# 误差数据处理和习题

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## 1 练习题

1. 用游标卡尺 (量程 125 mm, 分度值 0.02 mm, 允差请查教材表 7-1) 测量钢筒含钢体积并计算其结果的不确定度, 即 $V \pm \sigma_V$. 直接测量结果见下表, 每个量在不同位置测 6 次, 其中字母含义: 外径 $D$, 内径 $d$, 高 $H$.

表 1: 测量钢筒含钢体积的数据.

| 项目 | $D/\mathrm{cm}$ | $d/\mathrm{cm}$ | $H/\mathrm{cm}$ |
|---|---|---|---|
| 零点读数 | $D_0 = 0.000$ | $d_0 = 0.000$ | $H_0 = 0.000$ |
| 1 | 2.514 | 1.680 | 4.210 |
| 2 | 2.518 | 1.682 | 4.216 |
| 3 | 2.512 | 1.678 | 4.214 |
| 4 | 2.516 | 1.680 | 4.212 |
| 5 | 2.514 | 1.680 | 4.210 |
| 6 | 2.514 | 1.678 | 4.210 |
| 平均值 | 2.5147 | 1.6797 | 4.212 |
| 平均值的标准差 | 0.0008 | 0.0006 | 0.001 |
| 考虑仪器允差后的标准差 | 0.0014 | 0.0013 | 0.002 |
| 修正零点后的平均值 | 2.5147 | 1.6797 | 4.212 |

游标卡尺允差为 0.002 cm.

测量结果:

$\overline{D} \pm \sigma_D = (2.5147 \pm 0.0014)\,\mathrm{cm}$,

$\overline{d} \pm \sigma_d = (1.6797 \pm 0.0013)\,\mathrm{cm}$,

$\overline{H} \pm \sigma_H = (4.212 \pm 0.002)\,\mathrm{cm}$.
计算结果:
$V = \frac{\pi}{4}\left(\overline{D}^2 - \overline{d}^2\right)\overline{H} = 48.800\,\mathrm{cm}^3$,
$\sigma_V = \sqrt{\left(\frac{\partial V}{\partial D}\sigma_D\right)^2 + \left(\frac{\partial V}{\partial d}\sigma_d\right)^2 + \left(\frac{\partial V}{\partial H}\sigma_H\right)^2} = 0.028\,\mathrm{cm}^3$,
$V \pm \sigma_V = (48.800 \pm 0.028)\,\mathrm{cm}^3$.

2. 用螺旋测微器 (千分尺)(允差请查教材表 7-1) 测量钢球体积并计算结果的不确定度, 即 $V \pm \sigma_V$. 在不同位置测 6 次直径 $d$, 测量结果如下:
零点读数 $d_0 = -0.0003\,\mathrm{cm}$.

表 2: 测量小钢球直径的数据.

| 测量次数 | $d/\mathrm{cm}$ |
|---|---|
| 1 | 1.4690 |
| 2 | 1.4691 |
| 3 | 1.4693 |
| 4 | 1.4690 |
| 5 | 1.4694 |
| 6 | 1.4693 |
| 平均值 | 1.46918 |
| 平均值的标准差 | 0.00008 |
| 考虑仪器允差的标准差 | 0.00024 |
| 修正零点后的平均值 | 1.46948 |

螺旋测微器允差为 $0.0004\,\mathrm{cm}$.
测量结果:
$\overline{d} \pm \sigma_d = (1.46948 \pm 0.00024)\,\mathrm{cm}$,
计算结果:
$V = \frac{\pi}{6}\overline{d}^3 = 1.66147\,\mathrm{cm}^3$,
$\sigma_V = \sqrt{\left(\frac{\partial V}{\partial d}\sigma_d\right)^2} = 0.00083\,\mathrm{cm}^3$,
$V \pm \sigma_V = (1.66147 \pm 0.00083)\,\mathrm{cm}^3$.

# 2 课后习题

1. (1) 1 位有效数字; (2) 4 位有效数字; (3) 2 位有效数字; (4) 6 位有效数字.

2. (1) $\sigma_a = 0.1\,\mathrm{cm}$, $\sigma_b = 0.1\,\mathrm{cm}$. $\sigma_c = \sqrt{\left(\frac{\partial c}{\partial a}\sigma_a\right)^2 + \left(\frac{\partial c}{\partial b}\sigma_b\right)^2} \approx 0.1\,\mathrm{cm}$. $c = \frac{ab}{b-a} \approx 10.0\,\mathrm{cm}$. 故 $c \pm \sigma_c = (10.0 \pm 0.1)\,\mathrm{cm}$.

   (2) $\sigma_x = 0.01$. $\sigma_y = \sqrt{\left(\frac{\partial y}{\partial x}\sigma_x\right)^2} \approx 1.5 \times 10^{-38}$. $y = \mathrm{e}^{-x^2} \approx 8.3 \times 10^{-38}$. 故 $y \pm \sigma_y = (8.3 \pm 1.5) \times 10^{-38}$.

   (3) $\sigma_x = 0.1$. $\sigma_y = \sqrt{\left(\frac{\partial y}{\partial x}\sigma_x\right)^2} \approx 0.002$. $y = \ln x \approx 4.038$. 故 $y \pm \sigma_y = 4.038 \pm 0.002$.

   (4) $\sigma_x = 1'$. $\sigma_y = \sqrt{\left(\frac{\partial y}{\partial x}\sigma_x\right)^2} \approx 0.00005$. $y = \cos x \approx 0.98657$. 故 $y \pm \sigma_y = 0.98657 \pm 0.00005$.

3. (b) $\sigma_\rho = \sqrt{\left(\frac{\partial \rho}{\partial m_1}\sigma_{m_1}\right)^2 + \left(\frac{\partial \rho}{\partial m_2}\sigma_{m_2}\right)^2} = \sqrt{\frac{m_2^2\rho_0^2}{(m_1-m_2)^4}\sigma_{m_1}^2 + \frac{m_1^2\rho_0^2}{(m_1-m_2)^4}\sigma_{m_2}^2}$.

   (c) $\sigma_y = \sqrt{\left(\frac{\partial y}{\partial a}\sigma_a\right)^2 + \left(\frac{\partial y}{\partial b}\sigma_b\right)^2} = \sqrt{\frac{b^2}{a^2(a+b)^2}\sigma_a^2 + \frac{a^2}{b^2(a+b)^2}\sigma_b^2}$.

4. 测量 $L$ 有三种方法:

   ① $L = L_1 + \frac{d_1}{2} + \frac{d_2}{2}$, $\sigma_L = \sqrt{\sigma_{L_1}^2 + \left(\frac{1}{2}\sigma_{d_1}\right)^2 + \left(\frac{1}{2}\sigma_{d_2}\right)^2}$. 代入数据得 $\sigma_L \approx 0.9\,\mu\mathrm{m}$.

   ② $L = L_2 - \frac{d_1}{2} - \frac{d_2}{2}$, $\sigma_L = \sqrt{\sigma_{L_2}^2 + \left(\frac{1}{2}\sigma_{d_1}\right)^2 + \left(\frac{1}{2}\sigma_{d_2}\right)^2}$. 代入数据得 $\sigma_L \approx 1.1\,\mu\mathrm{m}$.

   ③ $L = \frac{L_1 + L_2}{2}$, $\sigma_L = \sqrt{\frac{\sigma_{d_1}^2}{4} + \frac{\sigma_{d_2}^2}{4}}$. 代入数据得 $\sigma_L \approx 0.6\,\mu\mathrm{m}$.

   $\sigma_{L_3} < \sigma_{L_1} < \sigma_{L_2}$, 故采用第 3 种方法能使 $\sigma_L$ 最小.

5. 上表面面积 $S = L_1L_2 - \frac{\pi d_1^2}{4} - \frac{\pi d_2^2}{4}$. 由题意,

$$\frac{\sigma_S}{S} = \sqrt{\left(\frac{\partial \ln S}{\partial L_1}\sigma_{L_1}\right)^2 + \left(\frac{\partial \ln S}{\partial L_2}\sigma_{L_2}\right)^2 + \left(\frac{\partial \ln S}{\partial d_1}\sigma_{d_1}\right)^2 + \left(\frac{\partial \ln S}{\partial d_2}\sigma_{d_2}\right)^2} \leqslant 0.5\%.$$

   解得 $\sigma_{d_2} \leqslant 1.2\,\mathrm{cm}$. 故需使测量方案的 $\sigma_{d_2} \leqslant 1.2\,\mathrm{cm}$, 才能使 $\frac{\sigma_S}{S} \leqslant 0.5\%$. 用米粗尺测量即可满足这一要求.

7. (1) $g = \frac{2h}{t^2}$. 设实际长度和时间为 $h, t$, 则实际重力加速度 $g = \frac{2h}{t^2}$; 设所测得长度和时间为 $h', t'$, 则所测得重力加速度 $g' = \frac{2h'}{t'^2}$.

由题意, $h = h'(1+10^{-5})$, $t' = 0.9999t$. 解得 $g' = \dfrac{1}{0.9999^2(1+10^{-4})} g \approx 980.1\,\mathrm{cm/s^2}$.

(2) 欲使测定的单摆周期准确度不低于 0.5%, 则有 $\dfrac{|T-T_0|}{T} \leqslant 0.5\%$. 解得 $\theta \leqslant 16.2°$.

欲使测定的单摆周期准确度不低于 0.05%, 则有 $\dfrac{|T-T_0|}{T} \leqslant 0.05\%$. 解得 $\theta \leqslant 5.13°$.

10. $y_i$ 的标准差为 $\sigma_{y_i} = \sqrt{\left(\dfrac{e}{\sqrt{3}}\right)^2 + \left(\dfrac{e_{y_i}}{\sqrt{3}}\right)^2}$.

由最小二乘法的相关性质, $\lambda$ 的标准差 $\sigma_\lambda = \dfrac{\sigma_{y_i}}{\sqrt{\sum_{k=1}^{10}(i_k - \bar{i})^2}}$, 其中 $i_k = k$.

$c$ 的标准差为 $\sigma_c = \sqrt{\lambda^2\sigma_f^2 + f^2\sigma_\lambda^2}$, 其中 $\sigma_f = \dfrac{e}{\sqrt{3}} \approx 0.029\,\mathrm{kHz}$.

由最小二乘法, $\lambda = \dfrac{\sum_{j=1}^{10}(i_j - \bar{i})(y_j - \bar{y})}{\sum_{j=1}^{10}(i_j - \bar{i})^2}$, 其中 $i_j = j$.

联立解得 $c \pm \sigma_c = (346.44 \pm 0.25)\,\mathrm{m/s}$.

11. ① $m - \dfrac{1}{t^2}$ 关系图如图所示.

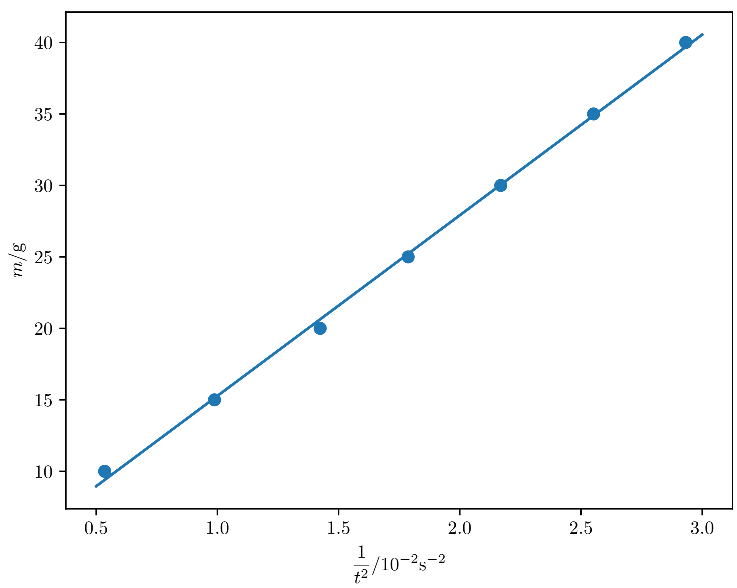


图 1: $m - \dfrac{1}{t^2}$ 关系图. 由图可知, $m - \dfrac{1}{t^2}$ 满足线性关系.

② $k_2 = \dfrac{\overline{m \cdot \frac{1}{t^2}} - (\overline{m})(\overline{\frac{1}{t^2}})}{\overline{m^2} - (\overline{m})^2}$, $b_2 = \overline{\dfrac{1}{t^2}} - k_2\overline{m}$, $r_2 = \dfrac{\overline{m \cdot \frac{1}{t^2}} - (\overline{m})(\overline{\frac{1}{t^2}})}{\sqrt{\left(\overline{m^2} - (\overline{m})^2\right)\left(\overline{(\frac{1}{t^2})^2} - (\overline{\frac{1}{t^2}})^2\right)}}$.

代入数据解得 $k_2 \approx 0.079$, $b_2 \approx -0.206$, $r_2 \approx 0.999$.

③ $k_1 = \dfrac{\overline{m\cdot\frac{1}{t^2}} - (\overline{m})(\overline{\frac{1}{t^2}})}{\overline{(\frac{1}{t^2})^2} - (\overline{\frac{1}{t^2}})^2}$, $b_1 = \overline{m} - k_1\overline{\dfrac{1}{t^2}}$, $r_1 = \dfrac{\overline{m\cdot\frac{1}{t^2}} - (\overline{m})(\overline{\frac{1}{t^2}})}{\sqrt{\left(\overline{m^2} - (\overline{m})^2\right)\left(\overline{(\frac{1}{t^2})^2} - (\overline{\frac{1}{t^2}})^2\right)}}$.

代入数据解得 $k_1 \approx 12.637$, $b_1 \approx 2.638$, $r_1 \approx 0.999$.

②和③得到的相关系数相同, 因为相关系数 $r = \dfrac{\overline{xy} - \overline{x}\overline{y}}{\sqrt{\left(\overline{x^2} - (\overline{x})^2\right)\left(\overline{y^2} - (\overline{y})^2\right)}}$ 关于自变量和因变量是对称的. $k_1, k_2$ 和相关系数的关系为 $k_1k_2 = r^2$.