \alpha_i \varepsilon_i^m\}_{m=1}^\infty$, where $\alpha_i \in \mathbb{Z}$, $\varepsilon_i \in \mathbb{C}$ and $\varepsilon_i^k = 1$ for some $k \in \mathbb{N}$ or there exists $S \subset \{1, \ldots, r\}$ such that $\{|\sum_{i=1}^r \chi_j \exp(2\pi\imath m\theta_j)|\}_1^\infty$ is dense in $[0, a_{\mathcal{S}}]$.

Suppose first that $\dim_{\mathbb{Q}}\{\theta_1, \ldots, \theta_r, 1\} = 1$, i.e. all θ_j are rational, say $\theta_j = p_j/q_j$. Then θ_j is a root of unity of degree q_j, thus all θ_j are roots of unity of degree $k = \mathrm{LCM}(q_1, \ldots, q_r)$, the sequence $\{\sum_{i=1}^r \chi_j \exp(2\pi\imath m\theta_j)\}_1^\infty$ is k-periodic and we have the case (3.1.53.2).

Assume finally that $\dim_{\mathbb{Q}}\{\theta_1, \ldots, \theta_r, 1\} = s > 1$, e.g. $\dim_{\mathbb{Q}}\{\theta_{j_1}, \ldots, \theta_{j_s}, 1\} = s$. The latter means that if $\sum_{i=1}^s l_{j_i}\theta_{j_i} = l$, $l_{j_i}, l \in \mathbb{Z}$, then $l_{j_1} = \cdots = l_{j_s} = l = 0$. From the Kronecker theorem (cf. [Ch, Chapter VIII, Theorem 6]) it follows that the sequence $(m\theta_{j_1}, \ldots, m\theta_{j_s})$ is dense in $\mathbb{T}^s_{\mathcal{S}} \subset \mathbb{T}^r$. This gives the case of the statement, with $[0, a_{\mathcal{S}}]$, and proves the theorem. □

(3.1.54) Remark. Observe that $\mathrm{sp}_{\mathrm{es}}(f) \leq 1$ implies either $\mathrm{sp}_{\mathrm{es}}(f) = 0$ or $\mathrm{sp}_{\mathrm{es}}(f) = 1$ and then we have case (3.1.53.2). Indeed, from the proof of implication (3.1.23.4) ⇒ (3.1.23.1) of Theorem (3.1.23) it follows that if $\lambda_i \in \mathbb{C}$ appears in the sum (3.1.23.4) of Theorem (3.1.23) with non-zero coefficient χ_i, then every λ algebraically conjugated to it appears in this sum with the same coefficient. Once more the Kronecker theorem yields that all non-zero λ appearing in the sum (3.1.23.4) of Theorem (3.1.23) are then the roots of unity.

(3.1.55) Remark. We would like to emphasize that there are known lower estimates for a maximal module of all conjugated algebraic number which depend on their algebraic degree. One of the best results is given [BM], and states that if $\lambda_1, \ldots, \lambda_n$ are all roots of a monic polynomial $w(x) \in \mathbb{Z}(x)$ and $\rho := \max_{1 \leq j \leq n} |\lambda_j|$, then

$$\rho \geq 1 + \frac{1}{30n^2 \ln 6n},$$

provided $\rho > 1$.

This allows us to give an estimate of the essential radius of a map f of a space X with the finitely generated real homology spaces. Note that the degree of the characteristic polynomial of the map $H_i(f)$, $0 \leq i \leq d = \dim X$, divides the Betti number $b_i(X)$. Put $b := \max_{1 \leq i \leq d} b_i(X)$. Consequently, if $\mathrm{sp}_{\mathrm{es}}(f) > 1$, then

$$\mathrm{sp}_{\mathrm{es}}(f) \geq 1 + \frac{1}{30b^2 \ln 6b},$$

by the above estimate.

(3.1.56) Example. We show that all three cases of Theorem (3.1.53) happen if we study Lefschetz numbers of iterations of maps of the torus.

Let $w_1(\lambda) = \lambda - 1$, $w_2(\lambda) = \omega_d(\lambda)$ be the d-th cyclotomic polynomial of index $d > 1$, $w_3(\lambda) \in \mathbb{Z}[\lambda]$ be any irreducible monic polynomial having a root λ_0 of $|\lambda_0| = 1$, λ_0 is not a root of unity, and finally let $w_4(\lambda)$ be an irreducible monic polynomial with all roots $|\lambda_i| > 1$. By Exercise (3.1.13) we know that for any monic polynomial $w(\lambda)$ of degree d there exists an integral $d \times d$ matrix A such that $\chi_A(\lambda) = w(\lambda)$. Every such a matrix induces a map $f_A \colon \mathbb{T}^d \to \mathbb{T}^d$ of the torus (see Subsection 4.3.3 of Chapter IV). Moreover for such a map f_A we have $L(f_A^m) = \det(\mathrm{I} - A^m)$ by Theorem (4.3.14). But the latter is equal to $\prod_{j=1}^d (1 - \lambda_j^m)$ over all roots of the polynomial $w(\lambda)$.

Now if we take $w(\lambda) = w_1(\lambda) w'(\lambda)$ with any monic w', then we have case (3.1.53.1) of Theorem (3.1.53). If we take $w(\lambda) = w_2(\lambda)$ or $w(\lambda) = w_2(\lambda) w_4(\lambda)$, then we have the case (3.1.53.2) and $\mathrm{sp}_{\mathrm{es}} > 1$ for the second polynomial. Finally if we take either $w(\lambda) = w_3(\lambda)$ or $w(\lambda) = w_2(\lambda) w_3(\lambda)$ we have case (3.1.53.3) of this theorem.

It is of importance to know not only the rate of growth of the sequence $\{L(f^m)\}_1^\infty$ but also the frequency with the largest Lefschetz number is encountered. The following theorem shows that this sequence grows relatively dense.

In what follows we assume that $\mathrm{sp}_{\mathrm{es}}(f) > 0$ and put $n(f) := \#\{j : |\lambda_j| = \mathrm{sp}_{\mathrm{es}}(f) \text{ and } \chi_j \neq 0\}$.

(3.1.57) THEOREM. *Suppose that $f \colon X \to X$ is as in Theorem* (3.1.53). *Then there exist $\gamma > 0$ and a natural number N such that whenever $m \geq N$, at most one of the numbers $\{L(f^i)/\mathrm{sp}_{\mathrm{es}}(f)^i\}_{i=m}^{m+n(f)-1}$ is greater than γ.*

PROOF. By Theorem (3.1.23), for $m \in \mathbb{N}$ we can write $L(f^m) = \Gamma_m + \Omega_m$, where $\Gamma_m := \sum_{i=1}^{n(f)} \chi_i \lambda_i^m$ with $|\lambda_i| = \mathrm{sp}_{\mathrm{es}}(f)$ and $\Omega_m := \sum_{j=1}^s \chi_j \lambda_j^m$ with $|\lambda_j| < \mathrm{sp}_{\mathrm{es}}(f)$. Put $\theta_i := \lambda_i/\mathrm{sp}_{\mathrm{es}}(f)$, $i = 1, 2, \ldots, n(f)$. If $\mathrm{sp}_{\mathrm{es}}(f) \leq 1$, then $\mathrm{sp}_{\mathrm{es}}(f) = 1$ and $\Omega_m = 0$ by Remark (3.1.54). We have $\Gamma_m/\mathrm{sp}_{\mathrm{es}}(f)^m = \sum_{i=1}^{n(f)} \chi_i \theta_i^m$, and we can write $n(f)$ consecutive equations

$$\frac{\Gamma_{m+j}}{\mathrm{sp}_{\mathrm{es}}(f)^{m+j}} = \sum_{i=1}^{n(f)} (\chi_i \theta_i^m) \theta_i^j, \quad j = 1, \ldots, n(f) - 1.$$

Let $W = W(\theta_1, \ldots, \theta_{n(f)})$ be the Vandermonde operator in $\mathbb{C}^{n(f)}$ given by the Vandermonde matrix. If we put $2\gamma = \|W^{-1}\| \sqrt{n(f)}$, then the vector $\overrightarrow{\chi} = (\chi_1 \theta_1^m, \ldots, \chi_{n(f)} \theta_{n(f)}^m)$ satisfies $\|W \overrightarrow{\chi}\| \geq (2\gamma/\sqrt{n(f)}) \|\overrightarrow{\chi}\| \geq 2\gamma$, since $\|\overrightarrow{\chi}\| \geq \sqrt{n(f)}$. If $\mathrm{sp}_{\mathrm{es}}(f) = 1$ the statement is shown. If $\mathrm{sp}_{\mathrm{es}}(f) > 1$, choose N so large that $|\Omega_m/\mathrm{sp}_{\mathrm{es}}(f)^m| < \gamma$ for $m > N$. The proof is complete. $\square$

(3.1.58) PROPOSITION. *Let f be such that* $\mathrm{sp}_{\mathrm{es}}(f) > 1$. *Then for a given* $\gamma > 0$ *there exists N such that for $m > N$ the algebraic multiplicity satisfies*

$$|a_m(f)| \geq \frac{\gamma}{2}\,\mathrm{sp}_{\mathrm{es}}(f)^m$$

provided $|L(f^m)| \geq \gamma \mathrm{sp}_{\mathrm{es}}(f)^m$. *Correspondingly, the integral algebraic multiplicity* $i_m(f)$ *then satisfies* $|i_m(f)| \geq m(\gamma/2)\mathrm{sp}_{\mathrm{es}}(f)^m$.

PROOF. By the definition of algebraic multiplicity of map (3.1.17) we have

$$|ma_m(f)| = \Big|\sum_{k|m} \mu(n/k)L(f^k)\Big| \geq |L(f^m)| - \Big|\sum_{k|m,\,k\neq m} \mu(n/k)L(f^k)\Big|.$$

Note that for any $c \geq 2\mathfrak{h}(f)$ we have $|L(f^m)| \leq c\,\mathrm{sp}_{\mathrm{es}}(f)$, and consequently

$$\begin{aligned} |ma_m(f)| &\geq |L(f^m)| - \sum_{k|m,\,k\neq m} |L(f^k)| \geq |L(f^m)| - c\sum_{k|m,\,k\neq m} \mathrm{sp}_{\mathrm{es}}(f)^k \\ &\geq |L(f^m)| - c\tau(m)\mathrm{sp}_{\mathrm{es}}(f)^{m/2} \geq [\gamma - c\tau(m)\mathrm{sp}_{\mathrm{es}}(f)^{-m/2}]\mathrm{sp}_{\mathrm{es}}(f)^m. \end{aligned}$$

Since $\tau(m) \leq 2\sqrt{m}$, obviously $\tau(m)\rho^{-m/2} \overset{n\to\infty}{\longrightarrow} 0$ for $\rho > 1$, and consequently there exists N such that $c\tau(m)\mathrm{sp}_{\mathrm{es}}(f)^{-m/2} < \gamma$ for every $m > N$. Moreover, increasing N if necessary we achieve $c(\tau(m)/m)\mathrm{sp}_{\mathrm{es}}(f)^{-m/2} < \gamma$, which is the desired inequality. □

As a direct consequence of Theorem (3.1.57) and Proposition (3.1.58) we get the following.

(3.1.59) COROLLARY. *Suppose that for $f\colon X \to X$, X as in Theorem* (3.1.57) *we have* $\mathrm{sp}_{\mathrm{es}}(f) > 1$. *Then there is a number N such that for every $m > N$ there exists $m \leq k \leq m + n(f) - 1$ such that* $|a_k(f)| \geq (\gamma/2)\mathrm{sp}_{\mathrm{es}}(f)^k$, *where γ is as in Theorem* (3.1.57). *In particular $a_k(f) \neq 0$.*

(3.1.60) REMARK. In fact we can state a litle bit more about the density of the set of algebraic periods $\mathcal{A}$, i.e. $m \in \mathbb{N}$ for which $a_m(f) \neq 0$. To formulate it we need the notion of the lower natural density of a set $\mathcal{A} \subset \mathbb{N}$ (cf. [Ch]). We say that $\mathcal{A} \subset \mathbb{N}$ has the lower density $\mu(\mathcal{A}) \geq \alpha$ if

$$\liminf \frac{\#\mathcal{A} \cap [1, m]}{m} \geq \alpha.$$

From Corollary (3.1.59) it follows that $\mu(\mathcal{A}) \geq 1/n(f)$. On the other hand from Theorem (3.1.53) it follows that $\mu(\mathcal{A}) \geq 1/k$ where either $k = \mathrm{LCM}(q_1, \dots, q_r)$ in the case (3.1.53.2) or k is any real number in $(0, 1)$ in the case (3.1.53.3). The first follows from the fact that $\{\sum_{i=1}^{r} \chi_j \exp(2\pi\imath m\theta_j)\}_1^\infty$ is k-periodic and nonzero by

Theorem (3.1.57). The second is a consequence of the Weyl version of Kronecker's theorem (cf. [Ch]), which states that the sequence $\{m\overrightarrow{\theta}\} = \{(m\theta_{j_1}, \dots, m\theta_{j_s})\}$ is then uniformly distributed in $\mathbb{T}^s_{\mathcal{S}}$. It means that for every closed subset $P \subset \mathbb{T}^s_{\mathcal{S}}$ the density of $\{m \in \mathbb{N} : m\overrightarrow{\theta} \in P\}$ is equal to the measure $\mu(P)$ with the normalization $\mu(\mathbb{T}^s_{\mathcal{S}}) = 1$. Since the set $\{(\overrightarrow{\theta}) \in \mathbb{T}^s_{\mathcal{S}} : h(\overrightarrow{\theta}) = 0\}$ has zero measure as the intersection of a torus with a hyperplane, we can take as P the set $\{\overrightarrow{\theta} : h(\overrightarrow{\theta}) \geq \delta\}$ for every $\delta > 0$. Summing up, if we know which case of Theorem (3.1.53) occurs, then we can say more about the lower density of $\mathcal{A}$.

3.2. Fixed point index of iterations of a smooth map

3.2.1. Lefschetz numbers of iterations as a k-periodic expansion. In this subsection we use an obvious observation that for the sequence $L = \{l_m\}_{m=0}^{\infty}$ or equivalently the power series $\sum_{m=1}^{\infty} l_m z^m$, $l_m = L(f^m)$ one can use the sequence of $\{a_m\}_0^{\infty}$ of its integral algebraic multiplicities as follows from the Möbius formula. This allows us to look at the sequence $\{L(f^m)\}_0^{\infty}$ as an arithmetic function given as a virtual character of finite-dimensional representation of $\mathbb{Z}$ (cf. [Mar]). To do so, we introduce a formal language of so called k-periodic expansion of an arithmetic function, which resembles the Fourier expansion (power expansion). The k-th coefficient is equal to $a_k(f)$ and describes the k-periodic part of this function. Truly speaking, all the presented facts, and proofs, can be stated without this language, but it has a very natural interpretation. Namely, under some geometrical assumptions the non-vanishing of $a_k(f)$ implies the existence of k-periodic points of f. Moreover it allows, expanding also the sequence $\{I_m\}_1^{\infty}$ of the fixed point indices $I_m = \operatorname{ind}(f^m)$ of iterations of f into this k-periodic expansion with the coefficients $\{i_k\}$. Then we can reformulate the Lefschetz–Hopf theorem for periodic points as the equality $a_k = i_k$ of the coefficients of expansions. Then a general scheme of an approach to getting periodic points by the Lefschetz theory is to put global assumptions on X and f (e.g. on the homology) which give $a_k(f) \neq 0$ and then pose geometrical, local assumptions which guarantee that $i_k(f) \neq 0$ implies $P_k(f) \neq \emptyset$.

(3.2.1) Definition. Let $A \in M_n(\mathbb{Z})$. By the character of a representation (integral representation) of $\mathbb{Z}$ associated to A, we mean the function $\chi(A)\colon \mathbb{N} \to \mathbb{Z} \subset \mathbb{C}$ assigned to A by the formula $m \mapsto \operatorname{tr} A^m$.

Let $A \in M_n(\mathbb{C})$, or $M_n(\mathbb{Z})$, a linear operator, given by the matrix A. Denote by $N\colon E_0 \to E_0$ the nilpotent operator $A_{|E_0}$ where E_0 is the generalized eigenspace of A corresponding to 0. Then $A = \widetilde{A} \oplus N$, where $\widetilde{A} \in GL(\widetilde{n}, \mathbb{C})$, $(\in GL(\widetilde{n}, \mathbb{Z}))$, $\widetilde{n} \leq n$, is a nonsingular operator and $\widetilde{n} + n_0 = n$. If $A = N$, then $\widetilde{A} = 0$ by definition. This means that $\widetilde{A} \in M_{\widetilde{n}}(\mathbb{C})$, $(\in M_{\widetilde{n}}(\mathbb{Z}))$, $N \in M_{n_0}(\mathbb{C})$, $(\in M_{n_0}(\mathbb{Z}))$

and $\operatorname{tr} A = \operatorname{tr} \widetilde{A}$. Moreover, for every $m \in \mathbb{N}$, we have

$$\operatorname{tr} A^m = \operatorname{tr} \widetilde{A}^m, \tag{3.2.2}$$

since $\widetilde{A}$ has the same nonzero eigenvalues as A.

We say that an arithmetic function $\phi\colon \mathbb{N} \to \mathbb{C}$ or $\mathbb{Z}$, equivalently a sequence $\{l_m\}$, is a finite-dimensional character of (or finite-dimensional integral character) if it can be represented as the difference of two characters (integral characters) i.e.

$$\phi(m) = l_m = \operatorname{tr} A_1^m - \operatorname{tr} A_2^m, \quad \text{for every } m \in \mathbb{N},$$

where $A_i \in M_{n_i}(\mathbb{C})$, (or $M_{n_i}(\mathbb{Z})$), $i = 1, 2$.

Observe that the mapping $m \mapsto \widetilde{A}^m$ is a homomorphism from $\mathbb{N} \cup \{0\}$ (with the additive structure) to $GL(n; \mathbb{Q})$. Note that $\chi(A)(m) \in \mathbb{Z}$ if $m \geq 0$, and $\chi(A)(0) = \operatorname{tr} \widetilde{A}^0 = \widetilde{n}$ is the dimension of the subspace on which A is nonsingular. The number $\chi(A_1)(0) - \chi(A_2)(0) = \operatorname{tr} \widetilde{A}_1^0 - \operatorname{tr} \widetilde{A}_2^0$ is called the dimension (virtual dimension) of a finite-dimensional character $\{\operatorname{tr} \widetilde{A}_1^m - \operatorname{tr} \widetilde{A}_2^m\}$.

(3.2.3) Remark. Note that equivalent conditions on an integral valued arithmetic function ϕ (an integral sequence $\{l_m\} = \{\phi(m)\}$) to be a finite-dimensional integral character, provided it satisfies the Dold relations, are given in Theorem (3.1.23). One of them is $\{l_m\} := \{L(f^m)\}$ given by of the Lefschetz numbers of iterations of a map f of compact ENR, and such a sequence is denoted by $\mathcal{L}(f)$.

Note also that $\operatorname{tr} \widetilde{A}_1^0 - \operatorname{tr} \widetilde{A}_2^0$ is equal to $n^+ - n^- = \sum_{j=1}^{r^+} \chi_j^+ - \sum_{j=1}^{r^-} \chi_j^-$ in the notation of Subsection 3.2.2. Consequently this integer does not depend on A_1, A_2, and we can define $\chi(0) := n^+ - n^-$, which is a correspondent of the dimension of such a character.

Our aim is to represent any arithmetic function χ, in particular $\mathcal{L}(f)$, as a decomposition (sum) of characters of some elementary representations of $\mathbb{Z}$, which correspond to periodic points of a given period.

(3.2.4) Definition. For a given $k \in \mathbb{N}$ the arithmetic function of $\operatorname{reg}_k\colon \mathbb{N} \to \mathbb{Z}$ given by the formula

$$\operatorname{reg}_k(m) = \begin{cases} k & \text{if } k|m, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2.5}$$

is called the k-th *regular representation*.

Note that reg_k is the elementary periodic sequence $(0, \dots, 0, k, 0, \dots, 0, k, \dots)$, where the nonzero entries are for indices divisible by k.

Furthermore observe that this a character of $\mathbb{Z}$ in the sense of Definition (3.2.1). Indeed, take as the vector space $E := \mathbb{R}^k$ spanned by the basis $\{e_0, \dots, e_{k-1}\}$. Next take as the matrix

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

i.e. $A(e_i) = e_{i+1}$, where i is taken mod k.

By the definition $A \in O(n, \mathbb{R}) \subset GL(n, \mathbb{R})$. It is easy to check that $\operatorname{reg}_k(m) = \operatorname{tr} A^m$ for $m \geq 1$. The notation reg_k is taken from the theory of representation of finite groups (cf. [Ser]). Indeed, the mapping $m \mapsto A^m$ forms a homomorphism from $(\mathbb{Z}_k, +)$ to $O(n, \mathbb{R})$, thus a representation of the cyclic group of order k called the regular representation and reg_k is its character.

It is easy to check that we have the following rule of the multiplication of characters of regular representations. For $k, l \in \mathbb{N}$ we have

$$\operatorname{reg}_k \cdot \operatorname{reg}_l = (l, k)\operatorname{reg}_{[k,l]}, \tag{3.2.6}$$

where (k, l) is the greatest common divisor and $[k, l]$ is the least common multiple of k and l.

(3.2.7) Proposition. *Every arithmetic function $\psi \colon \mathbb{N} \to \mathbb{C}$ can be written in the form*

$$\psi = \sum_{k=1}^{\infty} a_k(\psi)\operatorname{reg}_k,$$

where $a_k(\psi) = i_k(\psi)/k = 1/k \sum_{d|k} \mu(k/d)\psi(d)$. Moreover ψ is integral valued and satisfies the Dold congruences if and only if $a_k(\phi) \in \mathbb{Z}$ for every $k \in \mathbb{N}$, or equivalently $i_k(\psi) \cong 0 \bmod k$.

Proof. Obviously, if ψ is integral valued and satisfies the Dold congruences, then $a_k \in \mathbb{Z}$. Conversely, from the Möbius formula and definition of reg_k it follows that $\psi(k) = \sum_{d|k} i_d(\psi) = \sum_{d|k} d a_d(\psi) \in \mathbb{Z}$, and $a_k(\psi) \in \mathbb{Z}$ means $i_k(\phi) \cong 0$ (mod k) as required. $\square$

(3.2.8) Definition. We say that the expansion of (3.2.7) is a *k-periodic expansion* of ψ and $\{a_k(\psi)\}_1^{\infty}$ the coefficients of this expansion.

(3.2.9) Remark. Note that Theorem (3.1.23) gives a list of necessary and sufficient conditions on an arithmetic function to be a virtual character in the sense of Definition (3.2.1). Analogously Theorem (3.1.46) gives a list of necessary and sufficient conditions on a virtual character to have finite k-periodic expansion.

Let $\mathbb{A}$ denote the linear space (over $\mathbb{C}$) of all arithmetic complex functions. From the definition of a_k we have the following.

(3.2.10) PROPOSITION. *For every $k \in \mathbb{N}$ the map $\psi \mapsto a_k(\psi)$ is a linear functional from $\mathbb{A}$ to $\mathbb{C}$.*

(3.2.11) PROPOSITION. *Let $\psi' = \{l_n^{'}\}$, $\psi'' = \{l_n^{''}\}$ be two arithmetic functions and $\{a_k^{'}\}$, $\{a_k^{''}\}$ be the coefficients of k-periodic expansions of ψ' and ψ'' respectively. Let $\{l_n\} = \theta = \psi\psi$, i.e. $l_n = l_n^{'} l_n^{''}$, for every $n \in \mathbb{N}$. Then $\theta = \sum_{k=1}^{\infty} a_k \mathrm{reg}_k$ where $a_k = \sum_{[p,q]=k} (p,q) a_p^{'} a_q^{''}$.*

PROOF. The statement follows from formula (3.2.6). □

Expanding the arithmetic function given by the sequence of Lefschetz numbers of iterations of a map we get.

(3.2.12) PROPOSITION. *Let $f\colon X \to X$ be a map of a space X with*

$$\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) < \infty.$$

Then there exists $a_k^e(f)$, $a_k^o(f) \in \mathbb{Z}$, $a_k(f) := a_k^e(f) - a_k^o(f)$ such that, for every $m \in \mathbb{N}$,

$$\mathcal{L}(f)(m) := L(f^m) = \sum_{k=1}^{\infty} a_k(f) \mathrm{reg}_k(m).$$

Moreover, the expansion is finite if and only if $\{L(f^m)\}$ is bounded.

PROOF. The last point of the statement is exactly the equivalence (3.1.46.1) $\Leftrightarrow$ (3.1.46.2) of Theorem (3.1.46). □

We need additional notation. Let $\mathcal{U} \quad X$ be an open subset of a space X and $f\colon \mathcal{U} \quad X$ a continuous map. We use the following notation $P^m(f,\mathcal{U}) = P^m(f) \cap \mathcal{U}$, $P_m(f,\mathcal{U}) = P^m(f) \cap \mathcal{U}$. Finally $P(f,\mathcal{U}) = \bigcup_m P^m(f,\mathcal{U})$ denote the set of periodic points of f in $\mathcal{U}$.

We denote by $\mathrm{Per}(f,\mathcal{U}) = \{m \in \mathbb{N} : P_m(f,\mathcal{U}) \neq \emptyset\}$ the set of all minimal periods of f in $\mathcal{U}$.

From now on we assume that X is an ENR, and $f\colon \mathcal{U} \quad X$ is a map such that for every $m \in \mathbb{N}$ the set $P^m(f,\mathcal{U})$ is compact and $P^m(f,\mathcal{U}) \cap \partial\mathcal{U} = \emptyset$.

Let $f\colon \mathcal{U} \quad X$ be as above. By $\mathcal{I}(f,\mathcal{U})$ we denote the integral arithmetic function $\mathcal{I}(f,\mathcal{U})\colon \mathbb{N} \to \mathbb{Z}$ given by the formula

$$\mathcal{I}(f,\mathcal{U})(m) = \mathrm{ind}\,(f^m,\mathcal{U}), \tag{3.2.13}$$

where $\mathrm{ind}\,(f^m,\mathcal{U})$ is the fixed point index of f in $\mathcal{U}$.

It is known that the k-periodic coefficients a_k of the sequence $\mathcal{I}(f,\mathcal{U})(m) = \mathrm{ind}\,(f^m,\mathcal{U})$ are integral (cf. [Do2] and Theorem (3.1.10)). A proof of it is by approximation of a map by a smooth transversal map and we do not present it. From this and Theorem (3.0.1) we get the following.

(3.2.14) THEOREM. *Let $f\colon \mathcal{U} \quad X$ be a map of a subset of an* ENR X *such that for every $m \in \mathbb{N}$ the set $P^m(f,\mathcal{U})$ is compact and $P^m(f,\mathcal{U}) \cap \partial\mathcal{U} = \emptyset$. Then the function $\mathcal{I}(f,\mathcal{U})$ has an integral k-periodic expansion, i.e. $\mathcal{I} = \sum_{k=1}^{\infty} a_k(f,\mathcal{U})\mathrm{reg}_k$, where $a_k(f) \in \mathbb{Z}$ are given by the formula of Proposition* (3.2.7). *Moreover, if there exists a finite complex $K \subset \mathcal{U}$,* $\dim_{\mathbb{Q}} H^*(K;\mathbb{Q}) < \infty$, *such that $f(\mathcal{U}) \subset K$, then $\mathcal{I}(f,\mathcal{U}) = \mathcal{L}(f_{|K})$, i.e. $a_k(f,\mathcal{U}) = a_k(f|K)$ for every $k \in \mathbb{N}$. Consequently if $a_k(f,\mathcal{U}) \neq 0$, then $P^k(f,\mathcal{U}) \neq \emptyset$.*

PROOF. $a_k(f) = \sum_{m|k} \mu(k/m)L(f^m) \neq 0$ implies that there exists $m|k$ such that $L(f^m) \neq 0$. $\square$

The coefficient $a_k(f,\mathcal{U}) \in \mathbb{Z}$ of $\mathcal{I}(f,\mathcal{U})$ sometimes is called the *k-periodic index*. Roughly speaking it measures an impact, to $\mathcal{I}(f,\mathcal{U})$, which comes from the k-periodic points, but unfortunately the precise statement of Theorem (3.2.14) states the existence of points of period k only.

(3.2.15) EXAMPLE. Let us consider a map $f\colon S^d \to S^d$ of degree r. Then $L(f^m) = (-1)^d r^m + 1$, and the k-periodic expansion of $\mathcal{L}(f)$ has the below form. For $|r| \leq 1$

- $1\mathrm{reg}_1$ if $r = 0$,
- $(1 + (-1)^d)\mathrm{reg}_1$ if $r = 1$,
- $(1 - (-1)^d)\mathrm{reg}_1 + 2(-1)^d\mathrm{reg}_2$ if $r = -1$.

If $|r| > 1$, then the coefficient

$$a_k(f) = \sum_{m|k} \mu(k/m) r^m.$$

because for every $k > 1$ we have $\sum_{m|k} \mu(m) = 0$ (cf. [Ch] and Exercise (3.2.16)). Using the estimate $\tau(k) = \#\{m|k\} \leq 2\sqrt{k}$ (cf. [Ch] and Exercise (3.2.17)) it is easy to show that for a given r, $|r| > 1$, there exists k_0 such that $a_k(f) \neq 0$ if $k \geq k_0$. In particular the expansion is infinite.

On the other hand for the map of Example (1.0.20) we have $P_k(f) = \emptyset$ for any $k > 1$ which shows that the claim about the existence of k-periodic points is not true in general.

(3.2.16) EXERCISE. Show that the Möbius function has the following property

$$\sum_{m|k} \mu(m) = \begin{cases} 0 & \text{if } k > 1, \\ 1 & \text{if } k = 1. \end{cases}$$

(3.2.17) EXERCISE. Show that for every natural number k we have the estimate $\tau(k) \leq 2\sqrt{k}$ for the number of divisors of k.

3.2.2. Periodicity of local index of iterations for C^1-map. Let $f\colon\mathcal{U}\quad\mathbb{R}^n$ be a C^1-map of the Euclidean space, such that for every $k \in \mathbb{N}$ and for each $k \in \mathbb{N}$, x_0 is an isolated fixed point of f^k. In [ShSul] Shub and Sullivan showed that the sequence $\mathcal{I} = \{\operatorname{ind}(f^k)\}$ is bounded under the above assumption. This theorem had been sharpened by Chow, Mallet-Paret and Yorke ([ChM-PY]) who showed that this sequence is periodic and described the period. On the other hand, the periodicity follows from the Shub–Sullivan theorem by Theorem (3.1.46). The only problem is to describe the period as it was given by an analytic argument in [ChM-PY].

In this section we present a complete and detailed algebraic proof of the Chow, Mallet-Paret and Yorke theorem taken from the paper [MarPrz] of Marzantowicz and Przygodzki. The basic geometric idea is the same as in [ShSul], [ChM-PY], [BaBo], [HaKa], namely to approximate $\mathrm{I} - f^k$ by $(\mathrm{I} + f + \cdots + f^{k-1})(\mathrm{I} - f)$ so that if $\mathrm{I} + f + \cdots + f^{k-1}$ is a diffeomorphism, then $|\deg_0(\mathrm{I} - f^k)| = |\deg_0(\mathrm{I} - f)|$. Unlike [ChM-PY] our considerations will be purely algebraic – we use the notion of k-periodic expansion as the algebraic framework. We show that the k-periodic expansion of $\mathcal{I}(f, x_0)$ is finite with indices k for which coefficients $a_k(f)$ are nonzero, completely determined by the derivative and of a form also expressing the period-doubling outcome.

We need further notation. Assume that $x_0 = 0$ and set $A = Df(0)$.

(3.2.18) Definition. The operator A defines arithmetic functions

$$\nu_+(m) = \#\{\sigma(A^m)) \cap (1,\infty)\}, \qquad \nu_-(m) = \#\{\sigma(A^m) \cap (-\infty,-1)\},$$

where $\#$ denotes the cardinality with multiplicities of eigenvalues included. We use the convention $\nu_\pm = \nu_\pm(1)$.

It is easy to check the following properties of the arithmetic functions $\nu_\pm$.

(3.2.19) Lemma.

(3.2.19.1) $$\nu_+(m) \underset{\text{mod } 2}{\cong} \begin{cases} \nu_+ & \textit{if } m \textit{ is odd,} \\ \nu_+ + \nu_- & \textit{if } m \textit{ is even.} \end{cases}$$

(3.2.19.2) $$\nu_-(m) \underset{\text{mod } 2}{\cong} \begin{cases} \nu_- & \textit{if } m \textit{ is odd,} \\ 0 & \textit{if } m \textit{ is even.} \end{cases}$$

(3.2.19.3) $$\nu_-(m) \underset{\text{mod } 2}{\cong} m\,\nu_-.$$

Now we introduce a notion that is necessary for studying the local fixed point indices of a smooth map.

(3.2.20) Definition. For a given linear map (a matrix) A let $\mathcal{O}(A)$, or shortly $\mathcal{O}$, is defined as

$$\mathcal{O} = \{m \in \mathbb{N} : \exists y \in \mathbb{R}^n \text{ s.t. } y, A(y), \dots , A^{m-1}(y) \text{ are distinct and } A^m(y) = y\}.$$

For fixed point x_0 of f and $A = Df(x_0)$, the set $\mathcal{O}(A)) \subset \mathbb{N}$ is called the set of *virtual periods* of f at x_0 ([ChM-PY]). We decompose $\mathcal{O} = \mathcal{O}_{\mathrm{e}} \cup \mathcal{O}_{\mathrm{o}}$ into even and odd elements.

The main theorem about the local fixed point index of iterations of a smooth map is formulated in the term of k-periodic expansion and has the following form.

(3.2.21) Theorem. *Let $f{:}\,\mathcal{U} \quad \mathbb{R}^n$ be a C^1- map such that 0 is an isolated point of $P^k(f,\mathcal{U})$ for every $k \in \mathbb{N}$. Then there exist $c_k(f) \in \mathbb{Z}$ for $k \in \mathcal{O}$ such that*

$$\mathcal{I}(f, 0) = \sum_{k\in\mathcal{O}} c_k(f)(\mathrm{reg}_k + 1/2[(-1)^{\nu_-(k)} - 1]\mathrm{reg}_{2k}).$$

Equivalently for $m \in \mathbb{N}$ (cf. [ChM-PY]),

$$I_m = \mathcal{I}(f^m, 0) = \begin{cases} \displaystyle\sum_{k\in\mathcal{O}} c_k \mathrm{reg}_k(m) & \text{for } \nu_- \text{ even},\\ \displaystyle\sum_{k\in O_e} c_k \mathrm{reg}_k(m) + \sum_{k\in O_o} c_k(\mathrm{reg}_k(m) - \mathrm{reg}_{2k}(m)) & \text{for } \nu_- \text{ odd.} \end{cases}$$

To shorten notation we use I_m for $\mathrm{ind}\,(f^m, x_0)$ till the end of this subsection. First we state a fact which allows us to compare consecutive terms of $\mathcal{I} = \{I_m\}$ to the first one as in ([ShSul], [ChM-PY]).

(3.2.22) Remark. We should remark that the statement of Theorem (3.2.21) holds also for a simplicial map of a smooth type in the sense of Siegberg (cf. [GrII]).

(3.2.23) Lemma. *For any $m \in \mathbb{N}$ if $\det(\sum_{i=0}^{m-1} A^i(0)) \neq 0$, then*

$$I_m = \mathrm{sign}\left(\det\left(\sum_{i=0}^{m-1} A^i(0)\right)\right) \cdot I_1.$$

Our task is to deduce when the above determinant is different from zero and to describe a regularity of its sign depending on m.

For a given $m \in \mathbb{N}$ and $\mathcal{O}$ as in (3.2.20) we define the set

$$\mathcal{O}^m := \{k : k \in \mathcal{O},\ k|m\}. \tag{3.2.24}$$

Observe that $\mathcal{O}^m$ is the set of virtual periods of A dividing m. Let us find out when $\mathrm{I} + \dots + A^{m-1}$ is nonsingular.

(3.2.25) LEMMA. *For any $m \in \mathbb{N}$ we have*

$$\det\left(\sum_{j=0}^{m-1} A^j\right) \neq 0 \Leftrightarrow \mathcal{O}^m = \{1\}.$$

PROOF. ($\Rightarrow$) Let $k \in \mathcal{O}^m$. Then $k|m$ and there exists $v \in \mathbb{R}^n$ such that $v, A(v), \dots, A^{k-1}(v)$ are distinct and $A^k(v) = v$. Since $k|m$, we have $A^m(v) = v$ and consequently $(\sum_{i=0}^{m-1} A^i)(\mathrm{I} - A)(v) = 0$. By assumption we get $A(v) = v$, and therefore $k = 1$.

($\Leftarrow$) For the opposite, assume $(\sum_{i=0}^{m-1} A^i)(v) = 0$. This implies $A^m(v) = v$, and $A(v) = v$, by the assumption. Consequently $0 = \sum_{i=0}^{m-1} A^i(v) = \sum_{i=1}^{m-1} v = mv$, which gives $v = 0$ and proves the statement. $\square$

Now we define another set that is defined by the linear map A:

$$\mathcal{O}_n^m := \left\{\frac{k}{(k,n)}\right\} \quad \text{where } k \in \mathcal{O},\ k|m, \tag{3.2.26}$$

and (k,n) is the greatest common divisor. Notice that $\mathcal{O}_1^m = \mathcal{O}^m$. It is easy to notice that if $n|m$, then $\mathcal{O}_n^m$ is the set of virtual periods of A^n dividing m/n. As an easy consequence of Lemma (3.2.25) we obtain

(3.2.27) LEMMA. *Let $n|m$. Then*

$$\det\left(\sum_{i=0}^{m/n-1} A^{ni}\right) \neq 0 \Leftrightarrow \mathcal{O}_n^m = \{1\}.$$

Let us denote by q_m the least common multiple of the set $\{k \in \mathcal{O} : k|m\}$. We have the following description of the sequence $\{I_m\}$.

(3.2.28) PROPOSITION. *For every $m \in \mathbb{N}$, $\mathcal{O}_{q_m}^m = \{1\}$, thus*

$$\det\left(\sum_{i=1}^{m/(q_m-1)} A^{q_m i}\right) \neq 0.$$

Consequently

$$I_m = \operatorname{sign}\det\left(\sum_{i=1}^{m/(q_m-1)} A^{q_m i}\right) I_{q_m} \quad \textit{for every } m \in \mathbb{N}.$$

PROOF. From definition of q_m it follows that if $k \in \mathcal{O}$, then $q_m|k$ so $k/(q_m,k) = 1$. Combining the above with Lemma (3.2.27) we get the second condition of the statement. Substituting f^{q_m} for f to Lemma (3.2.23) we get the last part of Proposition (3.2.28). $\square$

Now we describe periodicity of the sign of $\{I_m\}$.

(3.2.29) LEMMA. *If* $\det(\sum_{j=0}^{n} A^j) \neq 0$, *then* $\operatorname{sign}\det(\sum_{j=0}^{n} A^j) = (-1)^{n\nu_-}$.

PROOF. Let $f(t) := \det(t\mathrm{I} - A)$, $g(t) := \det(t^n\mathrm{I} + \cdots + A^n)$, $h(t) := \det(t^{n+1}\mathrm{I} - A^{n+1})$. Then $f(t)g(t) = h(t)$ and $\lim_{t\to\infty} g(t) = \infty$. Assume t is sufficiently large and move towards 1. Then $f(t)$ would change the sign ν_+ times, while $h(t)$, regarding point (3.2.19.1) of Lemma (3.2.19) would change the sign ν_+ or $\nu_+ + \nu_-$ times if n is even or odd, respectively. As $g(t) = h(t)/f(t)$ the statement is proven. □

The above leads to the following.

(3.2.30) PROPOSITION. *For every* $m \in \mathbb{N}$

$$I_m = \begin{cases} -I_{q_m} & \text{for } m \text{ even and } q_m \text{ odd, if } \nu_- \text{ is odd,} \\ I_{q_m} & \text{otherwise.} \end{cases}$$

PROOF. From (3.2.19.1) and Proposition (3.2.28) it follows that

$$\begin{aligned} \operatorname{sign}\det\left(\sum_{i=0}^{m/q_m - 1} A^{q_m i}\right) &= \begin{cases} (-1)^{\nu_-(q_m)} & \text{for } m/q_m \text{ even,} \\ 1 & \text{for } m/q_m \text{ odd,} \end{cases} \\ &= \begin{cases} (-1)^{\nu_-} & \text{for } m \text{ even, } q_m \text{ odd,} \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$ □

Now we describe a relation between the set $\mathcal{O} \subset \mathbb{N}$ and the spectrum $\sigma(A)$ of A.

Let us denote by $\sigma_{(1)}(A)$ the set of degrees of primitive roots of unity contained in $\sigma(A)$ by $\Delta(\sigma(A))$, the set of their degrees.

(3.2.31) PROPOSITION. *Let* $A\colon \mathbb{R}^d \to \mathbb{R}^d$ *be a linear map. Then*

$$\mathcal{O} = \{[\mathcal{K}] : \mathcal{K} \subset \Delta(\sigma(A))\}$$

where $[\mathcal{K}]$ *is the lowest common multiple of a finite set* $\mathcal{K} \subset \mathbb{N}$ *and* $\mathcal{K}$ *ranks all subsets of* $\Delta(\sigma(A))$, *with the convention* $[\emptyset] = 1$.

PROOF. Let us take the spectral decomposition $A = A_1 \oplus A_2$, where A_1 corresponds to $\sigma_{(1)}(A)$ and A_2 to the remaining part of $\sigma(A)$. Note that A_2 has no periodic points except $\{0\}$. On the other hand $A_1 = \bigoplus_{i=1}^{r} A_{1,i}$, where $A_{1,i}$ corresponds to a given $q_i \in \{q_1, \dots, q_r\} = \sigma_{(1)}(A)$. Since $\mathcal{O}(A_{1,i}) = [1, q_i]$ and $\mathcal{O}(A_{1,i} \oplus A_{1,j}) = [1, q_i, q_j] = [q_i, q_j]$ the statement follows by an induction argument. □

(3.2.32) REMARK. From Proposition (3.2.31) it follows that $\mathcal{O} \subset \mathbb{N}$ is closed with respect to taking the least common multiple, i.e. $\mathcal{O} = \{[\mathcal{K}] : \mathcal{K} \subset \mathcal{O}\}$. Moreover, $\mathcal{O} \cup 2\mathcal{O}$ has the same property.

It is well known that for $m \neq 1$ we have $\sum_{k|m} \mu(k) = 0$ (cf. Exercise (3.2.16)). To prove Theorem (3.2.21) we need a more general formula.

Let $\mathcal{K} \in \mathbb{N}$, $k_0, i \in \mathbb{N}$. Denote $\mathrm{LCM}(\mathcal{K}) = \{[Q] : Q \subset \mathcal{K}\}$, $\mathcal{K}(k_0) := \{[k, k_0] : k \in \mathcal{K}\}$. We will write relation $i \sim \mathcal{K}$ if and only if there exists $k \in \mathcal{K}$ such that $k|i$.

(3.2.33) Lemma. *If $m \in \mathbb{N}$ does not belong to $\mathrm{LCM}(\mathcal{K})$, then*

$$\sum_{i \sim \mathcal{K},\, i|m} \mu(m/i) = 0.$$

Proof. (Induction on $\#\mathcal{K}$) For $\#\mathcal{K}$ equal to 0 or 1 it is obvious. Suppose the formula is proven for the sets of cardinality of the set $\mathcal{K}$. Let $k_0 \in \mathbb{N} \setminus \mathcal{K}$. Note that

$$(i \sim k) \wedge (i \sim \{k_0\}) \Leftrightarrow i \sim \mathrm{LCM}(\mathcal{K}(k_0)).$$

Since $m \notin \mathrm{LCM}(\mathcal{K} \cup \{k_0\}$,

$$\sum_{i|m, i \sim \mathcal{K} \cup \{k_0\}} \mu(m/i) = \Bigg(\sum_{i|m,\, i \sim \mathcal{K}} + \sum_{k_0|i|m} + \sum_{i|m,\, i \sim \mathcal{K}(k_0)} \Bigg) \mu(m/i) = 0. \qquad \square$$

In fact we need a more refined version. Just for the lemma below put

$$p_n := [i \in \mathcal{K}, i|n].$$

(3.2.34) Lemma. *If $m, p, k_0 \in \mathbb{N}$ and $m \notin \mathrm{LCM}(\mathcal{K} \cup \{k_0\})$, then*

$$\sum_{k_0|i|m,\, q_i = q} \mu(m/i) = 0.$$

Proof. First note that for $\mathbb{S} \subset \mathbb{N}$, and $k \in \mathbb{N}$

$$(i \sim \mathbb{S}) \wedge (k|i) \Leftrightarrow i \sim \mathbb{S}(k).$$

Set $\mathbb{S}\mathcal{K}(q) \setminus \{q\}$. If $p \in \mathrm{LCM}(\mathcal{K})$, then

$$\sum_{k_0|i|m, p_i = p} \mu(m/i) = \Bigg(\sum_{[p,k_0]|i|m} - \sum_{i|m,\, i \sim \mathbb{S}(k_0)} \Bigg) \mu(m/i) = 0.$$

If $p \notin \mathrm{LCM}(\mathcal{K})$, then the formula is obvious. $\square$

Now we are in position to prove Theorem (3.2.21). Again put $p_n := [i \in \mathcal{O} : i|n]$.

Proof of Theorem (3.2.21). Let $\mathcal{I} = \{I_m\} = \sum_{k=1}^{\infty} d_k \operatorname{reg}_k$ be the k-periodic expansion.

(1) First assume that ν_- is even. We show that $d_m = 0$ for $m \notin \mathcal{O}$. Indeed, if $m \notin \mathcal{O}$, then from the inversion formula, Proposition (3.2.30) and Lemma (3.2.34) ($\mathcal{K} = \mathcal{O}$, $k_0 = 1$) follows

$$md_m = \sum_{k|m} \mu(m/k) l_k = \sum_{k|m} \mu(m/k) I_{q_k} = \sum_{q\in\mathcal{O}} I_q \sum_{k|m, q=q_k} \mu(m/k) = 0.$$

(2) Suppose now that ν_- is odd. First we show that $d_m = 0$ for $m \notin \mathcal{O} \cup 2\mathcal{O}_{\mathrm{o}}$. From the formula of Lemma (3.2.34) with ($\mathcal{K} = \mathcal{O}$, $k_0 = 2$) we have

$$\sum_{i|m,\, q_i=q} \mu(m/i) = \sum_{2|i|m,\, q_i=q} \mu(m/i) = \sum_{2\nmid i|m,\, q_i=q} \mu(m/i) = 0.$$

Using Proposition (3.2.30) we get

$$md_m = \sum_{i|m,\, q_i=q} \mu(m/i) I_i = \left(\sum_{2|q_i,\, i|m} + \sum_{2\nmid q_i,\, 2\nmid i|m} - \sum_{2\nmid q_i,\, 2|i|m} \right) \mu(m/i) I_{q_i} = 0.$$

Next we check that $d_{2m} = -d_m$ for $m \in \mathcal{O}_{\mathrm{o}}$ such that $2m \notin \mathcal{O}_{\mathrm{e}}$. Assume that $i|m$. Then $q_{2i} = q_i$ and m are odd. Hence $I_{2i} = -I_{q_{2i}} = -I_{q_i} = -I_i$. Therefore

$$\begin{aligned}
2md_{2m} &= \sum_{i|2m} \mu(2m/i) I_i = \left(\sum_{2|i|2m} + \sum_{2\nmid i|2m} \right) \mu(2m/i) I_i \\
&= \sum_{i|m} \mu(m/i) I_{2i} + \sum_{2\nmid i|2m} \mu(2m/i) I_i = \sum_{i|m} \mu(m/i) \cdot (I_{2i} - I_i) \\
&= -2 \sum_{i|m} \mu(m/i) I_i = -2md_m.
\end{aligned}$$

From the above it follows that

$$\begin{aligned}
\mathcal{I} = \{I_k\} = \sum d_m \mathrm{reg}_m &= \sum_{m\in\mathcal{O}_{\mathrm{e}}} d_m \mathrm{reg}_m + \sum_{m\in\mathcal{O}_{\mathrm{o}}} d_m \mathrm{reg}_m + \sum_{\substack{m\in\mathcal{O}_{\mathrm{e}} \\ 2m\notin\mathcal{O}_{\mathrm{e}}}} d_{2m} \mathrm{reg}_{2m} \\
= \sum_{m\in\mathcal{O}_{\mathrm{e}}} d_m \mathrm{reg}_m &+ \sum_{m\in\mathcal{O}_{\mathrm{o}}} d_m (\mathrm{reg}_m - \mathrm{reg}_{2m}) + \sum_{\substack{m\in\mathcal{O}_{\mathrm{e}} \\ 2m\in\mathcal{O}_{\mathrm{e}}}} d_m \mathrm{reg}_{2m} \sum_{m\in\mathcal{O}_{\mathrm{e}}} c_m \mathrm{reg}_m \\
&+ \sum_{m\in\mathcal{O}_{\mathrm{o}}} c_m (\mathrm{reg}_m - \mathrm{reg}_{2m}),
\end{aligned}$$

where

$$c_m = \begin{cases} d_{2m} + d_m & \text{for } m \in \mathcal{O}_{\mathrm{e}} \text{ and } (m/2) \in \mathcal{O}_{\mathrm{o}}, \\ d_m & \text{otherwise.} \end{cases}$$

This ends the proof of Theorem (3.2.21). $\square$

We must say that it is difficult to compute coefficients $c_k(f)$ in general. Note that $c_1(f) = \operatorname{ind}(f, 0)$. This gives $c_1(f) = (-1)^{\nu_+}$ if $\mathrm{I} - A$ is nonsingular by the classical Hopf formula (see Example (2.2.4)).

However in [ChM-PY] the coefficients $c_1(f)$, and $c_2(f)$ correspondingly, are described provided $\mathrm{I} - A$ has a one-dimensional kernel, or $\mathrm{I} - A$ is nonsingular and $\mathrm{I} - A^2$ has a one-dimensional kernel (see Proposition (3.3.29)).

Similar consideration allows us to describe the sequence $\{I(f^m, 0)\}$ in the case when f is a C^1-map of the plane [BaBo] (see Theorem (3.3.30)). Also one can estimate for how many k the k-periodic coefficient of $\mathcal{I}(f, 0)$ is not zero. We will discuss this later.

3.2.3. Applications – finding periodic points of smooth mappings. In Section 3.2.1 we formulated a general Lefschetz–Hopf theorem (Theorem (3.2.14)) for the index function $\mathcal{I}(f, \mathcal{U})$. A most natural approach to get its applications is to restrict the class of maps in the problem by some geometric condition. This allows us to represent $\mathcal{I}(f, X)$ in a nicer form. To get a simpler form of $\mathcal{L}(f, X)$ we have to put some assumption on $H^*(f)$ or $H^*(X; \mathbb{Q})$. At the beginning of this section we present the result of Chow, Mallet-Paret and Yorke ([ChM-PY], see also [BaBo]) on the fixed point index of iteration of a smooth map near an orbit. In the original exposition there is a lack of an algebraic framework. The k-periodic expansion presented here seems to be a natural one. Next we study periodic points of transversal maps, refining Matsuoka's results and reproving the Franks theorem. Finally, by use of the k-periodic expansion we show the existence of infinite prime periods for a class of smooth maps. The approach is based on the work [MarPrz] of Marzantowicz and Przygodzki.

(3.2.35) Definition (cf. [ChM-PY], [Do2], [Fr1]). Let $f\colon \mathcal{U} \quad X$ be a C^1-map of an open subset of a manifold X and let $x \in P_m(f)$. Analogously as in Definition (3.2.18) we define arithmetic functions

$$\nu_+(x)(k) = \#\{\sigma(Df^{mk}(x)) \cap (1, \infty)\}, \quad \nu_-(x)(k) = \#\{\sigma(Df^{mk}(x)) \cap (-\infty, -1).$$

As previously we use convention $\nu_\pm(x) = \nu_\pm(x)(1)$. It is clear that the functions $\nu_\pm(x)$ have the same properties as $\nu_\pm(\ )$ defined in Section 3. Analogously, for $x \in P_m(f)$, we define $\mathcal{O}(x) = \mathcal{O}(A)$, where $A = Df^m(x)$. Let $x \in P_m(f) \subset P(f)$. By $[x]$ we denote the orbit of x, i.e. the set $\{x, f(x), f^2(x), \dots, f^{m-1}\}$.

We begin with a description of the k-periodic expansion of the fixed point index of iterations of a C^1-map at an isolated orbit $[x]$ of x.

(3.2.36) Theorem. *Let $f\colon \mathcal{U} \quad \mathbb{R}^n$ be a C^1-map and $x \in P_m(f)$ an isolated point of $P^m(f, \mathcal{U})$, for every $m \in \mathbb{N}$. Then there exist $c_k(f, x) \in \mathbb{Z}$, $k \in \mathcal{O}(x)$ such* *that*

$$\mathcal{I}(f,[x]) = \sum_{k\in\mathcal{O}(x)} c_k(f,x)\Big(\operatorname{reg}_{mk} + \frac{1}{2}[(-1)^{k\nu_-(x)}-1]\operatorname{reg}_{2mk}\Big).$$

Equivalently, the k-periodic expansion

$$\mathcal{I}(f,[x]) = \sum_{k\in\mathcal{O}(x)} a_{mk}(f,x)\operatorname{reg}_{mk} = \sum_{l=mk,\,k\in\mathcal{O}(x)} a_l(f,x)\operatorname{reg}_l,$$

where $a_l = a_{mk} = c_k + (1/2)((-1)^{(k/2)\nu_-(x)}-1)c_{k/2}$, with the convention that $c_k = 0$ if $k\notin\mathcal{O}(x)$ or $k/2\notin\mathbb{Z}$.

PROOF. Note that $P^q(f)=\emptyset$ if q is not divided by m. It is sufficient to apply Theorem (3.2.21) to the map f^m at every $x_i = f^i(x)\in[x]$ and compare the values of both sides of the equality at each $q\in\mathbb{N}$. □

Theorem (3.2.36) and direct computation give an expression of $\mathcal{I}(f,\mathcal{U})$ in the case when f is a C^1-map and all its periodic points are isolated.

(3.2.37) THEOREM. *Let $f:\mathcal{U}\to X$ be a C^1-map of an open bounded subset $\mathcal{U}$ of a compact smooth manifold X. Assume that $P(f)$ consists of isolated points only, i.e. for each $m\in\mathbb{N}$ the set $P^m(f)$ consists of isolated points. Then*

$$\mathcal{I}(f,\mathcal{U}) = \sum_{m=1}^{\infty}\sum_{[x]\subset P_m(f)}\sum_{k\in\mathcal{O}(x)} c_k(x)\Big(\operatorname{reg}_{mk} + \frac{1}{2}[(-1)^{\nu_-(x)(k)}-1]\operatorname{reg}_{2mk}\Big),$$

with the convention that $[x]\subset P_m(f)$ means that $x\in P_m(f)$, $P_{m/2}(f)=\emptyset$, if m is odd, and $c_k(x)$ are defined as in Theorem (3.2.36)

Applying Theorem (3.2.37) in the case when $\mathcal{U}=X$ and using the Lefschetz–Hopf theorem we get the following.

(3.2.38) PROPOSITION. *Let $f: X\to X$ be a C^1-map of a compact manifold X such that $P(f)$ consists of isolated points only. Then the coefficient $a_l(f)$ of $\mathcal{L}(f) = \mathcal{I}(f,X)$ at reg_l is equal to*

$$\sum_{mk=l}\sum_{x\in P_m(f)} c_k(x) + \sum_{m2k=l}\sum_{x\in P_m(f)}\frac{1}{2}[(-1)^{\nu_-(x)(k)}-1]c_k(x),$$

with the convention that $c_k(x)=0$ if $k\notin\mathcal{O}(x)$.

As a direct consequence of Theorem (3.2.37) we get the famous Shub–Sullivan theorem of [ShSul].

(3.2.39) Corollary. *Let $f: X \to X$ be a C^1-map of a compact manifold X. If the sequence $\{L(f^m)\}$ is unbounded, then f has infinitely many periodic points, i.e. $\#P(f) = \infty$.*

Proof. If $P(f)$ is finite, then it consists of isolated points and from Theorem (3.2.37) the k-periodic expansion of $\mathcal{I}(f, X) = \mathcal{L}(f)$ is finite. By Proposition (3.2.12) the sequence $l_m = L(f^m)$ is bounded, which contradicts the assumption. □

Note that the main part of the proof of Corollary (3.2.39) is the finiteness of the k-periodic expansion of $\mathcal{I}(f, X)$. Indeed, by the Möbius formula the sequence $\{l_m\}$, $l_m = \sum_{k|m} a_k(f)$, is bounded if $a_k(f) \neq 0$ for only finitely many indices k.

Observe that the Shub–Sullivan theorem does not gives any information, neither about the the cardinalities $\#P_m(f)$, $\#P^m(f)$ nor about the set $\mathrm{Per}(f)$.

However, concerning the second question we have an obvious consequence of Corollary (3.2.39).

(3.2.40) Corollary. *Let $f: X \to X$ be a C^1-map of a compact closed manifold. Suppose that $\{L(f^m)\}$ is unbounded.*

If for every m, $P^m(f)$ consists of isolated points only, then $\mathrm{Per}(f)$ *is infinite.*

Proof. Since $P^m(f)$ consists of isolated points, it is finite as a closed, i.e. compact, subset of X. The same for $P_m(f)$. Now finiteness of $\mathrm{Per}(f)$ implies finiteness of $P(f)$ contrary to Corollary (3.2.39). □

As a finer application of Theorems (3.2.36), (3.2.37) and Proposition (3.2.38) we give an estimate from below of the number of periodic orbits of a C^1 map (cf. [BaBo]). For a given map $f: X \to X$ we denote by $\mathrm{Or}(f, m)$ the set of all orbits of f of a length not greater than m. In other words it is the quotient set of $P^m(f)$ by the relation

$$x \sim y \Leftrightarrow \exists i \in \{0, \dots, m-1\} \quad \text{such that } y = f^i(x).$$

(3.2.41) Theorem. *Let $f: X \to X$ be a C^1-map of a compact manifold X of $\dim X = d$. Suppose that the sequence $\mathcal{L} = \{L(f^m\}$ of the Lefschetz number of iterations is unbounded. Then there exists $n_0(f) \in \mathbb{N}$ such that for every $m \geq n_0$,*

$$\#\mathrm{Or}(f, m) \geq \frac{(m - n_0)}{D \cdot 2^{[(d+1)/2]}},$$

where $D = \dim_{\mathbb{Q}} H_*(X; \mathbb{Q})$ and $[(d+1)/2]$ is the integral part of $(d+1)/2$.

Proof. Let us begin with new notation. For a given $m \in \mathbb{N}$ let

$$\mathcal{A}(f, m) := \{k \leq m : a_k(f) \neq 0\}$$

be the set of algebraic periods up to m. Analogously for a given isolated periodic point $x \in P(f)$ let

$$\mathcal{A}(f,[x]) := \{k \in \mathbb{N} : a_k(f,[x]) \neq 0\}$$

be the set of local algebraic periods at the orbit $[x]$.

Now observe that Corollary (3.1.59) says that there exist $n_0(f)$ and $n(f) \in \mathbb{N}$, (equal to $\Upsilon(f)-1$) such that for every $m > n_0$, there exists at least one $m \leq i \leq m+n(f)$ such that $a_i(f) \neq 0$. Note that $n(f) \leq \dim_{\mathbb{R}} H_*(X) := D$. This leads to the estimate

$$\#\mathcal{A}(f,m) \geq \frac{m-n_0}{n(f)} \geq \frac{m-n_0}{D}, \quad \text{for } m \geq n_0. \tag{3.2.42}$$

On the other hand, from Proposition (3.2.38) it follows that a nonzero coefficient $a_l(f)$ appears in the k-periodic expansion of f only if for at least one divisor $n|l$, $l = nk$, there exists $x \in P_n(f)$, with $k \in \mathcal{O}(x)$, i.e. $l \in \mathcal{A}(f,[x])$ or writing shortly

$$\mathcal{A}(f,m) \subset \bigcup_{n\leq m}\ \bigcup_{[x]\in \mathrm{Or}_n(f)} \mathcal{A}(f,[x]) = \bigcup_{[x]\in\mathrm{Or}(f,m)} \mathcal{A}(f,[x]), \tag{3.2.43}$$

where $\mathrm{Or}_n(f) \subset \mathrm{Or}(f,n)$ is the subset consisting of orbits $[x]$ of points $x \in P_n(f)$.

To show the statement we need the following estimate of cardinality of the set $\mathcal{A}(f,[x])$ ([BaBo])

(3.2.44) Lemma. *With the above notation we have* $\#\mathcal{A}(f,[x]) \leq 2^{[(d+1)/2]}$.

Proof. First we have to estimate the cardinality of $\mathcal{O}(x)$ for a fixed $n \in \mathbb{N}$. By Proposition (3.2.31), $\#\mathcal{O}(x)$ is smaller than or equal to the cardinality of all subsets $\mathcal{K} \subset \Delta(\sigma(Df^n(x)))$, where $\Delta(\sigma(Df^n(x)))$ is the set of all degrees of roots of unity in the spectrum of $Df^n(x)$. Since a root of unity, different from 1 and -1, is a root of a polynomial of degree 2, any $d \times d$ matrix has at most $\tau \leq [(d+1)/2]$ eigenvalues being distinct roots of unity. The latter means that $\#\Delta(\sigma(Df(x))) \leq \tau \leq [(d+1)/2]$, which shows that $\#\mathcal{O}(x) \leq 2^{[(d+1)/2]}$. From Theorems (3.2.21) and (3.2.37) it follows that $\mathcal{A}(f,[x])$ consists of numbers of the following form:

(a) $\{nk\} : k \in \mathcal{O}(x)$ if n is odd,
(b) $\{nk\} : k \in \mathcal{O}(x)$ if n is even and ν_- is even,
(c) $\{nk, 2nk\} : k \in \mathcal{O}(x)$ if n is even and ν_- is odd.

In the cases (a) and (b) the statement of Lemma (3.2.44) follows directly from the above estimate of $\#\mathcal{O}(x)$. In the case (c) the characteristic polynomial $\chi(t)$ of $Df^n(x)$ decomposes into $\chi'(t)\chi''(t)$ of degree $1 \leq d'$, $d'' < d$, where $\chi'(t)$

corresponds to the real eigenvalues $\in (-\infty, -1)$ and $\chi''(t)$, to the remaining eigenvalues. Thus we have at most $[(d_2+1)/2]$ distinct roots of unity in the spectrum. Consequently $\#\mathcal{O}(x) \leq 2^\tau \leq 2^{[d/2]}$, which gives a weaker inequality

$$\#\mathcal{A}(f,[x]) \leq 2\#\mathcal{O}(x) \leq 2^{[d/2]+1} \leq 2^{[(d+2)/2]}.$$

To get the sharper inequality of the statement we have to repeat the detailed discussion of Corollary 3.6 of [BaBo]. To do it we need new notation. For a given $x_0 \in P_n(f)$ let $A = Df^n(x_0)$ be the differential operator and

$$\sigma(Df^n(x_0)) = \{\varepsilon_1, \dots, \varepsilon_l, \lambda_1, \dots, \lambda_{\nu_+}, \lambda'_1, \dots, \lambda'_{\nu_-}, \mu_1, \dots, \mu_r\}.$$

where ε_i is a root of unity of degree $d_i \geq 1$, $\lambda_i \in (1,\infty)$ and $\lambda'_i \in (-\infty,-1)$ are real eigenvalues of module greater then 1, and $\{\mu_i\}_1^r$ are the remaining eigenvalues. Note that $l+\nu_++\nu_-+r = d$. Recall that the set $\{d_1, \dots, d_l\}$ is denoted by $\Delta(\sigma(A))$ and by $\mathcal{O}$ the set of the lowest common multiplies of finite subsets of Δ with the convention $\mathcal{O}(\emptyset) = 1$. Suppose the case (c) is equivalent to the assumption $\nu_- \geq 1$ odd. Now, by the dimension argument we have:

(1) If $2 \in \Delta$, then for every virtual period $k \in \mathcal{O}_{\mathrm{o}}$ the number $2k \in \mathcal{O}_{\mathrm{e}}$ and in the second sum of Theorem (3.2.21) the summand reg_{2k} appears twice with opposite coefficients, thus disappears. Consequently we have the estimate $\#\mathcal{A}(f,[x]) \leq \#\mathcal{O}(x) \leq 2^\tau$.
Suppose now additionally that $1 \notin \Delta$; then $2(\tau-1) \leq d-1$, i.e. $\tau \leq d+1/2$ and $\#\mathcal{A}(f,[x]) \leq 2^\tau \leq 2^{[(d+1)/2]}$.
If additionally $1 \in \Delta$, then $2(\tau-2) \leq d-3$, since $\nu_- \geq 1$, i.e. $\tau \leq [(d+1)/2]$, and consequently $\#\mathcal{A}(f,[x]) \leq 2^{[(d+1)/2]}$.

(2) If $1 \notin \Delta$, $2 \notin \Delta$ and $\nu_-0 \geq 1$, then $2\tau \leq d-1$, i.e. $\tau \leq d-1/2$, and consequently $\#\mathcal{A}(f,[x]) \leq 2\#\mathcal{O}(x) \leq 2^{[(d-1)/2]+1} \leq 2^{[(d+1)/2]}$.

(3) If $1 \in \Delta$, $2 \notin \Delta$ and $\nu_-0 \geq 1$, then $2(\tau-1) \leq d-2$. Note that $\#\mathcal{O}(x) \leq 2^{\tau-1}$ then, because $\mathrm{LCM}(1,k) = k$. This gives $\#\mathcal{A}(f,[x]) \leq 22^{\tau-1} \leq 2^{[d/2]}$ in this case. □

The inclusion (3.2.43) and Lemma (3.2.44) give

$$\#\mathcal{A}(f,m) \leq \#\mathrm{Or}(f,m) \cdot 2^{[(d+1)/2]}. \tag{3.2.45}$$

Comparing inequalities (3.2.42) and (3.2.45) we get the statement of Theorem (3.2.41). □

(3.2.46) Remark. In the paper [ShSul] Shub and Sullivan posed the following conjecture about the rate of growth of periodic points of a C^1-map:

$$\limsup_m \frac{\log \#P^m(f)}{m} \geq \limsup_m \frac{\log |L(f^m)|}{m}.$$

The mentioned above result of Babienko and Bogatyi ([BaBo]) (Theorem (3.2.41)) does not give an answer to this conjecture, because it provides the linear rate of growth of $\#P^m(f)$. (We recall that the rate of growth of $|L(f^m)|$ is exponential – Theorem (3.1.57)). On the other hand also the statement of Theorem (3.2.41) does not follow from this conjecture, because Theorem (3.2.41) provides a monotonic growth in m of the number of m-periodic orbits.

The formula of Theorem (3.2.41) seems be the best estimate from below of $\#\mathrm{Or}(f,m)$ (perhaps with another constant) in the smooth homotopy class.

Now we show how one can apply Theorems (3.2.36) and (3.2.37) to get more information about the set $\mathrm{Per}(f)$ for a smooth map f, namely to show the existence of infinitely many primes in $\mathrm{Per}(f)$.

We would like to give a sufficient condition on $\sigma(f)$ for $\mathcal{A}(f)$ to include an infinite sequence of primes, or weaker infinite sequence of multiples of the form bp, where $b \in \mathbb{N}$ is a fixed and p a prime number.

By $|\sigma(f)| \subset [0,\infty)$ we denote the set of modules of all eigenvalues $\lambda \in \sigma(f)$. For given $\rho \in |\sigma(f)|$, $\rho \neq 0$ we put

$$L_\rho(f) := \sum_{|\lambda_i|=\rho} \chi_i \lambda_i,$$

i.e. $L^\rho(f)$ is the part of $L(f)$ which corresponds to $\lambda \in \sigma(f)$, $|\lambda| = \rho$.

Next put $\widetilde{L}^\rho(f) = (1/\rho)L^\rho(f)$. Let $\rho_0 = \mathrm{sp}_{\mathrm{es}}(f)$, and $\rho_1 = \max\{\rho \in |\sigma(f)|, \rho < \rho_0\}$, or $\rho_1 = 0$ if $\rho_0 = 0$. Finally we put

$$\chi_\rho(f) := \sum_{|\lambda_i|=\rho} \chi_i$$

and denote $D = \dim_{\mathbb{Q}} H^*(X;\mathbb{Q})$.

As a first step we show the following lemma.

(3.2.47) Lemma. *We have* $\limsup_m |L(f^m)| \cdot \rho_0^{-m} = \limsup_m |\widetilde{L}^{\rho_0}(f^m)|$.

Proof. Note that if $\rho_0 \leq 1$, then we can assume that $\rho_1 = 0$ in respect of Remark (3.1.54). We have $L(f^m) = \widetilde{L}^{\rho_0}(f^m)\rho_0^m + \sum_{\rho<\rho_0} L^\rho(f^m)$. This gives

$$||L(f^m)/\rho_0^m| - |\widetilde{L}^{\rho_0}(f^m)|| \leq D(\rho_1/\rho_0)^m$$

which gives the statement, because $\lim_m (\rho_1/\rho_0)^m = 0$. □

For given $q \in \mathbb{N}$ we denote the set of all roots of unity of degree q by $\mathcal{C}_q$. We are in position to formulate our next theorem.

(3.2.48) THEOREM. *Let $f\colon X \to X$ be a C^1-map of a compact manifold X. Assume that every $P^m(f)$ consists of isolated points only. Suppose that the sequence $\{L(f^m)\}$ is unbounded and there exists $q \in \mathbb{N}$ such that $\sigma_{\rho_0}(f) \subset \rho_0 \mathcal{C}_q$ for $\rho_0 = \rho(f)$. Then there exist a number $b \in \mathbb{N}$ and an infinite sequence $p_1 < \cdots < p_j < \cdots$ of prime numbers such that $\{bp_j\} \subset \mathrm{Per}(f^b)$.*

PROOF. Note that our assumption $\sigma_{\rho_0}(f) \subset \rho_0\mathcal{C}_q$ means that if $\lambda \in \sigma_{\rho_0}(f)$, then $\lambda = \rho_0 \exp(2\pi\imath p/q)$, $p \in \mathbb{N}$, i.e. this is the case (3.1.53.2) of Theorem (3.1.53). With the above assumption the sequence $\{\widetilde{L}^{\rho_0}(f^m)\}$ is periodic and nonzero, because $\limsup|\widetilde{L}(f^m)| > 0$, as follows from Lemma (3.2.47) and the assumption about $\{L(f^m)\}$. Consequently, there exists $1 \le b \le q$ such that $\widetilde{L}^{\rho_0}(f^b) = \widetilde{\gamma} \neq 0$. Take $h = f^b$.

By the assumption we have $\widetilde{L}^{\rho_0}(f^{b+kq}) = \widetilde{L}^{\rho_0}(f^b) \neq 0$, i.e. h satisfies all the assumptions of Theorem (3.2.48) and $\widetilde{L}(h^{1+kq}) = \widetilde{L}(h) = \widetilde{\gamma} \neq 0$. By Lemma (3.2.47), for some positive γ we have $|L((f^{1+kq})| \ge \gamma\rho_0^{1+kq} > 0$ for all k sufficiently large. From the Dirichlet theorem ([Ch]) it follows that there are infinitely many primes, say $p_1 < p_2 \cdots < p_i, \cdots$, in the sequence $n_k = qk + 1$. From Proposition (3.1.58) it follows that $|a_{1+kq}| \ge (1/2)\gamma\rho_0^m$ if k is sufficiently large.

Assume that $\{p_i\}$ consists of such p_i that $a_{p_i} \ge (1/2)\gamma\rho_0^{p_i}$ and choose the sequence $\{p_j\}$ as

$$\{p_i\} \setminus \bigcup_{x \in P^1(f)} \mathcal{O}(x).$$

Note that $P_1(h) = P^b(f)$ is finite and each $\mathcal{O}(x)$ is finite, by Proposition (3.2.31). By the choice of $\{p_j\}$ and Theorem (3.2.37) we have $\#P_{p_j}(f) > 0$. Indeed the corresponding k-periodic coefficient $a_{p_j}(f) \neq 0$, p_j is prime and we excluded the k-periodic coefficients that come from the virtual periods of at $x \in \mathrm{Fix}\,(h)$. We have shown that every such p_j is a minimal period of h, thus bp_j is a period (not necessarily minimal) of f. This means that b_j, p_j with some $1 \le b_j | b$ is the minimal period of f. Consequently there exists a fixed $b_0 | b$ for which the sequence $\{b_0 p_{j_k}\}_1^\infty$ is infinite and consists of minimal periods of f only. This proves the statement. □

As a direct consequence of the proof of Theorem (3.2.48) we get the following corollary that shows that the algebraic period is the minimal period if it is a prime number.

(3.2.49) COROLLARY. *Let $f\colon M \to M$ be a C^1-map such that all periodic points are isolated. Then there exists an $N > 0$ such that for all $p \in \mathbb{N}$ prime and $p > N$, if $a_p(f) \neq 0$ then $p \in \mathrm{Per}(f)$.*

PROOF. Indeed, it is sufficient to take $N := \max_{x \in \mathrm{Fix}\,(f)}\{d \in \mathcal{O}(x)\}$ and use the argument in the proof of Theorem (3.2.48). □

Note that Theorem (3.2.48) provides a global sufficient condition which provides nonvanishing of $a_p(f)$ or $a_{bp}(f)$. We conjecture that the statement of Theorem (3.2.48) holds always if the sequence $\{L(f^m)\}$ is unbounded.

Recall that the i-th Betti number $b_i(X)$of a finite CW-complex is by definition $\dim_{\mathbb{Q}} H_i(X;\mathbb{Q})$.

(3.2.50) COROLLARY. *Let $f\colon X \to X$ be a C^1-map as in Theorem* (3.2.48). *If X is such that the Betti numbers $b_i(X) \le 1$ for every $0 \le i \le d = \dim X$, then for every C^1-map with $P(f)$ consisting of isolated points only and with unbounded $\{L(f^m)\}$, the set* $\operatorname{Per}(f)$ *contains an infinite sequence $\{p_j\}$ of prime numbers or an infinite sequence of the form $2p_j$, where p_j are primes.*

PROOF. If $b_i(X) = 1$, then $\sigma(f) \subset \mathbb{Z}$ which yields $\sigma_{\rho_0}(f) \subset \{\rho_0\} \cup \{-\rho_0\}$. Now the statement follows from the argument of the proof of Theorem (3.2.48) □

A direct application of the method used in the proof of Theorem (3.2.48) allows us to show the existence of infinitely many primes if $\lim L(f^m) = \pm\infty$.

(3.2.51) PROPOSITION. *Let $f\colon X \to X$ be a C^1-map of a compact manifold X. Assume that for every m the set $P^m(f)$ consists of isolated points only. Suppose that $L(f^m) \le L(f^{m+1})$ and $\limsup L(f^m) = \infty$, or dually $L(f^m) \ge L(f^{m+1})$ and $\liminf L(f^m) = -\infty$. Then there exists an infinite sequence of primes $\{p_j\} \subset$* $\operatorname{Per}(f)$.

PROOF. We prove only the first case. Note that $L(f^m) \le L(f^{m+1})$ and $\limsup L(f^m) = \infty$ gives $\lim L(f^m) = \infty$. In particular $\rho_0 = \mathrm{sp}_{\mathrm{es}}(f) > 1$. We show that there exists m_0 such that $|L(f^m)|/\rho_0^m \ge c > 0$ for $m \ge m_0$. Then the statement follows from Proposition (3.1.58) and Corollary (3.2.49). Let $m \ge m_0$ be such that $L(f^{m-n(f)+1}) > 0$ where $n(f)$ is the number of the statement of Theorem (3.1.57), and such that $m \ge N$, where N is the constant of Theorem (3.1.57). By Theorem (3.1.57) there exist $0 \le i_0 \le n(f) - 1$ and γ such that $L(f^{m-i_0})/\rho_0^{m-i_0} \ge \gamma$. By monotonicity $L(f^m)/\rho_0^{m-i_0} \ge L(f^{m-i_0})/\rho_0^{m-i_0}$ and consequently $L(f^{m-i_0})/\rho_0^m \ge \gamma/\rho^{i_0} \ge \gamma/\rho^{n(f)}$ which is the desired inequality. □

For $X = S^d$ we have already discussed the sequence $\{l^m\}$ and showed that if $|r = \deg(f)| > 1$, then $a_k(f) \ne 0$ for every $k \ge 1$ (Example (3.2.15)).

Let $\mathbb{R}P(d)$ be the real projective space, i.e. the quotient space $S^d/\mathbb{Z}_2$ by $\mathbb{Z}_2 = \{1,-1\}$ acting on $S^d \subset \mathbb{R}^{d+1}$ by multiplying coordinates. It is known that if d is odd, then $H_i(\mathbb{R}P(d);\mathbb{Q}) = \mathbb{Q}$ for $i = 1, d$ and $H_i(\mathbb{R}P(d);\mathbb{Q}) = 0$ otherwise. If d is even, then $H_i(\mathbb{R}P(d);\mathbb{Q}) = \mathbb{Q}$ for $i = 0$ and $H_i(\mathbb{R}P(d);\mathbb{Q}) = 0$ otherwise (see [Sp] for details). Note that if d is odd, then the image $H_d(f)(1) \in \mathbb{Q}$ is integral and equal to $\deg(f)$.

(3.2.52) Exercise. Let $f\colon \mathbb{R}P(d) \to \mathbb{R}P(d)$ be a map of real projective space of degree r if d is odd. Show that

(3.2.52.1) If d is odd, then the sequences $\{L(f^m)\}$ and $\{a_k(f)\}$ are the same as the corresponding sequences for a map of degree r of the sphere S^d.

(3.2.52.2) If d is even, then $L(f^m) = 1$ for every m and consequently $a_k(f) = 0$ if $k > 1$.

Let $\mathbb{C}P(d)$ be the complex projective space, i.e. the quotient space S^{2d+1}/S^1, where S^1 acts on $S^{2d+1} \subset \mathbb{C}^{d+1}$ by multiplying the coordinates. It is known that the cohomology algebra (over $\mathbb{Z}$) of $\mathbb{C}P(d)$ is isomorphic to the quotient polynomial ring $\mathbb{Z}[z]/(z^d)$, where the generator $z \in H^2(\mathbb{C}P(d)) = \mathbb{Z}$. In particular $H^*(\mathbb{C}P(d);\mathbb{Z}) = \bigoplus_{i=1}^d H^{2i}(\mathbb{C}P(d);\mathbb{Z}) \simeq \bigoplus_{i=1}^d H^{2i}(\mathbb{C}P(d);\mathbb{Z})$ (see [Sp] for details).

(3.2.53) Exercise. Let $f\colon \mathbb{C}P(d) \to \mathbb{C}P(d)$ be a map of complex projective space and $a = a_f \in \mathbb{Z}$ the image of the generator of $\mathbb{Z} = H^2(\mathbb{C}P(d);\mathbb{Z})$ by the homomorphism $H^2(f)$. Show that

(3.2.53.1)
$$L(f) = \sum_{i=0}^{d-1} a^i = \begin{cases} \dfrac{a^d-1}{a-1} & \text{if } a \neq 1, \\ d+1 & \text{if } a = 1, \end{cases} \qquad \text{thus always } L(f) \neq 0.$$

(3.2.53.2) $\deg(f) = a^{d-1}$.

(3.2.53.3) $\limsup |L(f^m)| = \infty$ if and only if $|a| > 1$, and then for every k we have $a_k(f) \neq 0$.

Note that if $X = S^d$, or $\mathbb{R}P(d)$, $\mathbb{C}P(d)$ is a sphere or projective space, then the assumption of Corollary (3.2.50) is satisfied.

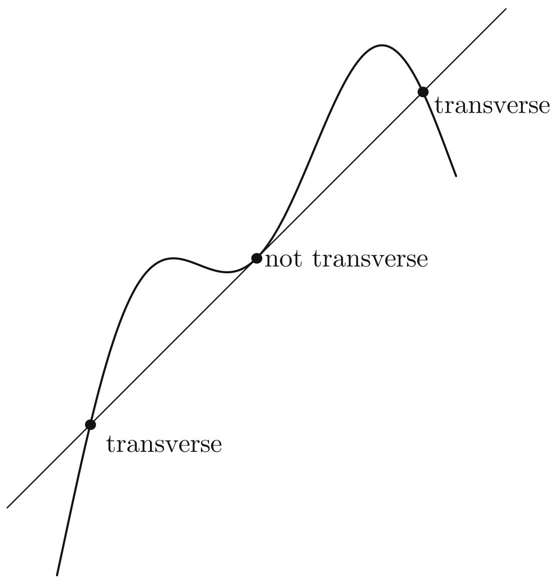
3.3. Periodic points of special classes of smooth maps

In this section we shall discuss applications of the general theory of periodic points of smooth maps developed in the previous sections of this chapter. It can be done by imposing an additional local property of a smooth map. Due to local analytical properties of these maps, the k-periodic expansion of the fixed point index at an isolated periodic point is of a special form which gives information about the minimal periods and cardinalities of the sets $P^m(f)$, and $P_m(f)$. For the opposite, a condition posed on a map can be global, e.g. a special form of the (co)homology spaces or the dimension.

3.3.1. Transversal maps. In this subsection we will study special class of C^∞-maps that have a nice property of the local fixed point index at each periodic point. This property allows us to prove theorems that are not true in the general case of any smooth map.

(3.3.1) Definition ([Do2], [Fr1]). Let $f\colon\mathcal{U} \quad X$ be a C^∞-map of an open subset $\mathcal{U}$ of a manifold X, $\dim X = d$. We say that $f \in \mathcal{T}$, or f is transversal if for any $m \in \mathbb{N}$ and $x \in P^m(f)$,

$$1 \notin \sigma(D^m f(x)).$$



The main property of the class of transversal maps is given in the following theorem which is widely used but whose rigorous proof is complicated (see [Do2] if $X = \mathbb{R}^d$, and [Sei] for the general case).

(3.3.2) Theorem. *The set $\mathcal{T}$ is generic in $C^0(\mathcal{U}, X)$, i.e. is the intersection $\mathcal{T} = \bigcap_{n=1}^\infty G_n$ where G_n is open and dense. In particular every map $f\colon\mathcal{U} \quad X$ is homotopic to a transversal map $h\colon\mathcal{U} \quad X$.*

Note that the second part of the statement follows from the first, because two maps close in C^0-topology are homotopic.

Another geometric property of a transversal map states the following.

(3.3.3) Proposition. *For any $f \in \mathcal{T}$ and every $m \in \mathbb{N}$ the set $P^m(f)$ consists of isolated points.*

Proof. For a given m let us put $g := f^m$. We will prove the statement if we show that $P(g)$ consists of isolated points. Recall that for a map $g\colon X \to X$ of a manifold of dimension d the graph of g, defined as $\mathrm{graph}(g) := \{(x, g(x))\} \subset X \times X$, is a a submanifold of $X \times X$ of dimension d. First observe that for a transversal map g, $\mathrm{graph}(g)$ and the diagonal $\Delta = \{(x, x)\} \subset X \times X$ are transversal at each fixed point $x \in \operatorname{Fix}(g)$. We recall that two submanifolds $X_1, X_2 \subset X$, of dimension d_1, and d_2 respectively, are transversal if for every $x \in X_1 \cap X_2$ the tangent space $T_x X$ at x is the direct sum of $T_x X_1 \oplus T_x X_2$ of tangent spaces. Note that then $d_1 + d_2 = d$. Next, if $x \in X_1 \cap X_2$ and X_1, X_2 are transversal, then x is isolated point of $X_1 \cap X_2$, because the exponential map $\exp\colon T_x X \to X$ is a local diffeomorphism and exp maps $T_x X_i$ into X_i (cf. [Mil]). Now it is sufficient to check that for a transversal graph(g) and Δ are transversal at any $x \in \operatorname{Fix}(g) = \operatorname{graph}(g) \cap \Delta$. Indeed, $T_x\Delta = \{(v, v)\}$ and $T_x\operatorname{graph}(g) = \{(w, Dg(x)w)\}$. Every vector (u_1, u_2) is represented as $(v+w, v+Dg(x)w)$ if and only if the equation $w - Dg(x)w = u_1 - u_2$ has a unique solution w. But the latter is equivalent to $\det(\mathrm{I} - Dg(x)) \neq 0$, which is satisfied for the transversal map by definition. $\square$

(3.3.4) Remark. Recall that for a map $f\colon X \to X$, $\operatorname{Fix}(f) = \operatorname{graph}(f) \cap \Delta$, thus the observation of the proof of Proposition (3.3.3) justifies the name "transversal" for the class $\mathcal{T}$.

Assume that $f \in \mathcal{T}$ and $x \in P_m(f)$. As a direct consequence of the definition of index and Lemma (2.1.24) we get the classical *Hopf index formula*,

$$I(f^m, x) = \operatorname{sign}\det(1 - Df^m(x)). \tag{3.3.5}$$

Consequently, we can split $P_m(f)$ into a disjoint sum $P_m^E(f) \cup P_m^O(f)$ depending on whether the index is equal to 1 or -1. We say that $x \in P_m(f)$ is a *twisted* m-periodic point ([Fr1]), or *inverting* ([Do2], [Mat]), if $I(f^m, x) = -I(f^{2m}, x)$, and is called *untwisted* (or *noninverting*) in the opposite case. This gives the splitting of $P_m^E(f)$ and $P_m^O(f)$ into

$$P_m^E(f) = P_m^{EE}(f) \cup P_m^{EO}(f), \qquad P_m^O(f) = P_m^{OE}(f) \cup P_m^{OO}(f). \tag{3.3.6}$$

(See [Mat] for a more detailed description.) Set $P_m^{tw}(f) = P_m^{EO}(f) \cup P_m^{OO}(f)$, called the set of twisted points.

It is easy to check that $x \in P_m^O(f)$ if and only if $\nu_+(x)(m) \cong 1 (\operatorname{mod} 2)$, and $x \in P_m^{tw}(f)$ if and only if $\nu_-(x)(m) \cong 1 (\operatorname{mod} 2)$, which gives another description of the definition of the above decomposition of $P_m(f)$, and allows us to extend the definition of twisted point onto the C^1 case.

By Theorem (3.2.37), for the orbit $[x] = \{x, f(x), \dots, f^{m-1}(x)\}$ of x we have consequently

$$\mathcal{I}(f, [x]) = (-1)^{\nu_+(x)(m)} \left(\operatorname{reg}_m + \frac{1}{2}[(-1)^{\nu_-(x)} - 1]\operatorname{reg}_{2m} \right) \tag{3.3.7}$$

in particular $\mathcal{I}(f, [x])$ is of the finite type.

Note that $I(f^k, x) = \#P_k^{EE}(f) + \#P_k^{EO}(f) - \#P_k^{OE}(f) - \#P_k^{OO}(f)$, and $I(f^k, x) = \#P_k^{EO}(f) - \#P_k^{OO}(f)$ if $x \in P_k^{tw}(f)$. Substituting (3.3.7) into $\mathcal{I}(f, \mathcal{U})$ with x ranking over all $x \in P(f, \mathcal{U})$ we get the following.

(3.3.8) Proposition. *If $f\colon \mathcal{U} \quad X$ is a transversal map. then*

$$(3.3.9) \quad \mathcal{I}(f,\mathcal{U}) = \sum_{k=1}^{\infty} a_k(f)\mathrm{reg}_k, \quad \textit{with } a_k(f) = \sum_{x\in P_k(f)} c_1(x) - \sum_{x\in P^{tw}_{k/2}(f)} c_1(x),$$

where $c_1(x) = (-1)^{\nu_+(x)} = I(f^k, x)$ if $x \in P_k(f)$, and by convention $P^{tw}_{k/2}(f) = \emptyset$ if k is odd.

Equality (3.3.9) was shown in paper [Do2], independently of [ChM-PY], and is called the Dold formula.

The Dold equality means that the k-th coordinate $a_k(f)$ of $\mathcal{I}(f,\mathcal{U})$ measures the algebraic multiplicity of k-orbits of f if k is odd. If k is even, there is a term which comes from twisted $k/2$-periodic orbits.

As a direct consequence of the formula (3.3.9) of Proposition (3.3.8) we get the following.

(3.3.10) Corollary. *Let $f\colon \mathcal{U} \quad X$ be a transversal map. If $a_m(f) \neq 0$, then*

$$\begin{cases} P_m(f) \cup P_{m/2}(f) \neq \emptyset & \textit{if } m \textit{ is even,} \\ P_m(f) \neq \emptyset & \textit{if } m \textit{ is odd.} \end{cases}$$

In other words Corollary (3.3.8) says that for a transversal map, if m is an algebraic period then m or $m/2$ is the minimal period of f, i.e. the global information (calculation and nonvanishing of the k-periodic coefficient) guarantees the existence of a k-periodic point (cf. Theorem (3.2.14) for the general case).

We show now another application of the Dold formula. It is a finer formulation of a result of Matsuoka ([Mat]). We need further notation. Let X be a compact smooth manifold of dimension d and $b_i = b_i(X)$, $0 \le i \le d$ its i-th Betti number. Next put $\beta_i(X) = 2(2^{b_i} - 1)(2^{b_i - 1} - 1)$, which is equal to the order of $GL(b_i, \mathbb{Z}_2)$. Put next $\beta(X) = \{\beta_i(X)\} = [\{\beta_i(X)\}]$, $0 \le i \le d$, and $\beta_{\mathrm{od}}(X)$ is the odd factor of an integer (i.e. $\beta(X) = 2^r \beta_{\mathrm{od}}(X)$, $\beta_{\mathrm{od}}(X)$-odd). Let $A \in M_n(\mathbb{Z})$ and $A_{(2)} \in M_n(\mathbb{Z}_2)$ be its reduction modulo 2. As in the complex (integral) case we can write $A_{(2)}$ as $\widetilde{A}_{(2)} \oplus N$, where $\widetilde{A}_{(2)}$ is nonsingular, or the zero matrix, and N is nilpotent restriction of $A_{(2)}$ to its generalized kernel. Define $\mathrm{rk}\, A_{(2)} := \mathrm{rk}\, \widetilde{A}_{(2)}$, as the order of the element $\widetilde{A}_{(2)}$ in $GL(n, \mathbb{Z}_2)$, and $\mathrm{rk}_{\mathrm{od}}(A_{(2)})$ equal to its odd factor. If $\widetilde{A}_{(2)} = 0$, then we set $\mathrm{rk}\, \widetilde{A} = 1$. For a given map $f\colon X \to X$ we set

$$\alpha(f) = [\mathrm{rk}_{\mathrm{od}} H^{\mathrm{ev}}(f), \mathrm{rk}_{\mathrm{od}} H^{\mathrm{od}}(f)].$$

(3.3.11) THEOREM. *Let* $f\colon X \to X$, $f \in \mathcal{T}$ *be a map of a compact smooth manifold* X. *Suppose that* m *is an odd number such that* $\alpha(f)^2|m$. *Then the number of* m*-periodic orbits of* f *is even. In particular, if* $m \cong 0 \pmod{\beta^2_{\mathrm{od}}(X)}$, *then the number of* m*-periodic orbits of* f *is even for every* f *as above.*

PROOF. It is sufficient to show that $\alpha(f)$ is equal to the odd factor of the smallest $l \in \mathbb{N}$ such that $L(f^{i+l}) \cong L(f^i) \pmod 2$ for every $i \in \mathbb{N}$, which we left to the reader. By this, the statement of (3.3.11) is a part of the hypothesis of Theorem 2 of the quoted paper of Matsuoka ([Mat]). He defined and used $\alpha(f)$ equal to the odd part of the smallest period of the sequence $\{L(f^i)_{(2)}\}$. The remaining part of his proof is natural: from the Möbius formula we get $a_m(f) \cong 0 \pmod 2$, and the statement follows from the Dold formula. $\square$

Next we give another condition which implies that for any m odd the set of m-periodic orbits is of even cardinality.

(3.3.12) THEOREM. *Let* $f\colon X \to X$, $f \in \mathcal{T}$ *be a map of a compact smooth manifold* X. *Suppose that* $\sigma(f) \subset 2\mathbb{Z} \cup \{1\}$. *Then for every odd* $m \in \mathbb{N}$ *we have*

$$\frac{\#P_m(f)}{m} \cong 0 \pmod 2.$$

PROOF. First note that for every $m \in \mathbb{N}$ we have $i_m(f) \cong 0 \pmod 2$, by the definition of $i_m(f)$ and a property of the Möbius function. This implies $a_m(f) = i_m(f)/m \cong 0 \pmod 2$ if m is odd, which gives the statement in respect of (3.3.9). $\square$

Now we would like to present a new proof of the Franks period- doubling cascade theorem. This proof gives a little bit more general statement of the theorem.

(3.3.13) DEFINITION. Let $f\colon X \to X$ be a C^1-map of a compact manifold X, $f \in \mathcal{T}$. We say that f satisfies Franks assumptions if

(3.3.13.1) there exists an odd $k \in \mathbb{N}$, such that for every $r \geq 0$, $P^{EE}_{2^r k}(f) = P^{OO}_{2^r k}(f) = \emptyset$ (the dual assumption: $P^{EO}_{2^r k}(f) = P^{OE}_{2^r k}(f) = \emptyset$);

(3.3.13.2) $P^{tw}_k(f) \neq \emptyset$ or $a_k(f) \geq 0$, (and $a_k(f) \leq 0$ for the dual assumption (3.3.13.1));

(3.3.13.3) for every $r \geq 1$, $a_{2^r k}(f) = 0$ in the expansion of $\mathcal{I}(f, X) = \mathcal{L}(f)$.

(3.3.14) THEOREM. *Suppose that for a given* $k \in \mathbb{N}$ *a map* f *satisfies Franks assumptions. Then, for every* $r \geq 1$, $P^{tw}_{2^r k}(f) \neq \emptyset$.

PROOF. We consider only the first case of assumption (3.3.13.1) in Definition (3.3.13). From the Dold formula we get:

(1) $0 \leq a_k(f) = \#P^{EO}_k(f) - \#P^{OE}_k(f)$, which shows that $\#P^{EO}_k(f) \neq 0$, if it is not given by the assumption;

(2) $0 = a_{2k}(f) = \#P^{EO}_{2k}(f) - \#P^{OE}_{2k}(f) - 2\#P^{EO}_{k}(f)$, which shows that $\#P^{EO}_{2k}(f) \neq 0$;
(3) we continue this argument for $r = 2, 3, \dots$, which proves the statement. □

(3.3.15) REMARK. Originally in [Fr1] this theorem was formulated for $X = D^n$, the disc. Franks ([Fr1]) proved also a version of this theorem in which assumption (3.3.13.3) was replaced by the following (say (3.3.13.4)):

(3.3.13.4) For a given $k \in \mathbb{N}$ assume that $f\colon X \to X$ is a map such that every nonzero eigenvalue of $H^*(f)$ is a root of unity of an odd degree q such that $k \nmid q$.

The assumption (3.3.13.4) can be replaced by a general one (cf. [Mar]). In view of Theorem (3.1.46) we can assume, instead of (3.3.13.4), that the sequence $\{L(f^m)\}$ is bounded. Roughly speaking all these assumptions give some algebraic restrictions on $H^*(f)$, or $\sigma(f)$, which imply the Assumption (3.3.13.3). Recalling that for $q \in \mathbb{N}$ by $\mathcal{C}_q$, we denoted the set of all roots of unity of degree q, we set $\mathcal{C} := \bigcup_1^\infty \mathcal{C}_q$. To illustrate it we show the following.

(3.3.16) PROPOSITION. *Suppose that $f\colon X \to X$, $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) < \infty$, is such a map that $\sigma(f) \subset \mathcal{C}_q \cup \{0\}$, or $L(f) = \sum_{\lambda \in \sigma(f) \cap \mathcal{C}_q} L^\lambda(f)$, with q odd. Then for every k odd and every $r \geq 1$ we have $a_{2^r k}(f) = 0$.*

For a given polynomial $\omega \in \mathbb{Z}[x]$ we denote the set of all its roots by $\sigma(\omega)$. We begin with the following lemma.

(3.3.17) LEMMA. *Let $\omega(z) \in \mathbb{Z}[z]$ be a monic polynomial such that $\sigma(\omega(z)) \subset \mathcal{C}_q$, q-odd (i.e. all roots of ω are the q-th roots of unity). Then, for every $r \geq 1$ and every odd $l \in \mathbb{N}$,*

$$S(2^r l) = \sum_{\lambda_j \in \sigma(\omega)} \lambda_j^{2^r l} = \sum_{\lambda_j \in \sigma(\omega)} \lambda_j^l = S(l).$$

PROOF. Let $\omega = \omega_1 \cdots \omega_s$ be a decomposition of ω into a product of irreducible factors. We have $\sigma(\omega_i) \subset \mathcal{C}_{q_i}$, $q_i | q$, and $S(2^r l) = \sum_{i=1}^s S_i(2^r l)$, $S(l) = \sum_{i=1}^s S_i(l)$. For a given ω_i, $\sigma(\omega_i)$, consists of all primitive roots of unity of degree $q_i | q$ (i.e. ω_i is the q_i-th cyclotomic polynomial). If $\lambda_j \in \sigma(\omega_i)$ then $\lambda_j^l \in \sigma(\widetilde{\omega})$, where $\widetilde{\omega}_i(z)$ is the $q_i/(l, q_i)$-th cyclotomic polynomial. Moreover

$$\sum_{\lambda_i \in \sigma(\omega)} \lambda_j^l = \sum_j (l, q_i) \lambda_j^{l/(l,q_i)} = (l, q_i) \sum_{\mu_j \sigma(\widetilde{\omega}_i)} \mu_j,$$

where μ_j are all primitive roots of unity of degree $q_i/(l, q_i)$.

This shows that the sum $S_i(l)$ and consequently $S(l)$ is a sum of the sums of all roots of the cyclotomic polynomials of odd orders. For any such a polynomial $\widetilde{\omega}(z)$ we have

$$\sum_{\mu \in \sigma(\widetilde{\omega})} \mu_j = \sum_{\mu_j} \mu_j^{2^r},$$

$\mu_j \subset \mathcal{C}_{\widetilde{q}_j} \subset \mathcal{C}_{q_j} \subset \mathcal{C}_q$, because $(2^r, \widetilde{q}_j)|(2^r, q) = 1$ and consequently the power map $(\mu \mapsto \mu^{2^r}) \in Gal(\widetilde{\omega}(\mathbb{Q}), \mathbb{Q}) = \mathbb{Z}^*_{q_j}$ is a permutation of all roots of $\widetilde{\omega}(z)$. This proves the lemma. □

Proof of Proposition 3.3.16. We have

$$a_{2^r k}(f) = (2^r k)^{-1} i_{2^r k}(f) = (2^r k)^{-1} \sum_{l|k} \mu(l)(L(f^{2^r l}) - L(f^{2^{r-1} l})).$$

It is enough to apply Lemma (3.3.17) to the characteristic polynomials of $H^{\mathrm{ev}}(f)$ and $H^{\mathrm{od}}(f)$. □

3.3.2. Periodic points of holomorphic maps. In this section we show that the local degree, and consequently the local fixed point, of a holomorphic map and its iteration has special properties. These local properties combined with the already discussed global properties of the sequence of the Lefschetz numbers of iterations give information about the set $\mathrm{Per}(f)$ of a holomorphic map of a compact complex manifold.

(3.3.18) Definition. Let $f{:}\,\mathcal{U} \quad \mathbb{C}$ be a continuous complex-valued function on an open subset $\mathcal{U} \quad \mathbb{C}^d$. We say that f is holomorphic in U if for every $z_0 = (z_0^1, \dots, z_0^d) \in \mathcal{U}$ there exists a polydisc $\{|z - z_0^1| < \rho_1 \times |z - z_0^2| < \rho_2 \times \cdots \times |z - z_0^d| < \rho_d\}$ in which f is represented by the series

$$f(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha (z - z_0)^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \geq 0$ is the multi-index, $a_\alpha \in \mathbb{C}$, and $(z - z_0)^\alpha$ means $(z^1 - z_0^1)^{\alpha_1}(z^2 - z_0^2)^{\alpha_2} \cdots (z^d - z_0^d)^{\alpha_d}$.

We say that a map $f{:}\,\mathcal{U} \quad \mathbb{C}^d$, $f = (f_1, \dots, f_d)$ is homolomorphic if for every $1 \leq i \leq d$, the function f_i is holomorphic.

Let us recall well-known some facts about the local degree and fixed point index of a holomorphic map at an isolated zero, a an isolated fixed point, respectively (cf. [Cr1], [Pal]).

(3.3.19) THEOREM. *Let* $f\colon \mathcal{U} \quad \mathbb{C}^d \to \mathbb{C}^d$ *be a holomorphic map of an open subset such that* $z_0 \in \mathcal{U}$ *is an isolated zero. Then* $\deg(f, z_0) > 0$.

PROOF. For $d = 1$ from the Weierstrass theorem it follows that locally near z_0 the function f is represented as $f(z) = (z - z_0)^k g(z)$, where $k > 0$ and $g(z_0) \neq 0$. It is easy to verify that $\deg(f, z_0) = k > 0$. In the case $d \geq 2$ we have to use the Cronin theorem [Cr1]. Suppose for simplicity that $z_0 = 0$, and for $1 \leq i \leq d$ let $f_i(z) = \sum_{|\alpha|>0} a(i)_\alpha z^\alpha$, $\alpha = (\alpha_1, \dots, \alpha_d)$ be the series expansion of the i-th coordinate f at 0. Let next k_i be the smallest module of the multiindex α for which $a(i)_\alpha \neq 0$. The Cronin theorem says that $\deg(f, 0) \geq \prod_{i=1}^d k_i > 0$. □

It is worth of emphasizing that $\deg(f, z_0)$ is equal to the dimension μ of the quotient algebra $\mathbb{C}[(z - z_0)/I$, of the local algebra $\mathbb{C}[(z - z_0)]$ by the ideal I generated by $f_1, \dots, f_d$ (cf. [Pal]).

(3.3.20) EXERCISE. Show directly, using the differential definition of degree (see Subsection 2.1.1) that $\deg(f, z_0) \geq 0$ for every holomorphic map $f\colon \mathcal{U} \quad \mathbb{C}^d$, $z_0 \in \mathcal{U}$.

Hint. Note that the derivative $Df(z_0) : \mathbb{C}^d = \mathbb{R}^{2d} \to \mathbb{R}^{2d} = \mathbb{C}^d$ of a holomorphic map is given by a real matrix A which comes from a complex matrix by considering its coefficients as the real numbers. Show that for such a matrix we have $\det A \geq 0$ (cf. Example (2.1.7)).

As a consequence of Theorem (3.3.19) we get the following.

(3.3.21) PROPOSITION. *Let* $z_0 \in P_m(f)$ *be an isolated periodic point of a holomorphic map* $f\colon \mathcal{U} \quad \mathbb{C}^d \to \mathbb{C}^d$. *Then* $\operatorname{ind}(f^m, z_0) > 0$. *Moreover, for every divisor* $k|m$, *if* $z_0 \in P^k(f)$ *then* $\operatorname{ind}(f^k, z_0) \leq \operatorname{ind}(f^m, z_0)$.

PROOF. By the definition of local fixed point index, $\operatorname{ind}(f, z_0) = \deg(z - f(z), z_0) > 0$, which is positive by Theorem (3.3.19). Let $k|m$ and z_0 be an isolated fixed point of f^k and f^m. We have $\mathrm{id} - f^m = (\mathrm{id} - f^k)(f^{m/k} + \dots + f^2 + f)$. Since z_0 is an isolated zero of $\mathrm{id} - f^m$ it is an isolated zero of $h := f^{m/k} + \dots + f^2 + f$. Using the composition property of degree and Theorem (3.3.19) we get $\deg(\mathrm{id} - f^m, z_0) = \deg(\mathrm{id} - f^k) \cdot \deg(h, z_0) \geq \deg(\mathrm{id} - f^k, z_0)$. □

Now we need to recall the notion of complex manifold. Let M be a smooth manifold of dimension $2d$. We say that M is the complex manifold if there exists an atlas $\{(\mathcal{U}_\gamma, \phi_\gamma)\}$ of M such that the compositions of coordinates functions $\phi_\gamma \phi_{\gamma'}^{-1}\colon \phi_\gamma^{-1}(\mathcal{U}_\gamma \cap \mathcal{U}_{\gamma'}) \subset \mathbb{R}^{2d} = \mathbb{C}^d \to \phi_{\gamma'}(\mathcal{U}_{\gamma'}) \subset \mathbb{R}^{2d} = \mathbb{C}^d$ are holomorphic.

If the map $f\colon M \to M$ is holomorphic, then Proposition (3.3.21) leads naturally to an estimate from above of the cardinalities $\#P^m(f)$, correspondingly $\#P_p(f)$, by $L(f^m)$ and $a_p(f)$ respectively (cf. [FgLl]).

(3.3.22) THEOREM. *Let $f\colon M \to M$ be a holomorphic map of a compact complex manifold with isolated periodic points. Then $L(f^m) \geq \#P^m(f)$ for every $m \in \mathbb{N}$. Moreover, for every prime number p, we have $a_p(f) \geq \#P_p(f)$.*

PROOF. By the Lefschetz–Hopf theorem $L(f^m) = \sum_{x\in P^m(f)} \operatorname{ind}(f^m, x)$, and the first part of the statement follows from the first part of Proposition (3.3.21).

If p is prime then the Möbius formula defining $a_p(f)$ reduces to

$$\begin{aligned} a_p(f) = L(f^p) - L(f) &= \sum_{x\in P^p(f)} \operatorname{ind}(f^p, x) - \sum_{x\in P^1(f)} \operatorname{ind}(f, x) \\ &= \sum_{x\in P_p(f)} \operatorname{ind}(f^p, x) + \sum_{x\in P^1(f)} (\operatorname{ind}(f^p, x) - \operatorname{ind}(f, x)). \end{aligned}$$

By the second part of Proposition (3.3.21) the second term of the sum is nonnegative, and the statement follows from the first part of Proposition (3.3.21), because $\operatorname{ind}(f^p, x) > 0$ at each $x \in P^p(f)$. □

(3.3.23) COROLLARY. *Suppose that $f\colon M \to M$ is a holomorphic map of a compact complex manifold with isolated periodic points. Then there exists a constant $N > 0$ (depending on f) such that, for every prime number $p \geq N$, we have*

$$p \in \operatorname{Per}(f) \Leftrightarrow a_p(f) \neq 0.$$

PROOF. If $p \in \operatorname{Per}(f)$, then $a_p(f) \neq 0$ with respect to the second inequality of Theorem (3.3.22).

The converse implication follows from Corollary (3.2.49) and holds for any smooth map. □

As a consequence of Theorem (3.3.22) we get an exponential estimate from above of the number of periodic points of period m (cf. [FgLl]).

(3.3.24) COROLLARY. *Let $f\colon M \to M$ be a holomorphic map such that all periodic points are isolated. Then $\#\operatorname{Fix}(f^m) \leq D\rho_0^m$, where $D = \dim_{\mathbb{R}} H_*(M)$ and $\rho_0 = \operatorname{sp}_{\mathrm{es}}(f)$ is the essential radius of f.*

PROOF. With respect to Theorem (3.3.22) the statement follows by an observation that $|L(f^m)| \leq D\rho_0^m$. The latter is a consequence of the definition of essential spectral radius (Definition (3.1.37)) and the obvious estimate $|L(f)| \leq D\rho_0$ (see also Example (3.1.15)). □

(3.3.25) REMARK. In [Arn] V. I. Arnold posed the following question: "Is the number of periodic points of period m of a real analytic diffeomorphism $f\colon M \to M$ of a compact analytical manifold bounded above by an exponential function?" The above Corollary (3.3.24) gives the affirmative answer if M is a complex manifold and f is a holomorphic map (not only a diffeomorphism).

Theorem (3.3.22) and its corollaries have been discussed in the paper [FgLl] by Fagella and Llibre. The paper contains also other interesting applications as new proof of the Baker theorem on minimal periods of a rational map of S^2.

(3.3.26) REMARK. Observe that from Proposition (3.3.21) it follows that if a holomorphic map has isolated periodic points, then its Lefschetz number $L(f) > 0$. This shows that the set of holomorphic maps with isolated periodic points, e.g. the transversal maps, is *not* dense in the set of holomorphic maps as it is for smooth maps (Theorem (3.3.2)). In particular it is not true that every holomorphic map $f\colon X \to X$ of a compact complex manifold is homotopic to a transversal holomorphic map. As a simple example one can take $f = \mathrm{id}$ the identity map of a two-dimensional compact oriented surface X of genus $g > 1$. Every such surface has a complex structure, i.e. is a complex curve and obviously id is holomorphic. On the other side $L(\mathrm{id}) = \chi(X) = 2 - 2g < 0$.

3.3.3. Explicit forms of the sequence of local indices. In this subsection we present information about the coefficients of the k-periodic expansion of the local fixed point index at an isolated periodic point of a smooth map. As we have already said, the considerations of Subsection 3.2.2 do not give any information on it, but imposing additional conditions, e.g. on the dimension, it is possible to get lists of all appearing expansions. The material is based on the paper [BaBo] of Babenko and Bogatyi.

Recall that the local k-periodic expansion of the fixed point index of f at a fixed point $x = [x]$ isolated as the periodic point has the form

$$\mathcal{I}(f, [x]) = \mathcal{I}(f, x) = \sum_{k \in \mathcal{O}} c_k(f, x)\left(\mathrm{reg}_{mk} + \frac{1}{2}[(-1)^{k\nu_-(x)} - 1]\,\mathrm{reg}_{2mk}\right).$$

Equivalently, the k-periodic expansion

$$\mathcal{I}(f, [x]) = \mathcal{I}(f, x) = \sum_{k \in \mathcal{O}} a_{mk}(f, x)\,\mathrm{reg}_{mk} = \sum_{l = mk, k \in \mathcal{O}(x)} a_l(f, x)\,\mathrm{reg}_l, \tag{3.3.27}$$

where $a_l = a_{mk} = c_k + (1/2)((-1)^{(k/2)\nu_-(x)} - 1)c_{k/2}$, with the convention that $c_k = 0$ if $k \notin \mathcal{O}(x)$ or $k/2 \notin \mathbb{Z}$.

We begin with showing the following fact

(3.3.28) PROPOSITION. *Let* 0 *be an isolated periodic point of a* C^1*-map* $f :$ $\mathbb{R}^1 \supset \mathcal{U} \quad \mathbb{R}^1$ *of the real line. Then all possible sequences* $\{i_m = \mathrm{ind}\,(f^m, 0)\}_1^\infty$ *of the local fixed point of iterations of* f *are of the form*

$$i_m = a, \quad \textit{where } a = 1, -1, 0, \qquad i_m = \begin{cases} 1 & \textit{if } 2 \nmid m, \\ -1 & \textit{if } 2 | m. \end{cases}$$

Moreover, examples of maps giving a realization of each of the above sequences are the following

$$
\begin{aligned}
f(x) &= 0, & f(x) &= 2x + x^n, \quad |n| > 1,\\
f(x) &= x + x^n, \quad n = 2k,\ |n| > 1, \qquad & f(x) &= -2x.
\end{aligned}
$$

Proof. First recall that the degree of a one-dimensional map is contained in $\{1, -1, 0\}$ (cf. Exercise (2.2.5)). Next note that for a line map the set $\Delta(Df(0))$ is either $\emptyset$, or one of the one-point sets $\{1\}$, $\{2\}$. From the expansion formula (3.3.27) we have $\operatorname{ind}(f^m, 0) = \deg(\mathrm{id} - f^m, 0) = a_1\mathrm{reg}_1(m) + a_2\mathrm{reg}_2(m)$ with the condition that $a_2 = 0$ if $-1 \notin \sigma(Df(0))$. The mentioned condition on the degree of a line map applied to f and f^2 gives $a_1 \in \{1, -1, 0\}$, $a_1 + 2a_2 \in \{1, -1, 0\}$. If $a_2 = 0$, then we have one of the listed constant sequences. $a_1 = 1$ and $a_2 = -1$ give the listed nonconstant periodic sequence. We have to exclude the remaining algebraic solution of these equations with $a_2 \neq 0$, namely $a_1 = -1$ and $a_2 = 1$. But $\Delta = \{2\}$ is equivalent to $Df(0) = -1$ here. But then $f(x) = -x + h(x)$, where $h(x)/|x| \to 0$ if $x \to 0$. Consequently $a_1 = \operatorname{ind}(f, 0) = \deg(2x - h(x), 0) = 1$, which proves the statement. One can verify directly that the listed maps have the required fixed point indices. □

As a generalization of Proposition (3.3.28) we get the following statement already mentioned at the end of Subsection 3.2.2 (cf. [ChM-PY]).

(3.3.29) Proposition. *Let 0 be an isolated periodic point of a smooth map $f\colon\mathcal{U} \quad \mathbb{R}^d \to \mathbb{R}^d$, i.e. it is the isolated point of each of its iterations f^m, $m \geq 1$, and $A = Df(0)$. Then for the coefficients of k-periodic expansion* (3.3.27) *we have:*

(3.3.29.1) $a_1(f) = (-1)^{\nu_+}$ *if* $\mathrm{I} - A$ *is nonsingular.*
(3.3.29.2) $a_1(f) \in \{-1, 0, 1\}$ *if* $\mathrm{I} - A$ *has a one-dimensional kernel.*
(3.3.29.3) $a_2(f) \in \{0, (-1)^{\nu_+ + 1}\}$ *if* $\mathrm{I} - A$ *is nonsingular and* $\mathrm{I} - A^2$ *has a one-dimensional kernel.*

Proof. As we already mentioned the item (3.3.29.1) is a direct consequence of the Hopf index formula (3.3.5). To prove (3.3.29.2) it is enough to show that $\operatorname{ind}(f, 0) \in \{-1, 0, 1\}$, with this assumption. Let $E_1 := \operatorname{Ker}(\mathrm{Id} - Df(0))$ and E_2 be a complementing linear subspace, i.e. $E_1 \oplus E_2 = \mathbb{R}^m$ and $x = (x_1, x_2)$ coordinates corresponding to this decomposition. Let $\Phi(x_1, x_2) = (\Phi_1(x_1, x_2), \Phi_2(x_1, x_2)) := (x_1, x_2) - f(x_1, x_2) = (x_1 - f_1(x_1, x_2), x_2 - f_2(x_1, x_2))$. Then $\operatorname{ind}(f, 0) = \deg(\Phi, 0)$ and $\partial\Phi_2/\partial x_2(0, 0)$ is nonsingular.

Near 0 we introduce a local change of coordinates

$$
s\colon\mathcal{U} \quad \mathbb{R}^m, \qquad s = (\widetilde{x}_1(x_1, x_2), \widetilde{x}_2(x_1, x_2)),
$$

where $\widetilde{x}_1 = x_1$, $\widetilde{x}_2 = x_2 - f_2(x_1, x_2)$. Then the Jacobi matrix at $(0,0)$ of this change of coordinates has the form:

$$\frac{D(\widetilde{x}_1, \widetilde{x}_2)}{D(x_1, x_2)} = \begin{bmatrix} 1 & 0 \\ c & \frac{\partial \Phi_2(0,0)}{\partial x_2} \end{bmatrix}$$

with the determinant equal to $D = \det(\mathrm{I} - \partial f_2(0,0)/\partial x_2) = \pm 1$. Thus $\deg(s, 0) = \operatorname{sgn} D = D$.

We have from the Multiplicativity Property of degree:

$$\deg(\Phi, 0) = \deg(s\Phi s^{-1}, 0) = \deg(s, 0)\deg(\Phi s^{-1}, 0) = D \deg(\Phi s^{-1}, 0).$$

Notice now that the following equality holds: $\Phi s^{-1}(\widetilde{x}_1, \widetilde{x}_2) = (\Phi_1 s^{-1}(\widetilde{x}_1, \widetilde{x}_2), \widetilde{x}_2)$ and for each fixed $\widetilde{x}_2$, $\Phi_1 s^{-1}(\widetilde{x}_1, \widetilde{x}_2)$ is a self-map of the real line.

By the assumption 0 is isolated in the set $\{x : \Phi(x) = 0\}$. On a small enough neighbourhood $\mathcal{V}$ of 0, we define homotopy h_t by the formula: $h_t(\widetilde{x}_1, \widetilde{x}_2) = ((1-t)\Phi_1 s^{-1}(\widetilde{x}_1, \widetilde{x}_2) + t\Phi_1 s^{-1}(\widetilde{x}_1, 0), \widetilde{x}_2)$. There is $h_0 = \Phi s^{-1}$ and $h_1(\widetilde{x}_1, \widetilde{x}_2) = (\Phi_1 s^{-1}(\widetilde{x}_1, 0), \widetilde{x}_2)$. As the zeros of h_t lie only on the line $\widetilde{x}_2 = 0$, we see that $0 \notin h_t(\partial\mathcal{V})$. Thus, by homotopy invariance of degree, we get $\deg(\Phi s^{-1}, 0) = \deg(h_1, 0) = \deg((\Phi_1 s^{-1}(\widetilde{x}_1, 0), \widetilde{x}_2), 0)$. On the other hand the product formula for degree gives

$$\deg(\Phi s^{-1}, 0) = \deg(\Phi_1 s^{-1}(\widetilde{x}_1, 0), 0) \in \{-1, 0, 1\}$$

because $\Phi_1 s^{-1}(\widetilde{x}_1, 0)$ is a one-dimensional map. Finally we obtain $\deg(\Phi, 0) = D\deg(\Phi_1 s^{-1}(\widetilde{x}_1, 0), 0) \in \{-1, 0, 1\}$, which ends the proof of (3.3.29.2).

To prove (3.3.29.3) assume that $1 \notin \sigma(Df(0))$ and -1 is the eigenvalue of $Df(0)$ with multiplicity 1. Then $c_1 = \operatorname{ind}(f, 0) = (-1)^{\sigma_+}$ by (3.3.29.1), and $\operatorname{ind}(f^2, 0) \in \{-1, 0, 1\}$ by (3.3.29.2). On the other hand, if σ_- is even then $\operatorname{ind}(f^2, 0) = c_1 \mathrm{reg}_1(2) + c_2 \mathrm{reg}_2(2)$. Consequently, $c_2 = (1/2)(\operatorname{ind}(f^2, 0) - \operatorname{ind}(f, 0))$ and by Dold's relations either c_2 is equal to 0 or $-\operatorname{ind}(f, 0) = (-1)^{1+\sigma_+} = (-1)^{1+\sigma_+ + \sigma_-}$. If σ_- is odd then $\operatorname{ind}(f^2, 0) = c_1 \mathrm{reg}_1(2) + (c_2 - c_1)\mathrm{reg}_2(2)$, thus $c_2 = (1/2)(\operatorname{ind}(f^2, 0) + \operatorname{ind}(f, 0))$ and either c_2 is equal to 0 or $c_2 = (-1)^{\sigma_+} = (-1)^{1+\sigma_+ + \sigma_-}$. □

Now we describe all possible k-periodic expansions of the local fixed point index of smooth maps of the plane (cf. [BaBo]).

(3.3.30) THEOREM. *Suppose that* $f\colon \mathbb{R}^2 \supset \mathcal{U} \quad \mathbb{R}^2$, $0 \in \mathcal{U}$, *is a* C^1*-map such that* 0 *is its isolated periodic point. Then the* k*-periodic expansion of the local fixed index of* f *is one of the following forms:*

(3.3.30.1) $\mathcal{I}(f, 0) = c\,\mathrm{reg}_1$, $c \in \mathbb{Z}$,

(3.3.30.2) $\mathcal{I}(f,0) = \mathrm{reg}_1 + c\,\mathrm{reg}_m$, $c \in \mathbb{Z}$, $m \in \mathbb{N}$,
(3.3.30.3) $\mathcal{I}(f,0) = c\,\mathrm{reg}_2$, $c \in \mathbb{Z}$,
(3.3.30.4) $\mathcal{I}(f,0) = -\mathrm{reg}_1 + c\,\mathrm{reg}_2$, $c \in \mathbb{Z}$.

Moreover, each type of the above periodic expansions is realized as the k-periodic expansion of a local polynomial diffeomorphism.

Proof. First remark that for $A = Df(0)$ the set $\Delta(A)$, (see the definition before Proposition (3.2.31) is one of the following:

(1) $\Delta = \emptyset$,
(2) $\Delta = \{1\}$,
(3) $\Delta = \{2\}$,
(4) $\Delta = \{1, 2\}$, or
(5) $\Delta = \{d\}$, where $d \geq 3$.

In the first two cases $\mathcal{O} = \{1\}$ and Theorem (3.2.21) leads to the two cases: either $\mathcal{I}(f,0) = c_1\mathrm{reg}_1$ if ν_- is even, or $\mathcal{I}(f,0) = c_1(\mathrm{reg}_1 - \mathrm{reg}_2)$ if ν_- is odd. The first case is covered by (3.3.30.1). If ν_- is odd, thus equal to 1 here, we have $c_1 = \mathrm{ind}\,(f,0) \in \{-1,0,1\}$ by Proposition (3.3.29.2). Consequently we get $\mathcal{I}(f,0) = c_1(\mathrm{reg}_1 - \mathrm{reg}_2)$, where $c_1 \in \{-1,0,1\}$. If $c_1 = 0$, then we have case (3.3.30.1) with $c = 0$, if $c_1 = 1$ then (3.3.30.2) with $m = 2$ and $c = -1$, if $c_1 = -1$ then (3.3.30.4) with $c = 1$.

In cases (3) and (4) we have $\mathcal{I}(f,0) = c_1\mathrm{reg}_1 + c_2\mathrm{reg}_2$ for ν_- even. Since $-1 \in \sigma(A)$, 1 can be at most a simple eigenvalue. Using once more Proposition (3.3.29.2) we get $c_1 \in \{-1,0,1\}$. Then the expansion is one of the cases (3.3.30.4), (3.3.30.3), or (3.3.30.2) correspondingly, with $c = c_1$ and $m = 2$ in (3.3.30.2), respectively. If ν_- is odd, thus equal to 1, then $\mathcal{I}(f,0) = c_1(\mathrm{reg}_1 - \mathrm{reg}_2) + c_2\,\mathrm{reg}_2$. On the other hand $1 \notin \sigma(A)$ and $\nu_+ = 0$, then by the dimension argument and consequently, $c_1 = \mathrm{ind}\,(f,0) = (-1)^{\nu_+} = 1$. Furthermore, $\mathrm{I} - A^2$ has one-dimensional kernel then and by Proposition (3.3.29.3) we get $c_2 \in \{0, (-1)^2\} = \{0,1\}$. If $c_2 = 0$, then we have the case (3.3.30.4) with $c = 1$. If $c_2 = 1$, then we have the case (3.3.30.4) with $c = 2$.

Finally, in case (5) the set $\sigma(A)$ consists of one not real root of unity of degree $d \geq 3$. Consequently $c_1 = \mathrm{ind}\,(f,0) = 1$, and then $\nu_- = 0$. By Theorem (3.2.21) $\mathcal{I}(f,0) = \mathrm{reg}_1 + c_d\mathrm{reg}_d$ which is the case (3.3.30.2) with $c = c_d$.

We are left with the task to show examples of maps giving particular expansions at $x = 0$. We shall use the complex coordinate on the plane.

(a) If $c \in \{-1,0,1\}$, then it reduces to the line maps and examples of local maps are given in Proposition (3.3.28). If $c \geq 2$, then $f(z) = \bar{z} + z^c$ gives this expansion at $z = 0$. Similarly, if $c \leq -2$, then $f(z) = z + (\overline{z})^c$ has this expansion.

(b) If $c \geq 1$, then $f(z) = \exp(2\pi\imath/m)z + z^{cm+1}$. If $c = -1$ then it once more reduces to a line map – the second of the statement of Proposition (3.3.28). If $c \leq -1$, $m \geq 3$ or $c \leq -2$, $m = 2$, then $f(z) = \exp(2\pi\imath/m) + (\overline{z})^{cm+1}$.

(c) If $c \geq 1$, then $f(z) = \overline{z} + (\overline{z})^{2c}$. If $c \leq -1$, then $f(z) = \overline{z} + z^{2|c|}$.

(d) The case $c = 1$ has a realization as the hyperbolic point of the map. If $c \geq 2$, then $f(z) = \overline{z} + \overline{z}^{-1+2c}$; if $c \leq -1$, then $f(z) = \overline{z} + z^{|-1+2c|}$. □

3.4. Global cohomology conditions

All the previous sections of this chapter presented theorems on the existence of periodic points based on Theorem (3.0.1), which used assumptions about the analytical, and local structure of a map, e.g. smooth, holomorphic, transversal. These assumptions provided a good behaviour of the sequence of the local fixed point index of such a map. In all the mentioned situations employing the Lefschetz–Hopf formula of Theorem (3.0.1) we did not use any special information about the behaviour of the sequence of Lefschetz numbers $\{L(f^m)\}_1^\infty$ of the iterations besides the general theory presented in Sections 3.1–3.3. The exception is Proposition (3.2.51) but we are able only to give a few particular examples of spaces for which its hypothesis is satisfied (see Example (3.2.15) and Exercises (3.2.52), (3.2.53)). It is desirable to find any class of space for which the sequence $\{L(f^m)\}_1^\infty$ has some additional regularity.

There are a natural classes of spaces (compact nilmanifolds, NR-solvmanifolds) which contain the tori and which we will discuss in Chapter VI in detail. From our point of view they have the property that for every self-map there exists an integral matrix A such that $L(f^m) = \det(\mathrm{I} - A^m)$.

Haibao Duan [Dua] observed that the mentioned above formula for the Lefschetz number of the iterated map holds for self-maps of so-called rational exterior spaces.

3.4.1. Lefschetz numbers for maps on rational exterior spaces. We now briefly sketch the main result of Duan's paper [Dua].

As previously we assume that the space X in the problem is a compact ENR, however the properties of X discussed below are the algebraic requirements on $H(X;\mathbb{Q})$ and could be imposed for more general spaces.

For a given space X and an integer $r \geq 0$, let $H^r(X;\mathbb{Q})$ be the r-th singular cohomology space with rational coefficients. Let $H^*(X;\mathbb{Q}) = \bigoplus_{r=0}^{s} H^r(X;\mathbb{Q})$ be the cohomology algebra with multiplication given by the cup product (cf. [Sp] for the definition of the cup product and cohomology algebra). An element $x \in H^r(X;\mathbb{Q})$ is *decomposable* if there are pairs $(x_i, y_i) \in H^{p_i}(X;\mathbb{Q}) \times H^{q_i}(X;\mathbb{Q})$ with $p_i, q_i > 0$, $p_i + q_i = r > 0$ so that $x = \sum x_i \cup y_i$. Let $A^r(X) = H^r(X)/D^r(X)$, where D^r is the linear subspace of all decomposable elements. For a continuous map $f\colon X \to X$, let f^* be the induced homomorphism on cohomology and $A(f)$ the induced homomorphism on $A(X) := \bigoplus_{r=0}^{s} A^r(X)$.

(3.4.1) Definition. Let f be a self-map of a space X and let $\mathrm{I}\colon A(X) \to A(X)$ be the identity homomorphism. The polynomial

$$A_f(t) = \det(t\mathrm{I} - A(f)) = \prod_{r\geq 1} \det(t\mathrm{I} - A^r(f))$$

will be called the *cohomology characteristic polynomial* of f. The zeros of this polynomial: $\lambda_1(f), \dots, \lambda_k(f)$, $k = \operatorname{rank} X$, where $\operatorname{rank} X$ is the dimension of $A(X)$ over $\mathbb{Q}$, will be called the *quotient eigenvalues* of f.

We have the following theorem proved in [Dua].

(3.4.2) Theorem. *If f is a self-map of a space X, then $A_f(t) \in Z[t]$, i.e. is a polynomial with integral coefficients. Moreover, if $\dim A^r(X)$ is either 1 or 0 for all $r \geq 1$, then the quotient eigenvalues $\lambda_1(f), \dots, \lambda_k(f)$ are all integers and $A_f(t) = \prod_{i=1}^{k}(t - \lambda_i(f))$.*

Now we introduce the class of rational exterior spaces.

(3.4.3) Definition. A connected topological space X is called *rational exterior* if there are some homogeneous elements $x_i \in H^{\mathrm{od}}(X;\mathbb{Q})$, $i = 1, \dots, k$, such that the inclusions $x_i \hookrightarrow H^*(X;\mathbb{Q})$ give rise to a ring isomorphism

$$\Lambda_{\mathbb{Q}}(x_1, \dots, x_k) = H^*(X;\mathbb{Q}),$$

where Λ denotes the exterior algebra generated by $\{x_i\}_1^k$ (cf. [La]). Additionally if the set $\{x_i\}_{i=1}^k$ can be ordered so that $\dim x_1 < \dots < \dim x_k$, we call X a *simple rational exterior space.* The number k is called the *rank of rational exterior space.*

The rational exterior spaces are a wide class of spaces that encompass: finite H-spaces (see [Sp] for a definition of the H-space), including all finite-dimensional Lie groups and some real Stiefel manifolds (cf. [Wh] for the definition), and spaces that admit a filtration of the form

$$X = X_0 \xrightarrow{p_0} X_1 \xrightarrow{p_1} \cdots \xrightarrow{p_{k-1}} X_k \xrightarrow{p_k} X_{k+1} = \{\text{point}\}$$

where p_i is the projection of an odd-dimensional sphere bundle (cf. [Dua]).

The Lefschetz number for self-maps of a rational exterior space can be expressed in terms of quotient eigenvalues. It was also shown in [Dua]. The proof is not difficult but it uses the property of cup product in the cohomology algebra. Since we have not discussed it we skip the proof, referring the reader to the quoted paper.

(3.4.4) THEOREM. *Let f be a self-map of a rational exterior space and $A_f(t)$ be the cohomology characteristic polynomial of f. Then $L(f) = A_f(1)$.*

We can repeat the construction of $A(f)$, given at the beginning of this section, for cohomology with integer coefficients. Consider the cohomology group $H^r(X;\mathbb{Z})$ and its subgroup $B^r(X;\mathbb{Q})$ generated by all r-dimensional decomposable elements. Define $\widetilde{A}^r(X) = H^r(X)/B^r(X)$, $r > 0$. Let $\widetilde{A}(f)$ be the homomorphism induced by f on $\widetilde{A}(X) = \bigoplus_{r=0}^s \widetilde{A}^r(X)$, and $\widetilde{A}_f(t)$ be the cohomology characteristic polynomial of f on $\widetilde{A}(X)$. Then (cf. [Dua, Lemmas 4.2 and 4.3]) $\widetilde{A}^r(X)$ is a free $\mathbb{Z}$-module, $\mathrm{rank}_{\mathbb{Z}}\widetilde{A}^r(X) = \dim_{\mathbb{Q}} A^r(X)$ and $A_f(t) = \widetilde{A}_f(t)$. As a consequence we get

(3.4.5) THEOREM. *Let f be a self-map of a rational exterior space, and let $\lambda_1, \dots, \lambda_k$ be the quotient eigenvalues of f. Let A denote the integral matrix of $\widetilde{A}(f)$. Then $L(f^m) = \det(\mathrm{I} - A^m) = \prod_{i=1}^k (1 - \lambda_i^m)$.*

3.4.2. Periodic points of smooth maps of rational exterior powers. The algebraic properties of the sequence $\{\det(\mathrm{I} - A^m)\}_{m=1}^{\infty}$, where A is an integral square matrix, are described in Section 6.4 of Chapter VI. There we present the theory developed by Jiang and Llibre [JiLb] for the study of periodic points of self-maps of tori. The topological part of their work, and its generalization (cf. Chapter VI) is based on the fact that for self-maps of tori we have: $|L(f^m)| = N(f^m) \geq 0$, where $N(f^m)$ is the Nielsen number (cf. Definition (4.1.2)) of f^m which yields more information about the periodic points of f. Nevertheless the algebraic and combinatorics considerations and statements are independent of the topological meaning of the notion, thus can be used to study the algebraic periods of a self-map of rational exterior space. It was observed by G. Graff in [GrI].

For a square matrix $A \in M_{d\times d}(\mathbb{Z})$ we define $T_A := \{m \in \mathbb{N} : \det(\mathrm{I} - A^m) \neq 0\}$ (cf. Definition (6.4.1)).

Let ρ be the spectral radius of A. Following the Jiang and Llibre result (see the proof of Theorem (6.4.4) case (G)), there exists $m_0(d)$ such that for every $A \in M_{d\times d}(\mathbb{Z})$ with $\rho > 1$ and all $m, n \in T_A$ with $n|m$, $m > m_0(r)$ we have

$$|\det(\mathrm{I} - A^m)|/|\det(\mathrm{I} - A^n)| > 1.$$

For a discussion of the effectiveness of derivation of the constant $m_0(d)$ see Section 6.4, e.g. Problem (6.4.24). Moreover for every such m we have (see Theorem (6.4.5), Remark (6.4.6))

$$\sum_{k|m} \mu(m/k)\det(\mathrm{I} - A^k) \neq 0.$$

As an application we get the following information about the behaviour of the Lefschetz number of iteration of a self-map f of a rational exterior space, and consequently also about the algebraic multiplicities $a_k(f)$ of the sequence $\{L(f^m)\}$ and the algebraic periods of f.

(3.4.6) PROPOSITION. *Suppose that $f\colon X \to X$ is a continuous map of a rational exterior power space of rank d such that the sequence $\{L(f^m)\}$ is unbounded. Then there exists a number $m_0(d)$ (i.e. depending on d only) such that for every $m \in T_{A_f}$, $m > m_0(d)$,*

$$i_m(f) = \sum_{k|m} \mu(m/k) L(f^k) = \sum_{k|m} \mu(m/k) \det(\mathrm{I} - A_f^k) \neq 0.$$

Consequently, for every $m > m_0(d)$, $a_m(f) \neq 0$ and m is an the algebraic period of f.

Note that now we can apply Proposition (3.4.6) to any theorem on the existence of periodic points of previous sections which based on the supposition that the periodic coefficient $a_m(f)$ is nonzero.

We include only some of them. First note that the set $T_A := \{m \in \mathbb{N} : \det(\mathrm{I} - A^m) \neq 0\}$ consists of all multiplicities of all d_j, where d_j, is the degree of a root unity $\varepsilon_j \in \sigma(A)$. Consequently T_A contains only a finite number of prime numbers. This leads to the following corollary of Proposition (3.4.6) and Corollary (3.2.49).

(3.4.7) COROLLARY. *Let X be a rational exterior power space of rank d which is a closed manifold and $f\colon X \to X$ be a C^1-map such that $\{L(f^m)\}$ is unbounded.*

Then the set $\mathrm{Per}(f)$ *contains infinitely many primes. More precisely, it contains all primes contained in the set $T_{A_f} \setminus [1, m_0(d)]$.*

Now we show that the estimate of the number of periodic points for C^1 self-map $f\colon X \to X$ given in Theorem (3.2.41) is independent of f if the manifold X is a rational exterior space (cf. [GrI]).

(3.4.8) THEOREM. *Suppose that X is a compact manifold which is rational exterior power of rank d. Let $f\colon X \to X$ be a C^1-map such that the sequence $\mathcal{L} = \{L(f^m\}$ of the Lefschetz number of iterations is unbounded. Then there exists $n_0(d) \in \mathbb{N}$ such that for every $m \geq n_0(d)$,*

$$\#\mathrm{Or}(f, m) \geq \frac{(m - n_0)}{D \cdot 2^{[(d+1)/2]}},$$

where $D = \dim_{\mathbb{Q}} H_*(X; \mathbb{Q})$ and $[(d+1)/2]$ is the integral part of $(d+1)/2$.

PROOF. We follow the proof of Theorem (3.2.41). First remember that to get inequality (3.2.42) we used Corollary (3.1.59). We retain the first constant appearing in the consideration of this inequality, namely $n(f) \in \mathbb{N}$ (equal to $\Upsilon(f) - 1$), and estimate $n(f) \leq \dim_{\mathbb{R}} H_*(X) := D$ as we did previously. This showes that in any interval $m \leq i \leq m + n(f)$ there exists at least one i such that $a_i(f) \neq 0$, provided m is sufficiently large i.e. $m > n_0(f)$. Thus the second $n_0(f)$ describes the range of the inequality. Note that X is a rational exterior power, thus algebraic periods are contained in T_A. Moreover the theorem provides a constant $m_0(d)$ such that every $m \in T_{A_f}$ and $m > m_0(d)$ is an algebraic period, because $\limsup |L(f^m)| = \infty$. This leads to the estimate (with $n_0(d) = m_0(d)$)

$$\#\mathcal{A}(f,m) \geq \frac{m - n_0(d)}{D}, \quad \text{for } m \geq n_0. \tag{3.4.9}$$

The remaining part of the proof is the same as that of Theorem (3.2.41). $\square$

Note that since the set $\mathcal{A}(f,m) \setminus [1, m_0(d)]$ is equal to the $T_{A_f}(f) \setminus [1, m_0(d)]$, asymptotically its cardinality $\#\mathcal{A}(f,m)$ can be estimated by $m - m_0(d)$ multiplied the lower density $\mu(T_{A_f})$ (cf. Remark (3.1.60)). Moreover, here we can use the natural density instead of the lower density.

(3.4.10) Lemma. *Let $d_1, \dots, d_s$ be all distinct degrees of roots of unity belonging to $\sigma(A)$. Then the natural density of the set T_A is well-defined and we have*

$$\mu(T_A) := \lim_{m \to \infty} \frac{\#\{T_A \cap [1,m]\}}{m} \geq \prod_{j=1}^{s} \left(1 - \frac{1}{d_j}\right).$$

Proof. Recall that $T_A = \mathbb{N} \setminus \bigcup_{j=1}^{s} (d_j)$, where (d_j) is the principal ideal consisting of all multiplicities of d_j. Since the natural density of the principal ideal is $\mu(d_j) = 1/d_j$, we get $\mu(\mathbb{N} \setminus (d_j)) = 1 - (1/d_j)$. Since $\mathbb{N} \setminus \bigcup_{j=1}^{s} (d_j) = \bigcap_{j=1}^{s} (\mathbb{N} \setminus (d_j))$, we get the statement, because $\mu(S_1 \cap S_2) \geq \mu(S_2)\mu(S_2)$. $\square$

Note that the estimate of Lemma (3.4.10) is not sharp.

(3.4.11) Exercise. Derive the value of natural density $\mu(T_A)$ of the set T_A.

This let us to get rid the factor D from the denominator of (3.4.9). The density of T_f can be computed effectively (cf. Exercise (3.4.11)). It depends on A, but on the other hand is essentially larger then the factor $1/D$. Consequently we get the following asymptotic estimate of the number of periodic orbits of a C^1-map of rational exterior power manifold.

(3.4.12) Proposition. *Suppose that X and $f \colon X \to X$ are as in* (3.4.8). *Then*

$$\lim_{m \to \infty} \frac{\#\mathrm{Or}(f,m)}{m} \geq \frac{\mu(T_{A_f})}{2^{[(d+1)/2]}} \geq \frac{1}{2^{[(d+1)/2]}} \prod_{j=1}^{s} \left(1 - \frac{1}{d_j}\right).$$

3.4.3. Rational Hopf spaces. Note that Theorem (3.4.5) does not cover the cases when the generators of $H^*(X;\mathbb{Q})$ are in even-dimensional cohomology, so it does not embrace the case of S^{2d} and other similar spaces. However, it is possible to extend Duan's method to find a formula for the Lefschetz number for a wider class of spaces as it was observed in [GrI].

(3.4.13) DEFINITION. A connected topological space X is called a *simple rational Hopf space* if there are homogeneous elements $x_i \in H^{\mathrm{od}}(X;\mathbb{Q})$, $y_j \in H^{\mathrm{ev}}(X;\mathbb{Q})$, $i=1,\dots,k$, $j=1,\dots,l$, such that the inclusions $x_i \hookrightarrow H^*(X;\mathbb{Q})$, $y_j \hookrightarrow H^*(X;\mathbb{Q})$ give rise to an algebra isomorphism $H_{\mathbb{Q}}(x_1,\dots,x_k,y_1,\dots,y_l) = H^*(X;\mathbb{Q})$, where $H_{\mathbb{Q}}$ is the free skew-commutative graded algebra with the additional relations $y_j^{d_j+1}=0$, and the set $\{z_i\}_{i=1}^{k+l} = \{x_i\}_{i=1}^{k} \cup \{y_j\}_{i=1}^{l}$ can be ordered so that $\dim z_1 < \dots < \dim z_{k+l}$.

Let $1 \in H^0(X;\mathbb{Q})$ be the unit cocycle. Then $\{x_i\}_{i=1}^{k} \cup \{y_j\}_{i=1}^{l}$ is a vector space basis for $A(X)$ and $B=\{1, x_{i_1}\cup\dots\cup x_{i_n}\cup y_{j_1}^{p_{j_1}}\cup\dots\cup y_{j_m}^{p_{j_m}} : 1\le i_1<\dots<i_n\le k,\ 1\le j_1<\dots<j_m\le l,\ 1\le p_{j_t}\le d_{j_t}\}$ is a vector space basis for $H^*(X;\mathbb{Q})$. We will use the following notation: $D=k+\sum_{j=1}^{l} d_j$, $\dim\lambda_i = p$ if $A(f)(z_i)=\lambda_i z_i$ and $z_i \in A^p(X)$. The following theorem is a consequence of computation by [Dua] (cf. [GrI]).

(3.4.14) THEOREM. *If f is a self-map of a simple rational Hopf space X with the nonzero quotient eigenvalues $\lambda_1,\dots,\lambda_k$ having odd-dimensional eigenvectors and $\lambda_{k+1},\dots,\lambda_{k+l}$ having even-dimensional eigenvectors, then*

$$L(f^m) = 1+\dots+(-1)^{\sum_{r=1}^{s}\dim\lambda_{g_r}}(\lambda_{g_1}\dots\lambda_{g_s})^m+\cdots \\ +(-1)^D\left(\lambda_1\dots\lambda_k\lambda_{k+1}^{d_{k+1}}\dots\lambda_{k+l}^{d_{k+l}}\right)^m,$$

where the sum extends over all $1\le g_1,\dots,g_s\le k+l$ such that if $g_{t_1}=\dots=g_{t_w}$ then, $\dim\lambda_{t_j}$ is even and $d_{t_w}\le w$.

Now we point out that the even-dimensional spheres and projective spaces discussed in Example (3.2.15) and Exercise (3.2.52) are simple rational Hopf space

(3.4.15) EXAMPLE. The following spaces are simple rational Hopf spaces.

(3.4.15.1) The sphere $X=S^{2d}$, and then $L(f^m)=1+r^m$, where $r=\deg f$.

(3.4.15.2) The complex projective space $X=\mathbb{C}\mathrm{P}^{\mathrm{d}}$. It is known (cf. [Sp]) that $H^*(\mathbb{C}\mathrm{P}^{\mathrm{d}};\mathbb{Q}) = \mathrm{Span}\,\{1,y,y^2,\dots,y^d\}$ where $0\ne y\in H^2(\mathbb{C}\mathrm{P}^{\mathrm{d}};\mathbb{Q})$. Consequently if $a_f\in\mathbb{Z}$ is defined by $a_f y = H^2(f)(y)$, then $L(f^m)=1+r^m+r^{2m}+\dots+r^{dm}$ (cf. Exercise (3.2.53)).

(3.4.15.3) The product $X=S^q\times S^q$, where q is even. Then $H^n(X;\mathbb{Q})=\mathbb{Q}$ if $n=0,\,2q$, $H^n(X;\mathbb{Q})=\mathbb{Q}\oplus\mathbb{Q}$ if $n=q$, and $H^n(X;\mathbb{Q})=0$ otherwise. Moreover, multiplicatively the cohomology algebra is generated by y_1, y_2 being generators of $H^q(X;\mathbb{Q})$ and $y_1 \cup y_2$ is the generator of $H^{2q}(X;\mathbb{Q})$. Consequently $L(f^m) = 1 + \lambda_1^m + \lambda_2^m + (\lambda_1\lambda_2)^m$, where λ_1, λ_2 are the eigenvalues of f^* on $H^q(X;\mathbb{Q})$.

By a combination of methods used in Subsection 3.1.3 of this chapter and the number theory estimates of [JiLb] (see the proof of Theorem (6.4.4)) it is possible to show that Theorem (3.4.5) extends onto the case of simple rational Hopf space (cf. [GrI]). We skip its proof.

(3.4.16) THEOREM. *Let X be a simple rational Hopf space of rank d, $f\colon X \to X$ be a map of X and A_f the integral matrix associated with f Then there exists a number $m_0(d)$ which depends only on the rank of space X, such that for every self-map f of X with unbounded $\{L(f^m)\}$, each $m \in T_{A_f}$ with $m > m_0(d)$ is an algebraic period of f.*

(3.4.17) COROLLARY. *The statements of Corollary* (3.4.7) *and Theorem* (3.4.8) *hold for the simple rational Hopf spaces.*

CHAPTER IV

NIELSEN FIXED POINT THEORY

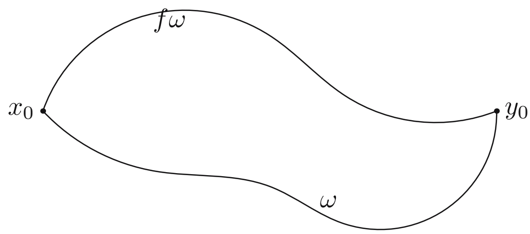
4.1. Nielsen and Reidemeister numbers

The Lefschetz Fixed Point Theorem gives a sufficient condition for the existence of fixed points of the considered $f\colon X \to X$ and any map homotopic to it. It is natural to ask whether this condition is also necessary. More exactly: does $L(f) = 0$ imply the possibility to deform f to a fixed point free map?

The answer is in general negative even if we confine ourselves to such fine spaces as manifolds. We will see that, roughly speaking, the fixed point set splits into the sum $\mathrm{Fix}(f) = A_1 \cup \cdots \cup A_n$ of mutually disjoint subsets (Nielsen classes) which may change but can not glue during any homotopy. Now to remove $\mathrm{Fix}(f)$ means to remove each class separately thus we need to know that the fixed point index of each class disappears. The information that their sum, which equals the Lefschetz number, is zero may be not sufficient. The investigation of the above splitting and its behaviour during homotopies is the subject of the Nielsen fixed point theory.

This theory was initiated by Jakob Nielsen in 1921 [Nie] by the observation that every self-map of the two-dimensional torus $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ has at least $|\det(\mathrm{I} - A)|$ fixed points (here I, $A \in \mathcal{M}_{2\times 2}(\mathbb{Z})$ are respectively the identity matrix and the matrix representing the induced homotopy homomorphism $f_\#$ of $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$). Let us emphasize that at this time neither the fixed point index nor the Lefschetz number were known. This confined the domain of Nielsen's research to surfaces: the covering maps from (Euclidean and non-Euclidean) planes onto the surfaces were used to split $\mathrm{Fix}(f)$ into classes. Later the notion of the fixed point index allowed extension of the ideas of Nielsen onto larger classes of spaces: polyhedra (Heinz Hopf, Kurt Reidemeister, Franz Wecken in the 1930s) and then onto neighbourhood retracts ENRs, (Solomon Lefschetz). For a long time the approach of Hopf and Reidemeister (using homotopy of paths) was dominated the literature [Br2]. For the last twenty years, many papers have used covering spaces for the definition of basic notions of Nielsen theory. Since both approaches have their advantages, we show that they are equivalent and then we will be able to use them alternatively.

4.1.1. Nielsen and Reidemeister relations. Let us consider a self-map $f\colon X \to X$ of a topological space X. We define the *Nielsen relation* on $\mathrm{Fix}(f)$. Two points $x, y \in \mathrm{Fix}(f)$ are *Nielsen related* if and only if there is a path $\omega\colon [0,1] \to X$ satisfying $\omega(0) = x$, $\omega(1) = y$ and the paths ω, $f\omega$ are fixed end point homotopic, i.e. there is a map $h\colon I \times I \to X$ satisfying $h(t,0) = \omega(t)$, $h(t,1) = f\omega(t)$, $h(0,s) = x$, $h(1,s) = y$. This is an equivalence relation, hence $\mathrm{Fix}(f)$ splits into disjoint *Nielsen classes*.



Let us notice that if X is path-connected and simply-connected then all fixed points are Nielsen related. In fact any two points $x, y \in \mathrm{Fix}(f)$ can be joined by a path ω. Then the paths ω, $f\omega$ are homotopic in X, hence x and y are Nielsen related. In this chapter we assume that the considered spaces are compact connected ENRs, unless otherwise stated. Then

(4.1.1) Lemma. *The Nielsen relation is locally constant (i.e. every fixed point x admits a neighbourhood $\mathcal{V}$ x such that all fixed points from $\mathcal{V}$ are Nielsen related to x).*

Proof. We recall that ENR is locally contractible (i.e. every point admits an arbitrarily small contractible neighbourhood). Let $x \in \mathrm{Fix}(f)$. We fix contractible neighbourhoods $\mathcal{U}, \mathcal{V}$ satisfying $x \in \mathcal{V}$ and $f(\mathcal{V}) \subset \mathcal{U}$. We may assume that $\mathcal{V}$ is path-connected. Then every $x' \in \mathrm{Fix}(f) \cap \mathcal{V}$ is Nielsen related to x. In fact we may join these points with a path $\omega\colon [0,1] \to \mathcal{V}$ and then the paths ω, $f\omega$ are homotopic in the contractible set $\mathcal{U}$. □

By Lemma (4.1.1) every Nielsen class is open in $\mathrm{Fix}(f)$. Since $\mathrm{Fix}(f)$ is closed in the compact space X, the number of Nielsen classes is finite $\mathrm{Fix}(f) = \mathbb{A}_1 \cup \ldots \cup \mathbb{A}_n$ hence each class is compact. Thus the fixed point index $\mathrm{ind}(f; \mathbb{A}_i)$ of each class is defined. A class $\mathbb{A}$ is called *essential* if and only if $\mathrm{ind}(f; \mathbb{A}) \neq 0$. Otherwise $\mathbb{A}$ is called *inessential*.

(4.1.2) Definition. The number of essential classes is called the *Nielsen number*. We denote this number by $N(f)$.

(4.1.3) Example.

(4.1.3.1) If f is a constant map, then $N(f) = 1$.

(4.1.3.2) If X is simply-connected, then for arbitrary f,

$$N(f) = \begin{cases} 1 & \text{if } L(f) \neq 0, \\ 0 & \text{if } L(f) = 0. \end{cases}$$

(4.1.3.3) For every compact connected ENR space X,

$$N(\mathrm{id}_X) = \begin{cases} 1 & \text{if } \chi(X) \neq 0, \\ 0 & \text{if } \chi(X) = 0. \end{cases}$$

To verify the above formulae it is enough to notice that in each case there is at most one Nielsen class. Moreover $\chi(X) = \mathrm{L(id)}$.

Now we will show a map with a nontrivial Nielsen number.

(4.1.4) EXAMPLE. Let $f\colon S^1 \to S^1$ be the *flip map* $f(z) = \overline{z}$. We will show that $N(f) = 2$. We notice that $\mathrm{Fix}(f) = \{-1, +1\}$ and the fixed point index at each of these points equals $+1$ (since near each of these points f is locally of the form $f(t) = -t$). It remains to show that the points -1, $+1$ are not Nielsen related. Let $\omega\colon I \to S^1$ be a path from $+1$ to -1. We will show that ω is not homotopic to $f\omega = \overline{\omega}$. We recall that two paths (with common ends) in S^1 are homotopic if and only if their lifts to the covering $p\colon \mathbb{R} \to S^1$, $p(t) = \exp(2\pi\imath t)$, starting from the same point have common ends. Let $\widetilde{\omega}\colon [0,1] \to \mathbb{R}$ be the lift of ω starting from $\widetilde{\omega}(0) = 0$. Then $\omega(1) = -1$ implies $\widetilde{\omega}(1) = (1/2) + k$ for a $k \in \mathbb{Z}$. On the other hand $-\widetilde{\omega}(t)$ is the lift of $\overline{\omega}(t)$. Thus $-\widetilde{\omega}(1) = -(1/2) - k \neq (1/2) + k = \widetilde{\omega}(1)$ hence the paths ω and $\overline{\omega}$ are not homotopic. □

The above example is a special case of a self-map of the circle S^1. The formula for $N(f)$ of an arbitrary self-map of S^1 will be given in the next section.

During the homotopy $H(x,t) = f_t(x)$ the set $\mathrm{Fix}(f)$ may vary. What is happening then to the Nielsen classes? Let us denote (overusing the notation) $\mathrm{Fix}\, H = \{(x,t) \in X \times I : H(x,t) = x\}$. We define *$H$-relation* on $\mathrm{Fix}\, H$ as follows: $(x,t) \sim (x',t')$ if and only if there is a path $\omega\colon [0,1] \to X$ satisfying $\omega(0) = x$, $\omega(1) = x'$ and the path $s \to H(\omega(s), (1-s)t + st') \in X$ is homotopic (rel ends) to ω. It is easy to check that this is an equivalence relation and splits $\mathrm{Fix}\, H$ into a finite number of compact classes.

(4.1.5) LEMMA. *If $\mathbb{A} \subset \mathrm{Fix}\, H$ is an H-class and $t_0 \in [0,1]$, then the set $\mathbb{A}_{t_0} = \{x \in X : (x,t_0) \in \mathbb{A}\}$ is a Nielsen class of f_{t_0} or is the empty set.*

PROOF. Let $x, x' \in \mathbb{A}_{t_0}$. Then $(x,t_0), (x',t_0) \in \mathbb{A}$, hence there exists a path $\omega\colon I \to X$ from x to x' such that the path $H(\omega(s), t_0) = f_{t_0}\omega(s)$ is homotopic to $\omega(s)$. Thus ω gives the Nielsen relation between $x, x' \in \mathrm{Fix}(f_{t_0})$.

On the other hand if $\omega\colon I \to X$ gives the Nielsen relation between $x, x' \in \mathrm{Fix}(f_{t_0})$, then the path $(\omega(s), t_0)$ gives an H-relation between $(x, t_0), (x', t_0) \in \mathrm{Fix}\, H$. $\square$

(4.1.6) THEOREM. *The Nielsen number is the homotopy invariant, i.e.* $N(f_0) = N(f_1)$ *for homotopic maps* f_0, f_1.

PROOF. We will show a bijection between the sets of essential classes of f_0 and f_1. Let $H(t, s)$ be a homotopy from f_0 and f_1. By the above lemma for every Nielsen class $\mathbb{A}_0 \subset \mathrm{Fix}(f_0)$ there is exactly one H-class $\mathbb{A} \subset \mathrm{Fix}\, H$ containing $\mathbb{A}_0$. Let $\mathbb{A}_1 = \{x \in X : (x, 1) \in \mathbb{A}\}$. Then Lemma (4.1.5) implies that $\mathbb{A}_1$ is a Nielsen class of f_1 or is empty. We notice that $\mathrm{ind}(f_0, \mathbb{A}_0) = \mathrm{ind}(f_1, \mathbb{A}_1)$ by the Homotopy Invariance index property (cf. Lemma (2.2.19). Thus if $\mathbb{A}_0$ is essential then so is $\mathbb{A}_1$, and we get a map from the set of essential classes of f_0 to the set of essential classes of f_1. It remains to notice that the homotopy $H(x, 1-t)$ gives the inverse map. $\square$

(4.1.7) THEOREM. *Every map g homotopic to f has at least $N(f)$ fixed points.*

PROOF. By Theorem (4.1.6) $N(f) = N(g)$. On the other hand the inequality $\#\mathrm{Fix}(g) \geq N(g)$ is evident since each essential Nielsen class of g is nonempty. $\square$

Thus the Nielsen number brings the very important geometric information that in every moment of the deformation of the map f there must be at least $N(f)$ fixed points. Two natural problems appear.

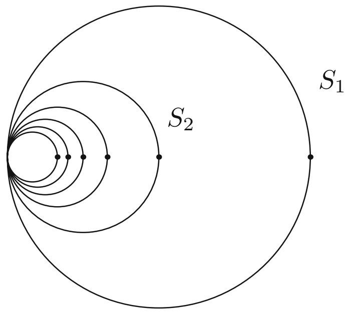
(1) Is $N(f)$ the optimal bound of the number of fixed points in the homotopy class of f? More exactly: does there exist a map f' homotopic to f with $\#\mathrm{Fix}(f') = N(f)$?
(2) How to compute $N(f)$?

The first question was answered positively for a wide class of polyhedra. We discuss this in Section 4.2.1. Unfortunately the effective computation of the Nielsen number seems to be much harder than the computation of the Lefschetz number and relatively few results have been obtained. To find $N(f)$ we have first to find the splitting of $\mathrm{Fix}(f)$ into Nielsen classes and this is usually easier. However the problems arise in finding out which classes are essential.

We end this subsection with an example showing that beyond locally contractible spaces the number of Nielsen classes may be infinite.

(4.1.8) EXAMPLE. Let $S = \bigcup_{k\in\mathbb{N}} S_k$ be the union of circles $S_k = S((1/k, 0), 1/k) \subset \mathbb{R}^2$ and let $f\colon S \to S$ be the flip map $f(z) = \overline{z}$.

Then $\mathrm{Fix}(f) = \{(0,0); (2,0), (1,0), (2/3,0), (2/4,0), \dots, (2/n,0), \dots\}$. First we show that the points $(2,0) \in S_1$ and $x_2 = (1,0) \in S_2$ are not Nielsen related.



Suppose otherwise. Let $\omega\colon I \to S$ be a path joining these points and such that there is a fixed end points homotopy $H_t\colon I \to S$ joining the paths ω and $\overline{\omega}$. Let $r\colon S \to S_1$ be the retraction sending all S_k (for $k \geq 2$) into the point $(0,0)$. Then the homotopy rH_t establishes the Nielsen relation between the two fixed points of the flip map $rf\colon S_1 \to S_1$. This contradicts Example (4.1.4). Similarly one can show that any other two fixed points are not Nielsen related.

Let us notice that S is not ENR since the point $(0,0)$ has no contractible neighbourhood.

4.1.2. Nielsen classes and the universal covering. To define the Nielsen relation we used the homotopy of paths. Since there is the well known connection between the fundamental group and the universal covering of a space, one may expect that the results of the previous section can be also obtained or interpreted by the lifts of the considered maps to the universal coverings (see [Sp] for the definition and information about coverings). In fact, as we have mentioned, Jakob Nielsen in his pioneering works considered the covering of Euclidean or non-Euclidean planes over a torus or other surfaces respectively.

Let us fix a universal covering $p\colon \widetilde{X} \to X$. Let $\mathcal{O}_X = \{\alpha \in \widetilde{X} \to \widetilde{X}; p\alpha = p\}$ denote the group of *deck transformations* of this covering. Any map $f\colon X \to X$ admits a *lift* $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$, i.e. the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \widetilde{X} \arrow[d, "p"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

commutes. The set of all lifts will be denoted by $\mathrm{lift}(f)$. If $\widetilde{f}_0$ is a fixed lift, then the map

$$\mathcal{O}_X \ni \alpha \to \alpha\widetilde{f}_0 \in \mathrm{lift}(f)$$

is the bijection.

(4.1.9) LEMMA. *For every lift* $\widetilde{f}\colon\widetilde{X}\to\widetilde{X}$ *of* f *the set* $p(\mathrm{Fix}(\widetilde{f}))\subset\mathrm{Fix}(f)$ *is either a Nielsen class or is empty. Moreover,* $\mathrm{Fix}(f)=\bigcup_{\widetilde{f}'}p(\mathrm{Fix}(\widetilde{f}'))$ *where* $\widetilde{f}'$ *runs the set of lifts of* f.

PROOF. Let $x,y\in p(\mathrm{Fix}(\widetilde{f}))$. Then $x=p\widetilde{x}$, $y=p\widetilde{y}$ for some $\widetilde{x},\widetilde{y}\in\mathrm{Fix}(\widetilde{f})$. Let $\widetilde{\omega}$ be a path joining $\widetilde{x}$ with $\widetilde{y}$ in $\widetilde{X}$. Since $\widetilde{X}$ is simply-connected, there is a homotopy $\widetilde{H}$ from the path $\widetilde{\omega}$ to $\widetilde{f}\widetilde{\omega}$. The projection $H=p\widetilde{H}$ is a homotopy from the path $\omega=p\widetilde{\omega}$, joining the points $x,y\in\mathrm{Fix}(f)$, to the path $f\omega=p\widetilde{f}\widetilde{\omega}$. Thus the points $x,y\in\mathrm{Fix}(f)$ are Nielsen related.

Now we suppose that $x\in p(\mathrm{Fix}(\widetilde{f}))$ and that $y\in X$ is another fixed point of f Nielsen related to x. We have to show that $y\in p(\mathrm{Fix}(\widetilde{f}))$. Let ω be a path joining x and y and establishing a Nielsen relation. Let $H\colon I\times I\to X$ be a homotopy between ω and $f\omega$. Let $\widetilde{H}$ be the lift of H satisfying $\widetilde{H}(0,s)=\widetilde{x}\in\mathrm{Fix}(\widetilde{f})$ for all $s\in I$. Let us denote $\widetilde{y}=\widetilde{H}(1,0)$. Let us notice that $p\widetilde{y}=y$ and $\widetilde{y}=\widetilde{H}(1,s)$ for each $s\in[0,1]$ (since $\widetilde{H}(1,\cdot\,)$ is the lift of constant path $H(1,\cdot\,)$). It remains to show that $\widetilde{f}\widetilde{y}=\widetilde{y}$. We notice that $\widetilde{H}(\cdot\,,1)$ and $\widetilde{f}\widetilde{H}(\cdot\,,0)$ are lifts of the same path $f\omega$. In fact $p\widetilde{H}(\cdot\,,1)=H(\cdot\,,1)=f\omega$ and $p\widetilde{f}\widetilde{H}(\cdot\,,0)=fp\widetilde{H}(\cdot\,,0)=fH(\cdot\,,0)=f\omega$. Since $\widetilde{f}\widetilde{H}(0,0)=\widetilde{f}\widetilde{x}=\widetilde{x}$ and $\widetilde{H}(0,1)=\widetilde{H}(0,0)=\widetilde{x}$ ($H(0,\cdot\,)$ is a lift of a constant path), the two lifts are equal. Now $\widetilde{f}\widetilde{y}=\widetilde{f}\widetilde{H}(1,0)=\widetilde{H}(1,1)=\widetilde{H}(1,0)=\widetilde{y}$.

To prove that $\mathrm{Fix}(f)=\bigcup_{\widetilde{f}'}\mathrm{Fix}(\widetilde{f}')$ we fix an $x\in\mathrm{Fix}(f)$ and a point $\widetilde{x}\in p^{-1}(x)$. Then there exists a lift $\widetilde{f}'\in\operatorname{lift}f$ satisfying $\widetilde{f}'\widetilde{x}=\widetilde{x}$. Now $x=p\widetilde{x}\in p(\mathrm{Fix}(\widetilde{f}'))$. □

Since Nielsen classes are mutually disjoint, $p(\mathrm{Fix}(\widetilde{f}))$ and $p(\mathrm{Fix}(\widetilde{f}'))$ are either equal or disjoint. The next lemma makes precise when the classes are equal.

(4.1.10) LEMMA. *Let* $\widetilde{f},\widetilde{f}'\in\operatorname{lift}f$. *Then* $p(\mathrm{Fix}(\widetilde{f}))=p(\mathrm{Fix}(\widetilde{f}'))\neq\emptyset$ *implies the existence of an* $\alpha\in\mathcal{O}_X$ *satisfying* $\alpha\widetilde{f}=\widetilde{f}'\alpha$.

PROOF. By the assumption there exist $\widetilde{x}\in\mathrm{Fix}(\widetilde{f})$ and $\widetilde{x}'\in\mathrm{Fix}(\widetilde{f}')$ satisfying $p\widetilde{x}=p\widetilde{x}'$. Let $\alpha\in\mathcal{O}_X$ satisfy $\alpha(\widetilde{x})=\widetilde{x}'$. Then $\widetilde{f}'\alpha(\widetilde{x})=\widetilde{f}'\widetilde{x}'=\widetilde{x}'=\alpha(\widetilde{x})=\alpha(\widetilde{f}\widetilde{x})$. Since the lifts $\alpha\widetilde{f}$, $\widetilde{f}'\alpha$ coincide at the point $\widetilde{x}$, they are equal. □

(4.1.11) LEMMA. *Let* $\widetilde{f}'=\alpha\widetilde{f}\alpha^{-1}$ *for an* $\alpha\in\mathcal{O}_X$. *Then* $p(\mathrm{Fix}\widetilde{f})=p(\mathrm{Fix}(\widetilde{f}'))$.

PROOF. The restriction of α gives a homeomorphism between $\mathrm{Fix}(\widetilde{f})$ and $\mathrm{Fix}(\widetilde{f}')$. □

Lemma (4.1.10) says that for $\widetilde{f},\widetilde{f}'\in\operatorname{lift}(f)$ the Nielsen fixed point classes $p(\mathrm{Fix}(\widetilde{f}))$ and $p(\mathrm{Fix}(\widetilde{f}'))$ are equal if $\widetilde{f}'=\alpha\widetilde{f}\alpha^{-1}$ for an $\alpha\in\mathcal{O}_X$. This suggests the action

$$\alpha\circ\widetilde{f}=\alpha\widetilde{f}\alpha^{-1}$$

of $\mathcal{O}_X$ on $\operatorname{lift}(f)$. We will denote the quotient set by $\mathcal{R}(f)$ and we will call its elements *Reidemeister classes*.

(4.1.12) THEOREM. *There is a natural inclusion from the set of Nielsen classes into the set of Reidemeister classes given by*

$$\mathcal{N}(f) \ni \mathbb{A} \to [\widetilde{f}] \in \mathcal{R}(f) \quad \textit{where } \mathbb{A} = p(\mathrm{Fix}(\widetilde{f})).$$

PROOF. Let us fix a Nielsen class $\mathbb{A} \in \mathcal{N}(f)$. By Lemma (4.1.9) there exists a lift $\widetilde{f}$ satisfying $\mathbb{A} = p(\mathrm{Fix}(\widetilde{f}))$. If $\widetilde{f}'$ is another lift with the same property, then by Lemma (4.1.10), $\widetilde{f}' = \alpha\widetilde{f}\alpha^{-1}$ for an $\alpha \in \mathcal{O}_X$ hence both $\widetilde{f}$, $\widetilde{f}'$ represent the same Reidemeister class and the considered map is well defined. On the other hand if $\mathbb{A} = p(\mathrm{Fix}(\widetilde{f}))$, $\mathbb{A}' = p(\mathrm{Fix}(\widetilde{f}'))$ and $[\widetilde{f}'] = [\widetilde{f}]$, then by Lemma (4.1.11), $\mathbb{A} = \mathbb{A}'$ and the map is injective. □

The above theorem gives the inclusion from the set of Nielsen classes into the set of Reidemeister classes. This allows us to identify each Nielsen class of the given map f (a subset of X) with a Reidemeister class (an element of an algebraic object). The advantage is that this algebraic object does not change during the homotopy, in particular its elements do not disappear as nonessential classes.

We end this section by applying the above theory to compute the basic nontrivial case: the Nielsen number of self-maps of the unit circle S^1.

(4.1.13) PROPOSITION. *If $f\colon S^1 \to S^1$ is a map of degree $n \in \mathbb{Z}$, then $N(f) = |n-1|$.*

PROOF. Recall that each self-map of S^1 of degree n is homotopic to $f(z) = z^n$, $n \in \mathbb{Z}$. If $n = 1$ then $f = \mathrm{id}$ and the last is homotopic (by a small twist of the circle) to a fixed point free map, hence $N(f) = 0$. On the other hand $n = 0$ gives the constant map, hence $N(f) = 1$.

Let $p\colon \mathbb{R} \to S^1$, $p(t) = \exp(2\pi ti)$ be the universal covering. Then every lift of $f(z) = z^n$ is of the form $\widetilde{f}_k(t) = tn + k$ $(k \in \mathbb{Z})$. We notice that for $n \neq 1$, $\mathrm{Fix}(\widetilde{f}_k) = \{-k/(n-1)\}$ is a singleton.

Now we will concentrate on the case $n \geq 2$. By Lemma (4.1.9) every Nielsen class has the form $p(\mathrm{Fix}(\widetilde{f}_k))$ hence it is a single point $\varepsilon_{n-1}^k = \exp(2\pi ki/(n-1))$ for $k = 1, \dots, n-1$. In other words $\mathrm{Fix}(f) = \{\varepsilon_{n-1}^{-1}, \dots, \varepsilon_{n-1}^{n-1}\}$ and each point is a Nielsen class. It remains to notice that the map z^n is expanding (since $n \geq 2$) hence $\mathrm{ind}(f, \varepsilon_{n-1}^i) = -1$ and each class is essential. Thus $N(f) = n - 1 = |n-1|$ for $\deg f \geq 2$.

If $n < 0$ then we repeat the above and we get $\mathrm{Fix}(f) = \{\varepsilon_{-n+1}^1, \dots, \varepsilon_{-n+1}^{-n+1}\}$ where each fixed point is a Nielsen class of index $+1$. Now $N(f) = -n + 1 = |n-1|$. □

4.1.3. Independence of the universal covering. To define the set of Reidemeister classes $\mathcal{R}(f)$ we used a universal covering of the considered space $p\colon \widetilde{X} \to X$. Now we will show that the use of another universal covering $p'\colon \widetilde{X}' \to X$ leads to the set $\mathcal{R}'(f)$ which is in the *canonical* bijection with $\mathcal{R}(f)$. This discussion could be omitted if we considered spaces with prescribed universal coverings. However in exploration of the fibre maps many different coverings of the same fibre appear and we have to know the relation among the Reidemeister sets they give.

Consider two universal coverings of the same space $p\colon \widetilde{X} \to X$, $p'\colon \widetilde{X}' \to X$. Then there is a homeomorphism $\widetilde{h}\colon \widetilde{X} \to \widetilde{X}'$ satisfying $p'\widetilde{h} = p$. Let $f\colon X \to X$ be a given map and let $\operatorname{lift}(f)$, $\operatorname{lift}'(f)$ denote the sets of lifts of f to the coverings p, p' respectively. Let $\mathcal{R}(f)$, $\mathcal{R}'(f)$ denote the sets of Reidemeister classes obtained from these coverings.

(4.1.14) THEOREM. *All sets of Reidemeister classes of the given space are canonically isomorphic: if $p\colon \widetilde{X} \to X$, $p'\colon \widetilde{X}' \to X$ are universal coverings and $\widetilde{h}\colon \widetilde{X} \to \widetilde{X}'$ is an isomorphism of these coverings (i.e. $p\widetilde{h} = p$) then the formula*

$$\mathcal{R}(f) \ni [\widetilde{f}] \to [\widetilde{h}\widetilde{f}\widetilde{h}^{-1}] \in \mathcal{R}'(f)$$

is the canonical bijection which does not depend on $\widetilde{h}$. Moreover, this bijection is compatible with the inclusion from Theorem (4.1.12), *i.e. the diagram*

$$\begin{tikzcd}
 & \mathcal{R}(f) \arrow[dd] \\
N(f) \arrow[ur] \arrow[dr] & \\
 & \mathcal{R}'(f)
\end{tikzcd}$$

is commutative.

PROOF. We define the map $A\colon \operatorname{lift}(f) \to \operatorname{lift}'(f)$ by the formula $A(\widetilde{f}) = \widetilde{h}\widetilde{f}\widetilde{h}^{-1}$. This map depends, in general, on the choice of $\widetilde{h}$. Now we notice that A preserves the Reidemeister relation: if α is a deck transformation of p ($p\alpha = p$), then $\alpha' = \widetilde{h}\alpha\widetilde{h}^{-1}$ is a deck transformation of p'. Moreover,

$$A(\alpha\widetilde{f}\alpha^{-1}) = \widetilde{h}(\alpha\widetilde{f}\alpha^{-1})\widetilde{h}^{-1} = \alpha'(\widetilde{h}\widetilde{f}\widetilde{h}^{-1})\alpha'^{-1} = \alpha' A(\widetilde{f})\alpha'^{-1}.$$

Thus A induces the map $A(\widetilde{h})\colon \mathcal{R}(f) \to \mathcal{R}'(f)$. We will show that this map does not depend on the choice of the homeomorphism $\widetilde{h}$. Let $\widetilde{k}\colon \widetilde{X} \to \widetilde{X}'$ be another homeomorphism satisfying $p'\widetilde{k} = p$. Then

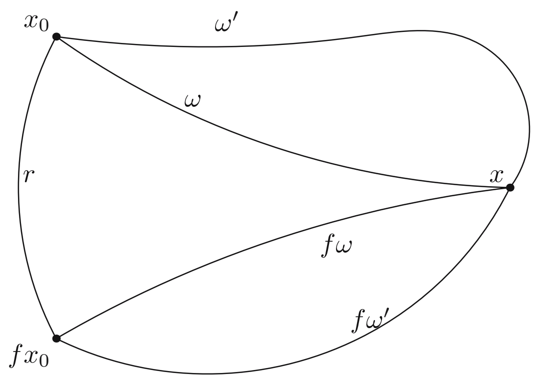
$$A(\widetilde{h})[\widetilde{f}] = [\widetilde{h}\widetilde{f}\widetilde{h}^{-1}] = [\widetilde{h}\widetilde{k}^{-1}(\widetilde{k}\widetilde{f}\widetilde{k}^{-1})(\widetilde{h}\widetilde{k}^{-1})^{-1}] = [\widetilde{k}\widetilde{f}\widetilde{k}^{-1}] = A(\widetilde{k})[\widetilde{f}]$$

since $\widetilde{h}\widetilde{k}^{-1}$ is a deck transformation of the covering $p' \colon \widetilde{X}' \to X$.

Now we prove the commutativity of the diagram. Let A be a Nielsen class of f and let $\widetilde{f}$ be a lift of f to the covering $\widetilde{X}$ satisfying $p(\mathrm{Fix}(\widetilde{f})) = A$. We will show that the corresponding lift $\widetilde{f}' = \widetilde{h}\widetilde{f}\widetilde{h}^{-1}$ of $\widetilde{X}'$ also gives $p'(\mathrm{Fix}(\widetilde{f}')) = A$. It follows from the observation that $\widetilde{h} \colon \widetilde{X} \to \widetilde{X}'$ maps $\mathrm{Fix}(\widetilde{f})$ homeomorphicaly onto $\mathrm{Fix}(\widetilde{f}')$. In fact if $\widetilde{f}(\widetilde{x}) = \widetilde{x}$, then $\widetilde{f}'\widetilde{h}(\widetilde{x}) = (\widetilde{h}\widetilde{f}\widetilde{h}^{-1})\widetilde{h}(\widetilde{x}) = \widetilde{h}\widetilde{f}(\widetilde{x}) = \widetilde{h}(\widetilde{x})$, hence $\widetilde{h}(\widetilde{x}) \in \mathrm{Fix}(\widetilde{f}')$ and similarly we prove the opposite inclusion. $\square$

4.1.4. Reidemeister classes and the fundamental group. In this section we will show that the same algebraic object $\mathcal{R}(f)$ can be obtained as a quotient set of the fundamental group of X (without the use of a universal covering). The proof of the equivalence of the two definitions will require a longer but routine calculation. As soon as this is done, we will be able to apply elements of the fundamental group as "coordinates" in describing Reidemeister and Nielsen classes.

Now we recall the definitions from [Yu1].



We introduce the Reidemeister classes using the fundamental group. To each point $x_0 \in X$ and a path r joining in X the points x_0 and fx_0, we define a quotient set of the group $\pi_1(X; x_0)$ and we denote it by $\mathcal{R}(f; x_0, r)$. We will see that there exists a canonical bijection between all these sets (for different pairs (x_0, r)). Moreover, all these sets are canonically bijective with $\mathcal{R}(f)$: the set of Reidemeister classes obtained in the previous section by the means of universal coverings.

Let us fix a point $x_0 \in X$ and a path $r \colon I \to X$ such that $r(0) = x_0$ and $r(1) = fx_0$. We will call (x_0, r) *a reference pair.* Let $x \in \mathrm{Fix}(f)$ and let ω be a path from x_0 to x. Then $\omega * (f\omega^{-1}) * r^{-1}$ is a loop based at x_0, hence it defines an element in $\pi_1(X; x_0)$. Let ω' be another path joining the points x_0 and x. Then $\omega' = \alpha * \omega$ for an $\alpha \in \pi_1(X; x_0)$. Moreover,

$$\begin{aligned}\omega' * (f\omega')^{-1} * r^{-1} &= \alpha * \omega * (f(\alpha * \omega)^{-1})) * r^{-1} = \alpha * \omega * (f\omega^{-1}) * f(\alpha^{-1}) * r^{-1} \\ &= \alpha * (\omega * (f\omega^{-1}) * r^{-1}) * (r * (f\alpha^{-1}) * r^{-1}).\end{aligned}$$

This suggests to consider the action of $\pi_1(X;x_0)$ on itself given by the formula

$$\alpha\circ\beta=\alpha*\beta*(r*(f\alpha)^{-1}*r^{-1}).$$

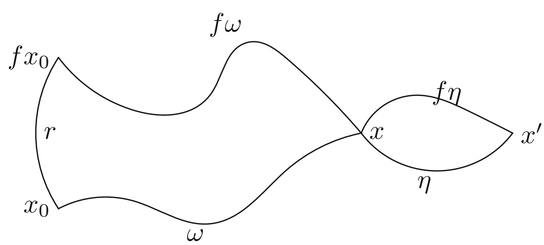
The orbits of this action are called *Reidemeister classes* and will be denoted $\mathcal{R}(f;x_0,r)$. We use this name since we will show that this orbit set is canonically isomorphic to $\mathcal{R}(f)$, the set of Reidemeister classes from the previous sections. The above considerations show that each $x\in \mathrm{Fix}\,(f)$ determines an element in $\mathcal{R}(f;x_0,r)$. Moreover,

(4.1.15) LEMMA. *Let $\mathbb{A}\subset \mathrm{Fix}(f)$ be a Nielsen class. Let $x\in\mathbb{A}$ and let ω be a path from x_0 to x. Then $[\omega*(f\omega^{-1})*r^{-1}]\in\mathcal{R}(f;x_0,r)$ depends neither on the choice of $x\in\mathbb{A}$ nor on the path ω. Thus we get a map $\mathcal{N}(f)\to\mathcal{R}(f;x_0,r)$ which is an injection.*

PROOF. We have already noticed above that any two paths from x_0 to x give the same Reidemeister class. Now let $x,x'\in\mathbb{A}$ and let η be a path from x to x' establishing the Nielsen relation ($\eta\sim f\eta$). For a path ω from x_0 to x we define $\omega'=\omega*\eta$, a path from x_0 to x'. Now

$$\begin{aligned}\omega'*(f\omega')^{-1}*r^{-1}&=\omega*\eta*(f(\omega*\eta)^{-1})*r^{-1}\\&=\omega*\eta*(f\eta)^{-1}*(f\omega)^{-1}*r^{-1}=\omega*(f\omega)^{-1}*r^{-1}\in\pi_1(X)\end{aligned}$$

so we get the same Reidemeister class.



Thus we have defined a map $\mathcal{N}(f)\to\mathcal{R}(f;x_0,r)$. It remains to show that this map is injective. Suppose that $x,x'\in\mathrm{Fix}(f)$, ω, ω' are paths from the base point x_0 to x, x' respectively and $[\omega*(f\omega)^{-1}*r^{-1}]=[\omega'*(f\omega')^{-1}*r^{-1}]\in\mathcal{R}(f;x_0,r)$. Then there exists a loop γ satisfying:

$$\omega'*(f\omega')^{-1}*r^{-1}=\gamma*(\omega*(f\omega)^{-1}*r^{-1})*r*(f\gamma)^{-1}*r^{-1}=\gamma*\omega*(f(\gamma*\omega))^{-1}*r^{-1}$$

hence $\omega^{-1}*\gamma^{-1}*\omega'=f(\omega^{-1}*\gamma^{-1}*\omega')$. Thus the path $\omega^{-1}*\gamma^{-1}*\omega'$ makes the points x, x' Nielsen related. $\square$

If $\pi_1(X;x_0)$ is abelian, then $[\alpha]=[\beta]\in\mathcal{R}(f;x_0,r)$ if $\beta=\gamma+\alpha-f^r_{\#}\gamma$ for a $\gamma\in\pi_1(X;x_0)$ where $f^r_{\#}\colon\pi_1(X;x_0)\to\pi_1(X;x_1)$ is given by

$$f^r_{\#}(\alpha)=r*f(\alpha)*r^{-1}.$$

Thus we get the following proposition.

(4.1.16) PROPOSITION. *If $\pi_1(X; x_0)$ is an abelian group, then*

$$\mathcal{R}(f; x_0, r) = \pi_1(X; x_0)/\mathrm{Im}(\mathrm{id} - f^r_\#) = \mathrm{Coker}\,(\mathrm{id} - f_\#).$$

(4.1.17) REMARK. In particular, in this case $\mathcal{R}(f; x_0, r)$ inherits the abelian group structure and all Reidemeister classes have the same cardinality. The last is not true in general. For example if we take the identity map as f, then $[\alpha] = \{\gamma\alpha\gamma^{-1} \colon \gamma \in \pi_1(X; x_0)\}$ hence the cardinality of this set is 1 if and only if α belongs to the center of $\pi_1(X; x_0)$. Thus for every space X with a nonabelian fundamental group, the Reidemeister classes in $\mathcal{R}(\mathrm{id})$ are not all of the same cardinality.

4.1.5. Canonical isomorphism of Reidemeister sets $\mathcal{R}(f; x, r)$. It turns out that $\mathcal{R}(f, x_0; r_0)$ does not depend on the choice of the reference pair (x_0, r_0). We will see that there is a canonical bijection between any two Reidemeister sets (of the same map f). Let (x_1, r_1) be another reference pair. To establish the promised canonical bijection we fix a path u from x_0 to x_1.

(4.1.18) LEMMA ([Yu1]). *The map $\kappa \colon \pi_1(X; x_0) \to \pi_1(X; x_1)$ given by the formula $\kappa(\alpha) = u^{-1} * \alpha * r_0 * fu * {r_1}^{-1}$ induces a map $\kappa \colon \mathcal{R}(f; x_0, r_0) \to \mathcal{R}(f; x_1, r_1)$ which is a bijection. Moreover κ does not depend on the choice of the path u.*

PROOF. We show that κ induces the map $\mathcal{R}(f; x_0, r_0) \to \mathcal{R}(f; x_1, r_1)$. Let $[\alpha] = [\beta] \in \mathcal{R}(f; x_0, r_0)$. Now $\beta = \gamma * \alpha * r_0 * f\gamma^{-1} * r_0^{-1}$ implies

$$\begin{aligned}
\kappa(\beta) &= u^{-1} * (\gamma * \alpha * r_0 * f\gamma^{-1} * r_0^{-1}) * r_0 * fu * r_1^{-1} \\
&= (u^{-1} * \gamma * u) * (u^{-1} * \alpha * r_0 * fu * r_1^{-1}) * r_1 * (fu^{-1} * f\gamma^{-1} * fu) * r_1^{-1} \\
&= (u^{-1} * \gamma * u) * (u^{-1} * \alpha * r_0 * fu * r_1^{-1}) * r_1 * f(u^{-1} * \gamma * u)^{-1} * r_1^{-1} \\
&= \gamma' * \kappa(\alpha) * r_1 * f{\gamma'}^{-1} * r_1^{-1}
\end{aligned}$$

(where $\gamma' = u^{-1} * \gamma * u$). Thus $[\kappa(\alpha)] = [\kappa(\beta)] \in \mathcal{R}(f; x_1, r_1)$.

To see that κ is the bijection we notice that $\kappa' \colon \pi_1(X; x_1) \to \pi_1(X; x_0)$, given by $\kappa'(\beta) = u * \beta * r_1 * (fu^{-1}) * r_0^{-1}$, induces the inverse map.

Independence of u. Let v be another path from x_0 to x_1. Let $\kappa' \colon \pi_1(X; x_0) \to \pi_1(X; x_0)$ denote the above κ with u replaced with v. Then

$$\begin{aligned}
\kappa'(\alpha) &= v^{-1} * \alpha * r_0 * fv * {r_1}^{-1} \\
&= (v^{-1} * u * u^{-1} * \alpha * r_0 * fu * {r_1}^{-1}) * (r_1 * fu^{-1} * fv * r_1^{-1}) \\
&= (v^{-1} * u) * (u^{-1} * \alpha * r_0 * fu * {r_1}^{-1}) * (r_1 * f(v^{-1} * u)^{-1} * r_1^{-1}) \\
&= (v^{-1} * u) * \kappa(\alpha) * r_1 * f(v^{-1} * u)^{-1} * r_1^{-1}.
\end{aligned}$$

Thus $[\kappa(\alpha)] = [\kappa'(\alpha)] \in \mathcal{R}(f; x_1, r_1)$. $\square$

(4.1.19) Definition. Let $\gamma\colon [0,1] \to X$ be a path from $\gamma(0) = x_0$ to $\gamma(1) = x_1$. Let $p\colon \widetilde{X} \to X$ be a universal covering. We define $\tau_\gamma\colon p^{-1}(x_0) \to p^{-1}(x_1)$ putting $\tau_\gamma(\widetilde{x}_0) = \widetilde{\gamma}(1)$ where $\widetilde{\gamma}$ denotes the lift of γ starting from $\widetilde{\gamma}(0) = \widetilde{x}_0$.

We notice that:

(1) if γ, γ' have common ends and are homotopic then $\tau_\gamma = \tau_{\gamma'}$,
(2) if $\gamma(1) = \gamma'(0)$ then $\tau_{\gamma * \gamma'} = \tau_{\gamma'}\tau_\gamma$,
(3) $\tau_c = \mathrm{id}$ for the constant path c.

In particular the map τ_γ is inverse to $\tau_{\gamma^{-1}}$ hence τ_γ is a bijection.

Now we show that all sets $\mathcal{R}(f; x_0, r_0)$ are in the natural bijection with $\mathcal{R}(f)$ (the set of Reidemeister classes defined by lifts). We fix a point $\widetilde{x}_0 \in p^{-1}(x_0)$ and $[\alpha] \in \mathcal{R}(f, x_0, r_0)$. We define $\phi[\alpha] = [\widetilde{f}] \in \mathcal{R}(f)$ where $\widetilde{f} \in \mathrm{lift}(f)$ satisfies $\widetilde{f}(\widetilde{x}_0) = \tau_{\alpha * r_0}(\widetilde{x}_0)$.

The rest of this section is the proof of the following theorem establishing the canonical bijection.

(4.1.20) Theorem. *The map $\phi\colon \mathcal{R}(f; x_0, r_0) \to \mathcal{R}(f)$ is a bijection and the diagram*

$$\begin{tikzcd}
\mathcal{R}(f; x_0, r_0) \arrow[dd, "\kappa"'] \arrow[dr, "\phi"] & \\
 & \mathcal{R}(f) \\
\mathcal{R}(f; x_1, r_1) \arrow[ur, "\phi"'] &
\end{tikzcd}$$

is commutative.

Scheme of the Proof. We prove the correctness of ϕ (Lemma (4.1.22)) then we find the inverse map ψ (Lemma (4.1.23)) and we prove that the diagram commutes (Lemma (4.1.24)). $\square$

We begin with a technical lemma.

(4.1.21) Lemma. *Let us fix points $x_0 \in X$ and $\widetilde{x}_0 \in p^{-1}(x_0)$. Let $\gamma\colon I \to X$ be a loop based at x_0 and let $\widetilde{\gamma}\colon I \to \widetilde{X}$ be its lift starting from $\widetilde{x}_0$. We denote by $h_\gamma \in \mathcal{O}_X$ the transformation satisfying $h_\gamma(\widetilde{x}_0) = \tau_\gamma(\widetilde{x}_0)$. Let $\widetilde{u}$ be a path joining $\widetilde{x}_0$ with $\widetilde{f}\widetilde{x}_0$ in $\widetilde{X}$ and let $u = p\widetilde{u}$. Then*

(4.1.21.1) $h_\gamma(\tau_\beta(\widetilde{x}_0)) = \tau_{\gamma*\beta}(\widetilde{x}_0)$, $h_\gamma^{-1}(\tau_\beta(\widetilde{x}_0)) = \tau_{\gamma^{-1}*\beta}(\widetilde{x}_0)$,
(4.1.21.2) $\widetilde{f}(\tau_\gamma(\widetilde{x}_0)) = \tau_{u*f\gamma}(\widetilde{x}_0)$.

PROOF.

$$(4.1.21.1)\quad h_\gamma(\tau_\beta(\widetilde{x}_0)) = h_\gamma(\widetilde{\beta}(1)) = \text{(end of the lift of the path } \beta \text{ starting from } h_\gamma(\widetilde{\beta}(0)) = h_\gamma(\widetilde{x}_0) = \tau_\gamma(\widetilde{x}_0)).$$

On the other hand

$$\begin{aligned}\tau_{\gamma*\beta}(\widetilde{x}_0) &= \text{(end of the lift of the path } \gamma*\beta \text{ starting from } \widetilde{x}_0) \\ &= \text{(end of the lift of the path } \beta \text{ starting from } \widetilde{\gamma}(1) = \tau_\gamma(\widetilde{x}_0)).\end{aligned}$$

$$\begin{aligned}(4.1.21.2)\quad \tau_{u*f\gamma}(\widetilde{x}_0) &= \text{(end of the lift of the path } u*f\gamma \text{ starting from } \widetilde{x}_0) \\ &= \text{(end of the lift of the path } f\gamma \\ &\qquad \text{starting from } \widetilde{u}(1) = \widetilde{f}\widetilde{x}_0 = \widetilde{f}\widetilde{\gamma}(0))) \\ &= \text{(end of the path } \widetilde{f}\widetilde{\gamma}) = \widetilde{f}\widetilde{\gamma}(1) = \widetilde{f}\tau_\gamma(\widetilde{x}_0).\end{aligned}$$

□

(4.1.22) LEMMA. *The formula defining ϕ induces the map $\phi\colon \mathcal{R}(f; x_0, r_0) \to \mathcal{R}(f)$ which does not depend on the choice of $\widetilde{x}_0$.*

PROOF. We show that (for a fixed reference pair (x_0, r_0)) the map $\phi\colon \pi_1(X; x_0) \to \mathcal{R}(f)$ induced by the above formula does not depend on $\widetilde{x}_0$. Let us fix $\widetilde{x}_0, \widetilde{x}'_0 \in p^{-1}(x_0)$ and $\alpha \in \pi_1(X; x_0)$. Let $\widetilde{\gamma}$ be a path joining $\widetilde{x}_0, \widetilde{x}'_0 \in \widetilde{X}$ and let $\gamma = p(\widetilde{\gamma})$. We have to show that if the lifts $\widetilde{f}, \widetilde{f}'$ satisfy $\widetilde{f}(\widetilde{x}_0) = \tau_{\alpha*r_0}(\widetilde{x}_0)$, $\widetilde{f}(\widetilde{x}'_0) = \tau_{\alpha*r_0}(\widetilde{x}'_0)$ then $[\widetilde{f}] = [\widetilde{f}'] \in \mathcal{R}(f)$. It is enough to show that $h_\gamma \widetilde{f} h_\gamma^{-1} = \widetilde{f}'$ where $h_\gamma \in \mathcal{O}_X$ is determined by $h_\gamma(\widetilde{x}_0) = \tau_\gamma(\widetilde{x}_0)$. In fact

$$\begin{aligned}h_\gamma \widetilde{f} h_\gamma^{-1}(\widetilde{x}'_0) &= h_\gamma \widetilde{f}(\widetilde{x}_0) = h_\gamma \tau_{\alpha*r_0}(\widetilde{x}_0) \\ &= \tau_{\gamma*(\alpha*r_0)}(\widetilde{x}_0) = \tau_{\alpha*r_0}(\tau_\gamma(\widetilde{x}_0)) = \tau_{\alpha*r_0}(\widetilde{x}'_0) = \widetilde{f}'(\widetilde{x}'_0),\end{aligned}$$

where the middle equality comes from Lemma (4.1.21.1).

Now we will show that $[\alpha] = [\alpha'] \in \mathcal{R}(f; x_0, r_0)$ implies $\phi(\alpha) = \phi(\alpha') \in \mathcal{R}(f)$. By the assumption $\alpha' = \gamma^{-1} * \alpha * (r_0 * f\gamma * r_0^{-1})$ for a $\gamma \in \pi_1(X; x_0)$. Let us fix an $\widetilde{x}_0 \in p^{-1}(x_0)$ and let $h_\gamma \in \mathcal{O}_X$ satisfy $h_\gamma(\widetilde{x}_0) = \tau_\gamma(\widetilde{x}_0)$ as above. Let $\phi(\alpha) = \widetilde{f}$, $\phi(\alpha') = \widetilde{f}'$, i.e. the lifts $\widetilde{f}$, $\widetilde{f}'$ satisfy $\widetilde{f}(\widetilde{x}_0) = \tau_{\alpha*r_0}(\widetilde{x}_0)$, $\widetilde{f}'(\widetilde{x}_0) = \tau_{\alpha'*r_0}(\widetilde{x}_0)$. It remains to show that $h_\gamma^{-1} \widetilde{f} h_\gamma = \widetilde{f}'$. In fact

$$\begin{aligned}h_\gamma^{-1} \widetilde{f} h_\gamma(\widetilde{x}_0) &= h_\gamma^{-1} \widetilde{f}(\tau_\gamma(\widetilde{x}_0)) = h_\gamma^{-1}(\tau_{\alpha*r_0*f\gamma}(\widetilde{x}_0)) \\ &= \tau_{\gamma^{-1}*\alpha*r_0*f\gamma}(\widetilde{x}_0) = \tau_{\alpha'*r_0}(\widetilde{x}_0) = \widetilde{f}'(\widetilde{x}_0)\end{aligned}$$

where the middle equalities come from Lemma (4.1.21) for $u = \alpha r_1$. □

(4.1.23) Lemma. *$\phi\colon \mathcal{R}(f;x_0,r_0) \to \mathcal{R}(f)$ is the bijection.*

Proof. We define the opposite map $\psi\colon \mathcal{R}(f) \to \mathcal{R}(f;x_0,r_0)$. Let us fix a point $\widetilde{x}_0 \in p^{-1}(x_0)$. Let $[\widetilde{f}] \in \mathcal{R}(f)$. We define $\psi[\widetilde{f}] = [u * r_0^{-1}] \in \mathcal{R}(f;x_0,r_0)$ where $u = p\widetilde{u}$ and $\widetilde{u}$ is a path joining $\widetilde{x}_0, \widetilde{f}\widetilde{x}_0 \in \widetilde{X}$. We show that

(1) ψ is correctly defined (with a given $\widetilde{x}_0$),
(2) ψ does not depend on the choice of $\widetilde{x}_0$,
(3) $\phi\psi = 1_{\mathcal{R}(f)}$, $\psi\phi = 1_{\mathcal{R}(f;x_0,r_0)}$.

(1) We have to show that $\psi(h_\gamma^{-1}\widetilde{f}h_\gamma) = \psi(\widetilde{f}) \in \mathcal{R}(f;x_0,r_0)$.

We notice that $h_\gamma^{-1}\widetilde{f}h_\gamma(\widetilde{x}_0) = h_\gamma^{-1}\widetilde{f}\tau_\gamma(\widetilde{x}_0) = h_\gamma^{-1}(\tau_{u*f\gamma}(\widetilde{x}_0)) = \tau_{\gamma^{-1}*u*f\gamma}(\widetilde{x}_0)$. This implies $\psi(h_\gamma^{-1}\widetilde{f}h_\gamma) = [\gamma^{-1} * u * f\gamma * r_0^{-1}] \in \mathcal{R}(f;x_0,r_0)$. It remains to notice that

$$\gamma^{-1} \circ (u * r_0^{-1}) = \gamma^{-1} * u * r_0^{-1} * r_0 * f\gamma * r_0^{-1} = \gamma^{-1} * u * f\gamma * r_0^{-1}.$$

(2) If we use in the definition of ψ another point $\widetilde{x}_0' \in p^{-1}(x_0)$ then we get the formula $\phi'[\widetilde{f}] = [u' * r_0^{-1}]$ where $u' = p(\widetilde{u}')$ and where $\widetilde{u}'$ is a path joining the points $\widetilde{x}', \widetilde{f}(\widetilde{x}') \in \widetilde{X}$. If $\widetilde{\gamma}$ joins the points $\widetilde{x}, \widetilde{x}'$, then the paths $\widetilde{\gamma}^{-1} * \widetilde{u} * \widetilde{f}\widetilde{\gamma}$ and $\widetilde{u}'$ have common ends $\widetilde{x}'$, $f\widetilde{x}'$, hence are homotopic in the simply-connected space $\widetilde{X}$. Thus their projections $\gamma^{-1} * u * f\gamma$ and u' are homotopic in X. This implies

$$\phi'[\widetilde{f}] = [u' * r_0^{-1}] = [\gamma^{-1} * u * f\gamma * r_0^{-1}] = [\gamma^{-1} * (u * r_0^{-1}) * r_0 * f\gamma * r_0^{-1}] = [u * r_0^{-1}].$$

(3) Let $\widetilde{f} \in \operatorname{lift}(f)$. Then $\psi[\widetilde{f}] = [p(\widetilde{u}) * r_0^{-1}] \in \mathcal{R}(f;r_0,x_0)$, where $\widetilde{u}$ joins the points $\widetilde{x}_0, \widetilde{f}\widetilde{x}_0 \in \widetilde{X}$. This implies $\phi(\psi[\widetilde{f}])(\widetilde{x}_0) = \tau_{u*r_0^{-1}*r_0}(\widetilde{x}_0) = \tau_u(\widetilde{x}_0) = \widetilde{f}(\widetilde{x}_0)$. Thus $\psi\phi = 1_{\mathcal{R}(f)}$.

Now let $\alpha \in \pi_1(X;x_0)$. Then $\phi[\alpha] = [\widetilde{f}]$ where $\widetilde{f}(\widetilde{x}_0) = \tau_{\alpha*r_0}(\widetilde{x}_0)$, hence $\psi\phi[\alpha] = \psi[\widetilde{f}] = [(\alpha * r_0) * r_0^{-1}] = [\alpha]$. Thus $\phi\psi = 1_{\mathcal{R}(f;x_0,r_0)}$. □

(4.1.24) Lemma. *The diagram in Theorem* (4.1.20) *is commutative.*

Proof. Let us fix $[\alpha] \in \mathcal{R}(f;x_0,r_0)$, a point $\widetilde{x}_0 \in p^{-1}(x_0)$, and a path u joining the points $x_0, x_1 \in X$. Let $\widetilde{x}_1 = \tau_u(\widetilde{x}_0)$. Then $\phi[\alpha] = [\widetilde{f}]$, where $\widetilde{f}(\widetilde{x}_0) = \tau_{\alpha*r_0}(\widetilde{x}_0)$. This implies

$$\widetilde{f}(\widetilde{x}_1) = \widetilde{f}(\tau_u(\widetilde{x}_0)) = \tau_{\alpha*r_0*fu}(\widetilde{x}_0) = (*).$$

On the other hand $\kappa[\alpha] = [\alpha']$ where $\alpha' = u^{-1} * \alpha * r_0 * fu * r_1^{-1}$. Thus $\phi\kappa[\alpha] = \phi[\alpha'] = [\widetilde{f}']$, where $\widetilde{f}' \in \operatorname{lift}(f)$ satisfies $\widetilde{f}'(\widetilde{x}_1) = \tau_{\alpha'*r_1}(\widetilde{x}_1)$. But the last equals

$$\begin{aligned}\tau_{u^{-1}*\alpha*r_0*fu*r_1^{-1}*r_1}(\widetilde{x}_1) &= \tau_{u^{-1}*\alpha*r_0*fu}(\widetilde{x}_1)\\ &= \tau_{\alpha*r_0*fu}\tau_{u^{-1}}(\widetilde{x}_1) = \tau_{\alpha*r_0*fu}(\widetilde{x}_0) = (*).\end{aligned}$$

Thus $\widetilde{f}(\widetilde{x}_1) = (*) = \widetilde{f}'(\widetilde{x}_1)$ which implies $\widetilde{f}' = \widetilde{f}$. □

4.1.6. Commutative diagrams. We will show that a commutative diagram of path-connected spaces admitting universal coverings

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

induces a map $\mathcal{R}_h : \mathcal{R}(f) \to \mathcal{R}(g)$.

We fix universal coverings $p_X : \widetilde{X} \to X$, $p_Y : \widetilde{Y} \to Y$, a map $h: X \to Y$ and a lift $\widetilde{h} : \widetilde{X} \to \widetilde{Y}$ ($p_Y \widetilde{h} = h p_X$). Then for a given lift $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ ($p_X \widetilde{f} = f p_X$) there exists a unique $\widetilde{g} : \widetilde{Y} \to \widetilde{Y}$ such that the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{h}"'] & \widetilde{X} \arrow[d, "\widetilde{h}"] \\ \widetilde{Y} \arrow[r, "\widetilde{g}"'] & \widetilde{Y} \end{tikzcd}$$

commutes. This defines the map $l_{\widetilde{h}} : \mathrm{lift}(f) \to \mathrm{lift}(g)$ given by $l_{\widetilde{h}}(\widetilde{f}) = \widetilde{g}$. We will show that $l_{\widetilde{h}}$ induces the promised map of the sets of Reidemeister classes.

More precisely let $\mathcal{SM}$ denote the category whose objects are continuous self-maps $f: X \to X$ of compact ENRs and morphisms from $f: X \to X$ to $g: Y \to Y$ are continuous maps $h: X \to Y$ satisfying $hf = gh$, i.e. making the above diagram commutative. Under these assumptions:

(4.1.25) THEOREM. *The map $l_{\widetilde{h}}$ induces the map $\mathcal{R}_h : \mathcal{R}(f) \to \mathcal{R}(g)$ which depends neither on the choice of the lift $\widetilde{h}$ nor on the universal coverings. This gives a functor between the categories $\mathcal{SM} \to \mathcal{S}et$.*

PROOF. We will show that

(1) $l_{\widetilde{h}}$ induces the map $\mathcal{R}_h : \mathcal{R}(f) \to \mathcal{R}(g)$,
(2) $\mathcal{R}_h$ does not depend on the choice of the lift $\widetilde{h}$,
(3) $\mathcal{R}_h$ does not depend on the choice of the universal coverings,
(4) $\mathcal{R}_h$ is functorial.

(1) We notice that for every deck transformation of $\alpha : \widetilde{X} \to \widetilde{X}$ there exists a unique deck transformation $\beta : \widetilde{Y} \to \widetilde{Y}$ such that the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\alpha"] \arrow[d, "\widetilde{h}"'] & \widetilde{X} \arrow[d, "\widetilde{h}"] \\ \widetilde{Y} \arrow[r, "\beta"'] & \widetilde{Y} \end{tikzcd}$$

is commutative. This gives the commutative diagram

$$
\begin{tikzcd}
\widetilde{X} \arrow[r, "\alpha"] \arrow[d, "\widetilde{h}"'] & \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{h}"'] & \widetilde{X} \arrow[r, "\alpha^{-1}"] \arrow[d, "\widetilde{h}"] & \widetilde{X} \arrow[d, "\widetilde{h}"] \\
\widetilde{Y} \arrow[r, "\beta"'] & \widetilde{Y} \arrow[r, "\widetilde{g}"'] & \widetilde{Y} \arrow[r, "\beta^{-1}"'] & \widetilde{Y}
\end{tikzcd}
$$

hence $l_{\widetilde{h}}(\alpha^{-1}\widetilde{f}\alpha) = \beta^{-1}\widetilde{g}\beta$. Thus $l_{\widetilde{h}}$ induces the map $\mathcal{R}_{\widetilde{h}} : \mathcal{R}(f) \to \mathcal{R}(g)$.

(2) We will show that this map does not depend on the choice of the lift $\widetilde{h}$. Let $\widetilde{k}$ be another such lift. Then

$$[l_{\widetilde{k}}(\widetilde{f})] = [\widetilde{k}\widetilde{f}\widetilde{k}^{-1}] = [(\widetilde{k}\widetilde{h}^{-1})\widetilde{h}\widetilde{f}\widetilde{h}^{-1}(\widetilde{k}\widetilde{h}^{-1})^{-1}] = [\widetilde{h}\widetilde{f}\widetilde{h}^{-1}] = [l_{\widetilde{h}}(\widetilde{f})] \in \mathcal{R}(g).$$

(3) We will show also that this map does not depend on the choice of the universal coverings $p_X \colon \widetilde{X} \to X$ and $p_Y \colon \widetilde{Y} \to Y$. Let us fix other coverings $p'_X \colon \widetilde{X}' \to X$ and $p'_Y \colon \widetilde{Y}' \to Y$. We will show the commutativity of the diagram

$$
\begin{tikzcd}
\mathcal{R}(f) \arrow[r, "\mathcal{R}_h"] \arrow[d, "\eta"'] & \mathcal{R}(g) \arrow[d, "\eta"] \\
\mathcal{R}'(f) \arrow[r, "\mathcal{R}'_h"'] & \mathcal{R}'(g)
\end{tikzcd}
$$

where the Reidemester set in the upper row arise from the coverings p_X, p_Y while in the lower row from p'_X, p'_Y. Vertical arrows mean canonical bijections of Subsection 4.1.3. We fix homeomorphisms $k_X \colon \widetilde{X} \to \widetilde{X}'$, $k_Y \colon \widetilde{Y} \to \widetilde{Y}'$ satisfying $p'_X k_X = p_X$, $p'_Y k_Y = p_Y$. Now for given $\widetilde{f}$, k_X, k_Y, $\widetilde{h}$ we find unique $\widetilde{g}$, $\widetilde{f}'$, $\widetilde{g}'$, $\widetilde{h}'$ making the following diagram commutative

$$
\begin{tikzcd}
\widetilde{X} \arrow[rrr, "\widetilde{f}'"] \arrow[ddd, "\widetilde{h}'"'] & & & \widetilde{X}' \arrow[ddd, "\widetilde{h}'"] \\
& \widetilde{X} \arrow[ul, "k_X"'] \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{h}"'] & \widetilde{X} \arrow[ur, "k_X"] \arrow[d, "\widetilde{h}"] & \\
& \widetilde{Y} \arrow[dl, "k_Y"] \arrow[r, "\widetilde{g}"'] & \widetilde{Y}' \arrow[dr, "k_Y"'] & \\
\widetilde{Y} \arrow[rrr, "\widetilde{g}'"'] & & & \widetilde{Y}'
\end{tikzcd}
$$

Thus

$$\eta\mathcal{R}_h[\widetilde{f}] = \eta[\widetilde{g}] = [\widetilde{g}'] = \mathcal{R}_h[\widetilde{f}'] = \mathcal{R}_h\eta[\widetilde{f}]$$

where η denotes the canonical identification of Reidemeister sets $\mathcal{R}(f)$ and $\mathcal{R}'(f)$ obtained from different universal covers (see Subsection 4.1.3).

(4) Let the diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f_1"'] & Y \arrow[r, "k"] \arrow[d, "f_2"] & Z \arrow[d, "f_3"] \\ X \arrow[r, "h"'] & Y \arrow[r, "k"'] & Z \end{tikzcd}$$

be commutative. By the above for a given lift $\widetilde{f}_1\colon \widetilde{X} \to \widetilde{X}$ of f_1 there exist lifts $\widetilde{f}_2$, $\widetilde{f}_3$ making the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{h}"] \arrow[d, "\widetilde{f}_1"'] & \widetilde{Y} \arrow[r, "\widetilde{k}"] \arrow[d, "\widetilde{f}_2"] & \widetilde{Z} \arrow[d, "\widetilde{f}_3"] \\ \widetilde{X} \arrow[r, "\widetilde{h}"'] & \widetilde{Y} \arrow[r, "\widetilde{k}"'] & \widetilde{Z} \end{tikzcd}$$

commutative. Then

$$\mathcal{R}_{kh}[\widetilde{f}_1] = [\widetilde{f}_3] = \mathcal{R}_k[\widetilde{f}_2] = \mathcal{R}_k\mathcal{R}_h[\widetilde{f}_1]. \qquad \square$$

It turns out that not only strictly commutative diagrams induce maps of the set of the Reidemeister classes: it is enough to assume that the diagram is only homotopy commutative.

Suppose that we are given a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

with a fixed homotopy $H\colon hf \sim gh$. Let us fix universal coverings $p_X\colon \widetilde{X} \to X$, $p_Y\colon \widetilde{Y} \to Y$ and a lift $\widetilde{h}\colon \widetilde{X} \to \widetilde{Y}$ of the map h. Let $\widetilde{f} \in \operatorname{lift}(f)$. Since $H(\,\cdot\,,0) = hf = p_Y(\widetilde{h}\widetilde{f})$, there exists a lift $\widetilde{H}\colon X \times I \to \widetilde{Y}$ of the homotopy H satisfying $\widetilde{H}(\,\cdot\,,0) = \widetilde{h}\widetilde{f}$. Then $p_Y(\widetilde{H}(\,\cdot\,,1)) = gh$, hence there exists exactly one $\widetilde{g} \in \operatorname{lift}(g)$ satisfying $\widetilde{H}(\,\cdot\,,1) = \widetilde{g}\widetilde{h}$, i.e. the homotopy $\widetilde{H}$ makes the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{h}"] \arrow[d, "\widetilde{f}"'] & \widetilde{Y} \arrow[d, "\widetilde{g}"] \\ \widetilde{X} \arrow[r, "\widetilde{h}"'] & \widetilde{Y} \end{tikzcd}$$

commutative. We define the map $l_{\widetilde{h}}\colon \operatorname{lift}(f) \to \operatorname{lift}(g)$ putting $l_{\widetilde{h}}(\widetilde{f}) = \widetilde{g}$.

(4.1.26) THEOREM. *The map $l_{\widetilde{h}}$ and a homotopy $H\colon X \to Y$ between the maps hf and gh induce the map $\mathcal{R}_{h,H}\colon \mathcal{R}(f) \to \mathcal{R}(g)$ which depends neither on the choice of the lift $\widetilde{h}$ nor on the universal coverings.*

PROOF. Follow the proof of Theorem (4.1.25). □

(4.1.27) REMARK. The map $\mathcal{R}_{h,H}\colon \mathcal{R}(f) \to \mathcal{R}(g)$ is functorial in the following sense. Suppose that we are given a homotopy commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{h} & Y & \xrightarrow{k} & Z \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ X & \xrightarrow[h]{} & Y & \xrightarrow[k]{} & Z \end{array}$$

with the fixed homotopies $H\colon hf_1 \sim f_2h$, $K\colon kf_2 \sim f_3k$. Then we define the homotopy $L\colon X\times I \to Z$ as the concatenation $L = kH * K(h\times \mathrm{id}\ )$ where $kH(x,t) = k(H(x,t))$ and $K(h\times \mathrm{id})(x,t) = K(h(x),t)$. Then $\mathcal{R}_{kh,L} = \mathcal{R}_{kK}\mathcal{R}_{hH}$. Thus $\mathcal{R}_{h,H}$ give rise to a functor from the category of homotopy commutative diagrams with fixed homotopies, where the composition of homotopies is described above, to the set category.

4.1.7. Nielsen theory modulo a subgroup. For a given topological space X we consider the category whose objects are the points of X and morphisms from x to y are homotopy classes of paths between these points. Then the homotopy groups $\{\pi_1(X,x)\}_{x\in X}$ (with the usual action $\omega\circ\alpha = \omega * \alpha * \omega^{-1}$) can be regarded as a functor on this category. We will denote by H a functor which assigns to each $x\in X$ a normal divisor $H(x) \triangleleft \pi_1(X,x)$ (i.e. $\omega_*(H(x)) \subset H_*(y)$ for any path ω from x to y).

(4.1.28) EXAMPLE. Let $X \subset Y$ be path-connected spaces. Then $H(x_0) = \mathrm{Ker}\,[i_{\#}\colon \pi_1(X;x_0) \to \pi_1(Y;x_0)]$ is such a functor.

Paths $c, d\colon I \to X$ satisfying $c(0) = d(0)$, $c(1) = d(1)$ are called *H-homotopic* if and only if $c * d^{-1} \in H(c(0))$ which will be denoted $c \overset{H}{\simeq} d$. We notice that $c \overset{H}{\simeq} d$ yields $d \overset{H}{\simeq} c$, $c^{-1} \overset{H}{\simeq} d^{-1}$, $u*c \overset{H}{\simeq} u*d$, $c*v \overset{H}{\simeq} d*v$ (if only these paths are defined).

For $x\in X$ we consider the group $\pi_H(X;x) = \pi_1(X;x)/H(x)$ and we denote its elements by $\langle a\rangle_H$. We denote by $\omega_H\colon \pi_H(X;x) \to \pi_H(X;y)$ the homomorphism $\omega_H(\langle a\rangle_H) = \langle \omega^{-1} * a * \omega\rangle_H$ for a path ω from x to y. Notice that for $H = 1$ (the trivial subgroup) we get $\pi_1(X;x_0)$ hence the notation is formally compatible.

Let $f\colon X \to X$ be a map satisfying $f_{\#}(H) \subset H$ (i.e. $f_{\#}(H(x)) \subset H(f(x))$ for all $x\in X$). Then we have the induced map $f_{\#H}\colon \pi_H(X;x) \to \pi_H(X;fx)$.

Now we are in a position to generalize the Nielsen theory onto the relative case. We say that two points in $\mathrm{Fix}(f)$ are Nielsen related mod H if $c \overset{H}{\simeq} fc$ for a path c

joining these points. This relation splits $\mathrm{Fix}(f)$ into disjoint *H-Nielsen classes.* The set of these classes will be denoted by $\mathcal{N}_H(f)$. A class $\mathbb{A} \in \mathcal{N}_H(f)$ will be called *essential* if its index is nonzero and the number of essential H-Nielsen classes will be called *H-Nielsen number* and will be denoted $N_H(f)$.

Similarly we define Reidemeister classes modulo H. For a fixed point $x \in X$ we fix a path r joining x with $f(x)$ and we define the action of $\pi_H(X;x)$ on itself putting

$$\langle a\rangle_H \circ \langle b\rangle_H = \langle a * b * r * (fa)^{-1} * r^{-1}\rangle_H.$$

The orbits of this action are called *H-Reidemeister classes* and denoted $[\langle a\rangle]_H$. Their set will be denoted $\mathcal{R}_H(f;x,r)$. As in the absolute case we get the natural identification of the sets $\mathcal{R}_H(f;x,r)$ which yields the set $\mathcal{R}_H(f)$ and the injection $\mathcal{N}_H(f) \to \mathcal{R}_H(f)$.

We may also get $\mathcal{R}_H(f)$ as in Section 4.1.4 using instead of the universal covering the regular covering determined by the normal subgroup $H \subset \pi_1(X)$. Checking this is a good but boring exercise for the reader.

4.1.8. Commutativity of the Nielsen number. We have already seen that two homotopic maps have the same Nielsen number. Now we will show that $N(f)$ is also invariant in a wider sense: the maps of the same homotopy type also have equal Nielsen numbers.

(4.1.29) LEMMA. *Let $f\colon X \to Y$, $g\colon Y \to X$ be two maps. Then the restrictions of these maps give homeomorphisms between the sets* $\mathrm{Fix}(gf)$ *and* $\mathrm{Fix}(fg)$*. Moreover, the restrictions preserve the Nielsen relation, hence define a bijection between the sets of Nielsen classes $\mathcal{N}(f)$ and $\mathcal{N}(g)$. If moreover X, Y are compact* ENR*, then for every Nielsen class $\mathbb{A} \subset \mathrm{Fix}(gf)$ the set $f(\mathbb{A}) \subset \mathrm{Fix}(fg)$ is also the Nielsen class and moreover* $\mathrm{ind}(gf;\mathbb{A}) = \mathrm{ind}(fg; f(\mathbb{A}))$.

PROOF. The equality of indices follows from the Commutativity Property of index (Lemma (2.2.11)). □

(4.1.30) COROLLARY. *Under the above assumptions $N(fg) = N(gf)$.*

PROOF. Lemma (4.1.29) gives a preserving index bijection between the sets of essential Nielsen classes. □

In particular we get

(4.1.31) COROLLARY. *Let $X_0 \subset X$ be compact* ENR*s and $f\colon X \to X$ satisfy $f(X) \subset X_0$. Then denoting by $f_0\colon X_0 \to X_0$ the restriction of f we get $N(f_0) = N(f)$.*

Proof. Let $i\colon X_0 \to X$ denote the inclusion and let $f'\colon X \to X_0$ be given by $f'(x) = f(x)$. Since $if' = f$ and $f'i = f_0$, by the above corollary we get

$$N(f) = N(if') = N(f'i) = N(f_0). \qquad \square$$

On the other hand we may prove

(4.1.32) Theorem. *Let $h\colon X \to Y$ be a homotopy equivalence between two* ENR*s and let the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

be homotopy commutative. Then $N(f) = N(g)$.

Proof. Let $k\colon Y \to Y$ denote the homotopy inverse to h (i.e. $kh \sim 1_X$ and $hk \sim 1_Y$). Then

$$\begin{aligned} N(f) &= N(f(kh)) = N((fk)h) \\ &= N(h(fk)) = N((hf)k) = N((gh)k) = N(g(hk)) = N(g), \end{aligned}$$

where the middle equality comes from Lemma (4.1.30). $\square$

4.2. Wecken theorem

In this section we show that for self-maps of manifolds of dimension $\neq 2$, the Nielsen number is the best homotopy invariant being the lower bound for the cardinality of $\mathrm{Fix}(f)$. In other words we prove that a map f is homotopic to g for which $\#\mathrm{Fix}(g) = N(f)$.

This question was raised by Jakob Nielsen in the 1920s but only for self-maps of two-dimensional manifolds (surfaces) since only these were in the domain of his research. In the 1930s when the notion of fixed point index appeared the problem was automatically extended onto a much larger class of spaces (compact polyhedra). In 1942 Franz Wecken [We] proved that the Nielsen number can be realized for every self-map of a compact manifold of dimension different than 2. In 1966 Shi [Shi] extended this result onto a very large class of polyhedra. Then in 1979 Boju Jiang extended this class onto all finite polyhedra that satisfy two conditions. The first condition: "there are no locally separating points in the polyhedron" seems to be natural, since separating points make the space to "wild" for such a nice theorem. Then if we restrict to the polyhedra without locally separating points the second assumption is that "the polyhedron is not a surface of negative Euler characteristic" (the original domain of Nielsen's research!). In the 1980s (forty years after Wecken) it turned out that Wecken's theorem is not true exactly for these surfaces. The first counterexamples were given by Fadell and Husseini (1982) [FaHu1] (in the noncompact case), Boju Jiang in [Ji5] and [Ji6]. Then there appeared many papers proving that in the case of surfaces the cardinality of the fixed point set is usually much larger than the Nielsen number. Finally in [Ke] Michael Kelly showed that this difference can be arbitrarily large. The last result was also shown by a different method by Xingguo Zhang ([Zha]). On the other hand if f is an embedding of a surface, on the contrary, $\max_{g\sim f} \#\mathrm{Fix}(g) = N(f)$. However this equality is not true if we assume only that f is a local embedding [FeGo].

An analogy with the famous Poincaré conjecture (every closed orientable n-manifold M with $\pi_k(M) = 0$ for $0 \leq k \leq n-1$ is homeomorphic to S^n) is evident. This conjecture was originally formulated by Poincaré in dimension $n = 3$, was solved in higher dimensions after many decades and only the dimension 3 was open until recent years when Perelmann presented its proof (not yet published in a complete form). The analogy is not casual since in both cases the proofs do not work in low dimension since then Whitney lemma does not hold.

Thus one can say that the problem of realizing the Nielsen number is almost solved for polyhedra. Surprisingly almost no progress was made in extending this theorem beyond polyhedra. It is still open whether the fixed point set can be made finite after a deformation of a self-map of an arbitrary ENR.

(4.2.1) THEOREM. *Every self-map $f\colon M \to M$ of a compact manifold of dimension $d \neq 2$ is homotopic to a map g realizing the Nielsen number: $\#\mathrm{Fix}(g) = N(f)$.*

PROOF. In dimension $d = 1$ there are only two compact manifolds: interval $[a, b]$ and the sphere S^1. Every self-map of the interval is homotopic to the constant which realizes $N(f) = 1$. The case of S^1 was discussed in Section 4.1.2 where it was shown that a Nielsen number is realized by the maps $f(z) = z^k$ for $k \neq 1$, and $f(z) = z$ is homotopic to a fixed point free map by a small twist for $k = 1$.

Let $\dim M \geq 3$. If M has the boundary, then it may be deformed into $\operatorname{int} M$ so we may assume that the image of f, hence all fixed points, are contained in $\operatorname{int} M$. We apply Lemma (4.2.3) to make $\mathrm{Fix}(f)$ finite. Then we use successively Lemma (4.2.4) to reduce each Nielsen class to a single point. At last Lemma (2.2.30) removes inessential classes. Thus all Nielsen classes of the obtained map are essential and each of them contains exactly one point, hence the cardinality of the fixed point set equals $N(f)$.

The proof will be complete once the below lemmas are proved. □

(4.2.2) Lemma. *Let $A \subset M$ be a compact subset disjoint from* $\mathrm{Fix}(f)$. *Then there is a number $\varepsilon > 0$ such that* $\mathrm{d}(f, f') < \varepsilon$ *and* $f'(x) = f(x)$ (*for all* $x \notin A$) *imply* $\mathrm{Fix}(f') = \mathrm{Fix}(f)$.

Proof. Since A is compact, $\eta = \inf\{\mathrm{d}(x, f(x)) : x \in A\} > 0$. We put $\varepsilon = \eta/2$ and we assume that $\mathrm{d}(f(a), f'(a)) < \varepsilon$ for $a \in A$ and $f(x) = f'(x)$ for $x \notin A$. It is evident that, for any $x \notin A$, $x \in \mathrm{Fix}(f)$ if and only if $x \in \mathrm{Fix}(f')$. On the other hand for $x \in A$,

$$\mathrm{d}(x, f'(x)) \geq \mathrm{d}(x, f(x)) - \mathrm{d}(f(x), f'(x)) \geq 2\varepsilon - \varepsilon = \varepsilon > 0,$$

hence no new fixed point in A appears. □

Transversality arguments give us the following.

(4.2.3) Lemma. *Any self-map of a compact manifold $f\colon M \to M$ is homotopic to a map with finite fixed point set. Moreover, the homotopy may be arbitrarily small.*

(4.2.4) Lemma. *Let* $\dim M \geq 3$, *let* $\mathrm{Fix}(f)$ *be finite and contained in* $\mathrm{int}\, M$. *Suppose that $x_0 \neq x_1 \in \mathrm{Fix}(f)$ are Nielsen related. Then there is a homotopy f_t constant in a neighbourhood of* $\mathrm{Fix}(f) \setminus \{x_0, x_1\}$ *satisfying $f_0 = f$ and* $\mathrm{Fix}(f_1) = \mathrm{Fix}(f_0) \setminus \{x_1\}$.

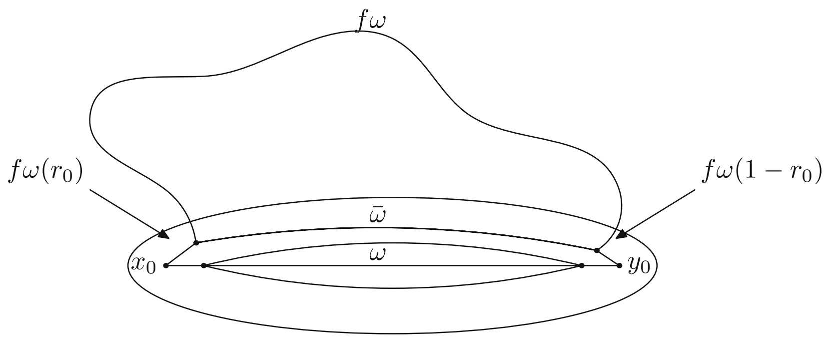
Proof. Since $\dim M \geq 3$, we may assume that a path ω realizing the Nielsen relation is a flat arc in M. We may also fix a Euclidean neighbourhood $\mathcal{V}$ M satisfying $\mathrm{Fix}(f) \cap \mathcal{V} = \{x_0, x_1\}$ and ω is a segment in $\mathcal{V} = \mathbb{R}^m$.

First we consider a special case: we assume moreover that $f\omega \subset \mathcal{V}$, i.e. ω and $f\omega$ are close.

Since $f(\omega) \subset \mathcal{V}$, there is a smaller convex neighbourhood $\mathcal{V}_0 \subset \mathcal{V} = \mathbb{R}^m$ satisfying $\omega[0,1] \subset \mathcal{V}_0 \subset \mathrm{cl}\,\mathcal{V}_0 \subset \mathcal{V}$ and $f(\mathrm{cl}\,\mathcal{V}_0) \subset \mathcal{V}$. Since $\mathcal{V} = \mathbb{R}^m$, the restriction $f\colon \mathrm{cl}\,\mathcal{V}_0 \to \mathcal{V}$ is homotopic (rel. boundary) to the map $f'((1-t)x + tx_0) = (1-t)f(x) + tx_0$ ($x \in \mathrm{bd}\,\mathcal{V}_0$). The homotopy may be extended, by the constant, to the whole M and we obtain the map $f'\colon M \to M$ satisfying $\mathrm{Fix}(f') = \mathrm{Fix}(f) \setminus \{x_1\}$. In fact since $x \neq f'(x)$ for all $x \in \mathrm{bd}\,\mathcal{V}_0$, x_0 is the only fixed point of f' in $\mathrm{cl}\,\mathcal{V}_0$.

Now we consider the general case: The paths ω and $f\omega$ may not be contained in a Euclidean manifold. We will show that there is a homotopy reducing this case to the previous one. The main difficulty in the construction of this homotopy is to avoid producing new fixed points. In fact this is the only place when the assumption on dimension (≥ 3) is essential.

Let us fix a Euclidean neighbourhood $\mathcal{V}$ $\omega[0,1]$. Then there is an $r_0 > 0$ such that $f\omega[0, r_0] \cup f\omega[1-r_0, 1] \subset \mathcal{V}$. We may assume that $f\omega[r_0, 1-r_0] \cap \omega[0,1] = \emptyset$. In fact, a small homotopy with the carrier in a small neighbourhood of $\omega[r_0, 1-r_0]$ (set



disjoint from $\mathrm{Fix}(f)$) takes $f\omega[r_0, 1-r_0]$ off the segment $\omega[0,1]$. Let $\overline{\omega}\colon [r_0, 1-r_0] \to \mathcal{V} \quad \omega[0,1]$ ($\approx \mathcal{V} \qquad = \mathbb{R}^n \setminus *$) be a path joining the point $f\omega(r_0)$ with $f\omega(1-r_0)$.

Now the homotopy between $f\omega$ and ω implies a fixed end point homotopy $h\colon [r_0, 1-r_0] \times I \to M$ between the restriction $f\omega_{|[r_0,1-r_0]}$ and $\overline{\omega}$. Since $h(t,0)$, $h(t,1) \notin \omega[0,1]$ and $\dim M \geq 3$, the homotopy h may be deformed rel. $\mathrm{bd}\,([r_0, 1-r_0] \times [0,1])$ to a homotopy into $M \setminus \omega[0,1] \approx M \setminus *$.

Now the compact sets $h([r_0, 1-r_0] \times [0,1])$ and $\omega[0,1]$ are disjoint, hence there is a convex neighbourhood $\mathcal{V}_1$ satisfying $\omega[0,1] \cap \mathcal{V}_1 = \omega(r_0, 1-r_0)$, $\mathrm{cl}\,\mathcal{V}_1 \subset \mathcal{V}$, $h([r_0, 1-r_0] \times [0,1]) \cap \mathrm{cl}\,\mathcal{V}_1 = \emptyset$. If moreover $\mathcal{V}_1$ is sufficiently thin, $\mathrm{cl}\,\mathcal{V}_1 \cap f(\mathrm{cl}\,\mathcal{V}_1) = \emptyset$. We extend the homotopy h onto $\mathrm{bd}\,\mathcal{V}_1 \cup \omega[r_0, 1-r_0] \to M$ putting $h(x,t) = f(x)$ for $x \in \mathrm{bd}\,\mathcal{V}_1$. The obtained map is continuous since $h(\omega(r_0), \cdot\,)$ and $h(\omega(1-r_0), \cdot\,)$ are constant. Let us notice that $h((\mathrm{bd}\,\mathcal{V}_1 \cup \omega[r_0, 1-r_0]) \times I) \subset M \setminus \mathrm{cl}\,\mathcal{V}_1$. On the other hand also $f(\mathrm{cl}\,\mathcal{V}_1) \subset M \setminus \mathrm{cl}\,\mathcal{V}_1$. Now the Homotopy Extension Property gives an extension $h'\colon \mathrm{cl}\,\mathcal{V}_1 \times I \to M \setminus \mathrm{cl}\,\mathcal{V}_1$. The last homotopy may be extended to $h''\colon M \times I \to M$ by $h''(x,t) = f(x)$ for $x \notin \mathcal{V}_1$. Now $f_1 = h''(\,\cdot\,, 1)$ is the desired map.

It remains to notice that

(1) $\mathrm{Fix}(h(\,\cdot\,, t))$ does not depend on t,
(2) f_1 satisfies the special case i.e. $f_1(\omega[0,1]) \subset \mathcal{V}$.

In fact $h(x,t) \notin \mathrm{cl}\,\mathcal{V}_1$, for all $x \in \mathrm{cl}\,\mathcal{V}_1$ (carrier of the homotopy h) which implies (1). On the other hand $f_1(\omega(s)) = h(s,1) \subset \mathcal{V}$ hence f_1 is the special case. □

4.2.1. Wecken theorem on polyhedra.

(4.2.5) Definition. A point $x \in X$ of a topological space X is called a local separating point if there is an open connected subset $\mathcal{U} \quad X$ such that $\mathcal{U} \quad x$ is not connected.

It turns out that to get a Wecken theorem it is reasonable to assume that the considered polyhedron has no local separating points. In general such points may be an obstruction to the Wecken Theorem: see Section 4 of Chapter IV in [Ki].

(4.2.6) LEMMA. *Let X be a polyhedron. A point $x \in X$ is a local separating point if and only if $H_1(X, X \setminus x) \neq 0$.*

PROOF. With no lack of generality we may assume that x is connected. The lemma is obvious for $X=$ a point, hence we may assume that the dimension of each component is ≥ 1.

($\Rightarrow$) Let $\mathcal{U} \quad X$ be a connected neighbourhood with $\mathcal{U} \quad x$ disconnected. From the homology exact sequence of the pair $(\mathcal{U}, \mathcal{U} \quad x)$ we have

$$H_1(\mathcal{U}, \mathcal{U} \quad x) \xrightarrow{\partial_*} H_0(\mathcal{U} \quad x) \xrightarrow{i_*} H_0(\mathcal{U})$$

where $H_0(\mathcal{U}) = \mathbb{Z}$, $H_0(\mathcal{U} \quad x) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ with the number of summands ≥ 2, since $\mathcal{U} \quad x$ is not connected. Thus $0 \neq \operatorname{Ker} i_* = \operatorname{im} \partial_*$ hence $H_1(\mathcal{U}, \mathcal{U} \quad x) \neq 0$ and the excision isomorphism gives $H_1(X, X \setminus x) \neq 0$.

($\Leftarrow$) Let $H_1(X, X \setminus x) \neq 0$. We will show that for any contractible neighbourhood $\mathcal{U}$ of x, (for example for the open star: $\mathcal{U} = \operatorname{st} x$) $\mathcal{U} \quad x$ is disconnected. By the excision $H_1(\mathcal{U}, \mathcal{U} \quad x) \neq 0$. We prove by a contradiction: we assume that $\mathcal{U} \quad x$ is connected. We consider again the exact sequence

$$H_1(\mathcal{U}) \xrightarrow{j_*} H_1(\mathcal{U}, \mathcal{U} \quad x) \xrightarrow{\partial_*} H_0(\mathcal{U} \quad x) \xrightarrow{i_*} H_0(\mathcal{U}).$$

By connectedness of $\mathcal{U} \quad x$, $i_* \colon H_0(\mathcal{U} \quad x) \to H_0(\mathcal{U})$ is the isomorphism hence $\operatorname{Im} \partial_* = \ker i_* = 0$. On the other hand $H_1(\mathcal{U}) = 0$ implies ∂_* mono, hence $H_1(\mathcal{U}, \mathcal{U} \quad x) = 0$ and we get a contradiction. □

The next lemma gives a useful characterization of polyhedra without local separating points. Let X again denote a compact polyhedron with a fixed triangulation.

(4.2.7) LEMMA. *X has no local separating points if and only if each maximal simplex has dimension ≥ 2 and the boundary of the star of each vertex is connected.*

PROOF. ($\Rightarrow$) X can not have a maximal simplex of dimension 1, since each point of a maximal 1-simplex is the local separating point. Let v be a vertex. Since v is not a local separating point,

$$0 = H_1(X, X \setminus v) = H_1(\operatorname{st} v, \operatorname{st} v \setminus v) = H_1(\operatorname{cl}(\operatorname{st} v), \operatorname{bd}(\operatorname{st} v)),$$

hence from the exact homology sequence of $(\operatorname{cl}(\operatorname{st} v), \operatorname{bd}(\operatorname{st} v))$ the natural homomorphism $H_0(\operatorname{bd}(\operatorname{st} v)) \to H_0(\operatorname{cl}(\operatorname{st} v)) = \mathbb{Z}$ is mono, hence the free group $H_0(\operatorname{bd}(\operatorname{st} v))$ may not have more than one generator, hence $\operatorname{bd}(\operatorname{st} v)$ is connected.

($\Leftarrow$) Let $x \in X$ be not a vertex and let the simplex σ be the carrier of x. Let $\mathcal{U}$ be an open connected set containing x. Since σ is a (possibly not proper) face of a simplex σ' of dimension ≥ 2, each path in $\mathcal{U}$ joining two points different than x can be deformed near x to a path avoiding x which proves that $\mathcal{U}$ x is connected.

Let $v \in X$ be a vertex. Then by the above lemma $\mathrm{bd}(\mathrm{st}\, v)$ is connected $\Leftrightarrow$ $H_1(X, X \setminus v) = 0 \Leftrightarrow v$ is not a local separating point. $\square$

Our aim is to prove

(4.2.8) THEOREM (Wecken Theorem for polyhedra). *Let X be a compact connected polyhedron without local separating points. We moreover assume that X is not a 2-manifold. Then each self-map $f\colon X \to X$ is homotopic to a map g satisfying $\#\mathrm{Fix}(g) = N(f)$.*

SCHEME OF THE PROOF. By Lemma (4.2.16) we may assume that $\mathrm{Fix}(f)$ is finite and each fixed point is lying inside a maximal simplex. Suppose for a moment that no two different fixed points are Nielsen related. If a class represented by a fixed point is inessential, its index is zero, hence we may remove this point by a local homotopy. Then the remaining fixed points represent different essential classes, hence $\#\mathrm{Fix}(f) = N(f)$.

Thus it is enough to show that each Nielsen class may be reduced to a point. As in the case of manifolds we will be coalescing successively two Nielsen related fixed points. First we will show that the last is possible if a path ω, establishing the Nielsen relation, and its image $f\omega$ are near. Here we will use only the assumption that X has no local separating points. Then we will show that the general case can be reduced to the special case. Here the assumption that X is not a 2-manifold is essential.

We start by reformulating the assumptions on the polyhedron X.

(4.2.9) LEMMA. *Let X be a connected polyhedron without local separating points. Assume that X is different from a point. Then the following three conditions are equivalent:*

(4.2.9.1) *X is not a 2-manifold.*

(4.2.9.2) *There is a 1-simplex which is a common face of at least three 2-simplices (in a triangulation).*

(4.2.9.3) *the space $\top \times I$ (here $\top$ denotes three segments with a common end) can be imbedded into X.*

PROOF. Recall that each maximal simplex in a polyhedron without local separating points must be at least two dimensional.

(4.2.9.1) $\Rightarrow$ (4.2.9.2) is obvious if $\dim X \geq 3$. Now let $\dim X = 2$. If each 1-simplex were a face of at most two 2-simplices then X would be a 2-pseudomanifold hence a 2-manifold.

(4.2.9.2) $\Rightarrow$ (4.2.9.3) obvious.

(4.2.9.3) $\Rightarrow$ (4.2.9.1) Let us fix the point $x = (v, 1/2) \in \top \times I$ where v is the common vertex of three segments in $\top$. Suppose that $j\colon \top\times I \to X$ is an imbedding into a 2-manifold. This would induce an isomorphism between $H_1(\top\times I, \top\times I\backslash x) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_1(j(\top \times I), j(\top \times I) \setminus j(x)) = H_1(X, X \setminus j(x)) = \mathbb{Z}$ which gives a contradiction. □

Let us fix the notation. We consider a connected polyhedron X with a finite triangulation K. Let $\{v_i\}_{i\in J}$ be the family of all vertices of K. Then each point $x \in X$ can be written uniquely as $x = \sum_{i\in J} \lambda_i v_i$ where $\lambda_i \geq 0$ and $\sum_{i\in J} \lambda_i = 1$ and where $\{i \in J : \lambda_i \neq 0\}$ is a simplex in K. When the triangulation K is fixed, we may identify a point $x \in X$ with the sequence of its coordinates: $x = (\lambda_i)_{i\in J}$.

Let

$$\sigma(x) = \Big\{ \sum_{i\in J} \lambda'_i v_i : \lambda'_i \neq 0 \leftrightarrow \lambda_i \neq 0 \Big\}$$

denote the (open) simplex containing x. Then its closure equals

$$\mathrm{cl}\,(\sigma(x)) = \Big\{ \sum_{i\in J} \lambda'_i v_i : \lambda'_i \neq 0 \Rightarrow \lambda_i \neq 0 \Big\}.$$

We define the star of a point x, as the open subset

$$\mathrm{st}\, x = \{y \in X : \sigma(x) \subset \mathrm{cl}\,(\sigma(y))\} = \Big\{ \sum_{i\in J} \lambda'_i v_i : \lambda_i \neq 0 \Rightarrow \lambda'_i \neq 0 \Big\}.$$

Let us denote subsets

$$\mathcal{V}(x) := \Big\{ \sum_{i\in J} \lambda'_i v_i : \lambda_i \neq 0 \text{ and } \lambda'_i \neq 0 \text{ for an } i \in J \Big\}.$$

We notice that for $x, y \in \sigma$ we have the equality $\mathcal{V}(x) = \mathcal{V}(y)$ which allows to define $\mathcal{V}(\sigma) = \mathcal{V}(x)$ for an $x \in \sigma$.

(4.2.10) Lemma. *Let σ be a maximal simplex in X. Then there is a map $\alpha\colon \mathrm{cl}\,(\sigma) \times \mathcal{V}(\sigma) \times I \to \mathcal{V}(\sigma)$ satisfying:*

(4.2.10.1) $\alpha(x, y, 0) = x$, $\alpha(x, y, 1) = y$, $\alpha(x, x, t) = x$.

(4.2.10.2) *If $x \neq y$ then the map $\alpha(x, y, \cdot)\colon I \to X$ is mono.*

(4.2.10.3) *If $x, y \in \sigma(x)$, $z \in \mathcal{V}(\sigma)$ and $0 < t \leq 1$, then $\alpha(x, y, t) = \alpha(x, z, t)$ implies $y = z$.*

Proof. We start with defining a natural deformation retraction $r_t\colon \mathcal{V}(\sigma) \to \mathcal{V}(\sigma)$. We identify the point $x \in \mathcal{V}(\sigma)$ with the sequence of its coordinates (with respect to a fixed triangulation) $x = (x_i)_{i\in J}$. Let $J_\sigma \subset J$ be the set of vertices spanning the simplex σ. We put $r_t(x_i) = (x_i')$ where

$$x_i' = \begin{cases} \dfrac{x_i}{\sum_{j\in J_\sigma} x_j + \sum_{j\notin J_\sigma} t\cdot x_j} & \text{for } i \in J_\sigma, \\ \dfrac{t\cdot x_i}{\sum_{j\in J_\sigma} x_j + \sum_{j\notin J_\sigma} t\cdot x_j} & \text{for } i \notin J_\sigma. \end{cases}$$

Now $r_1 = \mathrm{id}_{\mathcal{V}(\sigma)}$, r_0 is a projection onto $\mathrm{cl}\,\sigma$.

For $x, y \in \mathrm{cl}\,\sigma$ we define $\alpha(x,y,t) = (1-t)x + ty$. If $y \in V(\sigma)\setminus\sigma$, we define $\alpha(x,y,\,\cdot\,)$ as the path running through the broken line $[x, r_0(y), y]$ with constant velocity. Since $r_0(y) = y$ for $y \in \mathrm{bd}\,\sigma$, α is well defined and continuous. The properties are evident. □

We say that a self-map $f\colon X \to X$ is a *proximity map* on a subset $A \subset X$ if $f(a) \in \mathcal{V}(a)$ for all $a \in A$.

Before we start to unite the points from the same Nielsen class we show that if f is a proximity map on a convex set lying inside a maximal simplex, then all the fixed points in this set can be replaced with one single fixed point. For a subset $A \subset X$ of a compact polyhedron and $\eta > 0$ we denote $\mathcal{U}(A,\eta) = \{x \in X : \mathrm{dist}(x,A) < \eta\}$, where the distance is taken with the respect to the metric $\mathrm{d}(x,y) = \sqrt{\sum(x_i - y_i)^2}$. The below lemma was shown by Shi.

(4.2.11) Lemma. *Let $a \in X$ be a point lying in a maximal simplex $\sigma(a)$ and let $\eta > 0$ satisfy:* $\mathrm{cl}\,(\mathcal{U}(a,\eta)) \subset \sigma(a)$, $\mathrm{Fix}(f) \cap \mathrm{bd}\,\mathcal{U}(a,\eta) = \emptyset$, *$f$ is a proximity map on* $\mathrm{cl}\,\mathcal{U}(a,\eta)$. *Then there is a map $f'\colon X \to X$ satisfying:*

(4.2.11.1) *f, f' are homotopic on* $\mathrm{cl}\,(\mathcal{U}(a,\eta))$.

(4.2.11.2) $\mathrm{Fix}(f') \cap \mathrm{cl}\,(\mathcal{U}(a,\eta)) = \{a\}$.

(4.2.11.3) *f' is a proximity map on* $\mathrm{cl}\,(\mathcal{U}(a,\eta))$.

Proof. We start with the special case assuming that $f(\mathrm{bd}\,\mathcal{U}(a,\eta)) \subset \sigma(a)$. Now we may define the map $f'\colon X \to X$ by the formula

$$f'(x) = \begin{cases} (1-t)a + tf(y) & \text{for } x = (1-t)a + ty \in \mathrm{cl}\,\mathcal{U}(a,\eta)),\ y \in \mathrm{bd}\,\mathcal{U}(a,\eta), \\ f(x) & \text{for } x \notin \mathcal{U}(a,\eta). \end{cases}$$

Then the restriction of f and f' to $\mathrm{cl}\,\mathcal{U}(a,\eta)$, as mappings into $\sigma(x)$, are homotopic relative the boundary. Since $f(y) \neq y$ for $y \in \mathrm{bd}\,\mathcal{U}(a,\eta)$, (4.2.11.2) follows. On the other hand $f'(\mathrm{cl}\,\mathcal{U}(a,\eta)) \subset \sigma(a)$ implies (4.2.11.3).

Now we reduce the general situation to the above special case. Since $f(\mathrm{cl}\,\mathcal{U}(a,\eta)) \subset \mathcal{V}(a)$, $f(x) \in \mathcal{V}(a) = \mathcal{V}(x)$ for each $x \in \mathrm{cl}\,\mathcal{U}(a,\eta)$, hence $\alpha(x, f(x), t)$ is defined. The compactness of $\mathrm{cl}\,\mathcal{U}(a,\eta) \subset \sigma(a)$ implies the existence of an $\varepsilon > 0$ such that $\alpha(x, f(x), t) \in \sigma(a)$ for $0 \le t \le \varepsilon$ and $x \in \mathrm{cl}\,\mathcal{U}(a,\eta)$. Since $\mathrm{Fix}(f) \cap (\mathrm{bd}\,\mathcal{U}(a,\eta)) =$

$\emptyset$, there exists $0 < \eta' < \eta$ such that $\mathrm{Fix}(f) \cap (\mathrm{cl}\,\mathcal{U}(a,\eta) \setminus \mathcal{U}(a,\eta')) = \emptyset$. Let $\lambda\colon [0,\eta] \to \mathbb{R}$ be an increasing function satisfying $\lambda(0) = 0$, $\lambda(\eta') < \varepsilon$, $\lambda(\eta) = 1$. Now we consider the map

$$f'(x) = \begin{cases} f(x) & \text{for } x \notin \mathcal{U}(a,\eta), \\ \alpha(x, f(x), \lambda(\mathrm{d}(x,a)) & \text{for } \mathrm{cl}\,\mathcal{U}(a,\eta). \end{cases}$$

Now f, f' are homotopic on $\mathrm{cl}\,\mathcal{U}(a,\eta)$ (as the maps into the contractible subspace $V(\sigma)$). Moreover, f' is a proximity map on $\mathrm{cl}\,\mathcal{U}(a,\eta)$ and $f'(\mathrm{cl}\,\mathcal{U}(a,\eta')) \subset \sigma(a)$. Moreover, for $x \in \mathrm{cl}\,\mathcal{U}(a,\eta) \setminus \mathcal{U}(a,\eta')$, we have $\mathrm{d}(a,x) > 0$ and $f(x) \neq x$ which implies $f'(x) = \alpha(x, f(x), \lambda(\mathrm{d}(a,x)) \neq x$ (property (4.2.10.3)), there are no fixed points of f' in $\mathrm{cl}\,\mathcal{U}(a,\eta) \setminus \mathcal{U}(a,\eta')$. Now we may apply the first part to $\mathrm{cl}\,\mathcal{U}(a,\eta')$. □

(4.2.12) Remark. The above lemma remains valid if we replace the ball $\mathcal{U}(a,\eta)$ with an open convex subset C whose closure is contained in $\sigma(a)$. We fix a point $a \in D$ and define $D\colon C \to \mathbb{R}$ by the formula $D(x) = \mathrm{d}(x,a)/\mathrm{d}(y,a)$ where y is the intersection point of $\in \mathrm{bd}\, C$ with the half-line starting from a and passing through x. Then we follow the above proof using $D(x)$ instead of $\mathrm{d}\,(a,x)$.

(4.2.13) Lemma. *Let σ_1 and σ_2 be open maximal simplices (of dimension ≥ 2), $\dim(\mathrm{cl}\,\sigma_1 \cap \mathrm{cl}\,\sigma_2) \geq 1$, $a \in \sigma_1$, $b \in \mathrm{int}(\mathrm{cl}\,\sigma_1 \cap \mathrm{cl}\,\sigma_2)$. Let $f\colon X \to X$ be a continuous map satisfying:*

(4.2.13.1) *a is an isolated fixed point,* $\mathrm{Fix}(f) \cap [a,b] = \{a\}$, $\mathrm{ind}(f;a) = j \neq 0$,
(4.2.13.2) *f is a proximity map on $[a,b]$.*

Then there exist $\varepsilon > 0$ and a map $f'\colon X \to X$ satisfying:

(4.2.13.3) *f, f' are homotopic on $\mathcal{U}([a,b],\varepsilon)$,*
(4.2.13.4) $\mathrm{Fix}(f') \cap \mathrm{cl}\,(\mathcal{U}([a,b],\varepsilon)) = \{c\}$ *a point in σ_2,*
(4.2.13.5) *f' is a proximity map on* $\mathrm{cl}\,(\mathcal{U}[a,b],\varepsilon)$,
(4.2.13.6) *the Nielsen classes of the fixed points $a \in \mathrm{Fix}(f)$ and $c \in \mathrm{Fix}(f')$ correspond by the above homotopy.*

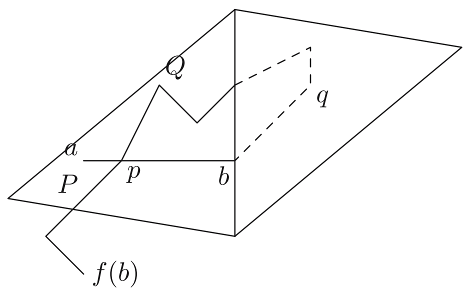
Proof. We notice that for a sufficiently small $\varepsilon > 0$:

(1) $\mathrm{cl}\,(\mathcal{U}(b,\varepsilon)) \subset \mathrm{st}\, b$,
(2) $f(\mathrm{cl}\,(\mathcal{U}(b,\varepsilon))) \cap \mathrm{cl}\,(\mathcal{U}(b,\varepsilon)) = \emptyset$,
(3) $\mathrm{Fix}(f) \cap \mathrm{cl}\,(\mathcal{U}(a,\varepsilon)) = a$,
(4) f is a proximity map on $\mathrm{cl}\,(\mathcal{U}[a,b],\varepsilon)$.

Let us fix points $p \in (a,b] \setminus \mathrm{cl}\,(\mathcal{U}(b,\varepsilon))$, $q \in \sigma_2 \setminus \mathcal{U}(b,\varepsilon)$.

Since $\mathrm{st}(b) \setminus \mathcal{U}(b,\varepsilon)$ is path-connected, there is a broken line Q joining the points q with p in this set. Moreover, there is a path P joining $f(b)$ and p in $\mathcal{V}(b)$. We define the path $r = P * ([p, b, q] * Q)^j$ and the map

$$f'(x) = \begin{cases} f(x) & \text{for } x \notin \mathcal{U}(b, \varepsilon), \\ f((2\mathrm{d}(x, b)/\varepsilon - 1)x) & \text{for } \mathcal{U}(b, \varepsilon) \setminus \mathcal{U}(b, \varepsilon/2), \\ r[1 - 2\mathrm{d}(x, b)/\varepsilon] & \text{for } x \in \mathcal{U}(b, \varepsilon/2). \end{cases}$$



We notice that this map has no fixed point in $\mathcal{U}(b, \varepsilon) \setminus \mathcal{U}(b, \varepsilon/2)$ since $f'(\mathcal{U}(b, \varepsilon) \setminus \mathcal{U}(b, \varepsilon/2)) \subset f(\mathcal{U}(b, \varepsilon))$ and the last set is disjoint from $\mathcal{U}(b, \varepsilon)$.

It remains to consider $\mathcal{U}(b, \varepsilon/2)$. We notice that $f'(\mathcal{U}(b, \varepsilon)) \cap \mathcal{U}(b, \varepsilon/2) \subset [a, b, q] \cap \mathcal{U}(b, \varepsilon/2)$. The last set consists of two segments $S = (S \cap [a, b]) \cup (S \cap [b, q])$. The analysis of the map r shows that the restriction of f' to the segment $S \cap (a, b]$ runs $|j|$ times through it. Each time we have an extending map, hence each time we have exactly one fixed point. Moreover, the fixed point index (of the restriction of f to the segment) at this point equals $-\mathrm{sgn}(j)$ (for $j > 0$ the restriction near the fixed point is reversing the orientation). By the Commutativity Property, the index of the map $f'\colon X \to X$ near the fixed point is the same. An analysis of the restriction of f' to $[b, q]$ gives $|j|$ fixed points in $[b, q]$, each of index $\mathrm{sgn}(j)$ (here f' preserves the orientation of the segment).

Now we apply Remark (4.2.12) to the convex subset of $\mathcal{U}([a, b], \varepsilon) \cap \sigma_1$ containing a and $|j|$ new fixed points. We get a single fixed point of index $j - \mathrm{sgn}(j)|j| = 0$ hence this point may be removed by a small local deformation. Similarly we find a convex set in σ_2 containing new fixed points which can be united to a point $c \in \sigma_2$ (of index $= j$).

The maps f, f' are homotopic on $\mathcal{U}(b, \varepsilon)$ as the proximity maps, since $f(\mathcal{U}(b, \varepsilon) \cup f'(\mathcal{U}(b, \varepsilon)) \subset \mathcal{V}(b)$ and the last set is contractible. $\square$

The above lemma allows us to shift a fixed point from one maximal simplex to a neighboring one, along a path on which f is a proximity map. As a consequence we get the main result of this subsection.

(4.2.14) LEMMA. *Let X be a compact polyhedron without local separating points. Let $f\colon X \to X$ be a self-map with* $\mathrm{Fix}(f)$ *finite and let $a, b \in$* $\mathrm{Fix}(f)$ *lie inside* *maximal simplices. We assume moreover that there is a path* ω *joining these points such that the restriction of* f *to* $\omega[0,1]$ *is a proximity map. Then there is a homotopy* f_t *starting from* $f_0 = f$ *and satisfying* $\mathrm{Fix}(f_1) = \mathrm{Fix}(f) \setminus a$. *Moreover, the carrier of the homotopy can be contained in a prescribed neighbourhood of* ω.

PROOF. We may approximate ω by a broken line, avoiding vertices and such that $\omega(t)$ belongs to maximal simplices, for all but a finite number of $t \in I$. Moreover, in these exceptional t, ω is passing from one maximal simplex into another. Now we may use the above lemma to shift the fixed point a along ω to the simplex $\sigma(b)$. Finally we use Lemma (4.2.11) to unite the two fixed points. □

Thus Wecken's theorem will be proved once we show that for any two fixed points from the same Nielsen class there is a homotopy $\{f_t\}$ such that $\mathrm{Fix}(f_t)$ is constant and that f_1 is a proximity map on a path joining these two points.

(4.2.15) COROLLARY. *Let* X *be a compact polyhedron without local separating points. If moreover* $\chi(X) = 0$, *then the identity map* id_X *is homotopic to a fixed point free map. Otherwise* id_X *is homotopic to a map with exactly one fixed point.*

PROOF. By Lemma (4.2.14) we get a small homotopy from the identity to a map f with $\mathrm{Fix}(f)$ finite and each fixed point is lying inside a maximal simplex. Then we find a normal PL-arc running over $\mathrm{Fix}(f)$. Now we may apply successively Lemma (4.2.13) to shift these points to one point. If moreover $\chi(X) = 0$ then the index at this point is zero, hence the fixed point can be removed by a small homotopy. □

4.2.2. A fixed point set can be made finite. We are going to prove

(4.2.16) LEMMA. *Each self-map of a compact polyhedron* $f\colon X \to X$ *is homotopic to a map* g *such that* $\mathrm{Fix}(g)$ *is finite and each fixed point is contained inside a maximal simplex of a fixed triangulation. Moreover, the homotopy may be arbitrarily small.*

This result was proved by Hopf. We refer the reader to [Br2] for a proof based on so called Hopf construction. Here we present an alternative proof that may be a little more difficult, but which is more geometric. We consider the polyhedron rather as a CW-complex, so to get the deformation we do not need to pass several times to the baricentric subdivision as in the original Hopf method. We start with an auxiliary lemma.

(4.2.17) LEMMA. *Each self-map* $f\colon X \to X$ *of a compact* CW-*complex is homotopic to a map* g *satisfying:*

(4.2.17.1) *For each* $k = 1, \dots, \dim X$ *there is a neighbourhood* $\mathcal{U}_k \subset X$ *of the skeleton* $X^{(k)}$ *such that* $g(\mathcal{U}_k) \subset X^{(k)}$.

(4.2.17.2) *The fixed point set splits into the mutually disjoint closed-open subsets*

$$\mathrm{Fix}(g) = \bigcup_\sigma \mathrm{Fix}(g) \cap \sigma,$$

where the summation runs throughout the family of (*open*) *cells in* X.

(4.2.17.3) *There exist open sets* $\{\mathcal{V}_\sigma\}$ *satisfying* $\mathrm{cl}\,\mathcal{V}_\sigma \cap \mathrm{cl}\,\mathcal{V}_{\sigma'} = \emptyset$ (*for* $\sigma \neq \sigma'$), $\mathrm{Fix}(g) \cap \sigma \subset \mathcal{V}_\sigma$, $g(V_\sigma) \subset \sigma$.

Proof. We may assume that f is a cellular map.

(4.2.17.1) We define $g = f \cdot R$ where $R\colon X \to X$ is the deformation from the proof of Theorem (2.4.1). Then $g(\mathcal{U}_k) = fR(\mathcal{U}_k) \subset f(X^{(k)}) \subset X^{(k)}$.

(4.2.17.2) Let σ be a cell of dimension s. We show that $\mathrm{Fix}(g) \cap \sigma$ is closed and open in $\mathrm{Fix}(g)$. By the above $\mathrm{Fix}(g) \cap \sigma \subset \sigma \setminus \mathcal{U}_{s-1}$ implies that $\mathrm{Fix}(g) \cap \sigma = \mathrm{Fix}(g) \cap (\sigma \setminus \mathcal{U}_{s-1})$ is closed in X. Now $\mathrm{Fix}(g) = \bigcup_\sigma \mathrm{Fix}(g) \cap \sigma$ is the sum of mutually disjoint closed summands. Since the sum is finite, all summands are also open in $\mathrm{Fix}(g)$.

(4.2.17.3) Since the sets $\mathrm{Fix}(f) \cap \sigma$ are isolated, we can take neighbourhoods $\mathcal{V}'_\sigma$ satisfying $\mathrm{Fix}(f) \cap \sigma \subset \mathcal{V}'_\sigma$, $f(\mathcal{V}'_\sigma) \subset X^{(s)}$ and $\mathrm{cl}\,\mathcal{V}'_\sigma \cap \mathrm{cl}\,\mathcal{V}'_{\sigma'} = \emptyset$, for $\sigma \neq \sigma'$. We put $\mathcal{V}_\sigma = \mathcal{V}'_\sigma \cap f^{-1}\sigma$. □

Proof of Lemma(4.2.16). Let $\{\mathcal{V}_\sigma\}$ be the family of open subsets from Lemma (4.2.17) (we regard the polyhedron X as a CW-complex). Then $\mathrm{Fix}(f) = \bigcup_\sigma \mathrm{Fix}(f) \cap \mathcal{V}_\sigma$ is the mutually disjoint sum of closed-open subsets (σ runs throughout the family of all cells in X). For each non-maximal simplex σ

- we define a Urysohn function $\lambda_\sigma\colon X \to [0,1]$, satisfying

$$\lambda_\sigma(\mathrm{Fix}(f) \cap \mathrm{cl}\,\mathcal{V}_\sigma) = 1, \qquad \lambda_\sigma(X \setminus \mathcal{V}_\sigma) = 0,$$

- we fix a maximal simplex $\sigma_{\max}$ such that $\sigma \subset \mathrm{cl}\,\sigma_{\max}$,
- we fix a point $v_\sigma \in \sigma_{\max}$.

Now we define the map $g\colon X \to X$ putting

$$g(x) = \begin{cases} (1-\lambda_\sigma(x))f(x) + \lambda_\sigma(x)v_\sigma & \text{for } x \in \mathrm{cl}\,\mathcal{V}_\sigma \ (\sigma \text{ non-maximal}),\\ f(x) & \text{otherwise.} \end{cases}$$

To see that the map is well defined we notice that for $x \in \mathrm{bd}\,\mathcal{V}_\sigma$ we have $\lambda_\sigma(x) = 0$ which implies $g(x) = f(x)$. Moreover $g(\mathrm{cl}\,\mathcal{V}_\sigma) \cup f(\mathrm{cl}\,V_\sigma) \subset \sigma_{\max}$ implies that f, g are homotopic. We notice that if $\lambda_\sigma(x) = 0$ (for all non-maximal σ) then $g(x) = f(x)$. But then $x \notin \bigcup_\sigma \mathcal{V}_\sigma \supset \mathrm{Fix}(f)$, hence $g(x) = f(x) \neq x$. Thus $g(x) = x$ implies $x \in \mathcal{V}_\sigma$ and $\lambda_\sigma > 0$ for a σ. But then $x = g(x) \in \sigma_{\max}$ hence each fixed point of g lies inside a maximal simplex.

Thus $\mathrm{Fix}(g)$ splits into mutually disjoint sum $\mathrm{Fix}(g) = \bigcup_\sigma \mathrm{Fix}(g) \cap \mathrm{cl}\,\mathcal{V}_\sigma$ where now σ runs through the family of all maximal simplices. In each maximal simplex we find an open set $\mathcal{U}_\sigma$ satisfying: $\mathrm{Fix}(g) \cap \sigma \subset \mathcal{U}_\sigma \subset \mathrm{cl}\,\mathcal{U}_\sigma \subset \sigma$ and $f(\mathcal{U}_\sigma) \subset \sigma = \mathbb{R}^m$. Now the transversality gives a homotopy relative to $X \setminus \mathcal{U}_\sigma$ to a map with $\mathrm{Fix}(g) \cap \mathcal{U}_\sigma$ finite. Repeating this in all maximal simplices we get Lemma (4.2.16).

The homotopy can be small: let us notice that the image of a point x during the homotopy does not leave the star of $f(x)$. It is enough to take the triangulation sufficiently fine. □

4.2.3. Reduction to a proximity map. Shi proved the Wecken theorem for polyhedra without local separating points which are at least three dimensional. The idea of this proof is to shift all fixed points of a Nielsen class to a 3-simplex and then to unite them there (see [Shi], [Br2]. In 1981 Boju Jiang gave an alternative proof of this result, extending it by the way onto a larger class of polyhedra [Ji2]. Here we will present a proof based on this approach.

We start with some notation inspired by the geometry of surfaces. We consider a polyhedron X without local separating points. A path $p\colon I \to X$ is called a PL-path if it is affine for some subdivisions of I and X. The image of each vertex of I is called a *corner*. A path p is called a *normal* PL-*arc* if

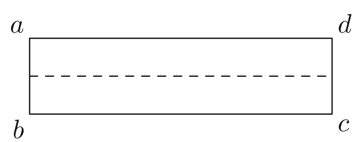
(1) p avoids vertices of X,
(2) p has only double intersections and no intersection is a corner,
(3) $p(s)$ is in a maximal simplex for all but a finite number of points. In these exceptional points, $p(s)$ leaves a maximal simplex and enters another.

A PL-path with different end points and without self-intersections is called a PL-*arc*.

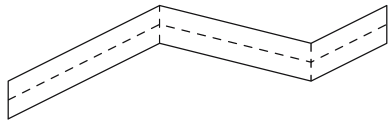
(4.2.18) Lemma. *Any path in X with different end points which are not vertices of X is ε-homotopic (relatively ends) to a normal* PL-*path.*

Proof. We approximate the path with a broken line. Since there are no local separating points, a small shift of corners makes (1)–(3) hold. □

(4.2.19) Lemma. *Let $q\colon I \to X$ be a normal* PL-*arc and let $S = [0,1] \times [-\eta, \eta]$ be a rectangular strip. Then there is a* PL-*imbedding $\widetilde{q}\colon S \to X$ extending q, i.e. $\widetilde{q}(t,0) = q(t)$.*



Proof. This is evident for X = a manifold (of dimension ≥ 2). In general we imbed successively the piece $[t_i, t_{i+1}] \times [-\eta, \eta]$ where t_i, t_{i+1} are consecutive corners. If q passes at the point t_{i+1} from one simplex to another one, we choose $\widetilde{q}(t_{i+1} \times [-\eta, \eta])$ as a segment in the common face of these simplices.



$\square$

(4.2.20) Lemma. *Each path whose end points are contained inside maximal simplices and are different is homotopic, relatively ends, to a normal* PL-*arc.*

Proof. By the above we may replace the path $p: I \to X$ by a normal PL-path. It remains to remove the intersections. If an intersection takes place in a maximal simplex of dimension ≥ 3, then a local small deformation allows to remove it. In general case we proceed as follows. Let $p(s') = p(s)$ where $s \in I$ is the greatest parameter such that $p(s)$ is the intersection point. We consider a strip Σ along $[p(s), p(1)]$ as in Lemma (4.2.19). We may assume that $[p(s' - \varepsilon), p(s' + \varepsilon)] = [a, b]$ is a side of this strip. If moreover $\varepsilon > 0$ is small enough, Σ has no other intersections with $p([0, 1])$. Now we may deform the segment $[a, b]$ into $[a, d, c, b]$ through the strip. Since by the assumption $p(s)$ is the last intersection point, no new intersections appear. Finally we can remove the intersection point $p(1)$, since $p(1)$ lies inside a maximal simplex. Following the procedure we eliminate all intersection points. $\square$

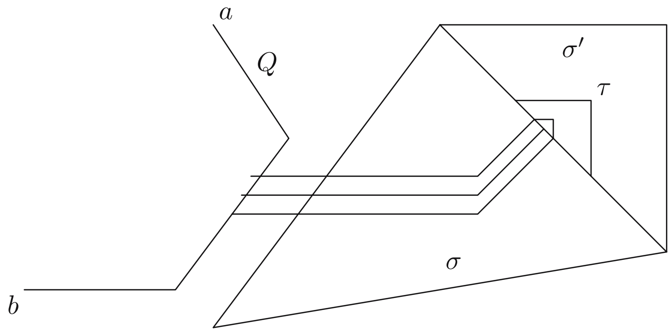
(4.2.21) Lemma. *Let X be a polyhedron without local separating points. Then each path $p: I \to X$, whose ends are different and lie inside maximal simplices, is homotopic to a normal* PL-*arc $q: I \to X$ satisfying:*

(4.2.21.1) *There is a maximal simplex τ and numbers $0 < s_1 < s_3 < s_2 < 1$ such that $q(s) \in \operatorname{bd} \tau$ if and only if $s = s_1$ or s_2, $q(s_3) \in \tau$ and q is affine on $[s_1, s_3]$ and $[s_3, s_2]$ (τ is a simplex in a suitable subdivision).*

(4.2.21.2) *$q(s_1)$, $q(s_2)$ may belong to a prescribed simplex σ_1 of dimension ≥ 1 which is assumed to be a commmon face of at least two maximal simplices of X.*

Proof. By Lemma (4.2.20) we may assume that p is a normal PL-arc q'. Let σ_1 be a simplex of dimension ≥ 1 which is a common face of at least two maximal simplices. Passing to a subdivision, if necessary, we may assume that the arc q avoids the star of σ_1. Now we fix the numbers $0 < s_1' < s_3' < s_2' < 1$ such that there are no corners in $q'[s_1', s_2']$. We find a normal PL-arc from $q'(s_3')$ to a point in σ_1 which has no other common points with $q'[0, 1]$ but $q'(s_3')$. Then we take a PL-imbedding of the strip along this arc. See the figure below.

Now we may homotope $q'[s_1', s_2']$ along the strip to get an arc q, $q(s) = q'(s)$ for $s \notin [s_1, s_2]$ and q satisfies the lemma for some $s_1 < s_2 \in (s_1', s_2')$ and $s_3' = s_3$.



□

Let us fix a normal PL-arc $q: I \to X$. We say that a path $p: I \to X$ is q-special if $p(0) = q(0)$, $p(1) = q(1)$ and $p(s) \neq q(s)$ for $0 < s < 1$. We say that two paths p_0, p_1 are q-specially homotopic if they can be connected by a homotopy $\{p_t\}$ where each p_t is q-special.

On the other hand let $A \subset X$ and let $h_t: A \to X$ be a homotopy. This homotopy is called *special* if $\mathrm{Fix}(h_s) = \{a \in A; h_s(a) = a\}$ does not depend on s.

(4.2.22) Remark. Let us notice that if q is a normal PL-arc, then the path fq is q-specially homotopic to p if and only if the restriction $f: q(I) \to X$ is specially homotopic to $pq^{-1}: q(I) \to X$.

(4.2.23) Lemma (Special Homotopy Extension Property). *Let X be a metric* ENR *and $A \subset X$ its closed subset. Let $f_0: X \to X$ be a map and $h_t: A \to X$ a special homotopy starting from $h_0 = f_{0|A}$. Then h_t can be extended to a special homotopy $f_t: X \to X$ of f_0.*

Proof. We define a map $H': X \times 0 \cup A \times I \to X$ by the formula

$$H'(x,t) = \begin{cases} f_0(x) & \text{for } t = 0, \\ h_t(x) & \text{for } x \in A. \end{cases}$$

Since X is ENR, there is an extension $H: X \times I \to X$. The set $C = \{x \in X : x \in H(x \times I)\}$ is closed since H is continuous and I is compact. We define

$$F(x,t) = \begin{cases} H(x,0) & \text{for } \mathrm{d}(x,A) \geq \mathrm{d}(x,C), \\ H(x, t - t\mathrm{d}(x,A))/\mathrm{d}(x,C)) & \text{for } \mathrm{d}(x,A) \leq \mathrm{d}(x,C) > 0. \end{cases}$$

The homotopy is continuous, since $\mathrm{d}(x,A) = \mathrm{d}(x,C) = 0$ means $x \in A \cap C$ is a fixed point of the special homotopy and $H(x,t)$ does not depend on t. This and the compactness of I imply the continuity of F for x satisfying $\mathrm{d}(x, A) = \mathrm{d}(x, C) = 0$. Now we check that the homotopy F is special. Since F is an extension of the special homotopy $\{h_t\}$, $\mathrm{Fix}F(\,\cdot\,, t) \cap A$ does not depend on t. Let $x \notin A$. If moreover $\mathrm{d}(x, A) \geq \mathrm{d}(x, C)$, then $F(x, t) = H(x, 0) = f_0(x)$ does not depend on t, hence x is a fixed point for all t or for none of them. On the other hand $\mathrm{d}(x, A) < \mathrm{d}(x, C)$ implies $F(x, t) \in H(x \times I)$ but the last set does not contain x since $x \notin C$. □

(4.2.24) LEMMA. *Each q-special path $p\colon I \to X$ is q-specially homotopic to a normal* PL-*path p'. The homotopy may be arbitrarily small.*

PROOF. Let us denote the ends $p(0) = q(0) = a$, $p(1) = q(1) = b$. Since q is a normal PL-arc, a and b are not vertices. Let $\varepsilon > 0$ be so small that q is linear on $[0, \varepsilon]$, $[1 - \varepsilon, 1]$ and that $p[0, \varepsilon] \subset \mathrm{st}(a)$, $p[1 - \varepsilon, 1] \subset \mathrm{st}(b)$. First we q-specially deform p to an affine map on $[0, \varepsilon]$. We define $h\colon [0, \varepsilon] \to X$ by the formula

$$h(s,t) = \begin{cases} \left(1 - \dfrac{s}{\varepsilon t}\right)a + \dfrac{s}{\varepsilon t} \cdot p(\varepsilon t) & \text{for } s \leq \varepsilon t \text{ and } t > 0, \\ p(s) & \text{for } s \geq \varepsilon t. \end{cases}$$

Let us notice that $h(s, t) \neq q(s)$ for $0 < s \leq \varepsilon, t \in I$. In fact since $q(s) \neq p(s)$ for $0 < s < 1$,

$$h(s,t) = \left(1 - \frac{s}{\varepsilon t}\right)a + \frac{s}{\varepsilon t} \cdot p(\varepsilon t) \neq \left(1 - \frac{s}{\varepsilon t}\right)q(0) + \frac{s}{\varepsilon t} \cdot q(\varepsilon t) = q(s).$$

Similarly we define $H\,(s, t)$ for $1 - \varepsilon \leq s \leq 1$. Since the above homotopies are constant for $s = \varepsilon$ and $1 - \varepsilon$, we may assume that p is affine on $[0, \varepsilon]$ and $[1 - \varepsilon, 1]$. Now $\inf\{\mathrm{d}(p(s), q(s)); \varepsilon \leq s \leq 1 - \varepsilon\} > 0$, hence we may extend the homotopy h onto $I \times I$ from $p = h(\,\cdot\,, x)$ to a PL-path and $h(s, t) \neq q(s)$ for $0 < s < 1$. □

Consider again $f\colon X \to X$ with $\mathrm{Fix}(f)$ finite and Nielsen related points $a_0, a_1 \in \mathrm{Fix}(f)$ lying in maximal simplices. By Lemma (4.2.20) we may assume that there is a normal PL-arc q joining these points avoiding other fixed points and $fq \sim q$. Suppose that the path $fq\colon I \to X$ is q-specially homotopic to a path close to q. Now the restriction $f\colon q(I) \to X$ is specially homotopic to a proximity map (Remark (4.2.22)). The last homotopy extends (Lemma (4.2.23)) to a special homotopy on X. The obtained map f_1 has the same fixed points as f but $f_1\colon q(I) \to X$ is a proximity map. Now Lemma (4.2.14) allows to join these points.

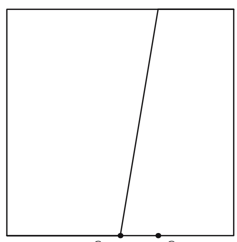
Thus to prove Wecken's theorem it is enough to show that for each pair of periodic points from the same Nielsen class there is a normal arc q such that fq is q-specially homotopic to a map close to q. We will show that this is possible for polyhedra satisfying the assumption of Wecken's Theorem (4.2.8).

(4.2.25) LEMMA. *Suppose that a normal* PL-*arc* q *satisfies* (4.2.21.1). *Then any* q*-special path* $p: I \to X$ *is* q*-specially homotopic to a path* p' *with* $p'(0,1) \cap (q(0,1) \cup \tau) = \emptyset$.

PROOF. By Lemma (4.2.24) we may assume that p is a normal PL-path. By a small correction we may reduce the intersections with $q(I)$ to a finite number (none at the corner). Now we will eliminate the intersections. Let $p(s') = q(s)$ for $s' < s$. As in the proof of Lemma (4.2.20) we deform $p(s' - \delta, s' + \delta)$ along the arc $[s, 1]$ and we remove the intersection. If $s' > s$ we do the same along $q[0, s]$. If p does not avoid τ we compose it with the radial projection $\tau \setminus q(s_3) \to \mathrm{bd}\, \tau$. Then $p(0,1) \cap \tau = \emptyset$. □

(4.2.26) LEMMA. *Suppose that the normal* PL-*arc satisfies* (4.2.21.1). *Let* $p_0, p_1: I \to X$ *be special paths whose images* $p_0(I)$ *and* $p_1(I)$ *do not meet* $q(0,1) \cup \tau$. *If* p_0 *and* p_1 *are homotopic in* $X \setminus \tau$, *then they are* q*-specially homotopic in* X.

PROOF. Let $\lambda: I \to I$ be given by the figure



(i.e. $\lambda[0, s_1] = 0$, $\lambda[s_2, 1] = 1$). The paths p_0 and $p_0 \cdot \lambda$ are connected by the homotopy $\{h_t\}_t = p_0((1-t)s + t\lambda(s))$ which is q-special. Similarly p_1 and $p_1 \cdot \lambda$ are q-specially homotopic.

Let $\{p_t\}$ be a homotopy connecting p_0 and p_1 as the paths in $X \setminus \tau$. Then $\{p_t \cdot \lambda\}$ is a homotopy connecting $p_0 \cdot \lambda$ and $p_1 \cdot \lambda$. This homotopy is q-special: $p_t \cdot \lambda(s) = q(0) \neq q(s)$ for $0 < s \leq s_1$, $p_t \cdot \lambda(s) = q(1) \neq q(s)$ for $s_2 \leq s < 1$ and for $s_1 < s < s_2$ we have $p_t \cdot \lambda(s) \in X \setminus \tau$ while $q(s) \in \tau$. □

(4.2.27) LEMMA. *Let the normal* PL-*arc* q *satisfy Lemma* (4.2.21) *where* σ_1 *is a* 1*-dimensional face of at least three* 2*-simplices of* X. *If two special paths* $p_0, p_1: I \to X$ *are homotopic, then they are* q*-specially homotopic.*

PROOF. In view of Lemma (4.2.25), we may assume that $p_0[0,1]$ and $p_1[0,1]$ are disjoint from $q(0,1) \cup \tau$. We choose a point $a = q(0)$ as the base point in X and in $X \setminus \tau$. Notice that $p_0 \sim p_1$ means $p_1 * p_0^{-1}$ is nullhomotopic in X.

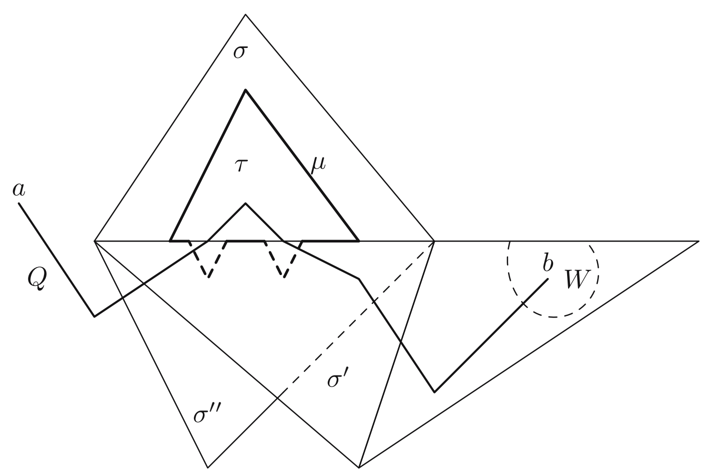
Suppose that moreover $\dim \tau \geq 3$. Then the inclusion $i: X \setminus \tau \subset X$ induces the isomorphism $i_\#: \pi_1(X \setminus \tau) \to \pi_1(X)$. Now $p_0 \sim p_1$ in X implies $p_0 \sim p_1$ in $X \setminus \tau$, hence by Lemma (4.2.26), p_0 and p_1 are q-specially homotopic.

Now we consider the case $\dim\tau = 2$.

We take a point $c \in \operatorname{bd}\tau$ different than $q(s_1)$, $q(s_2)$ and the vertices of X. Let u be a loop on $\operatorname{bd}\tau$, based in c, running once around $\operatorname{bd}\tau$. Then the group $\pi_1(X)$ is the factor group of $\pi_1(X \setminus \tau)$ modulo the normal subgroup generated by u. Now $p_1p_0^{-1} \sim 1$ in X implies $p_1p_0^{-1}$ is homotopic in $X \setminus \tau$ to a loop of the form $(\nu_1 u^{k_1}\nu_1^{-1})(\nu_2 u^{k_2}\nu_2^{-1})\dots(\nu_m u^{k_m}\nu_m^{-1})$, where $k_1,\dots,k_m$ are integers, $\nu_1,\dots,\nu_m$ are paths in $X \setminus \tau$ from a to c.

For every $j = 1,\dots,m$ we can find a PL-path ν_j' having only a finite number of intersections with Q such that $\nu_j' \sim \nu_j$ in $X \setminus \tau$. Then we can eliminate the intersections of ν_i' with $q(0,1)$ just as we did in the proof of Lemma (4.2.20). Thus we get a path w_j in $X \setminus (\tau \cup q(0,1))$ such that $w_j \sim \nu_j$ in $X \setminus \tau$.

Take a neighbourhood $\mathcal{W}$ of a such that $\operatorname{cl}\mathcal{W} \cap \operatorname{cl}\tau = \emptyset$. The condition (4.2.21.2) of Lemma (4.2.21) and the assumption that σ_1 is the common face of at least three 2-simplices make it possible to modify u into a loop μ in $X \setminus (\tau \cup q(0,1))$, based at c satisfying $\mu \sim u$ in $X \setminus (\tau \cup W)$.



Let $p_0' = (w_1 u^{k_1} w_1^{-1})\dots(w_m u^{k_m} w_m^{-1})p_0$. Now p_1 and p_0' are in $X \setminus (\tau \cup q(0,1))$ and they are homotopic in $X \setminus \tau$. By Lemma (4.2.26), p_1 and p_0' are q-specially homotopic. It remains to show that p_0 and p_0' are q-specially homotopic.

Let $\varepsilon > 0$ be so small that $q[0,\varepsilon]$ is contained in the neighbourhood $\mathcal{W} \ni a$ chosen above. In view of the reparametrization used in the proof of Lemma (4.2.26), we may assume that p_0 is constant on $[0,\varepsilon]$ and that p_0' makes up $(w_1 u^{k_1} w_1^{-1})\dots(w_m u^{k_m} w_m^{-1})$ on $[0,\varepsilon]$ and then follows p_0 on $[\varepsilon,1]$. Since $\mu \sim u$ in $X \setminus (\tau \cup W)$ and $u \sim 1$ in $\operatorname{cl}\tau \subset X \setminus W$, $\mu \sim 1$ in $X \setminus \mathcal{W} \subset X \setminus q(0,\varepsilon]$. But w_j is in $X \setminus q(0,\varepsilon]$, so $(w_j \mu^{k_j} w_j^{-1}) \sim (w_j w_j^{-1}) \sim 1$ in $X \setminus q(0,\varepsilon]$. Hence, the part of p_0' on $[0,\varepsilon]$ can be deformed to a constant without touching $q(0,\varepsilon]$. Thus p_0 and p_0' are q-specially homotopic. $\square$

Proof of Theorem (4.2.8). As we noticed in the scheme of the proof of Theorem (4.2.8) it is enough to show that f is homotopic to a map with all fixed points lying in different Nielsen classes. We may assume that $\mathrm{Fix}(f)$ is finite and each fixed point lies inside a maximal simplex. Let a, b be two fixed points lying in the same Nielsen class. We will show that there is a homotopy $\{f_t\}$ starting from f, $\{f_t\}$ is constant in a neighbourhood of $\mathrm{Fix}(f)\setminus a$ and $\mathrm{Fix}(f_1) = \mathrm{Fix}(f)\setminus a$.

Since a and b are Nielsen related, there is a path p joining these points and satisfying $fp \sim p$. By Lemma (4.2.20), p may be replaced by a normal PL-arc q satisfying moreover the assumption of Lemma (4.2.21) and that a, b are the only fixed points of f on $Q = q(I)$. We are going to apply Lemma (4.2.27) for this q.

Let $p'_\varepsilon = q\cdot h$ where $h\colon I\to I$ is a homeomorphism such that $h(0) = 0$, $h(1) = 1$ and $0 < |h(s) - s| < \varepsilon$ for $0 < s < 1$. Then, for a sufficiently small $\varepsilon > 0$, $p'_\varepsilon(s)$ and $q(s)$ are close, so that the map $\phi = p'_\varepsilon \cdot q^{-1}\colon Q\to X$ is a proximity map.

Now both p'_ε and $f\cdot q$ are q-special paths and they are homotopic, since they both are homotopic to q. By Lemma (4.2.27) they are q-specially homotopic. Now $f, p'_\varepsilon\cdot q^{-1}\colon Q\to X$ are specially homotopic maps. Then Lemma (4.2.23) extends this special homotopy to a special homotopy $f_t\colon X\to X$. Now $\mathrm{Fix}(f_1) = \mathrm{Fix}(f_0)$ and f_1 is a proximity map on q hence Lemma (4.2.14) yields a homotopy $\{f_t : 1\le t\le 2\}$ with $\mathrm{Fix}(f_2) = \mathrm{Fix}(f)\setminus a$. □

4.3. Some computations of the Nielsen number

4.3.1. Jiang group. Since the global fixed point index of a self-map equals the Lefschetz number, one may efficiently apply algebra to its computation. On the other hand the algebraic methods can be applied to establish the number of Reidemeister classes. When we know that all the Reidemeister classes of the considered map f have the same index, then $L(f)\neq 0$ implies that all classes are essential and $N(f) = R(f)$. Moreover, $L(f) = 0$ implies that all the classes are inessential and $N(f) = 0$. Unfortunately in general the fixed point index of Nielsen classes may be different. However in the proof of the homotopy invariance of the Nielsen number we showed that the Nielsen classes corresponding by a homotopy have the same index. Thus if we assume that for a given map $f\colon M\to M$ and for any two Reidemeister classes in $\mathcal{R}(f)$ there is a cyclic homotopy $h_t\colon X\to X$, $h_0 = h_1 = f$ such that the two classes correspond by the homotopy, then all Reidemeister classes have the same index and $N(f) = R(f)$. This led Boju Jiang in 1964 [Ji1] to consider the following class of spaces.

(4.3.1) Definition. Let X be a pathwise connected space. Then X is called *Jiang space* if for every element $\alpha\in\pi_1(X,x_0)$ there is a cyclic homotopy $h_t\colon X\to X$, $h_0 = h_1 = \mathrm{id}_X$, such that the path $[0,1]\ni t\to h_t(x_0)\in X$ represents α.

As one may expect, the above definition can be reformulated for universal covering spaces.

(4.3.2) LEMMA. *The space is Jiang if and only if all its deck transformations are fibrewise homotopic.*

PROOF. ($\Rightarrow$) We assume that X is Jiang. Let $\alpha \in \mathcal{O}_X$ be a deck transformation. We will show that α is fiberwise homotopic to the identity on $\widetilde{X}$. Let us fix a point $\widetilde{x}_0 \in p^{-1}(x_0)$. Let $\widetilde{\eta}$ be a path joining the points $\widetilde{x}_0, \alpha(\widetilde{x}_0) \in \widetilde{X}$. Then $\eta = p(\widetilde{\eta})$ is a loop in X. By the assumption there is a cyclic homotopy $h_t\colon X \to X$ such that $h_t(x_0) \sim \eta(t)$. Then the lift of this homotopy $\widetilde{h}_t(x_0)$ starting from $\widetilde{h}_0 = \mathrm{id}_{\widetilde{X}}$ is the desired fibrewise homotopy (since $\widetilde{h}_t(\widetilde{x}_0) \sim \widetilde{\eta}(t)$ implies $\alpha(\widetilde{x}_0) = \widetilde{\eta}(1) = \widetilde{h}_1(\widetilde{x}_0)$).

($\Leftarrow$) Now we assume that all deck transformations are fibrewise homotopic. We fix $\eta \in \pi_1(X; x_0)$. Let $\widetilde{\eta}$ be the lift of η starting from the point $\widetilde{x}_0$. Let α denote the deck transformation satisfying $\alpha(\widetilde{x}_0) = \widetilde{\eta}(1)$. Let $\widetilde{h}_t$ be the fibrewise homotopy from $\mathrm{id}_{\widetilde{X}}$ to α. This fibrewise homotopy induces the cyclic homotopy $h_t\colon X \to X$ such that $h_0 = h_1 = \mathrm{id}_X$ and its trace $h_t(x_0)$ is homotopic to η. $\square$

The next theorem gives the basic property of Jiang spaces.

(4.3.3) THEOREM. *If $f\colon X \to X$ is a self-map of a Jiang space, then all the Reidemeister classes of f have the same index.*

PROOF. Let us fix two lifts $\widetilde{f}$ and $\widetilde{f}'$ representing two Reidemeister classes in $\mathcal{R}(f)$. We have to show that

$$\mathrm{ind}(f; p(\mathrm{Fix}(\widetilde{f})) = \mathrm{ind}(f; p(\mathrm{Fix}(\widetilde{f}')).$$

Since $p\widetilde{f} = p\widetilde{f}'$, $\widetilde{f}' = \alpha\widetilde{f}$ for a deck transformation α. Let $\widetilde{h}_t$ be a fibrewise homotopy between $\mathrm{id}_{\widetilde{X}}$ and α. We define $\widetilde{H}(\widetilde{x}, t) = \widetilde{h}_t(\widetilde{f}(\widetilde{x}))$.

Since $\widetilde{H}$ is a fibrewise homotopy, $p(\mathrm{Fix}(\widetilde{H}))$ is a closed subset of $\mathrm{Fix}(f)$. Since $\mathrm{Fix}(H) = \bigcup_{\alpha \in \mathcal{O}_X} p(\mathrm{Fix}(\alpha\widetilde{H}))$ where all the summands are either disjoint or equal and their number is finite, they are also open.

Now we may apply the homotopy invariance of the fixed point index:

$$\begin{aligned}\mathrm{ind}(f; p(\mathrm{Fix}(\widetilde{f}))) &= \mathrm{ind}(f; p(\mathrm{Fix}(\widetilde{h}_0\widetilde{f}))) \\ &= \mathrm{ind}(f; p(\mathrm{Fix}(\widetilde{h}_1\widetilde{f}))) = \mathrm{ind}(f; p(\mathrm{Fix}(\widetilde{f}'))).\end{aligned} \qquad \square$$

(4.3.4) COROLLARY. *If $f\colon X \to X$ is a self-map of a Jiang space, then*

$$N(f) = \begin{cases} R(f) & \text{if } L(f) \neq 0, \\ 0 & \text{if } L(f) = 0. \end{cases}$$

(4.3.5) LEMMA. *Examples of Jiang spaces are:*

(4.3.5.1) *connected, simply-connected* ENR*s,*

(4.3.5.2) *topological groups,*

(4.3.5.3) *H-spaces,* (*see* [Sp] *for a definition*)*,*

(4.3.5.4) *homogeneous spaces G/G_0 where G is a Lie group and G_0 its closed connected subgroup.*

PROOF. (4.3.5.1) Obvious.

(4.3.5.2) Let α be loop based and the identity element $e \in G$. Then $h_t(x) = \alpha(t)x$ is a cyclic homotopy satisfying $h_t(e) = \alpha(t)$.

(4.3.5.3) For an H-group X we have a base point $e \in X$, multiplication $\mu\colon X \to X$, and a homotopy $F\colon X \times I \to X$ between id_X and $\mu(e, \cdot\,)$. For $\alpha \in \pi_1(X, e)$ we consider the concatenation of homotopies: F, $\mu(\alpha_t, \cdot\,)$ and the reverse homotopy F^{-1}.

(4.3.5.4) Since $p\colon G \to G/G_0$ is a locally trivial bundle with connected fibre G_0, the homomorphism $p_{\#}\colon \pi_1(G) \to \pi_1(G/G_0)$ is onto. Thus each homotopy class in G/G_0 is represented by $p\alpha$ where $\alpha\colon [0,1] \to G$ is a loop based at e. We define the cyclic homotopy $h_t\colon G/G_0 \to G/G_0$ by $h_t(xG_0) = \alpha(t)xG_0$. □

More generally we may define a *Jiang subgroup* of a self-map $f\colon X \to X$ as

$$J(f, x_0) = \{[\alpha] \in \pi_1(X, f(x_0)) : \text{there is a cyclic homotopy } h_t\colon X \to X \\ \text{such that } h_0 = h_1 = f \text{ and } h_t(x_0) = \alpha(t)\}$$

and the *Jiang subgroup of the space* X as $J(X; x_0) = J(\mathrm{id}_X; x_0)$. Of course X is a Jiang space if and only if $J(X) = \pi_1(X)$.

Modifying the proof of Theorem (4.3.3) we get

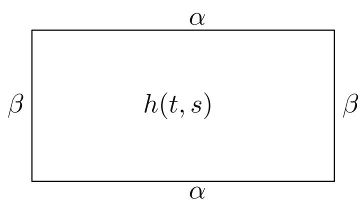
(4.3.6) THEOREM. *If $J(f; x_0) = \pi_1(X; fx_0)$, then all the Reidemeister classes have the same index. Moreover*

$$N(f) = \begin{cases} R(f) & \text{if } L(f) \neq 0, \\ 0 & \text{if } L(f) = 0. \end{cases}$$

Unfortunately the class of Jiang spaces is limited.

(4.3.7) LEMMA. *The Jiang subgroup is contained in the center of the fundamental group: $J(X) \subset Z(\pi_1(X))$.*

PROOF. Let $\alpha \in J(X; x_0)$ and $\beta \in \pi_1(X; x_0)$. We have to show that $\alpha * \beta$ and $\beta * \alpha$ are homotopic. Let $H_t\colon X \to X$ be a cyclic homotopy: $H_0 = H_1 = \mathrm{id}$, $H_t(x_0) = \alpha(t)$.



Then $h(t,s) = H_t(\beta(s))$ gives the desired homotopy. $\square$

(4.3.8) COROLLARY. *If X is a Jiang space, then $\pi_1(X)$ is abelian.*

(4.3.9) PROPOSITION. *If X is a connected compact* ENR *with the nonzero Euler characteristic then $J(X)$ is trivial.*

PROOF. Suppose that $0 \neq \alpha \in J(X)$. Then we have a deck transformation $\widetilde{h}\colon \widetilde{X} \to \widetilde{X}$, $\widetilde{h} \neq \mathrm{id}_{\widetilde{X}}$ which is fibrewise homotopic to $\mathrm{id}_{\widetilde{X}}$. Let $\widetilde{h}_t\colon \widetilde{X} \to \widetilde{X}$ be this homotopy: $\widetilde{h}_0 = \mathrm{id}_{\widetilde{X}}$, $\widetilde{h}_1 = \widetilde{h}$. Notice that then $\widetilde{h}(\widetilde{x}) \neq \widetilde{x}$ for all $\widetilde{x} \in \widetilde{X}$. Now

$$\begin{aligned}\chi(X) &= \mathrm{ind}(\mathrm{id}; X) = \mathrm{ind}(\mathrm{id}; p(\mathrm{Fix}(\widetilde{h}_0))) \\ &= \mathrm{ind}(\mathrm{id}; p(\mathrm{Fix}(\widetilde{h}_1))) = \mathrm{ind}(\mathrm{id}; p(\mathrm{Fix}(\widetilde{h}))) = 0,\end{aligned}$$

since by the above $\mathrm{Fix}(\widetilde{h})$ is empty. $\square$

(4.3.10) REMARK. Jiang spaces were the first nontrivial class of spaces for which the computation of the Nielsen number was available. However the formula from Corollary (4.3.4) ($N(f) = R(f)$ or 0) is equivalent to the fact that either all Nielsen classes have non-zero or (all) zero index. Similarly one may define the *weakly Jiang space* X as the space such that each self map $f\colon X \to X$ has all Nielsen classes simultaneously essential or inessential. We will show later that tori, nilmanifolds and solvmanifolds are weakly Jiang.

Peter Wong ([Wo2]) proved the following

(4.3.11) THEOREM. *The quotient space of a compact connected Lie group by its finite subgroup is weakly Jiang.*

4.3.2. Projective spaces. One may expect that the computation of the Nielsen number is easier when the fundamental group is small. Now we will consider the real projective spaces: the spaces with the smallest nontrivial fundamental group $\pi_1(\mathbb{R}P^d) = \mathbb{Z}_2$. Recall that the real projective space $\mathbb{R}P^d$ is the quotient space of the sphere S^d by the equivalence relation $x \sim -x$. We start with a description of the homotopy classes of self-maps of $\mathbb{R}P^d$.

(4.3.12) LEMMA. *Let d be odd. Then $\mathbb{R}P^d$ is the oriented manifold and the formula*

$$[\mathbb{R}P^d, \mathbb{R}P^d] \ni f \to \deg(f) \in \mathbb{Z}$$

establishes a bijection. Moreover $\deg(f)$ *is odd if* $f_{\#} = \mathrm{id}$ *and* $\deg(f)$ *is even if* $f_{\#} = 0$. *Let* d *be even. Then* $\mathbb{R}P^d$ *is not orientable and the set of homotopy classes splits*

$$[\mathbb{R}P^d, \mathbb{R}P^d] = H_{\mathrm{id}} \cup H_0$$

into the classes H_{id} *inducing* $f_{\#} = \mathrm{id}$ *and classes* H_0 *inducing* $f = 0$. *Moreover,* H_{id} *is in bijection with the set of positive odd integers by the correspondence* $k \to [f_k]$ *where* f_k *is the self-map of* $\mathbb{R}P^d$ *induced by the odd map* $\widetilde{f}_k \colon S^d \to S^d$ *of degree* k.

PROOF. Notice that then the other lift $-\widetilde{f}_k$ has degree $-k$. On the other hand H_0 consists of two elements $[f_0]$, $[f_2]$ where f_0 is the constant map and f_2 is the homotopy nontrivial element inducing zero on the fundamental group. The last map may be represented as the composition ps where $p\colon S^d \to \mathbb{R}P^d$ is the universal covering and $s\colon \mathbb{R}P^d \to S^d$ is the contraction of the $(d-1)$-skeleton of $\mathbb{R}P^d$ to the point. □

(4.3.13) THEOREM. *Let* $f\colon \mathbb{R}P^d \to \mathbb{R}P^d$ *and* $d \geq 2$. *Then*

$$N(f) = \begin{cases} 0 & \text{if } f \sim \mathrm{id} \text{ and } d \text{ is odd,} \\ 1 & \text{if } f_{\#} = 0 \quad \text{or } (f \sim \mathrm{id} \text{ and } d \text{ is even}), \\ 2 & \text{if } f_{\#} = \mathrm{id} \text{ and } f \text{ is not homotopic to } \mathrm{id}. \end{cases}$$

PROOF. Since $\pi_1(\mathbb{R}P^d) = \mathbb{Z}_2$, $\mathcal{R}(f) = \mathbb{Z}_2$ for $f_{\#} = \mathrm{id}$ and $\mathcal{R}(f)$ consists of one element if $f_{\#} = 0$.

Let $d = 2k+1$ be odd. Then $\mathcal{R}P^d$ is an orientable manifold. We will show that then $\mathbb{R}P^d$ is a Jiang space. In fact $S^{2k+1} \subset \mathbb{R}^{2k+2} = \mathbb{C}^{k+1}$ and the formula

$$\widetilde{h}_t(z_1, \dots, z_{k+1}) = (\exp(\pi t i) \cdot z_1, \dots, \exp(\pi t i) \cdot z_{k+1})$$

defines the isotopy $\widetilde{h}_t \colon S^{2k+1} \to S^{2k+1}$ from $\widetilde{h}_0(z) = z$ to $\widetilde{h}_1(z) = -z$, hence $\widetilde{h}_t$ is inducing a cyclic homotopy $h_t \colon \mathbb{R}P^d \to \mathbb{R}P^d$ based at the identity of $\mathbb{R}P^d$. Moreover $h_t[1, 0, \dots, 0] = [\exp(\pi t i), 0, \dots, 0]$ realizes the unique nontrivial element in $\pi_1(\mathbb{R}P^d)$ which proves that $\mathbb{R}P^d$ is a Jiang space.

On the other hand $H_k(\mathbb{R}P^d; \mathbb{Q}) = \mathbb{Q}$ if $k = 0, m$ and equals 0 otherwise. Thus $\mathrm{L}(f) = 1 - \deg(f)$. Now $\mathrm{L}(f) = 0$ if $\deg(f) = 1$, hence by Lemma (4.3.12), f is homotopic to the identity. Otherwise since $\mathbb{R}P^d$ is Jiang all the Reidemeister classes are essential. Thus

$$N(f) = \begin{cases} 0 & \text{if } f \sim \mathrm{id}, \\ 1 & \text{if } f_{\#} = 0, \\ 2 & \text{if } f_{\#} = \mathrm{id} \text{ and } f \text{ is not homotopic to id.} \end{cases}$$

More precisely if $f_\# = 0$, then we have one Reidemeister class whose index $= \mathrm{L}(f) = 1 - \deg(f)$. If $f_\# = \mathrm{id}$ we have two Reidemeister classes each of index $= \mathrm{L}(f)/2 = (1 - \deg(f))/2$.

Let d be even. Then $\mathbb{R}P^d$ is not orientable but is $\mathbb{Q}$-acyclic, i.e. $H_k(\mathbb{R}P^d;\mathbb{Q}) = 0$ for $k \in \mathbb{N}$. Now $\mathrm{L}(f) = 1$ for each self-map $f\colon \mathbb{R}P^d \to \mathbb{R}P^d$. If $f_\# = 0$, then $\mathcal{R}(f)$ contains the unique class and its index $= \mathrm{L}(f) = 1$. If $f_\# = \mathrm{id}$ then f lifts to a map $\widetilde{f}\colon S^d \to S^d$ of odd degree $2k+1$ $(k \geq 0)$ and to the map $-\widetilde{f}$ (of degree $-2k-1$). Then

$$\mathrm{ind}(p(\mathrm{Fix}(\widetilde{f})) = \frac{1}{2}(\mathrm{ind}(\widetilde{f})) = \frac{1}{2}\mathrm{L}(\widetilde{f}) = \frac{1}{2}(1 + 2k + 1) = k + 1,$$

and similarly

$$\mathrm{ind}(p(\mathrm{Fix}(-\widetilde{f})) = \frac{1}{2}\mathrm{L}(-\widetilde{f}) = \frac{1}{2}(1 - 2k - 1) = -k.$$

If $k = 0$ then one of the lifts is of degree 1, hence f is homotopic to the identity. Then there is one nonempty Nielsen class of index $\chi(\mathbb{R}P^m) = 1$ hence essential. The other class is empty. Thus $N(f) = 1$. If $k \geq 1$ then both classes are essential. Thus for m even,

$$N(f) = \begin{cases} 1 & \text{if } f_\# = 0 \text{ or } (f \sim \mathrm{id}), \\ 2 & \text{if } f_\# = \mathrm{id} \text{ and } f \text{ is not homotopic to id.} \end{cases}$$

Combining the formula for d odd and even we get the theorem. □

4.3.3. Nielsen number of self-maps of tori. The first nontrivial ideas in the Nielsen fixed point theory usually begin with a torus: the compact commutative Lie group.

In 1975 in [BBPT] R. Brooks, B. Brown J. Pak, and D. Taylor derived a nice formula for the Nielsen number for the torus map: the Nielsen number equals the absolute value of the Lefschetz number. Here we will give an elementary proof of this result. It is an interesting and open question about the class of the space for which this formula is true.

We consider the torus as the quotient Lie group $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Since $\mathbb{Z}^d$ is discrete, we get the covering map $p\colon \mathbb{R}^d \to \mathbb{T}^d$ and since $\mathbb{R}^d$ is contractible this covering is universal. Now $\pi_k(\mathbb{T}^d) = \pi_k(\mathbb{R}^d) = 0$ for $k \geq 2$ and $\pi_1(\mathbb{T}^d) = \mathbb{Z}^d$ shows that $\mathbb{T}^d$ is an Eilenberg–Mac Lane space $K(\mathbb{Z}^d, 1)$. Let $A \in \mathcal{M}_{m\times m}(\mathbb{Z})$ be a square matrix with integer entries. It defines a homomorphism $A\colon \mathbb{R}^d \to \mathbb{R}^d$ and the last induces the map $f_A\colon \mathbb{T}^d \to \mathbb{T}^d$. Since $\mathbb{T}^d$ is the Eilenberg–Mac Lane space with $\pi_1(\mathbb{T}^d) = \mathbb{Z}^d$, and each homomorphism of $\mathbb{Z}^d$ is given by the square $d \times d$ integral matrix, the correspondence

$$\mathcal{M}_{d\times d}(\mathbb{Z}) \ni A \to [f_A] \in [\mathbb{T}^d, \mathbb{T}^d]$$

is a bijection where $[\mathbb{T}^d, \mathbb{T}^d]$ denotes the set of homotopy classes of $\mathbb{T}^m$. The following theorem has been proved in [BBPT].

(4.3.14) Theorem. *For every self-map of the torus* $f\colon \mathbb{T}^d \to \mathbb{T}^d$,

$$L(f) = \det(\mathrm{I} - A) \quad \textit{and} \quad N(f) = |\mathrm{L}(f)|,$$

where $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$ *is the matrix representing the homotopy homomorphism*

$$f_{\#}\colon \pi_1(\mathbb{T}^d) \to \pi_1(\mathbb{T}^d) = \mathbb{Z}^d.$$

Proof. Since the Lefschetz and Nielsen numbers are homotopy invariants we may assume that $f = f_A$, i.e. f is induced by the linear map A.

We assume at first that $\det(\mathrm{I} - A) = 0$. Now it is enough to prove that f is homotopic to a fixed point free map. Since $\det(\mathrm{I} - A) = 0$, the linear map $\mathrm{I} - A$ is not onto, hence $\mathrm{Im}(\mathrm{I} - A) + \mathbb{Z}^d \neq \mathbb{R}^d$. We fix a $v \notin \mathrm{Im}(\mathrm{I} - A) + \mathbb{Z}^d$ and we consider the homotopy $h_t\colon \mathbb{R}^d \times I \to \mathbb{R}^d$ given by $h_t(x) = A(x) + tv$. This homotopy induces the homotopy $\overline{f}_t\colon \mathbb{T}^d \to \mathbb{T}^d$ starting from $\overline{h}_t(x) = f_A$. We will show that $\overline{h}_1$ has no fixed point. In fact $\overline{h}_1[x] = [x]$ implies $x \equiv Ax + v \mod \mathbb{Z}^d$ and $v \in \mathrm{Im}(\mathrm{I} - A) + \mathbb{Z}^d$, contradicting to the choice of v.

Now we assume that $\det\,(\mathrm{I} - A) \neq 0$. We will show that then

(1) f_A has exactly $|\det\,(\mathrm{I} - A)|$ fixed points,
(2) no two fixed points of f_A are Nielsen related,
(3) the index of each fixed point equals $\mathrm{sgn}(\det(\mathrm{I} - A))$.

Then

$$\begin{aligned} L(f_A) &= \mathrm{ind}(f_A) = \sum_{x \in \mathrm{Fix}(f_A)} \mathrm{ind}(f_A; x) \\ &\overset{(3)}{=} \sum_{x \in \mathrm{Fix}(f_A)} \mathrm{sgn}(\det(\mathrm{I} - A)) = (\#\mathrm{Fix}(f_A)) \cdot \mathrm{sgn}(\det(\mathrm{I} - A)) \\ &\overset{(1)}{=} |(\det(\mathrm{I} - A))| \cdot \mathrm{sgn}(\det(\mathrm{I} - A)) = \det(\mathrm{I} - A). \end{aligned}$$

Moreover,

$$N(f_A) \overset{(2),(3)}{=} \#\mathrm{Fix}(f_A) \overset{(3)}{=} |L(f_A)|.$$

It remains to prove (1)–(3).

(1) We notice that $p^{-1}(\mathrm{Fix}(f_A)) = (\mathrm{I} - A)^{-1}(\mathbb{Z}^d)$. In fact $x \in p^{-1}(\mathrm{Fix}(f_A))$ $\Leftrightarrow x \equiv Ax (\mathrm{mod} \mathbb{Z}^d) \Leftrightarrow (\mathrm{I} - A)x \in \mathbb{Z}^d$. Since $\mathrm{I} - A$ is an isomorphism, $M = (\mathrm{I} - A)^{-1}(\mathbb{Z}^d)$ is a uniform lattice in $\mathbb{R}^d$ (i.e. a discrete subgroup generating $\mathbb{R}^d$ as the real vector space). Now the restriction of the projection $p_|\colon M \to \mathrm{Fix}(f_A)$ is onto and $x, y \in \mathrm{Fix}(f_A)$ are sent to the same point if and only if $x - y \in \mathbb{Z}^d$, hence $p_|$ induces the bijection $M/\mathbb{Z}^d \to \mathrm{Fix}(f_A)$.

$$\#\mathrm{Fix}(f_A) = \#(M/\mathbb{Z}^d) = |\det(\mathrm{I} - A)|,$$

where the second equality follows from Proposition (4.3.16) for $\Lambda = \mathbb{Z}^d$, $e_1, \dots, e_d$ the canonical basis of $\mathbb{Z}^d$, $e'_1 = (\mathrm{I} - A)^{-1}(e_1), \dots, e'_d = (\mathrm{I} - A)^{-1}(e_d)$ and $P = (\mathrm{I} - A)$.

(2) Since each lift of f_A is of the form $A_v(x) = A(x) + v$ for a $v \in \mathbb{Z}^d$, each Nielsen class of f_A is of the form $p(\mathrm{Fix}(A_v))$. It remains to check that $\mathrm{Fix}(A_v)$ contains at most one element. Let $x, y \in \mathrm{Fix}(A_v)$. Then $x = A(x) + v$ and $y = A(y) + v$ imply $x - y = A(x - y)$. Now $x - y \in \mathrm{Ker}(\mathrm{I} - A) = 0$ hence $x = y$.

(3) Consider a point $x_0 \in \mathrm{Fix}(f_A)$. Then $x_0 = p(\widetilde{x}_0)$ for a point $\widetilde{x}_0 \in \mathrm{Fix}(A_v)$ and a $v \in \mathbb{Z}^d$. By 1, x_0 is an isolated fixed point (of f_A). Since p is a local homeomorphism, $\widetilde{x}_0$ is also an isolated fixed point of A and $\mathrm{ind}(f_A; x_0) = \mathrm{ind}(A_v; \widetilde{x}_0)$. But

$$\mathrm{ind}(A_v; \widetilde{x}_0) = \deg(\mathrm{I} - A_v; \widetilde{x}_0) = \deg(\mathrm{I} - A_v) = \deg(\mathrm{I} - A) = \mathrm{sgn}(\det(\mathrm{I} - A)). \quad \square$$

(4.3.15) Remark. The points (1)–(3) of the above proof say that the self-map f_A induced on $\mathbb{T}^d$ by the linear map $A\colon \mathbb{R}^d \to \mathbb{R}^d$, $A = f_\#$, has exactly $N(f)$ fixed points of the same index ± 1, provided $L(f) \neq 0$.

The fact presented below is well known in geometric number theory. We refer the reader to [Cas, Lemma I and Theorem I] for a proof of it. Let us recall that subset $e_1, \dots, e_d \subset \Lambda$ is called a *basis of lattice* Λ if every $v \in \Lambda$ can be uniquely presented as $v = k_1 e_1 + \dots + k_d e_d$ for $k_1, \dots, k_d \in \mathbb{Z}$.

(4.3.16) Proposition. *Consider uniform sublattices $\Lambda \subset \Lambda' \subset \mathbb{R}^d$. Let $e_1, \dots$, $e_d \subset \Lambda$, $e'_1, \dots, e'_d \subset \Lambda'$ be their bases. Then the cardinality of quotient group Λ'/Λ is equal to $\#(\Lambda'/\Lambda) = |\det P|$, where P denotes a transition matrix from the basis $e'_1, \dots, e'_d$ to $e_1, \dots, e_d$, i.e. $e'_i = P e_i$.*

It was natural to expect that this nice formula is true for other Lie groups. However as it is noticed in [BBPT], each compact nonabelian Lie group admits a map with $N(f) < L(f)$. In other words

(4.3.17) Remark. If G is a compact connected Lie group then: $N(f) = |L(f)|$ for any self-map of G if and only if G is a torus.

4.4. Nielsen relation of the fibre map

4.4.1. Fixed point index product formula.

(4.4.1) Theorem. *Let $p\colon E \to B$ be a locally trivial bundle where all involved spaces E, B and the fibre F are compact* ENR*s. Let $f\colon E \to E$ be a fibre map, $b_0 \in \mathrm{Fix}(\overline{f})$ an isolated fixed point and let A be a closed-open subset of* $\mathrm{Fix}(f_{b_0})$*. Then A is also closed-open subset of* $\mathrm{Fix}(f)$ *and*

$$\mathrm{ind}(f; A) = \mathrm{ind}(\overline{f}; b_0)\, \mathrm{ind}(f_{b_0}; A).$$

Proof. Let us fix a neighbourhood $\mathcal{V}$ B such that $\mathcal{V}$ $\mathrm{Fix}(\overline{f}) = \{b_0\}$ and the restriction of the bundle over $\mathcal{V}$ is trivial: $p^{-1}(V) = V \times E_{b_0}$. Then there is a fibre homotopy $f_t\colon E \to E$ satisfying: $f_0 = f$, $pf_t = \overline{f}$, f_t constant outside $p^{-1}(\mathcal{V})$, constant in $E_{b_0} = p^{-1}(b_0)$ and such that $f_1(b,x) = (\overline{f}(b), f_{b_0}(x))$ for $(b,x) \in \mathcal{V}_0 \times E_{b_0} = p^{-1}\mathcal{V}_0 \subset E$ where $\mathcal{V}_0$ is a smaller neighbourhood $b_0 \in \mathcal{V}_0 \subset \mathcal{V}$. Since the deformation is constant in E_{b_0} the fixed point set does not change during the homotopy so we may assume that $f = \overline{f} \times f_{b_0}$ near x_0. Now the formula holds because of the Multiplicativity of the fixed point index. □

We will use the shorthand IPF for the above Fixed Point Index Product Formula.

(4.4.2) Corollary. *If x_0, b_0 are isolated fixed points of f and $\overline{f}$ respectively, then*

$$\mathrm{ind}(f; x_0) = \mathrm{ind}(\overline{f}; b_0)\,\mathrm{ind}(f_{b_0}; x_0).$$

(4.4.3) Theorem. *Let $p\colon E \to B$ be a locally trivial fibre bundle where all involved spaces are compact connected* ENR*s. Assume moreover that* $\mathrm{Fix}(\overline{f}) = \{b_1, \dots, b_k\}$ *is finite. Then* $\mathrm{ind}(f) = \sum_{i=1}^k \mathrm{ind}(\overline{f}; b_i)\,\mathrm{ind}(f_{b_i})$. *In particular if* $\mathrm{ind}(f_{b_i})$ *does not depend on i we get the* Lefschetz number product formula

$$L(f) = L(\overline{f})L(f_b).$$

Proof. From the above Fixed Point Index Product Formula (IPF) in Theorem (4.4.1) and the Additivity of the index we get

$$L(f) = \mathrm{ind}(f) = \sum_{i=1}^k \mathrm{ind}(\overline{f}; b_i)\,\mathrm{ind}(f_{b_i}) = (*).$$

If moreover $\mathrm{ind}(f_{b_i})$ does not depend on the point b_i,

$$(*) = \sum_{i=1}^k \mathrm{ind}(\overline{f}; b_i)\,\mathrm{ind}(f_b) = \mathrm{ind}(f_b)\sum_{i=1}^k \mathrm{ind}(\overline{f}; b_i) = \mathrm{ind}(f_b)\,\mathrm{ind}(\overline{f}) = \mathrm{L}(\overline{f})\mathrm{L}(f_b).$$

□

(4.4.4) Remark. Since the map $\overline{f}$ is homotopic to a map with $\mathrm{Fix}\,(\overline{f})$ finite, the formula

$$L(f) = L(\overline{f}) \cdot L(f_b)$$

holds for any fibre map $f\colon E \to E$ (if only $L(f_b)$ does not depend on b). □

The next example shows that the Lefschetz number $L(f_b)$ may depend on the point $b \in \mathrm{Fix}(f)$.

(4.4.5) EXAMPLE. Let K be the Klein bottle, i.e. $[-1,+1]\times S^1$ with identifications $(-1,z)\sim(1,\overline{z})$. Then $p\colon K\to S^1=[-1,+1]/\{-1=+1\}$, $p([z,t])=[t]$, is a fibration. Consider the map $f[t,z]=[-t,\overline{z}]$. Then the map of the base space has exactly two fixed points $[0]$ and $[\pm 1]$. The restriction of f to the fibre over $[0]$ is the map of degree -1, hence $L(f_0)=2$. On the other hand the restriction over $[\pm 1]$ is the identity on S^1, hence $L(f_1)=0$.

In the next subsection we will show that the Lefschetz number $L(f_b)$ is the same for all $b\in \mathrm{Fix}(\overline{f})$ lying in the same Nielsen class.

4.4.2. Nielsen number naive product formula. At the end of 1960s, the Jiang spaces were the only nontrivial class of spaces with a formula of $N(f)$. In the monograph [Br2], which reflects the actual state of knowledge of that time, the computations are rather isolated although the theory is well developed. The next attempt to extend the area of spaces with a formula of the Nielsen number was done by Robert Brown in 1967. He raised the following question. Let $p\colon E\to B$ be a locally trivial fibre bundle with the fibre F and $f\colon E\to E$ a fibre map ($pf=\widetilde{f}p$). We fix a point $b\in \mathrm{Fix}(\overline{f})$. We denote by f_b the restriction of f to the fibre $E_b=p^{-1}(b)$. Under what assumptions does the product formula

$$N(f)=N(\overline{f})N(f_b)$$

hold?

It turned out that the connection among $N(f)$, $N(\widetilde{f})$, $N(f_b)$ is much more complex in general so the above formula is now called the *naive product formula.* Nevertheless it is desirable to know when this naive, or less naive, product formula holds. We start with an easy lemma.

(4.4.6) LEMMA. *The naive product formula holds for the product map $f=f_1\times f_2\colon E\to E$ where $E=B\times F$, $f(b,x)=(f_1(b),f_2(x))$ where B,F are compact* ENR*s.*

PROOF. We notice that in this special situation $\mathrm{Fix}(f)=\mathrm{Fix}(f_1)\times\mathrm{Fix}(f_2)$ and moreover the Nielsen relation splits: a path $\omega=\omega_1\times\omega_2$ establishes the Nielsen relation if and only if so do ω_1 and ω_2. Thus $\mathcal{N}(f)=\mathcal{N}(f_1)\times\mathcal{N}(f_2)$. Moreover, by the Index Product Formula (Theorem (4.4.1)), $\mathcal{E}(f)=\mathcal{E}(f_1)\times\mathcal{E}(f_2)$. Finally

$$N(f)=\#\mathcal{E}(f)=\#\mathcal{E}(f_1)\times\#\mathcal{E}(f_2)=N(f_1)N(f_2). \qquad \square$$

In a general the Nielsen number of the map f_b may depend on the fixed point b so the naive product formula makes no sense. Moreover, even if $N(f_b)$ is the same for all $b\in\mathrm{Fix}(\overline{f})$, then the Nielsen relations do not split as in the above lemma.

This problem was intensively studied in [Br1], [Fa], [Pak] where the necessary conditions for the naive Nielsen product formula were given. Finally in 1981 Cheng Ye You [Yu1] gave a necessary and sufficient condition for this formula. A similar result is also given in Section 4 of [Ji4]. See also [He]. In this section we prove the main result of [Yu1] using rather the covering approach.

Let us start with two counter examples to the naive product formula.

(4.4.7) Example. Let $p\colon S^3 \to S^2$ be the Hopf fibration $p(z_1, z_2) = z_1/z_2$. Then the formula $f(z_1, z_2) = (z_1^k/|z_1|^{k-1}, z_2^k/|z_2|^{k-1})$ gives a fibre map ($k \in \mathbb{Z}$). One may check that the restriction of f to a fibre has degree k, so $N(f_0) = |k-1|$. On the other hand $\deg(\overline{f}) = k$ and $\deg(f) = k^2$. Now $L(\overline{f}) = 1+k$, $L(f) = 1-k^2$ are nonzero for $k \neq \pm 1$. Since S^2, S^3 are simply-connected, $N(f) = N(\overline{f}) = 1$. Thus $N(f) = 1 \neq 1 \cdot |k-1| = N(\overline{f})N(f_0)$.

The next example shows that the Nielsen number of the restriction over two fixed points of $\overline{f}$ may be different.

(4.4.8) Example. Consider again the Klein bottle from Example (4.4.5) and the map $f[t, z] = [-t, \overline{z}]$. As we noticed there the Lefschetz numbers over the two fixed points of $\overline{f}$ equal $L(f_0) = 0$, $L(f_1) = 2$ respectively. Since the fibre is S^1, these Lefschetz numbers are equal to the Nielsen numbers, hence the last are also different. $\square$

Nevertheless we will show that for the points from the same Nielsen class the number $N(f_b)$ is the same and then we will try to express $N(f)$ by $N(\overline{f})$ and the Nielsen numbers of the restriction to the fibres.

We will concentrate on locally trivial bundles although we could consider more general objects such as Hurewicz fibrations. Regardless of the generality it is necessary to regard fibre bundles as Hurewicz fibrations. Let us recall the definition.

(4.4.9) Definition. The map $p\colon E \to B$ is called a *Hurewicz fibration* if it admits a *lifting map* $\lambda\colon P(E) \to E^I$ where

$$P(E) = \{(\overline{\omega}, x) \in B^I \times E : p(x) = \widetilde{\omega}(0)\}$$

and the two conditions are satisfied

(4.4.9.1) $p\lambda(\overline{\omega}, x) = \overline{\omega}$,

(4.4.9.2) $\lambda(\mathrm{const}, x) \equiv x$. $\square$

Let $p\colon E \to B$ be a Hurewicz fibration and let $\overline{\omega}\colon [0,1] \to B$ be a path. Then there exists a homotopy $h_t\colon E_{\overline{\omega}(0)} \to E$ satisfying $h_0 = \mathrm{id}_{E_{\overline{\omega}(0)}}$ and $ph_t(x) = \overline{\omega}(t)$ for all $x \in E_{\overline{\omega}(0)}$. Such a homotopy may be given by the formula $h_t(x) = \lambda(\overline{\omega}, x)(t)$ Then the map $h_1\colon E_{\overline{\omega}(0)} \to E_{\overline{\omega}(1)}$ is called *admissible* and is denoted $\tau_{\overline{\omega}}$.

Now it is easy to show that any two admissible maps (corresponding to the path $\overline{\omega}$) are homotopic and moreover any two such homotopies are homotopic. Thus $\tau_{\overline{\omega}}\colon E_{\overline{\omega}(0)} \to E_{\overline{\omega}(1)}$ will denote the homotopy class of an admissible map over the path $\overline{\omega}$ although sometimes we will use this symbol for an admissible map representing this class.

In particular each admissible map over a contractible loop is homotopic to the identity. On the other hand each admissible map $\tau_{\overline{\omega}}$ is a homotopy equivalence since the opposite path $\overline{\omega}'(t) = \overline{\omega}(1-t)$ defines the map $\tau_{\overline{\omega}'}\colon E_{\overline{\omega}(1)} \to E_{\overline{\omega}(0)}$ homotopy inverse to $\tau_{\overline{\omega}}$.

Although there is no canonical choice of an admissible map between two fibres, the Hurewicz lifting map guarantees a continuous choice of all functions $\tau_{\overline{u}}(e)$ with respect to $(\overline{u}, e) \in P(E)$.

(4.4.10) LEMMA. *Let $\overline{\omega}\colon I \to B$ be a path joining two fixed points $b_0, b_1 \in$* $\mathrm{Fix}(\overline{f})$. *Then the diagram*

$$\begin{tikzcd} E_{b_0} \arrow[r, "f_0"] \arrow[d, "\tau_{\overline{\omega}}"'] & E_{b_0} \arrow[d, "\tau_{\overline{f\omega}}"] \\ E_{b_1} \arrow[r, "f_1"'] & E_{b_1} \end{tikzcd}$$

is homotopy commutative.

PROOF. We write the homotopy $H\colon E_{b_0} \times I \to E_{b_1}$ by the formula

$$H(x,t) = \tau_{\overline{f\omega}_{[t,1]}}(f_{\overline{\omega}(t)}(\tau_{\overline{\omega}_{[0,t]}}(x))).$$

Here we use the above remark that the admissible maps can be chosen continuously. $\square$

(4.4.11) COROLLARY. *Let $b_0, b_1 \in$* $\mathrm{Fix}(\overline{f})$ *belong to the same Nielsen class. Let $\overline{\omega}\colon I \to B$ satisfy $\overline{\omega}(0) = b_0$, $\overline{\omega}(1) = b_1$ $\overline{f}\overline{\omega} \sim \overline{\omega}$. Then the diagram*

$$\begin{tikzcd} E_{b_0} \arrow[r, "f_{b_0}"] \arrow[d, "\tau_{\overline{\omega}}"'] & E_{b_0} \arrow[d, "\tau_{\overline{f\omega}}"] \\ E_{b_1} \arrow[r, "f_{b_1}"'] & E_{b_1} \end{tikzcd}$$

is homotopy commutative.

PROOF. Let $h_t\colon I \to B$ be the homotopy establishing the Nielsen relation $h_0 = \overline{f}\overline{\omega}$, $h_1 = \overline{\omega}$. It lifts to the homotopy $H_t = \tau_{h_t(\,\cdot\,)}\colon E_{b_0} \to E_{b_1}$ between $\tau_{\overline{f\omega}}$ and $\tau_{\overline{\omega}}$. Thus by the above lemma

$$f_1\tau_{\overline{\omega}} \sim \tau_{\overline{f\omega}} f_0 \sim \tau_{\overline{\omega}} f_0. \qquad \square$$

(4.4.12) REMARK. The homotopy making the diagram from the above lemma homotopy commutative is not canonical. It may depend on the choice of the homotopy between the paths $\overline{f\varpi}$ and $\overline{\varpi}$.

Now Theorem (4.1.32) implies

(4.4.13) COROLLARY. *The Lefschetz, correspondingly Nielsen, numbers of the restriction of the fibre map over the points from the same Nielsen class are equal:* $L(f_0) = L(f_1)$, *and respectively* $N(f_0) = N(f_1)$.

4.4.3. Nielsen classes on the fibres. Let $p: E \to B$ be a locally trivial fibre bundle where all involved spaces are compact connected ENRs. Let $p_E: \widetilde{E} \to E$, $p_B: \widetilde{B} \to B$ be universal coverings. Let us fix a base point $b_0 \in B$ and denote $F = p^{-1}(b_0)$. Then there is a map $\widetilde{p}$ making the diagram

$$\begin{tikzcd} \widetilde{E} \arrow[r, "\widetilde{p}"] \arrow[d, "p_E"'] & \widetilde{B} \arrow[d, "p_B"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

commutative. We fix such $\widetilde{p}$.

(4.4.14) LEMMA. $p_E^{-1}(F) = \bigcup_\beta \widetilde{F}_\beta$ *is the sum of connected components where* $\beta \in \pi_1(B)$. *Moreover, for each* $\beta \in \pi_1(B)$,

$$\operatorname{Im}[(p_{E|})_\#: \pi_1(\widetilde{F}_\beta) \to \pi_1(F)] = \operatorname{Ker}[i_\#: \pi_1(F) \to \pi_1(E)].$$

PROOF. Let us notice that

$$p_E^{-1}(F) = p_E^{-1}(p^{-1}(b_0)) = \widetilde{p}^{-1}(p_B^{-1}(b_0)) = \widetilde{p}^{-1}\left(\bigcup_\beta \{\widetilde{b}_\beta\}\right) = \bigcup_\beta \widetilde{p}^{-1}(\widetilde{b}_\beta),$$

where $\beta \in \pi_1(B) = p_B^{-1}(b_0)$.

To prove the first claim it remains to show that each $\widetilde{F}_\beta = \widetilde{p}^{-1}(\widetilde{b}_\beta)$ is connected. Let $\widetilde{x}, \widetilde{y} \in \widetilde{p}^{-1}(\widetilde{b}_\beta)$ and let $\widetilde{\omega}: I \to \widetilde{E}$ be a path joining these points. Then $\widetilde{p}\widetilde{\omega}$ is a loop in $\widetilde{B}$ based at $\widetilde{b}_\beta$. Since $\widetilde{B}$ is the universal covering space, $\widetilde{p}\widetilde{\omega}$ is homotopic to the constant map at $\widetilde{b}_\beta$. The last homotopy lifts to a homotopy in $\widetilde{E}$ from $\widetilde{\omega}$ to a path joining the points $\widetilde{x}$, $\widetilde{y}$ in $\widetilde{p}^{-1}(\widetilde{b}_\beta)$.

The proof of the second part of the lemma can be derived from the exactness of the homotopy sequence of the fibration (E, p, B). However we will give here a straightforward proof. Let us fix a component $\widetilde{F}_\beta = \widetilde{p}^{-1}(\widetilde{b}_\beta) \subset p_E^{-1}(F)$ and consider the commutative diagram

$$\begin{array}{ccc} \widetilde{F}_\beta & \xrightarrow{\widetilde{i}} & \widetilde{E} \\ {\scriptstyle p_{E|}}\downarrow & & \downarrow{\scriptstyle p_E} \\ F & \xrightarrow[i]{} & E \end{array}$$

$\subset$ follows from $i_\#(p_{E|})_\# = (ip_{E|})_\# = (p_E\widetilde{i})_\# = (p_E)_\#\widetilde{i}_\# = 0$ since $\mathrm{Im}\,\widetilde{i}_\# \subset \pi_1(\widetilde{E}) = 1$.

$\supset$ Let $\alpha \in \pi_1(F)$ satisfy $i_\#(\alpha) = 1 \in \pi_1(E)$. Then α extends to a map $h\colon [0,1]\times[0,1] \to E$ satisfying $h(t,0) = \alpha(t)$, $h(0,s) = h(t,1) = h(1,s) = x_\beta$. Since $\widetilde{E}$ is a covering space and $[0,1]\times[0,1]$ is simply-connected, there is a unique lift $\widetilde{h}\colon [0,1]\times[0,1] \to \widetilde{E}$ satisfying $p\widetilde{h} = h$, $\widetilde{h}(0,0) = \widetilde{x}_\beta$. Then $\widetilde{h}(0,s) = \widetilde{h}(t,1) = \widetilde{h}(1,s) = \widetilde{x}_\beta$, hence $\alpha = h(\,\cdot\,,0)$ lifts to a loop in $p_E^{-1}(F)$. Now $(p_{E|})_\#[\widetilde{h}(\,\cdot\,,0)] = [\alpha] \in \pi_1(F)$. $\square$

Let us denote $K = \mathrm{Im}\,[p_{E|\#}\colon \pi_1(\widetilde{F}_\alpha) \to \pi_1(F)] = \mathrm{Ker}\,[i_{\#}\colon \pi_1(F) \to \pi_1(E)]$ the normal subgroup of $\pi_1(F)$. We will consider

$\mathcal{R}_K(f_F) =$ the set of Reidemeister classes modulo the normal subgroup K.

Let us fix a component $\widetilde{F}_0 \subset p_E^{-1}(F)$. By the above the restriction of p_E gives the covering $\widetilde{F}_0 \to F$ corresponding to the normal subgroup $K \lhd \pi_1(F)$

Let $f\colon E \to E$ be a fibre map. We will consider its restriction to the fibre $f_F\colon F \to F$. Then the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{f_F} & F \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ E & \xrightarrow[f]{} & E \end{array}$$

gives the map $\mathcal{R}_i\colon \mathcal{R}_K(f_F) \to \mathcal{R}(f)$ (see Subsection 4.1.3). It may be described as follows.

Let us fix a component $\widetilde{F}_0 \subset p_E^{-1}(F)$. Then each natural transformation $\widetilde{f}_F\colon \widetilde{F}_0 \to \widetilde{F}_0$ of the covering $\widetilde{p}_{E|\widetilde{F}_0}\colon \widetilde{F}_0 \to F$ extends uniquely to a lift $\widetilde{f}\colon \widetilde{E} \to \widetilde{E}$ of the covering $p_E\colon \widetilde{E} \to E$. Then $\mathcal{R}_i[\widetilde{f}_0] = [\widetilde{f}]$.

(4.4.15) Remark. To define $\mathcal{R}_K(f_{F_0})$ we used as the covering space mod K an arbitrary component $\widetilde{F}_0 \subset p_E^{-1}(F)$. From the general theory (Section 4.1) we know that choosing another component $\widetilde{F}'_0 \subset p_E^{-1}(F)$ we get the Reidemeister set (temporarily denoted $\mathcal{R}'_K(f_{F_0})$) which is canonically isomorphic to $\mathcal{R}_K(f_F)$. We will need the explicit formula of this isomorphism. We notice that if $\alpha\colon \widetilde{E} \to \widetilde{E}$ is a deck transformation satisfying $\alpha(\widetilde{F}_0) = \widetilde{F}'_0$, then the diagram

$$\begin{tikzcd} \widetilde{F}_0 \arrow[r, "\alpha"] \arrow[d, "p_E"'] & \widetilde{F}'_0 \arrow[d, "p_E"] \\ F \arrow[r, equal] & F \end{tikzcd}$$

is commutative. Thus the natural bijection is given by

$$\mathcal{R}_K(f_{F_0}) \ni [\widetilde{f}_0] \to [\alpha \widetilde{f}_0 \alpha^{-1}] \in \mathcal{R}'_K(f_{F'_0})$$

for any transformation $\alpha\colon \widetilde{E} \to \widetilde{E}$ satisfying $\alpha(\widetilde{F}_0) = \widetilde{F}'_0$ (see Theorem (4.1.14)).

We will show that a path $\overline{\omega}$ establishing the Nielsen relation between the points $b_0, b_1 \in \operatorname{Fix}(\overline{f})$ yields a map of the Reidemeister sets $T_{\overline{\omega}}\colon \mathcal{R}_K(f_{F_0}) \to \mathcal{R}_K(f_{F_1})$. Let $\widetilde{\overline{\omega}}\colon I \to \widetilde{B}$ be the lift of the path $\overline{\omega}$ starting from the point $\widetilde{\overline{\omega}}(0) = \widetilde{b}_0 = \widetilde{p}(\widetilde{F}_0)$. Let $\widetilde{\overline{f}}\colon \widetilde{B} \to \widetilde{B}$ be a lift (of $\overline{f}\colon \overline{B} \to \overline{B}$) satisfying $\widetilde{\overline{f}}(\widetilde{b}_0) = \widetilde{b}_0$. Then $\widetilde{\overline{\omega}}, \widetilde{\overline{f}}\widetilde{\overline{\omega}}$ are lifts of the loops $\overline{\omega}$, $\overline{f}\overline{\omega}$ respectively starting at the same point $\widetilde{b}_0$. Since $\overline{\omega} \sim \overline{f}\overline{\omega}$, the ends of the lifts must be equal $\widetilde{\overline{\omega}}(1) = \widetilde{\overline{f}}\widetilde{\overline{\omega}}(1) \in p_B^{-1}(\omega(1)) = p_B^{-1}(b_1)$. Let us denote this common end by $\widetilde{b}_1$ and let $\widetilde{F}_1 = p_B^{-1}(\widetilde{b}_1)$.

Now we fix a lift $\widetilde{f}_0\colon \widetilde{F}_0 \to \widetilde{F}_0$ representing an element $[\widetilde{f}_0] \in \mathcal{R}_K(f_F)$ and let $\widetilde{f}\colon \widetilde{E} \to \widetilde{E}$ be its extension. Since $\widetilde{f}(\widetilde{F}_0) \subset \widetilde{F}_0$ and $\overline{f}\overline{\omega} \sim \overline{\omega}$, $\widetilde{f}(\widetilde{F}_1) \subset \widetilde{F}_1$. Denoting $\widetilde{f}_1 = \widetilde{f}_{\widetilde{F}_1}$ we get the map

$$\operatorname{lift}(f_{F_0}) \ni \widetilde{f}_0 \to \widetilde{f}_1 \in \operatorname{lift}(f_{F_1}).$$

This map preserves the Reidemeister relation: any deck transformation α_0 of $\widetilde{F}_0$ extends uniquely to a deck transformation α of $\widetilde{E}$, hence $\alpha_0 \widetilde{f}_0 \alpha_0^{-1} \in \operatorname{lift}(f_F)$ is sent onto $\alpha_1 \widetilde{f}_1 \alpha_1^{-1} \in \operatorname{lift}(f_F)$ where $\alpha_1 = \alpha_{|\widetilde{F}_1}$.

Thus we get a well-defined map $T_{\overline{\omega}}\colon \mathcal{R}_K(f_{F_0}) \to \mathcal{R}_K(f_{F_1})$ (for every path $\overline{\omega}$ joining the fixed points b_0, b_1 and satisfying $\overline{f}\overline{\omega} \sim \overline{\omega}$).

The following functorial properties follow straight from the construction of the map $T_{\overline{\omega}}$.

(4.4.16) LEMMA.

(4.4.16.1) $T_{\overline{\omega}} = T_{\overline{\omega}'}$ *for homotopic paths* $\overline{\omega}$, $\overline{\omega}'$ (*joining the fixed points* b_0, b_1 *of* $\overline{f}$ *and establishing the Nielsen relation*),

(4.4.16.2) $T_{\overline{\omega} * \overline{\omega}'} = T_{\overline{\omega}'} T_{\overline{\omega}}$ *provided* $\widetilde{\omega}(1) = \widetilde{\omega}'(0)$,

(4.4.16.3) $T_{\text{const}} = \text{id}$.

In particular for $b_0 = b_1$ we get the action of the group

$$\operatorname{Fix}(\overline{f}_{\#}; b_0) = \{\gamma \in \pi_1(B; b_0); \overline{f}_{\#}\gamma = \gamma\}$$

on the set $\mathcal{R}_K(f_{|E_{b_0}})$.

4.4.4. Transformation $T_{\overline{u}}$ in coordinates. To better understand the above action $T_{\overline{u}}$ we will interpret it "in coordinates". The canonical bijection between the sets of Reidemeister classes defined by coverings and by homotopy groups implies the map $T_{\overline{u}}^{(x_0,r_0)}$, making the diagram

$$\begin{tikzcd}
\mathcal{R}_K(f_{b_0}; x_0, r_0) \arrow[r, "T_{\overline{u}}^{(x_0,r_0)}"] \arrow[d, "\phi_0"'] & \mathcal{R}_K(f_{b_1}; x_1, r_1) \arrow[d, "\phi_1"] \\
\mathcal{R}_K(f_{b_0}) \arrow[r, "T_{\overline{u}}"'] & \mathcal{R}_K(f_{b_1})
\end{tikzcd}$$

commutative.

We fix a reference pair (x_0, r_0) in F_0. Now we make some choices: we fix an admissible map $\tau_{\overline{u}}\colon F_0 \to F_1$ and a path $u = \lambda(\overline{u}, b_0)$ where λ is a Hurewicz lifting function. We take as reference pair in F_1 the pair (x_1, r_1) where $x_1 = \tau_{\overline{u}}(x_0)$ and r_1 is a path in F_1 joining the point x_1 with fx_1 which is homotopic in E to the path $u^{-1} * r_0 * fu$. By Lemma (4.4.18) such a path exists and its homotopy type modulo K is determined. We define the map

$$T_{\overline{u}}^{(x_0,r_0)}\colon \mathcal{R}_K(f_0; x_0, r_0) \to \mathcal{R}_K(f_1; x_1, r_1)$$

by the formula

$$T_{\overline{u}}^{(x_0,r_0)}[a] = [\tau_{\overline{u}}(a)].$$

The next theorem establishes the canonical identification of $T_{\overline{u}}^{(x_0,r_0)}$ with $T_{\overline{u}}$. This implies that the map $T_{\overline{u}}^{(x_0,r_0)}$ does not depend on the choice of the admissible map and the path u. Moreover, the maps $T_{\overline{u}}^{(x_0,r_0)}$ corresponding to different reference pairs are coherent.

(4.4.17) THEOREM. *The map $T_{\overline{u}}^{(x_0,r_0)}$ is correctly defined and the diagram*

$$\begin{tikzcd}
\mathcal{R}_K(f_{b_0}; x_0, r_0) \arrow[r, "T_{\overline{u}}^{(x_0,r_0)}"] \arrow[d, "\phi_0"'] & \mathcal{R}_K(f_{b_1}; x_1, r_1) \arrow[d, "\phi_1"] \\
\mathcal{R}_K(f_{b_0}) \arrow[r, "T_{\overline{u}}"'] & \mathcal{R}_K(f_{b_1})
\end{tikzcd}$$

is commutative. Here ϕ_0, ϕ_1 denote the canonical identification of the sets of Reidemeister classes, see Section 4.1.5.

PROOF. We fix universal coverings $\widetilde{p}_E\colon \widetilde{E} \to E$, $\widetilde{p}_B\colon \widetilde{B} \to B$ and a point $\widetilde{b}_0 \in p_B^{-1}(b_0)$. We put $\widetilde{F}_0 = \widetilde{p}^{-1}(\widetilde{b}_0)$ and we fix $\widetilde{x}_0 \in \widetilde{F}_0 \cap p_E^{-1}(x_0)$. Let

$$\widetilde{x}_1 = \text{end of the lift of the path } u \text{ starting from } \widetilde{x}_0$$

with respect to the covering $p_E\colon \widetilde{E} \to E$, where u is the pair that was fixed in the definition of $T_{\overline{u}}^{(x_0,r_0)}$.

Let $[a] \in \mathcal{R}_K(f_{b_0}; x_0, r_0)$. Then $\phi_0[a] = [\widetilde{f}_0]$ where $\widetilde{f}_0\colon \widetilde{F}_0 \to \widetilde{F}_0$ satisfies

$$\widetilde{f}(\widetilde{x}_0) = \text{ end of the lift of the path } a * r_0 \text{ starting from } \widetilde{x}_0.$$

Now $T_{\overline{u}}\phi[a] = T_{\overline{u}}[\widetilde{f}_0] = [\widetilde{f}_1]$ where $\widetilde{f}_1\colon \widetilde{F}_1 \to \widetilde{F}_1$ is the restriction of the extension $\widetilde{f}\colon \widetilde{E} \to \widetilde{E}$, of the lift $\widetilde{f}_0$, to the fibre $\widetilde{F}_1 = \widetilde{p}^{-1}(\widetilde{b}_1)$ where $\widetilde{b}_1 = \widetilde{\overline{u}}(1)$ and $\widetilde{\overline{u}}$ is the lift of $\overline{u}$ starting from $\widetilde{b}_0$.

On the other hand $\phi \cdot T_{\overline{u}}^{(x_0,r_0)}[a] = \phi[\tau_{\overline{u}}(a)] = [\widetilde{f}'_1]$ where

$$\begin{aligned}
\widetilde{f}'_1(\widetilde{x}_1) &= \text{end of the lift of the path } \tau_{\overline{u}}(a) * r_1 \text{ starting from } \widetilde{x}_1\\
&= \text{end of the lift of the path } u * \tau_{\overline{u}}(a) * r_1 \text{ starting from } \widetilde{x}_0\\
&= \text{end of the lift of the path } (a * u) * (u^{-1} * r_0 * fu) \text{ starting from } \widetilde{x}_0\\
&= \text{end of the lift of the path } fu \text{ starting from } \widetilde{f}(\widetilde{x}_0)\\
&= \widetilde{f}(\text{end of the lift of the path } u \text{ starting from } \widetilde{x}_0) = \widetilde{f}(\widetilde{x}_1).
\end{aligned}$$

Here $\widetilde{f}$ denotes again the extension of $\widetilde{f}_0$. Let $\widetilde{f}'$ denote the extension of $\widetilde{f}'_1$. Then $\widetilde{f}'(\widetilde{x}_1) = \widetilde{f}'_1(\widetilde{x}_1) = \widetilde{f}(\widetilde{x}_1)$, hence the lifts $\widetilde{f}$, $\widetilde{f}'\colon \widetilde{E} \to \widetilde{E}$ are equal and

$$T_{\overline{u}}\phi[a] = [\widetilde{f}] = [\widetilde{f}'] = \phi T_{\overline{u}}^{(x_0,r_0)}[a]$$

which proves the commutativity of the diagram. □

(4.4.18) LEMMA. *Let $p\colon E \to B$ be a Hurewicz fibration with connected fibres and let ω be a path in E such that $\omega(0) = x_0$, $\omega(1) = x_1$, where $p(x_0) = p(x_1) = b_0$ and $p\omega = 1 \in \pi_1(B)$. Then there is a path $\omega'\colon I \to F_b$ satisfying: $\omega'(0) = x_0$, $\omega'(1) = x_1$ and $\omega \sim \omega'$ in E. Moreover, if ω'' is another such path, then $\omega' \sim \omega''$ modulo K in F_b.*

PROOF. Let us fix a path $v\colon I \to F_b$ satisfying $v(0) = x_0$, $v(1) = x_1$. Then $\omega * v^{-1}$ is a loop in E based at x_0. Moreover $p(\omega * v^{-1}) = p(\omega) = 1 \in \pi_1(B)$, hence from the homotopy exact sequence there is a loop ω_1 in F_b such that $\omega_1 \sim \omega * v^{-1} \in \pi_1(E)$. Now $\omega' = \omega_1 * v$ is the desired path in F_b.

Now let ω'' be another path in F_b satisfying $\omega'' \sim \omega$ in E. Then $\omega' * (\omega'')^{-1}$ is contractible in E, hence $[\omega' * (\omega'')^{-1}] \in \operatorname{Ker}[\pi_1(F) \to \pi_1(E)] = K$. □

4.4.5. Nielsen number product formula.

(4.4.19) THEOREM. *Let $\mathbb{A}, \mathbb{A}' \in \mathcal{R}_K(f_0)$. Then $\mathcal{R}_i(\mathbb{A}) = \mathcal{R}_i(\mathbb{A}')$ if and only if $\mathbb{A}$, $\mathbb{A}'$ belong to the same orbit of the action of* $\operatorname{Fix}(\overline{f}_\#; b_0)$ *on $\mathcal{R}_K(f_0)$, i.e. $\mathbb{A}' = T_{\overline{\omega}}(\mathbb{A})$ for an $\omega \in$* $\operatorname{Fix}(\overline{f}_\#; b_0)$.

PROOF. ($\Rightarrow$) Let $\mathbb{A} = [\widetilde{f}_0]$, $\mathbb{A}' = [\widetilde{f}'_0]$ where $\widetilde{f}_0, \widetilde{f}'_0\colon \widetilde{F}_0 \to \widetilde{F}_0$. Let $\widetilde{f}, \widetilde{f}'\colon \widetilde{E} \to \widetilde{E}$ be the extensions of $\widetilde{f}_0$, $\widetilde{f}'_0$ respectively to the lifts of f. Then $\mathcal{R}_i[\widetilde{f}_0] = \mathcal{R}_i[\widetilde{f}'_0]$

means $[\widetilde{f}] = [\widetilde{f}'] \in \mathcal{R}(f)$, hence there exists an $\alpha \in \mathcal{O}_E$ such that $\widetilde{f}' = \alpha^{-1}\widetilde{f}\alpha$. Let us denote $\widetilde{F}_1 = \alpha(\widetilde{F}_0) \subset p_E^{-1}(F)$ a connected component. Then $\widetilde{f}'(\widetilde{F}_0) \subset \widetilde{F}_0$ gives $\alpha^{-1}f\alpha(\widetilde{F}_0) \subset \widetilde{F}_0$, hence $\widetilde{f}\alpha(\widetilde{F}_0) \subset \alpha(\widetilde{F}_0)$ which means $\widetilde{f}(\widetilde{F}_1) \subset \widetilde{F}_1$. Let $\widetilde{\omega}$ be a path joining the components $\widetilde{F}_0$ and $\widetilde{F}_1$ in $\widetilde{E}$. Notice that then $T_{\overline{\omega}}[\widetilde{f}_0] = [\widetilde{f}_{\widetilde{F}_1}] \in \mathcal{R}_K(f_0)$, where $\overline{\omega}$ denotes the loop $p_B\widetilde{p}(\widetilde{\omega})$ and $\widetilde{f}_{\widetilde{F}_1}$ is the restriction of $\widetilde{f}$ to $\widetilde{F}_1$.

It remains to show that $[\widetilde{f}_{\widetilde{F}_1}] = [\widetilde{f}'_0] \in \mathcal{R}_K(f_0)$. But the two above lifts are defined on different coverings of F: $\widetilde{f}'_0\colon \widetilde{F}_0 \to \widetilde{F}_0$ while $\widetilde{f}_{\widetilde{F}_1}\colon \widetilde{F}_1 \to \widetilde{F}_1$. To compare them we notice that $\alpha\colon \widetilde{F}_0 \to \widetilde{F}_1$ is the homeomorphism satisfying $p_E\alpha = p_E$. Since the deck transformation $\alpha\colon \widetilde{E} \to \widetilde{E}$ satisfies $\alpha^{-1}(F_1) = F_0$, $[\widetilde{f}_{\widetilde{F}_1}] \in \mathcal{R}_K(f_{b_1})$ corresponds to $[\alpha^{-1}_{|F_0}\widetilde{f}_{F_1}\alpha_{|F_0}] = [\widetilde{f}'_0] \in \mathcal{R}_K(f_{b_0})$ (by Theorem (4.1.14) and Remark (4.4.15) $[\widetilde{f}_{\widetilde{F}_1}]$).

($\Leftarrow$) Let $\widetilde{f}_0\colon \widetilde{F}_0 \to \widetilde{F}_0$. $\mathbb{A} = [\widetilde{f}_0], \mathbb{A}' = T_{\overline{\omega}}[\widetilde{f}_0]$ where $\overline{\omega} \in \mathrm{Fix}(\overline{f}_\#; b_0)$. Let $\widetilde{f}\colon \widetilde{E} \to \widetilde{E}$ be the unique extension of $\widetilde{f}_0$. Then $T_{\overline{\omega}}[\widetilde{f}_0] = [\widetilde{f}_1]$ where $\widetilde{f}_1\colon \widetilde{F}_1 \to \widetilde{F}_1$ is the restriction of $\widetilde{f}$ to the component $\widetilde{F}_1$. Now $\mathcal{R}_i[\widetilde{f}_0] = \mathcal{R}_i[\widetilde{f}_1] = [\widetilde{f}] \in \mathcal{R}(f)$. $\square$

(4.4.20) THEOREM. *Let $\mathcal{R}_i\colon \mathcal{R}_K(f_F) \to \mathcal{R}(f)$, $\mathcal{R}_p\colon \mathcal{R}(f) \to \mathcal{R}(\overline{f})$ be the maps induced by the commutative diagrams*

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "f_F"'] & E \arrow[d, "f"] \\ F \arrow[r, "i"'] & E \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "p"] \arrow[d, "f"'] & B \arrow[d, "\overline{f}"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

where $F \subset E$ is the fibre over a fixed point $b_0 \in B$. Let $\overline{\mathbb{A}} \in \mathcal{R}(\overline{f})$ be the Nielsen class of $\overline{f}$ containing b_0. Then $\mathrm{Im}\,\mathcal{R}_i = \mathcal{R}_p^{-1}(\mathbb{A})$. In other words the sequence of sets, with the chosen element $\overline{\mathbb{A}} \in \mathcal{R}(\overline{f})$,

$$\mathcal{R}(f_F) \xrightarrow{\mathcal{R}_i} \mathcal{R}(f) \xrightarrow{\mathcal{R}_p} \mathcal{R}(\overline{f}) \longrightarrow 1$$

is exact.

PROOF. Let us recall that the Nielsen class $\overline{\mathbb{A}}$ as the Reidemeister class is represented by the map $\widetilde{\overline{f}}\colon \widetilde{B} \to \widetilde{B}$ if and only if $b_0 \in p_B(\mathrm{Fix}(\widetilde{\overline{f}}))$.

$\subset$ We have to show that $\mathrm{Im}\,\mathcal{R}_p\mathcal{R}_i = \overline{\mathbb{A}}$. But this is the consequence of the equality $\mathrm{Im}\,\mathcal{R}_p\mathcal{R}_i = \mathrm{Im}\,\mathcal{R}_{pi} = \mathrm{Im}\,\mathcal{R}_*$, where $*$ is the constant map into the point b_0.

$\supset$ Let $[\widetilde{f}] \in \mathcal{R}(f)$ satisfy $\mathcal{R}_p[\widetilde{f}] = \overline{\mathbb{A}}$. We recall that $\mathcal{R}_p[\widetilde{f}] = [\widetilde{\overline{f}}] \in \mathcal{R}(\overline{f})$, where

$$\begin{tikzcd} \widetilde{E} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{p}"'] & \widetilde{E} \arrow[d, "\widetilde{p}"] \\ \widetilde{B} \arrow[r, "\widetilde{\overline{f}}"'] & \widetilde{B} \end{tikzcd}$$

commutes. Now $\overline{\mathbb{A}} = p_B(\mathrm{Fix}(\overline{\widetilde{f}}))$ contains b_0, hence there exists $\widetilde{b}_0 \in \mathrm{Fix}(\overline{\widetilde{f}})$ satisfying $p_B(\widetilde{b}_0) = b_0$. Since $\widetilde{B}$ is simply-connected, $\widetilde{F} = \widetilde{p}^{-1}(\widetilde{b}_0)$ is a component of $p_E^{-1}(F)$. We show that $\widetilde{f}(\widetilde{F}) \subset \widetilde{F}$. Since $\widetilde{F} = \widetilde{p}^{-1}(\widetilde{b}_0)$, the last is equivalent to $\widetilde{f}\widetilde{p}^{-1}(\widetilde{b}_0) \subset \widetilde{p}^{-1}(\widetilde{b}_0)$, hence $\widetilde{p}\widetilde{f}\widetilde{p}^{-1}(\widetilde{b}_0) = \widetilde{b}_0$. But $\widetilde{p}\widetilde{f}\widetilde{p}^{-1}(\widetilde{b}_0) = \overline{\widetilde{f}}\widetilde{p}\widetilde{p}^{-1}(\widetilde{b}_0) = \overline{\widetilde{f}}(\widetilde{b}_0) = \widetilde{b}_0$. Now denoting by $\widetilde{f}_0$ the restriction of f to $\widetilde{F}$ we get the element $[\widetilde{f}_0] \in \mathcal{R}_K(f_F)$ such that $\mathcal{R}_i[\widetilde{f}_0] = [\widetilde{f}]$, hence $[\widetilde{f}] \in \operatorname{Im}\mathcal{R}_i$. □

By the Index Product Formula each essential Nielsen class in $\mathcal{E}(f)$ lies over an essential (hence nonempty) Nielsen class in $\mathcal{E}(\overline{f}) \subset \mathcal{R}(\overline{f})$. Thus

$$\mathcal{E}(f) \subset \bigcup_{\overline{\mathbb{A}}} \mathcal{R}_p^{-1}(\overline{\mathbb{A}}) = \bigcup_b \operatorname{Im}\mathcal{R}_{i_b},$$

where $\overline{\mathbb{A}}$ runs over $\mathcal{E}(\overline{f})$ and b runs over the set of *essential representatives* of $\overline{f}$, i.e. one point from each essential Nielsen class of $\overline{f}$.

Moreover, applying the Index Product Formula again, we notice that $\mathcal{R}_{i_b}(\mathbb{A})$ (for $\mathbb{A} \in \mathcal{R}_K(f_b)$) belongs to $\mathcal{E}(f)$ if and only if $\mathbb{A}$ is an essential class modulo K in $\mathrm{Fix}(f_b)$. Thus

$$\mathcal{E}(f) = \bigcup_b \operatorname{Im}(\mathcal{E}_K(f_b))$$

where b runs through the set of essential representatives of $\overline{f}$. Since the above sum is mutually disjoint,

$$N(f) = \#\mathcal{E}(f) = \sum_{j=1}^{s} \#\mathrm{Im}\,(\mathcal{E}_K(f_{b_j})).$$

Now $\#\mathcal{E}_K(f_b) = N_K(f_b)$ implies $N(f) \le N_K(f_{b_1}) + \cdots + N_K(f_{b_s})$ and the equality holds if and only if each map $\mathcal{R}_{i_b}$ is an injection for each b_j in the set of essential representatives of $\overline{f}$ (or equivalently for each b lying in an essential Nielsen class of $\overline{f}$).

(4.4.21) Theorem (Nielsen Number "Product" Formula). *The equality*

$$N(f) = N(f_{b_1}) + \cdots + N(f_{b_s})$$

holds if and only if the two following conditions are satisfied:

(4.4.21.1) $N_K(f_i) = N(f_i)$.

(4.4.21.2) *For every point $b \in \mathrm{Fix}(\overline{f})$ lying in an essential Nielsen class, the group $\mathrm{Fix}(\overline{f}_{\#}; b)$ acts trivially on $\mathcal{E}_K(f_b)$.*

Proof. Let us notice that in general

$$N(f) \le N_K(f_{b_1}) + \cdots + N_K(f_{b_s}) \le N(f_{b_1}) + \cdots + N(f_{b_s})$$

where the right-hand side inequality becomes the equality if and only if (4.4.21.1). Then we have the equality on the left if and only if (4.4.21.2) holds (Theorem (4.4.19)). □

(4.4.22) REMARK. The first assumption in Theorem (4.4.21) exactly means that each essential class in $\mathcal{E}_K(f_{b_i})$ contains exactly one ordinary Nielsen class of f_{b_i}. The second assumption is equivalent to: for each $b \in \mathrm{Fix}(\overline{f})$ lying in an essential Nielsen class the map $\mathcal{R}_{i_b}\colon \mathcal{E}_K(f_b) \to \mathcal{R}(f)$ is an injection.

(4.4.23) REMARK. The equality

$$N(f) = N(f_1) + \cdots + N(f_s)$$

holds if and only if the second condition of Theorem (4.4.21) is satisfied.

(4.4.24) COROLLARY. *The formula in Theorem* (4.4.21) *is satisfied if* $\pi_2(B) = 0$ *and* $\mathrm{Fix}(\overline{f}_\#; b) = 1$ *for every* $b \in \mathrm{Fix}(\overline{f})$.

PROOF. From the homotopy exact sequence of the fibration $p\colon E \to B$

$$\cdots \to \pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \cdots$$

$\pi_2(B) = 0$ implies $\pi_1(F) \to \pi_1(E)$ is mono, hence $K = \ker i_\#$ is trivial. This implies (4.4.21.1) in Theorem (4.4.21). On the other hand the trivial group $\mathrm{Fix}(\overline{f}_\#; b) = 1$ is acting trivially hence the second assumption of Theorem (4.4.21) is also satisfied. □

(4.4.25) COROLLATY. *Let* $p\colon E \to \mathbb{T}$ *be a fibration over the d-dimensional torus and let* $f\colon E \to E$ *be a fibre map. Then the formula* $N(f) = N_K(f_1) + \cdots + N_K(f_s)$ *holds.*

PROOF. Let $\overline{f}\colon \mathbb{T} \to \mathbb{T}$ be the induced map of base spaces. If $N(\overline{f}) = 0$, then $\overline{f}$ is homotopic to a map with no fixed points (see the proof of Theorem (4.3.14)). The homotopy lifts to a homotopy of f giving the map of total spaces without fixed points, hence $N(f) = 0$.

Now we assume that $N(\overline{f}) \neq 0$ and $b \in \mathbb{T}$ is a fixed point. We will show that the assumptions of Corollary (4.4.24) are satisfied. Since a torus is a $K(\mathbb{Z}^m, 1)$-space, $\pi_2(\mathbb{T}) = 0$. We denote by $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$ the matrix representing the induced homotopy homomorphism $f_\#\colon \pi_1(\mathbb{T}) \to \pi_1(\mathbb{T}) = \mathbb{Z}^d$. Suppose that $v \in \mathrm{Fix}(\overline{f}_\#; b) \subset \pi_1(\mathbb{T})$. Then $\overline{f}_\#(v) = v$ implies $0 = v - \overline{f}_\#(v) = (\mathrm{id} - \overline{f}_\#)v = (\mathrm{I} - A)v$. Since $\det(\mathrm{I} - A) = \pm N(\overline{f}) \neq 0$, $v = 0$ which proves that $\mathrm{Fix}(\overline{f}_\#; b)$ is trivial. □

The material given below precedes Theorem (4.4.28) that we will use to extend the Nielsen theory onto some classes of solvmanifolds, Section 6.2 of Chapter VI.

(4.4.26) PROPOSITION. *Let $p\colon E \to B$ be a Hurewicz fibration and let $f\colon E \to E$ be a fibre map. Let $\overline{u}$ be a loop in B based at a point $b \in \mathrm{Fix}(\overline{f})$ and satisfying $\overline{f}\overline{u} \sim \overline{u}$. If an admissible map $\tau_{\overline{u}}\colon E_b \to E_b$ is homotopic to the identity then the action of $T_{\overline{u}}$ on $\mathcal{R}_K(f_b)$ is trivial.*

PROOF. We may assume that $\tau_{\overline{u}} = \mathrm{id}_{E_b}$. We take as a reference pair (x_0, r_0) where $x_0 \in \mathrm{Fix}(f_b)$ and r_0 is the constant path. Then

$$T_{\overline{u}}^{(x_0,r_0)} : \mathcal{R}_K(f_0; x_0, r_0) \ni [a] \to [\tau_{\overline{u}}(a)] = [a] \in \mathcal{R}_K(f_0; x_0, r_1),$$

where r_1 is a loop in E_b homotopic in E to the loop $u^{-1} * fu$ and u is a loop based in x_0 satisfying $pu = \overline{u}$, $u^{-1} * a * u \sim \tau_{\overline{u}}(a)$ in E. It remains to notice that in the natural bijection of the sets of Reidemeister classes

$$\mathcal{R}_K(f_0; x_0, r_0) \ni [a] \to [u^{-1} * a * fu * r_1^{-1}] \in \mathcal{R}_K(f_0; x_0, r_1)$$

and that the following loops are homotopic in E:

$$u^{-1} * a * fu * r_1^{-1} \sim u^{-1} * a * fu * fu^{-1} * u \sim u^{-1} * a * u \sim \tau_{\overline{u}}(a) \sim a. \qquad \square$$

(4.4.27) LEMMA. *Let $p\colon E \to B$ be a locally trivial bundle with the trivial action of loops in B on the fibre. Then for any two fixed points $b, b' \in \mathrm{Fix}(\overline{f})$ the maps f_b and $f_{b'}$ have the same homotopy type. In particular the Nielsen numbers $N_K(f_b)$, $N_K(f_{b'})$ are equal.*

PROOF. Let $\overline{u}$ be a path joining the fixed points $b, b' \in \mathrm{Fix}(\overline{f})$. Then we have the homotopy commutative diagram (Lemma (4.4.10))

$$\begin{tikzcd} E_b \arrow[r, "f_b"] \arrow[d, "\tau_{\overline{u}}"'] & E_b \arrow[d, "\tau_{\overline{f}\overline{u}}"] \\ E_{b'} \arrow[r, "f_{b'}"'] & E_{b'} \end{tikzcd}$$

Since the loops act trivially, $\tau_{\overline{u}} \sim \tau_{\overline{f}\overline{u}}$ hence we get the homotopy commutative diagram

$$\begin{tikzcd} E_b \arrow[r, "f_b"] \arrow[d, "\tau_{\overline{u}}"'] & E_b \arrow[d, "\tau_{\overline{u}}"] \\ E_{b'} \arrow[r, "f_{b'}"'] & E_{b'} \end{tikzcd}$$

Since the admissible map $\tau_{\overline{u}}$ is a homotopy equivalence, f_b, $f_{b'}$ have the same homotopy type and the statement follows from Theorem (4.1.32) $\square$

(4.4.28) THEOREM. *If the hypothesis of Lemma* (4.4.27) *is satisfied, then for any fibre map* $f\colon E \to E$ *the Nielsen number product formula*

$$N(f) = N(\overline{f})N_K(f_b)$$

holds.

PROOF. Proposition (4.4.26) shows that the assumption (4.4.21.2) is satisfied. By Remark (4.4.23) the formula $N(f) = N_K(f_1) + \cdots + N_K(f_s)$ still holds. On the other hand by Lemma (4.4.27) all $N_K(f_i)$ are the same, hence $N(f) = N_K(f_b)N(\overline{f})$. □

4.5. Fixed point theory and obstructions

One may say that the aim of the Nielsen fixed point theory is to describe the Reidemeister set $\mathcal{R}(f)$ and the fixed point index $\mathrm{ind}(f;\mathbb{A})$ of each $\mathbb{A} \in \mathcal{R}(f)$. We may put all these data into the formal sum

$$\sum_{\mathbb{A}\in\mathcal{R}(f)} i_{\mathbb{A}}\mathbb{A} \in \mathbb{Z}[\mathcal{R}(f)],$$

where $\mathbb{Z}[\mathcal{R}(f)]$ denotes the free abelian group generated by the set of Reidemeister classes. It turns out that, in the case of a self-map of a manifold, such a sum arises as the obstruction to the deformation of $f\colon M \to M$ to a fixed point free map. This was shown in the paper of E. Fadell and S. Husseini [FaHu1] in 1981. This section is the presentation of the main result of this paper.

In fact we will define, for the sake of this section, another Reidemeister relation and we will denote the new Reidemeister set $\mathcal{R}'(f)$. Instead of the traditional action $\alpha \circ \omega = \alpha \cdot \omega \cdot f\alpha^{-1}$ we consider the action $\alpha \circ' \omega = f\alpha \cdot \omega \cdot \alpha^{-1}$. Of course both sets of Reidemeister classes are canonically isomorphic and even equal when the fundamental group is commutative. We could avoid this discrepancy by using the above definition from the beginning of this chapter, but the last would be not consistent with notation used in literature.

Let $f\colon M \to M$ be a self-map of a closed manifold. Let $F\colon M \to M \times M$ be given by the formula $F(x) = (x, f(x))$, i.e. F is the natural homeomorphism from M to the graph of f. It is easy to see that

$$\mathrm{Fix}(f) = \emptyset \;\Leftrightarrow\; F(M) \subset M \times M \setminus \Delta M.$$

Moreover, the next lemma says that f can be deformed to a fixed point free map if and only if F can be deformed into $M \times M \setminus \Delta M$. Then we convert the last question to the existence of a section of an appropriate Hurewicz fibration. Next we recall the necessary information about the obstruction theory and we compute the obstruction to the existence of this section. After several natural isomorphisms the obstruction will correspond to the above formal sum (Theorem (4.5.16)).

(4.5.1) LEMMA. *A self-map $f\colon M \to M$ of a closed manifold is homotopic to a fixed point free map if and only if the map $F\colon M \to M \times M$ $(F(x) = (x, f(x)))$ can be deformed into $M \times M \setminus \Delta M$.*

PROOF. ($\Rightarrow$) Let $f_t\colon M \to M$ be a homotopy from $f_0 = f$ to a fixed point free map f_1. Then $F_t(x) = (x, f_t(x))$ is the required deformation.

($\Leftarrow$) Let now $F_t(x) = (h_t(x), f_t(x))$ be a deformation of $F_0(x) = (x, f(x))$ to $F_1(x) \in M \times M \setminus \Delta M$. Then the map f_1 is homotopic to $f_0 = f$ but the inequality $f_1(x) \neq h_1(t)$ does not yet guarantee $\mathrm{Fix}(f_1) = \emptyset$. Since the manifold M is homogeneous, the projection $p_1\colon M \times M \setminus \Delta M \to M$ is a locally trivial fibre bundle. Now the commutative diagram

$$\begin{tikzcd}
M \times 1 \arrow[r, "F_1"] \arrow[d] & M \times M \setminus \Delta M \arrow[d, "p_1"] \\
M \times I \arrow[r, "h_t"'] & M
\end{tikzcd}$$

admits a lift $\widetilde{h}_t\colon M \times I \to M \times M \setminus \Delta M$. Then $p_2\widetilde{h}_t$ is a homotopy from the fixed point free map $p_2\widetilde{h}_0$ to f_1. $\square$

Thus the problem of removing fixed points reduces to the possibility of deforming the map $f\colon M \to M \times M$ into $M \times M \setminus \Delta M$. Let us consider the more general situation.

Let X be a topological space and $A \subset X$ be its closed subspace. We replace the inclusion $i\colon A \hookrightarrow X$ by a Hurewicz fibration as follows. Let $E = \{\omega\colon I \to X : \omega(0) \notin A\}$ and let $p\colon E \to X$ denote the projection $p(\omega) = \omega(1)$. Since M is uniform, p is a Hurewicz fibration. The diagram

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
X \setminus A \arrow[ur, "i"] \arrow[r] & X
\end{tikzcd}$$

where $i(x) = \overline{x}$ (constant path at x) is commutative and moreover, the map $j\colon E \to X \setminus A$ given by $j(\omega) = \omega(0)$ satisfies $ji = \mathrm{id}_{X\setminus A}$ and $ij \sim \mathrm{id}_E$ ($h_t\colon E \to E$, $h_t(\omega)(s) = \omega(ts)$ is the homotopy).

The fibre of the bundle E over the point $x_0 \in X$ is

$$F_{x_0} = \{\omega \in E : \omega(1) = x_0\} = \{\omega\colon (I, 0, 1) \to (X, X \setminus A, x_0)\}.$$

In particular if $x_0 \notin A$, then we have the homotopy group isomorphism

$$\pi_k(F_{x_0}, \overline{x}_0) = \pi_{k+1}(X, X \setminus A, x_0)$$

which can be described as follows. If $\phi\colon (I^k, \partial I^k) \to (F_{x_0}, \overline{x}_0)$ represents an element in $\pi_k(F_{x_0}, \overline{x}_0)$, then the map $\widetilde{\phi}\colon (I^{k+1}, (\partial I^k \times I) \cup (I^k \times 0), I^k \times 0) \to X, X \setminus A, x_0)$ given by $\widetilde{\phi}(x,t) = \phi(x)(t)$ where $(x,t) \in I^k \times I$ represents an element in $\pi_{k+1}(X, X \setminus A, x_0)$.

(4.5.2) Corollary. *If $(X, X \setminus A)$ is $(k+1)$-connected then fibre is k-connected.*

(4.5.3) Corollary. *Let M be a closed* PL*-manifold* $\dim M \geq 3$. *We put $X = M \times M$, $A = \Delta M$. Then $\pi_1(F_{x_0}) = \pi_2(M \times M, M \times M \setminus \Delta M) = 0$ hence the fibre is simply-connected. Moreover, $\pi_{m-1}(F_{x_0}) = \pi_m(M \times M, M \times M \setminus \Delta M)$ is the commutative group. In particular this group does not depend on the choice of the base point.*

(4.5.4) Lemma. *The map $f\colon Y \to X$ can be deformed into $X \setminus A$ if and only if there is a map $\widetilde{f}$ making the diagram*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutative.

Proof. If f_t is a homotopy satisfying $f_0 = f$, $f_1(Y) \subset X \setminus A$, then we define $\widetilde{f}(y)$ as the path $t \to f_t(y)$. □

Let $p^*\colon E^* \to Y$ denote the fibering induced from $p\colon E \to X$ by the map $f\colon Y \to X$, i.e. $E^* = \{(y,e) \in Y \times E : f(y) = p(e)\}$.

(4.5.5) Lemma. *The map $f\colon Y \to X$ admits a lift (from the previous lemma) $\widetilde{f}\colon Y \to E$ $(p\widetilde{f} = f)$ if and only if the fibering $p^*\colon E^* \to Y$ admits a section (i.e. the map $s\colon Y \to E^*$ satisfying $p^* s = \mathrm{id}_Y$).*

Proof. $(\Rightarrow)$ We define $s(y) = (y, \widetilde{f}(y))$.
$(\Leftarrow)$ We define $\widetilde{f}(y) = \rho_2(s(y))$ where $\rho_2\colon E^* \to E$ is given by $\rho_2(y,e) = e$. □

The two above lemmas imply:

(4.5.6) Corollary. *The map of topological spaces $f\colon Y \to X$ can be deformed into $X \setminus A$ if and only if the fibering $p^*\colon E^* \to Y$ has a section.*

4.5.1. Obstruction. Let us start with some general information on the obstruction theory. The obstruction is an element in a cohomology group. This element disappears if and only if the considered partial map (homotopy, cross section) can be extended onto the whole space. Here we will use the obstruction to extension of the section s from Lemma (4.5.5). Since the considered manifolds may be not simply-connected, we shall consider the cohomology with local coefficients [Wh].

Let $p\colon E \to B$ denote a Hurewicz fibration over a finite polyhedron B. Moreover, we assume that the fibre F is $(k-2)$ connected for a fixed $k \geq 3$. Then F is simply-connected and $\pi_{k-1}(F)$ is an abelian group. In particular $\pi_{k-1}(F)$ does not depend on the choice of the base point (cf. [Wh]). When does the bundle admit a cross-section $s\colon B \to E$?

We begin to construct a section putting arbitrary values on the vertices, hence on 0-skeleton $B^{(0)}$. Then the connectivity of the fibre allows us to extend the section on 1-skeleton and thanks to $k-2$ connectivity we may continue until we get the section on $B^{(k-1)}$. However the extension on k-simplices may be not possible in general.

We define a local system $\mathcal{F}$ on B putting $\mathcal{F}(b) = \pi_{k-1}(F_b)$. This is correct since by the above assumptions $\pi_{k-1}(F_b)$ does not depend on the base point in F, i.e. all $\pi_{k-1}(F_b, x)$ are canonically isomorphic. For a path $\omega\colon I \to B$ from b_0 to b_1 we define $\mathcal{F}(\omega)\colon \pi_{k-1}(F_{b_1}) \to \pi_{k-1}(F_{b_0})$ as the homotopy group homomorphism induced by an admissible map over the path ω. Any two admissible maps (over ω) are homotopic, hence $\mathcal{F}(\omega)$ is well defined.

We will show that a partial section s (on $(k-1)$-skeleton) defines a k-cochain in $C^*(B, \mathcal{F})$ (a simplicial cochain complex with coefficients in the local system $\mathcal{F}$, cf. [Sp]).

Let $\sigma\colon \Delta_k \to B$ be a simplicial simplex. Then the composition

$$p_2 s\sigma\colon \mathrm{bd}\Delta_k \xrightarrow{\sigma} \sigma(\mathrm{bd}\Delta_k) \xrightarrow{s} p^{-1}(\sigma(\Delta_k)) \simeq \sigma(\Delta_k) \times F_{\sigma(v_0)} \xrightarrow{p_2} F_{\sigma(v_0))}$$

defines an element of the group $\pi_{k-1}(F_{b_0})$. Thus we get a simplicial cochain $c^k(f) \in C^k(B; \mathcal{F})$.

The main theorem of the obstruction theory claims that under the above assumptions:

(4.5.7) THEOREM.

(4.5.7.1) *The cochain $c^k(f) \in C^k(B; \mathcal{F})$ is a cocycle.*

(4.5.7.2) *Using another partial section we get a cohomologous cochain so a cohomology element $o^k(f) \in H^k(B; \mathcal{F})$ is well defined.*

(4.5.7.3) *A section $s\colon B^{(k-1)} \to E$ can be extended onto $B^{(k)}$ if and only if $\langle c^k(f), \sigma\rangle = 0$ for each simplicial simplex $\sigma\colon \Delta_k \to B$.*

(4.5.7.4) *$o^k(f) = 0 \in H^k(B; \mathcal{F})$ (the cocycle $c^k(f)$ is the coboundary) if and only if the restriction of the section $s\colon B^{(k-2)} \to E$ admits an extension onto $B^{(k)}$.*

PROOF. See [Wh, Chapter V, Theorems 5.6, 5.1, 5.14]. $\square$

(4.5.8) COROLLARY. *The fibration $p: E \to B$ with $(k-1)$ connected fibre $(k \geq 3)$ admits a section over k-skeleton if and only if $o^k(f) = 0 \in H^k(B; \mathcal{F})$.*

PROOF. $(\Rightarrow)$ is evident.

$(\Leftarrow)$ Since the fibre is $(k-2)$-connected, there is a section $s: B^{(k-1)} \to E$. Then by Theorem (4.5.7.4) the restriction of s to $B^{(k-2)}$ admits an extension onto $B^{(k)}$ if and only if $o^k(f) = 0$. $\square$

We come back to the deformation of $f: M \to M$ to a fixed point free map (M is a closed PL-manifold of dimension ≥ 3). By Lemma (4.5.1) such deformation is possible if and only if the map $F = (\mathrm{id}, f): M \to M \times M$ can be deformed into $M \times M \setminus \Delta M$. Now we replace the inclusion $M \times M \setminus \Delta M \to M \times M$ by the Hurewicz fibration $p: E \to M \times M$ where $E = \{\omega: I \to M \times M : \omega(0) \notin \Delta M\}$ and $p(\omega) = \omega(1)$ and we ask about the existence of a cross section of this fibering (Corollary (4.5.6)).

The rest of this section is the description of the local system $\mathcal{F}(x, y)$ and the construction of the obstruction to a cross-section.

Since $\dim M \geq 3$, the homotopy group of the fibre $\pi_{k-1}(F) = \pi_k(M \times M, M \times M \setminus \Delta M)$ does not depend on the choice of the base point, hence the local system $\mathcal{F}(x, y) = \pi_{m-1}(F_{(x,y)})$ on $M \times M$ is defined.

(4.5.9) LEMMA. *If $\dim M \geq 3$, then the inclusion $M \ni x \to (x_1, x) \in M \times M$ induces the isomorphism $\pi_m(M, M \setminus x_1; x_2) \to \pi_m(M \times M, M \times M \setminus \Delta M; (x_1, x_2))$.*

On the other hand

(4.5.10) LEMMA. $\pi_m(M, M \setminus x_1; x_2) = \mathbb{Z}[\pi_1(M)]$.

PROOF. Let $p: \widetilde{M} \to M$ be the universal covering of M. Then we have the isomorphisms

$$\pi_m(M, M \setminus x_1; x_2)$$

{covering map induces the isomorphism of higher homotopy groups}

$$= \underbrace{\pi_m(\widetilde{M}, \widetilde{M} \setminus p^{-1}(x_1); \widetilde{x}_2)}_{\{\text{Hurewicz isomorphism}\}} = \underbrace{H_m(\widetilde{M}, \widetilde{M} \setminus p^{-1}(x_1))}_{\{\text{excision}\}}$$
$$= H_m(\oplus V_i, \oplus(V_i \setminus \widetilde{x}_i)) = \bigoplus_i H_m(V_i, V_i \setminus \widetilde{x}_i) = \mathbb{Z}[\pi_1(M)],$$

where $\widetilde{x}_i$ runs through the set $p^{-1}(x_1)$. $\square$

By the above two lemmas $\pi_m(M \times M, M \times M \setminus \Delta M; (x_1, x_2)) = \mathbb{Z}[\pi_1(M)]$.

Now we will describe the action of the group $\pi_{m-1}(F) = \pi_1(M \times M) = \pi_1(M \times M \setminus M)$ on $\pi_m(M \times M, M \times M \setminus \Delta M; (x_1, x_2)) = \mathbb{Z}[\pi_1(M)]$.

First we make some notation. We fix a tubular neighbourhood T of the diagonal: $\Delta M \subset T \subset M \times M$ and a Euclidean neighbourhood $\mathcal{W} \subset M$ so small that $\mathcal{W} \times \mathcal{W} \subset T$. We choose the base points $x_1 \neq x_2 \in \mathcal{W}$. The isomorphism

$$\pi_m(M, M \setminus x_1; x_2) = \mathbb{Z}[\pi_1(M; x_2)]$$

(from Lemma (4.5.10)) may be described as follows. We fix an imbedding $h'\colon D^m \to \mathcal{W} \subset M$ satisfying $h'(v_0) = x_2$, $x_1 \in h(\operatorname{int} D^m)$. Let $\theta'_1 \in \pi_m(M, M \setminus x_1; x_2)$ denote its homotopy class. Then we denote $\theta'_\alpha = \alpha \circ \theta'_1$ for $\alpha \in \pi_1(M; x_2)$. This yields the isomorphism

$$\theta'\colon \mathbb{Z}[\pi_1(M; x_2)] \to \pi_m(M, M \setminus x_1; x_2)$$

by Lemma (4.5.10). Finally the composition

$$\mathbb{Z}[\pi_1(M; x_2)] \xrightarrow{\theta'} \pi_m(M,\ M \setminus x_1; x_2) \xrightarrow{i_\#} \pi_m(M \times M, M \times M \setminus \Delta M; (x_1, x_2))$$

gives the isomorphism which we will denote by θ. We will write $\theta_\alpha = i_\# \theta'_\alpha$.

Now we will describe the action of the fundamental group $\pi_1(M \times M; (x_1, x_2)) = \pi_1(M \times M \setminus \Delta M; (x_1, x_2))$ on the fibre of the local system $\mathcal{F}_{(x_1,x_2)} = \pi_m(M \times M, M \times M \setminus \Delta M; (x_1, x_2))$. We will identify the groups $\pi_1(M; x_1) = \pi_1(M; x_2)$ by a path r joining these points in W.

$$\pi_1(M; x_1) \ni \alpha \to \alpha^{(r)} = r^{-1} * \alpha * r \in \pi_1(M; x_2).$$

(4.5.11) Lemma. *Let* $\sigma \in \pi_1(M; x_1)$. *Then*

$$(\sigma, \sigma^{(r)}) \circ \theta_1 = \operatorname{sgn}(\sigma)\theta_1 \in \pi_1(M \times M, M \times M \setminus \Delta M; (x_1, x_2))$$

where $\operatorname{sgn}(\sigma) = 1(-1)$ *if* σ *preserves* (*reverses*) *the local orientation of* M.

Proof. We may assume that the loops $\sigma, \sigma^{(r)}$ are close and satisfy

$$(\sigma(t), \sigma^{(r)}(t)) \in T \setminus \Delta M \quad (\text{for all } 0 \le t \le 1).$$

Let $h'_0\colon D^m \to \mathcal{W}$ be the imbedding representing θ'_1. Then we may shift $h'_0\colon D^m \to M$ along the path $\sigma^{(r)}$ to get a family of imbeddings $h'_t\colon D^m \to M$ satisfying $h'_t(v_0) = (\sigma^{(r)}(t), \sigma(t)) \in h'_t(\operatorname{int} D^m)$. We may choose these imbeddings so small that $h'_t(D^m) \times h'_t(D^m) \subset T$ for all t. We define $h_t(x) = (\sigma(t), h'_t(x))$. Now $h_t\colon (D^m, S^{m-1}; v_0) \to (T, T \setminus \Delta M; (\sigma(t), \sigma^{(r)}(t))$. In particular

$$h_0, h_1\colon (D^m, S^{m-1}, v_0) \to (M \times M, M \times M \setminus \Delta M; (x_1, x_2))$$

are represented by imbeddings, hence their homotopy classes are equal up to sign ± 1. More exactly $[h_1] = \operatorname{sgn}(\sigma)[h_0]$. $\square$

(4.5.12) LEMMA. *Let $\sigma,\tau\in\pi_1(M;x_1)$, $\alpha\in\pi_1(M;x_2)$. Then*

$$(\sigma,\tau^{(r)})\circ\theta_\alpha=\operatorname{sgn}(\sigma)\theta_{\tau^{(r)}\alpha(\sigma^{(r)})^{-1}}.$$

In other words identifying the groups $\pi_1(M;x_1)=\pi_1(M;x_2)$ as above (by a path in $\mathcal{W}$) we get the action of $\pi\times\pi$ on $\mathbb{Z}[\pi]$ by $(\sigma,\tau)\circ\alpha=(\operatorname{sgn}\sigma)\tau\alpha\sigma^{-1}$.

PROOF.

$$\begin{aligned}(\sigma,\tau^{(r)})\theta_\alpha&=(\sigma,\tau^{(r)})(1,\alpha)\theta_1=(\sigma,\tau^{(r)})(1,\alpha)(\sigma^{-1},(\sigma^{(r)})^{-1})(\sigma,\sigma^{(r)})\theta_1\\&=(\operatorname{sgn}\sigma)(1,\tau^{(r)}\alpha(\sigma^{(r)})^{-1})\theta_1=(\operatorname{sgn}\sigma)\theta_{\tau^{(r)}\alpha(\sigma^{(r)})^{-1}},\end{aligned}$$

where the third equality follows from the previous lemma. □

Now we are in a position to describe the cohomology group $H^m(M;\mathcal{F})$. By the above lemma the action of $\pi_1(M)$ on the fibre of the induced local system $\mathbb{Z}[\pi_1(M)]$ is given by

$$\sigma\circ\alpha=(\operatorname{sgn}\sigma)f\sigma*\alpha*\sigma^{-1}.$$

(4.5.13) REMARK. The above action suggests an alternative Reidemeister action of the set $\pi_1(X)$ on itself:

$$\sigma\circ'\alpha=(f\sigma)*\alpha*\sigma^{-1}$$

and we will call it *the second Reidemeister* action. The quotient set will be denoted $\mathcal{R}'(f)$. If $\circ$ denotes the Reideimeister action from Section 4.1 then

$$\sigma\circ'\alpha=(\sigma\circ\alpha^{-1})^{(-1)}.$$

In particular for $\pi_1(X)$ abelian, $\sigma\circ'\alpha=\sigma^{-1}\circ\alpha$.

We may represent $\mathbb{Z}[\pi_1(M)]=\bigoplus_{\mathbb{A}}\mathbb{Z}[\mathbb{A}]$ where the summation runs through the set of Reidemeister classes. The above formula shows that the homotopy elements $\sigma\circ'\alpha$ and α are Reidemeister related, hence the above action preserves each component of $\mathbb{Z}[\pi_1(M)]=\bigoplus_{\mathbb{A}}\mathbb{Z}[\mathbb{A}]$ which yields the splitting

$$H^m(M;\mathbb{Z}[\pi_1(M)])=\bigoplus_{\mathbb{A}}H^m(M;\mathbb{Z}[\mathbb{A}])$$

where $\mathbb{A}$ runs over the set $\mathcal{R}'(f)$.

(4.5.14) LEMMA. *$H^m(M;\mathbb{Z}[\mathbb{A}])=\mathbb{Z}$ for each class $\mathbb{A}$ of the second Reidemeister action.*

PROOF. We will use the Poincaré duality for the (co-)homology with the local coefficients. Let Orient_M be the local system over M with the fibre $\mathbb{Z}$ where the action of a loop ω is the identity if ω preserves the orientation of M and is multiplication by -1 if ω reverses the orientation. Then, by the Poincaré duality (see [Wh]),

$$H^k(M;\mathbb{Z}[\mathbb{A}]) = H_{m-k}(M;\mathbb{Z}[\mathbb{A}]\otimes \mathrm{Orient}_M).$$

Since the action of $\pi_1(M)$ on $f^*\mathcal{F}$ is given by $\sigma\circ\alpha = (\mathrm{sgn}\sigma) f\sigma * \alpha * \sigma^{-1}$, the action on the twisted system $f^*\mathcal{F}\otimes\mathrm{Orient}_M$ is given by $\sigma\circ\alpha = f\sigma*\alpha*\sigma^{-1}$, hence this is the permutation of the set of generators $\mathbb{A}$. Moreover the permutation is transitive. Thus by Poincaré duality and Theorem 3.2 from Chapter VI in [Wh],

$$H^m(M;\mathbb{Z}[\mathbb{A}]) = H_0(M;\mathbb{Z}[\mathbb{A}]\otimes\mathrm{Orient}_M) = \mathbb{Z}. \qquad \square$$

(4.5.15) COROLLARY. $H^m(M;f^*\mathcal{F}) = H^m(M;\mathbb{Z}[\pi_1(M)]) = \mathbb{Z}[\mathcal{R}'(f)]$.

4.5.2. The Reidemeister invariant as the obstruction.

(4.5.16) THEOREM. *Under the above isomorphism*

$$o_m(f) = (-1)^m \sum_{A\in\mathcal{R}'(f)} \mathrm{ind}(f;A)A \in H^m(M;f^*\mathcal{F}) = \mathbb{Z}[\mathcal{R}(f)].$$

PROOF. Since $o_m(f)$ is a homotopy invariant, we may assume that f is a PL-map with respect to a fixed triangulation $\mathcal{T}$ of M. We may assume that $\mathcal{T}$ is so fine that $\sigma\cap f\sigma\neq\emptyset$ implies $\sigma\times f\sigma\subset T$, where T denotes the fixed tubular neighbourhood of ΔM (for each simplex $\sigma\in\mathcal{T}$). Moreover, we may assume that $\mathrm{Fix}(f)$ is finite and each fixed point lies inside an m-simplex. We fix a simplex σ_0 such that $\sigma_0\times f\sigma_0\subset T$ and we choose as base points: $\sigma_0(v_0)$, $f\sigma_0(v_0)$, and we fix a path r joining these points. Let $C^* = C^*(M;f^*\mathcal{F})$ denote the complex of simplicial cochains. Let $c^m(f) = \sum_\sigma \lambda_\sigma\sigma \in C^* = C^*(M;f^*\mathcal{F})$ be the simplicial cochain representing the obstruction to the extension of the section onto the whole M (here σ runs over the set of all simplicial m-simplices and $\lambda_\sigma\in\mathcal{F}(\sigma(v_0),f\sigma(v_0))$). What is the contribution of a $\lambda_\sigma\sigma$ to $o^m(f) = H^m(M;f^*\mathcal{F}) = \mathbb{Z}[\mathcal{R}(f)]$?

By Lemma (4.5.17) if σ has no fixed points, then the contribution is zero and if σ contains a fixed point x, then the contribution equals $(-1)^m\mathrm{ind}(f;\sigma)[x]$ where $[x]$ denotes the class of the second Reidemeister action determined by the fixed point $x\in\sigma$. Finally

$$\begin{aligned} o(f) &= \sum_{x\in\mathrm{Fix}(f)} (-1)^m\mathrm{ind}(f;x)[x] = \sum_{\mathbb{A}\in\mathcal{N}(f)}\sum_{x\in\mathbb{A}} (-1)^m\mathrm{ind}(f;x)[x] \\ &= (-1)^m \sum_{\mathbb{A}\in\mathcal{N}(f)} \mathrm{ind}(f;\mathbb{A})\mathbb{A} = (-1)^m \sum_{\mathbb{A}\in\mathcal{R}'(f)} \mathrm{ind}(f;\mathbb{A})\mathbb{A} \in H^m(M;f^*\mathcal{F}) \\ &= \mathbb{Z}[\mathcal{R}'(f)]. \end{aligned} \qquad \square$$

It remains to prove the following

(4.5.17) LEMMA. *Let $f\colon M \to M$ be a self-map of a compact* PL *m-manifold. Assume that f is a* PL*-map with respect to a triangulation $\mathcal{T}$ of M. Assume that this triangulation is so fine that $\sigma \cap f\sigma \neq \emptyset$ implies $\sigma \times f\sigma \subset T$ (a fixed tubular neighbourhood of $\Delta M \subset M \times M$) for each simplex $\sigma \in \mathcal{T}$. Assume moreover that* Fix(f) *lies outside the $(m-1)$-dimensional skeleton of M and each m-simplex contains at most one fixed point.*

Let $\sigma \in \mathcal{T}$. If σ has no fixed points then its contribution to the obstruction $o^m(f) \in H^m(M; f^*\mathcal{F}) = \mathbb{Z}[\mathcal{R}(f)]$ is zero, and if σ contains a fixed point x then its contribution equals $(-1)^m \operatorname{ind}(f;\sigma)[x]$ where $[x]$ denotes the class of the second Reidemeister action determined by the fixed point $x \in \sigma$.

PROOF. Let us fix a basic simplex $\sigma_0 \in \mathcal{T}$. The m-dimensional cochain $c^m(f)$ representing the obstruction can be written as $c^m(f) = \sum_\sigma \lambda_\sigma \sigma$ $(\sigma \in \mathcal{T})$. We recall that $\lambda_\sigma \in f^*\mathcal{F}(\sigma(v_0)) = \pi_m(M \times M, M \times M \setminus \Delta M; (\sigma(v_0), f\sigma(v_0)))$ is represented by

$$(\mathrm{id}, f)\sigma\colon (\Delta_m, \operatorname{bd}\Delta_m; v_0) \to (M \times M, M \times M \setminus \Delta M; (\sigma(v_0), f\sigma(v_0))).$$

If $\sigma(x) \neq f\sigma(x)$ for all $x \in \Delta_m$, then

$$[(\mathrm{id}, f)\sigma] = [0] \in \pi_m(M \times M, M \times M \setminus \Delta M; (\sigma(v_0), f\sigma(v_0))),$$

hence the contribution of σ is zero.

Now we assume that the simplex σ containins a fixed point of f. To compare $[\lambda_\sigma]$ with the generators θ_α we fix a sequence of simplicial m-simplices $\sigma_0, \sigma_1, \ldots, \sigma_s = \sigma$ from the basic simplex σ_0 to the chosen simplex σ. We assume that σ_i, σ_{i+1} have a common $(m-1)$ face and moreover that their orientations are compatible. Consider the cocycle $\lambda_{\sigma_s}\sigma_s = \lambda_\sigma\sigma$. Then λ_{σ_s} is the homotopy class of the map

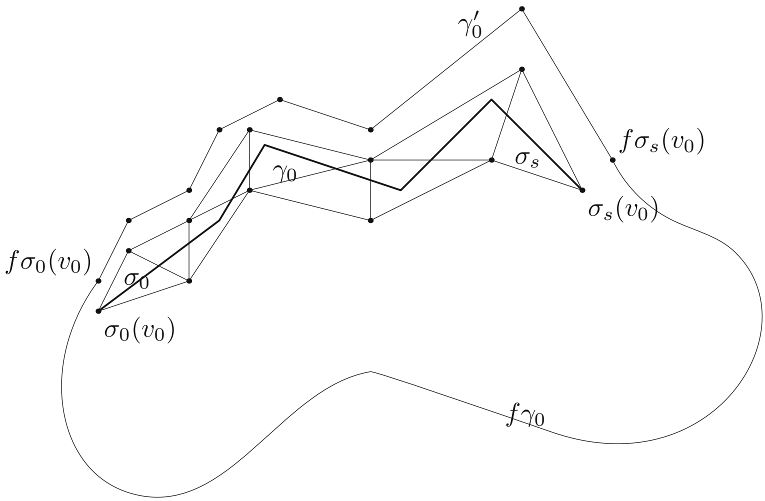
$$(\mathrm{id}, f)\sigma_s\colon (\Delta_m, \operatorname{bd}\Delta_m, v_0) \to (M \times M, M \times M \setminus \Delta M; (\sigma_s(v_0), f\sigma_s(v_0))).$$

Let us fix a Euclidean neighbourhood $\mathcal{W} \subset M$ satisfying $\sigma_s(\Delta_m) \cup f\sigma_s(\Delta_m) \subset \mathcal{W}$, $\mathcal{W} \times \mathcal{W} \subset T$ and giving the same local orientation as σ_s. We define the homotopy $h_t\colon \Delta_m \to \mathcal{W} \times \mathcal{W} = \mathbb{R}^m \times \mathbb{R}^m$,

$$h_t(x) = (\sigma_s(x) + t(\sigma_s(v_0) - \sigma_s(x)), f\sigma_s(x) + t(\sigma_s(v_0) - \sigma_s(x))).$$

Then $h_0(x) = (\sigma_s(x), f\sigma_s(x))$, $h_t(x) \notin \Delta M$ (for all $x \in \operatorname{bd}\Delta_m$) and $h_t(v_0) = (\sigma_s(v_0), f\sigma_s(x))$ does not depend on t. Thus λ_{σ_s} is represented by

$$h_1(x) = (\sigma_s(v_0), \sigma_s(v_0) + (f\sigma_s(x) - \sigma_s(x))).$$



Since the simplices σ_{s-1}, σ_s have a common $m-1$ face and give the same local orientation, $\lambda_s\sigma_s$ is cohomologous to $\lambda_{s-1}\sigma_{s-1}$, where

$$\lambda_{s-1} = (\gamma_{s-1}, f\gamma_{s-1}) \circ h_1 \in \pi_m(M \times M,\ M \times M \setminus \Delta M; (\sigma(v_0), f\sigma(v_0)))$$

and γ_{s-1} is a path from $\sigma_{s-1}(v_0)$ to $\sigma_s(v_0)$ in $\sigma_{s-1}(\Delta_m) \cup \sigma_s(\Delta_m)$. Continuing we obtain that $\lambda_s\sigma_s$ is cohomologous to $\lambda_0\sigma_0$ where $\lambda_0 = (\gamma_0, f\gamma_0) \circ h_1$ and γ_0 is a path from $\sigma_0(v_0)$ to $\sigma_s(v_0)$ along the simplices $\Delta_0, \dots, \Delta_s$. Let γ_0' be a path joining $f\sigma_0(v_0)$ with $f\sigma_s(v_0)$ in M and satisfying $(\gamma_0(t), \gamma_0'(t)) \in T \setminus \Delta M$ (for all $t \in I$). Then

$$\lambda_0 = (\gamma_0, f\gamma_0)h_1 = (\gamma_0, f\gamma_0) * (\gamma_0^{-1}, \gamma_0^{\prime -1}) * (\gamma_0, \gamma_0') \circ h_1 = (*).$$

Now we notice that

- The homotopy class $\lambda_{\sigma_s} = [h_1]$ equals $\deg(f\sigma_s(x) - \sigma_s(x))[h']$, where

  $$h' \colon (\Delta_m, \mathrm{bd}\Delta_m, v_0) \to (M \times M, M \times M \setminus \Delta M; (\sigma_s(v_0), f\sigma_s(v_0)))$$

  is given by $h'(x) = (\sigma_s(v_0), \overline{\sigma}(x))$ where $\overline{\sigma}$ is an imbedding of Δ_m giving the same local orientation as σ_s. But

$$\deg(f\sigma_s - \sigma_s) = (-1)^m \deg(\sigma_s - f\sigma_s) = (-1)^m \deg(\mathrm{id}_{\sigma_s} - f_{|\sigma_s}) = (-1)^m \mathrm{ind}(f; \sigma_s).$$

- Since $(\gamma_0, \gamma_0') \colon I \to T \subset M \times M$ is a path from (x_0, fx_0) to $(x_s; fx_s)$ and γ_0 is going through the compatibly oriented simplices $\sigma_0, \dots, \sigma_s$,

$$(\gamma_0, \gamma_0') \circ h_1 = (-1)^m \mathrm{ind}(f, \sigma_s)\theta_1 \in \pi_1((M \times M, M \times M \setminus \Delta M; (\sigma(v_0), f\sigma(v_0))).$$

Now

$$\lambda_0 = (*) = (-1)^m \mathrm{ind}(f;\sigma_s)(1, f\gamma_0 * (\gamma_0^{'-1}))\theta_1.$$

Thus the contribution of the simplex σ_s containing a fixed point equals

$$(-1)^m \mathrm{ind}(f;\sigma_s)\theta_{f\gamma_0 * \gamma_0^{'-1}}$$

where γ_0, γ_0', are paths from the point $\sigma_0(v_0)$ to $\sigma_s(v_0)$ and from $f\sigma_0(v_0)$ to $f\sigma_s(v_0)$ respectively such that $(\gamma_0(t), \gamma_0'(t)) \in T \setminus \Delta M$. It remains to notice that $f\gamma_0 * \gamma_0^{'-1}$ denotes the class of the points from σ_s with respect to the second Reidemeister action. $\square$

CHAPTER V

PERIODIC POINTS BY THE NIELSEN THEORY

5.1. Nielsen relation for periodic points

Now we ask about the least number of periodic points. We are given a self-map $f\colon X \to X$ and a natural number $n \in \mathbb{N}$. How large must be $\mathrm{Fix}(g^n)$ for any g homotopic to f? As it usually happens in the study of fixed points, the theory began with a result on self-maps of tori. In 1978 Benjamin Halpern [Ha1] proved a very nice theorem about the number of periodic points of a map $f\colon \mathbb{T}^d \to \mathbb{T}^d$. He noticed that f is homotopic to a map g whose *periodic point set* $P(g) = \bigcup_{m=1}^{\infty} \mathrm{Fix}(g^m)$ is finite if and only if the sequence of Nielsen numbers $\{N(f^m)\}_1^{\infty}$ is bounded. This result was generalized later in the Ph.D. thesis of Edward Keppelmann onto self-maps of nilmanifolds and some solvmanifolds [Kep]. Unfortunately the proof relies rather on special properties of these manifolds so it can not be generalized onto larger classes of spaces.

Here we will consider a similar problem. Given a self-map $f\colon X \to X$ and a number $m \in \mathbb{N}$, what is the lower bound of the cardinality of sets $P^m(g)$ and $P_n(g)$ where g runs through the set of all maps homotopic to f? Let us emphasis that we deform only the map f while we are investigating the fixed points of the iterate f^m. The first lower bounds of these cardinalities appeared in (unpublished) papers of Halpern [Ha3], [Ha4]. In [Ji4] Boju Jiang introduced general homotopy invariants which are estimates of the number of periodic points. These invariants were intensively explored in 1980s and 90s (see [HeYu], [HeKeI], [HeKeII], [HeKeIII], [HrKe]).

Since $\#\mathrm{Fix}(g^m) \geq N(f^m)$ for every map g homotopic to the given f, the number $N(f^m)$ is a lower bound of the cardinality of $P^m(f)$. But this estimation is not the best one.

(5.1.1) Example (Flip map). Let $f\colon S^1 \to S^1$ be given by $f(z) = \overline{z}$ (the complex conjugate) and let $n = 2$. Then $f^2 = \mathrm{id}$, hence $N(f^2) = N(\mathrm{id}) = 0$. But every map homotopic to f must have at least two fixed (hence also 2-periodic) points since $N(f) = |L(f)| = |1 - (-1)| = 2$ by Proposition (4.1.13).

5.1.1. Map action on the Reidemeister classes. The above example shows that we must not forget about the periodic points of periods smaller than n. But the sum $\sum_{k|n} N(f^k)$ may be too large as it is in the case of the constant map (since then the unique fixed point is counted in each period). Thus we must consider more carefully the relations among Reidemeister classes in $\mathcal{R}(f^k)$ for $k|n$.

Let $f: X \to X$ be a self-map of a compact ENR. For a fixed number $n \in \mathbb{N}$ we consider the iteration f^n. Then $\mathrm{Fix}(f) \subset \mathrm{Fix}(f^n)$. Let us notice that this inclusion preserves the Nielsen relation: if $x_0, x_1 \in \mathrm{Fix}(f)$ and the path ω satisfies $f\omega \sim \omega$, then also $f^n\omega \sim \omega$. Thus we get the map $\gamma: \mathcal{N}(f) \to \mathcal{N}(f^n)$. Notice that γ may be no longer an inclusion (as in the above example of the flip map).

Now we extend the map γ onto Reidemeister classes.

Let us fix a universal covering $p: \tilde{X} \to X$.

(5.1.2) Definition. We define $\gamma: \mathcal{R}(f) \to \mathcal{R}(f^n)$, putting $\gamma[\widetilde{f}] = [\widetilde{f^n}]$.

This definition is correct since $(\alpha\widetilde{f}\alpha^{-1})^n = \alpha\widetilde{f^n}\alpha^{-1}$. Now we show that the above map is really an extension of the map $\gamma: \mathcal{N}(f) \to \mathcal{N}(f^n)$.

(5.1.3) Lemma. *The diagram*

$$\begin{tikzcd} \mathcal{N}(f) \arrow[r, "\gamma"] \arrow[d, "j"'] & \mathcal{N}(f^n) \arrow[d, "j"] \\ \mathcal{R}(f) \arrow[r, "\gamma"'] & \mathcal{R}(f^n) \end{tikzcd}$$

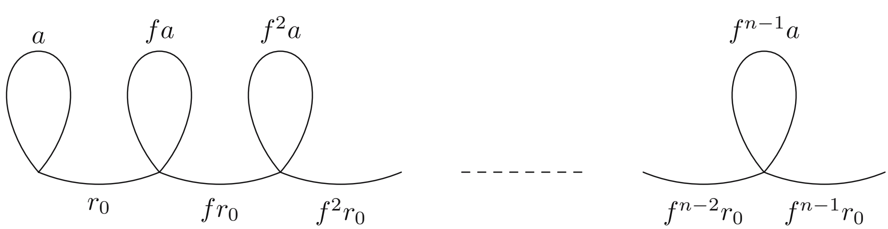
is commutative. Here the vertical arrows denote the canonical inclusion of the set of the Nielsen classes into the set of the Reidemeister classes.

Proof. Let us fix a Nielsen class $\mathbb{A} \in \mathcal{N}(f)$ and a point $x_0 \in \mathbb{A}$. Then $j(\mathbb{A}) = [\widetilde{f}]$ for a lift $\widetilde{f}$ satisfying $p(\mathrm{Fix}(\widetilde{f})) = \mathbb{A}$. Now $\gamma j(\mathbb{A}) = \gamma[\widetilde{f}] = [\widetilde{f^n}]$. On the other hand $\gamma(\mathbb{A}) = \mathbb{A}' \in \mathcal{N}(f^n)$ means $\mathbb{A} \subset \mathbb{A}'$, hence $j\gamma(\mathbb{A}) = j(\mathbb{A}')$. It remains to show that $j(\mathbb{A}') = [\widetilde{f^n}]$. But the Nielsen classes $\mathbb{A}', p(\mathrm{Fix}(\widetilde{f^n})) \in \mathcal{N}(f^n)$ contain x_0, hence are equal. Now $j(\mathbb{A}') = j(p(\mathrm{Fix}(\widetilde{f^n}))) = [\widetilde{f^n}]$. $\square$

Now we describe the map $\gamma: \mathcal{R}(f) \to \mathcal{R}(f^n)$ in "coordinates". We fix a reference pair (x_0, r_0) for the map $f: X \to X$. Then (x_0, r_1) where $r_1 = r_0 * fr_0 * \cdots * f^{n-1}r_0$ is a reference pair for f^n.

We define the map $\gamma^c: \pi_1(X; x_0) \to \pi_1(X; x_0)$ by

$$\gamma^c(a) = (a * r_0) * f(a * r_0) * \cdots * f^{n-1}(a * r_0) * (r_0 * fr_0 * \cdots * f^{n-1}r_0)^{-1}.$$



(5.1.4) LEMMA. *The above formula induces the map*

$$\gamma^c\colon \mathcal{R}(f;x_0,r_0)\to\mathcal{R}(f^n;x_0,r_1)$$

and the diagram

$$\begin{tikzcd}
\mathcal{R}(f;x_0,r_0) \arrow[r, "\gamma^c"] \arrow[d, "\phi"'] & \mathcal{R}(f^n;x_0,r_1) \arrow[d, "\phi"] \\
\mathcal{R}(f) \arrow[r, "\gamma"'] & \mathcal{R}(f^n)
\end{tikzcd}$$

where ϕ, a canonical isomorphism, is commutative.

PROOF. We prove that γ^c induces a map of Reidemeister sets. Let $a' = d*a*r_0*f(d^{-1})*r_0^{-1}$ (see Section 4.1.4). Then

$$\begin{aligned}
(a'*r_0)\ &*f(a'*r_0)*\cdots*f^{n-1}(a'*r_0)*(r_0*fr_0*\cdots*f^{n-1}r_0)^{-1}\\
&=(d*a*r_0*f(d^{-1}))*f(d*a*r_0*f(d^{-1}))*\cdots\\
&\quad *f^{n-1}(d*a*r_0*f(d^{-1}))*(r_0*fr_0*\cdots*f^{n-1}r_0)^{-1}\\
&=d*(a*r_0)*f(a*r_0)*\cdots*f^{n-1}(a*r_0)*f^n(d^{-1})\\
&\quad *(r_0*fr_0*\cdots*f^{n-1}r_0)^{-1}\\
&=d*[(a*r_0)*f(a*r_0)*\cdots*f^{n-1}(a*r_0)*(r_0*fr_0*\cdots*f^{n-1}r_0)^{-1}]\\
&\quad *((r_0*fr_0*\cdots*f^{n-1}r_0))*f^n(d^{-1})*(r_0*fr_0*\cdots*f^{n-1}r_0)^{-1}\\
&=d*[(a*r_0)*f(a*r_0)*\cdots*f^{n-1}(a*r_0)*(r_0*fr_0*\cdots*f^{n-1}r_0)^{-1}]\\
&\quad *r_1*f^nd^{-1}*r_1^{-1}.
\end{aligned}$$

To prove the commutativity of the diagram we fix a point $\widetilde{x}_0\in p^{-1}(x_0)$. Then $\phi[a]=[\widetilde{f}]$ where $\widetilde{f}\in\mathrm{lift}(f)$ satisfies

$$\widetilde{f}(\widetilde{x}_0) = \text{the end of the lift of the path } a*r_0 \text{ starting from } \widetilde{x}_0$$

(see Subsection 4.1.3 of Chapter IV). Thus $\gamma\phi[a]=\gamma[\widetilde{f}]=[\widetilde{f}^n]\in\mathcal{R}(f^n)$. On the other hand we denote

$$a_1=(a*r_0)*f(a*r_0)*\cdots*f^{n-1}(a*r_0)*(r_0*fr_0*\cdots*f^{n-1}r_0)^{-1}.$$

Notice that then

$$\widetilde{f}^n(\widetilde{x}_0) = \text{the end of the lift of the path } a_1*r_1 \text{ starting from } \widetilde{x}_0.$$

Since $\gamma^c[a]=[a_1]$, $\phi_n\gamma^c[a]=\phi[a_1]=[\widetilde{g}]$ where $\widetilde{g}\in\mathrm{lift}(f^n)$ is determined by the property

$$\widetilde{g}(\widetilde{x}_0) = \text{the end of the lift of the path } a_1*r_1 \text{ starting from the point } \widetilde{x}_0.$$

By the above $[\widetilde{g}] = [\widetilde{f^n}]$, hence $\phi\gamma^c[a] = \gamma\phi[a]$ which proves the commutativity of the diagram. $\square$

(5.1.5) REMARK. If x_0 is the fixed point and r_0 the constant path, then the map $\gamma^c\colon \mathcal{R}(f; x_0, r_0) \to \mathcal{R}(f^n; x_0, r_0)$ is given by $\gamma^c[a] = [a * fa * \cdots * f^{n-1}a]$. $\square$

Since $f^n = (f^m)^{n/m}$ for $m|n$, the above lemma gives the map $\gamma_{nm}\colon \mathcal{R}(f^m) \to \mathcal{R}(f^n)$ such that the diagram

$$\begin{tikzcd} \mathcal{N}(f^m) \arrow[r, "\gamma"] \arrow[d, "j"'] & \mathcal{N}(f^n) \arrow[d, "j"] \\ \mathcal{R}(f^m) \arrow[r, "\gamma_{nm}"'] & \mathcal{R}(f^n) \end{tikzcd}$$

is commutative, i.e. γ_{nm} sends the Nielsen class $\mathbb{A} \in \mathcal{N}(f^m)$ into the unique class $\mathbb{A}' \in \mathcal{N}(f^n)$ containing $\mathbb{A}$.

We will call γ_{nm} a *boosting* function. We notice that

(1) $\gamma_{nm}\gamma_{mk} = \gamma_{nk}$ for $k|m|n$,
(2) $\gamma_{mm} = \mathrm{id}$.

(5.1.6) THEOREM. *Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be a self-map of the torus such that the induced homomorphism $f_\#$ of the fundamental group $\pi_1(\mathbb{T}^d) = \mathbb{Z}^d$ is given by the matrix $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$. Then the map $\gamma\colon \mathcal{R}(f) \to \mathcal{R}(f^n)$ is given by*

$$\mathcal{R}(f) = \mathbb{Z}^d/(\mathrm{I}-A)(\mathbb{Z}^d) \ni \alpha \mapsto (\mathrm{I}+A+A^2+\cdots+A^{n-1})\alpha \in \mathbb{Z}^d/(\mathrm{I}-A^n)(\mathbb{Z}^d) = \mathcal{R}(f^n)$$

and is a homomorphism.

PROOF. Since any homotopy class of a self-map of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is represented by a linear map, we may assume that the point $x_0 = [0] \in \mathbb{T}^d$ is the fixed point and we may take as the reference pair $(x_0, \text{constant})$. Then by Proposition (4.1.16) we have the isomorphism $\mathcal{R}(f) = \mathbb{Z}^d/(\mathrm{I}-A)(\mathbb{Z}^d)$ and then Remark (5.1.5) gives the formula. $\square$

(5.1.7) LEMMA. *Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ satisfy $L(f^n) \neq 0$. Then for each $k|n$ the map $\gamma_{nk}\colon \mathcal{R}(f^k) \to \mathcal{R}(f^n)$ is mono.*

PROOF. Let the induced homotopy homomorphism $f_\#\colon \pi_1(\mathbb{T}^d) \to \pi_1(\mathbb{T}^d)$ be given by the matrix $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$. Then $\mathcal{R}(f^n) = \mathbb{Z}^d/(\mathrm{I}-A^n)\mathbb{Z}^d$ and the map $\gamma_{nk}\colon \mathcal{R}(f^k) \to \mathcal{R}(f^n)$ is given by $\gamma_{nk}[v] = [(\mathrm{I}+A^k+A^{2k}+\cdots+A^{n-k})v]$ for $v \in \mathbb{Z}^d$.

Let us assume that $\gamma_{nk}[v] = [0] \in \mathbb{Z}^d/(\mathrm{I}-A^n)\mathbb{Z}^d$. This means that $(\mathrm{I} + A^k + A^{2k} + \cdots + A^{n-k})v = (\mathrm{I}-A^n)w$ for a $w \in \mathbb{Z}^d$. But $(\mathrm{I}-A^n)w = (\mathrm{I} + A^k + A^{2k} + \cdots + A^{n-k})(\mathrm{I}-A^k)w$ which gives $(\mathrm{I} + A^k + A^{2k} + \cdots + A^{n-k})v = (\mathrm{I}+A^k+A^{2k}+\cdots+A^{n-k})(\mathrm{I}-A^k)w$. Since $L(f^n) \neq 0$, $0 \neq \det(\mathrm{I}-A^n) = \det(\mathrm{I}+A^k+A^{2k}+\cdots+A^{n-k})\det(\mathrm{I}-A^k)$. Now the matrix $\mathrm{I}+A^k+A^{2k}+\cdots+A^{n-k}$ is invertible and the earlier equality implies $v = (\mathrm{I}-A^k)w$ and $[v] = [0] \in \mathbb{Z}^d/(\mathrm{I}-A^n)\mathbb{Z}^d$. $\square$

(5.1.8) DEFINITION. We define the *depth* of a Reidemeister class $\mathbb{A} \in \mathcal{R}(f^n)$ as the smallest divisor k of n satisfying $\mathbb{A} \in \operatorname{Im} \gamma_{nk}$.

5.1.2. Orbits of Reidemeister classes. Let us notice that the restriction of f gives the homeomorphism $f_|\colon \operatorname{Fix}(f^n) \to \operatorname{Fix}(f^n)$ satisfying ${f_|}^n = \mathrm{id}$. In other words we get the action of the group $\mathbb{Z}_n$ on $\operatorname{Fix}(f^n)$. Moreover, this map preserves the Nielsen relation: if ω joins the points $x, y \in \operatorname{Fix}(f^n)$ and $f^n\omega \sim \omega$ then $f(\omega)$ joins the points $fx, fy \in \operatorname{Fix}(f^n)$ and $f^n(f\omega) \sim f(\omega)$. This yields the map $\mathcal{N}_f\colon \mathcal{N}(f^n) \to \mathcal{N}(f^n)$ satisfying $(\mathcal{N}_f)^n = \mathrm{id}$. Thus we get the action of $\mathbb{Z}_n$ on the set of Nielsen classes $\mathcal{N}(f^n)$. Now we are going to extend the map $\mathcal{N}_f\colon \mathcal{N}(f^n) \to \mathcal{N}(f^n)$ onto the set of Reidemeister classes. We start with a lemma about the set $\operatorname{lift}(f^n)$.

(5.1.9) LEMMA ([Ji4]). *Let $\widetilde{f}_1, \dots, \widetilde{f}_n$ be lifts of the given map f. Then the composition $\widetilde{f}_1, \dots, \widetilde{f}_n$ is a lift of f^n. Conversely each lift of f^n is of the above form.*

PROOF. The first part is obvious. Now for any lift $\widetilde{h}^{(n)} \in \operatorname{lift}(f^n)$ there exists a deck transformation $\alpha \in \mathcal{O}_X$ satisfying $\widetilde{h}^{(n)} = (\alpha\widetilde{f}_1)\widetilde{f}_2 \dots \widetilde{f}_n$. □

The commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f^n"] \arrow[d, "f"'] & X \arrow[d, "f"] \\
X \arrow[r, "f^n"'] & X
\end{tikzcd}$$

defines the map $\mathcal{R}_f\colon \mathcal{R}(f^n) \to \mathcal{R}(f^n)$. See Theorem (4.1.25).

(5.1.10) LEMMA ([Ji4]).
(5.1.10.1) $\mathcal{R}_f[\widetilde{f}_1, \dots, \widetilde{f}_n] = [\widetilde{f}_n\widetilde{f}_1, \dots, \widetilde{f}_{n-1}]$,
(5.1.10.2) $(\mathcal{R}_f)^n = \mathrm{id}$,
(5.1.10.3) $f(p(\operatorname{Fix}(\widetilde{f}_1 \dots \widetilde{f}_n)) = p(\operatorname{Fix}(\widetilde{f}_n\widetilde{f}_1 \dots \widetilde{f}_{n-1}))$,
(5.1.10.4) $\operatorname{ind}(f^n; p(\operatorname{Fix}(\widetilde{f}_1 \dots \widetilde{f}_n) = \operatorname{ind}(f^n; p(\operatorname{Fix}(\widetilde{f}_n\widetilde{f}_1 \dots \widetilde{f}_{n-1}))$.

PROOF. (5.1.10.1) follows from the commutativity of the diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}_1 \dots \widetilde{f}_n"] \arrow[d, "\widetilde{f}_n"'] & \widetilde{X} \arrow[d, "\widetilde{f}_n"] \\
\widetilde{X} \arrow[r, "\widetilde{f}_n\widetilde{f}_1 \dots \widetilde{f}_{n-1}"'] & \widetilde{X}
\end{tikzcd}$$

(5.1.10.2) We apply n-times (5.1.10.1).

(5.1.10.3) Notice that $\widetilde{f}_n(\mathrm{Fix}(\widetilde{f}_1 \dots \widetilde{f}_n)) = \mathrm{Fix}(\widetilde{f}_n \widetilde{f}_1 \dots \widetilde{f}_{n-1})$. Then we apply p onto both sides.

(5.1.10.4) Let us denote $\mathbb{A} = p(\mathrm{Fix}(\widetilde{f}_1 \dots \widetilde{f}_n)$ and $B = p(\mathrm{Fix}(\widetilde{f}_n \widetilde{f}_1 \dots \widetilde{f}_{n-1})$. Then by the above $f(\mathbb{A}) = \mathbb{B}$ and $f^{n-1}\mathbb{B} = \mathbb{A}$. The commutativity of the fixed point index implies $\mathrm{ind}(f^n; \mathbb{A}) = \mathrm{ind}(f^n; \mathbb{B})$. □

By the above lemma the map $\mathcal{R}_f \colon \mathcal{R}(f^n) \to \mathcal{R}(f^n)$ defines the action of the group $\mathbb{Z}_n$ on $\mathcal{R}(f^n)$. Moreover, the restriction of this action coincides with the earlier action of $\mathbb{Z}_n$ on the set of Nielsen classes.

Now we will describe this action in "coordinates".

We consider a self-map $f \colon X \to X$. We fix a reference pair (x_0, r_0) of f^n (i.e. $r_0(0) = x_0$, $r_0(1) = f^n(x_0)$). Then (fx_0, fr_0) is also a reference pair of f^n.

(5.1.11) Lemma. *The map $\pi_1(X, x_0) \ni a \to fa \in \pi_1(X, fx_0)$ defines a map $\mathcal{R}^c_f \colon \mathcal{R}(f^n; x_0, r_0) \to \mathcal{R}(f^n; fx_0, fr_0)$ such that the diagram*

$$\begin{tikzcd}
\mathcal{R}(f^n; x_0, r_0) \arrow[r, "\mathcal{R}^c_f"] \arrow[d, "\phi"'] & \mathcal{R}(f^n; fx_0, fr_0) \arrow[d, "\phi"] \\
\mathcal{R}(f^n) \arrow[r, "\mathcal{R}_f"'] & \mathcal{R}(f^n)
\end{tikzcd}$$

commutes. Here vertical lines mean the canonical bijections.

Proof. We check that the given map preserves the Reidemeister relation:

$$f(d * a * r_0 * f^n(d^{-1}) * r_0^{-1}) = fd * fa * (fr_0 * f^n(f(d^{-1})) * fr_0^{-1}).$$

Thus $\mathcal{R}^c_f$ is well defined. Now we show the commutativity of the diagram. We fix a point $\widetilde{x}_0 \in p^{-1}(x_0)$ and a lift $\widetilde{f} \colon \widetilde{X} \to \widetilde{X}$. We have $\mathcal{R}^c_f[a] = [fa] \in \mathcal{R}(f^n; fx_0, fr_0)$ hence $\phi\mathcal{R}^c_f[a] = \phi[fa] = [\widetilde{h}^{(n)}]$ where

$$\begin{aligned}
\widetilde{h}^{(n)}(\widetilde{f}\widetilde{x}_0) &= \text{the end of the lift of the path } fa * fr_0 \text{ starting from } \widetilde{f}\widetilde{x}_0 \\
&= f(\text{the end of the lift of the path } a * r_0 \text{ starting from } \widetilde{x}_0).
\end{aligned}$$

On the other hand $\phi[a] = [\widetilde{f}^{(n)}]$ where

$$\widetilde{f}^{(n)}(\widetilde{x}_0) = \text{the end of the lift of the path } a * r_0 \text{ starting from } \widetilde{x}_0.$$

Thus $\widetilde{h}^{(n)}(\widetilde{f}(\widetilde{x}_0)) = \widetilde{f}\widetilde{f}^{(n)}(\widetilde{x}_0)$ which implies the commutativity of the diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}^{(n)}"] \arrow[d, "\widetilde{f}"'] & \widetilde{X} \arrow[d, "\widetilde{f}"] \\
\widetilde{X} \arrow[r, "\widetilde{h}^{(n)}"'] & \widetilde{X}
\end{tikzcd}$$

which implies $\mathcal{R}_f[\widetilde{f}^{(n)}] = [\widetilde{h}^{(n)}]$. Finally we get

$$\mathcal{R}_f \phi[a] = \mathcal{R}_f[\widetilde{f}^{(n)}] = [\widetilde{h}^{(n)}] = \phi \mathcal{R}_f^c \phi[a]. \qquad \square$$

(5.1.12) REMARK. Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be a self-map of the torus with the fixed point $x_0 = 0$. Let $A \in \mathcal{M}_{d\times d}$ be the matrix representing the induced homotopy homomorphism. Let r_0 be the constant path. Then the map $\mathcal{R}_f^c\colon \mathcal{R}(f^n; x_0, r_0) \to \mathcal{R}(f^n; x_0, r_0)$ is given by

$$\mathcal{R}(f^n; x_0, r_0) = \mathbb{Z}^d/(\mathrm{I} - A^n)\mathbb{Z}^d \ni [\alpha] \mapsto [A\alpha] \in \mathbb{Z}^d/(\mathrm{I} - A^n)\mathbb{Z}^d = \mathcal{R}(f^n; x_0, r_0).$$

Now we consider the quotient set of $\mathcal{R}(f^n)$ under the action of the group $\mathbb{Z}_n$ defined in the previous section. We denote this quotient set by $\mathcal{OR}(f^n)$ and we call its elements *orbits of Reidemeister classes*. Let us notice that $\gamma_{nk}\colon \mathcal{R}(f^k) \to \mathcal{R}(f^n)$ sends an orbit into an orbit, hence we get the map $\gamma_{nk}\colon \mathcal{OR}(f^k) \to \mathcal{OR}(f^n)$. We may extend the notion of the depth onto the orbits of Reidemeister classes putting, for an $\mathcal{A} \in \mathcal{OR}(f^n)$,

$$\mathrm{d}(\mathcal{A}) = \text{the least divisor } k \text{ of } n \text{ satisfying } \mathcal{A} \in \mathrm{Im}\gamma_{nk}.$$

The Reidemeister class $\mathbb{A} \in \mathcal{R}(f^n)$ is called reducible if there is a $k|n$, $k < n$ and $\mathbb{B} \in \mathcal{R}(f^k)$ satisfying $\gamma_{nk}(\mathbb{B}) = \mathbb{A}$. By Remark (5.1.5) if $\mathbb{A}$, $\mathbb{B}$ are Nielsen classes then $\gamma_{nk}(\mathbb{A}) = \mathbb{B}$ means exactly $\mathbb{B} \subset \mathbb{A}$.

The orbit of Reidemeister classes is called *reducible* if it contains a reducible Reidemeister class. Otherwise the orbit is called *irreducible.*

We notice that the orbit $\mathcal{A} \in \mathcal{OR}(f^n)$ is irreducible if and only if $\mathrm{d}(\mathcal{A}) = n$.

By the Commutativity Property any two Reidemeister classes $\mathbb{A}$, $\mathbb{B}$ in the same orbit have the same index $\mathrm{ind}(f^n; \mathbb{A}) = \mathrm{ind}(f^n; \mathbb{B})$. We call an orbit $\mathcal{A} \in \mathcal{OR}(f^n)$ *essential* if a (hence any) Reidemeister class in $\mathcal{A}$ is essential.

(5.1.13) LEMMA. *If the orbit $\mathcal{A} \in \mathcal{OR}(f^n)$ is essential and irreducible, then it contains at least n periodic points.*

PROOF. Since the orbit is essential, all its classes are nonempty. Let us fix a point $a \in \mathbb{A} \in \mathcal{A}$. Then the length of the orbit of points $a, fa, \ldots$ must be n. In fact $f^n(a) = a$ since $a \in \mathrm{Fix}(f^n)$ and on the other hand $f^k a = a$ for a $k < n$ would imply $a \in \mathrm{Fix}(f^k)$ and the Nielsen class $\mathbb{B} \subset \mathrm{Fix}(f^k)$ containing a would satisfy $\gamma_{nk}(\mathbb{B}) = \mathbb{A}$, contradicting the irreducibility of $\mathcal{A}$. $\square$

5.1.3. The points of pure period n. Recall that $P_n(f) = \{x \in X : f^n(x) = x$ but $f^k(x) \neq x$ for$k|n,\ k < n\}$ denotes the set of points of the pure period n, i.e. the minimal period of x is n. We are going to define a homotopy invariant that is the lower bound of the cardinality of $P_n(f)$. We define

$$\begin{aligned} IEC_n(f) :=& \text{number of irreducible essential Reidemeister classes in } \mathcal{R}(f^n), \\ IEO_n(f) :=& (\text{number of irreducible essential orbits of Reidemeister classes} \\ & \text{in } \mathcal{OR}(f^n)) \times n. \end{aligned}$$

The factor n is explained in Lemma (5.1.13).

(5.1.14) Definition. We denote the above described number $IEO_n(f)$ by $NP_n(f)$ and call it the *prime Nielsen–Jiang periodic number.*

(5.1.15) Theorem. *$NP_n(f)$ is the homotopy invariant satisfying $NP_n(f) \leq \#P_n(f)$.*

Proof. The first part is evident since $IEO_n(f)$ is defined by the homotopy invariants. The inequality follows from Lemma (5.1.13). □

5.1.4. The estimation of $\#\mathrm{Fix}(f^n)$. We started the study of periodic points by presenting some naive bounds of the number $\#\mathrm{Fix}(f^n)$. Now we are going to define another Nielsen type number of periodic points introduced by Boju Jiang in 1983 in [Ji4].

Let us consider the disjoint sum $\bigcup_{k|n} \mathcal{OR}(f^k)$. A subset $\mathfrak{S} \subset \bigcup_{k|n} \mathcal{OR}(f^k)$ is called a *preceding system* if every essential orbit $\mathcal{A}$ in $\bigcup_{k|n} \mathcal{OR}(f^k)$ is preceded by an element from $\mathfrak{S}$. The preceding system $\mathfrak{S}$ is called a *minimal preceding system* (MPS for the shorthand) if the number $\sum_{\mathcal{A}\in\mathfrak{S}} \mathrm{d}(\mathcal{A})$ is minimal.

(5.1.16) Definition. We denote the cardinality of the above defined minimal system MPS by $NF_n(f)$, and call it the *full Nielsen–Jiang periodic periodic number.*

(5.1.17) Remark. Each preceding system contains all essential irreducible orbits of Reidemeister classes.

Proof. Let $\mathfrak{S}$ be a preceding system and let $\mathcal{A}$ be an irreducible essential orbit of Reidemeister classes. Since $\mathcal{A}$ is irreducible, it must be preceded by an orbit from $\mathfrak{S}$. But $\mathcal{A}$ as irreducible is preceded only but itself. Thus $\mathcal{A} \in \mathfrak{S}$. □

(5.1.18) Theorem. *$NF_n(f)$ is the homotopy invariant and a lower bound of the number of n-periodic points:*

$$NF_n(f) \leq \#\mathrm{Fix}(f^n).$$

Proof. The homotopy invariance is obvious. We prove the inequality. If $\mathrm{Fix}(f^n)$ is infinite then the inequality is evident. Assume that $\mathrm{Fix}(f^n)$ is finite and let us split it into the orbits of points

$$\mathrm{Fix}(f^n) = \{x_1^1, \dots, x_{s_1}^1; \dots, x_1^l, \dots, x_{s_l}^l\}.$$

The orbit $\{x_1^i, \dots, x_{s_i}^i\}$ determines an orbit of Nielsen (hence also Reidemeister) classes $\mathcal{A}_i \in \mathcal{OR}(f^{s_i})$. Then $\mathrm{d}(\mathcal{A}_i) \le s_i$ and we let $\mathcal{B}_i \in \mathcal{OR}(f^{\mathrm{d}(\mathcal{A}_i)})$ denote the orbit preceding $\mathcal{A}_i$. Then

$$\sum_{i=1}^{l} \mathrm{d}(\mathcal{B}_i) = \sum_{i=1}^{l} \mathrm{d}(\mathcal{A}_i) \le \sum_{i=1}^{l} s_i = \#\mathrm{Fix}(f^n).$$

It remains to show that the orbits $\mathcal{B}_1, \dots, \mathcal{B}_l$ form a Preceding System. In fact if $\mathcal{C}$ (f^k) is essential then it contains an orbit $\{x_1^i, \dots, x_{s_i}^i\}$ hence $\mathcal{B}_i$ precedes $\mathcal{C}$. $\square$

The next example shows that we may not confine the definition of MPS to the essential orbit only.

(5.1.19) Example. Let $f\colon S^{2n} \to S^{2n}$ be the antipodal map $f(x) = -x$. Since S^{2n} is simply-connected, $\mathcal{R}(f^k)$ consists of one class for each $k \in \mathbb{N}$. In particular the unique orbit in $\mathcal{R}(f^2)$ reduces to the only class in $\mathcal{R}(f^1)$ But the last one is inessential since $\mathrm{Fix}(f) = \emptyset$. Thus there is no essential irreducible orbit. Nevertheless $\mathrm{Fix}(g^2) \ne \emptyset$ for each $g \sim f$, since $g^2 \sim f^2 = \mathrm{id}$ and $L(\mathrm{id}) = \chi_{S^{2n}} = 2$.

Let us notice that in the above example $NF_2(f) = 1$, since the (unique) essential class in $\mathcal{R}(f^1)$ is the (unique) MPS.

Let us recall that

$$NP_k(f) = (\text{the number of essential irreducible orbits in } \mathcal{OR}(f^k)) \times n.$$

Since any Preceding System contains all essential irreducible orbits, we have

$$\sum_{k|n} NP_k(f) \le NF_n(f). \tag{5.1.20}$$

The above example shows that this inequality may be sharp.

Fortunately in many situations the equality $\sum_{k|n} NP_k(f) = NF_n(f)$ holds, which allowed us to make some computations of this invariant. Such computations were done for self-maps of tori nilmanifolds and some solvmanifolds in the papers of P. Heath and E. Keppelmann [HeKeI]–[HeKeIII].

5.1.5. Toroidal spaces. To present the mentioned results of Heath and Keppelmann we have to restrict ourselves to a class of spaces with some properties similar to those of the torus.

(5.1.21) Definition. We say that the map $f\colon X \to X$ is called *essentially reducible* provided that if an essential Nielsen $\mathbb{A} \in \mathcal{R}(f^n)$ and if $\mathbb{A}$ reduces to some class $\mathbb{B} \in \mathcal{R}(f^k)$ then $\mathbb{B}$ is also essential. If each self-map of X is essentially reducible, then X is called *essentially reducible.*

(5.1.22) Lemma. *If the map $f\colon X \to X$ is essentially reducible, then it has a unique* MPS. *This system consists of all irreducible essential orbits. In particular*

$$NF_n(f) = \sum_{k|n} NP_k(f).$$

Proof. By Remark (5.1.17) each MPS contains all irreducible essential orbits. Suppose that the considered MPS contains also a reducible orbit $\mathbb{A} = \gamma_{kl}(\mathbb{B})$, $l < k$. Then we may exchange in MPS the orbit $\mathbb{A} \in \mathcal{R}(f^k)$ with $\mathbb{B} \in \mathcal{R}(f^l)$. But $\mathrm{d}(\mathbb{B}) \leq k < n$ so we get an MPS with the smaller $\sum_{\mathbb{A}\in \mathrm{MPS}} \mathrm{d}(\mathbb{A})$ which contradicts the minimality of the given MPS. Now suppose that MPS contains an inessential orbit. Then by the essential reducibility it does not precede any essential orbit, hence it can be removed from MPS. The last also contradicts the minimality of the MPS. Now the formula follows since the depth of any irreducible orbit in $\mathcal{R}(f^k)$ equals k. □

(5.1.23) Lemma. *Tori are essentially reducible.*

Proof. Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be for a self-map of the torus $f\colon \mathbb{T}^d \to \mathbb{T}^d$ represented by the matrix $\mathbb{A}$ and let integers $k|n$. Then

$$\begin{aligned} L(f^n) &= \det(\mathrm{I} - A^n) = \det((\mathrm{I} - A^k)(\mathrm{I} + A^k + A^{2k} + \dots + A^{n-k})) \\ &= \det(\mathrm{I} - A^k) \cdot \det(\mathrm{I} + A^k + A^{2k} + \dots + A^{n-k}) \\ &= L(f^k) \cdot \det(\mathrm{I} + A^k + A^{2k} + \dots + A^{n-k}). \end{aligned}$$

Thus $L(f^n) \neq 0$ implies $L(f^k) \neq 0$ for $k|n$. Let $B \in \mathcal{OR}(f^n)$ be an essential orbit. Since a torus is a Jiang space, all classes preceding $\mathbb{B}$ must be essential. □

Now we pass to the next property.

(5.1.24) definition. Let $f\colon X \to X$ be a map. We say that f is *essentially reducible to the greatest common divisors* (GCD for the shorthand) if it is essentially reducible and whenever $[\alpha]^n \in \mathcal{E}(f^n)$ reduces to both $[\beta]^r \in \mathcal{R}(f^r)$, $[\gamma]^s \in \mathcal{R}(f^s)$ then there exists a $[\delta]^q \in \mathcal{R}(f^q)$ with $q = \mathrm{GCD}(r, s)$ to which both $[\beta]^r$, $[\gamma]^s$ reduce. If every f is essentially reducible to the GCD, then we say that X is *essentially reducible to the* GCD.

(5.1.25) LEMMA. *Tori are essentially reducible to the* GCD.

PROOF. We follow the proof from [Ji4]. By Lemma (5.1.23) tori are essentially reducible. Consider a self-map $f\colon \mathbb{T}^d \to \mathbb{T}^d$ and let A denote the matrix representing the induced homomorphism of homotopy groups. Let $[\alpha]^d$ be an essential Nielsen class of f^n. Suppose that $[\alpha]^n$ reduces to $[\beta]^r$, $[\gamma]^s$ (both classes are essential by Lemma (5.1.23)). We are looking for a $[\delta]^q$ to which both classes reduce. Let us recall that $\mathcal{R}(f^n) = \mathbb{Z}^d/(\mathrm{I} - A^n)$ and that $\gamma_{nk}\colon \mathcal{R}(f^k) \to \mathcal{R}(f^n)$ is given by $\gamma_{nk}[x]^k = [(1 + A^k + A^{2k} + \cdots + A^{n-k})x]^n$. In particular

$$[\alpha]^n = [(1 + A^r + A^{2r} + \cdots + A^{n-r})\beta]^n = [(1 + A^s + A^{2s} + \cdots + A^{n-s})\gamma]^n.$$

Let us assume that $\mathrm{GCD}(r, s) = 1$. Then the Euclidean algorithm gives the polynomials $B(t)$, $C(t)$ satisfying:

$$B(t)(1 + t + t^2 + \cdots + t^{r-1}) + C(t)(1 + t + t^2 + \cdots + t^{s-1}) = 1.$$

Let $\delta = B(A)\beta + C(A)\gamma$. We show that $[\alpha]^n$ reduces to $[\delta]^1$. In fact

$$\begin{aligned}
\gamma_{n1}[\delta]^1 &= [(1 + A + A^2 + \cdots + A^{n-1})\delta]^n \\
&= [(1 + A + A^2 + \cdots + A^{n-1})(B(A)\beta + C(A)\gamma)]^n \\
&= [(B(A)(1 + A + A^2 + \cdots + A^{r-1})(1 + A^r + A^{2r} + \cdots + A^{n-r})\beta) \\
&\quad + (C(A)(1 + A + A^2 + \cdots + A^{s-1})(1 + A^s + A^{2s} + \cdots + A^{n-s})\gamma)]^n \\
&= [(B(A)(1 + A + A^2 + \cdots + A^{r-1})\alpha \\
&\quad + (C(A)(1 + A + A^2 + \cdots + A^{s-1})\alpha)]^n = [\alpha]^n.
\end{aligned}$$

Now we will deduce that $\gamma_{r1}[\delta]^1 = [\beta]^r$. We notice that $\gamma_{r1}[\delta]^1, [\beta]^r \in \mathcal{R}(f^r)$ and $\gamma_{nr}(\gamma_{r1}[\delta]^1) = \gamma_{n1}[\delta] = \alpha$, hence $\gamma_{nr}(\gamma_{r1}[\delta]^1) = \gamma_{nr}[\beta]^r$. Since the class $[\alpha]^n$ is essential, by Lemma (5.1.7) γ_{nr} is mono which implies $\gamma_{r1}[\delta]^1 = [\beta]^r$.

Similarly we prove that $\gamma_{s1} = [\gamma]^s$ which ends the lemma for $q = \gcd(r, s) = 1$. If $q > 1$ we apply the above for $f = f^q$. □

(5.1.26) LEMMA. *If the map $f\colon X \to X$ is essentially reducible to the* GCD*, then for every essential Nielsen class $\mathbb{A}^n$ there is a unique irreducible class $\mathbb{B}^k$ preceding $\mathbb{A}^n$. Then $k = \mathrm{d}(\mathbb{A}^n)$ and the class $\mathbb{B}^k$ is essential.*

PROOF. Suppose that $\mathbb{A}^n$ reduces to two irreducible classes $\mathbb{B}^k$ and $\mathbb{C}^l$. By the reducibility to the GCD both $\mathbb{B}^k$ and $\mathbb{C}^l$ reduce to a class $\mathbb{D}^{\mathrm{GCD}(k,l)}$ which contradicts the irreducibility of $\mathbb{B}^k$ and $\mathbb{C}^l$. Since $\mathbb{B}^k$ precedes the essential class $\mathbb{A}^n$ and f is essentially reducible, $\mathbb{B}^k$ is also essential. □

(5.1.27) Definition. The self-map $f\colon X \to X$ is called *essentially toral* if it is essentially reducible and

(5.1.27.1) the length and the depth of each essential Nielsen class $[\alpha]^n \in \mathcal{N}(f^n)$ coincide: $\mathrm{d}[\alpha]^n = \mathrm{l}[\alpha]^n$,

(5.1.27.2) if $\gamma_{nk}[\beta]^k = \gamma_{nk}[\gamma]^k = [\alpha]^n \in \mathcal{E}(f^n)$ (i.e. $[\alpha]^n$ is essential), then $[\beta]^k = [\gamma]^k$.

(5.1.28) Remark. Let us notice that in general $\mathrm{d}[\alpha]^n \geq \mathrm{l}[\alpha]^n$. In fact if an orbit reduces to $\mathcal{R}(f^d)$, then the length of the orbit must be a divisor of d. Thus the equality holds if and only if $[\alpha]^n$ reduces to $\mathcal{R}(f^l)$ where $l = \mathrm{length}(\alpha)$.

(5.1.29) Lemma. *Tori are essentially toral.*

Proof. Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$. Then $\mathcal{R}(f^n) = \mathbb{Z}^d/(\mathrm{I} - A^n)\mathbb{Z}^d$.

(1) Let $[\alpha]^n \in \mathcal{E}(f^n)$ be essential. We show that then $[\alpha]^n$ reduces to a class in $\mathcal{R}(f^l)$ (see Remark (5.1.28)).

Since $[\alpha]^n$ is essential and $\mathbb{T}^d$ is a Jiang space, $L(f^n) \neq 0$. We may assume that f is induced by a linear map. Then (see the proof of Theorem (4.3.14)) $\mathrm{Fix}(f^n)$ is finite and each Nielsen class consists of exactly one point. If the length of the class $[\alpha]^n$ is l, then the unique fixed point (of f^n) in this class must be also a fixed point of f^l. Thus the class $[\alpha]^n$ reduces to a class in $\mathcal{R}(f^l)$.

(2) Let $[\alpha]^n \in \mathcal{E}(f^n)$. It is enough to show that if $k|n$, then γ_{nk} is mono. But this is implied by $L(f^n) \neq 0$ and Lemma (5.1.7). □

(5.1.30) Lemma. *If X is essentially toral, then $NP_n(f) = \#(IEC_n(f))$, where IEC denotes the set all irreducible essential classes in $\mathcal{R}(f^n)$.*

Proof. By the definition $NP_n(f) = (\#IEO(f^n)) \times n$. But each orbit in $IEO(f^n)$ has depth $= n$, hence by the definition of essential torality its length also equals n. Thus $\#(IEO(f^n)) \times n = \#(IEC(f^n))$. □

The next theorems give the formulae for $NF_n(f)$ for self-maps of the spaces satisfying the three introduced properties: essential reducibility, essential reducibility to the GCD and torality. In particular these formulas will be valid for all self-maps of tori (Lemmas (5.1.23), (5.1.25), (5.1.29) (5.1.30)).

(5.1.31) Theorem. *Suppose that X has essential torality and is essentially reducible to the* GCD. *If $f\colon X \to X$ is such that f^n is weakly Jiang and $N(f^n) \neq 0$, then $NF_n(f) = N(f^n)$. Moreover, the same formula holds for any divisor of n.*

Proof. By Lemmas (5.1.22), (5.1.30),

$$NF_n(f) = \sum_{k|n} NP_k(f) = \sum_{k|n} \#IEC_k.$$

We define the function $\phi\colon \bigcup_{k|n} IEC_k \to \mathcal{R}(f^n)$ putting $\phi(C^k) = \gamma_{nk}(C^k)$.

Now we define the inverse map. By Lemma (5.1.26) for each (essential) class $\mathbb{A}^n \in \mathcal{R}(f^n)$ there exists the unique irreducible essential class $\mathbb{B}^r$ preceding $\mathbb{A}^n$. We define $\psi\colon \mathcal{R}(f^n) \to \bigcup_{k|n} \mathrm{IEC}_k$ putting $\psi(\mathbb{A}^n) = \mathbb{B}^r$. Now it is easy to see that ψ is inverse to ϕ. Thus

$$N(f^n) = \#\mathcal{R}(f^n) = \#\Big(\bigcup_{k|n} IEC_k\Big) = \sum_{k|n} \#IEC_k = \sum_{k|n} NP_k(f) = NF_n(f).$$

The first equality follows from X being weakly Jiang and $N(f^n) \neq 0$ and the second from the bijectivity of ϕ. The same formula also holds for each $k|n$ since each divisor of n also satisfies the assumption of the lemma. □

Now we may drop the assumption that $N(f^n) \neq 0$.

(5.1.32) Definition. Let $f\colon X \to X$ be a map and $n \in \mathbb{N}$ a fixed number. We define $M(f,n)$ as the set of natural numbers k satisfying $N(f^k) \neq 0$ but $N(f^s) = 0$ for all $s > k$, $k|s|n$.

(5.1.33) Example. Consider the map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ given by the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

The characteristic polynomial $\chi_A(t) = t^2 - t + 1$, hence its eigenvalues are $t_1 = e^{i\pi/3}$ and $t_2 = e^{-i\pi/3}$ so $\chi_A(t) = (t - e^{i\pi/3})(t - e^{-i\pi/3})$. We notice that both eigenvalues are roots of unity of order 6. Let $n = 6$. Then $N(f^6) = |L(f^6)| = |\chi_{A^6}(1)| = |(1 - e^{6\pi/3})(1 - e^{-6i\pi/3})| = 0$. The similar equalities show that $N(f^2) \neq 0$ and $N(f^3) \neq 0$. Thus $M(f,6) = \{2,3\}$.

(5.1.34) Theorem. *Let $f\colon X \to X$ be a map with X essentially toral and essentially reducible to the* GCD. *Suppose that $n \in \mathbb{N}$ and f^m is weakly Jiang for all $m \in M(f,n)$. Then*

$$NF_n(f) = \sum_{\emptyset \neq \mu \subset M(f,n)} (-1)^{\#\mu - 1} N(f^{\xi(\mu)}),$$

where $\xi(\mu)$ is the greatest common divisor of all elements on μ.

Proof. Let us denote $\mathbf{W} = \bigcup_{m|n} IEC_m(f)$. Since f is toral, $NF_n(f) = \#\bigcup_{m|n} IEC_m(f)$ (Lemma (5.1.30)). Since f is essentially reducible to the GCD, any essential class of f^k (for $k|n$) reduces to exactly one class in $\mathbf{W}$. On the other hand each class $\mathbb{A} \in \mathbf{W}$ determines the set

$$\omega_A = \Big\{\mathbb{B} \in \bigcup_{k|n} \mathcal{E}(f^k) : \mathbb{B} \text{ reduces to } \mathbb{A}\Big\}.$$

Such a set will be called *a tree* and A will be called its *root*. Let Γ denote the set of all trees. By essential reducibility to the GCD, any two different trees are disjoint and any tree has a unique root. Let us notice that any class $\mathbb{B} \in \bigcup_{k|n} \mathcal{E}(f^k)$ belongs to a tree. Thus the map

$$\mathbf{W} \ni \mathbb{A} \to \omega_A \in \Gamma$$

is a bijection. Thus

$$NF_n(f) = \sum_{m|n} \#IEC_m(f) = \#\Gamma.$$

We say that a tree $\omega \in \Gamma$ bears *m-fruit* if ω contains an essential class at level m. For $m \in M(f,n)$ let Γ_m be the set of the trees bearing m-fruits.

To end the proof we will show a sequence of equalities

$$NF_n(f) = \#\Gamma = \sum_{\emptyset \neq \mu \subset \mathcal{M}(f,n)} (-1)^{\mu-1} \# \left(\bigcap_{i \in \mu} \Gamma_i \right) = \sum_{\emptyset \neq \mu \subset \mathcal{M}(f,n)} (-1)^{\mu-1} N(f^{\xi(\mu)}).$$

(1) The first equality is already proved.

(2) To prove the second equality we will show that

$$\Gamma = \bigcup_{m \in M(f,n)} \Gamma_m.$$

Then the equality will follow from the inclusion-exclusion principle:

Let G, A be finite sets and let $G = \sum_{i \in A} G_i$. Then this principle says that

$$G = \sum_{\emptyset \neq \mu \subset A} (-1)^{\mu-1} \# \left(\bigcap_{i \in \mu} G_i \right).$$

$\supset$ is evident. To prove $\subset$ we fix a tree $\omega \in \Gamma$. Let $\mathbb{A}^r$ be its root. Then $\mathbb{A}^r$ is essential. Let s_0 be the greatest number s of the property $r|s|n$ and the class $\mu_{sr}(\mathbb{A}^r)$ be essential. Then $s_0 \in M(f,n)$ and $\omega \in \Gamma_{s_0}$.

($3'$) We show that $\bigcap_{m \in \mu} \Gamma_m = \Gamma_{\xi(\mu)}$. To prove $\supset$ we have to show that for $m \in \mu$ any tree ω bearing $\xi(\mu)$-fruit bears also m-fruit. Let $\mathbb{A}^{\xi(\mu)} \in \omega$ be the $\xi(\mu)$ fruit. Since $\xi(\mu)$ is a divisor of m, we have the element $\gamma_{m,\xi(\mu)}(\mathbb{A}^{\xi(\mu)}) \in \mathcal{R}(f^m)$. Since $m \in M(f,n)$, $\mathcal{R}(f^m)$ contains an essential class. Moreover, f^m is weakly Jiang hence all classes in $\mathcal{R}(f^m)$ are essential, in particular $\gamma_{m,\xi(\mu)}(\mathbb{A}^{\xi(\mu)})$ is essential hence is m-fruit in ω.

To prove $\subset$ we consider a tree ω which bears an m fruit E^m for each $m \in \mu$. We have to show that ω bears also a $\xi(\mu)$ fruit. Let E^d be the unique root of the tree ω. Now d divides all $m \in \mu$, hence d divides $\xi(\mu)$ = the greatest common divisor of all $m \in \mu$. Consider the element $\gamma_{\xi(\mu),d}(E^d)$. It belongs to $\mathcal{R}(f^{\xi(\mu)})$ and to the tree ω. It remains to show that $\gamma_{\xi(\mu),d}(E^d)$ is essential: then it will be the $\xi(\mu)$-fruit. But each fruit E^m $(m \in \mu)$ (essential class) reduces to $\gamma_{\xi(\mu),d}(E^d)$ which by the essential reducibility must be also essential.

(3″) It remains to show that $\#\Gamma_m = N(f^m)$ for any $m \in M(f,n)$. We notice that each tree $\omega \in \Gamma_m$ bears exactly one m-fruit. In fact if $\mathbb{A}^m$, $\mathbb{B}^m$ were two essential classes (m-fruits) in ω then $\mathbb{A}^m$ and $\mathbb{B}^m$ would reduce to the same essential root, hence they would be equal. Since the trees are disjoint, the map sending each m-tree $\omega \in \Gamma_m$ to its unique m-fruit is the bijection between Γ_m and $\mathcal{E}(f^m)$. □

(5.1.35) Corollary. *The formulae from Theorems* (5.1.31) *and* (5.1.34) *hold for self-maps of tori.*

Proof. Tori are essentially reducible to the GCD (Lemma (5.1.25)) and essentially toral (Lemma (5.1.29)). Moreover, they are weakly Jiang since they are Jiang. Thus the assumptions of Theorems (5.1.31) and (5.1.34) are satisfied. □

5.1.6. Fibrations as toroidal spaces. The main theorems of the previous section (the formulae for the Nielsen–Jiang full number $NF_n(f)$) hold under the assumption that the considered spaces are essentially reducible to the GCD and essentially toral. Thus it is very desirable to prove that this class of spaces is large. Here we will prove that if the base space and the fibre satisfy these assumptions, then so does the total space. This will allow us to extend the mentioned formulae onto nilmanifolds and some solvmanifolds in Chapter VI. We begin with the following [HeKeI]

(5.1.36) Theorem. *Let $p\colon E \to B$ be a locally trivial fibre bundle where the base space B and the fibres are essentially reducible. Then any fibre map $f\colon E \to E$ is also essentially reducible.*

Proof. Let $[\alpha]^n \in \mathcal{R}(f^n)$ be an essential Reidemeister class reducing to a $[\beta]^k$ in $\mathcal{R}(f^k)$. We have to show that $[\beta]^k$ is also essential. Let us denote $[\overline{\alpha}]^n = \mathcal{R}_p([\alpha]^n)$, $[\overline{\beta}]^k = \mathcal{R}_p([\beta]^k)$. Then $[\overline{\alpha}]^n$ reduces to $[\overline{\beta}]^k$. By IPF $[\overline{\alpha}]^n$ is essential hence by the assumption (essential reducibility on B) $[\overline{\beta}]^k$ is also essential. Now $[\overline{\beta}]^k$ is nonempty, hence it contains a periodic point $b_0 \in \mathrm{Fix}(\overline{f}^k)$.

Since $\mathcal{R}_p([\beta]^k) = [\overline{\beta}]^k$, there is a class $[\delta]^k \in \mathcal{R}_K((f^k)_{b_0})$ satisfying $\mathcal{R}_i[\delta]^k = [\beta]^k$ where $i\colon E_{b_0} \to E$ denotes the inclusion of the fibre (see Theorem (4.4.20)).

On the other hand the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{R}_K((f^k)_{b_0}) \arrow[r, "\gamma_{nk}"] \arrow[d, "\mathcal{R}_i"'] & \mathcal{R}_K((f^n)_{b_0}) \arrow[d, "\mathcal{R}_i"] \\
\mathcal{R}(f^k) \arrow[r, "\gamma_{nk}"'] & \mathcal{R}(f^n)
\end{tikzcd}$$

gives $\mathcal{R}_i\gamma_{nk}[\delta]^k = \gamma_{nk}\mathcal{R}_i[\delta]^k = \gamma_{nk}[\beta]^k = [\alpha]^n$. Applying IPF again, the essentiality of $[\alpha]^n$ implies essentiality of $\gamma_{nk}[\delta]^k$. Since the fibre E_{b_0} is essentially reducible, $[\delta]^k \in \mathcal{R}((f^k)_{b_0})$ is also essential. Finally $[\overline{\beta}]^k = \mathcal{R}_p([\beta]^k)$ and $[\delta]^k$ essential and IPF make $[\beta]^k$ essential as required. $\square$

The next theorem gives a condition for essential reducibility to the GCD [HeKeI].

(5.1.37) THEOREM. *Suppose that $p: E \to B$ is a fibration where B and the fibre F are essentially reducible to the* GCD. *Then any fibre map such that for each iteration f^k the equality from Theorem* (4.4.21) *holds, is essentially reducible to the* GCD.

PROOF. By Remark (4.4.22) the ordinary Nielsen classes of f_{b_0} coincide with classes modulo $K: \mathcal{E}_K(f_b) \to \mathcal{E}(f_b)$ and $\mathcal{E}_K(f_b) \to \mathcal{R}(f)$ is mono. By Theorem (5.1.36) we know that f is essentially reducible. Let $[\alpha]^n \in \mathcal{R}(f^n)$ be an essential class which reduces to both $[\alpha]^k$ and $[\alpha]^l$. We have to show that they both reduce to an $[\alpha]^q$ where $q = \mathrm{GCD}(k,l)$.

Consider $[\overline{\alpha}]^n = \mathcal{R}_p[\alpha]^n$, $[\overline{\alpha}]^k = \mathcal{R}_p[\alpha]^k$, $[\overline{\alpha}]^l = \mathcal{R}_p[\alpha]^l$. Then $[\overline{\alpha}]^n$ is essential (by IPF) and reduces to $[\overline{\alpha}]^k$, $[\overline{\alpha}]^l$ so they reduce to an $[\overline{\alpha}]^q$ (essential reducibility to GCD on the base space B). The class $[\overline{\alpha}]^q$ is also essential (essential reducibility) hence it contains a point $b_0 \in \mathrm{Fix}(\overline{f}^q)$. We consider the restriction of f^q to the fibre E_{b_0}. Since $\mathcal{R}_i: \mathcal{R}(f_{b_0}) \to \mathcal{R}(f)$ is mono and $[\alpha]^n$ is essential, there is exactly one class $[\alpha_0]^n \in \mathcal{R}((f^n)_{b_0})$ (which is also essential by IPF) such that $\mathcal{R}_i[\alpha_0]^n = [\alpha]^n$. Similarly we find the unique $[\alpha_0]^k$, $[\alpha_0]^l$.

Since the fibre satisfies the essential reducibility to the GCD, $[\alpha_0]^k$, $[\alpha_0]^l$ are essential and they both reduce to an $[\alpha_0]^q$. Now we show that $[\alpha]^q = \mathcal{R}_i[\alpha_0]^q$ is the desired class. In fact

$$\gamma_{kq}[\alpha]^q = \gamma_{kq}\mathcal{R}_i[\alpha_0]^q = \mathcal{R}_i\gamma_{kq}[\alpha_0]^q = \mathcal{R}_i[\alpha_0]^k = [\alpha]^k$$

and similarly we show that $[\alpha]^l$ reduces to $[\alpha]^q$. $\square$

(5.1.38) LEMMA. *Suppose that $p: E \to B$ is a fibration satisfying the assumptions of the Nielsen Number "Product" Formula* (*Theorem* (4.4.21)) *i.e.*

(5.1.38.1) $N_K(f_i) = N(f_i)$,

(5.1.38.2) *for every point $b \in \mathrm{Fix}(\overline{f})$ lying in an essential Nielsen class the group* $\mathrm{Fix}(\overline{f}_\#; b)$ *acts trivially on $\mathcal{E}_K(f_b)$.*

If moreover, B and the fibre F are essentially toral then any fibre map $f: E \to E$ is essentially toral.

PROOF. We start with the remark that the above assumptions (from Theorem (4.4.21)) may be reformulated as

(1) each essential Nielsen class in $\mathcal{R}_K(f_b)$ contains exactly one class from $\mathcal{R}(f_b)$

(for any $b \in \mathrm{Fix}(\widetilde{f})$ lying in an essential class) i.e. the sets of essential classes are equal $\mathcal{E}_K(f_b) = \mathcal{E}(f_b)$.

(2) $\mathcal{R}_i : \mathcal{R}(f_b) \to \mathcal{R}(f)$ is mono on essential classes.

On the other hand, as we will see, the classes involved in the proof will be essential hence we may forget about the subgroup K and assume that $\mathcal{R}_i$ is mono.

(1) length = depth. Let $[\alpha]^n \in \mathcal{R}(f^n)$ be an essential class of length l. We will show that $[\alpha]^n$ reduces to a class in $\mathcal{R}(f^l)$.

Denote $[\overline{\alpha}]^n \in \mathcal{R}_p[\alpha]^n$. By IPF $[\overline{\alpha}]^n$ is essential. Let us denote $r = \mathrm{length}[\overline{\alpha}]^n$. By the essential torality of B, $[\overline{\alpha}]^n$ reduces to a class $[\overline{\alpha}]^r$, $(r|l)$. By the assumption that $\overline{f}$ is essentially reducible, the class $[\overline{\alpha}]^r$ is essential. Now $[\overline{\alpha}]^r$ considered as the Nielsen class of $\overline{f}$ contains a point b which is also contained in the Nielsen class $[\overline{\alpha}]^n = \gamma_{nr}[\overline{\alpha}]^r$. Since $\mathcal{R}_p[\alpha]^n = [\overline{\alpha}]^n$, there is (exactly one since $\mathcal{R}_i$ is mono) class $[\alpha_0]^n \in \mathcal{R}((f^n)_b)$ satisfying $\mathcal{R}_i[\alpha_0]^n = [\alpha]^n$.

We will show that length of $[\alpha_0]^n \in \mathcal{R}((f^r)_b^{n/r})$ equals l/r. In fact all elements of the sequence

$$[\alpha_0]^n, [(f^r)_b(\alpha_0)]^n, \dots, [(f^r)_b^{l/r-1}(\alpha_0)]^n \in \mathcal{R}((f^r)_b^{n/r})$$

are different since their images under $\mathcal{R}_i$ are

$$[\alpha]^n, [f^r(\alpha)]^n, \dots, [f^{l-r}(\alpha)]^n \in \mathcal{R}(f^n)$$

and these are different (since $\mathrm{length}[\alpha]^n = l$). Moreover, $[\alpha]^n = [f^l\alpha]^n \in \mathcal{R}(f^n)$ means $\mathcal{R}_i[\alpha_0]^n = \mathcal{R}_i[(f^r)_b^{l/r}(\alpha_0)]^n \in \mathcal{R}(f^n)$. Since $\mathcal{R}_i$ is mono, $[\alpha_0]^n = [(f^r)_b^{l/r}(\alpha_0)]$ $\in \mathcal{R}((f^r)^{l/r})$. Thus $[\alpha_0]^n$ reduces to an $[\alpha_0]^l \in \mathcal{R}((f^r)^{l/r})$ (essential torality on the fibre). Let us denote $\mathcal{R}_i[\alpha_0]^l = [\alpha]^l$. We will show that $[\alpha]^l$ is the desired element to which $[\alpha]^n$ reduces. In fact $\gamma_{nl}[\alpha_0]^l = [\alpha_0]^n$ implies $\gamma_{nl}[\alpha]^l = \gamma_{nl}\mathcal{R}_i[\alpha_0]^l = \mathcal{R}_i\gamma_{nl}[\alpha_0]^l = \mathcal{R}_i[\alpha_0]^n = [\alpha]^n$.

(2) Let $[\alpha]^n \in \mathcal{R}(f^n)$ be an essential class which reduces to $[\beta]^k, [\delta^k] \in \mathcal{R}(f^n)$. We have to show that $[\beta]^k = [\delta^k]$.

We denote $[\overline{\alpha}]^n = \mathcal{R}_p[\alpha]^n \in \mathcal{R}(\overline{f}^n)$. It obviously reduces to both $[\overline{\beta}]^k = \mathcal{R}_p[\beta]^k$, $[\overline{\delta}]^k = \mathcal{R}_p[\delta]^k \in \mathcal{R}(\overline{f}^k)$. By IPF, $[\overline{\alpha}]^n$ is essential. By the assumption that $\overline{f}$ is essentially toral, the classes are equal $[\overline{\beta}]^k = [\overline{\delta}]^k$ and essential. Now they contain a point b which is also contained in the Nielsen class $[\overline{\alpha}]^n = \gamma_{nk}[\overline{\beta}]^k$. There exist elements $[\alpha_0]^n \in \mathcal{R}((f^n)_b)$, $[\beta_0]^k$, $[\delta_0]^k \in \mathcal{R}((f^k)_b)$ which are mapped by $\mathcal{R}_i$ onto $[\alpha]^n, [\beta]^k$ and $[\delta]^k$, respectively. Since $\mathcal{R}_i$ is mono, $[\alpha_0]^n$ reduces to both $[\beta_0]^k$ and $[\delta_0]^k$. By the assumption (essential torality on fibres) $[\beta_0]^k = [\delta_0]^k$, which implies that the images under $\mathcal{R}_i$ are also equal $[\beta]^k = [\delta]^k$. $\square$

5.1.7. Periodic points on $\mathbb{R}P^d$. In section 4.3.2 we gave formulae for the Nielsen number of self-maps of real projective spaces. We will show that this number (with one exception) determines the numbers $NF_n(f)$. The considerations are based on [Je5] of the first author.

(5.1.39) THEOREM. *Let $f\colon \mathbb{R}P^d \to \mathbb{R}P^d$ be a self-map of a real projective space $(d \geq 2)$. Then for each $m \in \mathbb{N}$ the following formula holds (with one exception)*

$$NF_m(f) = \begin{cases} 0 & \text{if } N(f) = 0, \\ 1 & \text{if } N(f) = 1, \\ 2^{s+1} & \text{if } N(f) = 2. \end{cases}$$

where $m = l \cdot 2^s$ with l being odd. The exceptional case: for d odd and $\deg f = -1$,

$$N(f) = 2 \text{ and } NF_m(f) = 2 \text{ for all } m \in \mathbb{N}.$$

PROOF. In the proof we will refer, often implicitly, to Theorem (4.3.13).

Let $N(f) = 0$. Then d is odd and $f \sim \mathrm{id}$ (see Theorem (4.3.13)). Thus $N(f^k) = 0$ for all $k \in \mathbb{N}$, hence there are no essential classes and the empty set is the Preceding System. Thus $NF_k(f) = 0$.

Let $N(f) = 1$. We consider two subcases. First we suppose that $f_\#$ is constant. Then also $f_\#^m$ is constant hence $\mathcal{R}(f^k)$ consists of one point and the unique class in $\mathcal{R}(f^1)$ is the MPS which implies $NF_m(f) = 1$. Now we assume that $f_\# = \mathrm{id}$ and $N(f) = 1$. Then d is even and $f \sim \mathrm{id}$. We may assume that $f = \mathrm{id}$. Then the nonempty Reidemeister class of f is essential since its index equals $L(\mathrm{id}) = \chi(\mathbb{R}P^d) = 2$ (d is even) and the second class is empty hence inessential. Since $f = \mathrm{id}$ the same holds for all f^k. Now γ_{k1} sends the nonempty class into nonempty class hence the unique nonempty a class in $\mathcal{R}(f)$ makes an MPS. We get again $NF_k(f) = 1$.

Let $N(f) = 2$. Then $f_\# = \mathrm{id}$ but f is not homotopic to the identity map (see Theorem (4.3.13)). Consider a lift $\widetilde{f}\colon S^d \to S^d$ of the map f. We consider two subcases.

Assume moreover that for any lift $\widetilde{f}$ we have $\deg(\widetilde{f}) \neq \pm 1$. Notice that then no iteration f^k is homotopic to the identity map since otherwise the lift $\widetilde{f}^k$ would be of degree ± 1. Now $N(f^k) = 2$: see again Theorem (4.3.13). Thus all Nielsen classes are essential. On the other hand the action of $\mathbb{Z}_2$ on $\mathcal{R}(f^k)$ is trivial since $f_\# = \mathrm{id}$. Thus each orbit of Reidemeister classes consists of one element.

As we have noticed all the Nielsen classes are essential, hence there the unique MPS consists of the elements of all $IEOR(f^k) = IER(f^k)$ for $k|n$ (cf. Lemma (5.1.22)). For each $k \in \mathbb{N}$ we have two essential Nielsen classes. For $k = 1$ both are irreducible. Let $k \geq 2$ have an odd divisor. Then by Lemma (5.1.40) both classes are reducible. Now let k have no odd divisors, i.e. $k = 2^s$. Then using again Lemma (5.1.40) we get exactly one irreducible class in $\mathcal{R}(f^k)$. Finally (for $m = l \cdot 2^s$, l-odd)

$$\begin{aligned} NF_n(f) &= \#IER(f) \cdot 1 + \#IER(f^2) \cdot 2 + \cdots + \#IER(f^{2^s}) \cdot 2^s \\ &= 2 \cdot 1 + 1 \cdot 2 + \cdots + 1 \cdot 2^s = 2 + 2 + 2^2 + \cdots + 2^s = 2^{s+1}. \end{aligned}$$

Now we assume that a lift $\widetilde{f}^k$ of some iteration has degree ± 1 (and still $N(f) = 2$). This is possible only if $\deg(\widetilde{f}) = -1$ and d is odd, since for d even for the other lift $\deg(-\widetilde{f}) = 1$ and f would be homotopic to the identity. We recall that for d odd $\mathbb{R}P^d$ is a Jiang space and $L(f^k) = 1 - (-1)^k$. Thus for $k = 1$ we have two essential irreducible classes. For $k > 1$ odd all classes are reducible (as above). Finally for k even no class is essential. Thus in the exceptional case the unique MPS consists of the two classes in $\mathcal{R}(f^1)$ hence $NF_m(f) = 2$. It remains to show that $\deg f = \deg \widetilde{f}$ which follows from Lemma (5.1.41) □

(5.1.40) LEMMA. *Let $f\colon \mathbb{R}P^d \to \mathbb{R}P^d$ induce the identity map on fundamental groups. Then $\mathcal{R}(f^k) = \mathbb{Z}_2$ for all integers k. Moreover, the map $\gamma_{kl}\colon \mathcal{R}(f^l) \to \mathcal{R}(f^k)$ is identity if k/l is odd and is the constant map if k/l is even. In particular if $k = 2^s$ (for a natural s) then exactly one class in $\mathcal{R}(f^k)$ is irreducible. Otherwise (i.e. if k has an odd divisor) both classes in $\mathcal{R}(f^k)$ are reducible.*

PROOF. By Remark (5.1.5),

$$\gamma_{k,k/l}[\alpha] = [\alpha * f^{k/l}(\alpha) * \cdots * f^{k-k/l}(\alpha)] = [\alpha^l] = \begin{cases} [\alpha] & \text{for } l \text{ odd}, \\ [1] & \text{for } l \text{ even}. \end{cases}$$

Now if l is an odd divisor of k then $\gamma_{k,k/l}$ is onto, hence both classes in $\mathcal{R}(f^k)$ are reducible. If $k = 2^s$, then for each divisor $l|k$ the image of the map $\gamma_{k,l}$ equals the image of $\gamma_{k,1}$ which also is a singleton. Thus the other element in $\mathcal{R}(f^k)$ is irreducible. □

(5.1.41) LEMMA. *Suppose that we are given a self-map of the odd dimensional projective space and its lift.*

$$\begin{tikzcd} S^d \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & S^d \arrow[d, "p"] \\ \mathbb{R}P^d \arrow[r, "f"'] & \mathbb{R}P^d \end{tikzcd}$$

Then $\deg(\widetilde{f}) = \deg(f)$.

PROOF. Since $p\colon S^d \to \mathbb{R}P^d$ is a two-fold covering of orientable manifolds, $\deg(p) = 2$. Now $\deg(p\widetilde{f}) = \deg(fp)$ implies $\deg(p) \cdot \deg(\widetilde{f}) = \deg(f) \cdot \deg(p)$, hence $2 \cdot \deg(\widetilde{f}) = \deg(f) \cdot 2$ which gives the lemma. □

5.2. Weak Wecken's Theorem for periodic points

It is natural to ask whether the invariant $NF_n(f)$ can be realized in the homotopy class of the map f, i.e. is f homotopic to a map g with $\#\mathrm{Fix}(g^n) = NF_n(f)$. In other words is the analog of the Wecken theorem valid also for periodic points? We will show in the next sections that this is true in the case of manifolds of dimension different than 2.

We start with an easier question. Given a map $f\colon X \to X$ and the number $n \in \mathbb{N}$ when is f homotopic to a map g satisfying $\mathrm{Fix}(g^n) = \emptyset$?

Let us notice that if a map $f\colon X \to X$ is homotopic to a map g with $\mathrm{Fix}(g^n)$ empty, then also $\mathrm{Fix}(g^k) = \emptyset$ for each divisor $k|n$. This implies $N(f^k) = 0$ for all $k|n$. We will show that in the case of manifolds of dimension $\neq 2$ this is also the sufficient condition for the deformation of the map f to a map with no n-periodic points. The rest of this section is the proof of the following Weak Wecken's Theorem for Periodic Points. In fact, the proof is quickly reduced to the Cancelling Procedure, Theorem (5.2.3), which was introduced by Jezierski in [Je1]. The weak Wecken's theorem was proved in [Je1] for $\dim X \geq 4$, and next extended in [Je4] for $\dim X \geq 3$. A proof presented here is a modification of the original one.

(5.2.1) THEOREM (Weak Wecken's Theorem for Periodic Points). *Any self-map $f\colon X \to X$ of a compact connected* PL-*manifold of dimension ≥ 3 is homotopic to a map g without periodic points of period n (i.e.* $\mathrm{Fix}\,(g^n) = \emptyset$) *if and only if for any divisor k of n the Nielsen number $N(f^k) = 0$.*

PROOF. In dimension 1 we have only two compact connected manifolds: the interval $[0,1]$ and the sphere S^1. Each self-map of the interval has exactly one Nielsen class and this class is essential. A self-map of S^1 has no essential class only if $\deg(f) = 1$, since $N(f) = |L(f)| = |1 - \deg(f)|$. But then f is homotopic to the identity and the last map after a twist with an irrational angle becomes the map with no periodic points $f(z) = e^{\alpha i}z$, α/π irrational.

Now we may concentrate on the case $\dim X \geq 3$.

$\Rightarrow$ is evident. The rest of this section is the proof of $\Leftarrow$. We use induction with the respect to the divisors of the given $n \in \mathbb{N}$: for every $k|n$ we show that

$$f \text{ is homotopic to a map } g \text{ satisfying } g^l(x) \neq x \text{ for all } l|n,\ l \leq k.$$

For $k = 1$ the theorem follows from the classical Wecken Theorem for fixed points. Now we assume that the induction assumption holds for all divisors of n which are less than k. We will show how to remove k-periodic points. We will base our argument on the following technical result (for the proof we refer to [Je1]).

(5.2.2) THEOREM. *Let $X \subset \mathbb{R}^N$ be a compact* PL-*submanifold with the metric inherited after the Euclidean metric in $\mathbb{R}^N$. Let $n \in \mathbb{N}$ be a fixed number. Then* *any continuous map* $f\colon X \to X$ *is homotopic to a map* g *such that* $\mathrm{Fix}\,(g^n)$ *is finite and* g *is a* PL*-homeomorphism near any point* $x \in \mathrm{Fix}\,(g^n)$*. Moreover, for any* $\varepsilon > 0$*, we may choose a* g *satisfying* $d(f,g) < \varepsilon$. □

By the above theorem we may assume that $\mathrm{Fix}(f^n)$ is finite and f is a linear homeomorphism near each $x \in \mathrm{Fix}(f^n)$. In particular $\mathrm{ind}(f^k, x) = \pm 1$ at these points. Consider an orbit of Nielsen classes $\mathbb{A} \subset \mathrm{Fix}(f^k)$. Since by the induction assumption $f^l(x) \neq x$ ($l|n$, $l < k$, $x \in X$), all orbits of points in $\mathrm{Fix}(f^k)$ have the length k. Since $N(f^k) = 0$, $\mathrm{ind}(f^k; \mathbb{A}) = 0$ hence $\mathbb{A}$ splits into finite pairs of orbits $\{x_1, \dots, x_k\}$, $\{y_1, \dots, y_k\}$ such that there is a path $\omega\colon [-1,1] \to X$ establishing the Nielsen relation between x_1, y_1 and $\mathrm{ind}(f^k; x_1) + \mathrm{ind}(f^k; y_1) = 0$.

Thus the induction step will be done once we prove that the orbits $\{x_1, \dots, x_k\}$, $\{y_1, \dots, y_k\}$ can be removed by a homotopy which is constant in a neighbourhood of the other fixed points and which does not produce new fixed points. In other words it remains to show the following.

(5.2.3) Theorem (Cancelling Procedure). *Let* $f\colon X \to X$ *be a map with* $\mathrm{Fix}\,(f^k)$ *finite* ($\dim X \geq 3$)*. Assume that*

(5.2.3.1) $\{x_0, \dots, x_{k-1}\}$, $\{y_0, \dots, y_{k-1}\}$ *are disjoint orbits of length* k *which are Nielsen related, i.e. there is a path* $\omega\colon [-1,1] \to X$ *from* $f(-1) = x_0$ *to* $f(1) = y_0$ *such that* $f^k\omega$ *and* ω *are fixed end point homotopic.*

(5.2.3.2) *Near each point in* $\{x_0, \dots, x_{k-1}; y_0, \dots, y_{k-1}\}$*,* f *is a* PL*-homeomorphism.*

(5.2.3.3) $\mathrm{ind}(f^k; x_0) + \mathrm{ind}(f^k; y_0) = 0$.

Then there is a homotopy $\{f_t\}$ *starting from* $f_0 = f$ *constant in a neighbourhood of* $\mathrm{Fix}(f^k) \setminus \{x_0, \dots, x_{k-1}; y_0, \dots, y_{k-1}\}$ *and satisfying*

$$\mathrm{Fix}(f_1^k) = \mathrm{Fix}(f^k) \setminus \{x_0, \dots, x_{k-1}; y_0, \dots, y_{k-1}\}.$$

The rest of this section is the proof of the above Cancelling Procedure. Here is the outline of the proof.

Some technical lemmas allow us to make ω and their images $f\omega, \dots, f^{k-1}\omega$ flat arcs and f the homeomorphism in neighbourhoods of these arcs (more exactly in neighbourhoods of $f^i(\omega[-1,0))$ and $f^i(\omega(0,1])$ for $i = 0, \dots, k-2$).

If it moreover happens that $f^k\omega$ is close to ω then the use of a modified Hopf lemma allows us to remove the two orbits. Thus the main difficulty in the general case is to make $f^k\omega$ close to ω without adding new periodic points.

5.2.1. How to control the periodic points during a homotopy. In all deformations $\{f_t\}$ we have to leave the periodic point set $\mathrm{Fix}(f_t^k)$ unchanged or to control its changes. We will use (often implicitly) three methods.

(1) When the support of the homotopy

$$\mathrm{Supp}\{f_t\} = \{x \in X : f_t(x) \neq f(x) \text{ for a } t \in [0,1]\}$$

is isolated from $\mathrm{Fix}(f_0^k)$ and the homotopy $\{f_t\}$ is sufficiently small:

(5.2.4) LEMMA. *Let (M,d) be a metric space, $f\colon M \to M$ a continuous map, $C \subset M$ a compact subset disjoint from $\mathrm{Fix}\,(f^n)$. Then there exists an $\varepsilon > 0$ such that, for any map $g\colon M \to M$ satisfying $g(x) = f(x)$ for $x \notin C$ and $d(f(x), g(x)) < \varepsilon$ for $x \in C$, the equality $\mathrm{Fix}\,(g^n) = \mathrm{Fix}\,(f^n)$ holds.*

PROOF. It is enough to notice that $d(x, f^n(x)) > 0$, for $x \in C$ (compact set), implies

$$\inf\{d(x, f^n(x)) : x \in C\} > 0. \qquad \square$$

(2) When the homotopy does not send its carrier back to itself:

(5.2.5) LEMMA. *Suppose that there exist sets $A_0, \dots, A_k \subset X$ such that $f(A_i) \subset A_{i+1}$ $(i = 0, \dots, k-1)$ and moreover $A_1 \cup \dots \cup A_k$ is disjoint from A_0. Suppose that $\mathrm{Supp}\{f_t\} \subset A_0$, $f_0 = f$ and $f_t(A_0) \subset A_1$. Then $\mathrm{Fix}(f_0^k) = \mathrm{Fix}(f_1^k)$.*

PROOF. Since $A_1 \cup \dots \cup A_k$ is disjoint from $\mathrm{Supp}\{f_t\} \subset A_0$, $f_t^k(a) = f^{k-1}(f_t(a)) \in f^{k-1}A_1 \subset A_k$ for any $a \in A_0$. This implies $f_t^k(a) \neq a$ for any point from the support of the homotopy. Thus the k-periodic points of both maps lie outside the carrier of the homotopy, hence they must be equal. $\square$

(3) If we have to change f near the periodic points we will need a formula to control the set of periodic points. Since the majority of homotopies $\{f_t\}$ will be constant on $\mathrm{Fix}(f_0^k)$ we will use the following

(5.2.6) LEMMA. *Let $f_t\colon X \to X$ be a homotopy constant on $\mathrm{Fix}(f_0^k)$. Then*

$$\mathrm{Fix}(f_1^k) = \mathrm{Fix}(f_0^k) \cup \{\textit{orbits of } f_1^k \textit{ cutting } \mathrm{Supp}\{f_t\}\}.$$

PROOF. $\supset$ is obvious since $f_1^k(a) = f_0^k(a) = a$ for any $a \in \mathrm{Fix}(f_0^k)$.

$\subset$ Let $\{x_0, \dots, x_{k-1}\}$ be an orbit in $\mathrm{Fix}(f_1^k)$. If the orbit does not cut $\mathrm{Supp}\{f_t\}$, then $f_1(x_i) = f_0(x_i)$ for all $i = 0, \dots, k-1$, hence it is also the orbit in $\mathrm{Fix}(f_0^k)$. $\square$

We will also need a lemma making the inverse images of a neighbourhood of a point small in the homotopy sense explained in the next lemma.

(5.2.7) LEMMA. *Let $f\colon M \to M$ be a self-map of a compact manifold and let the iterations $x_0, fx_0, \dots, f^k x_0$ be different. Then f is homotopic to a map f_1 whose iterations f_1, f_1^2, …, f_1^k are transverse to x_0. Moreover:*

(5.2.7.1) *The homotopy may be arbitrarily small.*

(5.2.7.2) *If the iterations of f are already transverse to x_0 on a closed subset $A \subset M$, then the homotopy may be constant in a neighbourhood of A.*

(5.2.7.3) *The support of the homotopy may be contained in a prescribed neighbourhood of $x_0 \cup f^{-1}(x_0) \cup \cdots \cup f^{-k}(x_0)$.*

(5.2.7.4) *In particular, the homotopy may be constant in a neighbourhood of $\mathrm{Fix}(f^k)$.*

PROOF. We notice that the sets x_0, $f^{-1}(x_0)$, $\ldots$, $f^{-k}(x_0)$ are mutually disjoint. In fact if $0 \leq i < j \leq k$ and $y \in f^{-i}(x_0) \cap f^{-j}(x_0)$, then $x_0 = f^j(y) = f^{j-i}(f^i(y)) = f^{j-i}(x_0)$ which contradicts that the first k elements of the orbit of x_0 are different. We will denote $x_i = f^i(x_0)$. Since the below deformation can be arbitrarily small, we may assume that the values $x_0, \ldots, x_k$ remain different, hence the sets x_0, $f_t^{-1}(x_0)$, $\ldots$, $f_t^{-k}(x_0)$ are disjoint in each moment of this homotopy.

A small deformation constant outside a prescribed neighbourhood of $f^{-1}(x_0)$ makes f transverse to x_0.

Assume that $f, \ldots, f^{l-1}$ are transverse to x_0. We may correct f (after a homotopy with the support in a prescribed neighbourhood of $f^{-l}(x_0)$) to a map transverse to x_0, $f^{-1}(x_0)$, $\ldots$, $f^{-(l-1)}(x_0)$. This is possible since the considered inverse images are mutually disjoint. Now the obtained map satisfies the main claim of the lemma for $k = l$, hence the inductive step is done.

The points (5.2.7.1)–(5.2.7.3) can be satisfied since any correction to a transverse map may be arbitrarily small and its image may be constant outside a prescribed neighbourhood of the corresponding inverse image. $\square$

(5.2.8) COROLLARY. *Let the first k iterations of the map $f\colon M \to M$ be transverse to x_0 and let the sets x_0, $f^{-1}(x_0), \ldots, f^{-k}(x_0)$ be mutually disjoint. Then for a sufficiently small ball neighbourhood $\mathcal{U}_0 \ni x_0$, the sets $\mathcal{U}_0, f^{-1}(\mathcal{U}_0), \ldots, f^{-k}(\mathcal{U}_0)$ are disjoint and each $f^{-i}(\mathcal{U}_0)$ splits into the finite sum of connected components, each of them mapped by f^i homeomorphically onto $\mathcal{U}_0$.*

PROOF. By the compactness of M and the transversality, the sum $f^{-1}(x_0) \cup \cdots \cup f^{-k}(x_0)$ is finite and each of its points $z \in f^{-i}(x_0)$ admits a neighbourhood $\mathcal{U}_z$ which is mapped by f^i homeomorphically onto a neighbourhood of x_0. Thus the finite intersection $\mathcal{V} = \bigcap_{i=0}^{k} \bigcap_{z \in f^{-i}(x_0)} f^i(\mathcal{U}_z)$ is a neighbourhood of x_0. On the other hand x_0 does not belong to the compact set $\bigcup_{i=0}^{k} f^i(M \setminus \bigcup_{z \in f^{-i}(x_0)} \mathcal{U}_z)$, hence there is a ball neighbourhood $\mathcal{U}_0$ ($x_0 \in \mathcal{U}_0 \subset \mathcal{V}$) disjoint from this sum. It remains to prove that $\mathcal{U}_0$ satisfies our corollary. Let $z' \in f^{-i}(\mathcal{U}_0)$. Then $f^i(z') \in \mathcal{U}_0$ hence $f^i(z') \notin f^i(M \setminus \bigcup_{z \in f^{-i}(x_0)} \mathcal{U}_z)$. Now $z' \notin M \setminus \bigcup_{z \in f^{-i}(x_0)} \mathcal{U}_z$ so $z' \in \mathcal{U}_{z''}$ for a $z'' \in f^{-i}(x_0)$. By the above $f^i\colon \mathcal{U}_{z''} \to f^i(\mathcal{U}_{z''})$ is a homeomorphism, $f^i(\mathcal{U}_{z''}) \supset \mathcal{V}$ $_0$ and $\mathcal{U}_0$ is connected. Let $\mathcal{U}_0^{''}$ denote the connected component of $z'' \in f^{-i}(\mathcal{U}_0)$. Then the restriction of f^i to $\mathcal{U}_0'' \subset \mathcal{U}_{z''}$ is a homeomorphism onto $\mathcal{U}_0 \subset f^i(\mathcal{U}_{z''})$. □

(5.2.9) COROLLARY. *Let $f\colon M \to M$ be a map with* $\mathrm{Fix}(f^l)$ *finite and let $x_0, f(x_0), \dots, f^l(x_0)$ be different. Then f is homotopic to a map g and there is a neighbourhood $\mathcal{U}$ $\;x_0$ such that $\mathrm{cl}\,\mathcal{U}, g^{-1}(\mathrm{cl}\,\mathcal{U}), \dots, g^{-l}(\mathrm{cl}\,\mathcal{U})$ is contained inside a finite number of mutually disjoint closed balls in M. Moreover:*

(5.2.9.1) *The homotopy may be arbitrarily small.*

(5.2.9.2) *The support of the homotopy may be contained inside a prescribed neighbourhood of $x_0, f^{-1}(x_0), \dots, f^{-l}(x_0)$.*

(5.2.9.3) $\mathrm{Fix}(f^k)$ *does not change during the homotopy.*

(5.2.9.4) *The Corollary is stable in the following sense: if g satisfies the corollary then so also does any map sufficiently close to g.*

PROOF. Let $\mathcal{U}_0$ be the neighbourhood from Corollary (5.2.8). Then we may take as $\mathcal{U}$ any ball satisfying $x_0 \in \mathcal{U}$ $\;\mathrm{cl}\,\mathcal{U}$ $\;{}_0$. Now the property (5.2.9.4) follows from the compactness of M. □

5.2.2. Making $f^k(\omega)$ close to ω. The crucial step in the proof of the Cancelling Procedure (Theorem (5.2.3)) is the following theorem which makes the path $f^k\omega$ close to ω. This step corresponds to the reduction to the Euclidean case in the proof of the classical Wecken Theorem (Theorem (4.2.1)).

For $\mathcal{V}$ $\;\mathbb{R}^{m-1} \times \mathbb{R}$ we denote

$$\mathcal{V}^+ = \{(x,t) \in \mathcal{V} : t > 0\}, \quad \mathcal{V}^- = \{(x,t) \in \mathcal{V} : t < 0\}, \quad \mathcal{V}^0 = \{(x,t) \in \mathcal{V} : t = 0\}.$$

(5.2.10) THEOREM. *Under the assumption of the Cancelling Procedure* (*Theorem* (5.2.3)) *there exists a homotopy which does not change* $\mathrm{Fix}(f^k)$, *is constant in a neighbourhood of* $\mathrm{Fix}(f^k)$ *and after which the map f satisfies*

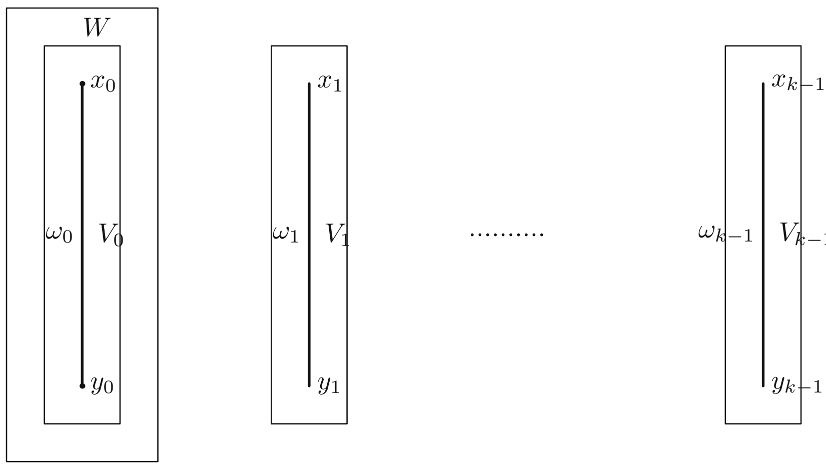
(5.2.10.1) *a path $\omega_0\colon [-1,1] \to M$ establishing the Nielsen relation between $x_0, y_0 \in \mathrm{Fix}(f^k)$ is a* PL-*arc avoiding other periodic points,*

(5.2.10.2) *there exist mutually disjoint Euclidean neighbourhoods $\mathcal{V}_0', \dots, \mathcal{V}_{k-1}'$ such that*

$$\begin{aligned} f^i\omega(t) &= (0,t) \in \mathcal{V}_i' = \mathbb{R}^{m-1} \times \mathbb{R}, \quad \textit{for } i = 0, \dots, k-1, \\ f(\mathcal{V}_i'^{+}) &\subset (\mathcal{V}_{i+1}'^{+}), \\ f(\mathcal{V}_i'^{-}) &\subset (\mathcal{V}_{i+1}'^{-}), \\ f((\mathcal{V}_i')^0) &= 0 \in \mathcal{V}_{i+1}' = \mathbb{R}^m, \qquad\qquad \textit{for } i = 0, \dots, k-2, \end{aligned}$$

(5.2.10.3) *the restriction of f to $\mathcal{V}_i' \setminus (\mathcal{V}_i')^0$ is a homeomorphism on its image, $i = 0, \dots, k-2$.*

(5.2.10.4) *there exists a Euclidean neighbourhood* $\mathcal{W}$ *such that* $\mathcal{V}'_0 \subset \mathcal{W}$, $f(\mathcal{V}'_{k-1}) \subset \mathcal{W}$ *and the restriction of* f^{k-1} *to* $\mathcal{W} \setminus \mathcal{W}^0$ *is a homeomorphism.*



The proof of Theorem (5.2.10) will be given after Lemma (5.2.16) and will follow from a sequence of lemmas.

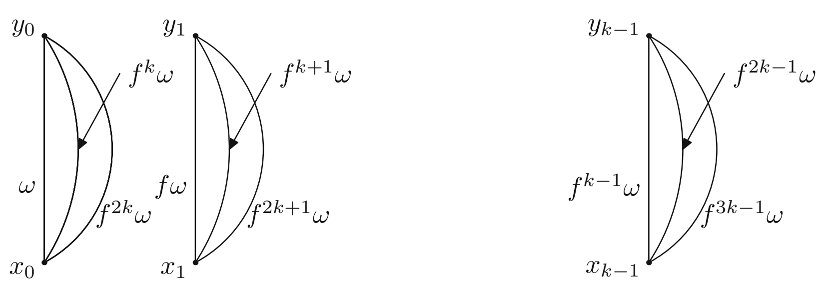
(5.2.11) LEMMA. *Under the assumption of the Cancelling Procedure, there is a homotopy* $\{f_t\}$ *starting from* $f_0 = f$ *and an arc* $\omega_0: [-1,1] \to M$, *from* $\omega_0(-1) = x_0$ *to* $\omega_0(+1) = y_0$, *satisfying* $f^k\omega_0 \sim \omega_0$ *and:*

(5.2.11.1) $\omega_0, \omega_1 = f\omega_0, \dots, \omega_S = f^S\omega_0$ $(S = 2k-1)$ *are* PL-*arcs whose interiors,* $\{\omega_i(-1,1)\}$ *are mutually disjoint and disjoint from* $\mathrm{Fix}(f^k)$.

(5.2.11.2) $\{f_t\}$ *is constant in a prescribed neighbourhood of* $\mathrm{Fix}(f^k)$.

(5.2.11.3) $\{f_t\}$ *can be arbitrarily small.*

(5.2.11.4) $\mathrm{Fix}(f_1^k) = \mathrm{Fix}(f^k)$.



PROOF. Let us concentrate on proving (5.2.11.1).

Since $\mathrm{Fix}(f^k)$ is finite, ω_0 may be chosen a PL-arc such that $\omega_0(t) \notin \mathrm{Fix}(f^k)$ for $-1 < t < +1$. Since f is a linear homeomorphism in neighbourhoods of x_0 and y_0, there exists an $\varepsilon > 0$ such that $f\omega_0[-1,-1+\varepsilon]$, $f\omega_0[1-\varepsilon,1]$ are segments in the corresponding Euclidean neighbourhoods. Since $\dim M \geq 3$, there exists a small homotopy (relatively ends) on $\omega_0[-1+\varepsilon, 1-\varepsilon]$ after which the path $f(\omega_0[-1+\varepsilon, 1-\varepsilon])$ becomes also a PL-arc and moreover the homotopy may be extended on the whole M by a homotopy with the carrier in a prescribed neighbourhood of $\omega_0[-1+\varepsilon, 1-\varepsilon]$. Thus we may assume that $\omega_1 = f(\omega_0)$ is also an arc and is disjoint from $\omega_0[-1, 1]$ (Lemma (5.2.4)).

We repeat the above construction to the arc ω_1 and we get that ω_2 is also an arc disjoint from ω_0 and ω_1. Thus we may assume that $\omega_0, \omega_1, \ldots, \omega_{k-1}$ are mutually disjoint arcs. Let us recall that f is still a PL-homeomorphism in neighbourhoods of the points $\{x_0, \ldots, x_{k-1}; y_0, \ldots, y_{k-1}\}$. We may continue this procedure to make the arcs $f^k[-1+\varepsilon, 1-\varepsilon], \ldots, f^S[-1+\varepsilon, 1-\varepsilon]$ mutually disjoint. Since f is a PL-homeomorphism near the points $\{x_i; y_j\}$ and $\varepsilon > 0$ may be arbitrarily small, we may assume that $i \neq j$ and $f^i\omega_0(t) = f^j\omega_0(s)$ imply $k|(j-i)$ and $t = s = \pm 1$ for $i, j = 0, \ldots, S = 2k-1$ (see the above figure).

It remains to notice that all the above deformations have the carrier isolated from $\mathrm{Fix}(f^k)$. Thus we have (5.2.11.2). On the other hand these deformations may be arbitrarily small. Now (5.2.11.3) implies (5.2.11.4) – see Lemma (5.2.4).□

Let us denote $z_i = \omega_i(0)$ for $i = 0, \ldots, k-1$.

(5.2.12) LEMMA. *Under the assumptions of the Cancelling Procedure, there is a homotopy $\{f_t\}$ starting from $f_0 = f$ satisfying:*

(5.2.12.1) *$\{f_t\}$ is constant in a prescribed neighbourhood of* $\mathrm{Fix}(f^k)$.
(5.2.12.2) *$\{f_t\}$ can be arbitrarily small.*
(5.2.12.3) *there exists a neighbourhood $\mathcal{U}_{k-1} \ni z_{k-1}$ such that the sum*

$$\mathrm{cl}\,\mathcal{U}_{k-1} \cup f_1^{-1}(\mathrm{cl}\,\mathcal{U}_{k-1}) \cup \cdots \cup f_1^{-(k-1)}(\mathrm{cl}\,\mathcal{U}_{k-1})$$

is contained inside a finite sum of mutually disjoint m-balls.
(5.2.12.4) *The interiors of the arcs ω_i are mutually disjoint for $i = 0, \ldots, 2k-1$.*

PROOF. We may assume that $f = f_1$ from Lemma (5.2.11). Since ω_i are mutually disjoint arcs, we may choose mutually disjoint Euclidean neighbourhoods $\mathcal{V}_i$ where $\omega_i(t) = (0, t) \in \mathcal{V}_i = \mathbb{R}^{m-1} \times \mathbb{R}$ $(i = 0, \ldots, k-1)$. Since the points $z_i = \omega_i(0)$ $(i = 0, \ldots, k-1)$ do not belong to $\mathrm{Fix}(f^k)$, we may deform f in small neighbourhoods $\mathcal{U}_i$ $(z_i \in \mathcal{U}_i \subset \mathcal{V}_i = \mathbb{R}^m)$ so that f takes there the form

$$\mathcal{U}_i \ni (x, t) \to (x, t) \in \mathcal{V}_{i+1}$$

for $(x, t) \in \mathbb{R}^{m-1} \times \mathbb{R}$ and $i = 0, \ldots, k-2$.

Now the lemma follows from Corollary (5.2.9) (for $x_0 = z_{k-1}$ and $l = k-1$).□

(5.2.13) LEMMA. *Under the assumptions of the Cancelling Procedure, there is a homotopy* $\{f_t\}$ *starting from* $f_0 = f$ *and satisfying:*

(5.2.13.1) *There exist mutually disjoint Euclidean neighbourhoods* $\mathcal{V}_0, \dots, \mathcal{V}_{k-1} \subset M$ *where* $\omega_i(t) = (0,t) \in \mathcal{V}_i = \mathbb{R}^{m-1} \times \mathbb{R}$ *for* $-1 \le t \le +1$ *and* $i = 0, \dots, k-1$.

(5.2.13.2) $f_1(\mathcal{V}_i^+) \subset \mathcal{V}_{i+1}^+$, $f_1(\mathcal{V}_i^-) \subset \mathcal{V}_{i+1}^-$, $f_1(\mathcal{V}_i^0) = z_{i+1} \in \mathcal{V}_{i+1}^0$, $i = 0, \dots, k-2$.

(5.2.13.3) *The restriction of* f_1 *to* $\mathcal{V}_i \setminus \mathcal{V}_i^0$ *is a homeomorphism on the image.*

(5.2.13.4) $\{f_t\}$ *is constant in a prescribed neighbourhood of* $\operatorname{Fix}(f^k)$.

(5.2.13.5) $\{f_t\}$ *can be arbitrarily small.*

(5.2.13.6) $\operatorname{Fix}(f_t^k)$ *does not depend on* t.

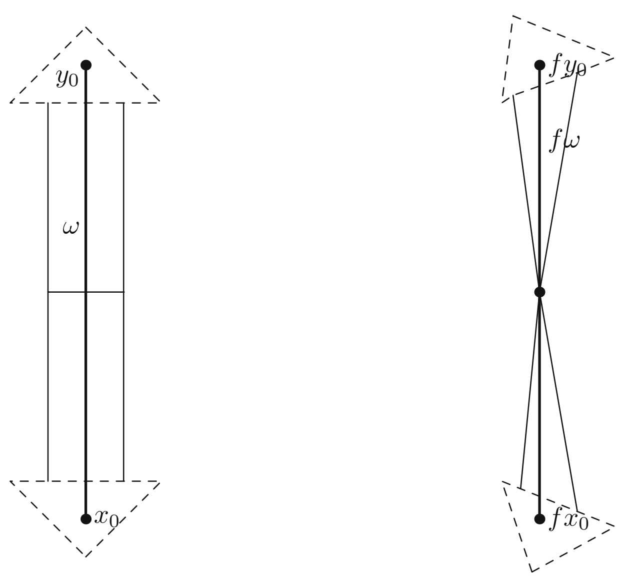
(5.2.13.7) *There exists a neighbourhood* $\mathcal{U}_{k-1} \ni z_{k-1}$ *such that the sum*

$$\operatorname{cl}\mathcal{U}_{k-1} \cup f_1^{-1}(\operatorname{cl}\mathcal{U}_{k-1}) \cup \dots \cup f_1^{-(k-1)}(\operatorname{cl}\mathcal{U}_{k-1})$$

is contained inside a finite sum of mutually disjoint m*-balls.*

PROOF. We may assume that $f = f_1$ from Lemma (5.2.12). We will correct f to make it a homeomorphism near $\omega_i[-1,0)$ and $\omega_i(0,1]$ for $i = 0, \dots, k-2$. We fix disjoint Euclidean neighbourhoods of ω_0 and $\omega_1 = f\omega_0$ where $\omega_0(t) = (0,t) \in \mathbb{R}^{m-1} \times \mathbb{R}$, $f\omega_0(t) = (0,t) \in \mathbb{R}^{m-1} \times \mathbb{R}$, respectively.

We fix m-simplices σ_x, σ_y containing x_0 and y_0 respectively on which f is a homeomorphism. Moreover, we assume that $(m-1)$-dimensional faces $\sigma_{-1+\varepsilon} \subset \sigma_x$, $\sigma_{1-\varepsilon} \subset \sigma_y$ belong to the hyperplanes $x_m = -1+\varepsilon$ and $x_m = 1-\varepsilon$, respectively.



Let $p' \colon \mathbb{R}^m \to \mathbb{R}^{m-1}$, $p'' \colon \mathbb{R}^m \to \mathbb{R}$ denote the projections $p'(x,t) = x$, $p''(x,t) = t$ for $(x,t) \in \mathbb{R}^{m-1} \times \mathbb{R} = \mathbb{R}^m$. Let σ_0 be an $(m-1)$-simplex contained in $p'(\sigma_{-1+\varepsilon}) \cap$

$p'(\sigma_{1-\varepsilon})$ and let $0 \in \mathrm{int}\sigma_0$. If σ_0 is chosen small enough, then $p''(f(x,-1+\varepsilon)) < 0 < p''(f(x,1-\varepsilon))$ for $x \in \sigma_0$.

We define the map $f'\colon \sigma_0 \times [-1+\varepsilon, 1-\varepsilon] \to \mathbb{R}^m$ by the formula

$$f'(x,t) = \begin{cases} \dfrac{t}{1-\varepsilon} f(x, 1-\varepsilon) & \text{for } 0 \le t \le 1-\varepsilon, \\ \dfrac{t}{-1+\varepsilon} f(x,-1+\varepsilon) & \text{for } -1+\varepsilon \le t \le 0. \end{cases}$$

Then the restrictions of f' to $\sigma_0 \times [-1+\varepsilon, 0)$ and to $\sigma_0 \times (0, 1-\varepsilon]$ are homeomorphisms. Let us fix a number $\eta > 0$. If the simplex σ_0 is small enough, then $d(f'(x,t), f(x,t)) < \eta$ for $(x,t) \in \sigma_0 \times [-1+\varepsilon,\, 1-\varepsilon]$. Moreover, the homotopy from f to f' (by segments) is still η-homotopy and admits an extension onto M which is constant outside a prescribed neighbourhood of $\sigma_0 \times [-1+\varepsilon, 1-\varepsilon]$. Thus we may assume that after this homotopy no new periodic point (of minimal period $l \le k, l|n$) appears (Lemma (5.2.4)).

On the other hand $f'(x, 1-\varepsilon) = f(x, 1-\varepsilon)$, $f'(x,-1+\varepsilon) = f(x,-1+\varepsilon)$ for $x \in \sigma_0$ so we may assume that the homotopy is constant on σ_x, σ_y. For a sufficiently small number $\varepsilon' > 0$; $\sigma_0 \times [1-\varepsilon', 1+\varepsilon'] \subset \sigma_y$, $\sigma_0 \times [-1-\varepsilon', -1+\varepsilon'] \subset \sigma_x$ hence the restriction of f to $\sigma_0 \times (0, 1+\varepsilon']$ and to $\sigma_0 \times [-1-\varepsilon', 0)$ is a homeomorphism. Finally $\mathcal{V}_0 = \mathrm{int}\sigma_0 \times (-1-\varepsilon', 1+\varepsilon')$ is the desired Euclidean neighbourhood.

In general we proceed as follows: we fix a Euclidean neighbourhood $\mathcal{V}_{k-1}$ of ω_{k-1} (in which $\omega_{k-1}(t) = (0,t)$). The above construction gives a neighbourhood $\mathcal{V}_{k-2} \supset \omega_{k-2}$ such that $f\colon \mathcal{V}_{k-2} \setminus \mathcal{V}^0_{k-2} \to \mathcal{V}_{k-1}$ is the homeomorphism on the image. Then we choose $\mathcal{V}_{k-3}$ and so on. The obtained map satisfies the points (5.2.13.1) and (5.2.13.2) of the lemma. The points (5.2.13.3) and (5.2.13.4) follow from the construction of $\mathcal{V}_i$. Since these sets may be arbitrarily thin, (5.2.13.5) is satisfied, which in turn implies (5.2.13.6). Since the homotopy may be small, (5.2.13.7) follows from (5.2.9.4) in Lemma (5.2.9). □

For a fixed number $\varepsilon > 0$ we put $\mathcal{V}^\varepsilon_{k-1} = \{(x,t) : -1+\varepsilon \le t \le 1-\varepsilon\}$.

(5.2.14) Remark. Since $\omega_k, \dots, \omega_{2k-1}$ are mutually disjoint, $f(\mathcal{V}^\varepsilon_{k-1}), \dots, f^k(\mathcal{V}^\varepsilon_{k-1})$ are also mutually disjoint for a sufficiently thin $\mathcal{V}_{k-1}$.

The next lemma will be used later for $a = 1-\varepsilon$.

(5.2.15) Lemma. *Let $f\colon M \to M$ be a self-map of a compact* PL*-manifold of dimension ≥ 3. We assume that:*

(5.2.15.1) $\mathrm{Fix}(f^k)$ *is finite.*

(5.2.15.2) $\omega_0\colon [-a, a] \to M$ *is a* PL*-arc such that $\omega_i = f^i\omega_0$, for $i = 0, \dots, 2k-1$, are mutually disjoint arcs, all disjoint from* $\mathrm{Fix}(f^k)$.

(5.2.15.3) $\mathcal{V}_0 \subset M$ *is a Euclidean neighbourhood such that $\omega_0(t) = (0,t) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathcal{V}_0$.*

(5.2.15.4) *$\omega_k(-a), \omega_k(+a) \in \mathcal{V}_0$ and ω_k is homotopic (in M, relatively end points) to a path lying in $\mathcal{V}_0$.*

(5.2.15.5) *There exists a neighbourhood $\mathcal{U}_{k-1} \ni z_{k-1} = \omega_{k-1}(0)$ such that the sum*

$$\mathrm{cl}\,\mathcal{U}_{k-1} \cup f_1^{-1}(\mathrm{cl}\,\mathcal{U}_{k-1}) \cup \cdots \cup f_1^{-(k-1)}(\mathrm{cl}\,\mathcal{U}_{k-1})$$

is contained inside a finite sum of mutually disjoint m-balls.

Then there is a partial homotopy $h_s\colon \omega_{k-1}[-a,a] \to M$ $(0 \le s \le 2)$ satisfying:

(5.2.15.6) *$h_0 = f_{|\omega_{k-1}}$, h_2 is a path in $\mathcal{V}_0 \setminus \omega[-a,a]$.*

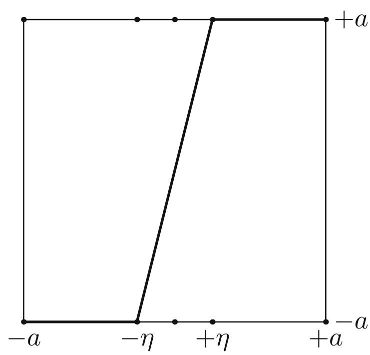
(5.2.15.7) *h_s is constant at the ends $\omega_{k-1}(-a)$ and $\omega_{k-1}(-a)$.*

(5.2.15.8) *$f^i h_s \omega_{k-1}[-a,a]$ is disjoint from $\omega_{k-1}[-a,+a]$ for $i = 0,\dots,k-1$.*

(5.2.15.9) *In particular $f^i h_s(x) \neq x$ for $x \in \omega_{k-1}[-a,+a]$ for $0 \le s \le 2$, $i = 0,\dots,k-1$.*

Proof. The partial homotopy $\{h_s\}$ will be obtained in two steps: reparametrization (for $0 \le t \le 1$) and the contraction of the path $f\omega_{k-1}$ to $\mathcal{V}_0$ (for $1 \le t \le 2$).

(1) Let $\eta > 0$ be so small that $0 \times [-\eta,\eta] \subset \mathcal{U}_{k-1} \subset \mathbb{R}^{m-1} \times \mathbb{R} = \mathcal{V}_{k-1}$ ($\mathcal{U}_{k-1}$ is the neighbourhood from assumption (5.2.15.5)). Let $r\colon [-a,a] \to [-a,a]$ be the map given by the figure below and let $r_t\colon \omega_{k-1}[-a,a] \to \omega_{k-1}[-a,a]$ be the segment homotopy from $r_0 = \mathrm{id}_{\omega_{k-1}[-a,a]}$ to $r_1(\omega_{k-1}(t)) = \omega_{k-1}(r(t))$ (reparametrization of the path ω_{k-1}).

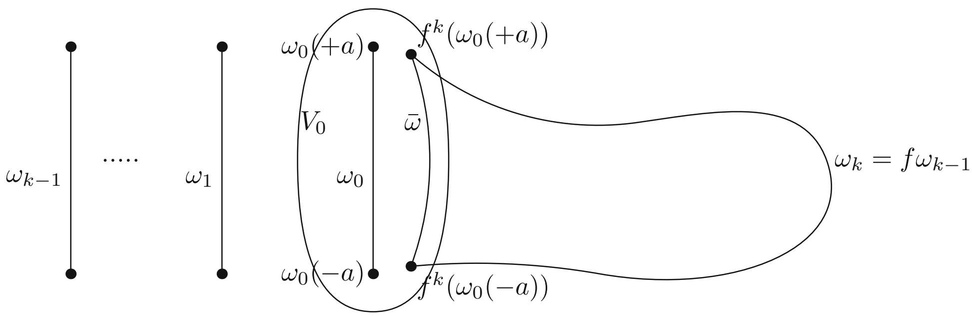


We define the partial homotopy $h_t\colon \omega_{k-1}[-a,a] \to M$ putting

$$h_t(x) = fr_t(x) \quad \text{for } x \in \omega_{k-1}[-a,a].$$

Since $f^i h_s \omega_{k-1}(x) \in \omega_{k+i}[-a,+a]$ and the arcs ω_j are mutually disjoint for $j = 0,\dots,2k-1$, $f^i h_s \omega_{k-1}(x) \neq x$ for $x \in \omega_{k-1}[-a,a]$, $i = 0,\dots,k-1$. Notice that $r_1(\omega_{k-1}[-a,-\eta])$, $r_1(\omega_{k-1}[\eta,a])$ are points, hence so are $h_1(\omega_{k-1}[-a,-\eta])$, $h_1(\omega_{k-1}[\eta,a])$.

(2) The homotopy h_s (for $1 \leq s \leq 2$) will be constant on $\omega_{k-1}[-a,-\eta]$ and on $\omega_{k-1}[\eta, a]$. We are going to define this homotopy on $\omega_{k-1}[-\eta,\eta]$. Let us fix a path $\overline{\omega}\colon [-a,+a] \to \mathcal{V}_0 \setminus \omega_0[-a,a]$ from $\omega_k(-a)$ to $\omega_k(+a)$.



Since $f\omega_{k-1}$ is homotopic relatively ends to a path $\overline{\omega}\colon \omega_{k-1}[-a,a] \to \mathcal{V}_0$, there is a homotopy between the maps $h_1, \overline{\omega}r_1$ considered as the maps from $\omega_{k-1}[-\eta,\eta]$ to M. We will show that the last homotopy may avoid the set

$$\mathrm{cl}\,\mathcal{U}_{k-1} \cup f_1^{-1}(\mathrm{cl}\,\mathcal{U}_{k-1}) \cup \cdots \cup f_1^{-(k-1)}(\mathrm{cl}\,\mathcal{U}_{k-1}).$$

In fact by the assumption (5.2.15.5) the sum is contained in a finite number of mutually disjoint balls $\bigcup_\alpha K_\alpha$. Since $\dim M \geq 3$,

$$\pi_2\Big(M, M \setminus \bigcup_\alpha K_\alpha\Big) = \pi_2(M, M \setminus \text{finite set}) = 0,$$

so the image of the (two-dimensional homotopy) $h_s\colon \omega_{k-1}[-\eta,+\eta] \times [1,2] \to M$ may be deformed (relatively boundary of $[-\eta,\eta] \times [1,2]$) outside $\bigcup_\alpha K_\alpha$.

Thus we may assume that $h_s(\omega_{k-1}(t)) \notin \bigcup_{i=0}^{k-1} f^{-i}(\mathrm{cl}\,\mathcal{U}_{k-1})$. Since $\omega_{k-1}[-\eta,+\eta] \subset \mathcal{U}$ we get (5.2.15.8) (for $x \in \omega_{k-1}[-\eta,+\eta]$). We extend the homotopy h_t onto $\omega_{k-1}[-a,a]$ by the constant homotopy outside $\omega_{k-1}[-\eta,+\eta]$ and $1 \leq t \leq 2$. Now the homotopy $\{h_t\}$ for $0 \leq t \leq 2$ satisfies the lemma. □

(5.2.16) Lemma. *The partial homotopy from Lemma* (5.2.15) *can be extended to* $f_t\colon M \to M$ *where*

(5.2.16.1) *the carrier of* $\{f_t\}$ *is contained in* $D \times [-a,a] \subset \mathcal{V}_{k-1} = \mathbb{R}^{m-1} \times \mathbb{R}$ *where* D *is any prescribed neighbourhood of* $0 \in \mathbb{R}^{m-1}$,

(5.2.16.2) $\mathrm{Fix}(f_1^k) = \mathrm{Fix}(f^k)$.

Proof. Let $D \subset \mathbb{R}^{m-1}$ be a closed ball centered in 0. Take an arbitrary extension f'_t of the partial homotopy $\{h_t\}$ on M. We consider the metric space $X = M \setminus \{\omega_{k-1}(-a), \omega_{k-1}(a)\}$. Then the sets $X \setminus D \times [-a,a]$ and $0 \times (-a,a)$ are disjoint closed subsets of X. Let $\lambda\colon X \to [0,1]$ be an Urysohn function satisfying $\lambda(X \setminus D \times [-a,a]) = 0$, $\lambda(0 \times (-a,a)) = 1$.

Then map $f_t\colon M \to M$

$$f_t(x) = \begin{cases} f'_{\lambda(x)t}(x) & \text{for } x \neq \omega_{k-1}(\pm a), \\ f(x) & \text{for } x = \omega_{k-1}(\pm a), \end{cases}$$

gives a homotopy satisfying (5.2.16.1) (f_t is continuous at the points $\omega_{k-1}(-a)$ and $\omega_{k-1}(+a)$ since the homotopy f'_t is constant there).

It remains to show that if D is small enough, then $\mathrm{Fix}(f_2^k) = \mathrm{Fix}(f^k)$. Suppose otherwise. Let D_n be a ball of radius $1/n$. Now we have $x_n \in D_n$ and $0 \le t_n \le 2$ such that $f'^k_{t_n}(x_n) = x_n$. The compactness gives subsequences convergent to $x_0 \in 0 \times [-a, a]$ and $0 \le t_0 \le 2$. Then $f^k_{t_0}(x_0) = x_0$ contradicts (5.2.15.9). $\square$

End of the proof of Theorem (5.2.10). We may assume that f satisfies Lemma (5.2.13). Then conditions (5.2.10.1)–(5.2.10.3) are satisfied. We will get (5.2.10.4) changing f only near $\omega_{k-1}[-1+\varepsilon, 1-\varepsilon]$. Let $\varepsilon > 0$ be so small that $f^k(\omega_0([-1,-1+\varepsilon] \cup [1-\varepsilon, 1])) \subset \mathcal{V}_0$. We may apply Lemma (5.2.15) to $a = 1-\varepsilon$ and we get $f^k(\omega_0) \subset \mathcal{V}_0$. Now we put $\mathcal{W} = \mathcal{V}_0$ and we find an open set $\mathcal{V}'_{k-1}$ satisfying $\omega_{k-1} \subset \mathcal{V}'_{k-1} \subset \mathcal{V}_{k-1}$ and $f\mathcal{V}'_{k-1} \subset \mathcal{W}$. Then we find $\mathcal{V}'_{k-2} \subset \mathcal{V}_{k-2}$ satisfying $\omega_{k-2} \subset \mathcal{V}'_{k-2} \subset \mathcal{V}_{k-2}$, $f\mathcal{V}'_{k-2} \subset \mathcal{V}_{k-1}$ and so on until we get $\mathcal{V}'_0 \subset \mathcal{V}_0$.

(5.2.17) Remark. Since the only condition on the path $\overline{\omega}$ in Lemma (5.2.15) was $\overline{\omega} \subset \mathcal{V}_0 \setminus \omega_0[-a, +a]$, we may assume that $\overline{\omega}(0) \in \mathcal{V}_0^-$. Then $f_2(z_{k-1}) = \overline{\omega}(0) \in \mathcal{V}_0^-$.

This ends the first step of the proof of the Cancelling Procedure: reduction to the local situation. $\square$

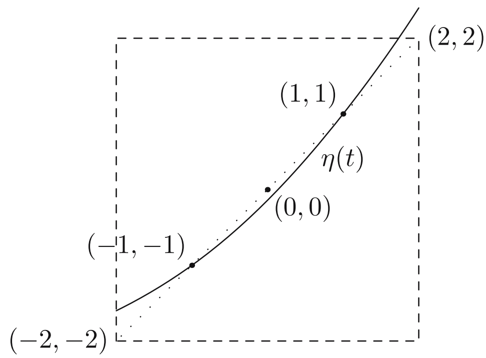
5.2.3. Extension of the partial homotopy. We will show that the map f satisfying Theorem (5.2.10) may be deformed so that f^k is given, near the arc $\omega_0[-1, 1]$, by a prescribed formula. We will do this first locally and then we will need an extension of this local deformation onto the whole M without adding new periodic points.

We may assume that $f\colon M \to M$ satisfies Theorem (5.2.10). We consider the orbits $\{x_0, \dots, x_{k-1}\}$, $\{y_0, \dots, y_{k-1}\}$. Since f is a PL-homeomorphism in a neighbourhood of each periodic point and $\mathrm{ind}(f^k; y_0) + \mathrm{ind}(f^k; x_0) = 0$, we may assume that $\mathrm{ind}(f^k; x_0) = +1$, $\mathrm{ind}(f^k; y_0) = -1$.

Let us denote $P = [-2, +2]^m$ and consider the map: $h\colon P \to \mathbb{R}^m$ given by the formula

$$h(x, t) = \left(\frac{|t|}{3} \cdot x, \eta(t) \right),$$

where $(x, t) \in \mathbb{R}^{m-1} \times \mathbb{R}$ and $\eta\colon [-2, 2] \to \mathbb{R}$ is an increasing function satisfying: $\eta(t) = t$ if and only if $t = \pm 1$, $\eta(-2) > -2$, $\eta(0) < 0$, $\eta(2) > 2$.



Now $\mathrm{Fix}(h) = \{(0,-1),(0,1) \in \mathbb{R}^{m-1} \times \mathbb{R}\}$. We identify $P = [-2,2]^m$ with a subset of $\mathcal{V}_0 = \mathbb{R}^m$. Then $\mathrm{ind}(h, x_0) = +1 = \mathrm{ind}(f^k; x_0)$, $\mathrm{ind}(h, y_0) = -1 = \mathrm{ind}(f^k; y_0)$.

Let us notice that putting $P_0 = [-2,2]^{m-1} \times 0$, both $h(P_0)$, $f^k(P_0)$ are points. We are going to show that f is homotopic to a map such that f^k equals h near the path ω_0.

We start by recalling a classical result. Let us denote $Q = [0,1]^m$ and $Q_0 = \{(x_1, \dots, x_m) \in Q : x_m = 0\}$.

(5.2.18) Theorem (Hopf Lemma). *Let $f_0, f_1 \colon Q \to \mathbb{R}^m$ be maps satisfying $f_0(z) \neq z \neq f_1(z)$ for $z \in \mathrm{bd}\, Q$ and $\mathrm{ind}\,(f_0) = \mathrm{ind}\,(f_1)$. Then there is a homotopy $h \colon Q \times I \to \mathbb{R}^m$ satisfying $h(z,i) = f_i(z)$ for $i = 0,1$, $z \in Q$ and $h(z,t) \neq z$ for $t \in I$, $z \in \mathrm{bd}\, Q$.*

We will need its modification:

(5.2.19) Lemma. *Let us add to the assumptions of the Hopf Lemma (5.2.18) the following three conditions:*

(5.2.19.1) *$f_0(Q_0)$ and $f_1(Q_0)$ are points,*
(5.2.19.2) *$\gamma : [0,1] \to \mathbb{R}^m \setminus Q_0$ is a path from $\gamma(0) = f_0(Q_0)$ to $\gamma(1) = f_1(Q_0)$.*
(5.2.19.3) *$\dim M = m \geq 3$.*

Then the homotopy h in the Hopf Lemma (Theorem (5.2.18)) may be chosen to satisfy $h(z,t) = \gamma(t)$ for $z \in Q_0$, $t \in I$.

Proof. We define the map $H_1 \colon Q \times 0 \cup Q \times 1 \cup Q_0 \times I \to \mathbb{R}^m$ by the formula

$$H_1(z,t) = \begin{cases} z - f_0(z) & \text{for } t = 0, \\ z - f_1(z) & \text{for } t = 1, \\ z - \gamma(t) & \text{for } z \in Q_0. \end{cases}$$

Then $\deg H_1 = \mathrm{ind}(f_0) - \mathrm{ind}(f_1) = 0$, hence there is an extension $H_2 \colon \mathrm{bd}\,(Q \times I) \to \mathbb{R}^m$ such that $H_2^{-1}(0) = H_1^{-1}(0)$. Let $H_3 \colon Q \times I \to \mathbb{R}^m$ be an arbitrary extension of H_2. Now $h(z,t) = z - H_3(z,t)$ is the desired homotopy. $\square$

We will use the following notation: $P = [-2, +2]^m \subset \mathbb{R}^m$ and let

$$\begin{aligned} P_0 &= \{(x_1, \dots, x_m) \in P : x_m = 0\}, & P_+ &= \{(x_1, \dots, x_m) \in P : x_m \geq 0\}, \\ P_- &= \{(x_1, \dots, x_m) \in P : x_m \leq 0\}, & P_{[a,b]} &= \{(x_1, \dots, x_m) \in P : a \leq x_m \leq b\}. \end{aligned}$$

(5.2.20) LEMMA. *Let the maps $f, g \colon P \to \mathbb{R}^m$ satisfy*

(5.2.20.1) $f(z) \neq z \neq g(z)$ *for* $z \in P_0 \cup \operatorname{bd} P$,
(5.2.20.2) $f(P_0)$, $g(P_0)$ *are points,*
(5.2.20.3) $\operatorname{ind}(f; P_+) = \operatorname{ind}(g; P_+)$, $\operatorname{ind}(f; P_-) = \operatorname{ind}(g; P_-)$.

Let $\gamma \colon [0,1] \to (\mathbb{R}^m \setminus P_0)$ be a path from $f(P_0)$ to $g(P_0)$. Then there is a homotopy $h \colon P \times I \to \mathbb{R}^m$ satisfying

(5.2.20.4) $h(z, 0) = f(z)$, $h(z, 1) = g(z)$ *for* $z \in P$,
(5.2.20.5) $h(z, t) \neq z$ *for* $z \in P_0 \cup (\operatorname{bd} P)$,
(5.2.20.6) $h(z, t) = \gamma(t)$ *for* $z \in P_0$.

PROOF. By Lemma (5.2.19) there is a homotopy $h_+ \colon P_+ \times I \to \mathbb{R}^m$ satisfying

(1) $h_+(z, 0) = f(z)$, $h_+(z, 1) = g(z)$ for $z \in P_+$,
(2) $h_+(P_0, t) = \gamma(t)$, $h_+(z, t) \neq z$ for $z \in \operatorname{bd} P_+$.

Then we may define a similar homotopy $h_- \colon P_- \times I \to \mathbb{R}^m$ and $h = h_+ \cup h_-$ will satisfy the lemma. □

The above lemma implies a homotopy $\{h_t\}$ from $h_0 = f^k$ (more precisely its restriction to P) to $h_1 = h$ (given above by the formula at the beginning of the section) such that P_0 is sent into a point and no point from $\operatorname{bd} P \cup P_0$ is fixed in any moment of this homotopy. This induces the partial homotopy $f'_t \colon f^{k-1}(P) \to M$ by the formula

$$f'_t(x) = h_t(y) \quad \text{for } x = f^{k-1}(y),\ y \in P.$$

In other words $f'_t(x) = h_t((f_|^{k-1})^{-1}(x))$ where $f_|^{k-1}$ denotes the restriction $f^{k-1}\colon P \to \mathcal{V}_{k-1} = \mathbb{R}^m$. Since $f_|^{k-1}$ is mono on $P \setminus P_0$ and $h_t(P_0)$ is a point (for any fixed t), the definition is correct. Moreover, we recall that the points $f(z_{k-1}) = f^k(z_0)$ and $h(z_0)$ belong to $\mathcal{V}_0^-$ (see Remark (5.2.17) and recall that $h(0,0) = (0, \eta(0))$ where $\eta(0) < 0$). Thus we may join these two points with a path $\gamma \colon [0,1] \to \mathcal{V}_0^-$ and we may assume that $f'_t(f^{k-1}P_0) = \gamma(t)$.

(5.2.21) LEMMA. *The partial homotopy $f'_t \colon f^{k-1}(P) \to M$ admits an extension $f_t \colon M \to M$ with the carrier contained inside an arbitrarily prescribed neighbourhood of $f^{k-1}(P)$ and such that* $\operatorname{Fix}((f_1)^k) = \operatorname{Fix}((f_0)^k)$.

PROOF. The lemma follows from Lemma (5.2.22) applied for $X = M$, $A = f^{k-1}(P)$. It remains to notice that then $\operatorname{Fix}(f_0^k) \cap f^{k-1}(P) = \operatorname{Fix}(f_1^k) \cap f^{k-1}(P) = \{x_{k-1}, y_{k-1}\}$. □

(5.2.22) LEMMA. *Let X be a compact* ENR *and let $A \subset X$ be its closed subset. Let $f\colon X \to X$ be a self-map and let $f'_t\colon A \to X$ be a partial homotopy satisfying*

(5.2.22.1) $f'_0(a) = f(a)$ *for* $a \in A$,
(5.2.22.2) $f^{l-1}(f'_t(A)) \cap A = \emptyset$ *for* $l = 1, \dots, k-1$,
(5.2.22.3) $f^{k-1}(f'_t(x)) \neq x$ *for* $x \in \operatorname{bd} A$.

Then for arbitrary neighbourhood $\mathcal{U}$ $\quad A$ *there exists a homotopy* $f_t\colon X \to X$ *satisfying*

(5.2.22.4) $f_t(a) = f'_t(a)$ *for* $a \in A$,
(5.2.22.5) $f_0(x) = f(x)$ *for* $x \in X$,
(5.2.22.6) $f_t(x) = f(x)$ *for* $x \notin \mathcal{U}$,
(5.2.22.7) *for any* $t_0 \in [0,1]$ *the orbits of* $\operatorname{Fix}(f^k_{t_0})$ *avoiding* A *coincide with the orbits of* $\operatorname{Fix}(f^k)$ *avoiding* A.

In particular, if $\operatorname{Fix}(f^k_{t_0}) \cap A = \emptyset$, *for a* $t_0 \in [0,1]$, *then*

$$\operatorname{Fix}(f^k_{t_0}) = \operatorname{Fix}(f^k) - \{\text{orbits of } f^k \text{ cutting } A\}.$$

PROOF. Since X is an ENR, the partial homotopy $f'_t\colon A \to A$ with $f'_0 = f$ admits an extension $f'_t\colon X \to X$. Then by the assumption (5.2.22.2) there exists an open neighbourhood $\mathcal{U}'$ such that $A \subset \mathcal{U}' \subset \mathcal{U}$ and $f^{l-1}(f'_t(\operatorname{cl}\mathcal{U}')) \cap \operatorname{cl}\mathcal{U}' = \emptyset$ for $l = 1, \dots, k-1$. Let $\mu\colon X \to [0,1]$ be a Urysohn function satisfying $\mu(A) = 1$, $\mu(M \setminus \mathcal{U}') = 0$. Then the map $f''_t(x) = f'_{\mu(x)t}(x)$ satisfies $(f''_t)^l(\operatorname{cl}\mathcal{U}')) \cap \operatorname{cl}\mathcal{U}' = \emptyset$ for $l = 1, \dots, k-1$ since for any $x \in \operatorname{cl}\mathcal{U}'$, $(f''_t)^l(x) = f^{l-1}(f'_{\mu(x)t})(x)$. Let $C = \{x \in X : (f''_t)^k(x) = x \text{ for a } t \in [0,1]\}$. This is a compact subset disjoint from $\operatorname{bd} A$ hence $C \setminus A$ is also compact. Let $\lambda\colon X \to [0,1]$ be an Urysohn function $\lambda(A) = 1$, $\lambda((M \setminus \mathcal{U}') \cup ((C \setminus A)) = 0$. We put $f_t(x) = f''_{t\lambda(x)}(x)$. Properties (5.2.22.4)–(5.2.22.6) are easy to check. Now we prove (5.2.22.7). Consider an orbit $\mathcal{O} \subset \operatorname{Fix}(f^k_{t_0})$ disjoint from A. Then $\mathcal{O} \subset C \setminus A$ hence $\lambda(x) = 0$ and the homotopy $\{f_t(x)\}$ is constant for any $x \in \mathcal{O}$. Thus $f_{t_0}(x) = f(x)$ and $\mathcal{O}$ turns out to be an orbit of f. Reversing this argument we prove that any orbit of f avoiding A is also the orbit of $\operatorname{Fix}(f^k_1)$ avoiding A. $\square$

5.2.4. End of the proof of the Cancelling Procedure. We may assume that $f\colon M \to M$ satisfies Theorem (5.2.10) and moreover,

$$f^k(x,t) = (|t| \cdot x/3, \eta(t)) \quad \text{for } (x,t) \in P \subset \mathcal{W} = \mathbb{R}^m.$$

(5.2.23) LEMMA. *There is a homotopy* $h_s\colon P \to \mathbb{R}^m$, $0 \le s \le 1$, $P = [-2,2]^m$, *satisfying*

(5.2.23.1) $h_0(x,t) = (|t| \cdot x/3, \eta(t))$,

(5.2.23.2) $h_s(P_0)$ *is a point, for each fixed* s,
(5.2.23.3) $h_s(z) \neq z$ *for* $z \in \operatorname{bd} P$,
(5.2.23.4) $h_s(P_0) \in \operatorname{int} P$,
(5.2.23.5) $h_1(z) \neq z$ *for all* $z \in P$.

PROOF. Let $\eta_s\colon I \to \mathbb{R}$ be a homotopy constant on the ends $\eta(-2) > -2$, $\eta(2) > 2$ from $\eta_0 = \eta$ to a fixed point free map η_1. Then the required homotopy is given by

$$h_s(x,t) = \left(\frac{|t|}{3} \cdot x, \eta_s(t)\right). \qquad \square$$

The homotopy from Lemma (5.2.20) induces a partial homotopy $h'_s\colon f^{k-1}(P) \to M$ by the formula: $h'_s(x) = h_s((f_|^{k-1})^{-1}(x))$. Then the Cancelling Procedure (Theorem (5.2.3)) follows from the application of Lemma (5.2.24) to

$$X = M, \quad A = f^{k-1}(P), \quad A' = f^{k-1}(P_0) = \{z_0\}.$$

It remains to show that the assumptions of Lemma (5.2.24) are satisfied. To see that the above $A' = \{z_0\}$ corresponds to A' in Lemma (5.2.24) we show that $f^{k-1}h'_t(v) \neq v$ for any $v \in \operatorname{bd} f^{k-1}(P) \setminus z_{k-1}$. Let $v \in \operatorname{bd} f^{k-1}(P) \setminus z_{k-1}$. Then $v = f^{k-1}(z)$ for a $z \in \operatorname{bd} P$, $z \notin P_0$. Suppose that $f^{k-1}h'_t(v) = v$. Then $f^{k-1}h'_t(v) = f^{k-1}(z)$. Since the restriction of f^{k-1} to $P \setminus P_0$ is mono, $h'_t(v) = z$. Thus $z = h'_t(v) = h'_t f^{k-1}(z) = h_t(z)$ contradicting (5.2.20.6) Now we are in a position to prove that the assumptions of the next lemma (Lemma (5.2.24)) are satisfied.

(1) This assumption is evident.
(2) Since $h'_t(A) = A'(f^{k-1}(A)) \subset \mathcal{W}$, $f^i(h'(A)) \subset f^i\mathcal{W}$ are disjoint from $\mathcal{W} \supset h'_t(A)$ for $i = 0, \dots, k-2$.
(3) Let $\mathcal{U}' = \operatorname{int} P$. By (5.2.23.4) $h_s(P_0) \subset \mathcal{U}'$, hence $h'_s(z_0) = h_s(P_0) \subset \mathcal{U}'$. This proves (5.2.24.3)(a). At last the equality $f^{k-1}(\operatorname{cl}\mathcal{U}') = f^{k-1}(P) = A$ proves (5.2.24.3)(b).

(5.2.24) LEMMA. *Let* X *be a compact* ENR, $A \subset X$ *be a closed subset,* $k \in \mathbb{N}$, $f\colon X \to X$ *a continuous map,* $h'_t\colon A \to X$ *a partial homotopy and let* $A' = \{x \in \operatorname{bd} A : x = f^{k-1}h'_t(x)$ *for some* $t \in [0,1]\}$. *Moreover, we assume that:*

(5.2.24.1) $h'_0(a) = f(a)$ *for* $a \in A$,
(5.2.24.2) *The sets* $A_0 = \{h'_t(a) : a \in A,\ 0 \leq t \leq 1\}$, $A_1 = f(A_0), \dots, A_{k-2} = f^{k-2}(A_0)$ *are disjoint from* A.
(5.2.24.3) *There exists an open subset* $\mathcal{U}' \subset X$ *satisfying*

(a) $h'_t(x) \in \mathcal{U}'$ *for* $x \in A'$ *and* $0 \leq t \leq 1$,
(b) $f^{k-1}(\operatorname{cl}\mathcal{U}') \subset A$.

Then there exists an extension of the partial homotopy $\{h'_t\}$ *to* $h_t\colon X \to X$ *satisfying*

(5.2.24.4) $h_0 = f$,

(5.2.24.5) *the carrier of the homotopy* $\{h_t\}$ *is contained in an arbitrarily prescribed neighbourhood of* A,

(5.2.24.6) *the set* $\mathrm{Fix}(h_t^k) \setminus$ (*the orbits cutting* A) *does not depend on* $t \in [0, 1]$.

Proof. Let $H'\colon X \times I \to X$ be an arbitrary extension of the partial homotopy h'_t onto the ENR X. We will write $H'(x,t) = h'_t(x)$ for all $x \in X$. Since $H(A' \times I) \subset \mathcal{U}'$, the compactness implies the existence of an open subset $\mathcal{U} \quad X$ containing A' and satisfying $H'(\mathrm{cl}\,\mathcal{U} \quad I) \subset \mathcal{U}'$. Moreover, if $\mathcal{U}$ is sufficiently small then, by assumption (5.2.24.2), the sets

$$B_0 = H'((\mathrm{cl}\,\mathcal{U} \quad A) \times I),\ B_1 = f(B_0), \dots, B_{k-2} = f^{k-2}(B_0)$$

are disjoint from $\mathrm{cl}\,\mathcal{U} \quad A$. Then we have $(h'_t)^k(x) = f^{k-1} h'_t(x)$ for all $x \in \mathrm{cl}\,\mathcal{U}$.

We will show that $(h'_t)^k(x) \neq x$ for all $x \in \mathrm{cl}\,\mathcal{U} \quad A$, $t \in [0,1]$. In fact $x \in \mathrm{cl}\,\mathcal{U} \quad A$ implies $x \in \mathrm{cl}\,\mathcal{U}$, hence $h'_t(x) \in \mathcal{U}'$. Now $(h'_t)^k(x) = f^{k-1} h'_t(x) \in f^{k-1}\mathcal{U}' \subset A$ implies $(h'_t)^k(x) \neq x$ since $x \notin A$.

Now we show that $(h'_t)^k(x) \neq x$ for all $x \in \mathrm{bd}\,(A \cup \mathrm{cl}\,\mathcal{U})$. Suppose the contrary, i.e. $(h'_t)^k(x) = x$. We notice that $\mathrm{bd}\,(A \cup \mathrm{cl}\,\mathcal{U}) \subset \mathrm{bd}\,A \cup \mathrm{bd}\,(\mathrm{cl}\,\mathcal{U})$. First we assume that $x \in \mathrm{bd}\,A$. Then the equality $(h'_t)^k(x) = x$ implies $x \in A'$ so $x \in A' \subset \mathcal{U} \quad \mathrm{int}(A \cup \mathrm{cl}\,\mathcal{U})$, hence $x \notin \mathrm{bd}\,(A \cup \mathrm{cl}\,\mathcal{U})$ contradicting to the assumption. If $x \in \mathrm{bd}\,(\mathrm{cl}\,\mathcal{U}) \setminus \mathrm{bd}\,A$ then $x \in \mathrm{cl}\,\mathcal{U} \quad A$ and $(h'_t)^k(x) \in A$ as above, hence $h'_t)^k(x) \neq x$.

Let $B = \{x \in X \setminus A; (h'_t)^k(x) = x \text{ for a } t \in I\}$. By the above, B is a closed subset of X disjoint from A. Let $\mathcal{V}$ be a neighbourhood of A disjoint from B. Let $\lambda\colon X \to [0,1]$ be an Urysohn function: $\lambda(A) = 1$, $\lambda(X \setminus \mathcal{V}) = 0$. We will show that $h_t(x) = h'_{\lambda(x)t}(x)$ is the desired homotopy. In fact (5.2.24.4) is evident. To see that (5.2.24.5) holds we notice that $h_t(x) = h'_{\lambda(x)t}(x) = h'_0(x) = f(x)$ for all $x \notin \mathcal{V}$. It remains to show (5.2.24.6). Let us fix $t \in [0,1]$ and consider an orbit of h_t avoiding A. This orbit must belong to B. Now $h_t(x) = h'_{\lambda(x)t}(x) = h'_0(x) = f(x)$ for any x from the orbit, which implies that this is also the orbit of f. The same arguments show that each orbit of $f = h_0$ avoiding A belongs to B and is the orbit of h_t which gives (5.2.24.6). □

This ends the proof of the Cancelling Procedure (Theorem (5.2.3)) which implies the Weak Wecken's Theorem for periodic points – see the scheme of the proof.

5.3. Wecken's Theorem for periodic points

The aim of this section is to prove that $NF_n(f)$ is the best lower bound of the number of periodic points for the self-maps of compact PL-manifolds of dimension not equal 2. It is called the Wecken theorem for periodic points and was first stated by Benjamin Halpern in [Ha2] with a hypothesis that $\dim X \geq 5$. Since that time it was also called a Halpern conjecture. The Wecken theorem was proved in [Je2] under the assumption that $\dim X \geq 4$, and next in [Je5] for $\dim X \geq 3$. Here we present the proof from [Je5] whose idea follows the proof of the (fixed point) Wecken theorem in [Ji3].

First we discuss the case dim = 1. Then there are only two compact manifolds: the interval and the circle. Any self-map of the interval is homotopic to the constant map which has exactly one periodic point.

Let $f\colon S^1 \to S^1$ be a self-map of degree d.

If $d \neq \pm 1$ then the map $f(z) = z^d$ realizes $NF_n(f) = N(f^n) = |d^n - 1|$.

If $d = 1$ then the twist by an irrational angle has no periodic points.

If $d = -1$ then again the map $f(z) = z^d$ realizes $NF_n(f) = N(f^n) = |d^n - 1|$ for $n \neq -1$. In the exceptional case $n = -1$ we denote the point $e^{\pi\phi i}$ by $[\phi]$ where $-1 \leq \phi \leq +1$. Then the map $f[\phi] = [-\phi^3]$ realizes $NF_n(f) = 2$.

Thus it remains to show

(5.3.1) Theorem (Wecken Theorem for periodic points). *If M is a* PL-*manifold of dimension ≥ 3, then each self-map $f\colon M \to M$ is homotopic to a map g satisfying*

$$\#\mathrm{Fix}(g^n) = NF_n(f).$$

5.3.1. Simply-connected case. To make the proof more transparent we assume at first that the manifold M is simply-connected. Then the Nielsen theory is trivial and Theorem (5.3.1) becomes

(5.3.2) Theorem. *Any self-map $f\colon M \to M$ of a compact connected simply-connected* PL-*manifold of dimension ≥ 3 is homotopic to a map g such that*

$$\mathrm{Fix}(g^n) = \begin{cases} \emptyset & \textit{if } L(f^k) = 0 \textit{ for all } k|n, \\ \textit{a point} & \textit{otherwise.} \end{cases}$$

The proof is based on the following corollary being a consequence of the Cancelling Procedure.

(5.3.3) Corollary. *Let M be a simply-connected* PL-*manifold of dimension ≥ 3 and let moreover*

(5.3.3.1) *B be a finite subset of $\mathrm{Fix}(f^n)$ which is f-invariant, i.e. $f(B) = B$,*
(5.3.3.2) *$\mathrm{Fix}(f^s) \setminus B$ be compact,*
(5.3.3.3) *$\mathrm{ind}(f^s; \mathrm{Fix}(f^s) \setminus B) = 0$ for any $s|n$.*

Then there is a homotopy $\{f_t\}$ *starting from* $f_0 = f$ *constant in a neighbourhood of* B *and* $\mathrm{Fix}(f_1^n) = B$.

Proof. Induction on the divisors of n. For any $k|n$ we prove:

$$f \textit{ is homotopic, relatively a neighbourhood of } B, \textit{ to a map } g_k \textit{ such that}$$
$$\mathrm{Fix}(g_k^l) \subset B \quad \text{for all } l|n,\ l \leq k.$$

Then $f_1 = g_n$ satisfies the corollary.

For $k = 1$. Since B is isolated from $\mathrm{Fix}(f^n) \setminus B$, B is also isolated from $\mathrm{Fix}(f) \setminus B$. After a small homotopy constant near B we may assume that $\mathrm{Fix}(f)$ is finite and $\mathrm{ind}(f; a) = \pm 1$ at each $a \in \mathrm{Fix}(f) \setminus B$. Now $\mathrm{ind}(f; \mathrm{Fix}(f) \setminus B) = 0$ means that $\mathrm{Fix}(f) \setminus B$ splits into the pairs of points with opposite indices. Since the manifold M is simply-connected, any two paths are homotopic, hence we apply the Cancelling Procedure to these pairs successively and we remove them all. The obtained map satisfies $\mathrm{Fix}(g) \subset B$.

Now we assume that $\mathrm{Fix}(f^l) \subset B$ for all $l|n$, $l < k$. We will remove $\mathrm{Fix}(f^k) \setminus B$. We notice that $\mathrm{Fix}(f^k) \setminus B$ contains only orbits of length k, indices ± 1 and we may assume that their number is finite (after a small local deformation if necessary). Since $\mathrm{ind}(f^k; \mathrm{Fix}(f^k) \setminus B) = 0$, these orbits split into pairs with opposite signs and we may apply the Cancelling Procedure to remove them all. This ends the inductive step. □

Next we use a result of [BaBo]. Let i_k be a sequence of integers satisfying Dold congruences ($\sum_{k|m} \mu(m/k) i_k \cong 0 \mod m$ for every $m \in \mathbb{N}$). If $d = \dim M \geq 2$, then there exists a map $\phi\colon \mathcal{U} \quad \mathbb{R}^d \to \mathbb{R}^d$ with 0 as an isolated periodic point and such that $\mathrm{ind}(\phi^m; 0) = i_m$ for each $m \in \mathbb{N}$. Moreover, if $d \geq 3$, then the map ϕ can be a homeomorphism from $\mathcal{U}$ onto $\phi(\mathcal{U})$.

Let $f\colon M \to M$ be a self-map of a manifold and let $\mathcal{A} \in \mathcal{OR}(f^n)$ be an orbit of Reidemeister classes. Then the numbers $i_k = \mathrm{ind}(f^k; \mathcal{A})$ satisfy Dold congruences for $k|n$. In fact, for the proof that $\mathrm{Fix}(f^k)$ is a Dold sequence, we replace f by a map with the finite $\mathrm{Fix}(f^n)$ and we show that the congruences hold for each orbit of periodic points. It remains to notice that now $\mathcal{A}$ is a finite sum of such orbits.

Proof of Theorem (5.3.2). Let $L(f^k) = 0$ for all $k|n$. Then putting $B = \emptyset$ in Corollary (5.3.3) we get a homotopy from f to a map g with $\mathrm{Fix}(g^n) = \emptyset$.

In the general case we make $\mathrm{Fix}(f^n)$ finite, we fix a point $x_0 \notin \mathrm{Fix}(f^n)$ and we consider the sequence $\alpha_k = \mathrm{ind}(f^k)$. Since the Dold congruences are satisfied, we may deform f near x_0 to a map f' such that $f'(x_0) = x_0$, x_0 is an isolated periodic point of f' and $\mathrm{ind}({f'}^s; x_0) = \alpha_s$ for all $s|n$. The carrier of the homotopy may be contained in any prescribed neighbourhood of x_0, hence it may be disjoint from the compact set $\mathrm{Fix}(f^n)$. Since the index is homotopy invariant, new periodic points appear. But then (for any $s|n$)

$$\begin{aligned}\alpha_s &= \mathrm{ind}(f^s) = \mathrm{ind}(f'^s) = \mathrm{ind}(f'^s; x_0) + \mathrm{ind}(f'^s; \mathrm{Fix}(f'^s) \setminus x_0) \\ &= \alpha_s + \mathrm{ind}(f'^s; \mathrm{Fix}(f'^s) \setminus x_0)\end{aligned}$$

which implies $\mathrm{ind}(f'^s; \mathrm{Fix}(f'^s) \setminus x_0) = 0$ for all $s|n$. Now we are in a position to apply Corollary (5.3.3) to $f = f'$, $B = \{x_0\}$. This yields a homotopy from f' to a map f_1 satisfying $\mathrm{Fix}(f_1^n) = B = \{x_0\}$. □

In the next section we will show the corresponding theorems in the non simply-connected case. The idea of the generalization is to apply the above to each orbit of Nielsen classes of f^k.

5.3.2. Non-simply-connected case.

(5.3.4) Lemma. *Let $f\colon M \to M$, $\dim M = m \geq 3$ and let n be a fixed natural number. We fix a minimal preceding system of Reidemeister classes and we denote it* MPS. *Suppose that f satisfies: each orbit of periodic points $\{x_0, \ldots, x_{l-1}\}$ (of length $l|n$) represents an orbit of Reidemeister classes from* MPS *of depth l which contains only the points $\{x_0, \ldots, x_{l-1}\}$. Then*

$$\#\mathrm{Fix}(f^n) = NF_n(f).$$

Proof. Since the inequality $\geq$ is the basic property of the number $NF_n(f)$, it remains to show $\leq$. The above condition gives an injection from the set of orbits of points into the orbits in MPS. Now

$$\#\mathrm{Fix}(f^n) = \sum_A \#A \leq \sum_B \mathrm{d}(B) = NF_n(f)$$

where A runs through the set of all orbits of points in $\mathrm{Fix}(f^n)$ and B runs through MPS. □

Now it remains to show that any self-map is homotopic to a map satisfying the assumptions of Lemma (5.3.4).

The next procedure makes a non-periodic point a periodic one (of index zero) in an arbitrarily prescribed Nielsen class.

(5.3.5) Theorem (Addition Procedure). *Given numbers $k, n \in \mathbb{N}$, $k|n$, a map $f\colon M \to M$ such that $\mathrm{Fix}(f^n)$ is finite and a point $x_0 \notin \mathrm{Fix}(f^n)$. Let moreover, $\dim M \geq 3$. Then there is a homotopy $\{f_t\}_{0 \leq t \leq 2}$ satisfying*

(5.3.5.1) $f_0 = f$,

(5.3.5.2) $\{f_t\}$ *is constant in a neighbourhood of* $\mathrm{Fix}(f^n)$,

(5.3.5.3) $f_2^k(x_0) = x_0$ *and* $f_2^i(x_0) \neq x_0$ *for* $i = 1, \dots, k-1$,

(5.3.5.4) $\mathrm{Fix}(f_2^n) = \mathrm{Fix}(f^n) \cup \{x_0, \dots, x_{k-1}\}$,

(5.3.5.5) *the points* $\{x_0, \dots, x_{k-1}\}$ *represent an arbitrarily prescribed orbit of Reidemeister classes in* $\mathcal{OR}(f_1)$.

PROOF. First we notice that $x_0 \notin \mathrm{Fix}(f^n)$ implies that after a small deformation the points $x_0, f(x_0), \dots, f^{2n}(x_0)$ are different. In fact $f(x_0) \neq x_0$, hence after a small change (if necessary) near $x_1 = f(x_0)$ we get $f(x_1) \neq x_0$ and $f(x_1) \neq x_1$. We may continue until we get the points $x_0, \dots, x_{2n}$ different. The deformations are small and their carrier is disjoint from $\mathrm{Fix}(f^n)$, hence no new periodic points appear.

Since the points $x_0, x_1, \dots, x_{n-1}$ do not belong to $\mathrm{Fix}(f^n)$, we may deform f near them to make f a local homeomorphism there. The deformation may be arbitrarily small, hence $\mathrm{Fix}(f^n)$ does not change. Let $\mathcal{V}_0 = \mathbb{R}^m$ be a Euclidean neighbourhood of x_0 where $f, f^2, \dots, f^{n-1}$ are homeomorphisms. We define $\mathcal{V}_i = f^i(\mathcal{V}_0)$ $(i = 1, \dots, 2n)$. For $i = 1, \dots, k-1$ we get Euclidean neighbourhoods $\mathcal{V}_i \ni x_i$ and if $\mathcal{V}_0$ is chosen small enough, the sets $\mathcal{V}_0, \dots, \mathcal{V}_{2n}$ are mutually disjoint. We may assume that the restriction of f near x_i is given in the coordinates by

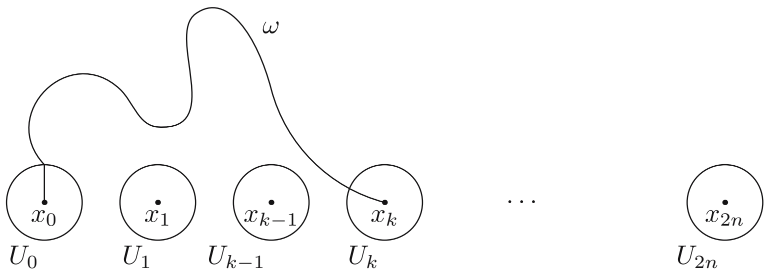
$$\mathcal{V}_i = \mathbb{R}^m \ni x \to x \in \mathbb{R}^m = \mathcal{V}_{i+1} \quad \text{for } i = 0, \dots, k-2.$$

We correct f so that $f, f^2, \dots, f^{n-1}$ become transverse to x_{k-1}. The deformation may be arbitrarily small and constant in a neighbourhood of $\mathrm{Fix}(f^n)$ and the points $x_0, \dots, x_{2n}$ still remain different.

Let us fix a ball neighbourhood $\mathcal{U}_0 = B(x_0, r_0) \subset \mathcal{V}_0 = \mathbb{R}^m$. We will denote $\mathcal{U}_i = f^i(\mathcal{U}_0) \subset \mathcal{V}_i$. By the above convention $\mathcal{U}_i = B(x_i, r_0) \subset \mathcal{V}_i = \mathbb{R}^m$ for $i = 1, \dots, k-1$. If $r_0 > 0$ is chosen small enough, then $\mathcal{U}_{k-1}$ is also small and $\mathrm{cl}\,\mathcal{U}_{k-1} \cup f^{-1}(\mathrm{cl}\,\mathcal{U}_{k-1}) \cup \dots \cup f^{-(k-1)}(\mathrm{cl}\,\mathcal{U}_{k-1})$ is a finite number of mutually disjoint balls in M: see Corollary (5.2.9). Let us fix a number $r_1 \in (0, r_0)$.

Let $\omega\colon [0, r_1] \to M$ be a path satisfying

(1) $\omega(0) = x_0$, $\omega(r_1) = x_k$,
(2) ω avoids $(\mathrm{cl}\,\mathcal{U}_{k-1} \cup f^{-1}(\mathrm{cl}\,\mathcal{U}_{k-1}) \cup \dots \cup f^{-(n-1)}(\mathrm{cl}\,\mathcal{U}_{k-1})) \setminus \mathrm{cl}\,\mathcal{U}_0$,
(3) $\omega(t) = (\lambda t, 0) \in \mathrm{cl}\,\mathcal{U}_0 \subset \mathbb{R} \times \mathbb{R}^{n-1}$ for $0 \le t \le \varepsilon$, where $\lambda > 1$ and $\varepsilon > 0$ satisfy $r_1 > \varepsilon = r_0/\lambda > 0$,
(4) $\omega(t) \notin \mathrm{cl}\,\mathcal{U}_0$ for $\varepsilon < t \le r_1$.



We define the map $f_1\colon M \to M$ deforming f only inside $\mathcal{U}_{k-1} = B(0, r_0) \subset \mathbb{R}^m$:

$$f_1(x) = \begin{cases} f(x) & \text{for } x \notin \mathcal{U}_{k-1} = \operatorname{cl} B(0, r_0), \\ f\left(\dfrac{\|x\| - r_1}{r_0 - r_1} \cdot x\right) & \text{for } r_1 \le \|x\| \le r_0, \\ \omega(\|x\|) & \text{for } \|x\| \le r_1. \end{cases}$$

We notice that f_1 is homotopic to f by a homotopy constant outside $\mathcal{U}_{k-1}$, hence $\operatorname{Fix}(f^n) \cap \mathcal{U}_{k-1} = \emptyset$ implies $\operatorname{Fix}(f^n) \subset \operatorname{Fix}(f_1^n)$. We will show that $\operatorname{Fix}(f_1^n) \cap B(0, r_0) = \{0\}$ ($= x_{k-1}$ a point of pure period k).

Assume that $x = f_1^n(x)$ for $r_1 \le \|x\| \le r_0$, i.e. x belongs to the annulus $A(0; r_1, r_0) = \{x \in \mathbb{R}^n : r_1 \le \|x\| \le r_0\}$. Now $x \in f_1^n(A(0; r_1, r_0)) = f^{n-1}(f(\operatorname{cl} B(0, r_0))) = f^n(\operatorname{cl} B(0, r_0)) = \operatorname{cl} \mathcal{U}_{n+k-1}$. This contradicts the assumption $x \in \mathcal{U}_{k-1}$, since $\operatorname{cl} \mathcal{U}_{k-1} \cap \operatorname{cl} \mathcal{U}_{n+k-1} = \emptyset$.

Now we assume that $0 \le \|x\| \le r_1/\lambda^{n/k}$ (in $\mathcal{U}_{k-1}$). Then $\|x\| \le r_1$, so $f_1(x) = \omega(\|x\|) = (\lambda\|x\|, 0)$ (in $\mathcal{U}_0$), hence $\|f_1(x)\| = \lambda\|x\|$. Since the restriction $f^{k-1}\colon \mathcal{U}_0 \to \mathcal{U}_{k-1}$ is an isometry in coordinates, $\|f_1^k(x)\| = \lambda\|x\| \le r_1/\lambda^{n/k-1} \le r_1$ (in $\mathcal{U}_{k-1}$). We may repeat the above and get $\|f_1^{2k}(x)\| = \lambda^2\|x\|, \dots, \|f_1^n(x)\| = \lambda^{n/k}\|x\|$. Since $\lambda > 1$, $f_1^n(x) = x$ if nd only if $x = 0 \in \mathcal{U}_{k-1}$ in this case.

Now let $r_1/\lambda^{n/k} < \|x\| \le r_1$ in $\mathcal{U}_{k-1}$.

Then the inclusion $f_1^{ik}(x) \in B(0, r_1) \subset \mathcal{U}_{k-1}$ does not hold for all $i = 1, \dots, n/k$. Let i be the smallest number for which the inclusion does not hold. Then we have one or other of two cases:

Case 1. $f_1^{ik}(x) \in A(0, r_1, r_0) \subset \mathcal{U}_{k-1}$, hence

$$f_1^{ik+1}(x) = f_1(f_1^{ki}(x)) \subset f_1(A(0, r_1, r_0)) = f(\operatorname{cl} \mathcal{U}_{k-1}) = \operatorname{cl} \mathcal{U}_k.$$

Then $f_1^{ik+2}(x) \in f(\operatorname{cl} \mathcal{U}_k) = \operatorname{cl} \mathcal{U}_{k+1}$ and so on. But the sets $\mathcal{U}_k, \mathcal{U}_{k+1}, \dots, \mathcal{U}_{2n}$ are disjoint from $\mathcal{U}_{k-1}$, hence $f_1^n(x) \ne x \in \mathcal{U}_{k-1}$.

Case 2. $f_1^{ik}(x) \in \omega[\varepsilon, r_1]$. Since i was the smallest number for which the inclusion did not take place, $f_1^{(i-1)k}(x) = (\lambda^{i-1}\|x\|, 0) \in B(0, r_1) \subset \mathcal{U}_{k-1}$ while

$f_1^{(i-1)k+1}(x) = \omega(\lambda^{i-1}\|x\|, 0) = (\lambda^i\|x\|, 0) \notin B(0, r_1) \subset \mathcal{U}_0$. We consider two subcases:

- if $f_1^{(i-1)k+1}(x) \in A(0, r_1, r_0) \subset \mathcal{U}_0$, then $f_1^{ik}(x) \in A(0, r_1, r_0) \subset \mathcal{U}_{k-1}$, hence $f_1^{ik+1}(x) \in f(A(0, r_1, r_0)) \subset \mathrm{cl}\,\mathcal{U}_k$ and successively as above $f_1^{ik+2}(x) \in \mathrm{cl}\,\mathcal{U}_{k+1}$ and so on. But these sets are disjoint from $\mathrm{cl}\,\mathcal{U}_{k-1} \ni x$;
- if $f_1^{(i-1)k+1}(x) \notin A(0, r_1, r_0) \subset \mathcal{U}_0$. Then it must belong to $\omega[\varepsilon, r_1] = \omega[0, r_1] \setminus \mathcal{U}_0$. But the last set is disjoint from $(\mathcal{U}_{k-1} \cup f^{-1}(\mathrm{cl}\,\mathcal{U}_{k-1}) \cup \cdots \cup f^{-(n-1)}(\mathrm{cl}\,\mathcal{U}_{k-1}))$, hence no iteration of this point comes back to $\mathcal{U}_{k-1} \ni x$.

Thus the map f_1 satisfies (5.3.5.1)–(5.3.5.4). To prove (5.3.5.5) we notice that the path $\omega\colon [0, r_1] \to M$ may represent any homotopy class (we require only that it avoids the finite sum of mutually disjoint balls $\mathrm{cl}\,\mathcal{U}_{k-1} \cup f^{-1}(\mathrm{cl}\,\mathcal{U}_{k-1}) \cup \cdots \cup f^{-(k-1)}(\mathrm{cl}\,\mathcal{U}_{k-1}))$, hence the obtained new periodic point may represent any prescribed Reidemeister class in $\mathcal{R}(f_1)$. □

The next procedure will be crucial in the proof of Wecken's Theorem. We will use it to create the desired periodic points and then the Cancelling Procedure will remove the remaining periodic points.

(5.3.6) Theorem (Creating Procedure). *Given numbers $k, n \in \mathbb{N}$, $k|n$ a map $f\colon M \to M$ such that* $\mathrm{Fix}(f^n)$ *is finite and a point $x_0 \notin \mathrm{Fix}(f^n)$, and moreover* $\dim M \geq 3$, *then there is a homotopy $\{f_t\}_{0 \leq t \leq 1}$ satisfying:*

(5.3.6.1) $f_0 = f$.

(5.3.6.2) *$\{f_t\}$ is constant in a neighbourhood of* $\mathrm{Fix}(f^n)$.

(5.3.6.3) $f_1^k(x_0) = x_0$ *and* $f_1^i(x_0) \neq x_0$ *for* $i = 1, \ldots, k-1$.

(5.3.6.4) *The map f_1^k is given near x_0 by an arbitrarily prescribed formula ϕ satisfying there $\phi(x) = x$ if and only if $x = x_0$.*

(5.3.6.5) *The orbit $x_0, f_1(x_0), \ldots, f_1^{k-1}(x_0)$ is isolated in* $\mathrm{Fix}(f_1^n)$.

(5.3.6.6) *All points in* $\mathrm{Fix}(f_1^n) \setminus \mathrm{Fix}(f^n)$ *represent the same, arbitrarily prescribed, orbit of Reidemeister classes in $\mathcal{OR}(f_1^n)$.*

Proof. Let $f = f_2$ in Theorem (5.3.5). Then (5.3.6.1), (5.3.6.3) hold. Let us denote $x_i = f^i(x_0)$. We fix neighbourhoods $\mathcal{U}_i \ni f^i(x_0)$ satisfying $\mathrm{cl}\,\mathcal{U}_i \cap \mathrm{cl}\,\mathcal{U}_j = \emptyset$ for $0 \leq i \neq j \leq k-1$. We notice that for a sufficiently small deformation f_1 of f, with the carrier in a sufficiently small neighbourhood $\mathcal{V}$, we have: $x_{k-1} \in \mathcal{V}$ $\quad {}_{k-1}$ $f_1^i(\mathrm{cl}\,V) \subset U_{(i+k-1) \bmod k}$, $f_1^i(x_0) = x_i$, f_1 is a homeomorphism near x_i, $(i = 0, \ldots, n)$, hence $\mathrm{Fix}(f_1^n) \subset \mathrm{Fix}(f^n) \cup U_0 \cup \cdots \cup U_{k-1}$. This implies (5.3.6.6). Now a small deformation near x_{k-1} makes $f^k(x) = \phi(x)$ near x_0 for a prescribed ϕ. In fact we denote by $\mathcal{W} \ni x_0$ a neighbourhood such that $f^{k-1}\colon \mathcal{W} \to f^{k-1}(\mathcal{W})$ is a homeomorphism and $\mathrm{cl} f^{k-1}(\mathcal{W}) \subset \mathcal{V}$. We define

$$f_1(x) = \begin{cases} \phi(f^{-(k-1)}(x)) & \text{for } x \in \mathrm{cl} f^{k-1}(\mathcal{W}), \\ f(x) & \text{outside } \mathcal{V}, \end{cases}$$

and we extend it arbitrarily onto $\mathrm{cl}\,\mathcal{V} \quad f^{k-1}(\mathcal{W})$ (by a small deformation). The obtained extension satisfies (5.3.6.4). □

(5.3.7) REMARK. Suppose that we are given integers $\{\alpha_s \in \mathbb{Z}\}$, satisfying Dold congruences, where s runs the set of all divisors of n/k. Then we may assume in the above Creating Procedure that $\mathrm{ind}(f_1^{sk}; x_0) = \alpha_s$. In fact: let $\phi\colon \mathbb{R}^m \to \mathbb{R}^m$ be a self-map having 0 as an isolated periodic point with $\mathrm{ind}(\phi^s; 0) = \alpha_s$ for every $s|(n/k)$. By (5.3.6.5) we may assume that $f_1^k = \phi$ near $x_0 = 0$. Now $\mathrm{ind}(f_1^{sk}; x_0) = \mathrm{ind}(\phi^s; 0) = \alpha_s$.

(5.3.8) COROLLARY. *Suppose that*

(5.3.8.1) $B \subset \mathrm{Fix}(f^n)$ *is a finite f-invariant subset* (*i.e.* $f(B) = B$),

(5.3.8.2) $\mathrm{Fix}(f^n) \setminus B$ *is compact,*

(5.3.8.3) $\mathrm{ind}(f^s; C \setminus B) = 0$ *for any $s|n$ and any orbit of Nielsen classes* $C \subset \mathrm{Fix}(f^s)$ (*or equivalently for any Nielsen class* $C \subset \mathrm{Fix}(f^s)$).

Then there is a homotopy $\{f_t\}$ starting from $f_0 = f$ constant in a neighbourhood of B and such that $\mathrm{Fix}(f_1^n) = B$.

PROOF. We prove by induction for all divisors k of n the following claim: *there is a homotopy, constant near B, from f to a map g such that* $\mathrm{Fix}(g^l) \subset B$ *for all divisors l of n, $l \le k$.*

We notice that since $B \subset \mathrm{Fix}(f^n)$ is f-invariant, then so also is $\mathrm{Fix}(f^n) \setminus B$.

Let $k = 1$. Then $\mathrm{ind}(f; C \setminus B) = 0$ for each Nielsen class $C \subset \mathrm{Fix}(f)$, hence after a small homotopy we may deform f to make $\mathrm{Fix}(f) \setminus B$ finite and $\mathrm{ind}(f; a) = \pm 1$ for each $a \in \mathrm{Fix}(f) \setminus B$. Since $\mathrm{ind}(f; C \setminus B) = 0$, $C \setminus B$ splits into a finite number of pairs of points with opposite signs. Since the points of each such pair belong to the same Nielsen class, we may use the Cancelling Procedure (for $k = 1$, i.e. the Whitney trick) to remove such a pair. Following this to all the Nielsen classes we make $\mathrm{Fix}(f) \subset B$.

Now we assume that our claim holds for all divisors of n which are less than k. We will deform f by a homotopy constant near B to make $\mathrm{Fix}(f^k) \subset B$. First we notice that $\mathrm{Fix}(f^k) \setminus B$ is the sum of orbits of length k and is isolated from B. We may deform f near $\mathrm{Fix}(f^k) \setminus B$ and we may assume that this set splits into a finite number of orbits of points each of length k. Now we consider an orbit of Nielsen classes $C \subset \mathrm{Fix}(f^k)$. Since $\mathrm{ind}(f; C \setminus B) = 0$, $C \setminus B$ splits into pairs of orbits of length k with opposite signs, we may apply the Cancelling Procedure to remove successively each such pair getting $\mathrm{Fix}(f^k) \subset B$. The Cancelling Procedure also guarantees that during these deformations $\mathrm{Fix}(f^l)$ (for $l < k$, $l|n$) do not vary, hence still $\mathrm{Fix}(f^l) \subset B$ for all $l < k$, $l|n$. This ends the inductive step. □

PROOF OF THEOREM (5.3.1). It remains to show that $f\colon M \to M$ is homotopic to a map satisfying the assumptions of Lemma (5.3.4). We prove inductively for all the divisors k of n:

f is homotopic to a map satisfying the assumption of Lemma (5.3.4) *for all $l|n$, $l \leq k$.*

For $k = 1$. By Wecken's Theorem f is homotopic to a map g where each essential class contains exactly one fixed point and inessential classes are empty. We notice that essential classes in $\mathcal{R}(f)$ belong to MPS. If moreover an inessential class belongs to MPS, then we apply the Creating Procedure (for $k = n = 1$) to add a fixed point (of zero index) to this class.

Now we assume that our claim holds for all divisors of n which are less than k. We will build a homotopy from f to a map satisfying the claim also for k. For each orbit of Reidemeister classes $\mathcal{A} \in \mathcal{OR}(f^k)$ we will either remove $\mathcal{A} \cap P_k(f)$ (if $\mathcal{A} \notin$ MPS) or we will reduce $\mathcal{A}$ to exactly k points (if $\mathcal{A} \in$ MPS). During these deformations we will not change $\bigcup_{l|n,\ l<k} \mathrm{Fix}(f^l)$; more precisely we may exchange an orbit of points representing an element in MPS preceding $\mathcal{A}$ with another orbit of points having the same number of elements. After performing the deformations corresponding to all orbits in $\mathcal{OR}(f^k)$ we will get the map g satisfying the claim for k. This will end the inductive step.

We consider an orbit $\mathcal{A} \in \mathcal{OR}(f^k)$.

Let $\mathcal{A}$ be irreducible. Then the compact set $\mathcal{A}$ is contained in $P_k(f)$, hence $\mathcal{A}$ is disjoint and isolated from the finite sum $\bigcup_{l|n,\ l<k} \mathrm{Fix}(f^l)$.

(1) Let $\mathcal{A}$ be irreducible and essential. Then $\mathcal{A} \in$ MPS. The Creating Procedure (for $k = n$) gives a local deformation $\{f_t\}$ from $f_0 = f$ to a map f_1 where the orbit $\mathcal{A}' \in \mathcal{R}(f_1^k)$ corresponding to $\mathcal{A}$ contains an isolated orbit of points $x_0, \dots, x_{k-1}$ satisfying $\mathrm{ind}(f_1; \{x_0, \dots, x_{k-1}\}) = \mathrm{ind}(f; \mathcal{A})$. Then $\mathrm{ind}(f_1; \mathcal{A}' \setminus \{x_0, \dots, x_{k-1}\}) = 0$, hence the Cancelling Procedure allows us to remove $\mathcal{A}' \setminus \{x_0, \dots, x_{k-1}\}$ reducing the orbit $\mathcal{A}'$ to the unique orbit of points $\{x_0, \dots, x_{k-1}\}$.

(2) Let $\mathcal{A}$ be irreducible and inessential. Then as above $\mathcal{A} \subset P_k(f)$. Moreover, $\mathrm{ind}(f^k; \mathcal{A}) = 0$ and $f(\mathcal{A}) = \mathcal{A}$, hence the Corollary (5.3.8) allows us to remove the orbit $\mathcal{A}$. If moreover $\mathcal{A} \in$ MPS then Theorem (5.3.5) adds an orbit of points $\{x_0, \dots, x_{k-1}\}$ with $\mathrm{ind}(f^k; \{x_0, \dots, x_{k-1}\}) = 0$.

Now we assume that the orbit $\mathcal{A} \in \mathcal{OR}(f^k)$ is reducible.

(3) Let $\mathcal{A} \in \mathcal{OR}(f^k)$ be reducible to an orbit in MPS. Let $\mathcal{B}_1, \dots, \mathcal{B}_h$ denote all orbits of Reidemeister classes in MPS preceding $\mathcal{A}$. Then $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_h \subset \mathcal{A}$ as the

Nielsen classes. By the inductive assumption each $\mathcal{B}_i$ consists of exactly one orbit of points of length $= \operatorname{depth}(\mathcal{B}_i) < k$. Moreover, $A_0 = \mathcal{A} \setminus (\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_h)$ splits into k-orbits. Let us fix one of these classes: say $\mathcal{B}_1 = \{b_1, \ldots, b_l\}$, $l|k$, $l < k$.

We consider a finite sequence of numbers $\alpha_s = 1/l \cdot \operatorname{ind}(f^{sl}; A_0 \cup \mathcal{B}_1)$ where s is a divisor of k/l. We will show that this sequence satisfies Dold congruences. We compare it with the sequence $\alpha'_s = 1/l \cdot \operatorname{ind}(f^{sl}; \mathcal{B}_1)$ which is a Dold sequence since

$$\alpha'_s = 1/l \cdot \operatorname{ind}(f^{sl}; \{b_1, \ldots, b_l\}) = \operatorname{ind}(f^{sl}; b_1)$$

and $b_1 \in \operatorname{Fix}(f^l)$. We notice that for $s < k/l$,

$$\alpha_s = 1/l \cdot \operatorname{ind}(f^{sl}; A_0 \cup \mathcal{B}_1) = 1/l \cdot \operatorname{ind}(f^{sl}; \mathcal{B}_1) = \alpha'_s,$$

since A_0 contains only orbits of length k. It remains to notice that k/l divides $\alpha_{k/l} - \alpha'_{k/l}$. In fact

$$\alpha_{k/l} - \alpha'_{k/l} = 1/l \cdot \operatorname{ind}(f^{sl}; A_0)$$

and k divides $\operatorname{ind}(f^k; A_0)$.

Since α_s is a Dold sequence, we may deform f to a map g by a homotopy constant near $\operatorname{Fix}(f^k)$, with an isolated orbit $\{b'_1, \ldots, b'_l\}$ satisfying $\operatorname{ind}(g^{ls}; b'_1) = \alpha_s$, (hence $\operatorname{ind}(g^{ls}; \{b'_1, \ldots, b'_l\}) = l \cdot \alpha_s$). Then $\operatorname{Fix}(g^k)$ splits into a disjoint sum

$$\operatorname{Fix}(g^k) = \operatorname{Fix}(f^k) \cup \{b'_1, \ldots, b'_l\} \cup B'$$

where $\operatorname{ind}(g^{ls}; B') = -l \cdot \alpha_s$. Moreover, we may assume that the points in $\{b'_1, \ldots, b'_l\} \cup B'$ represent orbits reducing to $\mathcal{B}_1$.

Consider a subset $D = (A_0 \cup \mathcal{B}_1) \cup B' \subset \operatorname{Fix}(g^k)$. This set is g invariant ($g(D) = D$) and all its points represent orbits reducing to $\mathcal{B}_1 \in \mathcal{R}(g^l)$. Thus $D \cap \operatorname{Fix}(g^r) = \emptyset$ for r not divisible by l and

$$\begin{aligned} \operatorname{ind}(g^{ls}; D) &= \operatorname{ind}(g^{ls}; A_0 \cup B_1) + \operatorname{ind}(g^{ls}; B') \\ &= \operatorname{ind}(f^{ls}; A_0 \cup B_1) + \operatorname{ind}(g^{ls}; B') = l \cdot \alpha_s - l \cdot \alpha_s = 0 \end{aligned}$$

(to get the second equality we notice that the homotopy between f and g is constant near $\operatorname{Fix}(f^k) \supset A_0 \cup B_1$).

Now we may apply Corollary (5.3.8) for $B = \operatorname{Fix}(g^k) \setminus D$ and we get a homotopy constant near $\operatorname{Fix}(g^k) \setminus D$ from g to a map h satisfying $\operatorname{Fix}(h^k) = \operatorname{Fix}(g^k) \setminus D$. Finally

$$\begin{aligned} \operatorname{Fix}(h^k) = \operatorname{Fix}(g^k) \setminus D &= (\operatorname{Fix}(f^k) \cup \{b'_1, \ldots, b'_l\} \cup B') \setminus (\{b_1, \ldots, b_l\} \cup A_0 \cup B') \\ &= (\operatorname{Fix}(f^k) \setminus (A_0 \cup \{b_1, \ldots, b_l\})) \cup \{b'_1, \ldots, b'_l\}. \end{aligned}$$

In other words we removed from $\mathrm{Fix}(f^k)$ all orbits of points of length k in $\mathcal{A} \in \mathcal{R}(f^k)$ and we replaced the orbit $\{b_1, \dots, b_l\}$ with $\{b'_1, \dots, b'_l\}$.

(4) Now we assume that $\mathcal{A}$ is reducible but there is no class in MPS preceding $\mathcal{A}$. Then $\mathcal{A}$ is inessential and moreover each orbit $\mathcal{B} \in \mathcal{OR}(f^l)$ preceding $\mathcal{A}$ is inessential. Now we apply Corollary (5.3.8) to $B = \mathrm{Fix}(f^k) \setminus \mathcal{A}$, $k = n$ and we remove the orbit $\mathcal{A}$.

Now the assumptions of Lemma (5.3.4) are satisfied, hence the final map realizes the number $NF_n(f)$. □

5.4. Least number of points of the given minimal period

The Procedures from the last two sections allow us to answer another natural question. One of the basic problems in Dynamical Systems is the existence of periodic points of the given minimal period. As we have seen, $NP_n(f) \neq 0$ gives such a point for each map homotopic to f. Is the above inequality the only obstruction to a deformation of f to a map with no periodic point of minimal period n? We will show that in dimensions ≥ 3 the answer is positive (cf. [Je2], [Je4]).

(5.4.1) THEOREM. *Let $f\colon M \to M$ be a self-map of a compact* PL*-manifold of dimension ≥ 3. Then f is homotopic to a map g satisfying $P_n(g) = \emptyset$ if and only if $NP_n(f) = 0$.*

PROOF. Let us recall that $NP_n(f) = 0$ if and only if there is no essential irreducible Reidemeister class in $\mathcal{R}(f^n)$.

$\Rightarrow$ is evident.

It remains to prove $\Leftarrow$. By Theorem (5.2.2) we may assume that $\mathrm{Fix}(f^n)$ is finite and moreover f is a local PL-homeomorphism near each fixed point of f^n. Let us denote $\mathrm{Fix}(f^n) = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{A}$ denotes the sum of all irreducible classes and $\mathcal{B}$ the sum of all reducible ones. Lemma (5.4.2) gives a homotopy $\{f_t\}_{0 \leq t \leq 1}$ which is constant in a neighbourhood of $\mathrm{Fix}(f^n) \setminus \mathcal{A} = \mathcal{B}$, $f_0 = f$ and $\mathrm{Fix}(f_1^n) = \mathrm{Fix}(f_0^n) \setminus \mathcal{A}$. Since the homotopy is constant in $\mathrm{Fix}(f_1^n)$, all classes in $\mathrm{Fix}(f_1^n)$ are reducible. Now Lemma (5.4.4) yields a homotopy from f_1 to a map g satisfying $P_n(g) = \emptyset$.

The proof of Theorem (5.4.1) will be complete once Lemmas (5.4.2) and (5.4.4) are proved. □

(5.4.2) LEMMA. *Let $f\colon M \to M$ be a self-map of a compact* PL*-manifold satisfying $NP_n(f) = 0$ where* $\mathrm{Fix}(f^n)$ *is finite and f is a local* PL*-homeomorphism near each fixed point of f^n. Let $\mathcal{A}$ be the sum of all irreducible Nielsen classes of f^n, i.e. classes of depth $= n$. Then there is a homotopy $\{f_t\}$ constant in a neighbourhood of* $\mathrm{Fix}(f^n) \setminus \mathcal{A}$ *and satisfying $f_0 = f$,* $\mathrm{Fix}(f_1^n) = \mathrm{Fix}(f_0^n) \setminus \mathcal{A}$.

PROOF. Since each orbit in $\mathcal{A}$ is irreducible, the length of each orbit of points in $\mathcal{A}$ is n. Since $NP_n(f) = 0$,the orbits of Nielsen classes are inessential and their sum splits into pairs of orbits of length n of opposite indices ± 1.

Consider a pair of orbits of points $\{x_1, \dots, x_n; y_1, \dots, y_n\} \subset \mathcal{A}$ of opposite indices. The Cancelling Procedure yields a homotopy constant in a neighbourhood of $\mathrm{Fix}(f^n) \setminus \{x_1, \dots, x_n; y_1, \dots, y_n\}$ such that $f_0 = f$ and $\mathrm{Fix}(f_1^n) = \mathrm{Fix}(f^n) \setminus \{x_1, \dots, x_n;\ y_1, \dots, y_n\}$. Following this procedure we can reduce the number of such orbits to zero, hence we get $\mathrm{Fix}(f_1^n) = \mathrm{Fix}(f_0^n) \setminus \mathcal{A}$ as required. □

(5.4.3) Remark. If f_1 satisfies the above lemma then all the Nielsen classes in $\mathrm{Fix}(f^n)$ are reducible.

(5.4.4) Lemma. *If all Nielsen classes in $P_n(f)$ are reducible then f is homotopic to a map g satisfying $P_n(g) = \emptyset$.*

Proof. Let $\mathcal{A} \in \mathcal{OR}(f^n)$ be an orbit reducing to $\mathcal{B} \in \mathcal{OR}(f^k)$. We define the numbers $\alpha_s = \mathrm{ind}(f^{sk}; i_{sk,k}(\mathcal{B}))$. The Creating Procedure gives a homotopy (constant near $\mathrm{Fix}(f^n)$) from f to a map f_1 such that the class $\mathcal{B}_1 \in \mathcal{OR}(f_1^k)$ (corresponding to $\mathcal{B}$) contains an isolated orbit $\{x_0, \dots, x_{k-1}\}$ with $\mathrm{ind}(f_1^{sk}; x_0) = \alpha_s/k$ for $s|(n/k)$. Now we are in a position to apply Corollary (5.3.8) to $B = (\mathrm{Fix}(f_1^n) \setminus \mathcal{A}_1) \cup \{x_0, \dots, x_{k-1}\}$. This yields a homotopy from f_1 to f_2 satisfying

$$\mathrm{Fix}(f_2^n) = B = (\mathrm{Fix}(f_1^n) \setminus \mathcal{A}_1) \cup \{x_0, \dots, x_{k-1}\} = (\mathrm{Fix}(f^n) \setminus \mathcal{A}) \cup \{x_0, \dots, x_{k-1}\}$$

hence the orbit $\mathcal{A}$ is replaced with $\{x_0, \dots, x_{k-1}\}$.

Repeating the above for all $\mathcal{A} \in \mathcal{OR}(f^n)$ we replace all orbits of points of length n with shorter ones. □

One might expect that the operation of realizing $NP_k(f)$ could be done simultaneously for two or more periods. But the example of the antipodal map on S^{2m} (Example (5.1.19)) shows that this is not possible in general: we can not remove points of periods 1 and 2 simultaneously although there is no essential irreducible orbit: $NP_k(f) = 0$ for all $k \in \mathbb{N}$.

The next theorem makes precise for which periods it is possible to remove periodic points simultaneously.

(5.4.5) Theorem. *Let $f\colon M \to M$ be a self-map of a compact* PL*-manifold of dimension ≥ 3. Let $N_0 \subset \mathbb{N}$ be finite. Then there is a homotopy $f_t\colon M \to M$ such that $f_0 = f$ and $\mathrm{P}_r(f_1) = \emptyset$ for all $r \in N_0$ if and only if for any $r \in N_0$ any essential Reidemeister class $\mathcal{A}^r \in \mathcal{R}(f^r)$ reduces to a class $\mathcal{B}^s \in \mathcal{R}(f^s)$ for an $s \notin N_0$.*

Proof. ($\Rightarrow$) Assume that $\mathrm{P}_r(f) = \emptyset$ for all $r \in N_0$. Consider an essential orbit of Reidemeister classes $\mathcal{A}^r \in \mathcal{OR}(f^r)$ where $r \in N_0$. Since $\mathcal{A}^r$ is essential, it contains an orbit of points $\{x_0, \dots, x_{s-1}\}$ for an $s|r$ and $s < r$. Then $P_s(f) \neq \emptyset$, hence $s \notin N_0$. Now the orbit of Reidemeister classes $\mathcal{A}^r$ reduces to the orbit $\mathcal{B}^s \in \mathcal{OR}(f^s)$ represented by the points $\{x_0, \dots, x_{s-1}\}$.

($\Leftarrow$) We use induction with respect to the number $l = \#N_0$. For $l = 1$ the theorem follows from Theorem (5.4.1). Now we assume that the theorem holds for $< l$. Let $N_0 \subset \mathbb{N}$ be a subset of cardinality l. Let r be the greatest element in N_0. By inductive assumption f is homotopic to a map f_1 satisfying $\mathrm{P}_s(f_1) = \emptyset$ for all $s \in N_0 \setminus \{r\}$. It remains to remove all orbits from $P_r(f_1)$. Let $\mathcal{A}^r \in \mathcal{OR}(f_1^r)$ be nonempty.

Suppose that $\mathcal{A}^r$ does not reduce (as a Reidemeister class) to any class $\mathcal{B}^s$ with $s \notin N_0$. Then $\mathcal{A}^r$ is inessential and each orbit of points in $\mathcal{A}^r$ must be of length r. Now $\mathcal{A}^r$ splits into the pairs of orbits of points of length r of opposite indices. We may apply the Cancelling Procedure to remove $\mathcal{A}^r$.

Now we suppose that $\mathcal{A}^r$ reduces to an orbit of Reidemeister classes $\mathcal{B}^s \in \mathcal{OR}(f_1^s)$ $(s \notin N_0)$. Then we may follow the proof of Theorem (5.4.1) and we get a map g homotopic to f such that

$$\mathrm{Fix}(g^r) \subset (\mathrm{Fix}(f^r) \setminus A_0) \cup B$$

where $A_0 = \mathcal{A} \cap P_n(f)$ and $B \subset P_s(f)$. Since r is the greatest element in N_0, no periodic point appears in $P_s(g)$ for any $s \in N_0$. □

CHAPTER VI

HOMOTOPY MINIMAL PERIODS

We have already pointed out in Chapter III that the study of two functions from $\mathbb{N}$ to $\mathbb{N} \cup 0 \cup \infty$ given as

$$(6.0.1) \qquad m \mapsto \#P^m(f) \quad \text{and} \quad m \mapsto \#P_m(f)$$

are of interest. The information that $P(f)$, correspondingly $\mathrm{Per}(f)$, are infinite, or that $\limsup \#P_m(f) = \infty$, respectively $\limsup \#P^m(f) = \infty$, says that the dynamics of f is rich. Unfortunately the above invariants, $\#P_m(f)$, $\#P^m(f)$, $P(f)$, of dynamics of f are not stable in the following sense:

We say that an invariant of a map is stable if for every g sufficiently close to a self-map f of a compact metric space X has the same value as for f.

Sufficiently close means here that the distance $\mathrm{dist}(f,g) := \sup \rho(f(x), g(x))$ is small. To illustrate it we give a simple example.

(6.0.2) Example. Let $f = \mathrm{id}_{|S^1}$ be the map of the circle $S^1 = \{\mathbb{R}/\mathbb{Z}\}$. Take $0 < \theta < 1$ an irrational number and consider $g_\theta \colon S^1 \to S^1$ a map of the circle defined as

$$g_\theta(s) := s + \theta \mod 1.$$

It is easy to check that $P^1(f) = P(f) = S^1$, $\mathrm{Per}(f) = \{1\}$ and $P(g_\theta) = \emptyset$, $\mathrm{Per}(g_\theta) = \emptyset$. On the other hand $\mathrm{dist}(f, g_\theta) = \theta$.

We would like to recall that if self-maps of a nice space, such as a manifold or polyhedron, are sufficiently close, then they are homotopic. We formulate it in a statement, not the most general, which proof we leave to the reader (it could be found in many places in the literature as well).

(6.0.3) Proposition. *Let $f \colon X \to X$ be a map of a compact smooth manifold. Then there exists $\varepsilon > 0$ such that for a map $g \colon X \to X$,*

$$\mathrm{dist}(f,\, g) < \varepsilon \Rightarrow g \sim f.$$

In the next definition, by an invariant we mean a general notion not specifying its values which could be elements of any object.

(6.0.4). We say that an invariant characterizes the *homotopy dynamics* of $f \in \mathrm{Map}(X, X)$ if it is derived from all iterations of f and has the same value for every g which is homotopic to f.

As a direct consequence of the definition and Proposition (6.0.3) we get the following.

(6.0.5) COROLLARY. *The functions $m \mapsto \#P^m(f)$, $m \mapsto \#P_m(f)$, and the sets $P(f)$, $\mathrm{Per}(f)$ are not homotopy dynamics invariants.*

Since for every m, $f \sim g$ implies $f^m \sim g^m$, the sequences $\{L(f^m)\}$ and $\{N(f^m)\}$ are homotopy dynamics invariants. To study the invariants mentioned in Corollary (6.0.5) by use of the sequence $\{L(f^m)\}$ one has to pose additional geometric assumptions on the map f such as smoothness, complex analyticity, or transversality. In particular Example (1.0.20) shows that without any such assumption we may have $L(f^m) \xrightarrow{m\to\infty} \infty$ but $P(f)$ consists of two fixed points. We shall discuss it later.

Also the use of the sequence $\{N(f^m)\}$ of the Nielsen numbers of iterations of $f\colon X \to X$ to study the dynamics of f requires a geometric assumption on the space X.

Anyway there is a direct consequence of the definition of a Nielsen number.

(6.0.6) PROPOSITION. *Let $f\colon X \to X$ be a continuous map of a compact* CW-*complex. If $\{N(f^m)\}$ is unbounded, then $P(f)$ is infinite.*

PROOF. By the main property of Nielsen number, $\#P^m(f) \geq N(f^m)$, and the statement follows. □

Now we give an example which illustrates more persuasively than Example (6.0.2) that the functions $m \mapsto \#P^m(f)$, $m \mapsto \#P_m(f)$, set $P(f)$ and the set $\mathrm{Per}(f)$, are not stable invariants.

To see this we can use the Shub Example (1.0.20). First consider the map $z \mapsto z^r$, $r \geq 2$, of the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. By the definition of f, $\#P^m(f) = \#\{z : z^{r^m} = z\} = r^m - 1$. Furthermore $P_m(f)$ is equal to the set of primitive roots of unity of degree $r^m - 1$. Consequently $\#P_m(f) = \phi(r^m - 1)$, where $\phi(m)\colon N \to N$ is the Euler function. It is known ([Ch]) that for $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$ we have $\phi(n) = \prod_{i=1}^s p_i^{\alpha_i}(1 - 1/p_i)$.

Now if we take the maps $h_r = z^r$ and $\zeta(t) = t - \varepsilon(t - \sqrt{t})$ in the construction of Example (1.0.20), then the defined map $f\colon S^2 \to S^2$ is an ε-perturbation of $\Sigma(z^r)$ of the suspension of z^r. Consequently, for every m, $P_m(\Sigma(z^r)) \neq \emptyset$ (here $P_m(z^r)$ is a subcomplex of dimension 1, thus infinite). This yields $\mathrm{Per}(\Sigma(z^r)) = \mathbb{N}$.

6.1. Definition of homotopy minimal period

All the above examples legitimate the introduction of the following notion which is stable and homotopy invariant by its definition.

(6.1.1) Definition. Define the *set of homotopy minimal periods* to be the set

$$\mathrm{HPer}(f) := \bigcap_{g \simeq f} \mathrm{Per}(g),$$

i.e. $m \in \mathbb{N}$ is a *homotopy minimal period* of f if it is the minimal period for every $g \sim f$ homotopic to f.

By definition $\mathrm{HPer}(f) \subset \mathrm{Per}(f)$ and the inclusion is proper in general.

(6.1.2) Example. For the map g_θ of (6.0.2) we have $\mathrm{Per}(g_\theta) = \emptyset$, thus

$$\mathrm{HPer}(\mathrm{id}_{|S^1}) \subsetneq \mathrm{Per}(\mathrm{id}_{|S^1}) = \{1\}.$$

Similarly for the map $\Sigma(z^r)$, $|r| > 1$, we have

$$\{1\} = \mathrm{HPer}(\Sigma z^r) \subsetneq \mathrm{Per}(\Sigma z^r) = \mathbb{N}$$

as follows from Example (1.0.20).

For a smooth manifold map, any homotopy dynamics invariant reflects information about the rigid part of dynamics, because a small perturbation of a map f is homotopic to it by Proposition (6.0.3). In particular

(6.1.3) Remark. If $f\colon X \to X$ is a map of a smooth compact manifold then $\mathrm{HPer}(f) = \mathrm{HPer}(h)$ for any small perturbation h of f.

Let us try to describe the set of homotopy minimal periods for the simplest non-trivial manifold: the circle. Since every map $f\colon S^1 \to S^1$ is homotopic to z^r it is enough to study $\mathrm{HPer}(z^r)$.

We have:

- for $r = 1$, $\mathrm{HPer}(f) = \emptyset$, by Example (6.1.2);
- for $r = 0$, $\mathrm{HPer}(f) = \{1\}$, because $\mathrm{Per}(*) = \{1\}$ for the constant map;
- for $r = -1$, $\mathrm{HPer}(f) = \{1\}$, because for $f(z) = \overline{z}$, $\mathrm{Per}(f) = \{1, 2\}$ and for the map:

$$f(\theta) = \begin{cases} 2\theta^2 & \text{if } 0 \le \theta \le 1/2, \\ 1/2 + 2(\theta - 1/2)^2 & \text{if } 1/2 \le \theta \le 1, \end{cases}$$

  $\theta \in \mathbb{R}/\mathbb{Z} = S^1$, we have $\mathrm{Per} = \{1\}$.

On the other hand we have already shown that for a map $f\colon S^1 \to S^1$ of degree r we have $N(f) = r$, thus $N(f^m) = r^m$. This yields $\#P^m(f) \geq r^m$, and together with the obvious $\mathrm{Per}(z^r) = \mathbb{N}$ for $|r| > 1$ makes reasonable the question whether $\mathrm{HPer}(z^r) = \mathbb{N}$ for $|r| > 1$.

The set of homotopy minimal periods (under another name) was first studied for the self-maps of the circle $M = S^1$ by L. S. Efremova in [Ef] and L. Block, J. Guckenheimer, M. Misiurewicz, L. S. Young in [BGMY]. They wanted to prove an analog of the Šarkovskii theorem of [Sr] for the circle maps. As a result they got the following theorem which states that the answer to the above question is almost affirmative except for only one case.

(6.1.4) THEOREM. *Let $f\colon S^1 \to S^1$ be a map of the circle and $A_f = r \in \mathbb{Z} = \mathcal{M}_{1\times 1}(\mathbb{Z})$ the degree of f. There are three types for the minimal homotopy periods of f:*

(E) $\mathrm{HPer}(f) = \emptyset$ *if and only if* $r = 1$.

(F) $\mathrm{HPer}(f) \neq \emptyset$ *and is finite if and only if* $r = -1$ *or* $r = 0$. $\mathrm{HPer}(f) = \{1\}$ *then.*

(G) $\mathrm{HPer}(f) = \mathbb{N}$ *for the remaining* r, *i.e.* $|r| > 1$, *except for one special case* $r = -2$ *when* $\mathrm{HPer}(f) = \mathbb{N} \setminus \{2\}$.

The examples of maps giving the minimal sets of periods for $r = \deg f \in \{-1, 0, 1\}$ have been presented before the theorem. Below we include an example of map which gives the special case of (G) of this theorem.

(6.1.5) EXAMPLE. Let us consider the map $f(z) = \overline{z}^2$ of the circle S^1. Then we have $\deg(f) = -2$, $\mathrm{Fix}\,(f) = \{z : z^3 = 1\}$, but the equation $\overline{\overline{z}^2}^2 = z$ is equivalent to $z^4 = z$, thus $\mathrm{Fix}\,(f) = \mathrm{Fix}\,(f^2)$ and $2 \notin \mathrm{Per}(f)$.

The original method of proof of Theorem (6.1.4) is elementary but complicated. We get it, as well as the next theorem stated below, as a consequence of a general statement. As the next L. Alsedá, S. Baldwin, J. Llibre, R. Swanson, and W. Szlenk examined the case $M = \mathbb{T}^2$ in [ABLSS2]. To give a description of the set of the homotopy minimal periods (which they called "the minimal set of periods") they used the Nielsen theory as the first. Their main theorem, after a reformulation to our terms, is presented in the Subsection 6.5.

A qualitative progress of methods had been made by B. Jiang and J. Llibre who gave a description of the set of homotopy minimal periods for the torus $M = \mathbb{T}^d$, with any $d \in \mathbb{N}$. ([JiLb]). To prove a general theorem (cf. (6.4.4)) they made use of a fine combinatorics argument and a deep algebraic number theory theorem that they proved (close to the A. Schinzel theorem on prime divisors cf. [Schi]), but also used a topological result of You ([Yu2], [Yu3]) on the periodic points on tori.

It was a natural question to extend this theorem onto larger classes of compact manifolds with similar structure as the tori, namely: compact nilmanifolds, or solvmanifolds. A general description of the set of homotopy minimal periods for the maps of compact NR-solvmanifolds is given in Theorem (6.4.4), which is based on the works [JeMr1], [JeKdMr]. Next we show its exemplifications in the form of detailed lists for the dimension 3 in Theorems (6.5.2), (6.5.14), with a specification for homeomorphisms in Theorem (6.5.8), and Corollary (6.5.15).

As applications of the main theorem and its specifications for 3-dimensional manifolds one can derive theorems which ensure that $\mathrm{HPer}(f) = \mathbb{N}$ provided a given homotopy period exists (Propositions (6.5.20) and (6.5.22)). We call them the Šarkovskii type theorems.

All these results are consequences of a fine geometrical structure of these classes of solvmanifolds which precipitate special properties of the Nielsen number, but that are not true in general. Nevertheless the set $\mathrm{HPer}(f)$ was also studied in other cases, e.g. it was derived for the self-maps of the real projective space in [Je5]. It is presented in Theorem (6.5.26).

6.2. Classes of solvmanifolds

6.2.1. Nil- and solvmanifolds. In this subsection we give an overlook on basic notions and definitions concerning nil- and solvmanifolds. This part of our presentation follows mainly [GOV], [Ra], [VGS]. We would like to avoid a long presentation of the Lie groups and their discrete subgroups. For a more detailed exposition of this subject we refer the reader to these books or to [Var].

A group G is called *nilpotent* if its central tower is finite, i.e.

$$G_0 = G \supset G_1 \supset \cdots \supset G_{k-1} \supset G_k = \mathrm{e}, \tag{6.2.1}$$

where $G_i := [G, G_{i-1}]$.

A group G is called *solvable* if its normal tower is finite, i.e.

$$G_0 = G \supset G_1 \supset \cdots \supset G_{k-1} \supset G_k = \mathrm{e}, \tag{6.2.2}$$

where $G_i := [G_{i-1}, G_{i-1}]$. Obviously every nilpotent group is solvable.

(6.2.3) Definition. A homogeneous space G/H of a nilpotent or solvable group G is called a *nilmanifold* or a *solvmanifold*, respectively.

Of course, every nilmanifold is also a solvmanifold.

(6.2.4) Example. Below we give some examples of compact nilmanifolds.
$\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d \equiv (S^1)^d$, the torus.
If $d \leq 2$ then it is the only example.

For a ring $\mathcal{R}$ with unity (e.g. $\mathcal{R} = \mathbb{R}$, $\mathcal{R} = \mathbb{C}$) let $\mathbf{N}_n(\mathcal{R})$ denote the group of all unipotent upper triangular matrices whose entries are elements of the ring $\mathcal{R}$, i.e.

$$\begin{bmatrix} 1 & r_{12} & \cdot & \cdots & \cdot & r_{1n} \\ 0 & 1 & r_{23} & \cdot & \cdots & r_{2n} \\ 0 & 0 & 1 & r_{34} & \cdots & r_{3n} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \cdot & \cdot & \cdot & \cdots & 1 & r_{n-1n} \\ 0 & 0 & \cdot & \cdots & 0 & 1 \end{bmatrix}.$$

– *Iwasawa manifolds*: $\mathbf{N}_n(\mathbb{R})/\mathbf{N}_n(\mathbb{Z})$ and $\mathbf{N}_n(\mathbb{C})/\mathbf{N}_n(\mathbb{Z}[\imath])$, where $\mathbb{Z}[\imath]$ is the ring of Gaussian integers, are examples of nilmanifolds of dimension 3 not diffeomorphic to the torus. The Iwasawa 3-manifold $\mathbf{N}_3(\mathbb{R})/\mathbf{N}_3(\mathbb{Z})$, is called “Baby Nil”.

It is known [Au] that for $d = 3$ every compact nilmanifold is, up to diffeomorphism, one of: $\mathbf{N}_3(\mathbb{R})/\Gamma_{p,q,r}$, where the subgroup $\Gamma_{p,q,r}$, with fixed $p, q, r \in \mathbb{N}$ consists of all matrices of the form

$$\begin{bmatrix} 1 & k/p & m/(p \cdot q \cdot r) \\ 0 & 1 & l/q \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where } k, l, m \in \mathbb{Z}.$$

Moreover, it is diffeomorphic to one of $\mathbf{N}_3(\mathbb{R})/\Gamma_{1,1,r}$. They are called Heisenberg manifolds.

As we noted above, in all our examples of nilmanifolds the subgroup $H \subset G$ is discrete. It is a consequence of a general fact. Recall that a discrete subgroup $\Gamma \subset G$ of a Lie group G is called a *lattice* if there exists a finite G-invariant measure on G/Γ [Ra]. A closed subgroup $H \subset G$ is called a uniform subgroup if G/H is compact. Due to the Mostow theorem, a closed subgroup $H \subset G$ of a solvable group G is uniform if and only if there exists a finite G-invariant measure on G/H (cf. [Ra, Theorem 3.1]). Consequently, a discrete subgroup $\Gamma \subset G$ of a solvable group G is a lattice if it is uniform. We shall use both names, a lattice or a discrete uniform subgroup.

It is known (cf. [GOV]) that every compact nilmanifold is diffeomorphic to a nilmanifold of the form G/Γ, where G is a simply-connected nilpotent Lie group and $\Gamma \subset G$ is a lattice. This statement leads to an equivalent definition of the nilmanifold.

(6.2.5) Theorem. *We say that a compact manifold X of dimension d is a nilmanifold if it is the quotient space G/Γ of a simply-connected nilpotent group G by a lattice $\Gamma \subset G$ of rank d* ([Au], [Mal], [Ra]).

By this definition a compact nilmanfold $X = G/\Gamma$ is a $K(\pi, \Gamma)$ space, i.e. $\pi_1(X) = \Gamma$ and $\pi_k(X) = 0$ for $k > 1$, because the nilpotent simply-connected group G is contractible (see the discussion in Section 6.2.2). As a matter of fact a simply-connected solvable Lie group G is diffeomorphic to the Euclidean space (cf. [GOV], [Ra] and [Var]).

(6.2.6) PROPOSITION. *Every connected, simply-connected, solvable Lie group G of dimension d is analytically diffeomorphic to the Euclidean space $\mathbb{R}^d$.*

Note that every compact nilmanifold X is parallelizable, i.e. its tangent bundle is isomorphic to the trivial bundle. Indeed, by Proposition (6.2.6), a connected simply-connected, nilpotent Lie group G is diffeomorphic to $\mathbb{R}^d$ thus parallelizable and a space $X = G/\Gamma$ is parallelizable too as being covered by G. (As a matter of fact every Lie group G of dimension d is paralleizable, because there exist d linearly independent vector fields $\mathcal{X}_1(g)$, $\mathcal{X}_2(g)$, $\ldots$, $\mathcal{X}_d(g)$ defined as $\mathcal{X}_i(g) = g^*(\mathcal{X}_i)$, where $\{\mathcal{X}_i\}$ are linearly independent vectors of the tangent space $T_e(G)$ at e.)

The same is not true for solvmanifolds. For example the Klein Bottle is a compact solvmanifold defined as the homogeneous space $SO(2) \ltimes \mathbb{R}^2/SO(1) \ltimes (\mathbb{Z} \times \mathbb{R})$ (cf. [GOV, p. 165]). It could be also represented as the $K(\pi, 1)$ space with fundamental group $\pi_1 = \pi = \mathbb{Z} \times_\phi \mathbb{Z}$, where $\phi\colon \mathbb{Z} \to \{1, -1\} = O(1) \subset \mathbb{Z}$ is given by the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}_2$ (cf. [Mo]), but it is not of the form G/Γ, for a connected, simply-connected solvable Lie group G and a lattice $\Gamma \subset G$. If it were of the form G/Γ it would be parallelizable. But the latter is not true (cf. [GOV], [Mil], [Sp]).

(6.2.7) DEFINITION. A solvmanifold of the form G/Γ where G is simply-connected and Γ is a lattice is called a *special* solvmanifold.

Note that every solvmanifold is finitely covered by a special solvmanifold by a theorem of Auslander [Au], [GOV].

(6.2.8) THEOREM. *Let X be a compact solvmanifold. Then there exists a special solvmanifold $X' = G/\Gamma$, which is a finite covering of X.*

This implies the following

(6.2.9) COROLLARY. *Every solvmanifold X is aspherical, that is $\pi_i(X) = 0$ for $i \geq 2$ for any solvmanifold X.*

PROOF. By the above theorem, X is finitely covered by X' which is covered by a simply-connected, connected Lie group G. Thus G covers X, hence $\pi_i(X) = \pi_i(G) = 0$ for all $i \geq 2$, as follows from Proposition (6.2.6). □

6.2.2. Completely solvable. Let $\mathcal{G}$ T_eG be the Lie algebra of a given Lie group G (cf. [GOV], [Var]). Now we designate a few classes of solvmanifolds that distinguish with respect to the spectrum of adjoint operator $\mathrm{ad}_X\colon \mathcal{G}$, $\mathcal{X}$

(cf. [GOV]). First we have to recall the notion of the adjoint operators of a Lie group G and its algebra $\mathcal{G}$.

Let $x \in G$ be an element. We assign to x the inner automorphism $g \mapsto xgx^{-1}$ of G, denoted by $\widetilde{\mathrm{Ad}}_x$. Note that $\widetilde{\mathrm{Ad}}_{x_1 \cdot x_2} = \widetilde{\mathrm{Ad}}_{x_1} \cdot \widetilde{\mathrm{Ad}}_{x_2}$.

Since for every $x \in G$, $\widetilde{\mathrm{Ad}}_x(e) = e$, the derivative $D\widetilde{\mathrm{Ad}}_x(e)$, denoted by Ad_x, is a linear map of $\mathcal{G}$ into itself. By the functoriality of derivative

$$\mathrm{Ad}_{x_1 \cdot x_2} = \mathrm{Ad}_{x_1} \cdot \mathrm{Ad}_{x_2},$$

thus Ad, $x \mapsto \mathrm{Ad}_x$, is a representation of G to $\mathrm{Aut}(\mathcal{G})$, called the *adjoint representation* of G in $\mathcal{G}$.

Taking the derivative of the map $\mathrm{Ad}\colon G \to GL(\mathcal{G})$ we get a linear mappping of the tangent spaces $\mathrm{ad}\colon \mathcal{G}$ $\mathrm{End}(\mathcal{G}, \mathcal{G})$, where $\mathrm{End}(\mathcal{G}, \mathcal{G})$ is the space of all linear maps of $\mathcal{G}$. The map ad, is called the adjoint representation of $\mathcal{G}$ in $L(\mathcal{G}, \mathcal{G})$ with respect to the property: for each $\mathcal{X}, \mathcal{Y} \in \mathcal{G}$,

$$[\mathcal{X}, \mathcal{Y}] = \mathrm{ad}_{\mathcal{X}}(\mathcal{Y}),$$

where $[\,\cdot\,,\,\cdot\,]$ is the Lie bracket in $\mathcal{G}$ and $\mathrm{ad}_{\mathcal{X}} = D\mathrm{Ad}(\mathcal{X})$.

The basic connection between these two maps is expressed as the commutativity of the diagram

(6.2.10)
$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathrm{End}(T_e) \arrow[d, "\exp"] \\
G \arrow[r, "\mathrm{Ad}"'] & \mathrm{Aut}(T_e)
\end{tikzcd}$$

where exp is the exponential map (cf. [Var]).

If for any $\mathcal{X}$ all eigenvalues of the operator $\mathrm{ad}_{\mathcal{X}}$ are real, then the group G and the Lie algebra $\mathcal{G}$ as well as every solvmanifold G/Γ is called *completely solvable*. Note that every nilmanifold belongs to this class, since $\mathrm{ad}_{\mathcal{X}}$ is a nilpotent matrix (cf. [Var], thus all the eigenvalues are equal to 0. The above remark says that every nilmanifold is completely solvable. The main property of completely solvable groups is the following property, called the *rigidity of lattices*:

(6.2.11) THEOREM. *Let G and G' be simply-connected completely solvable Lie groups and $\Gamma \subset G$ be a lattice. Then every homomorphism $f\colon \Gamma \to G'$ can be extended to a homomorphism $F\colon G \to G'$.*

(See [Mal], [Ra] for the nilpotent case and [VGS], [Sa] for the general case).

For our purposes, the following immediate consequence of Theorem (6.2.11) and the fact that G/Γ is a $K(\pi, 1)$-space with $\pi = \Gamma$ will be relevant.

(6.2.12) Corollary. *Every continuous map $G_1/\Gamma_1 \to G_2/\Gamma_2$ between compact special completely solvable solvmanifolds is homotopic to the map induced by a homomorphism $G_1 \to G_2$.*

An example of a completely solvable Lie group G of dimension 3 which is neither abelian nor nilpotent is given in Proposition (6.5.11). There are infinitely many non-isomorphic completely solvable compact solvmanifolds of dimension 3 which are not diffeomorphic to a torus or a nilmanifold (Example (6.5.16)).

Exponential. If for any $\mathcal{X}$ there is no pure imaginary eigenvalue of the operator $\mathrm{ad}_{\mathcal{X}}$, then the group G and the Lie algebra $\mathcal{G}$ as well as every solvmanifold G/Γ is called *exponential*. This is equivalent to the fact that for any $x \in G$ there is no eigenvalue of the operator Ad_x of the module equal to 1 other than one as follows from (6.2.10).

Furthermore, a geometric characterization of the exponential says that a group G is exponential iff the exponential map $\exp\colon \mathcal{G}$ G is injective or is a diffeomorphism, provided that G is simply-connected [Var]. Notice that every completely solvable group is exponential by the definition.

It is known that, if a Lie group is exponential, then it is solvable but not every solvable Lie group is exponential [VGS].

We call a solvmanifold X *exponential* if it is a homogeneous space G/H of an exponential solvable group G.

NR**-manifolds.** The class of *NR-solvmanifolds* was introduced by Keppelmann and McCord (see [KMC]). The definition is technical and we postpone it to the next subsection (Definition (6.3.18)) .

This class of solvmanifolds contains the class of special exponential solvmanifolds. The main property of NR-solvmanifold is that the Anosov theorem (6.3.13) can be generalized to it.

At the end we formulate a property of solvmanifolds we shall use in next.

(6.2.13) Proposition. *Let H be a closed connected subgroup of connected simply-connected solvable group G. Then H is also simply-connected.*

Proof. The space G/H is aspherical as a solvmanifold (see Corollary (6.2.9)). Then the statement follows from the long exact sequence of homotopy groups of the fibration $H \to G \to G/H$. Indeed for $i \geq 3$ all groups in this sequence are zero, and the exactness of $0 \to \pi_2(H) \to \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to 0$ gives

$$\pi_2(H) = \pi_1(H) = 0. \qquad \square$$

6.3. Properties of Nielsen numbers for solvmanifolds

In this subsection we first define the notion of linearization of a map which we will need to formulate of our theorems. Next we prove the Anosov Theorem (6.3.13) comparing the Lefschetz number of a map of a compact nilmanifold with the Nielsen number of the map. Then we extend the Anosov theorem from the class of compact nilmanifolds onto the class of compact NR-solvmanifolds (Theorem (6.3.21)).

This implies nice arithmetic properties of the sequence $\{N(f^m)\}$ for a map of such a manifold. Finally we recall briefly the summation formula for expressing the full Nielsen–Jiang m-periodic number $NF_m(f)$ as a sum of the prime Nielsen–Jiang m-periodic numbers $NP_k(k)$, $k|m$. This formula guarantees nice combinatorial properties of the sequence $\{N(f^m)\}$ for a map of an NR-solvmanifold which we will use for a proof of the main theorem of this chapter (Theorem (6.4.4)).

6.3.1. The linearization matrix. Let X be a compact connected nilmanifold of dimension d. We may present it as G/Γ where G is a connected simply-connected nilpotent Lie group of dimension d and Γ a uniform lattice in G. Let $G_0 = G$, $G_{i+1} = [G, G_i]$ be the central sequence with the last trivial group $G_k = e$. The number k is called the *length of nilpotency* of G.

We will base our argument on the following results of A. Malcev which was historically the first version of the already mentioned more general Theorem (6.2.11) (see [Mal], [Ra]).

(6.3.1) Theorem. *Let G be a nilpotent connected simply-connected Lie group, Γ its lattice. Then*

(6.3.1.1) *For each group G_i, $1 \leq i \leq k-1$ of the central sequence, the group $\Gamma_i := \Gamma \cap G_i$ is a uniform lattice in G_i.*

(6.3.1.2) *Every continuous self-map of the nilmanifold $X = G/\Gamma$ is homotopic to the factor map $\phi = \Phi_f/\Gamma$ induced by a preserving Γ homomorphism $\Phi_f\colon G \to G$.*

(6.3.1.3) *Every homomorphism $\phi\colon \Gamma \to \Gamma'$ to a lattice $\Gamma \subset G'$ of a connected simply-connected nilpotent group G' extends uniquely to a homomorphism $\Phi\colon G \to G'$.*

(6.3.2) Remark. A compact nilmanifold X determines uniquely the pair (G,Γ), G a connected simply-connected nilpotent Lie group, and Γ a lattice in it. To see this we notice that $\pi_1(X) = \Gamma$ hence if $X = G/\Gamma = G'/\Gamma'$ then Γ and Γ' are isomorphic. By (6.3.1.3) this isomorphism extends to a homomorphism, thus an isomorphism of $G \to G'$.

It is easy to check that all subgroups G_i of the central tower (6.2.1) and of the normal tower (6.2.2) are normal subgroups, and they are preserved by every homomorphism $\Phi\colon G \to G$. Moreover, if a homomorphism $\Phi\colon G \to G$ preserves Γ, i.e. $\Phi(\Gamma) \subset \Gamma$, then it preserves also every group Γ_i, $1 \le i \le k$. Note also that the quotient groups G_{i-1}/G_i, $1 \le i \le k$, of the normal and central towers are abelian by the definitions of towers. If we denote $d_i = \dim(G_{i-1}/G_i)$, then $\sum_{i=1}^k d_i = d$. By Proposition (6.2.13) each G_i is a connected, simply-connected Lie group, and so is every quotient G_{i-1}/G_i. This means that $G_{i-1}/G_i \simeq \mathbb{R}^{d_i}$ as the Lie group. Consequently the quotient groups $\Gamma_{i-1}/\Gamma_i \subset G_{i-1}/G_i$ are discrete abelian groups without torsion elements, i.e. free abelian group. Moreover, since $(G_{i-1}/G_i)/(\Gamma_{i-1}/\Gamma_i) = (G_{i-1}/\Gamma_{i-1)}/G_i/\Gamma_i)$ is compact, the group Γ_{i-1}/Γ_i is uniform, i.e. it is a lattice in G_{i-1}/G_i, consequently $\Gamma_{i-1}/\Gamma_i \approxeq \mathbb{Z}^{d_i}$.

Let us consider a continuous map $f\colon X \to X$ and let $\phi = f_\#$ be the induced homotopy homomorphism of $\pi_1(X) = \Gamma$. By the above and Theorem (6.3.1) the unique extension $\Phi\colon G \to G$ of ϕ induces the family of quotient homomorphisms

$$\phi_i\colon \Gamma_{i-1}/\Gamma_i \to \Gamma_{i-1}/\Gamma_i \quad 1 \le i \le k. \tag{6.3.3}$$

(6.3.4) Definition. Let $\phi\colon \Gamma \to \Gamma$ be a homomorphism preserving all Γ_i. We define the *linearization of a homomorphism* ϕ as the family $\{\phi_i\}_{i=1}^k$ of induced homomorphisms of abelian groups

$$\phi_i\colon \Gamma_{i-1}/\Gamma_i \to \Gamma_{i-1}/\Gamma_i.$$

Each ϕ_i as a homomorphism of the free abelian group $\mathbb{Z}^{d_i} \approxeq \Gamma_{i-1}/\Gamma_i$ is represented by an integral $d_i \times d_i$ matrix A_i in a fixed basis. The $d \times d$ integral matrix

$$A := \bigoplus_{i=1}^k A_i$$

is called the *linearization matrix* of the homomorphism of the nilpotent group $\phi\colon \Gamma \to \Gamma$. If we take $\phi = f_\#\colon \Gamma \to \Gamma$, where $f\colon X \to X$ is a map of the nilmanifold $X = G/\Gamma$, then the corresponding notions are called the *linearization of* and the *linearization matrix* of f.

(6.3.5) Remark. Each matrix A_i defined above depends on the choice of a basis of the abelian group Γ_{i-1}/Γ_i. In fact another choice of the basis gives a conjugated matrix $A_i' = P^{-1}AP$ with a unimodular matrix P. Consequently the characteristic polynomials (hence the traces and determinants) of the matrices A_i, A_i', thus also of A and A', are equal.

The linearization matrix can be interpreted as the direct sum of homomorphisms of fundamental groups of the below ladder of fibrations (see (6.3.7) in Proposition (6.3.6), and Theorem (6.3.11)). Consider a nilpotent connected simply-connected Lie group G and a lattice $\Gamma \subset G$. Let $X = G/\Gamma$, $G_0 = G$, $\Gamma_0 = \Gamma$

$G_1 = [G, G]$, $X_1 = G_1/\Gamma_1$ be as above. We have a natural fibration $X = G/\Gamma \to (G/G_1)/(\Gamma/\Gamma_1) = X/X_1$ where the fibre X_1 is the nilmanifold of length of nilpotency $k-1$, while the base space

$$(G/G_1)/(\Gamma/\Gamma_1) = \mathbb{R}^{d_1}/\mathbb{N}^{d_1} = \mathbb{T}^{d_1}$$

is a torus. As we have seen we may assume that a given continuous map $f\colon X \to X$ is induced by a homomorphism $\Phi_f\colon G \to G$. This map induces a map of the base space X/X_1 and the last induces the homomorphism of the fundamental group $\pi_1(X/X_1) = \Gamma/\Gamma_1$. The last is the restriction of the homomorphism Φ to Γ/Γ_1, hence it coincides with the linearization homomorphism ϕ_1 (A_1) from Definition (6.3.4).

Continuing this procedure we get

(6.3.6) PROPOSITION. *A nilpotent connected simply-connected Lie group G and a lattice $\Gamma \subset G$ give a sequence of fibrations*

$$p_i\colon X_i = G_i/\Gamma_i \to X_i/X_{i+1}, \quad i = 1, \dots, k-1 \tag{6.3.7}$$

where $X_0 = X = G/\Gamma$, X_i/X_{i+1} is a torus, and the fibre X_{i+1} is a nilmanifold of nilpotency length $k-i$. If moreover, $f\colon X \to X$ is a map, then the homomorphism $\Phi\colon G \to G$ determining f induces the map of X_i/X_{i+1} which in turn induces the homotopy group homomorphism of $\pi_1(X_i/X_{i+1}) = \Gamma_i/\Gamma_{i+1}$ which coincides with the linearization homomorphism ϕ_i, thus the linearization matrix A_i up to conjugacy of Remark (6.3.5), *from Definition* (6.3.4).

Now we will show that the linearization homomorphism can be obtained in a dual approach (cf. Remark (6.3.12)).

6.3.2. Anosov theorem. It was natural to expect that the nice formula $N(f) = |L(f)|$ could be generalized from tori onto other Lie groups. However, as we have seen (Remark (4.3.17)) it is not true for any noncommutative group. In 1985 A. Anosov conjectured that this formula holds for compact nilmanifolds: the quotient spaces of nilpotent Lie groups. He proved this later giving a straightforward formula for the self-map of a nilmanifold [An]. In the meantime E. Fadell and S. Husseini gave another proof of this fact, developing by the way a method of splitting the given map into a sequence of fiberings to which the Nielsen Number Product Formula, from the previous section, can be applied. This method was later widely used giving new formulae for the Nielsen number. In this section we present a proof of this theorem based on the method of the paper of E. Fadell and S. Husseini [FaHu2]. Connections between the Nielsen and Lefschetz numbers for maps of solvmanifolds and infrasolvmanifolds has been systematically studied by Ch. McCord in [MC1]–[MC4].

We begin with the following lemma which allows us to make use of the property of a Nielsen number of afibre map already discussed in Section 4.4 of Chapter IV.

(6.3.8) LEMMA. *Let G be a connected Lie group, G^0 its closed connected subgroup contained in the center $Z(G)$. Let $\Gamma \subset G$ be a uniform lattice such that $\Gamma^0 = \Gamma \cap G^0$ is also a uniform lattice in G^0. Then the compact manifold $E = G/\Gamma$ fibers over $B = (G/\Gamma)/(G^0/\Gamma^0)$ and the admissible map over each loop in B is homotopic to the identity.*

PROOF. Let us notice that if, for a chosen point $b_0 \in B$, the admissible maps over all loops based at b_0 are homotopic to the identity, then the same is true for all other points of B. We choose as b_0 the point $[e] \in B$ where e is the unit element in the group G. Let $\overline{u}$ be a loop in $B = (G/\Gamma)/(G^0/\Gamma^0) = (G/G^0)/(\Gamma/\Gamma^0)$ based at $[e]$. This loop lifts (uniquely) to a path u in G/G^0 starting from the point $[e] \in G/G^0$. The last lifts to a path $\widetilde{u}$ in G satisfying $\widetilde{u}(0) = e$, $\widetilde{u}(1) \in G^0 \cdot \Gamma$. Since G^0 is connected we may correct $\widetilde{u}$ to satisfy $\widetilde{u}(1) \in \Gamma$.

We define the translation map (over the loop $\overline{u}$) $\lambda_t\colon X_{\overline{u}(0)} \to X_{\overline{u}(t)}$ by the formula $\lambda_t[x] = [u(t)x]$. The definition is correct since $[x] = [x'] \in G/\Gamma$ implies $x' = x \cdot z$ for a $z \in \Gamma$, hence $[u(t)x'] = [u(t)xz] = [u(t)x] \in G/\Gamma$. Moreover, $p(\lambda_t[x]) = p[u(t)x] = \overline{u}(t)$, hence λ_1 is an admissible map. To see that λ_1 is the identity we notice that $X_{[e]} = \{[x] \in X; x \in G^0\}$. Now

$$\lambda_1[x] = [u(1)x] = [xu(1)] = [x] \in G/\Gamma,$$

where the middle equality follows from $x \in G^0 \subset Z(G)$ and the last equality from $u(1) \in \Gamma$. □

(6.3.9) DEFINITION. For a given compact nilmanifold $X = G/\Gamma$ the fibration of Lemma (6.3.8), with the base $B = (G/Z(\Gamma))/(\Gamma/\Gamma \cap Z(G))$ being a nilmanifold of dimension $< \dim X$ and the fibre $F = Z(G)/\Gamma \cap Z(G)$ equal to the torus, is called the *Fadell–Husseini fibration*. Note that here $Z(G) = G_{k-1}$ of the central tower (6.2.1) is connected, and $\Gamma \cap Z(G) = \Gamma_{k-1}$ is a uniform lattice by Theorem (6.3.1.1).

Now we are in a position to apply Theorem (4.4.28).

(6.3.10) LEMMA. *Each admissible map of the fibre $\mathbb{T}$ in the Fadell–Husseini fibration* (6.3.9) *$p\colon X \to B$ is homotopic to the identity map. In particular the Nielsen number product formula $N(f) = N(\overline{f})N(f_b)$ holds for every fibre-map $f\colon X \to X$.*

PROOF. By Lemma (6.3.8) we can apply Theorem (4.4.28) which gives $N(f) = N(\overline{f})N_K(f_b)$. It remains to prove that $N_K(f_b) = N(f_b)$. But the base space

$X' := B$ is a nilmanifold, hence it is a $K(\pi, 1)$-space and consequently $\pi_2(B) = 0$ implies $K = 0$ for the group K of Theorem (4.4.28). $\square$

Now we may follow the above procedure for the nilmanifold X' and we get a fibration $p'\colon X' \to X''$ over another nilmanifold $X'' = (G/G_{k-2})\big/(\Gamma/\Gamma_{k-2})$ with the fibre $\mathbb{T}' = (G_{k-2}/\Gamma_{k-2})\big/(G_{k-1}/\Gamma_{k-1})$. By the above lemma we also get $N(f') = N(f'')N(f_{\mathbb{T}'})$ for every fibre map $f'\colon X' \to X'$. Continuing this procedure we get

(6.3.11) Theorem. *Let X be a compact nilmanifold. Then there exists a sequence of fibrations*

$$X = X_0 \xrightarrow{p_1} X_1 \xrightarrow{p_2} \cdots X_{k-2} \xrightarrow{p_{k-1}} X_{k-1} \xrightarrow{p_k} *$$

where all X_i are nilmanifolds, the fibre of p_i is a torus $\mathbb{T}$ of dimension d_{k+1-i} (see the notation at page 249) and the Nielsen number product formula holds for each fibre map f_i. Moreover, each continuous map $f\colon X \to X$ is homotopic to a map $\widetilde{f}$ inducing the commutative ladder.

$$\begin{tikzcd}
X_0 \arrow[r, "p_1"] \arrow[d, "\widetilde{f}_0"'] & X_1 \arrow[r, "p_2"] \arrow[d, "\widetilde{f}_1"'] & \cdots \arrow[r, "p_{k-1}"] & X_{k-1} \arrow[d, "\widetilde{f}_{k-1}"] \\
X_0 \arrow[r, "p_1"'] & X_1 \arrow[r, "p_2"'] & \cdots \arrow[r, "p_{k-1}"'] & X_{k-1}
\end{tikzcd}$$

Proof. It remains to prove only the last claim. But it is enough to take the factor map Φ_f/Γ of a preserving Γ homomorphism $\Phi_f\colon G \to G$ homotopic to f by Theorem (6.3.1). This induces the commutative ladder since the subgroups G_i and the lattice Γ, thus Γ_i, are preserved by Φ_f. $\square$

(6.3.12) Remark. Theorem (6.3.11) can be used for another definition of the linearization matrix A of a map $f\colon X \to X$ of a nilmanifold X, $\dim X = d$, by an induction with respect to the length of nilpotency of X. Indeed, replacing the map f by $\widetilde{f}$ of Theorem (6.3.11) we see that $\widetilde{f}_0\colon X \to X$ is a fibre map $\widetilde{f} := \widetilde{f}_0 = (\widetilde{f_{\mathbb{T}}}, \widetilde{f}_1)$ where $\widetilde{f_{\mathbb{T}}}$ is the map of fibre $F = \mathbb{T}^{d_k}$, and $\widetilde{f}_1$ is a map of X_1. We define

$$A_f := A_{\widetilde{f}_{\mathbb{T}}} \oplus A_{\widetilde{f}_1},$$

where $A_{\widetilde{f}_{\mathbb{T}}}$ is the linearization matrix of the torus map (cf. Proposition (6.3.6)), and $A_{\widetilde{f}_1}$ is given by induction.

It is not difficult to show that the above defined matrix is equal to the linearization matrix of Definition (6.3.4), up to the conjugacy class as in Remark (6.3.5).

Now we may prove the fundamental theorem which allows us to extend the Nielsen fixed point theory from tori into nilmanifolds. This theorem was proved simultaneously by Anosov [An], and also Fadell and Husseini [FaHu2].

(6.3.13) THEOREM. *Let* $f\colon X \to X$ *be a self-map of a compact nilmanifold. Then*

$$N(f) = |L(f)| \quad \textit{and} \quad L(f) = \det(\mathrm{I} - A),$$

where A *denotes the linearization matrix of* f (*cf. Definition* (6.3.4), *Proposition* (6.3.6)).

PROOF. We use induction with respect to the length of nilpotency k of the group of G which equals the length of the ladder diagram of Theorem (6.3.11).

For $k = 1$, X is a torus, hence the above formula follows from Theorem (4.3.14). Now we assume that the above formula holds for the nilpotency length $< k$.

Consider a self-map $f\colon X \to X$ of a nilmanifold of length k. This gives a fibration $p\colon X \to \overline{X}$ as in Lemma (6.3.8), with torus fibre $\mathbb{T}$ and we may assume that $f = (f_{\mathbb{T}}, \overline{f})$ is a fibre map. By the assumption, the lemma holds for the base map $\overline{f}\colon \overline{X} \to \overline{X}$. On the other hand the fibre map $f\colon X \to X$ satisfies the Nielsen number product formula (Lemmas (6.3.8), (6.3.10)). Now

$$N(f) = N(f_{\mathbb{T}}) \cdot N(\overline{f}) = |L(f_{\mathbb{T}})| \cdot |L(\overline{f})| = |L(f)|$$

where the right hand side equality comes from Lefschetz product formula for a fibre map (cf. Remark (4.4.4)). On the other hand, by the induction assumption $L(\overline{f}) = \det(\mathrm{I} - \overline{A})$ where $\overline{A}$ is a linearization matrix of $\overline{f}$. Thus

$$L(f) = L(f_{\mathbb{T}}) \cdot L(\overline{f}) = \det(\mathrm{I} - A_{\mathbb{T}}) \cdot \det(\mathrm{I} - \overline{A}) = \det(\mathrm{I} - A)$$

where $A_{\mathbb{T}}$ is a linearization of map $f_{\mathbb{T}}\colon \mathbb{T} \to \mathbb{T}$ and $A = A_{\mathbb{T}} \oplus \overline{A}$ is a linearization of f. □

6.3.3. Linearization and Anosov theorem for NR-solvmanifolds. Now we are going to study whether the formulae $L(f) = \det(\mathrm{I} - A)$ and $N(f) = |L(f)|$ are also valid for solvmanifolds. We start with a classical result claiming that each solvmanifold fibres over a torus with a nilmanifold as the fibre.

(6.3.14) THEOREM (Mostow Theorem). *Let* X *be a compact connected solvmanifold. Let us denote* $\pi = \pi_1(X)$. *Then there is the* (*unique*) *nilpotent subgroup* $\Gamma \subset \pi$ *such that* $[\pi, \pi]$ *is of finite index in* π *and* $\Lambda_0 = \pi/\Gamma$ *is torsion free. Moreover, there is a fibration* $\mathbf{N} \subset X \xrightarrow{p} \mathbb{T}_0$ *where* $\mathbf{N}$ *is a nilmanifold with* $\pi_1(\mathbf{N}) = \Gamma$, *and* $\mathbb{T}_0$ *is a torus with* $\pi_1(\mathbb{T}_0) = \Lambda_0$. *Moreover, each self-map of* X *is homotopic to a fibre-map of the fibration.*

The fibration from the above theorem is called *Mostow fibration*.

Thus we may assume that a given self-map $f\colon X \to X$ of a solvmanifold is a fibre map, hence we have a commutative diagram

$$\begin{tikzcd}
\mathbf{N} \arrow[r] \arrow[d, "f_b"'] & X \arrow[r, "p"] \arrow[d, "f"] & \mathbb{T} \arrow[d, "f_0"] \\
\mathbf{N} \arrow[r] & X \arrow[r, "p"'] & \mathbb{T}
\end{tikzcd}$$

(6.3.15) Definition. Let $f\colon X \to X$ be a self-map of a compact connected solvmanifold. We may assume that f is a fibre map as above. Assume that $L(f_0) \neq 0$. Let us fix a point $e \in \mathrm{Fix}(f_0)$. We define the linearization matrix of f as $A = A_e \oplus A_0$ where A_0 and A_e are linearization matrices of the maps f_0 and f_e respectively.

The definition depends on the choice of the point $e \in \mathrm{Fix}(f_0)$. Hence we make the assumption

($*$) The Lefschetz number of the restriction of f to the fibre over each fixed point $L(f_b)$ is the same.

It will be easy to prove that under this assumption the required formulae hold. Later we will present a class of solvmanifolds for which the condition ($*$) holds.

(6.3.16) Lemma. *If ($*$) holds and $L(f_0) \neq 0$ then $L(f) = \det(\mathrm{I} - A)$.*

Proof. $L(f) = L(f_e)L(f_0) = \det(\mathrm{I} - A_e)\det(\mathrm{I} - A_0) = \det(\mathrm{I} - A)$.

The first equality follows from Remark (4.4.4) since by the assumption ($*$) $L(f_b)$ does not depend on $b \in \mathrm{Fix}(f_0)$. The second holds since the base space and the fibre are nilmanifolds. The last one follows from the definition of the linearization matrix $A = A_e \oplus A_0$. $\square$

(6.3.17) Lemma. *If ($*$) holds, then $N(f) = |L(f)|$.*

Proof. We may assume as above that $\#\mathrm{Fix}(f_0) = N(f_0)$.

Let $L(f_0) = 0$. Then $N(f_0) = L(f_0) = 0$, hence $\mathrm{Fix}(f_0) = \emptyset$ implies $\mathrm{Fix}(f) = \emptyset$ and $N(f) = L(f) = 0$.

Now we assume that $L(f_0) \neq 0$.

Since the index at each fixed point $b \in \mathrm{Fix}(f_0)$ is the same and equals ± 1,

$$L(f) = \sum_{b \in \mathrm{Fix}(f_0)} \mathrm{ind}(f; \mathbf{N}_b) = \sum_{b \in \mathrm{Fix}(f_0)} \mathrm{ind}(f_b) \cdot \mathrm{ind}(f_0; b) = \pm \sum_{b \in \mathrm{Fix}(f_0)} L(f_b).$$

On the other hand Corollary (4.4.25) and Anosov Theorem (6.3.13) imply

$$N(f) = \sum_{b \in \mathrm{Fix}(f_0)} N(f_b) = \sum_{b \in \mathrm{Fix}(f_0)} |L(f_b)|.$$

Thus the equality $N(f) = |L(f)|$ holds iff all $L(f_b)$ have the same sign where $b \in \mathrm{Fix}(f_0)$. But the last is implied by the assumption $(*)$. $\square$

We are going to present the class of solvmanifolds satisfying the condition $(*)$.

We consider a compact solvmanifold X and its Mostow fibration $\mathbf{N} \subset X \to \mathbb{T}$. We will consider the action of $\Lambda_0 = \pi_1(\mathbb{T})$ on the fibre $\mathbf{N}_e$ over a chosen point in $\mathbb{T}$. Each homotopy class $\lambda \in \Lambda_0$ determines, up to homotopy, an admissible map τ_λ of the fibre $\mathbf{N}_e$. However the admissible maps do not preserve basic points, hence the induced homotopy group homomorphism is not well defined (if only $\pi_1(\mathbf{N}_e) = \Gamma$ is not abelian). But τ_λ correctly defines the isomorphism of the quotient (abelian) groups $\Lambda_i = \Gamma_i/\Gamma_{i+1}$ where $\Gamma_i = \pi_1(\mathbf{N})_i$ and $\mathbf{N} \to \mathbf{N}_1 \to \cdots \to \mathbf{N}_s = 1$ is the Fadell–Husseini tower of the nilmanifold $\mathbf{N} = \mathbf{N}_e$. This action may be described as follows. Let $\lambda \in \Lambda_0 = \pi_1(\mathbb{T}) = \pi/\Gamma$ and let $\gamma \in \Gamma_i$. Let $\widetilde{\lambda} \in \pi$ represent $\lambda \in \pi/\Gamma$. One can check that $\widetilde{\lambda} \cdot \gamma \cdot \widetilde{\lambda}^{-1} \in \Gamma_i$ and the class $[\lambda \cdot \gamma \cdot \lambda^{-1}] \in \Gamma_i/\Gamma_{i+1} = \Lambda_i$ does not depend on the choices of γ and $\widetilde{\lambda}$. Thus each homotopy class $\lambda \in \Lambda_0 = \pi_1(\mathbb{T})$ determines an isomorphism of $\Lambda_i = \pi_1(X_i)/\pi_1(X_{i+1})$. This yields a homomorphism $A_i\colon \Lambda_0 \to \mathrm{Aut}(\Lambda_i)$.

(6.3.18) DEFINITION. The solvmanifold X is called a *NR-solvmanifold* (No Roots) if no $A_i(\lambda)$ has a root of unity other than 1 as its eigenvalue.

Let f be a continuous self-map of a solvmanifold. We may assume that $f = (f_0, f_e)$ is a fibre map of the Mostow fibration $\mathbf{N} \subset X \xrightarrow{p} \mathbb{T}$ and $\mathrm{Fix}(f_0)$ is finite. We take a point $b \in \mathrm{Fix}(f_0)$. We are going to compare the induced homotopy homomomorphisms induced by the restrictions of f to the fibres over the fixed points e and b.

(6.3.19) LEMMA. *Let $\{\phi_i\colon \Lambda_i \to \Lambda_i\}$ be the linearization homomorphisms of the fibre* (*nilmanifold*) $\mathbf{N}_e$ *where* $\Lambda_i = \Gamma_i/\Gamma_{i+1} = \pi_1(X)_i/\pi_1(X)_{i+1}$ (*see Definition* (6.3.4))*. Then the linearization homomorphisms at* $\mathbf{N}_b$ *are* (*up to a conjugation*) $\{A_i(\omega) \cdot \phi_i\}$ *where ω is a loop based at e determined by the fixed point b* (*i.e.* $\omega = c * fc^{-1}$ *for a path c from e to b.*)

PROOF. Let c be a path from e to b. We fix an admissible map $\tau_c\colon \mathbf{N}_e \to \mathbf{N}_b$. Then $f_b \cdot \tau_c \simeq \tau_{f_0 c} \cdot f_e$ (Lemma (4.4.10)) hence $f_b \simeq \tau_{f_0 c} \cdot f_e \cdot \tau_{c^{-1}}$ which implies $f_b \simeq \tau_c \cdot (\tau_{c^{-1}} \cdot \tau_{f_0 c}) \cdot f_e \cdot \tau_{c^{-1}} \simeq \tau_c \cdot (\tau_{(f_0 c) \cdot c^{-1}} \cdot f_e) \cdot \tau_{c^{-1}}$. Thus the linearization of f_b equals (is conjugated to) the linearization of $\tau_{(f_0 c)\cdot c^{-1}} \cdot f_e = \tau_{\omega^{-1}} \cdot f_e$ where $\omega = c * (f_0 c)^{-1}$. But the last map gives the homomorphisms $A_i(\omega^{-1})\phi_i\colon \Lambda_i \to \Lambda_i$. $\square$

It turns out that each self-map of NR-solvmanifolds satisfies the $(*)$ property. This will follow from an algebraic lemma of Keppelmann and McCord given below as Theorem (6.3.20).

We are going to compare the numbers $L(f_b)$ and $L(f_e)$. Under the above notation

$$L(f_b) = \det(\mathrm{I} - A_b) = \prod_i \det(\mathrm{I} - \phi_i) = \prod_i \det(\mathrm{I} - A_i(\omega^{-1})\phi_i),$$

where A_b is the matrix representing the linearization homomorphisms

$$\phi_i \colon \Gamma_i/\Gamma_{i+1} \to \Gamma_i/\Gamma_{i+1} \quad \text{and} \quad \Gamma = \pi_i(\mathbf{N}_i)$$

and ω represents in $\pi_1(\mathbb{T}, e)$ the fixed point class containing $b \in \operatorname{Fix}(f_0)$. On the other hand we have

$$L(f_e) = \prod_i \det(\,\mathrm{I} - \phi_i).$$

Now to have the equality $L(f_b) = L(f_e)$ it is enough to know that

$$\det(\mathrm{I} - A_i(\omega^{-1})\phi_i) = \det(\mathrm{I} - \phi_i)$$

for all i. The following theorem claims that this holds for all self-maps of NR-solvmanifolds.

(6.3.20) Theorem. *Let $B \in Gl_n(\mathbb{Z})$, $\Psi \in Gl_m(\mathbb{Z})$ be invertible matrices and $A \colon \mathbb{Z}^m \to SL_n(\mathbb{Z})$ a homomorphism satisfying $BA(v) = A(\Psi v)B$ for all $v \in \mathbb{Z}^m$. If moreover, no $A(v)$ has a root of unity as an eigenvalue and Ψ does not have* 1 *as an eigenvalue, then* $\det(A(v)B - \mathrm{I})$ *is independent of v:* $\det(A(v)B - \mathrm{I}) = \det(B - \mathrm{I})$ *for all $v \in \mathbb{Z}^n$.*

We get the desired equality putting $B = \phi_i \colon \Lambda_i \to \Lambda_i$, the fundamental group homomorphism induced by f_e, on $\mathbb{Z}^m = \pi_1(\mathbb{T}^m)$ and $A(v)$ is the homomorphism of $\Lambda_i = \pi_1(X_i)/\pi_1(X_{i+1})$ induced by the admissible map τ_v. Finally $\Psi = f_{0\#} \colon \pi_1(\mathbb{T}^m) \to \pi_1(\mathbb{T}^m)$. The assumptions of the Theorem are satisfied. In fact:

- $f\tau_v = \tau_{f_0 v} f$ implies $BA(v) = A(\Psi v)B$;
- since $L(f_0) \neq 0$, 0 is not the eigenvalue of $\Psi = f_{0\#}$, hence Ψ is invertible;
- since X is NR-solvmanifold, no a root of unity $\neq 1$ is an eigenvalue of $A(v)$.

As a corollary we get the main theorem of this subsection

(6.3.21) Theorem (Anosov Theorem for NR-solvmanifolds). *Let $f \colon X \to X$ be a map of a compact NR-solvmanifold and A its linearization matrix in the sense of Definition* (6.3.15). *Then*

$$N(f) = |L(f)| = |\det(\mathrm{I} - A)|.$$

For a map of a compact completely solvable manifold the matrix A_f of linearization of f can be defined in an analytic way described below (cf. [JeKdMr]).

We start with the following result of Hattori ([Ht], [Sa]) for completely solvable solvmanifolds that generalized a previous result of Nomizu [No] for nilmanifolds. We recall that for a given Lie algebra $\mathcal{G}$ the Chevalley–Eilenberg complex $(\Lambda^*\mathcal{G}^*, \delta)$ associated with $\mathcal{G}$ consists of the exterior algebra $\Lambda^*\mathcal{G}^*$ of the dual space $\mathcal{G}^*$ considered as a complex of vector spaces with the j-th, $0 \leq j \leq m$ gradation equal to $\wedge^j\mathcal{G}^*$ and the differential $\delta\colon \wedge^j\mathcal{G}^* \to \wedge^{j+1}\mathcal{G}^*$ defined as

$$\delta(\mathcal{X}^*)(\mathcal{X}_1, \dots, \mathcal{X}_{j+1}) := \sum (-1)^{s+t-1}\mathcal{X}^*([\mathcal{X}_s, \mathcal{X}_t], \mathcal{X}_1, \dots, \widehat{\mathcal{X}_s}, \dots, \widehat{\mathcal{X}_s}, \dots, \mathcal{X}_{j+1}),$$

where the sum is taken over all $1 \leq s \leq t \leq j+1$.

(6.3.22) THEOREM. *Let $(\Lambda^*\mathcal{G}^*, \delta)$ denote the Chevalley–Eilenberg complex associated to the Lie algebra $\mathcal{G}$ of a simply-connected completely solvable Lie group G. If $\Gamma \subset G$ is a discrete co-compact subgroup, then $H^*(G/\Gamma; \mathbb{R}) \cong H^*(\Lambda^*\mathcal{G}^*, \delta)$.*

Note that the Chevalley–Eilenberg complex can by identified with a subcomplex of the de Rham complex consisting of invariant forms. This result together with the Hopf Lemma for Traces (cf. Lemma (2.3.23) and [Sp]) leads to the following.

(6.3.23) PROPOSITION. *Let $f\colon G/\Gamma \to G/\Gamma$ be a self-map of a compact special completely solvable solvmanifold of dimension d. Let next $\Phi\colon G \to G$ be a homomorphism such that its factor map $\psi\colon G/\Gamma \to G/\Gamma$ is homotopic to f. Then for the linear operator $D\Phi_f(e)$ we have the inclusion of spectrum $\sigma(H^*(f)) \subset \sigma(\wedge D\Phi_f(e))$ and consequently the estimate $\mathrm{sp}(f) \leq \mathrm{sp}(\wedge D\Phi_f(e))$. Moreover, for the Lefschetz number we have $L(f^m) = \det(\mathrm{I} - (D\Phi(e))^m)$ for every $m \in \mathbb{N}$.*

PROOF. By the Nomizu and Hattori theorems the spectral radius and Lefschetz number of f can be derived by use of the map $D\Phi(e)^*$ of the Chevalley–Eilenberg complex. Since $D\Phi(e)$ is a homomorphism of the Lie algebra, the linear subspaces of co-boundaries, co-cycles are preserved by $\wedge D\Phi(e)^*$. Consequently the cohomology spaces can be identified with the factors of subspaces preserved by $\wedge D\Phi(e)^*$. The inclusion, and consequently inequality, follows from the fact that the spectrum of an operator restricted to an invariant subspace is a subset of the spectrum of an entire operator and the same for the factor operator induced on the factor of an invariant subspace. The second equality

$$L(f^n) = \sum_{k=0}^{k=m} (-1)^k \mathrm{tr}\, H^k(f^n) = \sum_{k=0}^{k=m} (-1)^k \mathrm{tr}\, \Lambda^k(D\Phi(e)^*)^n = \det(\mathrm{I} - (D\Phi(e))^n)$$

is a direct consequence of the Hopf Lemma (2.3.23) and linear algebra. $\square$

Now we show that the matrix $D\Phi(e)$ has the same spectrum as an integral $m \times m$-matrix A_f.

(6.3.24) Proposition. *Let $f\colon M \to M$ be a map of a compact completely solvable solvmanifold, e.g. a nilmanifold, which is induced by an endomorphism $\Phi\colon G \to G$. We have $\sigma(D\Phi(e)) = \sigma(A_f)$. Consequently* $\operatorname{sp}(D\Phi(e)) = \operatorname{sp}(A_f)$ *and* $\operatorname{sp}(\wedge D\Phi(e)) = \operatorname{sp}(\wedge A_f)$.

Proof. To shorten notation put $D := D\Phi(e)$. Assuming that M is a nilmanifold, we show that these two matrices have the same characteristic polynomial by using an induction over the length on nilpotency of G. If $M = \mathbb{T}^d$, thus $\mathcal{G} = \mathbb{R}^d$, then Φ is a linear map and $D = \Phi = A_f$.

In the general case note that the central tower of G gives a descending tower of ideals of $\mathcal{G}$:

$$\mathcal{G}_0 = \mathcal{G} \lhd \mathcal{G}_1 = [\mathcal{G}, \mathcal{G}] \lhd \mathcal{G}_2 \lhd \cdots \lhd \mathcal{G}_{k-1} \lhd \mathcal{G}_k = e,$$

which is preserved by any homomorphism of $\mathcal{G}$ where $\mathcal{G}_i$ is the Lie algebra of G_i. Since D is a homomorphism, the matrix of D in a basis $\mathcal{G}$ formed by $\exp^{-1}$ of generators Γ has the form

$$D = \begin{bmatrix} \widetilde{D}_1 & 0 \\ \breve{D}_1 & D_1 \end{bmatrix},$$

where $\widetilde{D}_1 = D\Phi(e)|_{\mathcal{G}_1}$ and D_1 is the matrix induced by D on the quotient (abelian) algebra $\mathcal{G}_0/\mathcal{G}_1$. Consequently, for the characteristic polynomial $\chi_D(t)$ we have $\chi_D(t) = \chi_{D_1}(t)\chi_{\widetilde{D}_1}(t)$, and the statement follows by the induction argument. $\square$

As a consequence of the above construction we get the following.

(6.3.25) Corollary. *Let $f\colon X \to X$ be a map of a compact nilmanifold $X = G/\Gamma$ and $\Phi_f\colon G \to G$ an endomorphism of G such that its factor map is homotopic to f. Let next $D = D\Phi(e)$ be the corresponding endomorphism of the Lie algebra $\mathcal{G} = T_eX$ of G and D_i, $1 \le i \le k$, the corresponding endomorphisms of the factor algebras $\mathcal{G}_i/\mathcal{G}_{i-1}$. Then $D_i = A_i$ and consequently $A_f = \bigoplus_{i=1}^k D_i$. Furthermore, if $f = (f_N, f_0)$ is a map of the Mostow fibration $\mathbf{N} \subset X \xrightarrow{p} \mathbb{T}_0$ of an NR-solvmanifold X, as in Theorem* (6.3.14), *then $A_f = \bigoplus_{i=0}^k D_i$, where D_i, $1 \le i \le k$, are matrices defined above for f_N, and D_0 is the linearization of f_0.*

6.3.4. Summation formula for maps of solvmanifolds. In this subsection we sum up properties of the Nielsen number of iteration, full Nielsen–Jiang periodic number, and prime Nielsen–Jiang number for a map of a solvmanifold. Our aim is to emphasize that they are relations between the mentioned invariants caused by the special structure of a manifold. It will be done by using the material already exposed in Chapters IV and V.

Let $f\colon X \to X$ be a self-map of a compact space. Then the full Nielsen–Jiang number $NF_m(f) \in \mathbb{N} \cup \{0\}$, (cf. Definition (5.1.16)) has the following properties

(Theorem (5.1.18)):

$$\begin{gathered} 0 \le N(f^m) \le NF_m(f) \le \#P^m(f), \\ NF_m(f) \text{ is a homotopy invariant.} \end{gathered} \tag{6.3.26}$$

The prime Nielsen–Jiang $NP_m(f) \in \mathbb{N} \cup \{0\}$, (cf. Definition (5.1.14)) has the following properties (Theorem (5.1.15)):

$$\begin{gathered} 0 \le NP_m(f) \le \#P_m(f) \quad \text{and} \quad NP_m(f) \le NF_m(f), \\ NP_m(f) \text{ is a homotopy invariant.} \end{gathered} \tag{6.3.27}$$

The Wecken theorem for periodic points says that $NF_m(f)$ is the best homotopy invariant estimating the number of points of period m, provided X is the PL-manifold of dimension $d \ge 3$ (Theorem (5.3.1)). A non-trivial property of the latter invariant $NP_m(f)$ is that if it vanishes, then f can be deformed to a map without the m-periodic points provided X is the PL-manifold of dimension $d \ge 3$ (Theorem (5.4.1)).

(6.3.28) Remark. The statement of these two theorems is not true in dimension 2, as we already mentioned.

Suppose for a while that X is a finite set. Then for $f\colon X \to X$ we have a summation over divisors of m $N(f^m) = \#P^m(f) = \sum_{k|m} \#P_k(f) = \sum_{k|m} NP_k(f)$, by an obvious combinatorial argument. One could expect the similar summation formula with the full Nielsen–Jiang periodic number and the prime Nielsen–Jiang periodic numbers. Unfortunately, we have only inequality (cf. (5.1.20)),

$$\sum_{k|m} NP_k(f) \le NF_m(f). \tag{6.3.29}$$

It is not difficult to show that the inequality is sharp in general. Indeed, it is enough to take the antipodal map $f(x) = -x$ of the sphere S^{2d} and $m = 2$ (cf. Example (5.1.19)).

Anyway, if X is a compact solvmanifold, then the equality, called the summation formula, holds due the fact that (see [HeKeI], also [HeKeII], [HeKeIII]).

(6.3.30) Theorem. *For a map $f\colon X \to X$ of compact solvmanifold we have*

$$NF_m(f) = \sum_{k|m} NP_k(f) \quad \text{and consequently} \quad NP_m(f) = \sum_{k|m} \mu(m/k) NF_k(f),$$

where $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$ is the Möbius function defined in (3.1.2).

Proof. It is enough to notice that compact solvmanifolds are essentially reducible and apply Lemma (5.1.22). In fact a solvmanifolds fibers over a torus with a nilmanifold as the fibre. This follows from Theorem (5.1.31). Indeed, tori are essentially reducible, hence the Fadell–Husseini fibration imply that so are nilmanifolds. Finally, each solvmanifold fibers over a torus with a nilmanifold as the fibre, hence using Theorem (5.1.31) we get that a solvmanifold is essentially reducible. □

For maps of NR-solvmanifolds the inequality $N(f^m) \leq NF_m(f)$ of (6.3.26) becomes the equality (cf. [HeKeI]).

(6.3.31) THEOREM. *For a map $X \to X$ of an NR-solvmanifold such that $N(f^m) \neq 0$ we have*

$$NF_m(f) = N(f^m).$$

The statement follows from Theorem (5.1.31) by the following proposition which confirms that the assumption of this theorem is satisfied.

(6.3.32) PROPOSITION. *Every compact solvmanifold X has essential torality and is essentially reducible to the* GCD. *Moreover, each self-map $f\colon X \to X$ of a compact NR-solvmanifold is weakly Jiang.*

PROOF. Since the essential reducibility to the GCD and toral essentiality are valid for tori (Corollary (5.1.35)) and each nilmanifold fibers over another nilmanifold with a torus as the fibre (cf. Definition (6.3.9)), Theorem (5.1.37), and Lemma (5.1.38) allow us to extend these properties onto each nilmanifold. Next, using the same argument to the Mostow fibration (Theorem (6.3.14)) we see that any compact solvmanifold has these properties.

On the other hand from the Anosov formula $N(f^m) = |L(f^m)|$ (see Theorem (6.3.21)) it follows that the indices of all Reidemeister classes of f^m are all equal (to 1, -1, or 0). Thus each NR-solvmanifold is weakly Jiang. □

Summing up, for a map f of an NR-solvmanifold we have $N(f^m) = |\det(\mathrm{I}-A^m)|$, where A the integral matrix of linearization by Theorem (6.3.21). Consequently the sequence $\{N(f^m)$ has nice arithmetical properties. Additionally it expresses the sequence $\{NP_m(f)\}$ by the the Möbius inversion as follows from Theorems (6.3.30), (6.3.31), because $NF_m(f) = N(f^m)$ provided $N(f^m) \neq 0$.

6.4. Main theorem for NR-solvmanifolds

In this section we present a theorem which describes the set of homotopy minimal periods for a map of an NR-solvmanifold. It is based on the proof of the corresponding theorem for tori maps given by Boju Jiang and Jaume Llibre in [JiLb] in 1998. The proof used all the properties of Nielsen numbers of iteration presented in our subsections 6.3.2 and 6.3.4, which were known for maps of torus earlier. A brilliant aspect of their approach was, for a given map $f\colon \mathbb{T}^d \to \mathbb{T}^d$, to pick up all m for which $NP_m(f) = 0$. It is given in terms of a condition on $N(f^m)$, by a fine combinatorial argument. Then they could use the You theorem (cf. [Yu2], [Yu3]) which states (among others facts) that for a torus map, $NP_m(f) = 0$ implies that f is homotopic to a map g such that $P_m(f) = \emptyset$ excluding such m from $\mathrm{HPer}(f)$. Remark that the mentioned You theorem is a special case of the general Theorem (5.4.1) for the torus map. Another distinction of the paper [JiLb] was the observation that there exists $Q(d)$, (i.e. depending on d only) such that for all $m > Q(d)$ the mentioned above condition can not be satisfied if $N(f^m) \neq 0$, i.e. $\mathrm{HPer}(f)$ is equal to the set $\{m : N(f^m) \neq 0\}$ outside of the interval $[1, Q(d)]$. To show this Jiang and Llibre used an estimate from algebraic number theory proved by them. Meantime their theorem was extended onto the case of maps of a nilmanifold [JeMr1] and consecutively maps of an NR-solvmanifold of dimension > 3 or a completely solvable solvmanifold of dimension ≥ 3 in [JeKdMr]. These generalization were possible due to extensions of the You theorem from tori onto the nilmanifold, and correspondingly completely solvable solvmanifolds of dimension ≥ 3 shown in [JeMr1] and [JeKdMr] respectively. Nowadays all these facts are consequences of the general Theorem (5.4.1) (cf. [Je2], [Je3], [Je4]) which does not need any special structure of a manifold. At this point it would be enough to say that to prove the below theorem one should use all the quoted properties of the sequence of Nielsen numbers $\{N(f^m)\}$, of a map of NR-solvmanifold (Theorems (6.3.31), (6.3.30)) and Theorem (5.4.1), and repeat almost verbatim the arguments of paper [JiLb]. Anyway we include these arguments, in a little bit extracted form, to expose the proof of the theorem.

Before stating the theorem we need the next notion.

(6.4.1) DEFINITION. Let $f \colon X \to X$ be a map of a compact NR-solvmanifold of dimension $d \geq 3$ and A its linearization $d \times d$ matrix (Definitions (6.3.4) and (6.3.15)). Put

$$\mathbb{N} \supset T_A := \{m : \det(\mathrm{I} - A^m) \neq 0\}.$$

We call T_A set of *algebraic minimal homotopy periods.*

If $m \notin T_A$ then $m \notin \mathrm{HPer}(f)$, i.e.

$$\mathrm{HPer}(f) \subset T_A. \tag{6.4.2}$$

Indeed, then $L(f^m) = N(f^m) = 0$ by the Anosov Theorem (6.3.21), thus $NP_k(f) = 0$, and Theorem (5.4.1) implies $m \notin \mathrm{HPer}(f)$. Observe also that

$$N(f) = 0 \Leftrightarrow L(f) = 0 \Leftrightarrow 1 \in \sigma(A_f). \tag{6.4.3}$$

We are in position to formulate our main theorem, which characterizes $\mathrm{HPer}(f)$ by T_{A_f}.

(6.4.4) THEOREM. *Let $f: X \to X$ be a map of a compact NR-solvmanifold of dimension $d \geq 3$, $A = A_f$ its linearization, and $T_A = \{m : \det(\mathrm{I} - A^m) \neq 0\}$. Then* $\mathrm{HPer}(f) \subset T_A$ *and it is in one of the following three* (*mutually exclusive*) *types:*

(E) $\mathrm{HPer}(f) = \emptyset$ *if and only if* $N(f) = 0 \Leftrightarrow 1 \in \sigma(A)$;

(F) $\mathrm{HPer}(f) \neq \emptyset$ *but finite* $\Leftrightarrow$ *all eigenvalues of A are either zero or roots of unity;*

(G) $\mathrm{HPer}(f)$ *is infinite and* $T_A \setminus \mathrm{HPer}(f)$ *is finite if there exists* $\lambda \in \sigma(A)$ *such that* $|\lambda| > 1$.

Moreover, for every d there exist constants $P(d), Q(d) \in \mathbb{N}$ depending on d only such that $\mathrm{HPer}(f) \subset [1, P(d)]$ *in Type* (F) *and* $T_A \setminus \mathrm{HPer}(f) \subset [1, Q(d)]$ *in Type* (G).

It is worth pointing out that the statement of this theorem is the same as that of its correspondent in the case when X is the torus $\mathbb{T}^d$ (cf. [JiLb]).

As we already mentioned, the description of $\mathrm{HPer}(f)$ is attainable due to an observation established by Boju Jiang and Llibre (cf. [JiLb]).

(6.4.5) THEOREM. *Let $f: X \to X$ be a map of a compact NR-solvmanifold. Then $NP_m(f) = 0$ if and only if either $N(f) = 0$ or $N(f^m) = N(f^{m/p})$ for some prime factor p of m.*

The proof of Theorem (6.4.5) will be discussed in Subsection 6.4.1.

(6.4.6) REMARK. We will see that the statement of Theorem (6.4.5) remains true for every self-map of a compact manifold for which the summation formula holds (Theorem (6.3.30)), the statement of Theorem (6.3.31), and the equality $N(f^m) = |\det(\mathrm{I} - A^m)|$, with an integral matrix A.

Finally, a nontrivial estimate from below of the rate of convergence of an algebraic number of module 1 is necessary to show that there exists $Q(d)$ such that $N(f^m) > N(f^{m/p})$ for all $m \in T_A$ satisfying $m > Q(d)$. To write it down we need a new notion.

(6.4.7) DEFINITION. Let α be an algebraic number of degree d and $a_0 x^d + a_1 x^{d-1} + \cdots + a_d$ its minimal polynomial with roots $\alpha_1, \ldots, \alpha_d$. The measure of α is defined as

$$M(\alpha) := a_0 \prod_{i=1}^{d} \max\{1, |a_i|\}.$$

The below characterization of an algebraic number given by Jiang–Llibre, and also Mignotte, (cf. [JiLb]) is crucial for the consideration.

(6.4.8) THEOREM. *For every algebraic number α of degree d and every $m \in \mathbb{N}$ such that $\alpha^m \neq 1$, we have*

$$|1-\alpha^m| > \frac{1}{2}e^{-9\alpha H^2},$$

where

$$a = \max\left\{20,\ 12.85|\log\alpha| + \frac{1}{2}\log M(\alpha)\right\}, \quad H = \max\left\{17, \frac{d}{2}\log m + 0.66d + 3.25\right\}.$$

As a consequence the following was derived in [JiLb] that allows us to prove the last part of Theorem (6.4.4).

(6.4.9) COROLLARY. *Let A be an integral $d \times d$ matrix and $\rho := \mathrm{sp}(A)$ its spectral radius. Suppose that* $\det(\mathrm{I} - A^m) \neq 0$, *$\rho > 1$ and $m \geq 5000$. Then for every $n|m$ we have*

$$\frac{|\det(\mathrm{I}-A^m)|}{|\det(\mathrm{I}-A^n)|} > \frac{\rho^{m/2}-1}{e^{9d(41.4+(d/2)\log\rho)(d\log m)^2}}.$$

We postpone a discussion of Theorem (6.4.5), (6.4.8), and Corollary (6.4.9) to next next subsections.

PROOF OF THEOREM (6.4.4). The proof is almost literally taken from [JiLb]. With respect to Theorem (5.4.1) we have $\mathrm{HPer}(f) = \{m : NP_m(f) \neq 0\}$. By inclusion (6.4.2) the set $\mathrm{HPer}(f)$ is equal to the set of all $m \in T_A$ for which the condition of Theorem (6.4.5) is violated.

Let $\psi_k(z)$ be the k-th cyclotomic polynomial, and let $\Psi_k := |\det\psi_k(A)|$. By the identity $z^m - 1 = \prod_{k|m}\psi_k(z)$ we have

$$N(f^m) = |\det(A^m - \mathrm{I})| = \prod_{k|m}|\det\psi_k(A)| = \prod_{k|m}\Psi_k. \tag{6.4.10}$$

Note the coefficients of ϕ_k are integers, A is an integer matrix and all these Ψ_k are nonnegative integers. Since $\mathrm{I} - A^m = (\mathrm{I} - A^n)(\mathrm{I} + A + \cdots + A^{m/n})$ if $n \mid m$ and det is a multiplicative function, we have

(6.4.11) PROPOSITION. *If $n|m$ and $N(f^n) = 0$, then $N(f^m) = 0$; if $n|m$ and $N(f^n) \neq 0$ then $N(f^n)|N(f^m)$.*

(6.4.12) PROPOSITION. *Let χ_A be the characteristic polynomial of an integral matrix A. Put $D_A := \{k \in \mathbb{N} : \psi_k|\chi_A\}$. Then*

(6.4.12.1) *D_A is a finite set, and $T_A = \mathbb{N} \setminus \bigcup_{k\in D_A} k\mathbb{N}$,*
(6.4.12.2) *T_A is empty when* $\det(\mathrm{I}-A) = 0$, *or is infinite when* $\det(\mathrm{I}-A) \neq 0$.

PROOF. (6.4.12.1) If $m \notin T_A$, then there exists $\lambda \in \sigma(A)$ such that $\lambda^m = 1$. Then λ is a (primitive) root of unity of degree $k|m$, i.e. a root of the cyclotomic polynomial ψ_k. Since ψ_k is irreducible, we have $\psi_k|\chi_A$, i.e. $k \in D_A$. It is known that ψ_k is of degree $\phi(k)$, where ϕ is the Euler function. It is also known that for a given d there is only a finite number of k such that $\phi(k) \leq d$, which shows that D_A is finite.

(6.4.12.2) When $\det(\mathrm{I} - A) = 0$, then $T_A = \emptyset$ since then $\det(\mathrm{I} - A^m) = 0$ for all m. If $\det(\mathrm{I} - A) \neq 0$ let m be the least common multiple of the set D_A. By (6.4.12.2) we see that $km + 1 \in T_A$ for every $k \in \mathbb{N}$. □

Suppose that the eigenvalues $\lambda_1, \dots, \lambda_d$ of A are indexed such that $\mathrm{sp}(A) = \rho = |\lambda_1| \geq \cdots \geq |\lambda_d|$. Then each λ_i is an algebraic integer of degree $k_i \leq d$ with the measure (cf. Definition (6.4.7)) $M(\lambda_i) \leq \rho^{k_i}$. The latter follows from the fact that for a monic polynomial (i.e. with the leading coefficient equal to 1) with integral coefficients we have this inequality.

Let us consider the cases of Theorem (6.4.4) depending on the form of spectrum of the linearization A.

Type (E). $\det(\mathrm{I} - A) = N(f) = 0$. Then, by Proposition (6.4.12) $T_A = \emptyset$ and from (6.4.2) it follows that $\mathrm{HPer}(f) = \emptyset$. Note that it is the first case of Theorem (6.4.5).

When f is not of Type E, then there are two possibilities, $\rho \leq 1$ or $\rho > 1$.

Type (F). $N(f) = \det(\mathrm{I} - A) \neq 0$ and $\rho \leq 1$. Then, by the Kronecker theorem [Nar] the all nonzero eigenvalues $\lambda_1, \dots, \lambda_q$ are roots of unity. The Kronecker theorem states that if the all roots of a monic polynomial with integral coefficients are in the unit disc, then they must be roots of unity or zero.

Each λ_i, $1 \leq i \leq q$, is a root of some cyclotomic polynomial $\phi_{k_i}(\lambda)$ of degree $k_i = \phi(d_i) \leq d$. Let $h = h(d)$ be the least common multiple of the set $\{k \in \mathbb{N}; \phi(k) \leq d\}$. Then $\lambda_i^h = 1$ for all $1 \leq i \leq q$. Note that $h \notin \mathrm{HPer}(f)$ by its definition and the sequence $|\det(\mathrm{I} - A^m) = \prod_{i=1}^q |1 - \lambda_i^m|$ is periodic of period h, i.e. $N(f^{m+h}) = N(f^m)$ for all m. We show that if $m \in \mathrm{HPer}(f)$, then $m|h$, hence $\mathrm{HPer}(f)$ is contained in the set of all proper divisors of $h = h(d)$, thus contained in the interval $[1, P(d)]$, where $P(d)$ is the largest proper divisor of h.

Now take $\delta = (m, h)$ the greatest common divisor of $m \in T_A$ and h. Then there exist natural numbers a, b such that $am - bh = \delta$. Since am is a multiple of n, by Proposition (6.4.11) $N(f^m)$ divides $N(f^{am}) = N(f^{bh+\delta}) = N(f^\delta)$. But $N(f^\delta)|N(f^m)$, because $\delta|m$, which leads to $N(f^m) = N(f^\delta)$. If $m \neq \delta$, then $NP_m(f) = 0$, as follows from the summation formula (Theorem (6.3.30)), since all the summands are nonnegative. Consequently $m \notin \mathrm{HPer}(f)$, which completes the proof of this case.

Type (G). Suppose $N(f) \neq 0$ and $\rho > 0$. Let $m \in T_A$ be such that $N(f^m) \neq 0$, thus also $N(f^n) \neq 0$ for every $n|m$. Applying the inequality of Corollary (6.4.9) we have

$$\frac{N(f^m)}{N(f^n)} > \frac{\rho^{m/2}-1}{e^{9d(41.4+(d/2)\log\rho)(d\log m)^2}} \quad \text{provided } m \geq 5000. \tag{6.4.13}$$

Consequently if

$$\frac{\rho^{m/2}-1}{e^{9d(41.4+(d/2)\log\rho)(d\log m)^2}} > 1, \tag{6.4.14}$$

then $N(f^m) > N(f^n)$ for every proper divisor n of m, so $m \in \mathrm{HPer}(f)$ by Theorem (6.4.5). Since the rate of growth of the numerator of the expression of (6.4.14) as a function of m is smaller than of its denominator, for any fixed $\rho > 1$ the inequality (6.4.14) is valid for sufficiently large m. Therefore the set $T_A \setminus \mathrm{HPer}(f)$ is finite for every matrix A with $\rho > 1$. We are left with the task to find a common bound valid for all such $d \times d$ matrices. Consider first the case $\rho \geq e^{82.8/d}$. Then $41.4 + \log(d/2)\rho \leq d\log\rho$, hence (6.4.14) is valid if

$$\frac{m}{2} > 9d^4(\log m)^2. \tag{6.4.15}$$

Let $m_1 \geq 5000$ be an integer that satisfies (6.4.15). Then (6.4.15) and consequently (6.4.14) holds for all $m \geq m_1$.

Finally consider the remaining case of $\rho < e^{82.8/d}$. Since the coefficients in the characteristic polynomial $\chi_A(\lambda)$ are their elementary symmetry polynomials in the eigenvalues we have

$$|a_i| \leq \binom{m}{i}\rho^i \quad \text{for all } 1 \leq i \leq d. \tag{6.4.16}$$

So there is only a finite number of possibilities for the integral sequences $(a_1, \dots, a_d)$ with $\rho < e^{82.8/d}$. Let $\rho > 1$ be the smallest of corresponding $\rho's$ and let m_2 be the smallest m that validates (6.4.14) for $\rho = \rho_1$. Thus, when $m \geq m_2$, the condition (6.4.14) is true whenever $\rho \geq e^{82.8/d}$.

Finally, let us put $m_0(d) := \max\{m_1, m_2\}$. By the above inequality (6.4.14) is satisfied if $m \geq m_0$, which shows that $T_A \setminus \mathrm{HPer}(f) \subset [1, \dots, m_0]$. Now take $Q(d) = m_0$. The proof of Theorem (6.4.4) is complete. □

6.4.1. Combinatorics and number theory. As we already said the proof of Theorem (6.4.5) is a combinatorics argument proved in ([JiLb, Proposition 3.3]) which we only quote here referring the reader to [JiLb] for details.

Let ω be a nonempty finite set. By $C := \{\omega, C_\tau\}$ we denote a function $C\colon 2^\omega \to [1, \infty) \subset \mathbb{R}$.

(6.4.17) PROPOSITION. *For a given $\tau \subset \omega$ and a function $C\colon 2^\omega \to [1,\infty)$ define*

$$D_\tau := \prod_{\xi\subset\tau} C_\xi, \quad \textit{for any } \tau\subset\omega, \quad E_\omega := \sum_{\tau\subset\omega}(-1)^{|\omega|-|\tau|}D_\tau,$$

where $|\tau|$ denotes here the cardinality of τ. Then

(6.4.17.1) $E_\omega \geq 0$,

(6.4.17.2) $E_\omega = 0$ *if and only if* $D_\omega = D_{\omega\setminus\{\alpha\}}$ *for some element* $\alpha\in\omega$.

PROOF OF THEOREM (6.4.5). We may assume that $N(f^m) > 0$. Let the prime factorization of m be

$$m = p_1^{\alpha_1}\cdots p_s^{\alpha_s}, \quad s\geq 1,\ p_i\text{-prime},\ \alpha_i\geq 1. \tag{6.4.18}$$

Let $\omega := \{1,\dots,s\}$ and $\tau\subset\omega$ its subset. Put $p_\tau := \prod_{i\in\tau} p_i$ and $m_\tau := m/p_{\omega\setminus\tau} = mp_\tau/p_\omega$. The Möbius inversion formula for $NP_m(f)$ can be written in the following equivalent form (cf. [HeKeI]),

$$NP_m(f) = \sum_{\tau\subset\omega}(-1)^{s-|\tau|}N(f^{m_\tau}). \tag{6.4.19}$$

In view of Theorem (5.4.1), $m\notin \mathrm{HPer}(f)$ if and only if $NP_m(f)=0$. Define

$$C_\tau := \prod_j \Psi_j \quad \text{where } j \text{ is such that } j = p_1^{\beta_1}\dots p_s^{\beta_s}, \tag{6.4.20}$$

with

$$\beta_i = \begin{cases} \alpha_i & \text{if } i\in\tau,\\ 0\leq\beta_i\leq\alpha_i & \text{if } i\notin\tau.\end{cases}$$

Note that $\det(\mathrm{I}-A^m)\neq 0$ implies $\Psi_k\geq 1$ for all $k|m$, hence all $C_k\geq 1$. It is easy to check that

$$\begin{aligned} D_\tau &= \prod_{\xi\subset\tau} C_\xi = \prod_{j|m_\tau}\Psi_j = N(f^{m_\tau}),\\ E_\omega &= \sum_{\tau\subset\omega}(-1)^{|\omega|-|\tau|}D_\tau = NP_m(f).\end{aligned} \tag{6.4.21}$$

Now applying Proposition (6.4.17), we conclude that $N(f^m) = E_\omega = 0$ if and only if $N(f^m) = D_\omega = D_{\omega\setminus\{i\}} = N(f^{m/p_i})$ for some $1\leq i\leq s$. $\square$

Let us turn back to the number theory part of the proof of Theorem (6.4.4). Since it is taken form [JiLb], we refer the reader there for details. In particular we are not going to show how Theorem (6.4.8) implies the inequality in Lemma (6.4.9). Note that Theorem (6.4.8) says that the powers of an algebraic number of absolute value equal 1, but not being a root of unity, must not tend to 1 too fast. However recall that the aim was to prove that $N(f^m)/N(f^n) > 1$ for $m > m_0(d)$, d the dimension of the matrix A. It was pointed out to the author by J. Browkin and A. Schinzel that the inequality $N(f^m)/N(f^n) > 1$ follows also from a result of Schinzel ([Schi]). To show it we need new notions and definitions. Let α, β be nonzero integers of an algebraic number field $\mathcal{K}$ of degree d. A prime ideal $\mathfrak{B}$ of $\mathcal{K}$ is called a primitive divisor of $\alpha^m - \beta^m$ if $\mathfrak{B}|(\alpha^m - \beta^m)$, but $\mathfrak{B}$ does not divide $\alpha^n - \beta^n$ if $n < m$. In 1974 A. Schinzel proved the following theorem (cf. [Schi, Theorem I]).

(6.4.22) THEOREM. *If $(\alpha, \beta) = 1$ and α/β is not a root of unity, then $\alpha^m - \beta^m$ has a primitive divisor for all $m > m_0(d)$, where d is the degree of α/β and $m_0(d)$ is effectively computable.*

Suppose that $N(f^m) = |\det(A^m - 1)|$, $A \in M_{d\times d}(\mathbb{Z})$ is an integer. For every eigenvalue $\lambda_j \in \sigma(A)$, $1 \le j \le d$, take $\alpha_j := \lambda_j$ and $\beta_j := 1$. By the definitions α_j, β_j are integers of the algebraic field given by the characteristic polynomial of A. If $m \in T_A$, i.e. $N(f^m) \ne 0$ then the hypothesis of the Schinzel theorem is satisfied. Note that if $n|m$, $k = m/n$, then

$$(\lambda^m - 1) = (\lambda^n - 1)(1 + \lambda^n + \lambda^{2n} + \cdots + \lambda^{(k-1)n}). \tag{6.4.23}$$

Consequently, for any $1 \le j \le d$ such that $|\lambda_|| > 1$ and $m \in T_A$, $m > m_0(d_j)$ there exists a primitive ideal $\mathfrak{B}_j \subset \mathcal{K}$ such that $\mathfrak{B}_j|1 + \lambda_j^n + \lambda_j^{2n} + \cdots + \lambda_j^{(k-1)n}$ as follows from the Schinzel theorem. Observe also that $d(\alpha_j) = d(\lambda_j)$ is a divisor of $d :=$ degree $\mathcal{K}$ and $m_0(d_j) \le m_0(d)$, by an argument in the proof of Theorem 3.4 of [Schi]. From this it follows that

$$\mathfrak{B}_j \quad \text{divides} \quad \frac{N(f^m)}{N(f^n)} = \prod_1^j (1 + \lambda_j^n + \lambda_j^{2n} + \cdots + \lambda_j^{(k-1)n}),$$

for every $1 \le j \le d$.

The above implies that then there exists a prime $q \in \mathcal{P} \subset \mathbb{N}$ such that q divides $N(f^m)/N(f^n)$ provided $m \in T_A$, $m > m_0(d)$. Indeed it is enough to take $q \in \mathcal{P}$, where q^f is the norm $|\mathfrak{B}_j|$ of the ideal $\mathfrak{B}_j$, since $\mathfrak{B}_j \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. This shows that $N(f^m)/N(f^n) > 1$ then.

It is worth pointing out that either the proof of the Schinzel theorem (6.4.22) or the Jiang–Llibre Theorem (6.4.8) are based on the Baker logarithm inequality which is a deep theorem of number theory. This leads to a natural question.

(6.4.24) PROBLEM. Compare the constant $m_0(d)$ of the proof of Theorem (6.4.8) of [JiLb] with $m_0(d)$ given by the proof of Theorem (6.4.22) of [Schi].

Let us only remark that an estimate of $m_0(d)$ of Theorem (6.4.22) is connected with the theory of Lehmer–Lucas numbers theory. Although the problem of effective estimate of $m_0(d)$ is of interest, it is a number theory problem and for a derivation of the set $\mathrm{HPer}([A])$, for a given A it is enough to use a weaker estimate based on the inequality in Corollary (6.4.9). We will discuss this in the next section.

6.4.2. Computation of the set $\mathrm{HPer}(f)$**.** In this subsection we would like to discuss the problems: How to get information about the set $\mathrm{HPer}(f)$ having a $d\times d$ integral matrix A of the linearization a self-map f of a compact NR-solvmanifold.

We begin with a well-known fact, given already in Exercise (3.1.13).

(6.4.25) PROPOSITION. *For any monic polynomial of degree d with integral coefficients $\lambda^d + a_{d-1}\lambda^{d-1} + \cdots + a_1\lambda^1 + a_0$, $a_i \in \mathbb{Z}$, there exists an integral $d \times d$ matrix A such that*

$$\lambda^d + a_{d-1}\lambda^{d-1} + \cdots + a_1\lambda^1 + a_0 = \chi_A(\lambda).$$

Proposition (6.4.25) shows that for a given dimension of a torus there is no other condition on the set $\mathrm{HPer}(f)$ if f, for a self-map f of a torus.

If $f\colon X \to X$ is a self-map of a compact nilmanifold $X = G/\Gamma$, then there is an algebraic condition on the form of A (cf . Definition (6.3.6)). Indeed for the decreasing filtration of $X = G/\Gamma$ by sub-nilmanifolds: $X_0 = X \supset X_1 \supset \cdots \supset X_{k-1} \supset X_k = *$, $X_i := G_i/\Gamma_i$, $0 \le i \le k$, of the length k, with $\dim X_i = s_i$. Let $d_i = s_{i-1} - s_i$. If X is an NR-solvmanifold, then A_0 is the linearization of the base map $f_0\colon \mathbb{T}^{d_0} \to \mathbb{T}^{d_0}$ (cf. Definition (6.3.15)). By definition we have

$$A = A_f = \bigoplus_{i=0}^{k} A_i, \quad \text{where } A_i \text{ is a } d_i \times d_i \text{ matrix.} \tag{6.4.26}$$

A direct consequence of (6.4.26) and the proof of Theorem (6.4.4) is the following

(6.4.27) COROLLARY. *The constants $P(d)$, $Q(d)$ of the statement of Theorem* (6.4.4) *depend only on $\widetilde{d}$, where $\widetilde{d} := \max_{0\le i\le k} d_i$.*

We have also another consequence of (6.4.26) (cf. [JeMr2] for the nilmanifold case).

(6.4.28) PROPOSITION. *Let $f\colon X \to X$ be a map of a compact NR-solvmanifold X of dimension d and $A = \bigoplus_{i=0}^{k} A_i$ the linearization matrix of f, where for $0 \le i \le k$, A_i is the linearization matrix of $f_i\colon \mathbb{T}^{d_i} \to \mathbb{T}^{d_i}$, induced by f as in Definition* (6.3.15)*. Then*

$$T_A = \bigcap_{i=0}^{k} T_{A_i}$$

and

$$T_A \cap \left(\bigcup_{i=0}^{k} \mathrm{HPer}(f_i) \right) = T_A \cap \left(\bigcup_{i=0}^{k} \mathrm{HPer}(A_i) \right) \subset \mathrm{HPer}(f).$$

PROOF. By the definition of $m \in T_A$ if and only if $\det(\mathrm{I} - A^m) = \chi_{A^m}(1) \neq 0$. But $\chi_A(1) = \prod_{i=0}^{k} \chi_{A_i}(1)$ which proves the first equality.

To prove the second formula, first note that $\chi_{A^n}(t) | \chi_{A^m}(t)$ if $n|m$, by the identity (6.4.23) or that used in Proposition (6.4.11). Thus $|\chi_{A^n}(1)|$ divides $|\chi_{A^m}(1)|$ if $n \mid m$ (provided $\chi_{A^n}(1) \neq 0$) for every integral matrix A. Consequently, by Theorem (6.4.5) it follows that $m \in \mathrm{HPer}(f)$ if $m \in T_A$ and there exists $0 \leq i_0 \leq k$ such that $|\chi_{A_{i_0}^m}(1)| > |\chi_{A_{i_0}^{m/p}}(1)|$ for every prime $p|m$, since $|\chi_{A_j^m}(1)| \geq |\chi_{A_j^{m/p}}(1)|$ for the remaining i. This shows the statement. $\square$

A property of $\mathrm{HPer}(f)$, derived in a similar way, says that (see [JiLb, Corollary 3.6] for a proof)

(6.4.29) PROPOSITION. *Suppose $m_1, m_2 \in \mathrm{HPer}(f)$ and their least common multiple $m \in T_A$. Then $m \in \mathrm{HPer}(f)$.*

Note that the formula (6.4.26) gives a necessary condition on the form of the linearization matrix A of a map of a given NR-solvmanifold X. Opposite to the torus case this condition is not sufficient in general, i.e. not every direct sum $A = \bigoplus_i A_i$ of integral $d_i \times d_i$ matrices is the linearization of a self-map of a nilmanifold $X = G/\Gamma$ (see the proof of Theorem (6.5.2), Lemma (6.5.12) for examples). The additional restriction is a consequence of the fact that A is derived from a homomorphism of the Lie algebra $\mathcal{G}$ of G (cf. Corollary (6.3.25)).

(6.4.30) REMARK. Since the set $\mathrm{HPer}(f)$ is determined by a purely algebraic procedure in terms of the linearization matrix, it was natural to make an effort implementing this scheme. In the article [KoMr] of R. Komendarczyk and the second author of this book is presented a computer program written as a notebook of "Mathematica", *deriving* the set $\mathrm{HPer}([A])$ for a given integral matrix A. It is based on Theorem (6.4.5) and the inequality in Corollary (6.4.9), since the dependence of $m_0(d) = Q(d)$ "only on" d is not necessary then. It uses also all the mentioned above information, e.g. Corollary (6.4.27), Proposition (6.4.28). As the first step it checks whether $\sigma(A) \cap \{|z| = 1\}$ consists of roots of unity only. Under this assumption one can use a modification (simplification) of the inequality in Corollary (6.4.9), which drastically cuts the interval of searching $m \in T_A \setminus \mathrm{HPer}([A])$ (see [KoMr]).

Summing up our discussion we can formulate a "metatheorem" which says that the homotopy dynamics of a map of any NR-solvmanifold is described by the homotopy dynamics of a map of the torus.

(6.4.31) THEOREM. *Let $f: X \to X$ be a map of a compact NR-solvmanifold of dimension d. Then there exists a self-map $[A]: \mathbb{T}^d \to \mathbb{T}^d$, of the torus induced by the linearization of f such that that* $\mathrm{HPer}([A]) = \mathrm{HPer}(f)$. *Consequently the set* $\mathrm{HPer}([A])$ *can be derived by the procedure described above.*

PROOF. Indeed, the integral $d \times d$ matrix A of linearization of f induces the quotient map $[A]: \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{T}^d$. Moreover, the linearization of $[A]$ is equal to A. This shows that the sets T_A, HPer are the same for f and $[A]$. $\square$

6.5. Lower dimensions – a complete description

It is natural to give a complete list of all sets of homotopy minimal periods of maps of a given NR-solvmanifold X in the case when the dimension d of X is small.

Case 1. A list for the case $d = 1$, i.e. $X = S^1$ is given in Theorem (6.1.4) and can be also derived from Theorem (6.4.4) by an easy computation.

Case 2. For the case $d = 2$ we have only one, up to isomorphism, compact NR-solvmanifold, namely the torus $\mathbb{T}^2$. As we already said the main theorem of [ABLSS] contains such a description. Here we present it in our own terms.

(6.5.1) THEOREM. *Let $f: \mathbb{T}^2 \to \mathbb{T}^2$ be a map of the torus, $A \in \mathcal{M}_{2\times 2}(\mathbb{Z})$ the linearization of f, and $\chi_A(\lambda) = \lambda^2 - a\,\lambda + b$ be its characteristic polynomial.*

There are three types for the minimal homotopy periods of f :

(E) $\mathrm{HPer}(f) = \emptyset$ *if and only if* $-a + b + 1 = 0$.

(F) $\mathrm{HPer}(f)$ *is nonempty and finite for six cases corresponding to one of the six pairs (a, b) listed below:*

$$(0,0),\ (-1,0),\ (-2,1),\ (0,1),\ (-1,1),\ (1,1).$$

We then have $\mathrm{HPer}(f) \subset \{1,2,3\}$. *Moreover, the sets T_A and* $\mathrm{HPer}(f)$ *are the following:*

(a,b)	T_A	$\mathrm{HPer}(f)$
$(0,0)$	$\mathbb{N}$	$\{1\}$
$(0,1)$	$\mathbb{N}\setminus 4\mathbb{N}$	$\{1,2\}$
$(-1,0)$	$\mathbb{N}\setminus 2\mathbb{N}$	$\{1\}$
$(-1,1)$	$\mathbb{N}\setminus 3\mathbb{N}$	$\{1\}$
$(-2,1)$	$\mathbb{N}\setminus 2\mathbb{N}$	$\{1\}$
$(1,1)$	$\mathbb{N}\setminus 6\mathbb{N}$	$\{1,2,3\}$

Cases of Type (F)

(G) $\mathrm{HPer}(f)$ *is infinite for the remaining* a, *and* b. *Furthermore,* $\mathrm{HPer}(f)$ *is equal to* $\mathbb{N}$ *for all pairs* $(a,b) \in \mathbb{Z}^2$ *except for the following special cases listed below. We say that a pair* $(a,b) \in \mathbb{Z}^2$ *satisfies condition*

(G1) *if* $a \neq 0$ *and* $a+b+1=0$,

(G2) *if* $a+b=0$,

(G3) *if* $a+b+2=0$, *respectively,*

and (a,b) *is not one of the pairs of case* (E) *and* (F).

We have the following table of special cases:

(a,b)	T_A	$\mathrm{HPer}(f)$
$(-2,2)$	$\mathbb{N}$	$\mathbb{N}\setminus\{2,3\}$
$(1,2)$	$\mathbb{N}$	$\mathbb{N}\setminus\{3\}$
$(0,2)$	$\mathbb{N}$	$\mathbb{N}\setminus\{4\}$
$(a,b):(a,b)$ *satisfies* (G1)	$\mathbb{N}\setminus 2\mathbb{N}$	$\mathbb{N}\setminus 2\mathbb{N}$
$(a,b):(a,b)$ *satisfies* (G2)	$\mathbb{N}$	$\mathbb{N}\setminus\{2\}$
$(a,b):(a,b)$ *satisfies* (G3)	$\mathbb{N}$	$\mathbb{N}\setminus\{2\}$

Case 3. In the paper [JiLb] Jiang and Llibre gave such a list for maps of $X = \mathbb{T}^3$ including a separate table for homeomorphisms. As we already emphasized (Theorem (6.4.31)) a discussion of this case is the most complicated from the algebraic point of view, because there is then no condition on the linearization matrix A. Consequently for the torus family of the sets of homotopy minimal periods it is the largest possible one (see Theorem (6.4.31)).

The aim of the work [JeMr2] was to give such a list for a three-dimensional nilmanifold not homeomorphic to the torus. Let X be a 3-dimensional compact nilmanifold. If X is not homeomorphic to $\mathbb{T}^3$, then by Example (6.2.4) $X = \mathbf{N}(\mathbb{R})/\Gamma_{1,1,r}$, $r \in \mathbb{N}$, is a Heisenberg manifold. Since $\dim[\mathbf{N}(\mathbb{R}), \mathbf{N}(\mathbb{R})] = 1$, X fibres over a nilmanifold of dimension 2 (thus the torus) with 1-dimensional fiber (thus the circle). Consequently the linearization matrix is of the form $A = A_1 \oplus \overline{A}$, where $A_1 \in \mathcal{M}_{1\times 1}(\mathbb{Z})$ and $\overline{A} \in \mathcal{M}_{2\times 2}(\mathbb{Z})$.

The corresponding theorem says the following.

(6.5.2) Theorem. *Let* $f\colon X \to X$ *be a map of three-dimensional compact nilmanifold* X *not diffeomorphic to* $\mathbb{T}^3$. *Let* $A = A_1 \oplus \overline{A} \in \mathcal{M}_{3\times 3}(\mathbb{Z})$ *be the matrix induced by the fibre map* $f = (f_1, \overline{f})$ *and* $\chi_A(\lambda) = \chi_{A_1}(\lambda)\cdot\chi_{\overline{A}}(\lambda) = (\lambda - c)(\lambda^2 - a\lambda + b)$ *be its characteristic polynomial. Then* $c = b$ *and there are three types for the minimal homotopy periods of* f:

(E) $\mathrm{HPer}(f) = \emptyset$ *if and only if or* $c = 1$ *or* $-a\ + c + 1 = 0$.

(F) $\mathrm{HPer}(f)$ *is nonempty and finite only for two cases corresponding to* $c = 0$ *combined with one of the two pairs* (a, b)*:*

$$(0,0) \quad \textit{and} \quad (-1,0).$$

We then have $\mathrm{HPer}(f) = \{1\}$*. Moreover, the sets* T_A *and* $\mathrm{HPer}(f)$ *are the following:*

(c,a,b)	T_A	$\mathrm{HPer}(f)$
$(0,0,0)$	$\mathbb{N}$	$\{1\}$
$(0,-1,0)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\{1\}$

(G) $\mathrm{HPer}(f)$ *is infinite for the remaining* $(c, a, b = c)$*.*

Furthermore, $\mathrm{HPer}(f)$ *is equal to* $\mathbb{N}$ *for all triples* $(c, a, b = c) \in \mathbb{Z}^3$ *except for the following special cases listed below.*

(c,a,b)	T_A	$\mathrm{HPer}(f)$
$a+c+1=0$ *with* $a \neq 0$ *and* $c \notin \{-1,-1,0,1\}$	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
$(0,-2,0)$	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$
$(-1,1,-1)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
$(-1,-1,-1)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
$(-2,1,-2)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
$(-2,0,-2)$	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$
$(-2,2,-2)$	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$

Special cases of type (G)

Moreover, for every pair subset $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathbb{N}$*, appearing as* $\mathrm{HPer}(f)$ *and* T_A *listed above, there exists a map* $f\colon X \to X$ *such that* $\mathrm{HPer}(f) = \mathcal{S}_1$ *and* $T_A = \mathcal{S}_2$*.*

A proof is based on a classification of all homomorphisms of the nilpotent group $\Gamma_{1,1,r}$ of Example (6.2.4).

With respect to the classification of three-dimensional manifolds (cf. Example (6.2.4)) and Theorem (6.3.1) it is enough to determine the set of matrices of linearization of all endomorphisms of $\Gamma = \Gamma_{1,1,r}$. We begin with a description of $\Gamma_{1,1,r}$. Then we give a description of all endomorphisms of $\Gamma_{1,1,r}$. We follow the approach of [HeKeI] where the case of $\Gamma_{1,1,1}$ was discussed.

Assigning with any matrix

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where } x, y, z \in \mathbb{R}, \quad \text{the vector } (x, y, z),$$

we get the homeomorphism between $\mathcal{N}_3(\mathbb{R})$ and $\mathbb{R}^3$. In these coordinates the multiplication has form

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy').$$

Using the coordinates we see that $\Gamma_{1,1,r} \subset \mathcal{N}_3(\mathbb{R})$ is generated by the matrices

$$a := (1, 0, 0), \quad b := (0, 1, 0), \quad c := (0, 0, 1/r),$$

since $(m, p, q/r) = a^m b^p c^{q-mp}$. Moreover, the only relations are

$$aba^{-1}b^{-1} = c^r, \quad aca^{-1}c^{-1} = e, \quad bcb^{-1}c^{-1} = e. \tag{6.5.3}$$

Let $\Phi \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a map and let

$$\Phi(a) = (\alpha_1, \beta_1, \gamma_1), \quad \Phi(b) = (\alpha_2, \beta_2, \gamma_2), \quad \Phi(c) = (\alpha_3, \beta_3, \gamma_3).$$

We look for a necessary and sufficient condition on Φ to extend to a homomorphism of $\Gamma_{1,1,r}$. Suppose that Φ extends to such a homomorphism. Then for some integer k

$$\Phi(c) = c^k, \tag{6.5.4}$$

because the cyclic group generated by c is equal to the center of $\Gamma_{1,1,r}$, consequently $\alpha_3 = 0$, $\beta_3 = 0$, $\gamma_3 = k$. Using the first equality of (6.5.3), (6.5.4), deriving $\Phi(a)\Phi(b)\Phi(a)^{-1}\Phi(b)^{-1}$, and comparing the coordinates we get

$$k = \alpha_1\beta_2 - \alpha_2\beta_1. \tag{6.5.5}$$

Note that $\phi(c) = c^k$ implies that the second and third relations of (6.5.3) are preserved, because $\phi(c)$ is in the center of $\Gamma_{1,1,r}$. Notice that γ_1, γ_2 may be arbitrary. Since (6.5.3) are the only relations we get the following fact.

(6.5.6) PROPOSITION. *A map* $\Phi\colon \Gamma_{1,1,r} \to \Gamma_{1,1,r}$ *defined in the coordinate system by its values on the generators* a, b, c *as*

$$\phi(a) = (\alpha_1, \beta_1, \gamma_1), \quad \Phi(b) = (\alpha_2, \beta_2, \gamma_2), \quad \Phi(c) = (\alpha_3, \beta_3, \gamma_3)$$

extends to an automorphism of $\Gamma_{1,1,r}$ *if and only if* $\alpha_3 = \beta_3 = 0$, *and* $\gamma_3 = \alpha_1\beta_2 - \alpha_2\beta_1$. *Consequently a* 3×3 *integral matrix* A *is the linearization matrix of a map of* X *given by an endomorphism of* $\Gamma_{1,1,r}$ *if and only if it is of the form*

$$A = A_1 \oplus \overline{A} = \begin{bmatrix} k & 0 & 0 \\ 0 & \alpha_1 & \beta_1 \\ 0 & \alpha_2 & \beta_2 \end{bmatrix} \quad \textit{where } \det \overline{A} = k.$$

Finally we formulate a topological consequence of Proposition (6.5.6).

(6.5.7) COROLLARY. *Let* $f\colon X \to X$ *be a map of three dimensional nilmanifold not diffeomorphic to the torus. Then there exists* $k \in \mathbb{Z}$ *such that* $\deg f = k^2$. *In particular if* $\deg f \neq 0$, *then* f *preserves the orientation.*

PROOF. Note that for a fibre-map $f = (f_1, \overline{f})$ we have $\deg f = \deg f_1 \deg \overline{f}$. On the other hand we have just shown that for a map induced by a homomorphism, thus for every map, we have $\deg f_1 = d = \det \overline{A} = \deg \overline{f}$, by Proposition (6.5.6).□

What is remarkable is that a condition on an integer 3×3 matrix A to be a linearization of a map does not depend on r, and consequently relies upon a condition on a matrix to be a homomorphism of the Heisenberg Lie algebra (cf. Example (6.2.4)). Note also that a consequence of Proposition (6.5.6) is the following algebraic condition on the matrix linearization $\chi_A(\lambda) = \chi_{A_1}(\lambda)\chi_{\overline{A}}(\lambda) = (\lambda - c)(\lambda^2 - a\lambda + b)$ here. But additionally the topology (i.e. that A comes from a homomorphism of the corresponding Lie algebra) yields that now $c = \deg f_1 = \deg \overline{f} = \det \overline{A} = b$, where f_1 is the fibre map, and $\overline{f}$ is the base map for a fibre map f of corresponding Fadell–Husseini fibration associated to $X = G/\Gamma_{1,1,r}$. The statement of Theorem (6.5.2) can be derived from Propositions (6.5.6) and (6.4.28) by an elementary analysis (cf. [JeMr2]).

As for the torus case (cf. (JiLb)) we specify the classification for homeomorphisms.

(6.5.8) THEOREM. *Let* $f\colon X \to X$ *be a homeomorphism of a three-dimensional compact nilmanifold* X *not diffeomorphic to* $\mathbb{T}^3$. *Let* $A = A_1 \oplus \overline{A} \in \mathcal{M}_{3\times3}(\mathbb{Z})$ *be the linearization matrix and* $\chi_A(\lambda) = (\lambda - c)(\lambda^2 - a\,\lambda + b)$ *its characteristic polynomial. Then* $c = b = \pm 1$ *and consequently* $\mathrm{HPer}(f) = \emptyset$ *if and only if* $c = 1$ (*i.e. if* f *preserves the orientation*), *or* $c = -1$ *and* $a = 0$. *In particular* $\mathrm{HPer}(f) = \emptyset$ *for every homeomorphism preserving orientation. For* $c = -1$ (*i.e.*

if f reverses the orientation) and the remaining a, we have $\mathrm{HPer}(f) = \mathbb{N}$ *with the only two exceptions being when $a = 1$ or $a = -1$. For these special cases* $T_A = \mathrm{HPer}(f) = \mathbb{N} \setminus 2\mathbb{N}$.

PROOF. The statement follows from Theorem (6.5.2) and the fact that $c = \pm 1$ (cf. [JeMr2]). □

We are left with a discussion for these 3-dimensional NR-solvmanifolds which are not the nilmanifolds. As a first step one can show that there is only one, up to isomorphism, three-dimensional solvable, but not nilpotent, Lie algebra $\mathcal{G}$ for which the corresponding unique simply-connected Lie group G admits a lattice but the quotient is an NR-solvmanifold. It is the following completely solvable Lie algebra $\mathcal{G}$ given by the generators $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and commutators

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{Y}, \quad [\mathcal{X}, \mathcal{Z}] = -\mathcal{Z}, \quad [\mathcal{Y}, \mathcal{Z}] = 0. \tag{6.5.9}$$

We begin with a brief summary of the information on three dimensional Lie groups and Lie algebras which is necessary for our consideration. All this material is contained in [Ki, Chapter I.6]. As follows from the Cartan theorem (cf. [Var]), there is one–one correspondence between Lie algebras and connected simply-connected Lie groups. Moreover, any connected Lie group G is of the form G_0/Δ where G_0 is the simply-connected group corresponding to a given Lie algebra $\mathcal{G}$ and Δ is a discrete normal subgroup.

It follows from the classification of Lie algebras of dimension 3 that all solvable 3-dimensional Lie algebras are of the following form:

- *Abelian*, i.e. $[\mathcal{X}, \mathcal{Y}] = 0$ for all $\mathcal{X}, \mathcal{Y} \in \mathcal{G}$. It corresponds to $G = \mathbb{R}^3$ with the structure of linear space.
- There exist linearly independent vectors $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathcal{G}$ such that $[\mathcal{X}, \mathcal{Y}] = \mathcal{Z}$, $[\mathcal{X}, \mathcal{Z}] = [\mathcal{Y}, \mathcal{Z}] = 0$. The algebra is *nilpotent* and corresponds to the Lie group of matrices
$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},$$
where $x, y, z \in \mathbb{R}$. We have $[\mathcal{G}, \mathcal{G}] = \langle \mathcal{Z} \rangle$, thus $\dim [\mathcal{G}, \mathcal{G}] = 1$ here.
- The last case consists of an infinite series of non-isomorphic solvable Lie algebras for which $\dim [\mathcal{G}, \mathcal{G}] = 2$. Let $\dim [\mathcal{G}, \mathcal{G}] = 2$ and $\mathcal{Y}$, $\mathcal{Z}$ be its generators. Any choice for $[\mathcal{X}, \mathcal{Y}] = a_{11}\mathcal{Y} + a_{12}\mathcal{Z}$, $[\mathcal{X}, \mathcal{Z}] = a_{12}\mathcal{Y} + a_{22}\mathcal{Z}$, or in other words any choice of 2×2 matrix $\mathbb{A} \in \mathrm{Aut}(\mathbb{R}^2)$ corresponding to the operator $\mathrm{ad}_{\mathcal{X}} \colon \mathcal{G}$ defines a solvable Lie algebra by the above formula. Moreover, two matrices $\mathbb{A}$, $\mathbb{B}$ give two isomorphic algebras if and only if there exists $0 \neq c \in \mathbb{R}$ and $\mathbb{C} \in \mathrm{Aut}(\mathbb{R}^2)$ such that $c\mathbb{A} = \mathbb{C}\mathbb{B}\mathbb{C}^{-1}$

with $c \in \mathbb{R}$. Every such algebra corresponds to a simply-connected solvable Lie group of the form

$$G := \mathbb{R} \times_{\phi(t)} \mathbb{R}^2, \tag{6.5.10}$$

where $\phi(t)\colon \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is one-parameter a group given by $t \mapsto e^{t\mathcal{A}}$. If $\mathbb{A}$ has purely imaginary eigenvalues, then it is not NR. Otherwise, $Z(G) = \{e\}$ and there is only one connected Lie group with the given Lie algebra.

By the above it is enough to study the Jordan forms, up to scalars, of real 2×2 matrices. Up to the above isomorphism, we have the following cases:

Case 1. $\mathbb{A}$ is real diagonal, i.e. either

$$\mathbb{A} = \mathrm{Id} \quad \text{or} \quad \mathbb{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In the first case the corresponding Lie algebra does not admit a lattice; the second gives an example we shall use below.

Case 2. $\mathbb{A}$ is a rotation matrix with purely imaginary eigenvalues $\imath\alpha$ and $-\imath\alpha$. This means that the corresponding connected simply-connected Lie group is not exponential and the resulting solvmanifold is not NR.

Case 3.

$$\text{Either} \quad \mathbb{A} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \ \alpha > 0 \quad \text{or} \quad \mathbb{A} = \begin{bmatrix} \beta & \alpha \\ -\alpha & \beta \end{bmatrix}, \ \alpha^2 + \beta^2 = 1, \ \alpha > 0, \ \beta > 0.$$

The corresponding Lie algebra $\mathcal{G}$, thus also the unique simply-connected Lie group G, is completely solvable. As we shall see in the proof of the next proposition, the group G does not admit a lattice. Consequently there is no special solvmanifold corresponding to this case.

As a consequence of this classification we have the following.

(6.5.11) Proposition. *There is only one, up to isomorphism, connected, simply-connected, completely solvable three-dimensional Lie group which admits a lattice and is neither abelian nor nilpotent:*

$$G := \mathbb{R} \times_{\phi(t)} \mathbb{R}^2, \quad \textit{where } \phi(t) := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

It corresponds to the Lie algebra defined by the matrix

$$\mathbb{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

or in other words it is defined by generators $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and commutators

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{Y}, \quad [\mathcal{X}, \mathcal{Z}] = -\mathcal{Z}, \quad [\mathcal{Y}, \mathcal{Z}] = 0.$$

Consequently, every three-dimensional special completely solvable solvmanifold, not being a torus or a nilmanifold, is, up to diffeomorphism, of the form G/Γ where $\Gamma \subset G$ is a lattice.

Proof. With respect to the classification given above of the solvable Lie algebras of dimension 3, it is enough to show that the simply-connected group G corresponding to the matrix $\mathbb{A} = \mathrm{Id}$

$$\mathbb{A} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \; a > 0, \quad \text{or respectively} \quad \mathbb{A} = \begin{bmatrix} \beta & \alpha \\ -\alpha & \beta \end{bmatrix}, \; \beta > 0$$

by the formula (6.5.10) does not have a lattice. We use the fact that if a solvable group has a lattice, then it is unimodular, i.e. every right-invariant measure μ on G is also left-invariant (cf. [Ra, Remark 1.9 and Theorem 3.1]). By the definition of twisted product G is the Euclidean space $\mathbb{R}^3$ with the multiplication given by the formula

$$(x, y, z)(x', y', z') := (x + x', y + a(x)y' + b(x)z', z + c(x)y' + d(x)z'),$$

where

$$\exp(t\mathbb{A}) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}.$$

A direct computation shows that the density form $dx \wedge dy \wedge dz$ gives a right-invariant measure on G which is not left-invariant, because the Jacobian of right multiplication, correspondingly left, by (x, y, z) at (x', y', z') is 1, respectively $\exp(x'\mathbb{A})$. The latter is equal to $\exp(2x')$, or $\exp(2\beta x')$ for the second form of the matrix $\mathbb{A}$, respectively. □

A direct computation shows (cf. [JeKdMr])

(6.5.12) Lemma. *Let $A\colon \mathcal{G}$ be any endomorphism of completely solvable Lie algebra* (6.5.9)*. Then it has the following form with respect to the basis $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$,*

$$A = \begin{bmatrix} a & 0 & 0 \\ b & r & s \\ c & u & v \end{bmatrix},$$

where the coefficients satisfy the following conditions: either $r = v = s = u = 0$ and $a \in \mathbb{Z}$ is an arbitrary integer, or there exists a coefficient r, u, s, v different from 0 and then $a \in \{-1, 1\}$. Moreover, we have:

(6.5.12.1) *if* $a = -1$, *then* $r = v = 0$,
(6.5.12.2) *if* $a = 1$, *then* $s = u = 0$.

This algebra is the Lie algebra of a connected, simply-connected, completely solvable three-dimensional Lie group G defined by the formula

$$G := \mathbb{R} \times_{\phi(t)} \mathbb{R}^2, \quad \text{where } \phi(t) := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}. \tag{6.5.13}$$

This leads to the correspondent specification of the theorem (6.4.4) for maps of three-dimensional special NR-solvmanifolds (cf. [JeKdMr]).

(6.5.14) Theorem. *Let $f\colon X \to X$ be a map of a compact three-dimensional special NR-solvmanifold, thus a completely solvable solvmanifold, which is not diffeomorphic to a nilmanifold. Next let $A \in \mathcal{M}_{3\times 3}(\mathbb{Z})$ be its linearization as in Lemma* (6.5.12). *Then we have three mutually disjoint cases:*

(E) $\mathrm{HPer}(f) = \emptyset$ *if and only if* $N(f) = 0$ *if anf only if* $a = 1$ *or* ($a = -1$ *and* $su = 1$).
(G) $\mathrm{HPer}(f) = \mathbb{N}$ *if and only if* $a \neq \{-2, -1, 0, 1\}$ *and* $r = s = u = v = 0$;
$\mathrm{HPer}(f) = \mathbb{N} \setminus \{2\}$ *if and only if* $a = -2$, $r = s = u = v = 0$,
$\mathrm{HPer}(f) = \mathbb{N} \setminus 2\mathbb{N}$ *if and only if* $a = -1$, $|su| \geq 2$ *and* $r = v = 0$.
(F) $\mathrm{HPer}(f) = \{1\}$ *in the remaining cases.*

(6.5.15) Corollary. *For any homeomorphism $f\colon X \to X$ of a compact special three-dimensional completely solvable solvmanifold which is not diffeomorphic to a nilmanifold* $\mathrm{HPer}(f)$ *is either empty or consists of the single number* 1.

At the end of this section we give an example of a countable family of non-isomorphic compact completely solvable solvmanifolds, each of them being the quotient of the group G defined in Proposition (6.5.11).

(6.5.16) Exercise. First for $a + a^{-1} = n$ and $2 < n \in \mathbb{N}$ we define a family of simply-connected solvable Lie groups by $G_n := \mathbb{R} \ltimes_{\kappa_n} \mathbb{R}^2$, where

$$\kappa_n(t) = \begin{bmatrix} \dfrac{a^{t+1} - a^{-t-1}}{a \;\; - a^{-1}} & \dfrac{a^{-t} - a^{t}}{a \;\; - a^{-1}} \\ \dfrac{a^{t} - a^{-t}}{a \;\; - a^{-1}} & \dfrac{a^{1-t} - a^{t-1}}{a \;\; - a^{-1}} \end{bmatrix}$$

is a family of 1-parameter subgroups of $SL_2(\mathbb{R})$. Each group G_n is isomorphic to G_3 [VGS] or equivalently to the group defined in Proposition (6.5.11). It is easy to see that each of them contains a lattice of the form $\Gamma_n := \mathbb{Z} \ltimes_{\kappa_n(1)} \mathbb{Z}^2$, where

$$\kappa_n(1) = \begin{bmatrix} n & -1 \\ 1 & 0 \end{bmatrix}.$$

Thus we obtain a family of solvmanifolds $X_n := G_n/\Gamma_n$. They are not homeomorphic, since the groups $\Gamma_n = \pi_1(X_n)$ are pairwise non-isomorphic. Indeed, representation of Γ_n on its commutator subgroup factors through $\mathbb{Z}$. If we consider this representation tensored by $\mathbb{Q}$, then it is generated by the matrix $\kappa_n(1)$. For the indices $n \neq m$ these representations are not equivalent because the traces of its generators are different.

6.5.1. Šarkovskii type theorems. We first remark that Theorem (6.4.4) allows us to state the existence of infinitely many homotopy minimal periods, thus infinitely many periodic points provided that for a map of an NR-solvmanifold we know that there exists a sufficiently large homotopy minimal period.

(6.5.17) THEOREM. *Let $f\colon X \to X$ be a map of an NR-solvmanifold X as in Theorem* (6.4.4)*, $\mathbf{N} \to X \to \mathbb{T}$ the Mostow fibration of X, and $\widetilde{f} = (f_{\mathbf{N}}, f_{\mathbb{T}})$ the fibre map homotopic to f. Then $A = A_{\mathbf{N}} \oplus A_{\mathbb{T}}$, $T_A = T_{A_{\mathbf{N}}} \cap T_{A_{\mathbb{T}}}$, and* $\mathrm{HPer}(f_{\mathbf{N}}) \cup \mathrm{HPer}(f_{\mathbb{T}}) \subset \mathrm{HPer}(f)$. (*More exactly the last inclusion makes sense if $L(f_{\mathbb{T}}) \neq 0$. Otherwise* $\mathrm{HPer}(f) = \emptyset$ *but $f_{\mathbf{N}}$ may not be defined*).

PROOF. The equality $A = A_{\mathbf{N}} \oplus A_{\mathbb{T}}$ follows from Definition (6.3.15). The remaining part of the statement follows from the first and Proposition (6.4.28).□

(6.5.18) COLLARY. *Let $f\colon X \to X$ be as in Theorem* (6.5.17)*. Then* $\mathrm{HPer}(f)$ *is nonempty and finite if and only if* $\mathrm{HPer}(f_{\mathbf{N}})$ *and* $\mathrm{HPer}(f_{\mathbb{T}})$ *are finite and nonempty.*

PROOF. For the characteristic polynomial of A we have $\chi_A(t) = \chi_{A_{\mathbf{N}}}(t)\chi_{A_{\mathbb{T}}}(t)$. It remains to notice that by Theorem (6.4.4), $\mathrm{HPer}(f)$ is nonempty and finite if and only if all eigenvalues are either zero or roots of unity different than 1. □

(6.5.19) THEOREM. *Let $f\colon X \to X$ be as in Theorem* (6.5.17)*, $\mathbf{N} \to X \to \mathbb{T}$ the Mostow fibration of X, and $\widetilde{f} = (f_{\mathbf{N}}, f_{\mathbb{T}})$ the fibre map homotopic to f. Let next $d_{\mathbf{N}} = \dim \mathbf{N}$, $d_{\mathbb{T}} = \dim \mathbb{T}$, and $\phi\colon \mathbb{N} \to \mathbb{N}$ be the Euler function. Set l equal to the least common multiple of all $\{k \in \mathbb{N} : \phi(k) \leq \max(d_{\mathbf{N}}, d_{\mathbb{T}})\}$.*

If there exists $n_0 \in \mathrm{HPer}(f)$ such that $n_0 \nmid l$, $n_0 < l$, then $\mathrm{HPer}(f)$ is infinite.

PROOF. By Theorem (6.4.4) if $\mathrm{HPer}(f)$ is finite then, $\chi_A(t)$ decomposes as the product of a power of t and cyclotomic polynomials each of which is of degree $\leq \max(d_{\mathbf{N}}, d_{\mathbb{T}})$. It follows that the sequence $\{N(f^n)\}$ is l-periodic. Then $n \in \mathrm{HPer}(f)$ if and only if $n \mid l$ by the argument used in the proof of Theorem (6.4.4), Type (F). Consequently $\mathrm{HPer}(f)$ is a subset of the set of all proper divisors of l.□

It is worth to noting that the statement of Theorem (6.5.19) illustrates that for self-maps of more complicated (nonabelian) solvmanifold (as in Theorem (6.4.4)) the family of all sets $\mathrm{HPer}(f)$ is essentially smaller than the corresponding family of all sets $\mathrm{HPer}(f)$ for self-maps of the torus of the same dimension. The reason is the following:

(a) The matrices are restricted to block diagonal form (and not the full set of $(d_{\mathbf{N}} + d_{\mathbb{T}}) \times (d_{\mathbf{N}} + d_{\mathbb{T}})$ integral matrices).

(b) The structure of the solvmanifolds, and nilmanifolds, is more complicated than the torus of the corresponding dimension so fewer maps hence linearizations are possible. Noteworthy is that the linearization of a map of a nilmanifold is a block matrix with the size of blocks corresponding to the quotients of the central tower, but additionally there are relations between the blocks.

In particular it is noticeable by a comparison of the complete description of all sets of homotopy minimal periods for the three-dimensional torus case ([JiLb, Theorem C]) and for the nonabelian three-dimensional nilmanifold ([JeMr2, Th. 3.1]).

Note that for a map of $\mathbb{T}^3$ (thus also any three dimensional special solvmanifold X) Theorem (6.5.19) guarantees that $\mathrm{HPer}(f)$ is infinite provided $m \geq 7$ belongs to $\mathrm{HPer}(f)$. However, reading the table of all possible sets of homotopy minimal periods of maps of the type (F) of $\mathbb{T}^3$ given in [JiLb], we see that if $4 \leq m \in \mathrm{HPer}(f)$, then $\mathrm{HPer}(f)$ is infinite.

In the case of nonabelian 3-dimensional NR-solvmanifolds we can say much more. A detailed description of all the possible sets of homotopy minimal periods appearing for maps of a given class of 3-dimensional NR-solvmanifolds led to the following theorems ([JeMr2], [JeKdMr]) which are proved by an analysis of the table of sets of homotopy minimal periods. (cf. Theorems (6.5.2), (6.5.8) for nonabelian nilmanifolds, Theorem (6.5.14), Corollary (6.5.15) for special not nilpotent NR-solvmanifolds).

(6.5.20) Proposition. *If a self-map of a* 3*-nilmanifold different than a* 3*-torus is such that* $3 \in \mathrm{HPer}(f)$, *then* $\mathbb{N} \setminus 2\mathbb{N} \subset \mathrm{HPer}(f) \subset \mathrm{Per}(f)$. *If* $2 \in \mathrm{HPer}(f)$ *then* $\mathbb{N} = \mathrm{HPer}(f) = \mathrm{Per}(f)$. *In particular, the first assumption is satisfied if* $L(f^3) \neq L(f)$ *and the second if* $L(f^2) \neq L(f)$.

(6.5.21) Corollary. *Let* $f\colon X \to X$ *be a homeomorphism, or more generally a homotopy equivalence, of a compact three-dimensional nilmanifold* X *not diffeomorphic to the torus. If* $\mathrm{HPer}(f) \neq \emptyset$, *then* $\mathbb{N} \setminus 2\mathbb{N} \subset \mathrm{HPer}(f)$. *Moreover, if* $2 \in \mathrm{HPer}(f)$, *e.g. if* $L(f^2) \neq L(f)$, (*or if any* $2k \in \mathrm{HPer}(f)$), *then* $\mathrm{HPer}(f) = \mathbb{N}$.

(6.5.22) Proposition. *For a map* $f\colon X \to X$ *of a special* NR*-solvmanifold of dimension* 3 *not diffeomorphic to a nilmanifold we have Šarkovski type implications:*

(6.5.22.1) $2 \in \mathrm{HPer}(f)$ *implies* $\mathrm{HPer}(f) = \mathbb{N}$.

(6.5.22.2) *If* $\mathrm{HPer}(f)$ *contains an even number, then* $\mathbb{N} \setminus \{2\} \subset \mathrm{HPer}(f)$.

(6.5.22.3) *If* $\mathrm{HPer}(f)$ *contains at least two numbers, then* $\mathbb{N} \setminus 2\mathbb{N} \subset \mathrm{HPer}(f)$.

We called all the above facts "Šarkovski type theorems", because they state that a given small homotopy period implies all, or all except for one, homotopy periods. Anyway we must emphasize that the nature of this phenomenon is completely different than that of the Šarkovski theorem [Sr]. It is worth pointing out that to check the supposition of a theorem above we have to find a periodic point which is not removable along a deformation. But as a consequence we get a statement about the set $\mathrm{HPer}(f)$, i.e. a stronger affirmation than in the Šarkovski theorem.

6.5.2. The Klein bottle and projective spaces. The theory of homotopy minimal periods is best developed for maps of the NR-solvmanifolds. One should remark that the case of compact solvmanifold which are not NR-solvmanifolds has been also discussed. First of all we should mention the work of J. Llibre [Ll] where he gave a description of the possible sets $\mathrm{HPer}(f)$ for maps f of the Klein bottle K, i.e. the nonabelian solvmanifold of the dimension 2. (Note that the Klein bottle is not an NR-solvmanifold, it is also not orientable and not special). It is based on a never published work of B. Halpern [Ha3]. We must add that there are doubts whether the statement and proof of the main theorem of this paper are correct.

Nevertheless, the computation of a formula for the Nielsen number for a map of the Klein bottle (6.5.24) had been shown in detail in the paper [DHT] of O. Davey, E. Hart, and K. Trapp. On the other hand the evidence that the vanishing of an algebraically defined (cf. (6.5.25)) numbers $\widetilde{N}P_m(f)$ implies that f is homotopic to a map without periodic points of the minimal period m has not yet been confirmed in a published form. Nevertheless we present this result.

(6.5.23) THEOREM. *Let* $f\colon K \to K$ *be a continuous map of the Klein bottle. There are three types for the minimal homotopy periods of* f :

(E) $\mathrm{HPer}(f) = \emptyset$,

(F) $\mathrm{HPer}(f) \neq \emptyset$ *and is finite and then* $\mathrm{HPer}(f) = \{1\}$.

(G) $\mathrm{HPer}(f) = \mathbb{N}$ *or* $\mathrm{HPer}(f) = \mathbb{N} \setminus 2\mathbb{N}$ *for the remaining maps, except for one special case when* $\mathrm{HPer}(f) = \mathbb{N} \setminus \{2\}$.

We have to recall the basic homotopy properties of the Klein bottle X. It is $K(\pi, 1)$-space with $\pi_1(X)$ being the group with two generators a, b and one relation $abab^{-1} = 1$. Moreover, every homomorphism of $\phi\colon \pi_1(X) \to \pi_1(X)$ is of the form $\phi(a) = a^u$, $\phi(b) = a^w b^v$, where $u, w, u \in \mathbb{Z}$. In [Ha3] B. Halpern showed that for every map $f\colon X \to X$ of the Klein bottle and every $m \in \mathbb{N}$,

$$N(f^m) = \begin{cases} |u^m(v^m - 1)| & \text{if } |u| > 1, \\ |v^m - 1| & \text{if } |u| \le 1, \end{cases} \tag{6.5.24}$$

where u, w, v are the above defined integers for homomorphism $f_\#$ induced on the fundamental group. Moreover, in the cited Halpern work he introduced a homotopy invariant of a map f of the Klein bottle, defined inductively as follows:

$$\widetilde{N}P_1(f) := N(f),$$

$$\widetilde{N}P_m(f) = \begin{cases} N(f^m) - \displaystyle\sum_{k<m,\, k|m} \widetilde{N}P_k, & \text{for } m > 1, \text{ if } u \neq 1, \\ N(f^m) - \displaystyle\sum_{k<m,\, k|m,\, m/k \,\mathrm{odd}} \widetilde{N}P_k, & \text{for } m > 1, \text{ if } u = 1. \end{cases} \tag{6.5.25}$$

Then he showed that this invariant has the same properties as discussed earlier for the invariant $NP_m(f)$ namely:

- $\widetilde{N}P_m(f) \le \#P_m(f)$,
- for every m in the homotopy class of f there exists a map g such that $\#P_m(g) = \widetilde{N}P_m(f)$.

To prove the theorem using the above property of $\widetilde{N}P_m(f)$, it is enough to find all those $m \in \mathbb{N}$ such that $\widetilde{N}P_m(f) \neq 0$. It have be done in [Ll] by an elementary analysis.

An additional consequence of the properties of the Nielsen numbers presented in the Section 6.3 is the fact that the fundamental group is then infinite and the rate of growth of the sequence $\{N(f^m)\}$ can then be exponential. Relatively less is known for maps of manifolds with a finite fundamental group. Anyway for the case of $X = \mathbb{RP}^d$ the real projective space there is a complete description of all sets of homotopy minimal periods given by J. Jezierski in [Je5]. It is based on the computation of the Nielsen number presented in Theorem (5.1.39).

As a consequence we have the following [Je5]:

(6.5.26) THEOREM. *Let $f\colon \mathbb{RP}^d \to \mathbb{RP}^d$ be a self-map of a real projective space, $d \ge 2$. Then the following formula holds* (*with one exception*)

$$\mathrm{HPer}(f) = \begin{cases} \emptyset & \text{if } N(f) = 0, \\ \{1\} & \text{if } N(f) = 1, \\ \{1, 2, 2^2, 2^3, \dots\} & \text{if } N(f) = 2. \end{cases}$$

The special case is for d odd and $\deg f = -1$ *when* $N(f) = 2$ *but* $\mathrm{HPer}(f) = \{1\}$.

We must add that a similar description of the set of homotopy minimal periods seems to be true for self-maps of the lens spaces and is just now being studied.

But, even so, an analysis in this direction led to an estimate of the number of periodic points of a map of a sphere which commutes with a free action of a group (see Section 7.1 of the next chapter).

CHAPTER VII

RELATED TOPICS AND APPLICATIONS

It is possible to use a similar approach to derive the existence of infinitely many periodic points for a continuous map of the sphere provided the map commutes with a free homeomorphism of a finite order (Theorems (7.1.4), (7.1.6)). The result gives also an asymptotic estimate of the functions defined in (6.0.1) (cf. [JeMR3]).

As the last group of applications we present relations between the topological entropy and the material discussed here. In particular the linearization matrix appears in a formula for the lower estimate of the entropy of a map of a special completely solvable solvmanifold (Theorems (7.2.3)–(7.2.5)). In particular the infiniteness of $\mathrm{HPer}(f)$ implies that entropy is nonzero (Proposition (7.2.13)).

7.1. Periodic points originated by a symmetry of a map

From Theorem (6.1.4) it follows that for a circle map if $|\deg f| > 1$, then $\mathrm{HPer}(f) = \mathbb{N}$ (or exceptionally $\mathbb{N} \setminus \{2\}$). Moreover, $\#P^m(f) \geq N(f^m) = |1 - \deg f^m|$, because S^1 is the torus (cf. Theorems (6.3.21), (6.3.31)) (see also [Ji4]). Furthermore, for a prime p we have $\#P_{p^\alpha}(f) \geq NP_{p^\alpha}(f) = p^\alpha - p^{\alpha-1}$ as follows from the Möbius formula. The same is not true for maps of S^d, $d \geq 2$, as follows from the Shub example (cf. Example (1.0.20)).

One can ask what condition on $f\colon S^d \to S^d$, together with the necessary $|\deg f| >$ 1 implies infinitely many periodic points. In [JeMR3] we showed that every continuous map $f\colon S^d \to S^d$, $n \geq 1$, of $\deg f = r$, where $|r| \geq 2$, has properties similar to the above provided it commutes with a free homeomorphism $g\colon S^d \to S^d$ of a finite order. The sequence $\{\#P^m(f)\}$ is unbounded and $\mathrm{Per}(f)$ is then infinite.

(7.1.1) Definition. Let X be a smooth manifold and $g\colon X \to X$ a homeomorphism of the finite order m. We say that g is free if for every $x \in X$, $g^k(x) = x$ and $1 \leq i \leq k$, $g^i(x) = x$ implies $k = m$. Equivalently we say that an action of the cyclic group $\{g\} \simeq \mathbb{Z}_k$ on X is given by $(i, x) \mapsto g^i(x)$. If g is free, then this action is called a *free action*.

(7.1.2) Definition. Let X be a smooth manifold with an action of a cyclic group $\mathbb{Z}_k$ defined by homeomorphisms $g\colon X \to X$. A map $f\colon X \to X$ is $\mathbb{Z}_k$-equivariant if $f\alpha = \alpha f$, for all $\alpha \in \mathbb{Z}_k$. Note that f is $\mathbb{Z}_k$-equivariant if it commutes with the generator of action i.e. $f(gx) = gf(x)$. A homotopy $H\colon X \times [0,1] \to X$ is equivariant, i.e. $z \in X$, $t \in [0,1]$, $\alpha \in \mathbb{Z}_k$ implies $H(\alpha x, t) = \alpha H(x,t)$.

Suppose that we have free action of $\mathbb{Z}_k$ on the sphere S^n, $n \geq 2$, i.e. given a free homeomorphism $g\colon S^n \to S^n$ of order k. To formulate our result we need new a notation.

(7.1.3) Definition. Let $k = p_1^{a_1} \cdots p_s^{a_s}$, $a_i > 0$, be the decomposition of m into prime powers. Let m be a natural number. We represent m as $m = p_1^{b_1} \cdots p_s^{b_s} p_{s+1}^{a_{s+1}} \cdots p_r^{a_r}$, where $p_1, \dots, p_r$ are all (distinct) primes satisfying $p_i | k \Leftrightarrow i \leq s$, $b_i \geq 0$. Finally we put $m' := p_1^{b_1} \cdots p_s^{b_s}$.

We are in position to formulate the main result of this section.

(7.1.4) Theorem. *Let $g\colon S^d \to S^d$, $d \geq 1$, be a free homeomorphism, and $f\colon S^d \to S^d$ a map commuting with g. Suppose that* $\deg f \neq -1, 0, 1$. *Then for every $m \in \mathbb{N}$ we have*

$$\#\mathrm{Fix}\, f^{mk} \geq k^2 m',$$

where m' is defined above. In particular, for $m = k^s$ we have

$$\#\mathrm{Fix}\, f^{k^{s+1}} \geq k^{s+2}.$$

To show this theorem we employed (see [JeMR3]) a fine modification of the full Nielsen–Jiang periodic number $NF_m(f)$ which estimates $\#\mathrm{Fix}\,(f^m)$ (cf. (6.3.26)). It can be applied to the map f/G induced by f on the quotient space X/G, which is then a generalized lens space. In this way we achive an estimate of $\#\mathrm{Fix}\,((f/G)^m)$ (this estimate is not good if a map of X/G is not of the form f/G). Finally, for a geometrical reason these fixed points of $(f/G)^m$ give fixed points of $f^{m\,k}$.

As a consequence we get:

(7.1.5) Corollary. *Under the above assumptions*

$$\limsup_{l \to \infty} \frac{\#\mathrm{Fix}(f^l)}{l} \geq k.$$

For a cyclic group of prime order the method also allows usto estimate the number of m-periodic points of f, with m being the minimal period. Fix a prime number $p|k$ and restrict the action to $\mathbb{Z}_p \subset \mathbb{Z}_k$.

(7.1.6) Theorem. *Let $f\colon S^d \to S^d$ be a continuous map which commutes with a free homeomorphism g of S^d of prime order p. If* $\deg(f) \neq \pm 1$, *then for each $s \in \mathbb{N}$ there exist at least $p-1$ mutually disjoint orbits of f of periodic points each* *of length p^s. Thus $\#P_{p^s}(f) \geq (p-1)p^s$. The same is true for any map homotopic to f by an equivariant homotopy.*

(7.1.7) Remark. In the above subsection we presented a situation when a symmetry of map implies the existence of periodic points. It is a natural another question: what is a minimal number of of fixed, or periodic points in the equivariant homotopy class of a given equivariant map $f\colon X \to X$ of a space X with an action of a compact Lie group. We would not like to discuss this problem. For classical lines on this subject we refer the reader to works of Dariusz Wilczyński, Peter Wong, and Davide Ferrario (e.g. [Wi], [FaWo], [Wo1], and [Fe]).

7.2. Relations to the topological entropy

We would like also to announce a relation between the linearization of the map of an NR-solvmanifold (Definition (6.3.15)) and the topological entropy of this map. Recall that for a continuous self-map $f\colon X \to X$ of a metric space X we can assign a real number $\mathbf{h}(f) \geq 0$, or ∞, which measures the dynamics of f and is called the *topological entropy*. Roughly speaking, if $\mathbf{h}(f) > 0$, then the dynamics of f is complex (rich). For the definition and more details on the entropy see [HaKa].

Let $H^*(f)\colon H^*(X;\mathbb{R}) \to H^*(X;\mathbb{R})$ be the linear map induced by f on the cohomology space

$$H^*(X;\mathbb{R}) := \bigoplus_{i=0}^{d} H^i(X;\mathbb{R}).$$

Recall that if X is a compact smooth manifold, then the singular, Čech, simplicial, cellular, or the de Rham cohomology theories are equivalent (cf. [Sp]).

Denote by $\sigma(f)$ the spectrum of the linear map $H^*(f)\colon H^*(X;\mathbb{R}) \to H^*(X;\mathbb{R})$ induced by $f\colon X \to X$. Next, by $\mathrm{sp}(f)$ we denote the spectral radius of map $H^*(f)$. Both are homotopy invariants.

In [Sh] Michael Shub asked under what assumption the topological entropy is estimated by the spectral radius of $H^*(f)$,

$$\mathrm{sp}(f) \leq \mathbf{h}(f). \tag{7.2.1}$$

He conjectured that is true for any C^1-map $f\colon X \to X$ of a compact smooth manifold. Estimate (7.2.1) was proved by Y. Yomdin ([Yo]) if f is a C^∞-map and for a few special cases under the C^1 assumption (cf. [HaKa], [Man]). Exempli gratia, M. Misiurewicz and F. Przytycki showed that $\mathbf{h}(f) \geq \log|\deg(f)|$ ([HaKa]). Note that (7.2.1) is not true for a C^0 map as follows from Example (1.0.20), because the entropy of that map is equal to 0.

In 1977 Misiurewicz and F. Przytycki showed that the estimate (7.2.1) holds for any continuous map of the torus $\mathbb{T}^d$ ([MiPr]). A conjecture that the same estimate holds for every continuous self-map, if we assume a special topological form of the manifold was posed by A. Katok in [Ka]:

Let X be a manifold with the universal cover homeomorphic to the Euclidean space $\mathbb{R}^d$.

$$\log \operatorname{sp}(f) \leq \mathbf{h}(f) \quad \textit{for all } C^0\textit{-maps } f\colon X \to X. \tag{7.2.2}$$

In [MarP] the estimate (7.2.2) was confirmed for the following class of manifolds.

(7.2.3) THEOREM. *Let $f\colon X \to X$ be a continuous self-map of a compact nilmanifold, X of dimension d. Then* $\log(\operatorname{sp}(f)) \leq \mathbf{h}(f)$.

Let G be a connected Lie group, Γ its subgroup. For a homomorphism $\Psi\colon G \to G$ such that $\Psi(\Gamma) \subset \Gamma$ by the factor map we mean the map induced on the quotient space G/Γ. For a given $d \times d$ matrix we put $\wedge A := \bigoplus_{l=0}^{d} \wedge^l A$, where $\wedge^l A$ is the l-th exterior power of A. A step in the proof of Theorem (7.2.3) is the following statement. (cf. [MarP]).

(7.2.4) THEOREM. *Let $X = G/\Gamma$ be a compact homogeneous space of a connected Lie group G by a discrete co-compact subgroup Γ, and $\phi\colon X \to X$ the factor map induced by a Γ preserving endomorphism $\Phi\colon G \to G$. Then*

$$\log \operatorname{sp}(\phi) \leq \log \operatorname{sp}(\wedge D\Phi(e)).$$

Recall that for a factor map $\phi = \Phi/G$ of a compact completely solvable solvmanifold X, the matrix $D\Phi(e)$ has the same spectrum as the linearization A_f of f (cf. Proposition (6.3.24)). Furthermore $\sigma(D\Phi(e)) = \sigma(A_f)$ implies $\sigma(\wedge D\Phi(e)) = \sigma(\wedge A_f)$. On the other hand every map f of a compact completely solvable solvmanifold X is homotopic to a factor map ϕ by the rigidity property (cf. Theorem (6.2.11)), and from Theorem (7.2.4) follows that $\log \operatorname{sp}(f) = \log \operatorname{sp}(\phi) \leq \operatorname{sp}(\wedge D\Phi(e)) = \operatorname{sp}(\wedge A_f)$. For a proof of Theorem (7.2.4) we refer the reader to [MarP]. It is elementary but needs some notions we do not use in this book. Note also that in the case when G is nilpotent, or completely solvable, it is sufficient to apply Proposition (6.3.23) to get the statement of Theorem (7.2.4).

Next one can modify the Misiurewicz argument of an unpublished proof of the main theorem of [MiPr] to show that (cf. [MarP])

(7.2.5) THEOREM. *For every map f of a compact nilmanifold $X = G/\Gamma$ and a homomorphism $\Phi\colon G \to G$, $\Phi(\Gamma) \subset \Gamma$, such that f is homotopic to the factor of f, we have* $\log \operatorname{sp}(\wedge D\Phi(e)) \leq \mathbf{h}(f)$.

A proof of Theorem (7.2.5) needs a fine analytical estimate of the growth of rate of iterations of map f in the stable and unstable direction of the foliation given on G by the homomorphism $\Phi\colon G \to G$. Below we present a proof, a weaker (under an additional assumption) version of Theorem (7.2.3) that is based on the Lefschetz and Nielsen periodic point theory (cf. [MarP]).

(7.2.6) THEOREM. *The estimate of* (7.2.1) *holds for any continuous map* $f\colon M \to M$ *of a completely solvable special solvmanifold which is not deformable to a fixed point free map.*

PROOF. First note that if $f\colon M \to M$ is a map of a compact special completely solvable solvmanifold, then $N(f) = 0$ is equivalent to the fact that f is deformable to a fixed point free map. The implication in one direction follows from the main property of Nielsen numbers. Conversely, if $f\colon M \to M$ is a map of a compact manifold and $\dim M \geq 3$, then the Wecken theorem says that $N(f) = 0$ implies that f is deformable to a fixed point free map (see Theorem (4.2.1)). If M is a special completely solvable compact solvmanifold of dim ≤ 2 it must be the torus and the statement holds by an elementary analysis (cf. [JeMr1]).

(7.2.7) DEFINITION. By the definition, the *asymptotic Nielsen number* $N^\infty(f)$ of f is defined as $N^\infty(f) := \limsup_n \sqrt[n]{N(f^n)}$, where $N(f)$ is the Nielsen number of f (cf. [Iv]).

In [Iv] Ivanov showed that

$$\log N^\infty(f) \leq \mathbf{h}(f), \tag{7.2.8}$$

for every continuous self-map f of a compact manifold. By the Anosov Theorem (6.3.21) we have $N(f) = |L(f)|$ for a map of a compact NR-solvmanifold. From this and Proposition (6.3.23) it follows that

$$N(f^n) = |L(f^n)| = |\det(I - D\Phi^n(e))|.$$

Now note that we can assume that $\operatorname{sp}(D\Phi(e)) > 1$, i.e. there exists at least $\lambda \in \sigma(D\Phi(e)$ with $|\sigma| > 1$. Indeed, otherwise $\operatorname{sp}(f) = 0$ by Theorem (7.2.4), and the inequality (7.2.1) reduces to $h(f) \geq 0$. On the other hand

$$\det(I - D\Phi^n(e)) = 1 - \left(\sum_{j=1}^{d} \lambda_j^n\right) + \sum_{j_1<j_2} \lambda_{j_1}^n \lambda_{j_2}^n + \cdots + (-1)^n(\lambda_1^n \cdots \lambda_d^n),$$

where $\{\lambda_1, \dots, \lambda_d\}$ are all eigenvalues of $D\Phi(e)$. Consequently, we have

$$\begin{aligned} \log N^\infty(f) &= \limsup \frac{1}{n} \log \left(\left|1 - \left(\sum_{1}^{d} \lambda_j^n\right)\right.\right. \\ &\qquad \left.\left. + \sum_{j_1<j_2} \lambda_{j_1}^n \lambda_{j_2}^n + \cdots + (-1)^n(\lambda_1^n \cdots \lambda_d^n)\right|\right) \\ &= \begin{cases} -\infty & \text{if } 1 \in \sigma(D\Phi(e)), \\ \log\left(\prod_{|\lambda_j|>1} |\lambda_j|\right) = \sum_{|\lambda_j|>1} \log|\lambda_j| & \text{otherwise,} \end{cases} \end{aligned} \tag{7.2.9}$$

since we have assumed that $\{j : |\lambda_j| > 1\} \neq \emptyset$. The latter equality follows by the following argument. Recall that the spectral radius of $\mathrm{sp}(D\Phi(e) > 1$. Let $\lambda_{j_1}, \ldots, \lambda_{j_k}$, $1 \leq k^+ \leq d$, be all eigenvalues of $D\Phi(e)$ of module greater then 1. Let λ_{j_i} $1 \leq i \leq k^-$, be all eigenvalues with $|\lambda_{j_i}| < 1$, and finally let λ_{j_i}, $1 \leq i \leq k$, be all eigenvalues with $|\lambda_{j_i}| = 1$, $k^- + k^+ + k = d$. Then we have

$$|\det(\mathrm{I} - D\Phi(e)^n)| = \left| \prod_{i=1}^{k^-}(1 - \lambda_{j_i}^n) \prod_{i=1}^{k}(1 - \lambda_{j_i}^n) \prod_{i=1}^{k^+}(1 - \lambda_{j_i}^n) \right|.$$

We estimate the asymptotic behaviour of each factor of the above multiplication. We have

$$\prod_{i=1}^{k^-}(1 - \lambda_{j_i}^n) = 1 - \left(\sum_1^{k^-} \lambda_j^n \right) + \sum_{j_1 < j_2} \lambda_{j_1}^n \lambda_{j_2}^n + \cdots + (-1)^n (\lambda_1^n \cdots \lambda_{k^-}^n) \xrightarrow{n \to \infty} 1,$$

because all $|\lambda_{j_i}| < 1$. Dividing the third factor by $\rho^n = \prod_{i=1}^{k^+} |\lambda_{j_i}|^n$ we get

$$\begin{aligned} \frac{|\prod_{i=1}^{k^+}(1 - \lambda_{j_i}^n)|}{\rho^n} =& \Bigg| \prod_{i=1}^{k^+} \frac{1}{\rho^n} - \sum_{j=1}^{k^+} \left(\frac{\lambda_j}{\rho}\right)^n + \sum_{j_1 < j_2} \left(\frac{\lambda_{j_1}\lambda_{j_2}}{\rho}\right)^n \\ &+ \cdots + (-1)^n \left(\frac{\lambda_1 \cdots \lambda_{k^+}}{\rho}\right)^n \Bigg|. \end{aligned}$$

Note that in each term except for the last of the sum of the right-hand side of the above equality we have a sum of m-powers of complex numbers of module < 1. The last term has module equal to 1. This shows that

$$\lim_n \sqrt[n]{\frac{|\prod_{i=1}^{k^+}(1 - \lambda_{j_i}^n)|}{\rho^n}} = 1.$$

We are left with a discussion of the mid-factor $\prod_{i=1}^{k}(1 - \lambda_{j_i}^n)$. Now observe that

$$\prod_{i=1}^{k}(1 - \lambda_{j_i}^n) = 1 - \left(\sum_1^{k} \lambda_j^n \right) + \sum_{j_1 < j_2} \lambda_{j_1}^n \lambda_{j_2}^n + \cdots + (-1)^n (\lambda_1^n \cdots \lambda_k^n)$$

is the sum discussed in the proof of Theorem (3.1.53). If at least one λ_{j_i}, $1 \leq i \leq k$, is not a root of unity, then we have case (3.1.53.3) and $\limsup_m |\prod_{i=1}^{k}(1 - \lambda_{j_i})|^n = \gamma > 0$. If all λ_{j_i}, $1 \leq i \leq k$, are the roots of unity, then the case when the product is 0 for every n is excluded by our assumption $1 \notin \sigma(D\Phi(e))$. Indeed if each λ_{j_i}, $1 \leq i \leq k$ is a primitive root of unity of degree $q_i > 1$, then taking $n = kq + 1$ where $q = \mathrm{LCM}\{q_i\}$ we have $1 - \lambda_{j_i}^n \neq 0$. This shows that we have the (3.1.53.2), and $\limsup_m \prod_{i=1}^{k} |(1-\lambda_{j_i})|^n = \gamma > 0$. Consequently in each of the above cases we have $\limsup_m \sqrt[n]{\prod_{i=1}^{k} |(1-\lambda_{j_i})|^n} = 1$. This shows that $\log N^\infty(f) = \log \mathrm{sp}(\wedge D\Phi(e))$ provided $1 \notin \sigma(D\Phi(e))$. Theorem (7.2.4), the rigidity property, and inequality (7.2.8) give the statement of Theorem (7.2.6). □

It is natural to ask whether the tools used for the proof of Theorem (7.2.6) work in a more general situation. We discuss it below.

(7.2.10) REMARK (Behavior of the asymptoptic Nielsen number). We recall that every compact solvmanifold $M = G/\Gamma$ can be represented (in many ways in general, as a fibration $\mathbf{N} \subset M \xrightarrow{p} \mathbb{T}$, where the fibre $\mathbf{N}$ is a nilmanifold and the base $\mathbb{T}$ is a torus. Moreover, every map $f\colon M \to M$ can be deformed to a fiber preserving map $f = (f_N, f_{\mathbb{T}})$ of this fibration (cf. Theorem (6.3.14)). Furthermore one can define the linearization matrix A_f as $A_{f_N} \oplus A_{f_{\mathbb{T}}}$ (cf. Definition (6.3.15)). But only for a map of an NR-solvmanifold the equality $\det(\mathrm{I} - A_f) = L(f)$ and Anosov Theorem (6.3.21) holds. Consequently, only for these solvmanifolds we get the estimate of $N^\infty(f)$ by $\mathrm{sp}(\wedge(A_f))$ by the above argument.

(7.2.11) REMARK (Inequality $\mathrm{sp}(f) \le \mathrm{sp}(\wedge(A_f))$). For the proof of this inequality we used the rigidity property replacing f by ϕ_f. On the other hand $f\colon M \to M$ is always, up to homotopy, a fibre map of the Mostow fibration. Using the Serre spectral sequence convergent to $H^*(M;\mathbb{R})$ one can show that $\mathrm{sp}(f) \le \mathrm{sp}(E_2^{p,q}(f))$, because in each step passing from $E_2^r(M)$, to $E_2^{r+1}(M)$, we either pass to a subspace or to a factor space. To get directly $\mathrm{sp}(E_2^{p,q}(f)) = \mathrm{sp}(f_{M_{\mathbf{N}}}) \cdot \mathrm{sp}(f_{\mathbb{T}})$, we need to assume that the fibration is orientable, i.e. that the system of local coefficients of this fibration is constant (cf. [Sp]). Indeed $E_2^{p,q} = H^p(\mathbb{T}) \otimes H^q(M_N)$ then, which yields the latter inequality. Unfortunately the Mostow fibration is not orientable in general. Summing up, the statement of Theorem (7.2.6) still holds for a map f of a compact special NR-solvmanifold G/Γ provided f is homotopic to a factor of an endomorphism of G.

In general Theorem (7.2.3) follows from Theorems (7.2.4), (7.2.5) (cf. [MarP]).

Note that in general Theorems (7.2.4) and (7.2.5) give a sharper estimate than Theorem (7.2.3).

Now we would like to discuss shortly a relation between the topological entropy and periodic points for a nilmanifold map. The claim that $\mathbf{h}(f) > 0$ implies a nonempty set of periodic points, or the set of minimal periods $\mathrm{Per}(f)$ of f, is false in view of the following example.

(7.2.12) EXAMPLE. Take $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the product map $f = (f_1, f_2)$, $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ where f_1 is of degree > 1 and f_2 has no periodic points. Conversely, for the map $f(x,y) = (x, x+y) \mod 1$ of the torus $\mathbb{T}^2$ we have $\operatorname{Per}(f) = \mathbb{N}$ but $\mathbf{h}(f) = 0$.

Moreover, there exists a homeomorphism f of the torus $\mathbb{T}^2$ with the above property (cf. [HaKa]).

However it seems that the set of homotopy minimal periods is an invariant of the dynamics of f which is related to the topological entropy with respect to the following proposition.

(7.2.13) Proposition. *Let $f\colon X \to X$ be a map of a compact nilmanifold. If* $\operatorname{HPer}(f)$ *is infinite, then* $\mathbf{h}(f) > 0$.

Proof. Note that $\operatorname{sp}(\wedge A) > 1$ if and only if $\operatorname{sp}(A) > 1$ by the linear algebra argument. Now the statement follows from the equality $\sigma(A) = \sigma(D\Phi(e))$, Theorems (7.2.4), (7.2.5) and Theorem (6.4.4) (Case (G)). $\square$

A direct consequence of Theorem (7.2.5) and estimates of the topological entropy from above, given earlier by S. Katok [KaS], Cz. Krzyżewski [Krz], D. Ruelle [Ru], and F. Przytycki [Pr] is the following fact (cf. [MarP])

(7.2.14) Theorem. *Let $M = G/\Gamma$ be a quotient of a connected Lie group by a uniform lattice Γ and $\phi\colon M \to M$ be the factor map of a Γ preserving endomorphism $\Phi\colon G \to G$. Then* $\mathbf{h}(\phi) = \log \operatorname{sp}(\wedge D\Phi(e))$. *If M is a nilmanifold, then ϕ minimizes the entropy in its homotopy class. If M is a completely solvable solvmanifold, then the statement holds provided the class does not contain a fixed point free map.*

Proof. It follows from [Ru], [Krz] or [KaS] (the latter in the case of diffeomorphism), see also [Pr] , that $\mathbf{h}(f) \le \limsup_{n\to\infty} n^{-1} \sup_{x\in M} \log \| \wedge (Df(x))\|$ for every C^1-mapping $f\colon M \to M$ of a compact manifold M. In our case $|| \wedge D\phi(x)|| = || \wedge D\Phi(e)||$ for every $x \in M = G/\Gamma$. Hence, by the definition of spectral radius, $\mathbf{h}(\phi) \le \log \operatorname{sp}(\wedge D\Phi(e))$.

The opposite inequality follows from Theorem (7.2.3) or (7.2.6). Indeed it can be proved like Theorem (7.2.5) but much simpler (see [MarP]).

Finally, since by Theorem (7.2.3), $\mathbf{h}(f) \ge \log \operatorname{sp}(\wedge D\Phi_f(e))$ in the nilmanifold case, $\mathbf{h}(\phi)$ minimizes $\mathbf{h}(f)$ in the homotopy class. $\square$

7.3. Indices of iterations of planar maps

(by Grzegorz Graff)

We would like to express our thanks to Grzegorz Graff for preparing this material.

Let $f\colon \mathcal{U}$ $\quad \mathbb{R}^2$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^2$, be a continuous map such that for each $m \in \mathbb{N}$ ($m \in \mathbb{Z}$ $m \neq 0$ for a homeomorphism) 0 is an isolated fixed point for f^m. In this case the fixed point index $\operatorname{ind}(f^m, 0)$ is well defined for f^m restricted to a small neighbourhood of 0. In this section we present recent results concerning the form of $\{\operatorname{ind}(f^n, 0)\}_{n=1}^{\infty}$ (or $\{\operatorname{ind}(f^n, 0)\}_{n\neq 0}$ if f is a homeomorphism).

As we know from previous sections, continuous differentiability of a map $g\colon \mathbb{R}^d \to \mathbb{R}^d$ implies periodicity of a sequence of indices of its iterations. What is more, the less is the dimension d, the simpler is the sequence, because there the set $\Delta(\mathcal{O})$ is smaller. Now we are interested in the question of what happens if we drop the assumption of differentiability but will consider low-dimensional maps, namely planar maps. In this section we give a survey (without proofs) of results on local fixed point indices of planar maps.

It is natural to ask whether in such a setting there are any additional constrainers, except for Dold relations, for $\{\operatorname{ind}(f^n, 0)\}_{n=1}^{\infty}$.

The following theorem states that, if we assume only continuity of f, the answer to that questions is negative (cf. [BaBo], [Gn2]).

(7.3.1) THEOREM. *For any sequence of integers $\{c_n\}_{n=1}^{\infty}$, with $n \mid \sum_{k|n} \mu(n/k) \cdot c_k$, there exists a self-map f of the unit disk D^2 in $\mathbb{R}^2$, such that $c_n = \operatorname{ind}(f^n, 0)$ for each n.*

The situation is essentially different for homeomorphisms. In 3-dimensional space Theorem (7.3.1) is still true if we replace f by g, where $g\colon D^3 \to D^3$ is a homeomorphism, D^3 is a unit disk in $\mathbb{R}^3$ (cf. [BaBo]), but is not true for planar homeomorphisms, for which there are strong restrictions on $\{\operatorname{ind}(f^n, 0)\}_{n\neq 0}$ which are a consequence of topological properties of the plane.

We present below the state of knowledge about indices of iterations for three important classes of homeomorphisms.

7.3.1. Orientation preserving homeomorphisms. Let H_n be the space of all orientation preserving planar homeomorphisms h such that the origin is the only fixed point and $\operatorname{ind}(h, 0) = n$. It was stated by B. Schmitt (cf. [Schm]) and reproved by M. Bonino (cf. [Bon1]) that H_n is path connected in compact-open topology. In 1990 Morton Brown used this fact to describe the behaviour of the sequence of indices of iterations $\{\operatorname{ind}(f^n, 0)\}_{n\neq 0}$.

(7.3.2) THEOREM (cf. [MBr]). *Let f be an orientation preserving local homeomorphism of the plane. Then there is an integer $p \neq 1$ such that for each $n \neq 0$:*

$$\operatorname{ind}(f^n, 0) = \begin{cases} p & \text{if } \operatorname{ind}(f, 0) = p, \\ 1 \text{ or } p & \text{if } \operatorname{ind}(f, 0) = 1. \end{cases}$$

(7.3.3) COROLLARY. *If* $\operatorname{ind}(f^n, 0) = p$ *then* $\operatorname{ind}(f^{kn}, 0) = p$ *for every* $k \in \mathbb{Z} \setminus \{0\}$.

We will consider only $n > 0$ since for orientation preserving planar homeomorphisms $\operatorname{ind}(f^n, 0) = \operatorname{ind}(f^{-n}, 0)$.

(7.3.4) DEFINITION. Let us define A, the set of generators for $\{\operatorname{ind}(f^n, 0)\}_{n=1}^{\infty}$, in the following way:

$$A = \{a \in \mathbb{N} : \operatorname{ind}(f^a, 0) = p \text{ and } \operatorname{ind}(f^b, 0) = 1 \text{ for all } b|a,\ b \neq a\}.$$

From Theorem (7.3.2) we have: if $A = \emptyset$, then for each n, $\operatorname{ind}(f^n, 0) = 1$.

(7.3.5) THEOREM (cf. [GN1]). *Let f be an orientation preserving local homeomorphism of the plane, $A \neq \emptyset$. Then A is finite and* $\operatorname{LCM}(A)|(p-1)$, *where* $\operatorname{LCM}(A)$ *denotes the lowest common multiplicity of all elements in A.*

(7.3.6) PROBLEM (cf. [GN1]). The open question is whether there are further restrictions on the set of generators A. It is likely that there is no room in $\mathbb{R}^2$ for more than one element in A, which is in accord with the hypothesis that indices of an iterated planar homeomorphism behave in the same way as indices of a planar C^1-map (cf. [BaBo]).

7.3.2. Orientation reversing homeomorphisms. A recent result of M. Bonino about the possible values of the fixed point index in the orientation-reversing case (announced earlier by M. Brown in [MBr]) provides also information about indices of iterations for this class of homeomorphisms.

(7.3.7) THEOREM (cf. [Bon2]). *Let f be an orientation reversing local homeomorphism of the plane. Then* $\operatorname{ind}(f, 0) \in \{-1, 0, 1\}$.

The theorem above and Dold relations determine the form of indices of odd iterations of an orientation reversing homeomorphism.

(7.3.8) THEOREM (cf. [GN1]). *Let f be an orientation reversing local homeomorphism of the plane. Then for every n odd,* $\operatorname{ind}(f^n, 0) \in \{-1, 0, 1\}$ *and*

$$\operatorname{ind}(f^n, 0) = \begin{cases} \operatorname{ind}(f, 0) & \textit{if } n > 0, \\ -\operatorname{ind}(f, 0) & \textit{if } n < 0. \end{cases}$$

What is more, by Dold relations one may easy deduce the form of indices of even iterations when $\operatorname{ind}(f, 0) = 0$, namely $\operatorname{ind}(f^{2n}, 0) = 2l$, where l is an integer.

(7.3.9) PROBLEM (cf. [GN1]). It is obvious that f^2 is always a homeomorphism which preserves the orientation. Is it true that for a homeomorphism f which reverses the orientation, $\{\operatorname{ind}(f^{2n}, 0)\}_{n=1}^{\infty}$ must be a constant sequence, i.e. the set of generators A for this sequence is either empty or equal to $\{1\}$?

7.3.3. le Calvez–Yoccoz homeomorphisms. We say that a local planar homeomorphism f belongs to the class of le Calvez–Yoccoz if the following two conditions are satisfied:

(A) There is no $\mathcal{V}$ – a neighbourhood of 0, such that $f(\mathcal{V}) \subset \mathcal{V}$ or $\mathcal{V} \quad f(\mathcal{V})$.
(B) There is $\mathcal{W}$ – a neighbourhood of 0, such that $\bigcap_{k\in\mathbb{Z}} f^k(\mathcal{W}) = \{0\}$.

(7.3.10) THEOREM (cf. [CaYo]). *Let f be an orientation preserving le Calvez–Yoccoz homeomorphism. Then there exist positive integers r, q, such that, for each $k \neq 0$,*

$$\operatorname{ind}(f^k, 0) = \begin{cases} 1 - rq & \text{if } q|k, \\ 1 & \text{if } q \nmid k. \end{cases}$$

Theorem (7.3.10) may be expressed in terms used in Theorem (7.3.2) and Definition (7.3.4) as: $A = \{q\}$ and $p \leq 0$, (namely: $p = 1 - rq$). This information was used by P. le Calvez and J. C. Yoccoz to show non-existence of minimal homeomorphisms of a punctered sphere (cf. [CaYo]), which was a classical problem of S. Ulam from *the Scottish Book* (cf. [Mau, Problem 115]). We recall that a homeomorphism $f\colon X \to X$ of a topological space X is called *minimal* if for every point $x \in X$ the orbit $\{f^n(x)\}_{n\in\mathbb{Z}}$ is dense in X.

It is easy to see that (B) is equivalent to the condition that $\{0\}$ is an isolated invariant set, which enables us to use the Conley index theory to study le Calvez–Yoccoz homeomorphisms. Following this line J. Franks obtained in a simpler way a part of Theorem (7.3.10): he showed that in the sequence $\{\operatorname{ind}(f^n, 0)\}_{n\neq 0}$ there are infinitely many negative terms (cf. [Fr2]).

The formula of le Calvez and Yoccoz from Theorem (7.3.10) was re-proved by an application of the Conley index by F. Ruiz del Portal and J. Salazar (cf. [RS1]). These authors use also the same methods to study the dynamics of le Calvez–Yoccoz homeomorphisms in more general settings (cf. [RS3], [RS2]). In [RS1] they described the indices of iterations in the orientation reversing case.

(7.3.11) THEOREM. *Let f be an orientation reversing a le Calvez–Yoccoz homeomorphism. Then there exist integers $\gamma \in \{-1, 0, 1\}$ and $r \geq 1$ such that for each $k > 0$:*

$$\operatorname{ind}(f^k, 0) = \begin{cases} \gamma & \text{if } 2 \nmid k, \\ \gamma - 2r & \text{if } 2|k. \end{cases}$$

By Theorem (7.3.8), $\gamma = \operatorname{ind}(f, 0)$. Notice also that in this case the hypothesis formulated in Problem (7.3.9) is true.

(7.3.12) PROBLEM. Let us consider a continuous map f which satisfies conditions (A) and (B). Is it true that $\{\operatorname{ind}(f^n, 0)\}_{n=1}^{\infty}$ is a periodic sequence?

7.4. Fixed and periodic points of multivalued maps

The topological fixed and periodic point theory has many applications in nonlinear analysis and it is well presented in the literature. To avoid expansion of our book we do not included them. A reader can find it in several places including [Br3], [Cr2], [Nir].

In this section we would like to present only a foundation of the Nielsen theory of fixed periodic points for multivalued maps. This recently developed theory has already a couple of applications. On the other hand, unlike to the classical fixed points theory of multivalued maps based on Lefschetz type theorems (see [Gorn2], [HandII]), the theory still lacks a full exposition in the literature.

7.4.1. Nielsen fixed point theory for multivalued maps. The homological methods of detecting of fixed points were extended also to the case of multivalued maps. In this subsection we will outline an extension of the Nielsen theory in this situation. For the details see [AnGrJr1]–[AnGrJr4]. The multivalued maps appear in finding solutions of differential equations (or differential inclusions) and in applications in economics (game theory, optimization). For a condensed survey of results in this direction as well as for the literature we refer to an article of J. Andres [An].(see also [CaW])

The natural assumption on the values is convexity and a kind of continuity. We consider a multivalued map $F\colon X \multimap Y$ (i.e. to each point $x \in X$ a nonempty subset $F(x) \subset Y$ is subordinated). The map F is called *upper semicontinuous* (u.s.c. for a shorthand) if for any point $x \in X$ and open subset $\mathcal{U} \quad Y$ containing $F(x)$ there is a neighbourhood $\mathcal{V} \quad X$ of x such that $F(\mathcal{V}) = \{f(x) : x \in \mathcal{V}$ is contained in $\mathcal{U}$. This condition is equivalent to

$$F^{-1}(\mathcal{U}) = \{x \in X : F(x) \subset \mathcal{U}$$

is open in X for each open set $\mathcal{U} \quad Y$.

We define the *graph* of a multivalued map $F\colon X \multimap Y$ as

$$\Gamma_F = \{(x, y) \in X \times Y : y \in F(x)\}.$$

Let $p_X\colon \Gamma_F \to X$, $p_Y\colon \Gamma_F \to Y$ be the natural projections $p_X(x, y) = x$, $p_Y(x, y) = y$. Then $F(x) = p_Y^{-1}(p_X(x))$. This suggests the following generalization due to Górniewicz (see [Gorn1]).

(7.4.1) Definition. Let X, Y, Γ be topological spaces. A pair of maps $p\colon \Gamma \to X$, $q\colon \Gamma \to Y$ will be called a *multivalued map*. We will denote it by $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ or briefly (p, q).

The pair (p, q) determines the multivalued map (in the common sense) $F(x) = p_Y^{-1}(p_X(x))$ which turns out to be upper semicontinuous. The convenient topological generalization of convexity is the acyclicity. Thus we will consider multivalued maps (p, q) satisfying

(1) X, Y, Γ are compact and moreover X, Y are polyhedra (or ENRs),

(2) p is a *Vietoris* map, i.e. p is onto and the inverse image of each point is *Q-acyclic*. The last means that $\check{H}_*(p^{-1}(x); \mathbb{Q}) = \check{H}_*(\text{point}; \mathbb{Q})$ (for each $x \in X$) where $\check{H}_*$ denotes Čech homology with rational coefficients (cf. [Sp]). We will write then $p\colon \Gamma \to X$.

The pair (p, q) satisfying the above will be called *admissible*.

(7.4.2) Remark. Thanks to the natural isomorphism $\check{H}_*(X; \mathbb{Q}) = \check{H}^*(X; \mathbb{Q})$ where X is a compact ENR, we may use the Čech cohomology functor instead of homology.

The following theorem was shown by Vietoris in 1927 and simplified by Begle in 1950.

(7.4.3) Theorem. *Let $p\colon \Gamma \to X$ be a Vietoris map between compact spaces. Then the induced cohomology homomorphism $p_*\colon \check{H}_*(\Gamma; \mathbb{Q}) \to \check{H}_*(X; \mathbb{Q})$ is an isomorphism.*

The below theorem of Eilenberg and Montgomery ([EiMt]) generalizes the Lefschetz Theorem on multivalued maps.

(7.4.4) Theorem. *Let $X \xleftarrow{p} \Gamma \xrightarrow{q} X$ be an admissible multivalued map. We consider the homomorphism $q_* p_*^{-1}\colon \check{H}_*(X; \mathbb{Q}) \to \check{H}_*(X; \mathbb{Q})$. If $L(q_* p_*^{-1}) \neq 0$ then (p, q) has a fixed point i.e. there exists a point $x \in X$ such that $x \in qp^{-1}(x)$.*

(7.4.5) Corollary (Brouwer Type Theorem). *Let $F\colon X \to X$ be an upper semicontinuous multivalued map with compact contractible values of a $\mathbb{Q}$-acyclic compact connected metric polyhedron. Then F has a fixed point.*

$L(q_* p_*^{-1})$ is a homotopy invariant with respect to admissible homotopies: by an admissible homotopy between multivalued pairs (p_0, q_0), (p_1, q_1) we mean a commutative diagram

$$\begin{tikzcd}
X \times 0 \arrow[d] & \Gamma_0 \arrow[l, "p_0"'] \arrow[r, "q_0"] \arrow[d] & Y \arrow[d, "\mathrm{id}"] \\
X \times I \arrow[d] & \Gamma \arrow[l, "p"'] \arrow[r, "q"] \arrow[d] & Y \arrow[d, "\mathrm{id}"] \\
X \times 1 & \Gamma_1 \arrow[l, "p_1"] \arrow[r, "q_1"'] & Y
\end{tikzcd}$$

where (p, q) is an admissible pair. Then $L(q_{0*} p_{0*}^{-1}) = L(q_{1*} p_{1*}^{-1})$. In particular:

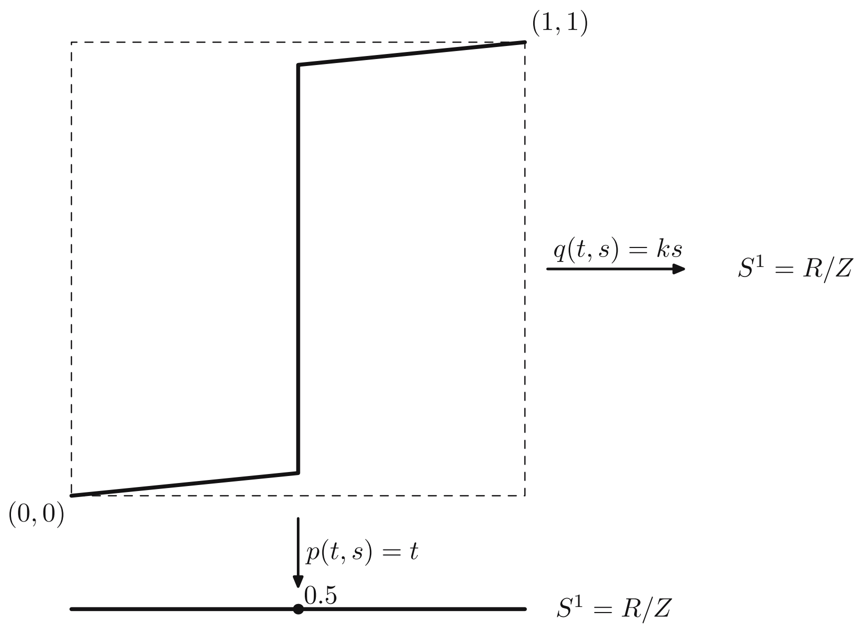
(7.4.6) Corollary. *If a multivalued map* (p, q) *is admissibly homotopic to a single-valued map* f *(more precisely to* (id, f)*) with* $L(f) \neq 0$ *then the map* (p, q) *has a fixed point.*

Estimate from below $\#\mathrm{Fix}(p, q)$**.** The last Corollary suggests the following question. Suppose that an admissible multivalued map (p, q) is admissibly homotopic to a single-valued map f. Does this imply the inequality

$$\#\mathrm{Fix}(p, q) \geq N(f)?$$

The next example shows that the answer is negative.

(7.4.7) Example ([AnGrJr3]). Let $S^1 = \mathbb{R}/\mathbb{Z} = [0,1]/\{0,1\}$. Consider the multivalued map (p, q) given by the figure:



$$p^{-1}[t] = \begin{cases} [2\varepsilon t] & \text{for } 0 \leq t < 1/2, \\ [\varepsilon, 1-\varepsilon] & \text{for } t = 1/2, \\ [1 - 2\varepsilon(1-t)] & \text{for } 1/2 < t \leq 1. \end{cases}$$

Notice that only two points $0 = 1$ and $0.5 \in [0,1]/\{0,1\}$, are fixed points of (p, q) (if $\varepsilon > 0$ is small and the number $k \in \mathbb{N}$ not too large). On the other hand our multivalued map is admissibly homotopic with a single-valued map $f \colon S^1 \to S^1$ of degree k (by the homotopy (p_τ, q) is given by the same formula as p with ε replaced by $\varepsilon_\tau = (1-\tau)\varepsilon + 1/2\tau$). Thus $N(f) = |k - 1|$ may be large while $\#\mathrm{Fix}(p, q) = 2$.

But on the other hand we have $p^{-1}(1/2) = [\varepsilon, 1-\varepsilon] \approx (0,1)$, hence $k-1$ points from $p^{-1}(1/2)$ are sent into $1/2$. In other words we have $k-1$ coincidences of the pair p, q.

This observation suggests that it is reasonable to estimate the number of coincidences of the pair (p,q). There is a well-developed Nielsen coincidence theory which works when the involved spaces are manifolds. Unfortunately the graph of a multivalued map can be a very strange topological space. The graph may be not locally connected, for example the Warsaw circle is a graph of an admissible multivalued map from the circle to the interval. Thus even the partition of the coincidence set into Nielsen classes can not be obtained by joining coincidences with paths. It requires new methods.

Coincidences of a multivalued map. Let $X \xleftarrow{p} \Gamma \xrightarrow{q} X$ be an admissible multivalued map. We define the *coincidence set* as

$$C(p,q) = \{z \in \Gamma : p(z) = q(z)\}.$$

Notice that $z \in C(p,q)$ implies $p(z) \in \mathrm{Fix}(p,q)$, more precisely $\mathrm{Fix}(p,q) = p(C(p,q))$. We will look for a lower bound of $\#C(p,q)$.

We are going to adjust the Nielsen fixed point theory to this setting. However both basic notions: the Nielsen relation and essential classes involve some problems.

Let us fix a universal covering $p_X\colon \widetilde{X} \to X$. We define (the pullback)

$$\widetilde{\Gamma} = \{(\widetilde{x}, z) \in \widetilde{X} \times \Gamma : p_X(\widetilde{x}) = p(z)\}$$

with the projections $p(\widetilde{x}, z) = \widetilde{x}$, $p_\Gamma(\widetilde{x}, z) = z$. Thus we get a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[d, "p_X"'] & \widetilde{\Gamma} \arrow[l, "\widetilde{p}"'] \arrow[d, "p_\Gamma"] \\ X & \Gamma \arrow[l, "p"] \end{tikzcd}$$

Let us notice that p_Γ is a covering map but not universal in general. We will try to lift the multivalued map to universal coverings. Then the projection of the coincidence sets of different liftings will split $C(f,g)$ into mutually disjoint subsets. These subsets will play the role of the Nielsen classes.

This theory requires the following assumption:

- There exists a lift $\widetilde{q}\colon \widetilde{\Gamma} \to \widetilde{X}$ making commutative the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[d, "p_X"'] & \widetilde{\Gamma} \arrow[l, "\widetilde{p}"'] \arrow[r, "\widetilde{q}"] \arrow[d, "p_\Gamma"] & \widetilde{X} \arrow[d, "p_X"] \\ X & \Gamma \arrow[l, "p"] \arrow[r, "q"'] & X \end{tikzcd}$$

A sufficient condition for the existence of such a lift is: *for any* $x \in X$ *the restriction* $q\colon p^{-1}(x) \to X$ *admits a lift*, i.e. there is a map $\widetilde{q}\colon p^{-1}(x) \to \widetilde{X}$ satisfying $p_X\widetilde{q} = q$ (see [AnGrJr1], [AnGrJr4]). The last is satisfied if for instance $p^{-1}(x)$ is simply-connected for each $x \in X$. This assumption can be regarded as the homotopic acyclicity.

The above commutative diagram induces homomorphisms: $\widetilde{p}^{!}\colon \mathcal{O}_X \to \mathcal{O}_\Gamma$, $\widetilde{q}_{!}\colon \mathcal{O}_\Gamma \to \mathcal{O}_X$ which are given by the formulae:

$$\widetilde{p}^{!}(\alpha)(\widetilde{x}, z) = (\alpha(\widetilde{x}), z), \qquad \widetilde{q}(\alpha \cdot (\widetilde{x}, z)) = \widetilde{q}_{!}(\alpha)\widetilde{q}(\widetilde{x}, z).$$

Now $\widetilde{q}_{!}\widetilde{p}^{!}\colon \mathcal{O}_X \to \mathcal{O}_X$ is a homomorphism hence we may consider the *Reidemeister set* $\mathcal{R}(\widetilde{q}_{!}\widetilde{p}^{!})$ defined as the quotient set of the action of $\mathcal{O}_X$ on itself:

$$\gamma \circ \alpha = \gamma \cdot \alpha \cdot (\widetilde{q}_{!}\widetilde{p}^{!})^{-1}(\gamma).$$

(7.4.8) Remark. Let us notice that the lift $\widetilde{p}$ was given with the definition of the pullback $\widetilde{\Gamma}$, hence we may consider it as a natural one. However the lift $\widetilde{q}$ is an arbitrary one. We will fix this lift. Let us remember that the homomorphism $\widetilde{q}_{!}$ depends on this choice.

Now one may check that

(7.4.9) Lemma. *We have:*

(7.4.9.1) $C(p, q) = \sum_{\alpha \in \mathcal{O}_X} p_\Gamma(C(\widetilde{p}, \alpha\widetilde{q}))$.

(7.4.9.2) *If* $p_\Gamma(C(\widetilde{p}, \alpha\widetilde{q})) \cap p_\Gamma(C(\widetilde{p}, \beta\widetilde{q})) \neq \emptyset$, *then* $\beta = \gamma \cdot \alpha \cdot (\widetilde{q}_{!}\widetilde{p}^{!})^{-1}(\gamma)$, *for a* $\gamma \in \mathcal{O}_X$.

(7.4.9.3) *The sets* $p_\Gamma(C(\widetilde{p}, \alpha\widetilde{q}))$ *are either disjoint or equal.*

Thus we get a disjoint sum $C(p, q) = \sum_\alpha p_\Gamma(C(\widetilde{p}, \alpha\widetilde{q}))$ where we sum over one element α from each Reidemeister class of $\mathcal{R}(\widetilde{q}_{!}\widetilde{p}^{!})$.

The above lemma gives the splitting of $C(p, q)$ into *Nielsen classes.*

All the above can be modified to the Reidemeister (hence) Nielsen classes modulo a normal subgroup $H \lhd \mathcal{O}_X$ satisfying $\widetilde{q}_{!}\widetilde{p}^{!}(H) \subset H$. We consider, instead of the universal covering $p_X\colon X \to X$, a covering $p_{XH}\colon \widetilde{X} \to X$ corresponding to $H \lhd \mathcal{O}_X = \pi_1(X)$.

Essential classes. It would be natural to define essential Nielsen classes of (p, q) as the classes with a nonzero local index. However such index does not (yet) exist since the space Γ (graph) may be quite arbitrary. To define essential classes we will need two auxiliary assumptions:

- $X \overset{p}{\Longleftarrow} \Gamma \xrightarrow{q} X$ is an admissible multivalued map where X is a compact connected polyhedron (ENR).
- There exists a normal subgroup $H \lhd \mathcal{O}_X = \pi_1(X)$ of finite index and such that $\widetilde{q}_{!}\widetilde{p}^{!}(H) \subset H$.

Then for each $\beta \in \mathcal{O}_H$ we get the commutative diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "p_X"'] & \widetilde{\Gamma} \arrow[l, "\widetilde{p}"'] \arrow[r, "\beta\widetilde{q}"] \arrow[d, "p_\Gamma"] & \widetilde{X} \arrow[d, "p_X"] \\
X & \Gamma \arrow[l, "p"] \arrow[r, "q"'] & X
\end{tikzcd}$$

The upper map $(\widetilde{p}, \beta\widetilde{q})$ is an admissible map of compact polyhedron $\widetilde{X}$ hence $L(\widetilde{p}, \beta\widetilde{q})$ is defined.

(7.4.10) Definition. The Reidemeister H-class corresponding to $(\widetilde{p}, \beta\widetilde{q})$ is called *essential* if and only if $L(\widetilde{p}, \beta\widetilde{q}) \neq 0$.

(7.4.11) Remark. Let us emphasize that we have defined essential classes and the Nielsen number only modulo a subgroup of finite index. We do not know if it is possible to extend the defintion without this assumption.

Thus we may define H-Nielsen number as

$$N_H(p, q) := \text{number of } H\text{-essential classes.}$$

Since the essential classes are nonempty, we get

(7.4.12) Theorem. $\#C(p, q) \geq N_H(p, q)$.

Since $N_H(p, q)$ is an admissible homotopy invariant, we get

(7.4.13) Lemma. *If a multivalued map (p, q) (satisfying the two above auxiliary assumptions) is admissibly homotopic to a single-valued map ρ, then $N_H(p, q) = N_H(\rho)$ where $N_H(\rho)$ denotes the Nielsen number of the single-valued map modulo the subgroup H.*

Recall that the similar formula does not hold for fixed points of the multivalued map in general! (see Example (7.4.7)).

(7.4.14) Theorem ([AnGrJr2]). *Any admissible multivalued map of a torus $\mathbb{T}^d$ is admissibly homotopic to a single-valued map ρ.*

(7.4.15) Corollary ([AnGrJr2]).

$$\#C(p, q) \geq N(\rho) = |\det(\mathrm{I} - A)|,$$

where A is the $d \times d$ integral matrix expressing the homomorphism $\rho_\#\colon \pi_1(\mathbb{T}^d) \to \pi_1(\mathbb{T}^d)$ induced by a single-map ρ admissibly homotopic to (p, q).

7.4.2. Periodic points of multivalued maps.

Orbits of Nielsen classes of coincidences. Since the ordinary multivalued maps $F\colon X \multimap Y$, $G\colon Y \to Z$ can be composed $G \circ F(x) = G(F(x))$, we may generalize the periodic point theory also to the multivalued case.

Let $F\colon X \multimap X$ be an ordinary multivalued self-map. A point $x \in X$ is called *k-periodic* if and only if $x \in F^k(x)$. The sequence of points (x_k) is called the *orbit* of F if and only if $x_{k+1} \in F(x_k)$. This motivates the following

(7.4.16) Definition. Let $X \xleftarrow{p} \Gamma \xrightarrow{q} X$ be a multivalued map. A sequence of points $z_1, z_2, \ldots \in \Gamma$ is called the *orbit of coincidences* if and only if $q(z_i) = p(z_{i+1})$.

$$\begin{tikzcd}
z_1 \arrow[dr, "q"] & & z_2 \arrow[dl, "p"'] \arrow[dr, "q"] & & z_3 \arrow[dl, "p"'] \arrow[dr, "q"] & \\
& \bullet & & \bullet & & \bullet
\end{tikzcd}$$

$$q(z_1) = p(z_2) \qquad q(z_2) = p(z_3) \qquad q(z_3) = p(z_4)$$

The orbit $(z_1, \ldots, z_k)$ is called *k-periodic* if and only if $pz_1 = qz_k$.

(7.4.17) Remark. The connection between the above two definitions is given by the equivalence:

(7.4.17.1) $x_1, \ldots, x_k$ is a periodic orbit of points of $F = qp^{-1}$ if and only if there is a k-periodic orbit of coincidences $z_1, \ldots, z_k \in \Gamma$ of (p, q) such that $pz_i = x_i$.

We denote the space of orbits

$$\Gamma_k = \{(z_1, \ldots, z_k) : z_i \in \Gamma,\ q(z_i) = p(z_{i+1})\}$$

and consider the multivalued map $X \xleftarrow{p_k} \Gamma_k \xrightarrow{q_k} X$ where $p_k(z_1, \ldots, z_k) = pz_1$, $q_k(z_1, \ldots, z_k) = qz_k$. Then a sequence of coincidences $z_1, \ldots, z_k \in \Gamma_k$ is k-periodic if and only if $(z_1, \ldots, z_k) \in C(p_k, q_k)$.

(7.4.18) Remark. It is easy to check that

(7.4.18.1) If $(p, q) = (\mathrm{id}, f)$ is a single-valued map, then $\Gamma = X$ and $(z_1, z_2, \ldots)$ is an orbit of (id, f) iff it is the orbit of the map f, i.e. $z_{i+1} = fz_i$. Thus the above definition can be regarded as the generalization of the notion of the periodic point into the multivalued case.

(7.4.18.2) In the single-valued case the first element determines the orbit $(x_1, x_2 = f(x_1), \ldots)$ but in the multivalued case this is not true.

(7.4.18.3) Now let (p, q) be an arbitrary multivalued map and let $F(x) = q(p^{-1}(x))$. Then $p_k(C(p_k, q_k)) = \mathrm{Fix}(F^k)$.

In the single-valued case k-orbits of periodic points were regarded as cyclic orbits without a fixed (first) element. However in our definition of the orbit of (p, q) the first point z_1 is distinguished (see Remark (7.4.18.2).

(7.4.19) Definition. Two k-orbits of coincidences $(z_1, \dots, z_k)$, $(z'_1, \dots, z'_k)$ are called *cyclically equal* if and only if $(z'_1, \dots, z'_k) = (z_i, \dots, z_k, z_1, \dots, z_{i-1})$ for an $i = 1, \dots, k$.

As we have noticed a k-periodic orbit of coincidences of (p, q) equals the coincidence of the multivalued map (p_k, q_k). Thus to detect k-periodic orbits of coincidences we may apply the above theory to the pair (p_k, q_k). This will be possible due to the following fact shown in Lemma 5.3 of [AnGrJr1].

(7.4.20) Proposition. *If the pair (p, q) is admissible, then so also is (p_k, q_k).*

Let us notice that for $l|k$ we have the natural map $j_{kl}: C(p_l, q_l) \to C(p_k, q_k)$ given by

$$j_{kl}(z_1, \dots, z_l) = (z_1, \dots, z_l; \dots; z_1, \dots, z_l).$$

We will call a k-orbit $(z_1, \dots, z_k) \in C(p_k, q_k)$ *reducible* if it belongs to the image of a j_{kl} for an $l < k$.

The map j_{kl} induces a map between the sets of Nielsen classes $j_{kl}: \mathcal{N}(p_l, q_l) \to \mathcal{N}(p_k, q_k)$.

On the other hand we notice that by an analogy to the single-valued case there is a natural action of the group $\mathbb{Z}_k$ on $C(p_k, q_k)$ given by $(z_1, \dots, z_k) \to (z_2, \dots, z_k, z_1)$. Let us notice that two cyclically equal sequences belong to the same orbit of this action.

Reidemeister classes. Let us start with an algebraic scheme.

Let $\phi: G \to G$ be a homomorphism of a group. We define the action $\alpha \circ \beta = \alpha \cdot \beta \cdot \phi(\alpha^{-1})$. The cosets are called *Reidemeister classes*. Their set is denoted by $\mathcal{R}(\phi)$ and its cardinality by $R(\phi) := \#\mathcal{R}(\phi)$.

Let $k, l \in \mathbb{N}$ and let $l|k$. We define $i_{kl} : \mathcal{R}(\phi^l) \to \mathcal{R}(\phi^k)$ putting

$$[\alpha] \to [\alpha \cdot \phi^l \alpha \cdot \dots \cdot \phi^{k-l}(\alpha)].$$

Then $i_{nk} i_{kl} = i_{nl}$, $i_{kk} = \mathrm{id}$. On the other hand ϕ induces an action of $\mathbb{Z}_k$ on $\mathcal{R}(\phi^k)$:

$$\mathcal{R}(\phi^k) \ni [\alpha] \to [\phi(\alpha)] \in \mathcal{R}(\phi^k).$$

The orbits of this action are called *orbits of Reidemeister classes.*

A class $\alpha \in \mathcal{R}(\phi^k)$ is called *reducible* if and only if there exist $l \in \mathbb{N}$ and $\beta \in \mathcal{R}(\phi^l)$ such that $l < k$, $l|k$ and $i_{kl}(\beta) = \alpha$. An orbit of Reidemeister classes is called reducible if a (hence any) class in this orbit is reducible.

Now we are in a position to apply the above algebraic formalism to the homomorphism $\phi = \widetilde{q}_!\widetilde{p}^!\colon \mathcal{O}_{XH} \to \mathcal{O}_{XH}$. This yields Reidemeister sets $\mathcal{R}(\phi^k) = \mathcal{R}((\widetilde{q}_!\widetilde{p}^!)^k)$. However to get a connection between these sets and the orbits of coincidences we will need an alternative description of the iterates of the homomorphism $\phi = \widetilde{q}_!\widetilde{p}^!$.

We consider again an admissible map $X \xleftarrow{p} \Gamma \xrightarrow{q} X$ and its lift

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "p_X"'] & \widetilde{\Gamma} \arrow[l, "\widetilde{p}"'] \arrow[r, "\widetilde{q}"] \arrow[d, "p_\Gamma"] & \widetilde{X} \arrow[d, "p_X"] \\
X & \Gamma \arrow[l, "p"] \arrow[r, "q"'] & X
\end{tikzcd}$$

We fix a lift $(\widetilde{p}, \widetilde{q})$ and we consider the diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "p_X"'] & \overline{\Gamma} \arrow[l, "\widetilde{p}_k"'] \arrow[r, "\widetilde{q}_k"] \arrow[d, "p_{\Gamma_k}"] & \widetilde{X} \arrow[d, "p_X"] \\
X & \Gamma_k \arrow[l, "p_k"] \arrow[r, "q_k"'] & X
\end{tikzcd}$$

where $\overline{\Gamma}_k = \{(\overline{z}_1, \dots, \overline{z}_k) : \overline{z}_i \in \widetilde{\Gamma},\ \widetilde{q}\overline{z}_i = \widetilde{p}\overline{z}_{i+1}\}$ and $\overline{p}_k(\overline{z}_1, \dots, \overline{z}_k) = \widetilde{p}\overline{z}_1$, $\widetilde{q}_k(\overline{z}_1, \dots, \overline{z}_k) = \widetilde{q}\overline{z}_k$.

(7.4.21) Remark. One can easily check that if $(p, q) = (\mathrm{id}, f)$ is a single-valued map, then we have a natural homeomorphism

$$\overline{\Gamma}_k = \{(\overline{z}_1, \dots, \overline{z}_k) : \widetilde{f}(\overline{z}_i) = \overline{z}_{i+1},\ i = 1, \dots, k-1\} \ni (z_1, \dots, z_k) \to z_1 \in \widetilde{\Gamma}.$$

Moreover, $(p_k, q_k) = (\mathrm{id}, f^k)$, $(\overline{p}_k, \overline{q}_k) = (\mathrm{id}, \widetilde{f}^k)$.

We fix the lifts $(\overline{p}_k, \overline{q}_k)$. In fact they are determined by the choice of $(\widetilde{p}, \widetilde{q})$. We get the homomorphisms of the groups of natural transformations: $\overline{p}_k^!\colon \mathcal{O}_X \to \mathcal{O}_{\Gamma_k}$, $\overline{q}_{k!}\colon \mathcal{O}_{\Gamma_k} \to \mathcal{O}_X$ by requiring the commutativity of the diagrams:

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "\alpha"'] & \overline{\Gamma}_k \arrow[l, "\overline{p}_k"'] \arrow[d, "\overline{p}_k^!(\alpha)"] \\
\widetilde{X} & \overline{\Gamma}_k \arrow[l, "\overline{p}_k"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\Gamma}_k \arrow[r, "\overline{q}"] \arrow[d, "\beta"'] & \widetilde{X} \arrow[d, "\overline{q}_{k!}(\beta)"] \\
\overline{\Gamma}_k \arrow[r, "\overline{q}"'] & \widetilde{X}
\end{tikzcd}$$

Now the crucial fact is the equality of self-homomorphisms of $\mathcal{O}_X$ given in [AnGrJr4].

(7.4.22) Corollary. *We have* $(\overline{q}_k)_!(\overline{p}_k)^!(\alpha) = [\widetilde{q}_!\widetilde{p}^!]^k(\alpha)$.

Using the above equality we may consider the sets $\mathcal{R}((\overline{q}_k)_!(\overline{p}_k)^!)$, the maps $i_{kl}\colon \mathcal{R}((\overline{q}_l)_!(\overline{p}_l)^!) \to \mathcal{R}((\overline{q}_k)_!(\overline{p}_k)^!)$ given by

$$i_{kl}(\alpha) = \alpha \cdot (\overline{q}_{l!}\overline{p}_l^!)(\alpha) \cdots (\overline{q}_{(k-l)!}\overline{p}_{(k-l)}^!)(\alpha),$$

and the action of $\mathbb{Z}_k$ on $\mathcal{R}((\overline{q}_k)_!(\overline{p}_k)^!)$ given by

$$\nu[\alpha] = [\widetilde{q}_!\widetilde{p}^!(\alpha)].$$

Using this description one can show (details in [AnGrJr4]) that there is the natural inclusion

$$\mathcal{ON}_H(p_k, q_k) \to \mathcal{OR}_H(p_k, q_k)$$

where reducible orbits of Nielsen classes are sent into reducible orbits of Reidemeister classes. Following the single-valued case we get

(7.4.23) Theorem. *Any multivalued map admissibly homotopic to* (p, q) *has at least* $\#\mathcal{IEOR}_H(p_k, q_k)$ *cyclically different periodic orbits of coincidences of length* k.

As a corollary we get that no multivalued deformation by an admissible homotopy, of a single-valued self-map of a torus $f\colon X \to X$, can remove a homotopy period

(7.4.24) Corollary. *Since each admissible multivalued self-map* (p, q) *of a torus is admissibly homotopic to a single-valued map* ρ *(see Theorem* (7.4.14)*),* (p, q) *is admissibly homotopic to a pair with no periodic orbits of coincidences of length* k *if and only if* $k \notin \mathrm{HPer}(\rho)$.

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AUTHORS

SYMBOLS

INDEX