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An obstruction theory for fibre-preserving maps

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TABLE OF CONTENTS

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I. INTRODUCTION AND PRELIMINARY NOTIONS

A. Introduction

Let $p: E \to B$ and $p': E' \to B'$ be fibrations. A continuous map $f: E \to E'$ is called a fibre-preserving map if for each $b \in B$ there is a $b' \in B'$ such that $f(p^{-1}(b)) \subset p'^{-1}(b')$. Suppose that there is given a fibre-preserving map $f: p^{-1}(A) \to E'$ where A is a subset of B. Then the main problems which will be investigated in this thesis can be stated as follows.

(1) Is there a fibre-preserving extension $g: E \to E'$ of f?

(2) If $g_o, g_1: E \to E'$ are two such extensions, does there exist a homotopy $\{g_t\}: E \to E'$ $(0 \leq t \leq 1)$ between them such that, for each t, g_t is a fibre-preserving extension of f?

The basic assumption we impose is that (B, A) is a connected relative CW-complex.

When p and p' are 'fibre bundles', similar problems have been studied by Dugundji, Heller, Conner and Floyd (see [11],[19],[7]). They require f and its extensions to be 'equivariant' with respect to the actions of the group of the fibre bundle. Heller [19] has shown that the equivariant extension problem can be reduced to the cross section problem in an appropriate fibre bundle. Such a reduction, however, is not likely to be applicable to problems (1) and (2).

If $E = B$, $E' = B'$ and p, p' are the identity maps, then (1) and (2) become the classical problems on 'free' extensions $B \to B'$ of the map $f: A \to B'$ to which the obstruction theory due to Eilenberg is applicable. Our main aim is to establish a generalization of Eilenberg's theory.

1. Outline of the method

We consider, briefly, problem (1). For simplicity of the description, let us assume (B, A) to be a pair of simplicial complexes. Let $\overline{B}^n$ denote the relative n-skeleton of (B, A) and let $\overline{E}_n = p^{-1}(\overline{B}^n)$. Assuming that f has a fibre-preserving extension $g: \overline{E}_n \to E'$, we will look for a fibre-preserving extension to $g': \overline{E}_{n+1} \to E'$; if g has no such extension, we will reconsider g in the hope of finding another extension of $g|\,\overline{E}_{n-1}$ which admits a fibre-preserving extension over $\overline{E}_{n+1}$.

For each $(n+1)$-simplex σ of $B - A$, the partial fibration $p^{-1}(\sigma) \to \sigma$ is equivalent to the projection $\sigma \times F \to \sigma$ where $F = p^{-1}(b_o)$ $(b_o \in B)$, and so we obtain the following diagram:

$$\begin{tikzcd} \sigma \times F \arrow[r, "\phi", "\simeq"'] & p^{-1}(\sigma) \arrow[dr, dashed] & \\ \dot{\sigma} \times F \arrow[u, phantom, "\cup"] \arrow[r, "\phi", "\simeq"'] & p^{-1}(\dot{\sigma}) \arrow[u, phantom, "\cup"] \arrow[r, "g_{\dot{\sigma}}"] & E' \end{tikzcd}$$

where ϕ denotes a fibre homotopy equivalence, $\dot{\sigma}$ is the boundary of σ, and $g_{\dot{\sigma}} = g|\,p^{-1}(\dot{\sigma})$. It will be shown that

$g_{\dot\sigma}$ has a fibre-preserving extension $g_\sigma \colon p^{-1}(\sigma) \to E'$ if and only if the map $g_{\dot\sigma} \phi | \dot\sigma \times F$ has a fibre-preserving extension to $\sigma \times F$ (Lemma 2.10). On the other hand, the obstruction to the latter property can be expressed by the homotopy class, say $c^{n+1}(g)(\sigma)$, of the adjoint map $\dot\sigma \to E'^{(F)}$ where $E'^{(F)}$ is an appropriately defined subspace of the mapping space E'^{F} (Corollary 1.11). Roughly speaking, the assignment $c^{n+1}(g) \colon \sigma \longmapsto c^{n+1}(g)(\sigma) \in \pi_n(E'^{(F)})$ becomes an (n+1)-cochain of (B, A) with coefficients in $\pi_n(E'^{(F)})$ and so we obtain a cohomology class

$$[c^{n+1}(g)] \in H^{n+1}(B,A;\ \pi_n(E'^{(F)})).$$

This class measures the obstruction for the fibre-preserving extensibility of $g | \bar{E}_{n-1}$ over $\bar{E}_{n+1}$ (Theorem 3.9).

In the arguments outlined above, however, we will soon encounter several difficulties. The first difficulty arises from the fact that the continuity of a map $\sigma \longrightarrow E'^{(F)}$ does not necessarily imply the continuity of its adjoint $\sigma \times F \to E'$. Second, in order to construct a global continuous map $g' \colon \bar{E}_{n+1} \to E'$ from maps $g_\sigma \colon p^{-1}(\sigma) \longrightarrow E'$ we need the 'weak topology' of $\bar{E}_{n+1}$. To overcome these difficulties we shall restrict the category in which we work to Steenrod's category $\underline{\underline{CG}}$ of compactly generated spaces (see Section B). Then, without any other assumption, we can formulate the (n+1)-extensibility of a fibre-preserving map $g \colon \bar{E}_n \to E'$ by the triviality of a certain homomorphism of homotopy groups (Theorem 2.14). This formulation will prove to be useful in applications. However, when we translate it into the cohomological language, other difficulties appear and force us to assume:

(1) the fibration $p: E \to B$ is orientable, and

(2) $E'^{(F)}$ is n-simple (for details, see Chapter III).

2. Applications

Let $p: E \to B$ be a fibration with fibre F and inclusion map $i: F \subset E$. Assume B and F to be path-connected.

(1) Hilton [21] and Ganea [14] gave a sufficient condition for p to be fibre homotopy equivalent to an 'induced principal fibration'. Using the generalized Eilenberg Theorem, we will reprove this result (Theorem 4.5). The method used there reproves also a result concerning extensions of a group (Theorem 4.7).

(2) Serre's result on the relation between $H^m(E,F; G)$ and $H^m(B, G)$ can also be proved in a generalized form (Theorem 4.10).

(3) If B is the suspension of a connected CW-complex, then for any H-space X (homotopy-associative and homotopy-invertible), there is a group-isomorphism $[E \cup CF, X]_o \cong [B, X^F]_o$ where $E \cup CF$ is the mapping cone of i and $[U, V]_o$ denotes the set

of homotopy classes of based maps $U \to V$ (Theorem 4.12).

The generalized obstruction theory gives us special information about the extension problem:

$$\begin{tikzcd} F \arrow[dr, "u"] \arrow[d, "i"'] & \\ E \arrow[r, dashed] & Y \end{tikzcd}$$

By using it, the following results will be obtained.

(4) If p is orientable and if F is a compact connected polyhedron with nonzero Euler number, then

$$\pi_1(E) \cong \pi_1(B) \times \pi_1(F) \quad \text{(direct product)},$$
$$H_1(E; G) \cong H_1(B; G) \oplus H_1(F; G), \text{ and}$$
$$H^1(E; G) \cong H^1(B; G) \oplus H^1(F; G)$$

where G is an abelian group (Theorem 3.12). This fact will reprove Gottlieb's result [18]: "If F is the even dimensional real projective space, then $H^*(E; \mathbb{Z}_2) \cong H^*(B; \mathbb{Z}_2) \otimes H^*(F; \mathbb{Z}_2)$ as $\mathbb{Z}_2$-vector spaces" (see Corollary 3.14).

(5) Suppose that B is n-connected. If F is a compact polyhedron and n is even, then for any commutative ring G with unit,

$$i^*: H^n(E; G) \cong H^n(F; G).$$

The isomorphism holds also for odd n if the Euler number of F is nonzero and G is a field (Theorem 4.14).

3. Notations

All spaces considered in this thesis (except in Section B below) are restricted to the category CG of compactly generated spaces. In 1967, Steenrod [34] called attention to the advantages of this category. A brief review of [34] will be given in Section B. There we will see also that, in many cases, a problem related to more general spaces can be reduced to a problem in the category CG. Attention should be paid to product spaces and spaces of continuous mappings (see Definition 1.2).

If $A, B, \cdots, X, Y, \cdots$ are pointed spaces, their base points will be denoted by $a_o, b_o, \cdots, x_o, y_o, \cdots$, respectively, or simply by $*$. The unit interval $\{t \mid 0 \leq t \leq 1\}$ will be denoted by I. The words 'a map' will be used instead of 'a continuous mapping'. Either a map $H: X \times I \to Y$ or the collection $\{h_t\}$ of the maps $h_t = H(\ , t): X \to Y$, $t \in I$, is called a homotopy from h_o to h_1. Maps and homotopies are not required to preserve base points unless specifically mentioned otherwise. The symbol 'Id' will be used to denote the identity map of a space. For spaces X, Y, the symbol $L(X, Y)$ will denote the set of (continuous) maps from X to Y. In Section B below (only there), $L(X, Y)$

is considered as a space with the compact-open topology. For convenient reference, we list the basic notations below.

Y^X	the space of maps $X \to Y$ (for topology, see p. 9),
${}^{X}Y$	the space of based maps $X \to Y$,
$f_{\#}: X^Z \to Y^Z$, $f^{\#}: Z^Y \to Z^X$	the maps induced by $f: X \to Y$,
$L^*(E, E')$	the set of fibre-preserving maps $E \to E'$,
$\{h_t\}: f \overset{*}{\simeq} g$	fibre-preserving homotopy,
$\bar{e}^n$	the closure of an n-cell e^n,
H^*EP	(see p. 30),
$\Gamma(g)$	(see p. 41),
$\Delta(h, k)$	(see p. 39),
K_*^*	(see p. 66),
(a) $\Longleftrightarrow$ (b)	statements (a) and (b) are equivalent,
$\mathbf{h}$	the Hurewicz homomorphism,
$\square$	the end of a proof.

B. The Category <u>CG</u> of Compactly Generated Spaces

We consider only Hausdorff spaces as spaces. A space X is called a <u>compactly generated space</u> or a <u>k-space</u> if it has the weak topology with respect to the collection of compact subsets of X, or equivalently if any subset of X which has a closed intersection with every compact subset of X is closed. A mapping defined on a k-space (with the image in any space) is continuous if and only if its restriction to each compact subset is continuous. The category consisting of k-spaces and their maps are denoted by <u>CG</u>. The following spaces are in <u>CG</u>:

(1) locally compact spaces,

(2) spaces satisfying the first axiom of countability (for example, metrizable spaces),

(3) CW-complexes,

(4) closed subsets of a k-space, and

(5) quotient spaces of a k-space.

There are spaces which are not k-spaces. In fact, Dowker [9] has constructed CW-complexes X, Y such that the cartesian product $X \times_c Y$ is not a k-space.

For a space Y, the <u>associated k-space</u> ${}^kY$ is the space obtained by retopologizing Y by the weak topology with respect to compact subsets. Clearly, the identity mapping η_Y: ${}^kY \to Y$ is continuous.

Lemma 1.1. (See [31] or [34].)

(1) If X is in CG, a mapping $f: X \to {}^kY$ is continuous if and only if $\eta_Y f: X \to Y$ is continuous.

(2) The map $\eta_Y: {}^kY \to Y$ is a weak homotopy equivalance, i.e., it induces isomorphisms of homotopy groups and singular (co)homology groups.

For a map $f: X \to Y$, let ${}^kf: {}^kX \to {}^kY$ denote the same function as f. By (1) of the above lemma, ${}^kf$ is continuous. So, $(\ ) \to {}^k(\ )$ is a functor from the category of all (Hausdorff) spaces and maps to the category CG.

As mentioned before, for spaces X, Y in CG, the cartesian product $X \times_c Y$ and, hence, the space $L(X, Y)$ (with the compact-open topology) are not in CG in general. So we shall use the following modified definition due to Steenrod [34].

Definition 1.2.

$$X \times Y = {}^k(X \times_c Y),$$
$$Y^X = {}^k(L(X, Y)).$$

The advantages of the category CG with the above definition can be found in the next theorem ([34], Th. 5.6, Lemma 5.2, Th. 5.9).

Theorem 1.3. Let X, Y, Z be in CG.

(1) $u: X \times Y \to Z$ is continuous if and only if

$\hat{u}: X \to Z^Y$ defined by $\hat{u}(x)(y) = u(x, y)$ is continuous. Each of $\hat{u}$ and u is called the adjoint map of the other.

(2) $Z^{X \times Y} = (Z^Y)^X$ (i.e., the mapping $u \longmapsto \hat{u}$ is a homeomorphism).

(3) The 'evaluation' $\tilde{w}: Y^X \times X \to Y$ defined by $\tilde{w}(u, x) = u(x)$ is continuous.

(4) The 'composition' $C: Z^Y \times Y^X \to Z^X$ defined by $C(u, v) = u \cdot v$ is continuous.

As noted in [34], the above modification of the concept of product space has an effect on other concepts based on products such as topological group G $(G \times G \to G)$, transformation group G of X $(G \times X \to X)$, etc. However, this fact does not cause a serious problem; as a matter of fact, many concepts are unaltered because of the following well-known lemma; in particular, the concept of homotopy $(X \times I \to Y)$ is not changed.

Lemma 1.4. If X is a k-space and K is a locally compact space, then $X \times_c K$ is a k-space and, hence, $X \times_c K = X \times K$.

Now we prove some elementary properties of the functor k.

Lemma 1.5. If $f \simeq g: X \to Y$, then ${}^k f \simeq {}^k g: {}^k X \to {}^k Y$.

Proof. Let $H: X \times_c I \to Y$ be a homotopy from f to g. Then ${}^kH: {}^k(X \times_c I) \to {}^kY$ is continuous. But,

$$\begin{aligned} {}^k(X \times_c I) &= ({}^kX) \times ({}^kI) \quad \text{(see [34], Lemma 4.5)} \\ &= ({}^kX) \times I \\ &= ({}^kX) \times_c I. \end{aligned}$$

Then, ${}^kH: ({}^kX) \times_c I \to {}^kY$ is a homotopy from ${}^kf$ to ${}^kg$. □

Proposition 1.6. If X has the homotopy type of a k-space, then the identity map $\eta_X: {}^kX \to X$ is a homotopy equivalence.

Proof. Take a k-space K and a homotopy equivalence $\varepsilon: K \to X$. By the above lemma and the functorial property of k, ${}^k\varepsilon: {}^kK \to {}^kX$ is a homotopy equivalence. Then the assertion follows easily from the commutative diagram:

$$\begin{array}{ccc} {}^kK & = & K \\ {\scriptstyle {}^k\varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ {}^kX & \xrightarrow{\eta_X} & X . \end{array} \quad \square$$

Proposition 1.7. If $p: E \to B$ is a (Hurewicz) fibration, then ${}^kp: {}^kE \to {}^kB$ is a fibration in the category CG, that means ${}^kp$ satisfies the covering homotopy property for all k-spaces.

Proof. Let X be in CG. Suppose that the diagram

$$\begin{array}{ccccc} X \times 0 & \xrightarrow{f} & k_E & \xrightarrow{\eta_E} & E \\ \cap & & \Big\downarrow {\scriptstyle k_p} & & \Big\downarrow {\scriptstyle p} \\ X \times I & \xrightarrow{G} & k_B & \xrightarrow{\eta_B} & B \end{array}$$

commutes, where f and G are given maps. Since p is a fibration, the map $\eta_E \cdot f$ extends to a homotopy $H'\colon X \times I \to E$ which covers $\eta_B \cdot G$. By Lemma 1.1 (1), $H = \eta_E^{-1} \cdot H'$ is continuous. Then H is a covering homotopy of G with the initial map f. This proves the lemma.□

Let $p\colon E \to B$ be a fibration with fibre $F = p^{-1}(b_o)$, and consider the commutative diagram:

$$\begin{array}{ccccc} k_F & \xrightarrow{k_i} & k_E & \xrightarrow{k_p} & k_B \\ {\scriptstyle \eta_F}\Big\downarrow & & {\scriptstyle \eta_E}\Big\downarrow & & {\scriptstyle \eta_B}\Big\downarrow \\ F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array}$$

where i is the inclusion map. For example, if B, E, F have the homotopy types of CW-complexes, then the three η's are homotopy equivalences (Proposition 1.6). So the homotopical properties of the fibration $p\colon E \to B$ are translated via η to those of $^k p\colon {}^k E \to {}^k B$ and vice versa. In many cases, this fact enables us to assume a fibration to be in the category <u>CG</u>, without losing generality.

We close this section by introducing a pair of notations. Let X, Y, Z be in <u>CG</u>. For a map $f\colon X \to Y$, we denote by

$$f_{\#}: X^Z \to Y^Z \quad \text{and} \quad f^{\#}: Z^Y \to Z^X$$

the maps defined by $f_{\#}(u) = f \cdot u$ and $f^{\#}(v) = v \cdot f$, respectively. The continuity of $f_{\#}$ and $f^{\#}$ and the following lemma can be verified easily by using Theorem 1.3 (4) (for a direct proof, see [6], p. 248).

Lemma 1.8. If $f \simeq g: X \to Y$, then $f_{\#} \simeq g_{\#}: X^Z \to Y^Z$ and $f^{\#} \simeq g^{\#}: Z^Y \to Z^X$. If $f: X \to Y$ is a homotopy equivalence, then $f_{\#}$ and $f^{\#}$ are also homotopy equivalences. □

C. The Set $L^*(E, E')$ and the Space $E^{(X)}$

From now on throughout, we work in the category CG. Let $p: E \to B$ and $p': E' \to B'$ be fixed fibrations (namely, p and p' satisfy the covering homotopy property for all spaces in CG). A map $f: E \to E'$ is called a fibre-preserving map if $p(x_1) = p(x_2)$ $(x_1, x_2 \in E)$ implies $p'f(x_1) = p'f(x_2)$. We denote by

$$L^*(E, E')$$

the subset of $L(E, E')$ consisting of all fibre-preserving maps from E to E'. Clearly, a map $f: E \to E'$ is in $L^*(E, E')$ if and only if there is a function $\bar{f}: B \to B'$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "\bar{f}"] & B' \end{tikzcd}$$

commutes. In general, $\bar{f}$ is not continuous. However, for example, if B is a CW-complex, then $\bar{f}$ is continuous (see Proposition 2.1). Let $f, g \in L^*(E, E')$. We say that f, g are <u>fibre-homotopic</u>, and we write

$$f \overset{*}{\simeq} g \ ,$$

if there is a homotopy $\{h_t\}$ from f to g such that $h_t \in L^*(E, E')$ for all $t \in I$. Then $\{h_t\}$ is called a <u>fibre-preserving homotopy</u> (from f to g). Furthermore, if $\{h_t\}$ satisfies $\bar{h}_t = \bar{h}_0 : B \to B'$ for all $t \in I$, then $\{h_t\}$ is called a <u>vertical homotopy</u>. For a map $H: I \times E \to E'$, the statement that $\{H(t, \ )\}$ is a fibre-preserving homotopy (resp. a vertical homotopy) is equivalent to

$$H \in L^*(I \times E, E')$$

where L^* is considered with respect to the fibration $\mathrm{Id} \times p: I \times E \to I \times B$ (resp. the fibration $0 \times p: I \times E \to \{0\} \times B$).

Suppose $p: E \to B$ and $p': E' \to B$ are fibrations with the same base space B. A fibre-preserving map $f: E \to E'$ is called a <u>fibre homotopy equivalence</u> if $\bar{f} = \mathrm{Id}: B \to B$ and if there is an $f' \in L^*(E', E)$ such that both of $f' \cdot f$ and $f \cdot f'$ are vertically homotopic to the identity maps. Note that $\bar{f}' = \mathrm{Id}$ holds. A useful criterion for f to be a fibre homotopy equivalence is given by the following theorem due to Dold ([8], p. 243).

<u>Theorem</u> 1.9. Let $p: E \to B$ and $p': E' \to B$ be fibrations. Assume that the common base space B is a connected CW-complex. Then a fibre-preserving map $f: E \to E'$ is a fibre homotopy equivalence if (and only if)

(1) $\bar{f} = \mathrm{Id}: B \to B$, and

(2) the restriction of f to some fibre $p^{-1}(b)$ is a homotopy equivalence from $p^{-1}(b)$ to some fibre of E'.

In Dold's proof of the theorem, the covering homotopy property of p and p' is repeatedly used. However, the essentially needed fact is that both of p and p' satisfy the covering homotopy property for the spaces E and E'. In our case, E, E' are in <u>CG</u> and so the theorem holds for fibrations in <u>CG</u>. $\square$

Now for a fibration $p': E' \to B'$ and a space X, we define a subspace $E'^{(X)}$ of E'^X by

$$E'^{(X)} = \{g \in E'^X \mid p'g(x_1) = p'g(x_2) \quad \text{for all} \quad x_1, x_2 \in X\}.$$

That $E'^{(X)}$ is in <u>CG</u> can be verified easily by showing that $E'^{(X)}$ is a closed subset of the k-space E'^X $(= {}^kL(X, E'))$. Also note that if $B' = b'_o$, then $E'^{(X)} = E'^X$.

Let $p: B \times X \to B$ be the projection. For a map $f: B \times X \to E'$, let $\hat{f}: B \to E'^X$ denote the adjoint map (i.e., $\hat{f}(b)(x) = f(b, x)$). Then Theorem 1.3 (1) tells us that the maps in $L(B \times X, E')$ are in one-to-one correspondence with the maps in $L(B, E'^X)$ by $f \longmapsto \hat{f}$. Observe now that the following statements are equivalent:

(1) f is fibre-preserving (with respect to p and p'),

(2) $p'f(b, x_1) = p'f(b, x_2)$ for all $b \in B$ and $x_1, x_2 \in X$,

(3) $p'(\hat{f}(b)(x_1)) = p'(\hat{f}(b)(x_2))$ for all $b \in B$ and $x_1, x_2 \in X$, and

(4) $\hat{f}(b) \in E'^{(X)}$ for all $b \in B$.

Then the equivalence of (1) and (4) gives the following.

<u>Proposition</u> 1.10. Regard $B \times X$ as a fibre space by the projection $B \times X \to B$. Then the following equivalences (denoted $\Longleftrightarrow$) hold.

(1) $f \in L^*(B \times X, E') \Longleftrightarrow \hat{f} \in L(B, E'^{(X)})$.

(2) $\{h_t\}: B \times X \to E'$ is a fibre-preserving homotopy if and only if $\{\hat{h}_t\}: B \to E'^{(X)}$ is a homotopy; hence, for $f, g \in L^*(B \times X, E')$,

$$f \overset{*}{\simeq} g\colon B \times X \to E' \Longleftrightarrow \hat{f} \simeq \hat{g}\colon B \to E'^{(X)}. \quad \square$$

The significance of Proposition 1.10 is that, in the case of a product fibre space, it enables us to reduce the problems on fibre-preserving maps to the problems on usual maps. In order to state a consequence of Proposition 1.10 in a convenient form, we introduce a notation. Let $g,h\colon I^n \to Y$ be maps such that $g|\,\partial I^n = h|\,\partial I^n$, where $I^n = I \times \cdots \times I$ is the n-cube and ∂I^n is the boundary of I^n. Using the equality $\partial I^{n+1} = \partial(I^n \times I) = \partial I^n \times I \cup I^n \times \partial I$, define a map $d(g,h)\colon \partial I^{n+1} \to Y$ by

$$d(g,h)(u,t) = \begin{cases} g(u) = h(u) & \text{if } (u,t) \in \partial I^n \times I \\ g(u) & \text{if } (u,t) \in I^n \times 0 \\ h(u) & \text{if } (u,t) \in I^n \times 1 . \end{cases}$$

This map is called the separation map of g, h, and has the property that

$$d(g,\ h) \simeq 0 \Longleftrightarrow g \simeq h \text{ rel } \partial I^n,$$

where 0 denotes a constant map. Then the next corollary follows easily from Proposition 1.10.

<u>Corollary</u> 1.11. Regard $I^n \times F$ as a fibre space by the projection $I^n \times F \to I^n$. Then the following 'equivalences' hold.

(1) $f \in L^*(\partial I^n \times F, E')$ has an extension

$$f' \in L^*(I^n \times F,\ E') \quad \text{if and only if}$$

$$\hat{f} \simeq 0\colon \partial I^n \to E'^{(F)}.$$

(2) Suppose that $f \in L^*(\partial I^n \times F,\ E')$ has two extensions $f', f'' \in L^*(I^n \times F,\ E')$. Then

$$f' \overset{*}{\simeq} f'' \text{ rel } \partial I^n \times F \Longleftrightarrow d(\hat{f}', \hat{f}'') \simeq 0\colon \partial I^{n+1} \to E'^{(F)}. \square$$

The expression "a fibration $F \xrightarrow{i} E \xrightarrow{p} B$" will be used to mean "a fibration $p\colon E \to B$ with fibre $F = p^{-1}(b_o)$ and inclusion map $i\colon F \to E$".

<u>Proposition</u> 1.12. For a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ and a space X, $F^X \xrightarrow{i_\#} E^{(X)} \xrightarrow{\bar{p}} B$ is a fibration, where $\bar{p}$ is the map defined by $\bar{p}(u) = pu(X)$ $(= pu(x_o))$.

<u>Proof</u>. Let K be an arbitrary space (in <u>CG</u>), and suppose that there are given a map $\hat{f}_o$ and a homotopy $\{g_t\}$, as indicated in the next diagram, such that $\bar{p} \cdot \hat{f}_o = g_o$:

$$\begin{tikzcd} & E^{(X)} \arrow[d, "\bar{p}"] \\ K \arrow[ur, "\hat{f}_o"] \arrow[r, "\{g_t\}"] & B \end{tikzcd} \qquad \begin{tikzcd} K \times X \arrow[r, "f_o"] \arrow[d, "\pi"'] & E \arrow[d, "p"] \\ K \arrow[r, "\{g_t\}"] & B \end{tikzcd}$$

Define the second diagram by letting π be the projection and f_o the adjoint map of $\hat{f}_o$. Theorem 1.3 (1) guarantees the continuity of f_o. That $pf_o = g_o\pi$ holds is easily verified. Then, since p is a fibration, the map f_o

extends to a homotopy $\{f_t\}$ such that $pf_t = g_t\pi$. But this relation indicates that $\{f_t\}$ is a fibre-preserving homotopy with respect to π and p. Hence, its adjoint $\{\hat{f}_t\}$ is a homotopy with the image in $E^{(X)}$ (see Proposition 1.10). A simple verification shows that $\{\hat{f}_t\}: K \to E^{(X)}$ covers $\{g_t\}$. Therefore, $\bar{p}: E^{(X)} \to B$ is a fibration. Clearly, $\bar{p}^{-1}(b_o) = F^X$. □

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration where B is path-connected. We recall the action of $\pi_1(B, b_o)$ on the fibre F. Suppose that $\alpha \in \pi_1(B, b_o)$ is represented by a closed path $\lambda: I \to B$, $\lambda(0) = \lambda(1) = b_o$. By the covering homotopy property of p, we find a homotopy $H: I \times F \to E$ such that $H(0, x) = x$ $(x \in F)$ and such that the following diagram (left) commutes (and, hence, the adjoint diagram (right) does).

$$\begin{tikzcd} I \times F \arrow[r, "H"] \arrow[d, "p_r"'] & E \arrow[d, "p"] \\ I \arrow[r, "\lambda"] & B \end{tikzcd} \qquad \begin{tikzcd} & E^{(F)} \arrow[d, "\bar{p}"] \\ I \arrow[ur, "\hat{H}"] \arrow[r, "\lambda"] & B \end{tikzcd}$$

Define $h_t: F \to p^{-1}(\lambda(t))$ by $h_t(x) = H(t, x)$. Then the following facts are well known (see [30], p. 101).

<u>Lemma</u> 1.13.

(1) For each t, $h_t: F \to p^{-1}(\lambda(t))$ is a homotopy equivalence.

(2) The homotopy class $[h_1]$ of $h_1: F \to p^{-1}(b_o) = F$ is determined only by $\alpha \in \pi_1(B, b_o)$.

(3) The correspondence $\alpha \longmapsto [h_1]$ defines a homomorphism

$$\theta: \pi_1(B, b_o) \to \varepsilon(F)$$

where $\varepsilon(F)$ is the group consisting of all the homotopy classes of homotopy equivalences $F \to F$ with the product given by 'composition'. □

The homomorphism θ is called the action of $\pi_1(B, b_o)$ on F.

Proposition 1.14. Let $F \to E \xrightarrow{p} B$ be a fibration. Let $\theta: \pi_1(B, b_o) \to \varepsilon(F)$ be the action of $\pi_1(B, b_o)$ on F. Then homotopy equivalences $u,v: F \to F$ are in the same path-component of $E^{(F)}$ if and only if

$$[u \cdot v^{-1}] \in \text{Image } \theta$$

where v^{-1} is a homotopy inverse of v and $[u \cdot v^{-1}]$ is the homotopy class of the composition $u \cdot v^{-1}$.

Proof. For maps $a,b: F \to F$, we use, temporarily, the notation '$a \sim b$' to mean that a and b are in the same path component of $E^{(F)}$. We show that the following statements are equivalent:

(1) $[u \cdot v^{-1}] = \theta(\alpha)$ for some $\alpha \in \pi_1(B, b_o)$,

(2) $u \cdot v^{-1} \sim Id$ (where Id is the identity map of F), and

(3) $u \sim v$.

Look at the two diagrams preceding Lemma 1.13. Each of them defines the other (see Proposition 1.10 (1)). Then, by the definition of $\theta(\alpha)$, we can easily see that statements (1) and (2) are equivalent. Now assume (2). The map $v^{\#}: E^{(F)} \to E^{(F)}$ sends $u \cdot v^{-1}$ and Id to $u \cdot v^{-1} \cdot v$ and v, respectively. Hence, $u \cdot v^{-1} \cdot v \sim v$. But, clearly $u \cdot v^{-1} \cdot v \sim u$. Therefore, (3) holds. Similarly, that (3) implies (1) can be seen by using the map $(v^{-1})^{\#}: E^{(F)} \to E^{(F)}$. □

<u>Definition</u> 1.15. A fibration $F \to E \xrightarrow{p} B$ (B path-connected) is called an <u>orientable fibration</u> if the action of $\pi_1(B, b_o)$ on F is trivial.

Consider the homotopy exact sequence associated to the fibration $F^F \xrightarrow{i_{\#}} E^{(F)} \xrightarrow{\bar{p}} B$:

$$\cdots \to \pi_{n+1}(B, b_o) \xrightarrow{\Delta_{n+1}} \pi_n(F^F, Id) \xrightarrow{(i_{\#})*} \pi_n(E^{(F)}, i) \xrightarrow{\bar{p}_*} \pi_n(B, b_o) \to \cdots$$

where Δ_{n+1} is given by the composition of

$$\pi_{n+1}(B, b_o) \underset{\cong}{\xrightarrow{\bar{p}_*^{-1}}} \pi_{n+1}(E^{(F)}, F^F, Id) \xrightarrow{\partial} \pi_n(F^F, Id) .$$

In particular, Δ_1 is defined in the following way (see

[30], p. 377). Let $\alpha \in \pi_1(B, b_o)$ be represented by a closed path $\lambda: I \to B$. Take a covering path $\tilde{\lambda}: I \to E^{(F)}$ of λ such that $\tilde{\lambda}(0) = \mathrm{Id}: F \to F \subset E$. Then, $\Delta_1(\alpha)$ is defined to be the path-component of F^F containing $\tilde{\lambda}(1)$. Now go back to the diagrams preceding Lemma 1.13. We can take $\hat{H}$ as $\tilde{\lambda}$. Hence,

$$\Delta_1(\alpha) = \text{the path component of } F^F \text{ containing } \hat{H}(1),$$

$$\theta(\alpha) = \text{the homotopy class } F \to F \text{ containing } \hat{H}(1).$$

Then, by Theorem 1.3 (1), we find that $\Delta_1(\alpha)$ contains the identity map if and only if $\theta(\alpha)$ does; hence, Δ_1 is trivial if and only if θ is also. Then, the exactness of

$$\pi_1(E^{(F)}, i) \xrightarrow{\bar{p}_*} \pi_1(B, b_o) \xrightarrow{\Delta_1} \pi_o(F^F, \mathrm{Id})$$

gives the following result.

<u>Proposition</u> 1.16. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration such that B is path-connected. Then the fibration is orientable if and only if

$$\bar{p}_*: \pi_1(E^{(F)}, i) \to \pi_1(B, b_o)$$

is an epimorphism. □

Now we shall look at the action of $\pi_1(E^{(F)}, i)$ on $\pi_n(E^{(F)}, i)$, $n \geq 1$. For convenience, we shall say that a based map $f: (X, x_o) \to (Y, y_o)$ <u>induces the trivial</u>

π_1-action if the image of $f_*: \pi_1(X, x_o) \to \pi_1(Y, y_o)$ acts trivially on $\pi_n(Y, y_o)$ for all $n \geq 1$. If (Y, y_o) is an H-space, then $\mathrm{Id}: (Y, y_o) \to (Y, y_o)$ induces the trivial π_1-action, or equivalently (Y, y_o) is n-simple for all $n \geq 1$. The proof of this well-known fact is based on the multiplication $m: Y \times Y \to Y$. The next lemma can be proved by the same method.

Lemma 1.17. Let $f: (X, x_o) \to (Y, y_o)$ be a based map. Suppose that there is a map $m: Y \times X \to Y$ such that

$$m(y, x_o) = y \quad \text{and} \quad m(y_o, x) = f(x).$$

Then f induces the trivial π_1-action. $\square$

Corollary 1.18. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration and let $\varepsilon: X \to F$ be a homotopy equivalence. Then the map $i_\#: (F^X, \varepsilon) \to (E^{(X)}, i \cdot \varepsilon)$ induces the trivial π_1-action. Hence, if B is simply connected, then $E^{(X)}$ is n-simple for all $n \geq 1$.

Proof. Consider the maps $(X^X, \mathrm{Id}) \xrightarrow{\varepsilon_\#} (F^X, \varepsilon) \xrightarrow{i_\#} (E^{(X)}, i \cdot \varepsilon)$. Because $\varepsilon_\#$ induces an isomorphism of the fundamental groups (see Lemma 1.8), it suffices to show that $i_\# \cdot \varepsilon_\#$ induces the trivial π_1-action. Consider the map $C: E^{(X)} \times X^X \to E^{(X)}$ given by $C(u, v) = u \cdot v$ (for the continuity of C, see Theorem 1.3). C satisfies the properties:

$$C(u, Id) = u \quad \text{and} \quad C(i \cdot \varepsilon, v) = i \cdot \varepsilon \cdot v = i_{\#}\varepsilon_{\#}(v).$$

Hence, by Lemma 1.17, $i_{\#} \cdot \varepsilon_{\#}$ induces the trivial π_1-action. For the second assertion, note that $(i_{\#})_*\colon \pi_1(F^X, \varepsilon) \to \pi_1(E^{(X)}, i \cdot \varepsilon)$ is epimorphic if B is simply connected. □

II. OBSTRUCTIONS TO EXTENDING FIBRE-PRESERVING MAPS

Throughout this chapter, $F \xrightarrow{i} E \xrightarrow{p} B$ denotes a fixed fibration such that B is a <u>connected CW-complex</u>. For a subset $K \subset B$, the subset $p^{-1}(K)$ of E will be denoted by E_K. If e^n is an n-dimensional cell of the CW-complex B, we shall call $E_{\bar{e}^n}$ the <u>n-block</u> over $\bar{e}^n$ where $\bar{e}^n$ is the closure of e^n.

In Section A, we study the topology of E through its blocks. It will be seen that the relationship between the space E and the collection $\{E_{\bar{e}}\}$ of its blocks is very analogous to the relationship between a CW-complex and the collection of its closed cells. A notion of a characteristic map for a block will be introduced in Section B and used to characterize the 'local' extensibility of a fibre-preserving map (Proposition 2.9). Section C describes our approach to the main problems on the extensions of fibre-preserving maps. The central result is Theorem 2.14. Some results on the extensibility in the lowest dimension are proved in Section D.

A. The Topology of a Fibre Space over a CW-Complex

<u>Proposition</u> 2.1. The map $p: E \to B$ is an identification map; namely, a subset K of B is closed whenever $p^{-1}(K)$ is closed.

<u>Proof</u>. Let K be a subset of B such that $p^{-1}(K)$ is closed. In order to prove that K is closed, it suffices to show that $K \cap \bar{e}^n$ is closed for every cell e^n of B, $n \geq 1$. Let $\psi\colon I^n \to \bar{e}^n$ be a characteristic map for an arbitrary cell e^n $(n \geq 1)$. We consider the following diagram:

$$\begin{array}{ccc} \xi^n & \xrightarrow{\tilde{\psi}} & E_{\bar{e}^n} \\ \pi \big\downarrow \big\uparrow s & & \big\downarrow p \\ I^n & \xrightarrow{\psi} & \bar{e}^n \end{array}$$

where π is the fibration induced from p by ψ, and $\tilde{\psi}$ is the canonical fibre-preserving map over ψ. Since I^n is contractible, π admits a cross section $s\colon I^n \to \xi^n$; $\pi \cdot s = \mathrm{Id}$. Let $K' = K \cap \bar{e}^n$. From the continuity and surjectivity of ψ, it follows that

$$\psi\, \overline{\psi^{-1}(K')} \subset \overline{\psi\psi^{-1}(K')} = \overline{K'}.$$

However, ψ is a closed map and, hence, $\psi\, \overline{\psi^{-1}(K')}$ is a closed set containing K'. Therefore,

$$\psi\, \overline{\psi^{-1}(K')} = \overline{K'}.$$

On the other hand,

$$\begin{aligned} \psi\, \overline{\psi^{-1}(K')} &= \psi\pi s\, \overline{\psi^{-1}(K')} \\ &= p\tilde{\psi} s\, \overline{\psi^{-1}(K')} \end{aligned}$$

$$\subset p\ \overline{\tilde{\psi}s\psi^{-1}(K')} \quad \text{(continuity of } \tilde{\psi}s\text{)}$$

$$\subset p\ \overline{p^{-1}(K')} \quad \text{(commutativity of the diagram)}$$

$$= p\,p^{-1}(K') \quad (p^{-1}(K') = p^{-1}(K) \cap E_{\bar{e}^n} \text{ is closed})$$

$$= K'.$$

Therefore, $\overline{K'} = K'$. $\square$

From this proposition, it follows that if $f\colon E \to E'$ is a fibre-preserving map (where E' is an arbitrary fibre space), then f induces a continuous mapping of the base spaces. However, when B is not a CW-complex, this assertion and Proposition 2.1 are generally not true even if B is path-connected. A counter-example for the proposition is the fibration of the half-line $[0, \infty)$ onto the 'Polish circle' (see [30], p. 77).

Proposition 2.2. E has the weak topology with respect to the collection $\{E_{\bar{e}}\}$ of blocks; that means, a subset M of E is closed if $M \cap E_{\bar{e}}$ is closed for every $E_{\bar{e}}$.

Proof. Let M be a subset of E such that $M \cap E_{\bar{e}}$ is closed for every $E_{\bar{e}}$. Recall that all spaces, in particular E, are assumed to be (Hausdorff) k-spaces. So it suffices to prove that, for any compact subset S of E, $M \cap S$ is closed. If S is a compact set of E, then

$p(S)$ is a compact set of B and is contained in the union of a finite number of closed cells;

$$p(S) \subset \bar{e}_1^{n_1} \cup \cdots \cup \bar{e}_k^{n_k}.$$

Then,

$$S \subset E_{\bar{e}_1^{n_1}} \cup \cdots \cup E_{\bar{e}_k^{n_k}},$$

and, hence,

$$M \cap S = \bigcup_{i=1}^{k} (M \cap E_{\bar{e}_i^{n_i}}) \cap S.$$

Therefore, $M \cap S$ is closed. $\square$

<u>Remark</u>. In general, if E is not a k-space, Proposition 2.2 is not true. Dowker [9] constructed CW-complexes B and F such that the (cartesian) product complex $B \times_c F$ is not a CW-complex. It is not difficult to show that Proposition 2.2 is false for the trivial fibration $B \times_c F \to B$. However, if B is a locally finite CW-complex, then the proposition is always true.

Let $\{h_t\}$ be a homotopy defined on a space X with the range in some space. A point $x_1 \in X$ is said to be stable under $\{h_t\}$ if $h_t(x_1)$ is constant as a function of $t \in I$. A fibration is called a <u>regular fibration</u> if any homotopy $\{g_t\}$ from an arbitrarily given space X to the base space has a lift $\{f_t\}$ with a prescribed initial map and also with the requirement that those points of X which are stable under $\{g_t\}$ be stable under $\{f_t\}$. We call such a homotopy $\{f_t\}$ a <u>regular covering homotopy</u> over $\{g_t\}$. A fibration is not necessarily a regular fibration (see [37]). However, a fibration with a metrizable base space is always regular ([23], [37]).

<u>Lemma</u> 2.3. Let $p^*: E^* \to B^*$ be a fibration, and let K be a strong deformation retract of B^*. Then there is a fibre-preserving map $R: E^* \to E^*_K$ $(= p^{*-1}(K))$ such that the restriction

$$R|\, E^*_K: E^*_K \to E^*_K$$

is fibre homotopic to the identity map. If p^* is a regular fibration, R can be chosen as a retraction (i.e., $R|\, E^*_K = \mathrm{Id}$).

<u>Proof</u>. By the assumption on K, there is a homotopy $\{g_t\}: B^* \to B^*$ such that $g_o = \mathrm{Id}$, $g_1(B^*) \subset K$ and $g_t(x) = x$ for $x \in K$. Let $\{f_t\}: E^* \to E^*$ be a covering homotopy over the homotopy $\{g_t \cdot p^*\}: E^* \to B^*$ such that f_o = the identity map of E^*. By the properties of $\{g_t\}$ we see:

$$f_1(E^*) \subset E^*_K \quad \text{and} \quad f_t(E^*_K) \subset E^*_K \quad \text{for all} \quad t \in I.$$

Note that $\{f_t|\, E^*_K\}: E^*_K \to E^*_K$ is a fibre-preserving homotopy since $p^* f_t = g_t p^*$. Then, $R = f_1$ is the desired map.

If p^* is a regular fibration, we can assume $\{f_t\}$ to be a regular covering homotopy. Then, since each point $\tilde{x} \in E^*_K$ is stable under $\{g_t \cdot p^*\}$, $\tilde{x}$ must be stable also under $\{f_t\}$; hence, $R(\tilde{x}) = f_1(\tilde{x}) = f_o(\tilde{x}) = \tilde{x}$. $\square$

Using the preceding lemma, we prove the following property of a block $E_{\bar{e}}$ of E. Let E' denote an arbitrary fibre space.

<u>Proposition</u> 2.4. Suppose that a fibre-preserving map $f_o\colon E_{\bar{e}} \to E'$ and a fibre-preserving homotopy $\{g_t\}\colon E_{\dot{e}} \to E'$ (where $\dot{e} = \bar{e} - e$) are given with the condition $g_o = f_o | E_{\dot{e}}$. Then f_o can be extended to a fibre-preserving homotopy $\{f_t\}\colon E_{\bar{e}} \to E'$ so that $g_t = f_t | E_{\dot{e}}$. In other words, the pair $(E_{\bar{e}}, E_{\dot{e}})$ satisfies the <u>fibre-preserving homotopy extension property</u> (abbreviated H^*EP).

<u>Proof</u>. Consider the diagram:

$$\begin{array}{ccccc}
E_{\bar{e}} \times I & \supset & E_{\bar{e}} \times 0 \cup E_{\dot{e}} \times I & \xrightarrow{G} & E' \\
{\scriptstyle p \times Id}\downarrow & & \downarrow{\scriptstyle p \times Id} & & \\
\bar{e} \times I & \supset & \bar{e} \times 0 \cup \dot{e} \times I & &
\end{array}$$

where G is the map defined by the obvious way from f_o and $\{g_t\}$. Note that $\bar{e} \times 0 \cup \dot{e} \times I$ is a strong deformation retract of $\bar{e} \times I$ (see [30], p. 400) and that $p \times Id$ is a regular fibration since $\bar{e} \times I$ is metrizable. Hence, by the previous lemma, there is a fibre-preserving retraction $R\colon E_{\bar{e}} \times I \to E_{\bar{e}} \times 0 \cup E_{\dot{e}} \times I$. Then the extension $G \cdot R$ of G gives us the desired extension $\{f_t\}$ of f_o and $\{g_t\}$.□

The next proposition follows from Proposition 2.4 and Proposition 2.2, by using induction.

Proposition 2.5. If K is a subcomplex of B, the pair (E, E_K) satisfies the H^*EP. □

Corollary 2.6. Let K be a subcomplex of B, and suppose fibre-preserving maps $f,g\colon E_K \to E'$ are fibre homotopic (i.e., $f \overset{*}{\simeq} g$). Then, if f has a fibre-preserving extension $E \to E'$, so does g. □

B. Characteristic Maps for Blocks

Let e^n be an n-cell of B, $n \geq 1$, and let $\psi\colon I^n \to \bar{e}^n \subset B$ be a characteristic map for e^n. By the contractibility of I^n and the covering homotopy property of the fibration $\bar{p}\colon E^{(F)} \to B$, there is a map $\hat{\phi}\colon I^n \to E^{(F)}_{\bar{e}^n} \subset E^{(F)}$ such that

(1) $\bar{p} \cdot \hat{\phi} = \psi$ (= a characteristic map for e^n), and

(2) $\hat{\phi}(I^n)$ is contained in the path-component of $E^{(F)}$ containing the inclusion map $i\colon F \to E$.

Note that if u and i are in the same path component of $E^{(F)}$, then u is a homotopy equivalence from F to some fibre $p^{-1}(b)$ (this fact can be seen easily by Lemma 1.13 (1)).

<u>Definition</u> 2.7. We call a map $\hat{\phi}: I^n \to E^{(F)}$ satisfying the above (1) and (2) a <u>characteristic map for the block</u> $E_{\bar{e}^n}$.

Let us look at the role of a characteristic map for a block in an example. Suppose that B is the n-sphere $S^n = e^o \cup e^n$. Since $\bar{e}^n = B$, E itself is a block. Let $\hat{\phi}$ be a characteristic map for E. From the diagram:

$$\begin{array}{ccc} & & (E^{(F)}, F^F) \\ & \overset{\hat{\phi}}{\nearrow} & \downarrow \bar{p} \\ (I^n, \partial I^n) & \xrightarrow{\psi} & (S^n, e^o) \end{array},$$

it is easy to see that $\hat{\phi}|\ \partial I^n$ represents the element

$$X(p) \in \pi_{n-1}(F^F, \hat{\phi}(x_o)) \qquad (x_o \in \psi^{-1}(e^o) = \partial I^n)$$

which is the image of the generator $[\psi] \in \pi_n(S^n, e^o)$ under the boundary homomorphism Δ_n in the exact sequence:

$$\cdots \to \pi_n(F^F, \hat{\phi}(x_o)) \xrightarrow{(i_\#)_*} \pi_n(E^{(F)}, \hat{\phi}(x_o)) \xrightarrow{\bar{p}_*} \pi_n(B, b_o) \xrightarrow{\Delta_n}$$

$$\pi_{n-1}(F^F, \hat{\phi}(x_o)) \to \cdots$$

If $\hat{\phi}|\ \partial I^n \simeq 0: \partial I^n \to F^F$ (i.e., $\Delta_n = 0$), we find easily a cross section $\hat{s}: B \to E^{(F)}$; $b_o \overset{\hat{s}}{\to} \hat{\phi}(x_o)$. Then the adjoint map $S: S^n \times F \to E$ is a fibre homotopy equivalence (by Theorem 1.9). It is also true that if there is a fibre homotopy equivalence $S^n \times F \to E$, then $\hat{\phi}|\ \partial I^n \simeq 0\colon \partial I^n \to F^F$.

To apply the notion of characteristic maps for blocks to the extension problem of fibre-preserving maps, the following elementary lemma is needed.

Lemma 2.8. Let $\phi\colon I^n \times F \to E_{\bar{e}^n}$ be the adjoint map of a characteristic map $\hat{\phi}\colon I^n \to E^{(F)}_{\bar{e}^n}$. Then ϕ can be factored as

$$(I^n \times F,\ \partial I^n \times F) \xrightarrow{h} (\xi^n,\ \partial\xi^n) \xrightarrow{\tilde{\psi}} (E_{\bar{e}^n}\ ,\ E_{\dot{e}^n})$$

where ξ^n is a fibre space over I^n, $\partial\xi^n$ is the part of ξ^n over ∂I^n, and $h, \tilde{\psi}$ are fibre-preserving maps with the following properties:

(1) h is a fibre homotopy equivalence, and

(2) $\tilde{\psi}$ is a relative homeomorphism; i.e., $\tilde{\psi}$ is onto and maps $\xi^n - \partial\xi^n$ homeomorphically onto $E_{\bar{e}^n} - E_{\dot{e}^n}$.

Proof. Let $\psi = \bar{p} \cdot \hat{\phi}\colon I^n \to \bar{e}^n$. By definition, ψ is a characteristic map for e^n, i.e., a relative homeomorphism from $(I^n, \partial I^n)$ onto $(\bar{e}^n, \dot{e}^n)$. Let $\pi\colon \xi^n \to I^n$ be the fibration induced by ψ from $p\colon E_{\bar{e}^n} \to \bar{e}$, and let $\tilde{\psi}\colon \xi^n \to E_{\bar{e}^n}$ be the canonical fibre-preserving map over ψ. Namely,

$$\xi^n = \{(s,\ z) \in I^n \times E_{\bar{e}^n} \mid \psi(s) = p(z)\},$$

$$\pi((s, z)) = s,$$

$$\tilde{\psi}((s, z)) = z.$$

Define $h: I^n \times F \to \xi^n$ by $h(s, x) = (s, \phi(s, x))$. Then h is a fibre-preserving map such that $\phi = \tilde{\psi} \cdot h$. By condition (2) in the definition of a characteristic map $\hat{\phi}$, the restriction of ϕ on each fibre $s \times F$ is a homotopy equivalence from $s \times F$ to some fibre of $E_{\bar{e}^n}$. Then property (1) follows from Theorem 1.9. To see property (2), first note that $\tilde{\psi}$ is onto and the restriction of $\tilde{\psi}$ is a one-to-one map of $\xi^n - \partial\xi^n$ onto $E_{\bar{e}^n} - E_{\dot{e}^n}$. So it suffices to show that $\tilde{\psi}$ is a closed map. Indeed, this can be proved easily from the following facts. Since I^n is compact, the projection $I^n \times E_{\bar{e}^n} \to E_{\bar{e}^n}$ is a closed map; and ξ^n is a closed subset of $I^n \times E_{\bar{e}^n}$. □

Suppose now that there is given a fibre-preserving map $f: E_{\dot{e}^n} \to E'$ where E' is any fibre space. We wish to know when f has an extension $f' \in L^*(E_{\bar{e}^n}, E')$ and when two extensions f', f'' are fibre homotopic rel $E_{\dot{e}^n}$. Let $\partial\hat{\phi}$ denote the restriction $\hat{\phi}|\ \partial I^n$ of a characteristic map $\hat{\phi}$ for the block $E_{\bar{e}^n}$, and let $f_\#: E_{\dot{e}^n}^{(F)} \to E'^{(F)}$ denote the map induced by f. Then the next proposition gives us the answers.

Proposition 2.9.

(1) f has an extension $f' \in L^*(E_{\bar{e}^n}, E')$ if and only if

$$f_{\#} \cdot \partial\hat{\phi} \simeq 0 \colon \partial I^n \to E'^{(F)}.$$

(2) Two extensions $f', f'' \in L^*(E_{\bar{e}^n}, E')$ of f are fibre homotopic if and only if

$$d(f'_{\#} \cdot \hat{\phi},\ f''_{\#} \cdot \hat{\phi}) \simeq 0 \colon \partial I^{n+1} \to E'^{(F)}$$

where $d(a, b)$ denotes the separation map of $a, b \colon I^n \to E'^{(F)}$.

To prove the proposition, factor the adjoint map ϕ as stated in the preceding lemma, and consider the diagram:

$$\begin{tikzcd}
I^n \times F \arrow[r, "h"] \arrow[rr, bend left, "\phi"] & \xi^n \arrow[r, "\tilde{\psi}"] & E_{\bar{e}^n} \arrow[drr, dashed] & & \\
\cup & \cup & \cup & & \\
\partial I^n \times F \arrow[r, "\partial h"] \arrow[rr, bend right, "\partial\phi"'] & \partial\xi^n \arrow[r, "\partial\tilde{\psi}"] & E_{\dot{e}^n} \arrow[rr, "f"] & & E'
\end{tikzcd}$$

where ∂h, $\partial\tilde{\psi}$, $\partial\phi$ are the restrictions of h, $\tilde{\psi}$, ϕ, respectively. Then the assertion follows from Corollary 1.11 and the next lemma.

Lemma 2.10.

(1) If $f \cdot \partial\phi$ has an extension $g' \in L^*(I^n \times F, E')$,

then f has an extension $f' \in L^*(E_{\bar{e}^n}, E')$ such that $f' \cdot \phi$ and g' are fibre-homotopic.

(2) Suppose that $f', f'' \in L^*(E_{\bar{e}^n}, E')$ are extensions of f. If $\phi \cdot f'$ and $\phi \cdot f''$ are fibre homotopic rel $\partial I^n \times F$, then f' and f'' are fibre homotopic rel $E_{\dot{e}^n}$.

Proof.

(1) Let $h': \xi^n \to I^n \times F$ be a vertical homotopy inverse of h; $h \cdot h' \overset{*}{\simeq} Id$, $h' \cdot h \overset{*}{\simeq} Id$ (vertically homotopic). For a given extension g' of $f \cdot \partial\phi$, clearly, $g' \cdot h' \in L^*(\xi^n, E')$. Observe that the restriction of $g' \cdot h'$ to $\partial\xi^n$ is fibre homotopic (in fact, vertically homotopic) to $f \cdot \partial\tilde{\psi}$;

$$\partial(g' \cdot h') = (\partial g') \cdot (\partial h') = (f \cdot \partial\phi) \cdot (\partial h')$$

$$= f \cdot \partial\tilde{\psi} \cdot \partial h \cdot \partial h' \overset{*}{\simeq} f \cdot \partial\tilde{\psi}.$$

Then, since $(\xi^n, \partial\xi^n)$ has the H^*EP (see Proposition 2.4), $f \cdot \partial\tilde{\psi}$ must have an extension (in $L^*(\xi^n, E')$) which is fibre homotopic to $g' \cdot h'$. Now the assertion follows easily from the fact $\tilde{\psi}: (\xi^n, \partial\xi^n) \to (E_{\bar{e}^n}, E_{\dot{e}^n})$ is a relative homeomorphism (see Lemma 2.8).

(2) It may be enough to mention, without further

comment, that assertion (2) can be reduced to (1), if we consider the following diagram:

$$
\begin{array}{ccccccc}
(I^n \times F) \times I & \xrightarrow{h \times Id} & \xi^n \times I & \xrightarrow{\tilde{\psi} \times Id} & E_{\bar{e}^n} \times I & & \\
\cup & & \cup & & \cup & & \\
(\partial I^n \times F) \times I \cup (I^n \times F) \times \partial I & \longrightarrow & \partial \xi^n \times I \cup \xi^n \times \partial I & \longrightarrow & E_{\dot{e}^n} \times I \cup E_{\bar{e}^n} \times \partial I & \xrightarrow{G} & E'
\end{array}
$$

where G is the map given by $G(z, 0) = f'(z)$, $G(z, 1) = f''(z)$, $G(w, t) = f'(z)$ for $z \in E_{\bar{e}^n}$, $w \in E_{\dot{e}^n}$, $t \in I$. $\square$

C. Extensions of Fibre-preserving Maps

We are now in a position to investigate our main problems about fibre-preserving extensions of a given fibre-preserving map f as indicated in the following picture:

$$
\begin{array}{ccc}
E \supset E_A & \xrightarrow{f} & E' \\
p \big\downarrow & & \big\downarrow p' \\
B & & B'
\end{array}
$$

where p and p' are fibrations and $E_A = p^{-1}(A)$, $A \subset B$. We assume that A is a connected subcomplex $(\neq \phi)$ of B.

For an integer $n \geq 0$, let B^n denote the n-skeleton of B. For $n \geq 1$, we will use the following notations:

$$\bar{B}^n = B^n \cup A,$$

$$\bar{E}_n = p^{-1}(\bar{B}^n).$$

However, for $n = 0$, we will use

$$\bar{B}^{o} = T(A),$$

$$\bar{E}_{o} = p^{-1}(\bar{B}^{o})$$

where $T(A)$ is a fixed 'maximal tree mod A' in B, that is, a connected subcomplex as in the following lemma.

Lemma 2.11. There is a subcomplex $T(A)$ of B such that $B^{o} \cup A \subset T(A) \subset B^{1} \cup A$ and such that A is a strong deformation retract of $T(A)$.

Proof. Let S be a maximal tree in A and extend it to a tree $T \subset B^{1}$ containing all vertices of B and $T \cap A = S$ (cf. [30], p. 139). It is easy to see that S is a strong deformation retract of T. Then A is a strong deformation retract of $T(A) = A \cup T$. □

Proposition 2.12. Any $f \in L^{*}(E_A, E')$ has an extension $f' \in L^{*}(\bar{E}_o, E')$. If $f', f'' \in L^{*}(\bar{E}_o, E')$ are extensions of f, then they are fibre homotopic rel E_A.

Proof. Since A is a strong deformation retract of $\bar{B}^{o}$ $(= T(A))$, there is a fibre-preserving map $R: \bar{E}_o \to E_A$ such that $R| E_A \overset{*}{\simeq} Id: E_A \to E_A$ (Lemma 2.3). Let $g = f \cdot (R| E_A)$. Then, $g \overset{*}{\simeq} f: E_A \to E'$, and g has the extension $f \cdot R \in L^{*}(\bar{E}_o, E')$. By Corollary 2.6, then f must also have an extension $f' \in L^{*}(\bar{E}_o, E')$. The second assertion is quite similar by using the fact that $\bar{B}^{o} \times \partial I \cup A \times I$ ($= T(A) \times \partial I \cup A \times I$) is a strong deformation retract of $\bar{B}^{o} \times I$ (for this fact, see [22], p. 33). □

For a given $f \in L^{*}(E_A, E')$, we say '<u>f is n-extensible</u>' if f has an extension in $L^{*}(\bar{E}_n, E')$. Let $f^{(o)} \in L^{*}(\bar{E}_o, E')$ be an arbitrary extension of f. If f is n-extensible, then by Proposition 2.12 and by Corollary 2.6, $f^{(o)}$ is also n-extensible. So we can replace our starting point f by $f^{(o)}$. Thus, the important step in solving our main problems is to find answers to the following questions. Given $g \in L^{*}(\bar{E}_n, E')$ $(n \geq 0)$, what is the obstruction to extending g to $\bar{E}_{n+1}$ in a fibre-preserving way? If $g', g'' \in L^{*}(\bar{E}_{n+1}, E')$ are extensions of g, what is the condition for them to be fibre homotopic rel $\bar{E}_n$? The answers will be given in Theorem 2.14 below.

First, we recall the definition and the properties of 'separation homomorphisms' defined by J. H. C. Whitehead [38]. Suppose that $h, k: (X, x_o) \to (Y, y_o)$ are based maps such that $h|X_o = k|X_o$, where X_o is a path-connected subset of X and $x_o \in X_o$. The <u>separation homomorphism of the pair (h, k)</u>,

$$\Delta(h, k): \pi_n(X, X_o, x_o) \to \pi_n(Y, y_o) \qquad (n \geq 1)$$

is defined as follows. Let $\alpha \in \pi_n(X, X_o, x_o)$ be represented by a based map $a: (I^n, \partial I^n, *) \to (X, X_o, x_o)$. Then

$\Delta(h, k)(\alpha) \in \pi_n(Y, y_o)$ is defined to be the homotopy class of the based map

$$b\colon \partial I^{n+1} = \partial I^n \times I \cup I^n \times \partial I \to Y$$

given by

$$b(u, t) = \begin{cases} ha(u) = ka(u) & \text{if } (u, t) \in \partial I^n \times I \\ ha(u) & \text{if } (u, 0) \in I^n \times 0 \\ ka(u) & \text{if } (u, 1) \in I^n \times 1, \end{cases}$$

where ∂I^{n+1} is furnished with an appropriate orientation. Note that, in our previous notation (used in Corollary 1.11), we can write $b = d(h \cdot a, k \cdot a)$. Also note that $\pi_1(X, X_o, x_o)$ has no group structure.

Lemma 2.13. (See [38], Appendix.)

(1) The above $\Delta(h, k)$ is well defined, and for $n \geq 2$ it is a homomorphism.

(2) If $h \simeq k$ rel X_o, then $\Delta(h, k) = 0$.

(3) If $g, h, k\colon (X, x_o) \to (Y, y_o)$ coincide on X_o, then

$$\Delta(g, h) + \Delta(h, k) = \Delta(g, k).$$

(4) The following diagram is commutative:

$$
\begin{tikzcd}
 & \pi_n(X, x_o) \arrow[dl, "q_*"'] \arrow[dr, "h_* - k_*"] & \\
\pi_n(X, X_o, x_o) \arrow[rr, "\Delta(h, k)"] & & \pi_n(Y, y_o)
\end{tikzcd}
$$

where q is the injection $(X, x_o, x_o) \to (X, X_o, x_o)$.

(5) For $n \geq 2$, $\Delta(h, k)$ is a $\pi_1(X_o, x_o)$-homomorphism; that means,

$$\Delta(h, k)(\alpha^{\xi}) = (\Delta(h, k)(\alpha))^{\xi}$$

for $\alpha \in \pi_n(X, X_o, x_o)$, $\xi \in \pi_1(X_o, x_o)$, where the action of $\pi_1(X_o, x_o)$ on $\pi_n(Y, y_o)$ is considered via the homomorphism

$$h_*(=k_*): \pi_1(X_o, x_o) \to \pi_1(Y, y_o). \quad \square$$

Now we prove the following theorem.

<u>Theorem</u> 2.14. Let $g \in L^*(\overline{E}_n, E')$, $n \geq 0$.

(1) g is $(n+1)$-extensible if and only if the composition

$$\Gamma(g): \pi_{n+1}(\overline{E}^{(F)}_{n+1}, \overline{E}^{(F)}_n, i) \xrightarrow{\partial} \pi_n(\overline{E}^{(F)}_n, i) \xrightarrow{(g_\#)_*} \pi_n(E'^{(F)}, u)$$

is trivial (where $u = g| F$).

(2) Suppose that $g', g'' \in L^*(\overline{E}_{n+1}, E')$ are extensions of g. Then they are fibre homotopic rel $\overline{E}_n$ if and only if the separation homomorphism

$$\Delta(g'_{\#}, g''_{\#}) : \pi_{n+1}(\overline{E}^{(F)}_{n+1}, \overline{E}^{(F)}_{n}, i) \to \pi_{n+1}(\overline{E}'^{(F)}, u)$$

is trivial.

Proof.

(1) Suppose $g' \in L^*(\overline{E}_{n+1}, E')$ is an extension of g. Then $\Gamma(g) = 0$ follows immediately from the diagram:

$$\begin{tikzcd}
\pi_{n+1}(\overline{E}^{(F)}_{n+1}, \overline{E}^{(F)}_{n}, i) \arrow[r, "\partial"] & \pi_n(\overline{E}^{(F)}_{n}, i) \arrow[r, "(g_{\#})_*"] \arrow[d] & \pi_n(E'^{(F)}, u) \\
 & \pi_n(E^{(F)}_{n+1}, i) \arrow[ur, "(g'_{\#})_*"'] &
\end{tikzcd}$$

To prove the converse assertion, let e^{n+1}_{λ} be an arbitrary $(n+1)$-cell of $B - A$, and let $\hat{\phi}_{\lambda}: (I^{n+1}, \partial I^{n+1}, *) \to (\overline{E}^{(F)}_{n+1}, \overline{E}^{(F)}_{n}, i_{\lambda})$ be a characteristic map for the block $E_{\overline{e}^{n+1}_{\lambda}}$, where $i_{\lambda} = \hat{\phi}_{\lambda}(*) \in \overline{E}^{(F)}_{n}$. We may assume that i_{λ} is connected with i by a path W in $\overline{E}^{(F)}_{n}$. Consider now the following diagram:

$$\begin{tikzcd}
\pi_{n+1}(\overline{E}^{(F)}_{n+1},\overline{E}^{(F)}_{n},i_{\lambda}) \arrow[r, "\partial"] \arrow[d, "w_{\#}", "\approx"'] & \pi_n(\overline{E}^{(F)}_{n},i_{\lambda}) \arrow[r, "(g_{\#})_*"] \arrow[d, "w_{\#}", "\approx"'] & \pi_n(E'^{(F)},u_{\lambda}) \arrow[d, "(g_{\#}\cdot w)_{\#}", "\approx"'] \\
\pi_{n+1}(\overline{E}^{(F)}_{n+1},\overline{E}^{(F)}_{n},i) \arrow[r, "\partial"] & \pi_n(\overline{E}^{(F)}_{n},i) \arrow[r, "(g_{\#})_*"] & \pi_n(E'^{(F)},u)
\end{tikzcd}$$

where $u_{\lambda} = g \cdot i_{\lambda}$, and the vertical arrows are the isomorphisms induced from the respective paths.

By assumption, the composition $(g_{\#})_* \cdot \partial$ in the lower part is trivial and, hence, by commutativity, $(g_{\#})_* \cdot \partial$ in the upper part is also trivial. In particular, the element $[\hat{\phi}_\lambda] \in \pi_{n+1}(\bar{E}^{(F)}_{n+1}, \bar{E}^{(F)}_n, i_\lambda)$ is carried to 0, i.e.,

$$[\hat{\phi}_\lambda] \xrightarrow{\partial} [\partial\hat{\phi}_\lambda] \xrightarrow{(g_{\#})_*} 0.$$

Thus, $g_{\#} \cdot (\partial\hat{\phi}_\lambda) \simeq 0\colon \partial I^{n+1} \to E'^{(F)}$ and, by Proposition 2.9, g can be extended over $\bar{E}_n \cup E_{\bar{e}^{n+1}_\lambda}$. Noting that $E_{\bar{e}^{n+1}_\lambda} \cap E_{\bar{e}^{n+1}_\mu} \subset \bar{E}_n$ for any two different $(n+1)$-cells e_λ, e_μ in $B - A$, and that $\bar{E}_{n+1}$ has the weak topology with respect to the blocks (Proposition 2.2), we obtain the result.

(2) If $g' \overset{*}{\simeq} g''$ rel $\bar{E}_n$, then $g'_{\#} \simeq g''_{\#}\colon \bar{E}^{(F)}_{n+1} \to E'^{(F)}$, rel $\bar{E}^{(F)}_n$ (see Lemma 1.8). Hence, by Lemma 2.13(2), we have $\Delta(g'_{\#}, g''_{\#}) = 0$.

The proof of the converse part is quite similar to the proof of (1) by Proposition 2.9(2). $\square$

D. 1-extensibility

Let $\theta\colon \pi_1(B, b_o) \to \varepsilon(F) \subset \pi_o(F^F, \mathrm{Id})$ be the action of $\pi_1(B, b_o)$ on the fibre F where $\varepsilon(F)$ denotes the collection of homotopy classes of homotopy equivalences $F \to F$ (see Lemma 1.13).

<u>Proposition</u> 2.15. Let $f \in L^*(E_A, E')$ and let $u = f|F$. Then f is 1-extensible if and only if

$$\pi_1(B, b_o) \xrightarrow{\theta} \pi_o(F^F, \mathrm{Id}) \xrightarrow{(u_\#)_*} \pi_o(E'^{(F)}, u)$$

is trivial. In particular, if $p\colon E \to B$ is an orientable fibration (i.e., $\theta = 0$), any $f \in L^*(E_A, E')$ is 1-extensible.

<u>Proof</u>. There is an extension $f^{(o)} \in L^*(\bar{E}_o, E')$ of f, and the 1-extensibility of f is equivalent to that of $f^{(o)}$. So, by Theorem 2.14, it suffices to show:

$$\Gamma(f^{(o)}) = 0 \Longleftrightarrow (u_\#)_* \cdot \theta = 0. \qquad (*)$$

Recall that the action θ can be identified with the boundary function $\partial\colon \pi_1(E^{(F)}, F^F, i) \to \pi_o(F^F, \mathrm{Id})$ (see the proof of Proposition 1.16). Then the equivalence (*) follows easily from the next commutative diagram:

$$\begin{tikzcd}
\pi_1(B, b_o) & \pi_1(E^{(F)}, F^F, i) \arrow[l, "\bar{p}_*"', "\cong"] \arrow[dr, "\partial = \theta"] & & \\
\pi_1(\bar{B}^1, b_o) \arrow[u, "\text{onto}"] \arrow[d, "\text{onto}"] & \pi_1(\bar{E}_1^{(F)}, F^F, i) \arrow[l, "\bar{p}_*"', "\cong"] \arrow[u] \arrow[d] \arrow[r, "\partial_1"] & \pi_o(F^F, \mathrm{Id}) \arrow[d] \arrow[dr, "(u_\#)_*"] & \\
\pi_1(\bar{B}^1, \bar{B}^o) & \pi_1(\bar{E}_1^{(F)}, \bar{E}_0^{(F)}, i) \arrow[l, "\bar{p}_*"', "\cong"] \arrow[r, "\partial_2"] \arrow[rr, bend right, "\Gamma(f^{(o)})"'] & \pi_o(\bar{E}_o^{(F)}, i) \arrow[r, "(f_\#^{(o)})_*"] & \pi_o(E'^{(F)}, u)
\end{tikzcd}$$

where the unlabelled arrows are the homomorphisms (or functions) induced from the injections. Recall that $\bar{B}^o$ is connected and so $\pi_1(\bar{B}^1, b_o) \to \pi_1(\bar{B}^1, \bar{B}^o)$ is onto. □

Proposition 2.16. Assume that the fibres of both the fibrations $p: E \to B$, $p': E' \to B'$ have the same homotopy type, and let $u: F \to F'$ $(= p'^{-1}(b_o'))$ be a homotopy equivalence. Then u has a fibre-preserving extension $E_1 \to E'$ if and only if for each $\alpha \in \pi_1(B, b_o)$ there is $\beta \in \pi_1(B', b_o')$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{u} & F' \\ {\scriptstyle \bar{\alpha}}\downarrow & & \downarrow{\scriptstyle \bar{\beta}} \\ F & \xrightarrow{u} & F' \end{array}$$

commutes in the homotopy sense where $\bar{\alpha}$, $\bar{\beta}$ are the homotopy equivalences determined from α, β, respectively.

Proof. In fact, the following statements are equivalent:

(1) u is 1-extensible,

(2) $(u_\#)_* \cdot \theta: \pi_1(B, b_o) \to \pi_o(F^F, \mathrm{Id}) \to \pi_o(E'^{(F)}, u)$ is trivial,

(3) for any $\alpha \in \pi_1(B, b_o)$, $u \cdot \bar{\alpha}$ and u are in the same path-component of $E'^{(F)}$, and

(4) for any $\alpha \in \pi_1(B, b_o)$, $u \cdot \bar{\alpha} \simeq \bar{\beta} \cdot u$ for some $\beta \in \pi_1(B', b_o')$.

That (1)$\Longleftrightarrow$(2) follows from the preceding proposition.

(2)$\Longleftrightarrow$(3) is clear. (3)$\Longleftrightarrow$(4) is from Proposition 1.14. $\square$

In particular, consider the case where p and p' are the same.

Corollary 2.17. A homotopy equivalence $u: F \to F$ has a fibre-preserving extension $E_1 \to E$ if and only if the homotopy class $[u]$ is in the normalizer of $\theta(\pi_1(B, b_o))$ in $\varepsilon(F)$.

In fact, by the proposition, 1-extensibility of u is equivalent to that $[u \cdot \bar{\alpha} \cdot u^{-1}]$ is in $\mathrm{Im}\ \theta$ for any $\bar{\alpha} = \theta(\alpha)$. $\square$

Let us look at a simple example. Let

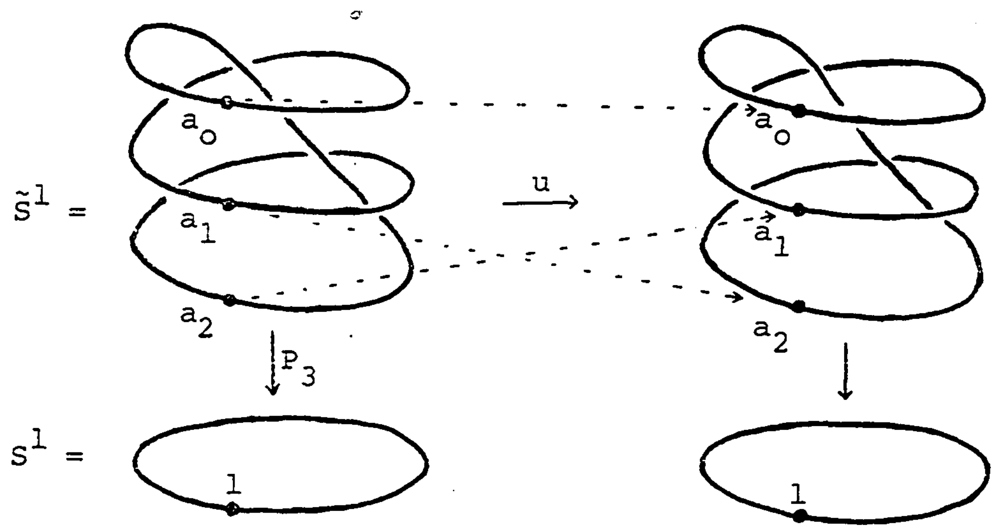
$$\tilde{S}^1 = S^1 = \text{the set of complex numbers with norm } 1.$$

Define $P_n: \tilde{S}^1 \to S^1$ by $P_n(e^{i\theta}) = e^{in\theta}$. Then P_n is a fibration (i.e., n-fold covering map). The fibre over the base point $1 \in S^1$ is given by

$$F = \{a_o, a_1, \cdots, a_{n-1}\}, \qquad a_k = e^{i\frac{2k\pi}{n}}$$
$$(k = 0, 1, \cdots, n-1).$$

$\varepsilon(F)$ is the permutation group of F and, as is easily seen, the action $\theta: \pi_1(S^1, 1) \to \varepsilon(F)$ is given by

$$\theta(\iota) = \begin{pmatrix} a_o, & a_1, & \cdots, & a_{n-1} \\ a_1, & a_2, & \cdots, & a_o \end{pmatrix}, \text{ i.e., } \theta(\iota)(a_k) = \begin{cases} a_{k+1} & k < n \\ a_o & k = n, \end{cases}$$



where $\iota \in \pi_1(S^1, 1)$ is the generator represented by the closed path $t \to e^{i2\pi t}$. Thus, Im θ is the cyclic group of order n generated by $\theta(\iota)$. For $n \geq 4$, Im θ is not a normal subgroup. So, not all permutations of F are 1-extensible. However, for $n \leq 3$, Im θ is normal (since index $[\varepsilon(F): \text{Im}\ \theta] = n!/n \leq 2$) and every permutation $u: F \to F$ is 1-extensible. The above figure indicates the case where $n = 3$ and $u = \begin{pmatrix} a_o, & a_1, & a_2 \\ a_o, & a_2, & a_1 \end{pmatrix}$. Observe that u has a fibre-preserving extension.

III. COHOMOLOGICAL EXPRESSION OF THE OBSTRUCTIONS

In this chapter an anology and/or generalization of the classical obstruction theory will be established; namely, for a given $g \in L^*(\bar{E}_n, E')$, $n \geq 1$, a necessary and sufficient condition for $g|\bar{E}_{n-1}$ to be $(n+1)$-extensible will be formulated in the cohomological language. Then the procedure in the classical theory can be followed, verbatim, leading to several results in the case of fibre-preserving maps.

Throughout the chapter we shall assume:

<u>Assumption</u> I. The fibration $\bar{p}: E \to B$ is orientable, and

<u>Assumption</u> II. The space $(E'^{(F)}, u)$ is n-simple where $u = g|F$.

For example, if E' is a fibre space over a simply connected space B', and $u: F \to F'$ (where F' is a fibre in E') is a homotopy equivalence, then Assumption II is satisfied (see Corollary 1.18). Another example is the case where E' is an H-space and a fibre space over the singleton space $B' = \{b_o'\}$. Note that if $n = 1$, then $\pi_1(E'^{(F)}, u)$ is an abelian group.

A. A Modification of $\pi_1(X, X_o, x_o)$

For $x_o \in X_o \subset X$, the fact that $\pi_1(X, X_o, x_o)$ has no group structure is inconvenient. So we shall define the following group:

$$\pi_1'(X, X_o, x_o) = \pi_1(X, x_o)/[i_*\pi_1(X_o, x_o)]$$

where $[i_*\pi_1(X_o, x_o)]$ denotes the minimal normal subgroup of $\pi_1(X, x_o)$ containing the image of $i_*: \pi_1(X_o, x_o) \to \pi_1(X, x_o)$ (i.e., $[i_*\pi_1(X_o, x_o)]$ is the intersection of all normal subgroups of $\pi_1(X, x_o)$ containing $\mathrm{Im}\ i_*$ (i being the inclusion map $X_o \subset X$)).

If $f: (X, X_o, x_o) \to (Y, Y_o, t_o)$ is a map, then the homomorphism $f_*: \pi_1(X, x_o) \to \pi_1(Y, y_o)$ defines a unique homomorphism of the quotient groups

$$f_{**}: \pi_1'(X, X_o, x_o) \to \pi_1'(Y, Y_o, y_o)$$

in the obvious way. Furthermore,

<u>Lemma</u> 3.1.

(1) Let $p: E \to B$ be any fibration such that $p_*: \pi_1(E, *) \to \pi_1(B, b_o)$ is epimorphic, and let K be a subset of B containing b_o. Then p_* induces the isomorphism

$$p_{**}: \pi_1'(E, E_K) \cong \pi_1'(B, K).$$

(2) Let X and its subset X_o be path-connected. Then the Hurewicz homomorphism $h: \pi_1(X, x_o) \to H_1(X)$ induces an epimorphism $h': \pi_1'(X, X_o) \to H_1(X, X_o)$ and the kernel of h' is the

commutator subgroup of $\pi_1'(X, X_o)$.

Proof.

(1) Consider the following diagram

$$\begin{tikzcd}
 & \pi_1(E_K) \arrow[r, "p_*"] \arrow[d, "i_*"'] & \pi_1(K) \arrow[d, "\bar{i}_*"] \\
\pi_1(p^{-1}(b_o)) \arrow[r, "\iota_*"] & \pi_1(E) \arrow[r, "p_*"] \arrow[d, "q"'] & \pi_1(B) \arrow[d, "q"] \\
 & \pi_1'(E, E_K) \arrow[r, "p_{**}"] & \pi_1'(B, K)
\end{tikzcd}$$

where q denotes the canonical homomorphism. Clearly, $p_*\colon \pi_1(E) \to \pi_1(B)$ induces the epimorphism p_{**} indicated in the diagram. To show that it is monomorphic, suppose $p_{**}(\bar{\alpha}) = 1$ where $\bar{\alpha} = q(\alpha)$, $\alpha \in \pi_1(E)$. Then $p_*(\alpha)$ belongs to the normal subgroup $[\mathrm{Im}\ \bar{i}_*]$, and so we can write

$$p_*(\alpha) = \prod_{k=1}^{n} (\bar{x}^{-1} \cdot \bar{y}_k \cdot \bar{x}_k)$$

for some $\bar{x}_k \in \pi_1(B)$ and $\bar{y}_k \in \mathrm{Im}\ \bar{i}_*$. Choose $x_k \in \pi_1(E)$ and $y_k \in \mathrm{Im}\ i_*$ so that $p_*(x_k) = \bar{x}_k$ and $p_*(y_k) = \bar{y}_k$, and let $\beta = \prod_{k=1}^{n} (x_k^{-1} \cdot y_k \cdot x_k)$. Then $p_*(\beta) = p_*(\alpha)$ and, hence, $\alpha\, \beta^{-1} \in \ker p_* = \mathrm{Im}\ \iota_* \subset \mathrm{Im}\ i_*$; i.e., $\alpha = y \cdot \beta$ for some $y \in \mathrm{Im}\ i_*$.

Hence, $\alpha \in [\mathrm{Im}\ i_*]$ and so $\bar{\alpha} = 1$.

(2) The proof of (2) is quite similar to (1) by using the following diagram:

$$\begin{tikzcd}
 & \pi_1(X_0) \arrow[r, "h", "\text{onto}"'] \arrow[d, "i_*"'] & H_1(X_0) \arrow[d, "\bar{i}_*"] \\
{[\pi_1(X), \pi_1(X)]} \arrow[r] & \pi_1(X) \arrow[r, "h", "\text{onto}"'] \arrow[d, "q"'] & H_1(X) \arrow[d, "\text{onto}"] \\
 & \pi_1(X, X_0) \arrow[r, dotted, "h'"] & H_1(X, X_0)
\end{tikzcd}$$

where $[\pi_1(X), \pi_1(X)]$ is the commutator subgroup of $\pi_1(X)$. Details are omitted. □

Now let $h,k\colon (X, x_0) \to (Y, y_0)$ be based maps satisfying $h|\ X_0 = k|\ X_0$ where X_0 is a path-connected subset of X containing x_0. Assume that $\pi_1(Y, y_0)$ is an abelian group (with the multiplication denoted +). Then, $h_*-k_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a homomorphism and its kernel contains the image of $i_*\colon \pi_1(X_0, x_0) \to \pi_1(X, x_0)$. Therefore, h_*-k_* induces a unique homomorphism $\Delta'(h, k)$ such that the diagram

$$\begin{tikzcd}
 & \pi_1(X, x_0) \arrow[dl, "q"'] \arrow[dr, "h_*-k_*"] & \\
\pi_1'(X, X_0, x_0) \arrow[rr, "\Delta'(h, k)"] & & \pi_1(Y, y_0)
\end{tikzcd}$$

commutes. Recall now the separation function

$$\Delta(h, k): \pi_1(X, X_o, x_o) \to \pi_1(Y, y_o)$$

(see Lemma 2.13). From the exactness of the sequence

$$\pi_1(X_o) \xrightarrow{i_*} \pi_1(X) \xrightarrow{j_*} \pi_1(X, X_o) \xrightarrow{\partial} \pi_o(X_o) = 0$$

we can easily see that the function j_* induces a bijection from the right quotient set $\pi_1(X)/\mathrm{Im}\ i_*$ onto $\pi_1(X, X_o)$. So there is a natural function (onto)

$$q': \pi_1(X, X_o) \to \pi_1'(X, X_o) \quad (= \pi_1(X)/[\mathrm{Im}\ i_*]).$$

Then from the diagram of Lemma 2.13(4) and the above diagram, we find that

$$\begin{tikzcd} \pi_1(X, X_o) \arrow[d, "q'"', "\text{onto}"] \arrow[dr, "\Delta(h, k)"] & \\ \pi_1'(X, X_o) \arrow[r, "\Delta'(h, k)"'] & \pi_1(Y) \end{tikzcd}$$

commutes. Hence, $\Delta(h, k) = 0$ if and only if $\Delta'(h, k) = 0$. In the next section we shall use the homomorphism $\Delta'(h, k)$ instead of the function $\Delta(h, k)$.

B. The Obstruction Cocycle and the Difference Cochain

We shall use the following 'chain complex' to define the cohomology groups of the pair (B, A):

$$\cdots \to H_{n+1}(\bar{B}^{n+1}, \bar{B}^n) \xrightarrow{d_{n+1}} H_n(\bar{B}^n, \bar{B}^{n-1}) \xrightarrow{d_n} \cdots \to H_1(\bar{B}^1, \bar{B}^o) \to 0$$

where H_n is the singular homology and d_{n+1} is the boundary homomorphism associated to the triple $(\bar{B}^{n+1}, \bar{B}^n, \bar{B}^{n-1})$. So an n-cochain c^n with coefficient group G is a homomorphism $H_n(\bar{B}^n, \bar{B}^{n-1}) \to G$, and the coboundary cochain of c^n is the homomorphism $c^n \cdot d_{n+1}: H_{n+1}(\bar{B}^{n+1}, \bar{B}^n) \to G$.

For a given $g \in L^*(\bar{E}_n, E')$, consider the diagram:

$$\begin{tikzcd} & \pi_{n+1}(\bar{E}^{(F)}_{n+1}, \bar{E}^{(F)}_n, i) \arrow[r, "\partial"] \arrow[d, "\bar{p}_*", "\cong"'] & \pi_n(\bar{E}^{(F)}_n, i) \arrow[r, "(g_\#)_*"] & \pi_n(E'^{(F)}, u) \\ H_{n+1}(\bar{B}^{n+1}, \bar{B}^n) & \pi_{n+1}(\bar{B}^{n+1}, \bar{B}^n, b_o) \arrow[l, "h"'] & & \end{tikzcd}$$

where h is the Hurewicz homomorphism. Note that $(g_\#)_* \cdot \partial$ is the obstruction homomorphism $\Gamma(g)$ defined in Theorem 2.14.

<u>Lemma</u> 3.2.

$$(g_\#)_* \cdot \partial \cdot \bar{p}_*^{-1} \cdot h^{-1}: H_{n+1}(\bar{B}^{n+1}, \bar{B}^n) \to \pi_n(E'^{(F)}, u)$$

is a well-defined homomorphism.

<u>Proof</u>. Since $(\bar{B}^{n+1}, \bar{B}^n)$ is n-connected, the Hurewicz isomorphism theorem tells us that h is an epimorphism and its kernel is generated by the elements of the form $\alpha - \alpha^\xi$ where $\alpha \in \pi_{n+1}(\bar{B}^{n+1}, \bar{B}^n, b_o)$ and $\xi \in \pi_1(\bar{B}^n, b_o)$. It suffices to prove that each element of this type is carried by $(g_\#)_* \cdot \partial \cdot \bar{p}_*^{-1}$ onto zero in $\pi_n(E'^{(F)}, u)$. Noting that

the orientability assumption implies the surjectivity of $\bar{p}_*: \pi_1(\bar{E}_n^{(F)}, i) \to \pi_1(\bar{B}^n, b_o)$, let $\eta \in \pi_1(\bar{E}_n^{(F)}, i)$ be such that $\bar{p}_*(\eta) = \xi$, and let $\beta = \bar{p}_*^{-1}(\alpha) \in \pi_{n+1}(\bar{E}_{n+1}^{(F)}, \bar{E}_n^{(F)}, i)$. Then,

$$\bar{p}_*(\beta - \beta^\eta) = \alpha - \alpha^\xi, \text{ i.e., } \bar{p}_*^{-1}(\alpha - \alpha^\xi) = \beta - \beta^\eta.$$

But,

$$(g_\#)_*\partial(\beta^\eta) = (g_\#)_*((\partial\beta)^\eta) = ((g_\#)_*\partial\beta)^{(g_\#)_*\eta} = (g_\#)_*\partial\beta$$

(where the last equality is due to Assumption II). Therefore,

$$(g_\#)_*\partial(\beta - \beta^\eta) = 0, \text{ i.e., } (g_\#)_*\partial\bar{p}_*^{-1}(\alpha - \alpha^\xi) = 0$$

This proves the lemma. □

We call the homomorphism defined in the previous lemma the <u>obstruction cocycle</u> for g, and denote it by $c^{n+1}(g)$. This name is justified by the following two lemmas.

<u>Lemma</u> 3.3. g is $(n+1)$-extensible if and only if $c^{n+1}(g) = 0$.

<u>Proof</u>. It is easy to see that $c^{n+1}(g) = 0$ if and only if $(g_\#)_* \cdot \partial = 0$. Then the lemma follows from Theorem 2.14. □

<u>Lemma</u> 3.4. $c^{n+1}(g)$ is a cocycle, i.e., $c^{n+1}(g) \cdot d_{n+2} = 0$.

Proof. This follows from the following diagram:

$$\begin{array}{ccccccc}
H_{n+2}(\bar{B}^{n+2},\bar{B}^{n+1}) & \underset{\text{onto}}{\overset{h}{\leftarrow}} & \pi_{n+2}(\bar{B}^{n+2},\bar{B}^{n+1}) & \underset{\cong}{\overset{\bar{p}_*}{\leftarrow}} & \pi_{n+2}(\bar{E}^{(F)}_{n+2},\bar{E}^{(F)}_{n+1}) & & \\
\downarrow{\scriptstyle d_{n+2}} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d''} & & \\
H_{n+1}(\bar{B}^{n+1},\bar{B}^{n}) & \overset{h}{\leftarrow} & \pi_{n+1}(\bar{B}^{n+1},\bar{B}^{n}) & \overset{\bar{p}_*}{\leftarrow} & \pi_{n+1}(\bar{E}^{(F)}_{n+1},\bar{E}^{(F)}_{n}) & \overset{\partial}{\rightarrow} & \pi_n(\bar{E}^{(F)}_n)
\end{array}$$

Note that $\partial \cdot d'' = 0$. □

Suppose now that $g', g'' \in L^*(\bar{E}_n, E')$ satisfy the condition $g' | \bar{E}_{n-1} = g'' | \bar{E}_{n-1}$ $(n \geq 1)$. We define the difference cochain $\delta^n(g', g'')$ for the pair (g', g'') to be the composition $\Delta(g'_\#, g''_\#) \cdot \bar{p}_\#^{-1} \cdot h^{-1}$ in the following diagram:

$$\begin{array}{ccccc}
 & & \pi_n(\bar{E}^{(F)}_n, \bar{E}^{(F)}_n, i) & \xrightarrow{\Delta(g'_\#,\, g''_\#)} & \pi_n(E'^{(F)}, u) \\
 & & \uparrow{\scriptstyle \bar{p}_*^{-1}}\ \cong & & \\
H_n(\bar{B}^n, \bar{B}^{n-1}) & \overset{h^{-1}}{\rightarrow} & \pi_n(\bar{B}^n, \bar{B}^{n-1}, b_o) & &
\end{array}$$

Here, for $n = 1$, we replace the two relative homotopy sets by the groups

$$\pi'_1(\bar{E}^{(F)}_1, \bar{E}^{(F)}_o, i), \quad \pi'_1(\bar{B}^n, \bar{B}^{n-1}, b_o), \text{ and } \Delta(g'_\#, g''_\#), h$$

by the homomorphisms $\Delta'(g'_\#, g''_\#)$, h', respectively (see Section A).

<u>Lemma</u> 3.5. $\delta^n(g', g'')$ is a well-defined homomorphism.

<u>Proof</u>. If $n = 1$, this follows from Lemma 3.1 which says that $\bar{p}_*$ is an isomorphism, h' is an epimorphism, and the kernel of h' is the commutator subgroup of $\pi_1'(\bar{B}^1, \bar{B}^o, b_o)$. The proof for $n > 1$ is similar to the proof of Lemma 3.2. □

<u>Lemma</u> 3.6. $g' \overset{*}{\simeq} g''$ rel $\bar{E}_{n-1} \Longleftrightarrow \delta^n(g', g'') = 0$.

<u>Proof</u>. Since $\delta^n(g', g'') = 0$ if and only if $\Delta(g_\#', g_\#'') = 0$, Theorem 2.14 proves this lemma. Notice that for $n = 1$, $\Delta'(g_\#', g_\#'') = 0$ is equivalent to $\Delta(g_\#', g_\#'') = 0$. □

<u>Lemma</u> 3.7. $\delta^n(g', g'')\cdot d_{n+1} = c^{n+1}(g') - c^{n+1}(g'')$.

<u>Proof</u>. This follows from the definitions of c^{n+1} and δ^n, and the following commutative diagram. The commutativity of the rightmost triangle is due to Lemma 2.13 (4).

$$\begin{tikzcd}
H_{n+1}(\bar{B}^{n+1},\bar{B}^n) \arrow[d, "d_{n+1}"'] & \pi_{n+1}(\bar{B}^{n+1},\bar{B}^n) \arrow[l, "h"'] \arrow[d] & \pi_{n+1}(\bar{E}^{(F)}_{n+1},\bar{E}^{(F)}_n) \arrow[l, "\bar{p}_*"', "\cong"] \arrow[d] \arrow[r, "\partial"] & \pi_n(\bar{E}^{(F)}_n) \arrow[dl] \arrow[dr, "(g_\#')_* - (g_\#'')_*"] & \\
H_n(\bar{B}^n,\bar{B}^{n-1}) & \pi_n(\bar{B}^n,\bar{B}^{n-1}) \arrow[l, "h"'] & \pi_n(\bar{E}^{(F)}_n,\bar{E}^{(F)}_{n-1}) \arrow[l, "\bar{p}_*"', "\cong"] \arrow[rr, "\Delta(g_\#',g_\#'')"] & & \pi_n(E'^{(F)},u)
\end{tikzcd}$$

For $n = 1$, the bottom line is altered appropriately. □

C. An Existence Lemma and a Generalization of Eilenberg's Theorem

To complete an analogy of Eilenberg's scheme, we must prove the following important lemma.

Lemma 3.8. For a given $g' \in L^*(\overline{E}_n, E')$ and an n-cochain $z^n\colon H_n(\overline{B}^n, \overline{B}^{n-1}) \to \pi_n(E'^{(F)}, u)$, there exists an extension $g'' \in L^*(\overline{E}_n, E')$ of $g' | \overline{E}_{n-1}$ such that

$$\delta^n(g', g'') = z^n.$$

Proof. We prove only the case $n > 1$ since the case $n = 1$ is quite similar (though the notations must be changed slightly).

For a given cochain z^n, define a homomorphism

$$\Delta\colon \pi_n(\overline{E}_n^{(F)}, E_{n-1}^{(F)}, i) \to \pi_n(E'^{(F)}, u)$$

by $\Delta = z^n \cdot \mathbf{h} \cdot \overline{p}_*$. We will show $\Delta = \Delta(g'_\#, g''_\#)$ for some extension $g'' \in L^*(\overline{E}_n, E')$ of $g' | \overline{E}_{n-1}$. If this is done then we have

$$\begin{aligned}\delta^n(g', g'') &= \Delta(g'_\#, g''_\#) \cdot \overline{p}_*^{-1} \cdot \mathbf{h}^{-1} \\ &= \Delta \cdot \overline{p}_*^{-1} \cdot \mathbf{h}^{-1} = z^n\end{aligned}$$

as desired.

Let e_λ^n be any n-cell in $B - A$, and let $\hat{\phi}_\lambda\colon (I^n, \partial I^n, *) \to (\overline{E}_n^{(F)}, \overline{E}_{n-1}^{(F)}, i_\lambda)$ be a fixed characteristic map for the block $E_{\overline{e}_\lambda^n}$. Let w be a path in $\overline{E}_{n-1}^{(F)}$

from i to i_λ. Then $\hat{\phi}_\lambda$ and w determine $w_\#[\hat{\phi}_\lambda] \in \pi_n(\overline{E}_n^{(F)}, \overline{E}_{n-1}^{(F)}, i)$ where $w_\#$ is the isomorphism induced by w. If w and w' are two such paths, $w_\#[\hat{\phi}_\lambda]$ and $w'_\#[\hat{\phi}_\lambda]$ differ only by the action of $\pi_1(\overline{E}_{n-1}^{(F)}, i)$. Therefore, the element $\Delta(w_\#[\hat{\phi}_\lambda]) \in \pi_n(E'^{(F)}, u)$ does not depend on the choice of w because $\Delta = \delta^n \cdot \mathbf{h} \cdot \overline{p}_*$, and $\mathbf{h}$ neglects the action of π_1. Let β_λ denote $\Delta(w_\#[\hat{\phi}_\lambda])$. Now, $g'_\# \cdot w$ is a path in $E'^{(F)}$ from u to u_λ (say). It induces an isomorphism $(g'_\# \cdot w)_\#\colon \pi_n(E'^{(F)}, u_\lambda) \to \pi_n(E'^{(F)}, u)$. Let $\beta'_\lambda = (g'_\# \cdot w)_\#^{-1}(\beta_\lambda)$.

$$\begin{tikzcd}
{[\hat{\phi}_\lambda] \in \pi_n(E^{(F)}_{\bar{e}^n_\lambda}, E^{(F)}_{\dot{e}^n_\lambda}, i_\lambda)} \arrow[d] \arrow[dr, dashed, "\Delta(g'_\#, g''_{\lambda\#})"] & \\
{[\hat{\phi}_\lambda] \in \pi_n(\overline{E}_n^{(F)}, \overline{E}_{n-1}^{(F)}, i_\lambda)} \arrow[d] & \pi_n(E'^{(F)}, u_\lambda) \ni \beta'_\lambda \arrow[d] \\
{w_\#[\hat{\phi}_\lambda] \in \pi_n(\overline{E}_n^{(F)}, \overline{E}_{n-1}^{(F)}, i)} \arrow[r, "\Delta"] & \pi_n(E'^{(F)}, u) \ni \beta_\lambda
\end{tikzcd}$$

We now have the map $g'_\# \cdot \hat{\phi}_\lambda\colon (I^n, *) \to (E'^{(F)}, u_\lambda)$ and the homotopy class $\beta'_\lambda \in \pi_n(E'^{(F)}, u_\lambda)$. By an elementary fact in homotopy theory, there is a map, say $\hat{h}''\colon (I^n, *) \to (E'^{(F)}, u_\lambda)$ such that $\hat{h}''|\,\partial I^n = g'_\# \cdot \hat{\phi}_\lambda|\,\partial I^n$ and such that the separation map $d(g'_\# \cdot \hat{\phi}_\lambda, \hat{h}'')$ represents β'_λ, i.e.,

$$\beta'_\lambda = [d(g'_\# \cdot \hat{\phi}_\lambda, \hat{h}'')] \tag{1}$$

Clearly, the adjoint map $h''\colon I^n \times F \to E'$ of $\hat{h}''$ is an extension of $g' \cdot \phi_\lambda | \partial I^n \times F$ (= the adjoint of $g'_\# \cdot \hat{\phi}_\lambda | \partial I^n$). So, by Lemma 2.10, there is a fibre-preserving extension, say $g''_\lambda : E_{\bar{e}^n_\lambda} \to E'$, of $g' | E_{\dot{e}^n_\lambda}$ such that h'' and $g''_\lambda \cdot \phi_\lambda$ are fibre-homotopic rel $\partial I^n \times F$, or equivalently (see Proposition 2.10)

$$\hat{h}'' \simeq g''_{\lambda\#} \cdot \hat{\phi}_\lambda : I^n \to E'^{(F)}, \text{ (rel } \partial I^n) \tag{2}$$

From (1) and (2), we have

$$\begin{aligned} \beta'_\lambda &= [d(g'_\# \cdot \hat{\phi}_\lambda, \hat{h}'')] = [d(g'_\# \cdot \hat{\phi}_\lambda, g''_{\lambda\#} \cdot \hat{\phi}_\lambda)] \\ &= \Delta(g'_\#, g''_{\lambda\#})([\hat{\phi}_\lambda]) \end{aligned} \tag{3}$$

Constructing such g''_λ for all n-cells, and pasting them together, we obtain a (continuous) fibre-preserving map $g'' : \bar{E}_n \to E'$. g'' is clearly an extension of $g' | \bar{E}_{n-1}$. Furthermore, for each λ and for an arbitrary choice of w for λ,

$$\begin{aligned} \Delta(g'_\#, g''_\#)(w_\#[\hat{\phi}_\lambda]) &= (g'_\# \cdot w)_\# \Delta(g'_\#, g''_\#)[\hat{\phi}_\lambda] \\ &= (g'_\# \cdot w)_\# \Delta(g'_\#, g''_{\lambda\#})[\hat{\phi}_\lambda] \\ &= (g'_\# \cdot w)_\# \beta'_\lambda \qquad \text{(by (3)} \\ &= \beta_\lambda = \Delta(w_\#[\hat{\phi}_\lambda]) \text{ (by definition).} \end{aligned}$$

Note that the set of $w_\#[\hat{\phi}_\lambda]$ for all λ and for all possible w, generates $\pi_n(\bar{E}_n^{(F)}, \bar{E}_{n-1}^{(F)}, i)$ because the p_*-image of

$\{w_\#[\hat{\phi}_\lambda]\}$ generates $\pi_n(\bar{B}^n, \bar{B}^{n-1}, b_o)$. Thus, the above calculation shows $\Delta = \Delta(g_\#', g_\#'')$. This completes the proof. □

Now we prove a generalization of the classical theorem of Eilenberg [12].

Theorem 3.9. For a given $g \in L^*(\bar{E}_n, E')$, $n \geq 1$, let

$$[c^{n+1}(g)] \in H^{n+1}(B, A; \pi_n(E'^{(F)}, u))$$

be the cohomology class of the cocycle $c^{n+1}(g)$.

(1) $[c^{n+1}(g)]$ is determined only by $g|\,\bar{E}_{n-1}$.

(2) $g|\,\bar{E}_{n-1}$ is (n+1)-extensible if and only if $[c^{n+1}(g)] = 0$.

Proof.

(1) Let $g' \in L^*(\bar{E}_n, E')$ be any other extension of $g|\,\bar{E}_{n-1}$. Then, by Lemma 3.7, $c^{n+1}(g') - c^{n+1}(g)$ is a coboundary and, hence, $[c^{n+1}(g')] = [c^{n+1}(g)]$.

(2) If $g'' \in L^*(\bar{E}_{n+1}, E')$ is an extension of $g|\,\bar{E}_{n-1}$, then $c^{n+1}(g''|\,\bar{E}_n) = 0$ by Lemma 3.3. Hence, $[c^{n+1}(g)] = [c^{n+1}(g''\ \bar{E}_n)] = 0$. Conversely, suppose that $[c^{n+1}(g)] = 0$, i.e., $c^{n+1}(g) = \partial z^n$ $(= z^n \cdot d_{n+1})$ for some n-cochain z^n. By Lemma 3.8, we can write $z^n = \delta^n(g, g')$

for some n-extension $g' \in L^*(\bar{E}_n, E')$ of $g|\bar{E}_{n-1}$. Then, by Lemma 3.7, $c^{n+1}(g) = \partial z^n = \partial\delta^n(g, g') = c^{n+1}(g) - c^{n+1}(g')$. So $c^{n+1}(g') = 0$ and, hence, g' is (n+1)-extensible (Lemma 3.3). □

If we take the special fibrations $* \to B \xrightarrow{Id} B$ and $* \to X \xrightarrow{Id} X$ as $F \to E \xrightarrow{p} B$ and $F' \to E' \xrightarrow{p'} B'$, respectively, then Theorem 3.9 gives rise to Eilenberg's Theorem. Observe that in this special case, the coefficient group $\pi_n(E'^{(F)}, u)$ becomes $\pi_n(X, x_o)$ where $x_o = g(b_o)$.

D. The Fibre-preserving Homotopy Problem

Now we turn to the fibre-preserving homotopy problem. Suppose that $g', g'' \in L^*(E, E')$ are given with the condition

$$g'|\, E_A = g''|\, E_A$$

where A is a nonempty connected subcomplex of B. Then, by Proposition 2.12,

$$g'|\, \bar{E}_o \overset{*}{\simeq} g''|\, \bar{E}_o \text{ rel } E_A.$$

So, assuming that, for an integer $n \geq 1$, there is a homotopy $\{h_t\} \subset L^*(\bar{E}_{n-1}, E')$ such that

$$h_o = g'|\, \bar{E}_{n-1} \quad\text{and}\quad h_1 = g''|\, \bar{E}_{n-1},$$

we will study the obstruction to extending $\{h_t\}$ to a fibre-preserving homotopy between $g'|\bar{E}_n$ and $g''|\bar{E}_n$.

Let $\mathbb{p}\colon \mathbb{E} \to \mathbb{B}$ denote the fibration $p \times \mathrm{Id}\colon E \times I \to B \times I$, and let $\mathbb{A} = B \times \dot{I} \cup A \times I$ (where $\dot{I}$ $(= \partial I) = \{0, 1\}$). Regarding I as a CW-complex consisting of the 1-cell $(0, 1)$ and 0-cells $0, 1$, we give $B \times I$ the product CW-complex structure. Then $\mathbb{A}$ is a connected subcomplex of $\mathbb{B}$ and, further, the relative n-skeleton of $(\mathbb{B}, \mathbb{A})$ is given by

$$\begin{aligned}\bar{\mathbb{B}}^n &\equiv (B \times I)^n \cup (B \times \dot{I} \cup A \times I)\\ &= (B^{n-1} \times I \cup B^n \times \dot{I}) \cup (B \times \dot{I} \cup A \times I)\\ &= (\bar{B}^{n-1} \times I) \cup (B \times \dot{I}).\end{aligned}$$

Note that the fibre $\mathbb{p}^{-1}(b_o \times 0)$ is $F \times 0$ (which we shall identify with F), and also that $\mathbb{p}$ is an orientable fibration since $\mathbb{p}$ is the fibration induced from the orientable fibration p by the projection $B \times I \to B$.

Now, given $\{h_t\}\colon g'|\bar{E}_{n-1} \overset{*}{\simeq} g''|\bar{E}_{n-1}$, define a fibre-preserving map $G\colon \bar{\mathbb{E}}_n = (\bar{E}_{n-1} \times I) \cup (E \times \dot{I}) \to E'$ by

$$G(z, t) = \begin{cases} g'(z) & \text{if } (z, t) \in E \times 0 \\ g''(z) & \text{if } (z, t) \in E \times 1 \\ h_t(z) & \text{if } (z, t) \in \bar{E}_{n-1} \times I. \end{cases}$$

Clearly, G is extensible to $\bar{\mathbb{E}}_{n+1}$ in a fibre-preserving way if and only if $\{h_t\}$ extends to a fibre-preserving homotopy between $g'|E_n$ and $g''|E_n$. The obstruction for these are measured by the cocycle

$$c^{n+1}(G): H_{n+1}(\bar{\mathbb{B}}^{n+1}, \bar{\mathbb{B}}^{n}) \to \pi_n(E'^{(F)}, u)$$

where $u = g'|F$ $(= g'|F \times 0)$.

If $\{e_\lambda^k\}_{\lambda\in\Lambda}$ is the set of k-cells of $\bar{B}^k - \bar{B}^{k-1}$, then $\{e_\lambda^k \times (0, 1)\}_{\lambda\in\Lambda}$ is the set of (k+1)-cells of $\bar{\mathbb{B}}^{k+1} - \bar{\mathbb{B}}^{k}$ (note that (k+1)-cells of the form $e^{k+1}\times 0$ or $e^{k+1}\times 1$ are in $B\times\dot{I}$ $(\subset \mathbb{A} \subset \bar{\mathbb{B}}^k)$). Therefore, there are two canonical isomorphisms

$$\pm\sigma_k': H_k(\bar{B}^k, \bar{B}^{k-1}) \underset{\cong}{\to} H_{k+1}(\bar{\mathbb{B}}^{k+1}, \bar{\mathbb{B}}^{k}).$$

It is not difficult to show that if one σ_k (σ_k' or $-\sigma_k'$) is chosen appropriately for each k, $k = 1, 2, \cdots$, then $\sigma = \{\sigma_k\}_{k=1}^{\infty}$ is an isomorphism between the two 'chain complexes' $\{H_k(\bar{B}^k, \bar{B}^{k-1})\}_{k=1}^{\infty}$ and $\{H_{k+1}(\bar{\mathbb{B}}^{k+1}, \bar{\mathbb{B}}^{k})\}_{k=1}^{\infty}$. Then, $c^{n+1}(G)\cdot\sigma_n: H_n(\bar{B}^n, \bar{B}^{n-1}) \to H_n(E'^{(F)}, u)$ is a cocycle. We denote this cocycle by

$$\delta^n(g', g'', \{h_t\}).$$

One can easily show that if $g'|\bar{E}_{n-1} = g''|\bar{E}_{n-1}$ and $\{h_t\}$ is the constant homotopy, i.e., $h_t = g'|E_{n-1}$ for all t, then $\delta^n(g', g'', \{h_t\})$ coincides with the difference cochain $\delta^n(g', g'')$ up to a sign.

Now consider the cohomology class $[\delta^n(g', g'', \{h_t\})] \in H^n(B, A; \pi_n(E'^{(F)}, u))$ of the cocycle $\delta^n(g', g'', \{h_t\})$. The next theorem is a translation of Theorem 3.9 via the chain-isomorphism $\sigma = \{\sigma_k\}$.

<u>Theorem</u> 3.10.

(1) If $\{h_t'\} \subset L^*(\overline{E}_{n-1}, E')$ is any other homotopy from $g'|\,\overline{E}_{n-1}$ to $g''|\,\overline{E}_{n-1}$ such that $h_t'|\,\overline{E}_{n-2} = h_t|\,\overline{E}_{n-2}$ (taking $\overline{E}_{-1} = E_A$), then

$$[\delta^n(g', g'', \{h_t'\})] = [\delta^n(g', g'', \{h_t\})].$$

(2) The homotopy $\{h_t|\,\overline{E}_{n-2}\}$ extends to a fibre-preserving homotopy between $g'|\,\overline{E}_n$ and $g''|\,\overline{E}_n$ if and only if $[\delta^n(g', g'', \{h_t\})] = 0$. $\square$

E. The Naturality Property and an Application

Let $g \in L^*(\overline{E}_n, E')$ be fixed. Let J be a fibre-preserving map from E' to another fibre space E'';

$$\overline{E}_{n+1} \supset \overline{E}_n \xrightarrow{g} E' \xrightarrow{J} E''.$$

Suppose that E'' satisfies Assumption II for the map $v = J \cdot g|\,F$. Then, as is easily seen from the definition of obstruction cocycles, the following 'naturality property' holds:

$$c^{n+1}(J \cdot g) = (J_{\#})_* c^{n+1}(g), \tag{4}$$

where $(J_{\#})_*: \pi_n(E'^{(F)}, u) \to \pi_n(E''^{(F)}, v)$ is the induced homomorphism. Similarly, if $g', g'' \in L^*(\bar{E}_n, E')$ satisfy $g'|\bar{E}_{n-1} = g''|\bar{E}_{n-1}$, then

$$\delta^n(J \cdot g', J \cdot g'') = (J_{\#})_*\delta^n(g', g''). \tag{5}$$

Now, consider a fibre-preserving map K as indicated in the following diagram:

$$\begin{array}{ccccc} 'E & \xrightarrow{K} & E \supset \bar{E}_n & \xrightarrow{g} & E' \\ {\scriptstyle 'p}\downarrow & & \downarrow{\scriptstyle p} & & \\ 'B & \xrightarrow{\bar{K}} & B & & \end{array}$$

Assume that $'p$ is an orientable fibration and $'B$ is a connected CW-complex. Furthermore, assume $\bar{K}$ is a cellular map which carries a connected subcomplex $'A$ $(\subset 'B)$ into A. We define $'\bar{B}^n$, $'\bar{E}^n$ by a way similar to that of $\bar{B}^n$, $\bar{E}^n$. Then, since K_n $(= K|\,'\bar{E}_n)$ maps $'\bar{E}_n$ into $\bar{E}_n$, we have $g \cdot K_n \in L^*('\bar{E}_n, E')$. What is the relation between the two classes $[c^{n+1}(g)]$ and $[c^{n+1}(g \cdot K_n)]$? Note that these classes belong to, respectively, $H^{n+1}(B, A; \pi_n(E'^{(F)}, u))$ and $H^{n+1}('B, 'A; \pi_n(E'^{('F)}, w))$ where $u = g|F$, $w = (g \cdot K_n)|F$. We assert:

<u>Theorem</u> 3.11.

$$[c^{n+1}(g \cdot K_n)] = K_*^*[c^n(g)]$$

Here, K_*^* denotes the homomorphism from $H^{n+1}(B, A; \pi_n(E'^{(F)}, u))$ to $H^{n+1}('B, 'A; \pi_n(E'^{('F)}, w))$ which is indicated by two possible paths through the following commutative diagram:

$$\begin{array}{ccc}
H^{n+1}(B,A;\pi_n(E'^{(F)},u)) & \xrightarrow{(K^{\#})_c} & H^{n+1}(B,A;\pi_n(E'^{('F)},w)) \\
\bar{K}^* \downarrow & & \downarrow \bar{K}^* \\
H^{n+1}('B,'A;\pi_n(E'^{(F)},u)) & \xrightarrow{(K^{\#})_c} & H^{n+1}('B,'A;\pi_n(E'^{('F)},w))
\end{array}$$

(where both the $(K^{\#})_c$ are induced from $(K|\ 'F)_*^{\#}:\pi_n(E'^{(F)},u)\to \pi_n(E'^{('F)}, w))$.

<u>Proof</u>. It suffices to show that the cocycles $c^{n+1}(g \cdot K_n)$ and $c^n(g)$ are related in such a way that makes the following diagram commute:

$$\begin{array}{ccc}
H_{n+1}('\bar{B}^{n+1}, '\bar{B}^n) & \xrightarrow{c^{n+1}(g \cdot K_n)} & \pi_n(E'^{('F)}, w) \\
\bar{K}_* \downarrow & & \uparrow (K|\ 'F)_*^{\#} \\
H_{n+1}(\bar{B}^{n+1}, \bar{B}^n) & \xrightarrow{c^{n+1}(g)} & \pi_n(E'^{(F)}, u).
\end{array} \qquad (6)$$

The commutativity of this diagram can be easily verified by recalling the definition of $c^{n+1}(\ )$ and by examining the following commutative diagram:

$$
\begin{tikzcd}[column sep=small]
 & & & \pi_{n+1}('\overline{E}^{('F)}_{n+1},\ '\overline{E}^{('F)}_{n}) \arrow[rr, "\partial"] \arrow[dd, "\cong"'] \arrow[dr, "(K_{\#})_*"] & & \pi_n('\overline{E}^{('F)}_{n}) \arrow[r, "(g\cdot K_n"] \arrow[dr, "(K_{\#})_*"] & {} \\
 & & & & \pi_{n+1}(\overline{E}^{('F)}_{n+1},\ \overline{E}^{('F)}_{n}) \arrow[rr, "\partial"] \arrow[dd, "\cong"] & & \pi_n(\overline{E} \\
 & H_{n+1}('\overline{B}^{n+1},\ '\overline{B}^{n}) \arrow[dr, "\overline{K}_*"'] & & \pi_{n+1}('\overline{B}^{n+1},\ '\overline{B}^{n}) \arrow[ll, "h"'] \arrow[dr, "\overline{K}_*"'] & & \pi_{n+1}(\overline{E}^{(F)}_{n+1},\ \overline{E}^{(F)}_{n}) \arrow[ul, "(K^{\#})_*"'] \arrow[dd, "\cong"] \arrow[r] & {} \\
 & & H_{n+1}(\overline{B}^{n+1},\ \overline{B}^{n}) \arrow[dr, equal] & & \pi_{n+1}(\overline{B}^{n+1},\overline{B}^{n}) \arrow[ll, "h"'] \arrow[dr, equal] & & \\
 & & & H_{n+1}(\overline{B}^{n+1},\ \overline{B}^{n}) & & \pi_{n+1}(\overline{B}^{n+1},\ \overline{B}^{n}) \arrow[ll, "h"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & \pi_{n+1}({}'\overline{E}^{({}'F)}_{n+1}, {}'\overline{E}^{({}'F)}_{n}) \arrow[rr, "\partial"] \arrow[dr, "(K_{\#})_{*}"] \arrow[dd, "\cong"'] & & \pi_{n}({}'\overline{E}^{({}'F)}_{n}) \arrow[rr, "(g\cdot K_{n})_{\#*}"] \arrow[dr, "(K_{\#})_{*}"] & & \pi_{n}(E'({}'F)) \arrow[dr, equals] & & \\
 & & \pi_{n+1}(\overline{E}^{({}'F)}_{n+1}, \overline{E}^{({}'F)}_{n}) \arrow[rr, "\partial"] \arrow[dd, "\cong"] & & \pi_{n}(\overline{E}^{({}'F)}_{n}) \arrow[rr, "(g_{\#})_{*}"] & & \pi_{n}(E'({}'F)) & \\
{}^{n}) & \pi_{n+1}({}'\overline{B}^{n+1}, {}'\overline{B}^{n}) \arrow[l, "h"'] \arrow[dr, "\overline{K}_{*}"'] & & \pi_{n+1}(\overline{E}^{(F)}_{n+1}, \overline{E}^{(F)}_{n}) \arrow[rr, "\partial"] \arrow[ul, "(K^{\#})_{*}"'] \arrow[dd, "\cong"] & & \pi_{n}(\overline{E}^{(F)}_{n}) \arrow[rr, "(g_{\#})_{*}"] \arrow[ul, "(K^{\#})_{*}"'] & & \pi_{n}(E'(F)) \arrow[ul, "(K^{\#})_{*}"'] \\
{}_{1}(\overline{B}^{n+1}, \overline{B}^{n}) \arrow[dr, equals] & & \pi_{n+1}(\overline{B}^{n+1}, \overline{B}^{n}) \arrow[ll, "h"'] \arrow[dr, equals] & & & & & \\
 & H_{n+1}(\overline{B}^{n+1}, \overline{B}^{n}) & & \pi_{n+1}(\overline{B}^{n+1}, \overline{B}^{n}) \arrow[ll, "h"'] & & & &
\end{tikzcd}
$$

where the notation $(K^{\#})_*$ is used instead of $(K|\ 'F)^{\#}_*$. □

The preceding diagram contains, as a part, the following commutative diagram:

$$\begin{tikzcd} \pi_{n+1}('\bar{E}^{('F)}_{n+1}, '\bar{E}^{('F)}_{n}) \arrow[r, "\Gamma(g\cdot K_n)"] \arrow[d, "\mathcal{K}"'] & \pi_n(E'^{('F)}) \\ \pi_{n+1}(\bar{E}^{(F)}_{n+1}, \bar{E}^{(F)}_{n}) \arrow[r, "\Gamma(g)"] & \pi_n(E'^{(F)}) \arrow[u, "(K|\ 'F)^{\#}_*"'] \end{tikzcd} \qquad (7)$$

where $\mathcal{K} = (K^{\#})^{-1}_* \cdot (K_{\#})_*$. Note that for this diagram, Assumptions I and II are not necessary.

We give an application of Diagram 7.

<u>Theorem</u> 3.12. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be an orientable fibration where F is path-connected. Suppose that the evaluation map $w: F^F \to F$ induces the trivial homomorphism $w_*: \pi_1(F^F, Id) \to \pi_1(F, *)$. Then,

(1) $i_*: \pi_1(F) \to \pi_1(E)$ is a monomorphism onto a direct summand of $\pi_1(E)$. Hence, $\pi_1(E) \cong \pi_1(B) \times \pi_1(F)$.

(2) For any abelian group G,
$i_*: H_1(F, G) \to H_1(E, G)$ is monomorphic, and
$i^*: H^1(E, G) \to H^1(F, G)$ is epimorphic.

<u>Remark</u>. The assumption on w_* is satisfied if F has the homotopy type of a compact polyhedron with nonzero Euler-Poincare number (see [16], Th. IV.1).

Proof. Without loss of generality, we can assume B to be a CW-complex because there exists a CW-complex $|B|$ and a weak homotopy equivalence $\varepsilon: |B| \to B$ and so, if necessary, we can replace the original problems by the ones on the fibration induced by ε.

Now, take a CW-complex pair $(|E|, |F|)$ and a weak homotopy equivalence $k: (|E|, |F|) \to (E, F)$ (for the existence of such a k, see [30], p. 420 G). Consider the map $pk: (|E|, |F|) \to (B, b_o)$ and approximate it by a cellular map $\bar{K}: (|E|, |F|) \to (B, b_o)$ such that $pk \simeq \bar{K}$. By the covering homotopy property, we find a map

$$K: (|E|, |F|) \to (E, F)$$

such that $K \simeq k$ and $pK = \bar{K}$.

To prove the required properties of the inclusion map $i: F \to E$, it suffices to show that the inclusion map $i': |F| \to |E|$ satisfies these properties (since K is a weak homotopy equivalence). Consider the following diagram:

$$\begin{tikzcd}
{|E|} \arrow[r, "K"] \arrow[d, equal] & E \supset F \arrow[r, "\mathrm{Id}"] \arrow[d, "p"] & F \arrow[d] \\
{|E|} \arrow[r, "\bar{K}"] & B & *
\end{tikzcd}$$

Since p is orientable, $\mathrm{Id}: F \to F$ has a 1-extension $g: E_1 \to F$. Denoting the n-skelton of $|E|$ by $|E|^n$, let

$$\overline{|E|^n} = |E|^n \cup |F| \quad \text{and} \quad K_n = K\Big| \ \overline{|E|^n}.$$

Then $g \cdot K_1 \colon \overline{|E|^1} \to F$ is an extension of $K\Big| \ |F| \colon |F| \to F$. Now, by Diagram 7 (on page 68), the obstruction $\Gamma(g \ K_1)$ factors through

$$(K| \ 'F)^{\#}_{*} \colon \pi_1(F^F, \mathrm{Id}) \to \pi_1(F^{'F}, *).$$

Since the fibre $'F$ of $'p\colon |E| \to |E|$ is a single point, we see that the map $(K| \ 'F)^{\#}$ is essentially the evaluation map $w\colon F^F \to F$. Therefore, by assumption, $\Gamma(g \cdot K_1) = 0$ and so there is an extension $f\colon |E|^2 \cup |F| \to F$ of the weak homotopy equivalence $K\colon |F| \to F$. Then, the commutative diagram

$$\begin{tikzcd}
\pi_1(|F|) \arrow[r, "i'_*"] \arrow[d, "K_*"', "\cong"] & \pi_1(|E|^2 \cup |F|) = \pi_1(|E|) \arrow[dl, "f_*"] \\
\pi_1(F) &
\end{tikzcd}$$

shows that there is a homomorphism $R\colon \pi_1(|E|) \to \pi_1(|F|)$ such that $R \cdot i'_* = \mathrm{Id}$. This proves assertion (1). If we consider the (co)homology diagram instead of the above homotopy diagram, we obtain assertion (2). $\square$

We make explicit the following result which was obtained in the process of the preceding proof.

<u>Corollary</u> 3.13. Let $F \to E \xrightarrow{p} B$ be an orientable fibration. Suppose that B and E are CW-complexes and F is

a connected finite subcomplex of E. If the Euler-Poincare number of F is nonzero, then there exists a retraction $E^2 \cup F \to F$ where E^2 is the 2-skelton of E. □

The following result on the even dimensional real projective space RP^{2n} is a consequence of Theorem 3.12(2). It was proved recently in [3] by a different method.

Corollary 3.14. Let $RP^{2n} \xrightarrow{i} E \xrightarrow{p} B$ (B path-connected) be a fibration. Then,

(1) $i^*: H^*(E, Z_2) \to H^*(RP^{2n}, Z_2)$ is epimorphic.

(2) $p^*: H^*(B, Z_2) \to H^*(E, Z_2)$ is monomorphic.

(3) $H^*(E, Z_2) \cong H^*(B, Z_2) \otimes H^*(RP^{2n}, Z_2)$ as Z_2-vector spaces.

Proof. We use two facts. First, since $\chi(RP^{2n}) \neq 0$, the homomorphism $w_*: \pi_1(RP^{2n^{RP^{2n}}}, Id) \to \pi_1(RP^{2n}, *)$ is trivial (see [16], Th. 4.1). Second, p is orientable; in fact, any self homotopy equivalence of RP^{2n} is homotopic to the identity map, i.e., $\varepsilon(RP^{2n}) = 0$ (see [3]). Then, by Theorem 3.12(2), $i^*: H^1(E, Z_2) \to H^1(RP^{2n}, Z_2)$ is onto. But, the cohomology ring $H^*(RP^{2n}, Z_2)$ is generated by $H^1(RP^{2n}, Z_2)$. So, (1) follows. (2) and (3) are consequences of (1) (see [4], p. 42). □

IV. APPLICATIONS

Spaces in this chapter are still assumed to be k-spaces. However, most of the results which will be obtained are true without this assumption because of the reason mentioned in Chapter I (see Propositions 1.6 and 1.7).

A. Preliminaries

In the preceding chapter, we expressed the obstruction to extending a fibre-preserving map by a cocycle (or a cohomology class) with coefficients in the group $\pi_n(E'^{(F)},u)$. In order to apply the theory established there, we need information about $\pi_n(E'^{(F)}, u)$. This preliminary section proposes to provide it.

The following notations will be used:

$L_o(X, Y)$ = the set of based maps $X \to Y$,

$[X, Y]_o$ = the set of (based) homotopy classes of based maps $X \to Y$, and

${}^XY$ = the subspace of Y^X consisting of based maps $X \to Y$.

For the topology of Y^X, see Definition 1.2.

<u>Proposition</u> 4.1. For a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ and a space X, let $w\colon E^{(X)} \to E$ denote the evaluation map (i.e., $w(g) = g(x_o)$). Then, for an arbitrary $u \in {}^XF$

$$w_*: \pi_n(E^{(X)}, {}^X F, u) \to \pi_n(E, *)$$

is an isomorphism for $n \geq 2$ and a monomorphism for $n = 1$.

Proof. It is easy to see that $w: F^X \to F$ is a fibration (in CG) with fibre ${}^X F$. (The proof is similar to that of Proposition 1.12 . Compare with [22], p. 83.) So we have

$$w_*: \pi_n(F^X, {}^X F, u) \cong \pi_n(F, *) \quad \text{for all} \quad n \geq 1.$$

On the other hand, by Proposition 1.12, we have the following commutative diagram

$$\begin{tikzcd}[column sep=small] & \pi_n(E^{(X)}, F^X, u) \arrow[dl, "w_*"'] \arrow[dr, "\cong\ \bar{p}_*"] & \\ \pi_n(E, F, *) \arrow[rr, "p_*", "\cong"'] & & \pi_n(B, b_o) \end{tikzcd}$$

whence

$$w_*: \pi_n(E^{(X)}, F^X, u) \cong \pi_n(E, F, *) \quad \text{for all} \quad n \geq 1.$$

Then the assertion of the proposition follows from the five-lemma applied to the following commutative ladder:

$$\begin{tikzcd}[column sep=small] \pi_{n+1}(E^{(X)}, F^X) \arrow[r] \arrow[d, "w_*"', "\cong"] & \pi_n(F^X, {}^X F) \arrow[r] \arrow[d, "w_*"', "\cong"] & \pi_n(E^{(X)}, {}^X F) \arrow[r] \arrow[d, dashed, "w_*"'] & \pi_n(E^{(X)}, F^X) \arrow[r] \arrow[d, "w_*"', "\cong"] & \pi_{n-1}(F^X, {}^X F) \arrow[d, "w_*"', "(\cong)"] \\ \pi_{n+1}(E, F) \arrow[r] & \pi_n(F, *) \arrow[r] & \pi_n(E, *) \arrow[r] & \pi_n(E, F) \arrow[r] & \pi_{n-1}(F, *) \end{tikzcd}$$

where the upper and lower rows are the exact sequence associated to $(E^{(X)}, F^X, {}^XF)$ and $(E, F, *)$, respectively. $\square$

Corollary 4.2. If E is n-connected, then

$$(i_{\#})_*: \pi_k({}^XF, u) \to \pi_k(E^{(X)}, u)$$

is an isomorphism for $1 \leq k < n$ and an epimorphism for $1 \leq k \leq n$. $\square$

A pointed space (Y, y_o) is said to be m-coconnected if $\pi_k(Y, y_o) = 0$ for all $k \geq m$.

Proposition 4.3. Suppose (X, x_o) has the homotopy type of a CW-complex. If (X, x_o) is (n-1)-connected, $n \geq 1$, and (Y, y_o) is m-coconnected, then $({}^XY, \theta)$ is (m-n)-coconnected where θ denotes the constant map. In addition, if Y is an H-space which is homotopy-associative and homotopy-invertible, then $({}^XY, u)$ is (m-n)-coconnected for any $u \in {}^XY$.

Proof. If $\varepsilon: (X, x_o) \to (X', x_o')$ is a (based) homotopy equivalence, then $(\varepsilon^{\#})_*: \pi_k({}^{X'}Y, \theta) \cong \pi_k({}^XY, \theta)$. So, without loss of generality, we may assume that X is a CW-complex. Now,

$$\pi_k({}^XY, \theta) = [S^k, {}^XY]_o \cong [S^k \wedge X, Y]_o$$

where $S^k \wedge X = (S^k \times X)/(S^k \times x_o \vee * \times X)$ (smash product). Suppose $k \geq m - n$. $S^k \wedge X$ is (n-1+k)-connected and, thus, is (m-1)-connected. Then,

$$H^i(S^k \wedge X, *; \pi_i(Y)) = 0 \quad \text{for all} \quad i \geq 0.$$

This fact implies, by the (classical) obstruction theory, that $[S^k \wedge X, Y]_o$ contains only the trivial homotopy class. Therefore,

$$\pi_k({}^X Y, \theta) = 0 \quad \text{if} \quad k \geq m - n.$$

If Y is an H-space with the stated property, so is ${}^X Y$. Then the assertion follows from the fact that all path-components of a homotopy-associative and homotopy-invertible H-space have the same homotopy type (see [10], p. 387). □

Proposition 4.4. Suppose X has the homotopy type of a connected CW-complex.

(1) Let $j\colon {}^X Y \to Y^X$ be the inclusion map, and let $\hat{s}\colon Y \to Y^X$ be the map given by $s(y)(x) = y$. Then

$$\hat{s}_* + j_*\colon \pi_k(Y, y_o) \oplus \pi_k({}^X Y, \theta) \cong \pi_k(Y^X, \theta).$$

(2) For an n-dimensional sphere S^n,

$$\pi_k(Y^{S^n}, \theta) \cong \pi_k(Y, y_o) \oplus \pi_{k+n}(Y, y_o).$$

(3) (Thom [36], Federer [13]). For an Eilenberg-Maclane space $K(\pi, n)$, π abelian, we have

$$\pi_k(K(\pi, n)^X, u) \cong H^{n-k}(X, \pi) \quad (u \in K(\pi, n)^X)$$

<u>Proof</u>.

(1) The evaluation map $w: Y^X \to Y$ is a fibration with fibre ${}^XY$, and $\hat{s}$ is a cross section for w. This proves (1).

(2) By (1),

$$\pi_k(Y^{S^n}, \theta) \cong \pi_k(Y, y_o) \oplus \pi_k({}^{S^n}Y, \theta).$$

But,

$$\pi_k({}^{S^n}Y, \theta) = [S^k, {}^{S^n}Y]_o \cong [S^k \wedge S^n, Y]_o$$

$$\cong [S^{k+n}, Y]_o = \pi_{k+n}(Y, y_o).$$

(3) Noting that $K(\pi, n)^X$ is an H-space, we have

$$\pi_k(K(\pi,n)^X,u) \cong \pi_k(K(\pi,n)^X,\theta)$$

$$\cong \pi_k(K(\pi,n),*) \oplus \pi_k({}^XK(\pi,n),\theta).$$

Then the following group-isomorphisms prove the assertion:

$$\pi_k({}^XK(\pi,n),\theta) \cong [S^k, {}^XK(\pi,n)]_o \cong [S^k \wedge X, K(\pi,n)]_o$$

$$\cong \tilde{H}^n(S^k \wedge X, \pi) \cong \tilde{H}^{n-k}(X,\pi)$$

(where $\tilde{H}$ is the reduced cohomology),

$$\pi_k(K(\pi, n), *) = \begin{cases} \pi, & n - k = 0 \\ 0, & n - k \neq 0 \end{cases}$$

$$= H^{n-k}(*, \pi), \text{ and}$$

$$H^{n-k}(*, \pi) \oplus \tilde{H}^{n-k}(X, \pi) = H^{n-k}(X, \pi). \quad \square$$

B. Induced Principal Fibrations

The path fibration $\pi: PY \to Y$ over a space Y is defined by $PY = \{u \in Y^I | \ u(0) = y_o\}$ and $\pi(u) = u(1)$. Clearly, the fibre over y_o is the loop space ΩY ($= \{u \in Y^I | \ u(0) = u(1) = y_o\}$). A fibration $p: E \to B$ is called an <u>induced principal fibration</u> if it is induced from some path fibration $\pi: PY \to Y$ by some map $f: B \to Y$. It is classical that if $p: E \to B$ is a fibration such that B is a 1-connected CW-complex and the fibre is an Eilenberg-Maclane space $K(\pi, n)$, π abelian, then p is fibre homotopy equivalent to an induced principal fibration.

The following theorem is a generalization of the above fact and has been proved by Ganea [14], Hilton [21], Nomura [26], Meyer [24], and Allaud [1] by different methods. We shall prove it as an application of the generalized Eilenberg Theorem (Theorem 3.9).

<u>Theorem</u> 4.5. Let $F \to E \xrightarrow{p} B$ be a fibration such that B is an n-connected CW-complex, $n \geq 1$, and F is path-connected and has at most n nonvanishing homotopy groups

in consecutive dimensions (namely, for some $q \geq 1$, $\pi_k(F) = 0$ unless $q \leq k < q + n$). Furthermore, assume that there is a homotopy equivalence

$$u\colon F \to \Omega Y$$

where Y is a space of the homotopy type of a CW-complex. Then p is fibre homotopy equivalent to an induced principal fibration.

<u>Proof</u>. Consider the fibre-preserving extension problem indicated by the following diagram

$$\begin{array}{ccc} E \supset F & \xrightarrow{u} & PY \\ p\downarrow & & \downarrow\pi \\ B & & Y \end{array}$$

where $\pi\colon PY \to Y$ is the path fibration over Y (with fibre ΩY). If u proves to have an extension $g \in L^*(E, PY)$ then it easily follows from Dold's criterion (Theorem 1.9) that p is fibre homotopy equivalent to the fibration induced by $\bar{g}\colon B \to Y$.

To show the ∞-extensibility of u, we will first check Assumptions I and II (see the beginning of Chapter III). The orientability assumption is obviously satisfied because $\pi_1(B) = 0$. Assumption II is guaranteed by Corollary 1.18 (note that Y is simply connected since ΩY is path-connected). Now, by Corollary 4.2,

$$\pi_k(PY^{(F)}, u) \cong \pi_k({}^F\Omega Y, u) \quad \text{for all} \quad k \geq 1.$$

But, by Proposition 4.3, $({}^F\Omega Y, u)$ is n-coconnected (since both the F and ΩY are $(q-1)$-connected and $(q+n)$-coconnected, and have the homotopy type of a CW-complex by Milnor's Theorem [25]). Therefore,

$$H^{k+1}(B, b_o; \pi_k(PY^{(F)}, u)) = 0 \quad \text{for all} \quad k \geq 1.$$

Hence, by Theorem 3.9, u is ∞-extensible. $\square$

A space X is said to be <u>aspherical</u> if $\pi_k(X, x_o) = 0$ except for $k = 1$.

<u>Theorem</u> 4.6. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration such that B, E, F have the homotopy type of aspherical CW-complexes. Then the following statements are equivalent:

(1) $i_*\pi_1(F)$ is contained in the center of $\pi_1(E)$,

(2) $\pi_1(F)$ is abelian and p is an orientable fibration, and

(3) p is fibre homotopy equivalent to an induced principal fibration.

<u>Remark</u>. That (1) implies (3) was proved by Hilton and by Ganea (see [15], p. 452). The proof given below differs from theirs.

<u>Proof</u>. (1)$\Longrightarrow$(2). If $i_*\pi_1(F)$ lies in the center

of $\pi_1(E)$, then the function $\phi\colon \pi_1(E \times F) = \pi_1(E) \oplus \pi_1(F) \to \pi_1(E)$ given by

$$\phi(\alpha, \beta) = \alpha + i_*(\beta) \qquad ((\alpha, \beta) \in \pi_1(E) \oplus \pi_1(F))$$

is a homomorphism. Therefore, since E is aspherical, there is an extension $m\colon E \times F \to E$ of $Id \vee i\colon E \vee F \to E$ such that $m_* = \phi$. Obviously, the next diagram (leftmost) commutes.

$$\begin{tikzcd}
\pi_1(E \times F) \arrow[rr, "\phi = m_*"] \arrow[dr, "(p\times 0)_*"'] & & \pi_1(E) \arrow[dl, "p_*"] \\
& \pi_1(B) &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times F \arrow[rr, "m"] \arrow[dr, "p\times 0"'] & & E \arrow[dl, "p"] \\
& B &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[rr, "\hat{m}"] \arrow[dr, "p"'] & & E^{(F)} \arrow[dl, "\overline{p}"] \\
& B &
\end{tikzcd}$$

Then, since B is aspherical, the middle diagram commutes in the homotopy sense. Since p is a fibration, we may assume that it commutes strictly and satisfies $m| e_o \times F = i$. Then we have the commutative 'adjoint diagram' (rightmost). Since $p_*\colon \pi_1(E, e_o) \to \pi_1(B, b_o)$ is onto, $\overline{p}_*\colon \pi_1(E^{(F)}, i)$ $\pi_1(B, b_o)$ must be onto and, hence, p is orientable (see Proposition 1.16).

(2) $\Longrightarrow$ (3). We solve the following extension problem:

$$\begin{tikzcd}
F \arrow[r, "u"] \arrow[d, "i"'] & \Omega Y \arrow[d] \\
E \arrow[r, dashed] \arrow[d, "p"'] & PY \arrow[d, "\pi"] \\
B \arrow[r, dashed] & Y
\end{tikzcd}$$

where π is the path fibration over $Y = K(\pi_1(F), 2)$ and u is a homotopy equivalence. We may assume that B is a CW-complex. Since p is orientable, u has a fibre-preserving extension $E_1 \to PY$. Now, for $k \geq 1$, we have

$$\pi_k(PY^{(F)}, u) \cong \pi_k({}^F\Omega Y, u) \qquad \text{(by Corollary 4.2)},$$

$$= 0 \qquad \text{(by Proposition 4.3)}.$$

Hence, u is ∞-extensible. This proves (3).

(3) $\Longrightarrow$ (1). To prove this, we may assume that $F \to E \to B$ is the fibration induced from a path fibration $\pi\colon PY \to Y$ by a (based) map $f\colon B \to Y$. So, by definition,

$$E = \{(b, \lambda) \in B \times PY |\ f(b) = \lambda(1)\},$$

$$F = \{(b_o, \mu) \mid \mu \in \Omega Y\} = b_o \times \Omega Y.$$

Define $m\colon E \times F \to E$ by

$$m((b, \lambda), (b_o, \mu)) = (b, \lambda * \mu)$$

where $\lambda * \mu$ is the path in Y given by

$$(\lambda*\mu)(t) = \begin{cases} \mu(2t), & 0 \leq t \leq 1/2 \\ \lambda(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

It is easy to see that for the natural inclusions $j_1\colon E \subset E \times F$ and $j_2\colon F \subset E \times F$, we have

$$mj_1 \simeq \mathrm{Id}\colon E \to E \ (\text{rel } e_o),$$

$$mj_2 \simeq i\colon F \to E \ (\mathrm{rel}\ e_o).$$

Then, the homomorphism $m_*\colon \pi_1(E) \oplus \pi_1(F) = \pi_1(E \times F) \to \pi_1(E)$ satisfies $m_*(\alpha, 0) = \alpha$ and $m_*(0, \beta) = i_*(\beta)$. Since $(\alpha, 0) \cdot (0, \beta) = (0, \beta) \cdot (\alpha, 0)$ in $\pi_1(E) \oplus \pi_1(F)$, α and $i_*(\beta)$ must commute. This proves (1). □

A short exact sequence of groups (not necessarily abelian)

$$0 \to H \xrightarrow{j} G \xrightarrow{q} \pi \to 0$$

is called a <u>group extension of H by π</u>. It is called a <u>central group extension</u> if jH is in the center of G. Two group extensions

$$0 \to H \to G_k \to \pi \to 0 \qquad (k = 1, 2)$$

are said to be equivalent if there exists an isomorphism $\theta\colon G_1 \to G_2$ such that

$$\begin{array}{ccccccccc} 0 & \to & H & \to & G_1 & \to & \pi & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\theta} & & \| & & \\ 0 & \to & H & \to & G_2 & \to & \pi & \to & 0 \end{array}$$

commutes. We give a topological proof to the following well-known result.

<u>Theorem</u> 4.7. Let H be an abelian group and π a group. Then the equivalence classes of central group extensions of H by π are in one-to-one correspondence with

elements of $H^2(K(\pi, 1), H)$.

Proof. Choose Eilenberg-Maclane spaces $K(H, 2)$ and $K(\pi, 1)$, and identify $\pi_1(\Omega K(H, 2))$ with H, $\pi_1(K(\pi, 1))$ with π and $H^2(K(\pi, 1), H)$ with $[K(\pi, 1), K(H, 2)]_o$. Let $\alpha \in H^2(K(\pi, 1), H)$ be represented by a based map $f: K(\pi, 1) \to K(H, 2)$. Then f induces a principal fibration

$$(\mathcal{F}_f) \qquad \Omega K(H, 2) \to E_f \to K(\pi, 1)$$

from the path fibration $\Omega K(H, 2) \to PK(H, 2) \to K(H, 2)$, where the fibre of $(\mathcal{F}_f)$ is canonically homeomorphic to $\Omega K(H, 2)$ and so we shall identify them. The 1-dimensional part of homotopy exact sequence of $(\mathcal{F}_f)$

$$(\mathcal{E}_f) \qquad 0 \to H \to \pi_1(E_f) \to \pi \to 0$$

is a central group extension of H by π (see the preceding theorem). Thus, it is sufficient to show the following two facts:

(1) if $f \simeq g: K(\pi, 1) \to K(H, 2)$ (rel $*$), then $(\mathcal{E}_f)$ and $(\mathcal{E}_g)$ are equivalent group extensions; and

(2) any central group extension $0 \to H \xrightarrow{j} G \xrightarrow{q} \pi \to 0$ is equivalent to $(\mathcal{E}_f)$ for some f, and the homotopy class of f is uniquely determined by the group extension.

Because (1) is easy, we prove only (2). First, find a fibration $F \xrightarrow{i} E \xrightarrow{p} K(\pi, 1)$ with aspherical F, E so that

$$\begin{tikzcd}
\pi_1(F) \arrow[r, "i_*"] & \pi_1(E) \arrow[r, "p_*"] & \pi_1(K(\pi, 1)) \\
H \arrow[u, "\tau", "\cong"'] \arrow[r, "j"] & G \arrow[u, "\theta", "\cong"'] \arrow[r, "q"] & \pi \arrow[u, equal]
\end{tikzcd}$$

commutes where τ, θ are some isomorphisms. Then, choose a homotopy equivalence $u\colon F \to \Omega K(H, 2)$ so that $u_* = \tau^{-1}\colon \pi_1(F) \to H = \pi_1(\Omega K(H, 2))$ (this is possible since $\Omega K(H, 2)$ is aspherical). Since $i_*\pi_1(F)$ is contained in the center of $\pi_1(E)$, we can construct the following commutative diagram (see the proof of Theorem 4.6):

$$\begin{tikzcd}
F \arrow[r, "u"] \arrow[d, "i"] & \Omega K(H, 2) \arrow[d] \\
E \arrow[r, "h"] \arrow[d, "p"] & PK(H, 2) \arrow[d] \\
K(\pi, 1) \arrow[r, "f"] & K(H, 2).
\end{tikzcd}$$

Then it is easy to see that $0 \to H \xrightarrow{j} G \xrightarrow{q} \pi \to 0$ is equivalent to $(\mathcal{E}_f)$. Finally, to show the uniqueness of the homotopy class of f, suppose that $(\mathcal{E}_f)$ and $(\mathcal{E}_g)$ are equivalent group extensions. Again, by the property of aspherical spaces, we can construct a homotopy-commutative diagram

$$\begin{array}{ccc} \Omega K(H,\ 2) & \xrightarrow{\theta'} & \Omega K(H,\ 2) \\ \downarrow & & \downarrow \\ E_f & \xrightarrow{\tilde{\theta}} & E_g \\ \downarrow & & \downarrow \\ K(\pi,\ 1) & \xrightarrow{Id} & K(\pi,\ 1) \end{array}$$

so that $\theta' \simeq Id$. Then, because of the covering homotopy property, we may assume that this diagram is strictly commutative. Recall that E_f and E_g are induced respectively by f and g, and let $\tilde{f}\colon E_f \to PK(H,\ 2)$ and $\tilde{g}\colon Eg \to PK(H,2)$ denote the canonical fibre-preserving maps. Thus, we have two fibre-preserving maps from E_f to $PK(H,\ 2)$;

$$\tilde{f},\ \tilde{g}\cdot\tilde{\theta}\colon E_f \to PK(H,\ 2).$$

Since $\tilde{\theta}|\ \Omega K(H,\ 2) = \theta' \simeq Id$, we see

$$\tilde{f}|\ \Omega K(H,\ 2) \overset{*}{\simeq} \tilde{g}\theta'\colon \Omega K(H,\ 2) \to PK(H,\ 2).$$

But, since

$$\pi_m(PK(H,\ 2)^{(\Omega K(H,2))}) \simeq \pi_m({}^{\Omega K(H,2)}\Omega K(H,\ 2)) = 0$$

for $m \geq 1$, there is no obstruction to extending the above fibre-preserving homotopy to

$$\tilde{f} \overset{*}{\simeq} \tilde{g}\tilde{\theta}\colon E_f \to PK(H,\ 2).$$

Therefore, it follows that

$$f \simeq g \cdot \mathrm{Id} = g\colon K(\pi, 1) \to K(H, 2) \ (\mathrm{rel}\ *).$$

This completes the proof of Theorem 4.7. $\square$

C. Homotopy Classes of Fibre-preserving Maps

Let $p\colon E \to B$ and $p'\colon E' \to B'$ be fibrations, and let $A \subset B$;

$$\begin{array}{ccc} (E, E_A) & \qquad & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ (B, A) & & B' \end{array}$$

For a given $f \in L^*(E_A, E')$, we will use the following notations:

$$L_f^*(E, E') = \{g \in L^*(E, E') \mid g| E_A = f\},$$

$$[E, E']_f^* = L_f^*(E, E') / \overset{*}{\simeq} \text{ rel } E_A.$$

Of course, $L_f^*(E, E')$ and $[E, E']_f^*$ may be empty. If $E = B$ (i.e., $p = \mathrm{Id}$) and $A = \{b_o\}$, then $L_f^*(E, E')$ and $[E, E']_f^*$ are nothing but the sets $L_o(E, E')$ and $[E, E']_o$, respectively.

Let $J\colon E' \to E''$ be a fibre-preserving map from E' to another fibre space E'';

$$\begin{array}{ccccc} (E, E_A) & \qquad & E' & \xrightarrow{\ J\ } & E'' \\ {\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p''} \\ (B, A) & & B' & \xrightarrow{\ \bar{J}\ } & B'' . \end{array}$$

Then J induces, in the obvious way, the functions

$$J_{\#}: L_f^*(E, E') \to L_{Jf}^*(E, E''), \text{ and}$$

$$J_{\square}: [E, E']_f^* \to [E, E'']_{Jf}.$$

Next, let K be a fibre-preserving map as indicated in the following diagram:

$$\begin{tikzcd}
('E, 'E_{'A}) \arrow[r, "K"] \arrow[d, "'p"'] & (E, E_A) \arrow[d, "p"] & E' \arrow[d, "p'"] \\
('B, 'A) \arrow[r, "\overline{K}"] & (B, A) & B'
\end{tikzcd} \qquad (\#)$$

where $'p$ is a fibration, $'A \subset 'B$, and $\overline{K}('A) \subset A$. Then K induces the functions

$$K^{\#}: L_f^*(E, E') \to L_{f'}^*('E, E'), \text{ and}$$

$$K^{\square}: [E, E']_f^* \to ['E, E']_{f'}^*,$$

where $f' = f \cdot (K|\ 'E_{'A})$.

<u>Example</u>.

(1) Consider the following special case of diagram (#):

$$\begin{tikzcd}
('B, 'b_o) \arrow[r, dash, "K"] \arrow[d, "Id"'] & (B, b_o) \arrow[d, "Id"] & X \arrow[d] \\
('B, 'b_o) \arrow[r, dash, "\overline{K}=K"] & (B, b_o) & *
\end{tikzcd}$$

Then $K^{\square}$ is the same as $K^*: [B, X]_o \to ['B, X]_o$.

(2) For a fibration $F \to E \xrightarrow{p} B$, consider the following

diagram:

$$\begin{array}{ccccc} (E,\ F) & \xrightarrow{K=p} & (B,\ b_o) & & X \\ \downarrow & & \| & & \downarrow \\ (B,\ b_o) & \overset{\overline{K}=Id}{=\!=\!=} & (B,\ b_o) & & * \end{array}$$

In this case, $K^{\square}$ can be identified with $p^*: [B, X]_o \to [E/F, X]_o$ where E/F is the space obtained from E by shrinking F to a point.

Now go back to diagram (#). We shall assume:

(1) B and $'B$ are connected CW-complexes, and A and $'A$ are connected subcomplexes;

(2) $\overline{K}$: $('B, 'A) \to (B, A)$ is a cellular map; and

(3) $p: E \to B$ and $'p: 'E \to 'B$ are orientable fibrations.

Under these assumptions, we consider the following problem.

<u>Problem</u>. When is $K^{\square}: [E, E']^*_f \to ['E, E']^*_{f'}$ an onto function?

Given a $k \in L^*_{f'}('E, E')$, let us try to find a $g \in L^*_f(E, E')$ such that $g \cdot K \overset{*}{\simeq} k$ rel $'E_{'A}$. Suppose that for some $n \geq 1$ we can find a map

$$g^{(n)} \in L^*_f(\overline{E}_n, E')$$

and a homotopy

$$\{h_t^{(n)}\}\colon g^{(n)}\, K_{n-1} \overset{*}{\simeq} k_{n-1} \text{ (rel } {}'E_{'A})$$

where K_{n-1}, k_{n-1} are the restrictions of K, k to $'\overline{E}_{n-1}$, respectively. (This is always possible for $n = 1$. See Proposition 2.15 and Proposition 2.12.)

Lemma 4.8. Let n be as above. Suppose the following conditions are satisfied:

(α_n) $(E'^{(F)}, u)$ and $(E'^{('F)}, w)$ are n-simple (where $u = f|\ F$ and $w = f'|\ 'F$),

(β_n) $K_*^*\colon H^{n+1}(B,A;\pi_n(E'^{(F)},u)) \to H^{n+1}('B,'A;\pi_n(E'^{('F)},w))$ is monomorphic, and

(γ_n) $K_*^*\colon H^n(B,A;\pi_n(E'^{(F)},u)) \to H^n('B,'A;\pi_n(E'^{('F)},w))$ is epimorphic.

(For the definition of K_*^*, see Theorem 3.11.) Then we can find a map

$$g^{(n+1)} \in L_f^*(\overline{E}_{n+1}, E')$$

and a homotopy

$$\{h_t^{(n+1)}\}\colon g^{(n+1)} \cdot K_n \overset{*}{\simeq} k_n \text{ (rel } {}'E_{'A})$$

so that

$$g^{(n+1)}|\ \overline{E}_{n-1} = g^{(n)}|\ \overline{E}_{n-1}, \qquad (8)$$

$$h_t^{(n+1)}|\ '\overline{E}_{n-2} = h_t^{(n)}|\ '\overline{E}_{n-2}.$$

If conditions (α_m), (β_m), (γ_m) are satisfied for all $m \geq n$, then there exists a $g \in L_f^*(E, E')$ such that $g \cdot K \overset{*}{\simeq} k$ (rel $'E_{,A}$).

Proof. The last assertion can be easily seen by repeated use of the first assertion. In fact, if (α_m), (β_m), (γ_m) are satisfied for all $m \geq n$, we can find sequences $\{g^{(m)}\}_{m=n}^{\infty}$ and $\{h_t^{(m)}\}_{m=n}^{\infty}$ with adjacent terms related by Relation 8. Then we can define g and a homotopy $\{h_t\}$: $g \cdot K \overset{*}{\simeq} k$ by

$$g(x) = g^{(m)}(x) \quad \text{if} \quad x \in \bar{E}_{m-1},$$

$$h_t(x') = h_t^{(m)}(x') \quad \text{if} \quad x' \in {}'\bar{E}_{m-2}.$$

Now, let us prove the first assertion. We show first that $g_{n-1}^{(n)}$ $(= g^{(n)} | E_{n-1})$ is (n+1)-extensible. Consider the obstruction class $[c^{n+1}(g^{(n)})] \in H^{n+1}(B,A;\pi_n(E'^{(F)},u))$. By the naturality property, we have

$$K_*^*[c^{n+1}(g^{(n)})] = [c^{n+1}(g^{(n)} \cdot K_n)].$$

Since $g^{(n)} \cdot K_{n-1} \overset{*}{\simeq} k_{n-1}$ and k_{n-1} is ∞-extensible, the H*EP (Corollary 2.6) implies that $g^{(n)} \cdot K_{n-1}$ is also ∞-extensible. Hence, $[c^{n+1}(g^{(n)} K_n)] = 0$. Then, by assumption (β_n), we see that $[c^{n+1}(g^{(n)})] = 0$, and so $g_{n-1}^{(n)}$ has an (n+1)-extension, say $g'^{(n+1)}$.

Note that

$$\{h_t^{(n)}\}: g'^{(n+1)} K_{n-1} \overset{*}{\simeq} k_{n-1} \ (\text{rel } 'E_{'A}) \tag{9}$$

holds. Then, by the H*EP, there is a $k' \in L^*_{f'}('E, E')$ such that

$$g'^{(n+1)} \cdot K_{n-1} = k'_{n-1} \quad \text{and} \quad k' \overset{*}{\simeq} k \ (\text{rel } 'E_{'A}), \tag{10}$$

and the homotopy in Relation 10 is an extension of $\{h_t^{(n)}\}$ in Relation 8. Consider now the class

$$\beta = [\delta^n(g'^{(n+1)} \cdot K_n, k'_n)] \in H^n('B, 'A; \pi_n(E'^{('F)}, w)).$$

By assumption (γ_n), there is an $\alpha \in H^n(B, A; \pi_n(E'^{(F)}, u))$ such that $K^*_*(\alpha) = \beta$. Let z^n be a cocycle in the class α. Then, by Lemma 3.8, there exists a $g'' \in L^*_f(\overline{E}_n, E')$ such that

$$g''_{n-1} = g'^{(n+1)}_{n-1} \quad \text{and} \quad \delta^n(g'', g'^{(n+1)}_n) = z^n. \tag{11}$$

For our purpose, it suffices to show

(1) g'' has an (n+1)-extension $g^{(n+1)}$, and

(2) $g'' \cdot K_n \overset{*}{\simeq} k'_n$ where the homotopy can be chosen as an extension of the identical homotopy $g'' \cdot K_{n-2} = k'_{n-2}$.

(By the condition on the homotopy $k' \overset{*}{\simeq} k$ in Relation 10, (2) implies the existence of a homotopy $\{h_t^{(n+1)}\}: g^{(n+1)} \cdot K_n = g'' \cdot K_n \overset{*}{\simeq} k_n$ which is an extension of $\{h_t^{(n)}\} | '\overline{E}_{n-2}$.)

From Relation 11 and Lemma 3.7, we have:

$$0 = \partial Z^n = \partial\delta^n(g'', g_n'^{(n+1)}) = c^{n+1}(g'') - c^{n+1}(g_n'^{(n+1)}) = c^{n+1}(g'').$$

This proves (1). For (2), look at

$$\begin{aligned}[\delta^n(g''\cdot K_n, k_n')] &= [\delta^n(g''\cdot K_n, g'^{(n+1)}\ K_n)] - [\delta^n(g'^{(n+1)}\cdot K_n, k_n')] \\ &= K_*^*[\delta^n(g'', g_n'^{(n+1)})] - \beta \\ &= K_*^*(\alpha) - \beta = 0.\end{aligned}$$

This proves (2) and completes the proof of the lemma. $\square$

Now we consider the problem: when is K^{Ω} one-to-one? Let $g', g'' \in L_f^*(E, E')$ satisfy $g'K \overset{*}{\simeq} g''K$ (rel $'E_{'A}$). We try to find the condition under which $g' \overset{*}{\simeq} g''$ (rel $\dot{E}_A$) holds. Consider $K \times Id$ indicated in the following diagram:

$$\begin{tikzcd}
('E\times I,\ 'E\times\dot{I} \cup 'E_{'A}\times I) \arrow[r, "K\times Id"] \arrow[d, "'p\times Id"'] & (E\times I,\ E\times\dot{I} \cup E_A\times I) \arrow[d, "p\times Id"] & E' \arrow[d] \\
('B\times I,\ 'B\times\dot{I} \cup 'A\times I) \arrow[r, "\overline{K}\times Id"] & (B\times I,\ B\times\dot{I} \cup A\times I). & B'
\end{tikzcd}$$

We rewrite this diagram in the following notations:

$$\begin{tikzcd}
('\mathbb{E},\ \mathbb{E}_{\mathbb{A}}) \arrow[r, "\mathbb{K}"] \arrow[d, "'\mathbb{P}"'] & (\mathbb{E},\ \mathbb{E}_{\mathbb{A}}) \arrow[d, "\mathbb{P}"] & E' \arrow[d] \\
('\mathbb{B},\ '\mathbb{A}) \arrow[r] & (\mathbb{B},\ \mathbb{A}) & B'
\end{tikzcd}$$

Define $H\colon \mathbb{E}_A = (E \times \dot{I}) \cup (E_A \times I) \to E'$ by

$$H(z,0) = g'(z), \quad H(z,1) = g''(z), \quad H(w,t) = f(w) \quad (= g'(w) = g''(w))$$

for $z \in E$, $w \in E_A$, $t \in I$. We wish to know when H has an extension $G \in L^*(\mathbb{E}, E')$ or equivalently when $L^*_H(\mathbb{E}, E')$ is nonempty. For $H' = H\cdot(\mathbb{K}\,|\,'\mathbb{E}_{'\mathbb{A}})$, $L^*_{H'}('\mathbb{E}, E')$ is not empty because $g'K \overset{*}{\simeq} g''K$ (rel $'E_{'A}$). Therefore, $L^*_H(\mathbb{E}, E')$ must be nonempty if

$$\mathbb{K}^{\square}\colon [\mathbb{E}, E']^*_H \to ['\mathbb{E}, E']^*_{H'} \quad \text{is onto.}$$

So the problem reduces to the previous problem. Observe that the homomorphism

$$\mathbb{K}^*_*\colon H^{p+1}(\mathbb{B}, \mathbb{A}; \pi_q(E'^{(F)}, u)) \to H^{p+1}('\mathbb{B}, '\mathbb{A}; \pi_q(E'^{('F)}, w))$$

can be identified with

$$K^*_*\colon H^p(B,A;\ \pi_q(E'^{(F)}, u)) \to H^p('B,'A;\ \pi_q(E'^{('F)}, w))$$

(cf. Section III D). Letting K^p_q denote the latter homomorphism, we can summarize the main points as follows.

<u>Theorem</u> 4.9. Suppose $(E'^{(F)}, u)$ and $(E'^{('F)}, w)$ are m-simple for all $m \geq 1$.

(1) If, for all $m \geq 1$, K^{m+1}_m are monomorphic and K^m_m are epimorphic, then

$$K^{\square}\colon [E, E']^*_f \to ['E, E']^*_{f'} \quad \text{is onto.}$$

(2) If, for all $m \geq 1$, K^m_m are monomorphic and

K_m^{m-1} are epimorphic, then $K^{\square}$ is one-to-one. □

We apply the preceding argument to prove the following result which was previously proved by Sugawara [35] by repeated use of Serre's Theorem (see the corollary below) in the Postnikov decomposition argument.

Theorem 4.10. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration satisfying the following conditions: B is r-connected $(r \geq 1)$, F is s-connected $(s \geq 0)$, and B, E, F have the homotopy types of CW-complexes. Let X be a space which is m-simple for all $m \geq 1$. Then,

$$p^*: [B, X]_o \to [E/F, X]_o$$

is onto if X is $(r+s+2)$-coconnected; and p^* is one-to-one if X is $(r+s+3)$-coconnected.

Corollary 4.11. (Serre). Let $F \to E \xrightarrow{p} B$ be a fibration satisfying the conditions in the above theorem. Then, for any abelian group G,

$$p^*: H^m(B, b_o; B) \to H^m(E, F; G)$$

is an isomorphism for $m \leq r + s + 1$; and a monomorphism for $m \leq r + s + 2$.

The corollary follows immediately from the theorem by taking $X = K(G, m)$. □

<u>Proof</u>. Without loss of generality we may assume B to be a CW-complex such that $B^r = b_o$. Consider the diagram:

$$\begin{tikzcd}
(E, F) \arrow[r, "\underline{K = p}"] \arrow[d, "p"'] & (B, b_o) \arrow[d, equal] & X \arrow[d] \\
(B, b_o) \arrow[r, equal] & (B, b_o) & *
\end{tikzcd}$$

In this case, $K^{\square}$ is the same as $p^*: [B, X]_o \to [E/F, X]_o = [(E, F), (X, x_o)]$.

(1) Suppose X is $(r+s+2)$-coconnected. Then, since F is s-connected, $({}^F X, \theta)$ is $(r+1)$-coconnected (Proposition 4.3), and hence the evaluation map $w: X^F \to X$ induces isomorphisms $w_*: \pi_m(X^F, \theta) \cong \pi_m(X, x_o)$ for all $m \geq r + 1$. Then it follows that (X^F, θ) is m-simple for $m \geq r + 1$ since X is. Now let $k: (E, F) \to (X, x_o)$ be any map. Let $g^{(r+1)}: B^{r+1} \to X$ be the constant map. Since $E_r = F$ (because $B^r = b_o$), we have $g^{(r+1)} \cdot (K| E_r) = k| E_r$ (both are a constant map). So we can use Lemma 4.8 in the case where $n = r + 1$. As seen above, condition (α_m) in Lemma 4.8 holds for every $m \geq r + 1$, i.e., $(X^{b_o}, *)$ and (X^F, θ) are m-simple for $m \geq r + 1$. Now consider the homomorphism:

$$K_*^*: H^k(B, b_o; \pi_m(X^{b_o})) \to H^k(B, b_o; \pi_m(X^F)).$$

By definition (see Theorem 3.10),

$$K_*^* = (\bar{K})^* \cdot (K|\ F)_c^\# = (K|\ F)_c^*$$

since $\bar{K} = Id$; that means, K_*^* is induced by the coefficient homomorphism

$$(K|\ F)_*^\#: \pi_m(X^{b_o}) \to \pi_m(X^F).$$

Clearly, this homomorphism can be identified with

$$\hat{s}_*: \pi_m(X,\ x_o) \to \pi_m(X^F,\ \theta)$$

where $\hat{s}: X \to X^F$ is given by $\hat{s}(x)(y) = x$. For $m \geq r + 1$, $\hat{s}_*$ is an isomorphism. Hence, conditions (β_m), (γ_m) in Lemma 4.7 are also satisfied for $m \geq r + 1$. Therefore, $K^\square = p^*$ is onto.

(2) Suppose now X is $(r+s+3)$-coconnected. The same argument as above shows that for every $m \geq r + 2$ $(X^F,\ \theta)$ is m-simple and $\hat{s}_*: \pi_m(X,\ x_o) \to \pi_m(X^F,\ \theta)$ is isomorphic. Let $\mathbb{p}: \mathbb{E} \to \mathbb{B}$ denote the fibration $p \times Id: E \times I \to B \times I$ and let $\mathbb{A} = B \times \dot{I}\ \ b_o \times I$. Consider the diagram:

$$\begin{tikzcd}
(\mathbb{E},\ \mathbb{E}_A) \arrow[r, "\mathbb{K} = \mathbb{P}"] \arrow[d, "\mathbb{P}"'] & (\mathbb{B},\ \mathbb{A}) \arrow[d, equal] & X \arrow[d] \\
(\mathbb{B},\ \mathbb{A}) \arrow[r, equal] & (\mathbb{B},\ \mathbb{A}) & *
\end{tikzcd}$$

In order to show p^* $(= K^\square)$ is one-to-one, we show that $\mathbb{K}^{\square}\colon [\mathbb{B}, X]^*_H \to [\mathbb{E}, X]^*_{H'}$ is onto, where $H\colon \mathbb{A} \to X$ is any fixed map and $H' = H\cdot(\mathbb{K}| \ \mathbb{E}_{\mathbb{A}})$. This follows easily from the first remarks on (X^F, θ) and $\hat{s}_*$, if Lemma 4.8 proves to be applicable <u>for $n = r + 2$</u>.

Let $G'\colon \mathbb{E} \to X$ be any extension of H'. Since the assumption $B^r = b_o$ implies $\bar{\mathbb{B}}^{r+1} = \mathbb{A}$, the map $H\colon \mathbb{A} \to X$ extends to $G^{(r+1)}$ $(= H)$: $\bar{\mathbb{B}}_{r+1} \to X$. Consider the obstruction homomorphism $\Gamma(G^{(r+1)})$. It is related with $\Gamma(G'_{r+1})$ $(G'_{r+1} = G'| \ \mathbb{E}_{r+1} = H')$ in the following commutative diagram (see Diagram 7 on page 68):

$$\begin{array}{ccc}
\pi_{r+2}(\bar{\mathbb{E}}^{(F)}_{r+2}, \ \mathbb{E}^{(F)}_{\mathbb{A}}) & \xrightarrow{\Gamma(G'_{r+1})} & \pi_{r+1}(X^F) \\
\Big\downarrow \mathcal{K} & & \Big\uparrow (\mathbb{K}|F)^{\#}_{*} \\
\pi_{n+2}(\bar{\mathbb{B}}^{(*)}_{r+2}, \ \mathbb{A}^{(*)}) & \xrightarrow{\Gamma(G^{(r+1)})} & \pi_{r+1}(X^{\{*\}})
\end{array}$$

where $*$ in the bottom line denotes the fibre of $Id\colon \mathbb{B} \to \mathbb{B}$. Observe that $(\mathbb{K}| \ F)^{\#}_{*}$ is equivalent to the homomorphism induced by $\hat{s}\colon X \to X^F$ given by $\hat{s}(x)(y) = x$ and, hence, it is a monomorphism. Also, note that $\mathcal{K}$ is an isomorphism in this case. Therefore, $\Gamma(G'_{r+1}) = 0$ implies $\Gamma(G^{(r+1)}) = 0$, and so $G^{(r+1)}$ has an $(r+2)$-extension $G^{(r+2)}$. Clearly,

$$G^{(r+2)}\cdot \mathbb{K}_{r+1} = G'_{r+1} \quad (= H').$$

Hence, Lemma 4.8 is applicable for $n = r + 2$. $\square$

Remark. Generally, the simplicity of X does not imply the simplicity of X^F. Federer [13] has shown that for $X = S^2$ and $F = S^1$, X^F is not 2-simple.

Theorem 4.9 can be used to prove the following result.

Theorem 4.12. Let $F \to E \xrightarrow{p} \Sigma B$ be a fibration where ΣB is the suspension of a connected CW-complex. Let X be an H-space. Then, there is a one-to-one correspondence

$$k\colon [E/F, X]_o \cong [\Sigma B, X^F]_o$$

which preserves the multiplications induced from the H-structure of X.

Proof. Let CB be the cone over B, and let $\psi\colon CB \to \Sigma B$ be the quotient map. Since CB is contractible, the fibration induced from p by ψ is fibre homotopy equivalent to the product. By this fact, we can construct the following commutative diagram:

$$\begin{tikzcd}
F \arrow[d] & F \arrow[l, "Id"'] \arrow[r, "Id"] \arrow[d] & F \arrow[d] \\
\Sigma B\times F \arrow[d, "\pi'"'] & CB\times F \arrow[l, "K'=\psi\times Id"'] \arrow[r, "K"] \arrow[d, "\pi"'] & E \arrow[d, "P"] \\
\Sigma B & CB \arrow[l, "\psi"'] \arrow[r, "\psi"] & \Sigma B
\end{tikzcd}$$

where π and π' are the projection maps. We apply Theorem 4.9 to

$$\begin{array}{ccccc}
(\Sigma B\times F,\ *\times F) & \xleftarrow{K'} & (CB\times F,\ B\times F) & \xrightarrow{K} & (E,\ F) \\
\pi'\downarrow & & \pi\downarrow & & \downarrow \\
(\Sigma B,\ *) & \xleftarrow{\overline{K}'=\psi} & (CB,\ B) & \xrightarrow{\overline{K}=\psi} & (\Sigma B,\ *).
\end{array}$$

Since $\psi^*\colon H^m(\Sigma B,*;\ G) \to H^m(CB,B;\ G)$ is an isomorphism for any G, and the three fibrations have the same fibre, K_*^* and K'^*_* are isomorphisms in every dimension. Also note that $(X^F,\ \theta)$ is m-simple for all $m \geq 1$ since it is an H-space. Hence,

$$k' = (K'^{\square})^{-1}\, K^{\square}\colon [(E,F),\ (X,x_0)] \to [(\Sigma B\times F,\ *\times F),(X,x_0)]$$

is one-to-one and onto. Clearly, k' preserves the multiplications. Now define a bijection

$$k''\colon [(\Sigma B\times F,\ *\times F),\ (X,\ x_0)] \cong [\Sigma B,\ X^F]_0$$

by $u \to \hat{u}$ (= the adjoint of u). To see that k'' preserves the multiplications is not difficult. Finally, identifying $[(E,F),\ (X,x_0)]$ with $[E/F,\ X]_0$ and letting $k = k''\cdot k'$, we obtain the result. $\square$

The fibration ${}^FX \to X^F \xrightarrow{w} X$ (w = evaluation map) admits the cross section $\hat{s}\colon X \to X^F$ defined by $\hat{s}(x)(y) = x$.

Let $\mu\colon X^F \times X^F \to X^F$ be the multiplication induced from the multiplication on X. Then we easily find that the map $\varepsilon\colon X \times {}^F X \to X^F$ given by

$$\varepsilon(x, g) = \mu(\hat{s}(x), g)$$

induces isomorphisms of homotopy groups. Hence, by the classical obstruction theory, we have a bijection

$$\varepsilon_*\colon [\Sigma B, X \times {}^F X]_o \simeq [\Sigma B, X^F]_o.$$

Corollary 4.13. Let $F \to E \to \Sigma B$ be a fibration as in Theorem 4.12. Assume that F and E have the homotopy types of CW-complexes. Then, for any abelian group G,

$$H^m(E, F; G) \cong \tilde{H}^m(\Sigma B, G) \oplus \tilde{H}^m(F \wedge \Sigma B, G),$$

where $\tilde{H}^m$ denotes the reduced cohomology.

Proof. In the preceding theorem, take $X = K(G, m)$. Then,

$$[E/F, X]_o \cong H^m(E, F; G).$$

On the other hand, noting that $X \times {}^F X \to X^F$ is an H-map,

$$\begin{aligned}[\Sigma B, X^F]_o &\cong [\Sigma B, X \times {}^F X]_o \cong [\Sigma B, X]_o \oplus [\Sigma B, {}^F X]_o \\ &\cong [\Sigma B, X]_o \oplus [F \wedge \Sigma B, X]_o \\ &\cong \tilde{H}^m(\Sigma B, G) \oplus \tilde{H}^m(F \wedge \Sigma B, G). \quad \square\end{aligned}$$

<u>Remark</u>.

(1) In the above corollary, if we take $\Sigma B = S^r$ (r-sphere, $r \geq 2$), then

$$\tilde{H}^m(S^r, G) \oplus \tilde{H}^m(F \wedge S^r, G) = H^{m-r}(F, G).$$

Hence, $H^m(E, F; G) \cong H^{m-r}(F, G)$. This fact is well known as the Wang isomorphism.

(2) In Corollary 4.13, the condition that the base space is a suspension space is necessary. Consider the well-known fibration

$$S^1 \to S^{2n+1} \to CP^n, \qquad n \geq 2,$$

where CP^n is the complex projective space. By the exact sequence associated to the pair (S^{2n+1}, S^1) we see

$$H^{2n}(S^{2n+1}, S^1; G) = 0.$$

But, $\tilde{H}^{2n}(CP^n, G) \oplus \tilde{H}^{2n}(S^1 \wedge CP^n, G) = G.$

D. A Consequence of Gottlieb's Theorem

Suppose that $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration such that F is path-connected and B is n-connected. Then, for any abelian group G, the Serre exact sequence

$$\cdots \to H^k(B,G) \xrightarrow{p_k^*} H^k(E,G) \xrightarrow{i_k^*} H^k(F,G) \xrightarrow{\tau_k} H^{k+1}(B,G) \to \cdots \to H^{n+1}(F,G)$$

tells us that i_k^* is isomorphic for $k < n$ and

$$i_n^*: H^n(E, G) \to H^n(F, G) \quad \text{is monomorphic.}$$

In general, i_n^* is not epimorphic as seen from the Hopf fibration $S^1 \to S^3 \to S^2$ (by taking $n = 1$) or the path fibration $\Omega S^3 \to PS^3 \to S^3$ $(n = 2)$. For any path-connected H-space X, we can convert the Hopf construction $X*X \to \Sigma X$ into a fibration $E \to \Sigma X$ such that E has the homotopy type of $X*X$ (join) and the fibre has the weak homotopy type of X (for example, see [33], Chapter 1). Unless X is acyclic, every such fibration offers an example of non-epimorphic i_n^*.

Assume that the integral homology group $H_*(F, Z) = \sum_{i \geq 0} H_i(F, Z)$ is finitely generated. Under this condition, if F is a noncontractible H-space, its Euler-Poincare number $\chi(F) = \sum_{i \geq 0} (-1)^i \cdot [\text{rank } H_i(F, Z)]$ is zero (see [17], p. 753). Roughly speaking, the next theorem says that the case where i_n^* is not epimorphic appears only if $\chi(F) = 0$ and n is odd.

Theorem 4.14. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration such that F has the homotopy type of a connected CW-complex. Suppose that B is n-connected, F is path-connected, and $H_*(F, Z)$ is finitely generated. Then $i_n^*: H^n(E, G) \to H^n(F, G)$ is epimorphic (hence isomorphic) if one of the following conditions holds:

(1) n is even and G is a commutative ring with unit, or

(2) n is odd, $\chi(F) \neq 0$, and G is a field.

(Note that any finitely generated abelian group admits a multiplicative structure of some commutative ring with unit.)

<u>Example</u>. Suppose $F \xrightarrow{i} E \xrightarrow{p} S^{n+1}$ is a fibration and F is a compact polyhedron. If n is even, then for any finitely generated abelian group G,

(1) $i^*: H^n(E, G) \to H^n(F, G)$ is isomorphic, and

(2) $p^*: H^{n+1}(S^{n+1}, G) \to H^{n+1}(E, G)$ is monomorphic.

If $\chi(F) \neq 0$ and G is a field, statements (1) and (2) are also true for odd n.

To prove Theorem 4.14 and its homology version (Theorem 4.18 below), we use the results of D. H. Gottlieb about evaluation subgroups. Let X be a space and let

$$w_*: \pi_k(X^X, \mathrm{Id}) \to \pi_k(X, x_o) \qquad (k \geq 1)$$

be the homomorphism induced by the evaluation map $w: X^X \to X$; $w(u) = u(x_o)$. The image of w_* is called the k-dimensional <u>evaluation subgroup</u> of X and is denoted by

$$G_k(X, x_o).$$

If X is an H-space, then $G_k(X, x_o) = \pi_k(X, x_o)$ for all $k \geq 1$ because the evaluation map $w: (X^X, \mathrm{Id}) \to (X, x_o)$ has a right homotopy inverse.

Among the properties of $G_k(X, x_o)$, one of the most interesting ones is its relationship to the Hurewicz homomorphism. Let G be a commutative ring with unit 1, and let h_G denote the composition of the following homomorphisms:

$$\pi_k(X, x_o) \xrightarrow{h} H_k(X, Z) \xrightarrow{\otimes 1} H_k(X, Z) \otimes G \subset H_k(X, G)$$

where h is the Hurewicz homomorphism and $\otimes 1$ is the homomorphism given by $\alpha \longmapsto \alpha \otimes 1$.

<u>Theorem</u> 4.15 (Gottlieb). Let X be a path-connected space such that $H_*(X, Z)$ is finitely generated.

(1) If n is even $(n \geq 2)$, then $G_n(X, x_o) \subset \ker h$ (see [18]). Hence, $G_n(X, x_o) \subset \ker h_G$ for any commutative ring G with 1.

(2) If n is odd and $\chi(X) \neq 0$, then $G_n(X, x_o) \subset \ker h_G$ for any field G (see [17], p. 739).

Using the above theorem, we prove the following lemma.

<u>Lemma</u> 4.16. Let F be a path-connected space such that $H_*(F, Z)$ is finitely generated. Assume either

(1) n is even and G is a commutative ring with 1,

or

(2) n is odd, $\chi(F) \neq 0$, and G is a field.

Then for any map $g\colon F \to K(G, n)$, $G_n(F, *)$ is contained in the kernel of $g_*\colon \pi_n(F) \to \pi_n(K(G, n))$.

Proof. Gottlieb's Theorem says $G_n(F, *) \subset \ker h_G$. On the other hand, the following diagram tells us $\ker h_G \subset \ker g_*$:

$$\begin{tikzcd}
\pi_n(F) \arrow[r, "h"] \arrow[d, "g_*"'] & H_n(F) \arrow[r, "\otimes 1"] \arrow[d, "g_*"'] & H_n(F) \otimes G \arrow[d, "g_* \otimes \mathrm{Id}"'] \\
\pi_n(K(G, n)) \arrow[r, "h", "\cong"'] & H_n(K(G, n)) \arrow[r, "\otimes 1"] \arrow[d, equal] & H_n(K(G, n)) \otimes G \arrow[d, equal] \\
 & G \arrow[r, "\otimes 1"] & G \otimes G
\end{tikzcd}$$

Note that $\otimes 1 : G \to G \otimes G$ is monomorphic since the function $h: G \otimes G \to G$ given by $\alpha \otimes \beta \to \alpha \cdot \beta$ is well defined and the composition $G \xrightarrow{\otimes 1} G \otimes G \xrightarrow{h} G$ is the identity homomorphism. $\square$

Proof of Theorem 4.14. Since B is n-connected, there is a weak homotopy equivalence $\varepsilon: |B| \to B$ where $|B|$ is a CW-complex such that its n-skelton is trivial (i.e., $|B|^n = *$). So, without loss of generality, we can assume B itself is such a CW-complex because, if necessary, we can replace the original fibration by the one induced by the weak homotopy equivalence ε.

To show $i^*: H^n(E, G) \to H^n(F, G)$ is epimorphic, pick $\bar{u} \in H^n(F, G)$ arbitrarily, and let $u: F \to K(G, n)$ be a representative map for $\bar{u}$. We consider the following extension problem:

$$\begin{array}{ccc} E \supset F & \xrightarrow{u} & K(G, n) \\ \downarrow & & \downarrow \\ B & & * \end{array}$$

Obviously, if it is shown that u has extension $E \to K(G, n)$, the proof will be completed.

Since $B^n = b_o$ and, hence, $E_n = F$, the first possible obstruction appears in dimension $n + 1$. Consider the following diagram:

$$\begin{array}{ccccc} & & & & \pi_n({}^F K(G, n), u) \\ & & & & \downarrow \\ \pi_{n+1}(E_{n+1}^{(F)}, F^F, i) & \xrightarrow{\partial} & \pi_n(F^F, Id) & \xrightarrow{(u_\#)_*} & \pi_n(K(G, n)^F, u) \\ & & \downarrow w_* & & \downarrow w'_* \\ & & \pi_n(F, *) & \xrightarrow{u_*} & \pi_n(K(G, n), *) \end{array}$$

where w and w' are the evaluation maps. Recall that $(u_\#)_* \cdot \partial$ is the $(n+1)$-obstruction homomorphism $\Gamma(u)$ (see Theorem 2.14). By the previous lemma, $u_* \cdot w_* = 0$. Hence, $w'_* \cdot (u_\#)_* = 0$. But,

$$\ker w'_* = \mathrm{Im}[\pi_n({}^F K(G, n), u) \to \pi_n(K(G, n)^F, n)].$$

$$= 0$$

because $\pi_n({}^F K(G, n), u) = 0$ (see Proposition 4.3). Hence,

$(u_{\#})_* = 0$, and $\Gamma(u) = (u_{\#})_* \cdot \partial = 0$. So, u is $(n+1)$-extensible. Now, since

$$\pi_k(K(G, n)^F, u) \cong H^{n-k}(F, G) = 0 \quad \text{for} \quad k \geq n + 1,$$

(Proposition 4.4(3)), all obstruction homomorphisms in the higher dimensions are trivial. Therefore, u is ∞-extensible. □

Now we consider the homology theorem dual to Theorem 4.14. This can be proved without using the obstruction theory. Instead, we will use the following property of evaluation subgroups:

<u>Proposition</u> 4.17 (Gottlieb). Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration, and let

$$\cdots \to \pi_{k+1}(B) \xrightarrow{\partial_{k+1}} \pi_k(F) \xrightarrow{i_*} \pi_k(E) \xrightarrow{p_*} \pi_k(B) \to \cdots$$

be the homotopy exact sequence. Then, the image of ∂_{k+1} is contained in the k-evaluation group of F.

This result follows from the definition of the evaluation group and from the exact ladder of the homotopy groups associated to the following commutative diagram:

$$\begin{array}{ccccc}
F^F & \xrightarrow{i_\#} & E^{(F)} & \xrightarrow{\bar{p}} & B \\
w\downarrow & & w\downarrow & & \| \\
F & \xrightarrow{i} & E & \xrightarrow{p} & B
\end{array}$$

where w denotes the evaluation map. $\square$

<u>Theorem</u> 4.18. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Suppose that B is n-connected, F is path-connected, and $H_*(F, Z)$ is finitely generated. Then

$$i_*: H_n(F, G) \to H_n(E, G)$$

is monomorphic (hence isomorphic) if one of the following conditions holds:

(1) n is even and G is a commutative ring with 1, or

(2) n is odd, $\chi(F) \neq 0$ and G is a field.

<u>Proof</u>. Note that in the Serre exact sequence

$$\cdots \to H_{n+1}(B, G) \xrightarrow{\tau} H_n(F, G) \xrightarrow{i_*} H_n(E, G) \to \underset{\substack{\| \\ 0}}{H_n(B, G)}$$

τ is defined by the composition of

$$H_{n+1}(B, G) \underset{\cong}{\xrightarrow{p_*^{-1}}} H_{n+1}(E, F; G) \xrightarrow{\partial} H_n(F, G).$$

We will show that $\tau = 0$ under the given assumptions. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi_{n+1}(B) & \overset{p_*''}{\underset{\cong}{\leftarrow}} & \pi_{n+1}(E, F) & \overset{\partial''}{\rightarrow} & \pi_n(F) \\
h\downarrow\cong & & h'\downarrow & & h''\downarrow \\
H_{n+1}(B) & \overset{p_*'}{\underset{\cong}{\leftarrow}} & H_{n+1}(E, F) & \overset{\partial'}{\rightarrow} & H_n(F) \\
T\downarrow & & T'\downarrow & & T''\downarrow \\
H_{n+1}(B, G) & \overset{p_*}{\underset{\cong}{\leftarrow}} & H_{n+1}(E,F;G) & \overset{\partial}{\rightarrow} & H_n(F, G)
\end{array}$$

where h, h', h'' are the Hurewicz homomorphisms and T, T', T'' are the homomorphisms defined by $\alpha \longmapsto \alpha \otimes 1$. Since B is n-connected, p_*, p_*', p_*'' and h are isomorphisms (by Serre's Theorem, and by Hurewicz's Theorem). Note that the G-module $H_{n+1}(B, G) = H_{n+1}(B) \otimes G$ is generated by elements of the form $\alpha \otimes 1$ $(= T(\alpha))$ where $\alpha \in H_{n+1}(B)$. Now, by the previous proposition, the image of $\pi_{n+1}(B)$ under $\partial''\cdot p_*''^{-1}$ is in $G_n(F, x_o)$, and by Theorem 4.15, $T''\cdot h''$ maps $G_n(F, x_o)$ to 0 in $H_n(F, G)$. Therefore,

$$0 = T''\cdot h''\cdot \partial''\cdot p_*''^{-1} = \partial\cdot p_*^{-1}\cdot T\cdot h.$$

Then, from the above remark on h and T, we conclude $\partial'\cdot p_*'^{-1}$ $(= \tau) = 0$. This proves the theorem. □

If F is simply connected, we can show the following theorem:

<u>Theorem</u> 4.19. Let $F \overset{i}{\rightarrow} E \overset{p}{\rightarrow} B$ be a fibration. Suppose that B is n-connected $(n \geq 1)$, F is simply connected and $H_*(F, Z)$ is finitely generated. Then

$$i_*: H_{n+1}(F, G) \to H_{n+1}(E, G)$$

is monomorphic if one of the following conditions holds:

(1') n is odd and G is a commutative ring with 1 such that $\mathrm{Tor}(H_{n+1}(B), G) = 0$, or

(2') n is even, $\chi(F) \neq 0$, and G is a field such that $\mathrm{Tor}(H_{n+1}(B), G) = 0$.

(Note that if G has no torsion, then $\mathrm{Tor}(H_{n+1}(B), G) = 0$.)

Proof. The proof is quite similar to the proof of Theorem 4.18. Consider the diagram:

$$\begin{tikzcd}
\pi_{n+2}(B) \arrow[d, "h"] & \pi_{n+2}(E, F) \arrow[l, "p''_*"', "\cong"] \arrow[r, "\partial''"] \arrow[d, "h'"] & \pi_{n+1}(F) \arrow[d, "h''"] \\
H_{n+2}(B) \arrow[d, "T"] & H_{n+2}(E, F) \arrow[l, "p'_*"', "\cong"] \arrow[r, "\partial'"] \arrow[d, "T'"] & H_{n+1}(F) \arrow[d, "T''"] \\
H_{n+2}(B, G) & H_{n+2}(E,F; G) \arrow[l, "p_*"', "\cong"] \arrow[r, "\partial"] & H_{n+1}(F, G)
\end{tikzcd}$$

Since B is n-connected and F is 1-connected, p'_*, p_* are isomorphisms. Since B is n-connected $(n \geq 1)$, h is epimorphic (see [22], p. 167). By the assumption on G,

$$H_{n+2}(B, G) \cong H_{n+2}(B) \otimes G \oplus \mathrm{Tor}(H_{n+1}(B), G) = H_{n+2}(B) \otimes G,$$

and so the image of T generates $H_{n+2}(B, G)$. Then the proof is completed by the same argument as in the proof of Theorem 4.18. $\square$

Example (Browder [5]). Let X be a path-connected H-space with finitely generated homology group, and let $\pi_n(X, x_o)$ be the first nonvanishing homotopy group. Then n is odd. If X is simply connected, then $H_{n+1}(X, Z) = 0$.

Proof. We may regard the Hopf construction $X*X \to \Sigma X$ as a fibration with fibre X (see the first paragraph of this section). The suspension space ΣX is n-connected. So if n were even, by Theorem 4.18, there would be a monomorphism $H_n(X, Z) \to H_n(X*X, Z)$. But the join $X*X$ is 2n-connected and, hence $0 = H_n(X, Z) \cong \pi_n(X, x_o)$ (contradiction). Therefore, n must be odd. Now if X is simply connected, we can apply Theorem 4.19 to obtain a monomorphism $H_{n+1}(X, Z) \to H_{n+1}(X*X, Z)$, whence $H_{n+1}(X, Z) = 0$. □

Note that the first assertion of the above example also follows immediately from Gottlieb's Theorem (Theorem 4.15(1)) and the Hurewicz isomorphism theorem.

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