Universitatea din Bucureşti | Analiză Numerică
Facultatea de Matematică şi Informatică | 18.11.2020

CURS#7

III. Interpolare polinomială

(a) Interpolarea Lagrange:

- (iv) metoda lui Neville: formula de recurenţă; algoritm;
- (v) metoda lui Newton; algoritm;
- (vi) metoda lui Newton cu diferenţe divizate (DD); algoritm.

3) ALGORITMUL LUI NEVILLE

DEFINIȚIE:

Fie $f:[a,b]\to\mathbb{R}$, $\mathcal{D}=\{(x_i,f(x_i))\mid i=\overline{0,n}\}$ $\subset\mathbb{R}^2$ a.î. $x_i\neq x_j$, $0\le i<j\le n$.

Fie $m_0,m_1,\dots,m_k\in\mathbb{N}$, $k\le n$, a.î.

(i) $0\le m_i\le n$, $i=\overline{0,k}$;

(ii) $m_i\neq m_j$, $0\le i<j\le k$.

Notăm cu $P_{m_0,m_1,\dots,m_k}(x)$ unicul polinom de interpolare Lagrange asociat funcției f și nodurilor $x_{m_0},x_{m_1},\dots,x_{m_k}$, i.e.

$$P_{m_0,m_1,\dots,m_k}(x_i)=f(x_i),\quad i=\overline{0,k}$$

TEOREMA 3 (formula de recurență):

Fie $f:[a,b]\to\mathbb{R}$ și setul de date $\mathcal{D}=\{(x_i,f(x_i))\mid i=\overline{0,n}\}\subset\mathbb{R}^2$, a.î.

$x_i \neq x_j$, $0 \leq i < j \leq n$.

Dacă $x_i, x_j \in \{x_0, x_1, \dots, x_n\}$ a.î. $x_i \neq x_j$ și

$$\boxed{P(x) = \frac{1}{x_i - x_j}\Big[(x-x_j)\,P_{0,1,\dots,j-1,j+1,\dots,n}(x) - (x-x_i)\,P_{0,1,\dots,i-1,i+1,\dots,n}(x)\Big], \quad \forall x \in [a,b]}$$

atunci $P \equiv P_n \in \mathbb{P}_n$ polinomul de interpolare Lagrange asociat funcției f și nodurilor $x_0, x_1, \dots, x_n$.

<u>Dem</u>: Notăm

$$\left.\begin{aligned} Q(x) &:= P_{0,1,\dots,j-1,j+1,\dots,n}(x), \quad \forall x \in [a,b] \\ \hat{Q}(x) &:= P_{0,1,\dots,i-1,i+1,\dots,n}(x), \quad \forall x \in [a,b] \end{aligned}\right\} \Rightarrow$$

$$P(x) = \frac{(x-x_j)\,Q(x) - (x-x_i)\,\hat{Q}(x)}{x_i - x_j}, \quad \forall x \in [a,b]$$

$\Rightarrow \operatorname{grad} P \leq \operatorname{grad} Q + 1 = n$

<u>Cazul I</u>: $x_k \in \{x_0, x_1, \dots, x_n\} \setminus \{x_i, x_j\}$

$Q(x_k) = \hat{Q}(x_k) = f(x_k) \Rightarrow$

$$P(x_k) = \frac{(x_k - x_j) f(x_k) - (x_k - x_i) f(x_k)}{x_i - x_j}$$

$$= \frac{(x_i - x_j) f(x_k)}{x_i - x_j} = f(x_k) \quad \checkmark$$

<u>Cazul II</u>: $x_k = x_i$

$Q(x_i) = \hat{Q}(x_i) = f(x_i) \Rightarrow$

$$P(x_i) = \frac{(x_i - x_j) f(x_i) - (x_i - x_i) f(x_i)}{x_i - x_j}$$

$$= \frac{(x_i - x_j) f(x_i)}{x_i - x_j} = f(x_i) \quad \checkmark$$

<u>Cazul III</u>: $x_k = x_j$

$Q(x_j) = \hat{Q}(x_j) = f(x_j) \Rightarrow$

$$P(x_j) = \frac{(x_j - x_j) f(x_j) - (x_j - x_i) f(x_j)}{x_i - x_j}$$

$= f(x_j)$ $\checkmark$ Deci $P = P_n$. $\square$

ALGORITM (Neville):

grad

x_i	$P_{m_i}(x)$ (0)	$P_{m_i,m_j}(x)$ (1)	$P_{m_i,m_j,m_k}(x)$ (2)
x_0	$P_0(x)=f(x_0)$		
x_1	$P_1(x)=f(x_1)$	$P_{0,1}(x)$	
x_2	$P_2(x)=f(x_2)$	$P_{1,2}(x)$	$P_{0,1,2}(x)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$P_n(x)=f(x_n)$	$P_{n-1,n}(x)$	$P_{n-2,n-1,n}(x)$

	$\ldots$	$P_{m_0,m_1,\ldots,m_n}(x)$ (n) ← grad
x_0		
x_1		
x_2		
$\vdots$		
x_n		$P_{0,1,\ldots,n}(x)$

4) METODA LUI NEWTON

<u>OBSERVAȚIE</u> (algoritmul lui Neville):

Date $D_n := \{(x_i, y_i) \mid i = \overline{0,n}\} \subset \mathbb{R}^2$, a.î. $x_i \neq x_j$, $0 \leq i < j \leq n$, determină $P_n \in \mathbb{P}_n$ în mod recursiv, prin determinarea polinoamelor Lagrange de grad $k = \overline{0,n}$ asociate cu seturile de date care conțin $(k+1)$ noduri succesive, $k = \overline{0,n}$, din totalul celor $(n+1)$ noduri $x_0, x_1, \dots, x_n$

<u>IDEEA</u> (metoda lui Newton):

$D_n := \{(x_i, y_i) \mid i = \overline{0,n}\} \subset \mathbb{R}^2$;

$x_i \neq x_j$, $0 \leq i < j \leq n$;

$\forall k = \overline{1,n}$, construiește $P_k \in \mathbb{P}_k$ care rezolvă problema de interpolare Lagrange pentru setul de date

$D_k := \{(x_i, y_i) \mid i = \overline{0,k}\} = D_{k-1} \cup \{(x_k, y_k)\}$, folosind $P_{k-1} \in \mathbb{P}_{k-1}$ care rezolvă problema de interpolare Lagrange pentru setul de date $D_{k-1} := \{(x_i, y_i) \mid i = \overline{0, k-1}\}$.

<u>$k=0$</u>: $D_0 = \{(x_0, y_0)\}$

$$P_0 \in \mathbb{P}_0 : \quad P_0(x_0) = y_0 \Rightarrow \begin{cases} P_0(x) = c_0 \\ P_0(x_0) = y_0 \end{cases}$$

$$\Rightarrow \boxed{P(x_0) = y_0}$$

<u>$k=1$</u>: $D_1 = D_0 \cup \{(x_1, y_1)\}$

$$P_1 \in \mathbb{P}_1 : \begin{cases} P_1(x_0) = y_0 \\ P_1(x_1) = y_1 \end{cases}$$

$$\underbrace{P_1(x)}_{\text{grad } 1} = \underbrace{P_0(x)}_{\text{grad } 0} + \underbrace{c_1(x - x_0)}_{\text{grad } 1}, \quad \underline{c_1 \in \mathbb{R}}$$

$$P_0(x_0) = y_0$$

Cum $P_1(x_0) = P_0(x_0) + c_1(x_0 - x_0) = P(x_0) = y_0$, trebuie ca:

$P_1(x_1) = y_1 \Rightarrow$

$P_0(x_1) + c_1(x_1 - x_0) = y_1 \Rightarrow \boxed{c_1 = \dfrac{y_1 - P_0(x_1)}{x_1 - x_0}}$

$\underline{k \mapsto k+1}$: $D_{k+1} = D_k + \{(x_{k+1}, y_{k+1})\}$

$P_{k+1} \in \mathbb{P}_{k+1}$: $P_{k+1}(x_i) = y_i$, $i = \overline{0,k+1}$

Dar $P_k \in \mathbb{P}_k$ și $P_k(x_i) = 0$, $i = \overline{0,k} \Rightarrow$

$$\underbrace{P_{k+1}(x)}_{\text{grad } k+1} = \underbrace{P_k(x)}_{\text{grad } k} + \underbrace{c_{k+1}(x-x_0)(x-x_1)\dots(x-x_k)}_{\text{grad } k+1}$$

$P_k(x_i) = y_i$, $i = \overline{0,k}$ $\qquad \underline{c_{k+1} \in \mathbb{R}}$

Cum $P_{k+1}(x_i) = P_k(x_i) = y_i$, $i = \overline{0,k}$,

mai trebuie ca

$P_{k+1}(x_{k+1}) = y_{k+1} \Rightarrow$

$$\boxed{c_{k+1} = \frac{y_{k+1} - P_k(x_{k+1})}{(x_{k+1} - x_0)\dots(x_{k+1} - x_k)}}$$

ALGORITM (metoda lui Newton):

Date: n, $D_n = \{(x_i, y_i) \mid i=\overline{0,n}\} \subset \mathbb{R}^2$

$x_i \neq x_j$, $0 \leq i < j \leq n$

$k=0$: $P_k(x) = c_k$; $c_k = y_k$

$k=\overline{1,n}$: $c_k = \dfrac{y_k - P_{k-1}(x_k)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})}$

$$P_k(x) = P_{k-1}(x) + c_k(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})$$

OBSERVAȚII

1) $P_n(x) = c_0 + \sum\limits_{k=1}^{n} c_k \prod\limits_{l=0}^{k-1}(x-x_l)$

2) În cazul metodei lui Newton, polinoamele de bază ale lui $\mathbb{P}_n$ sunt $\{1;\ (x-x_0);\ (x-x_0)(x-x_1);\ \dots,\ (x-x_0)(x-x_1)\dots(x-x_{n-1})\}$

3) Sist. de ecuații liniare pt determinarea c_k, $k=\overline{0,n}$, este inferior triunghiular.

5) METODA LUI NEWTON CU DIFERENȚE DIVIZATE (DD)

DEFINIȚIE:

Fie $f:[a,b]\to\mathbb{R}$, $\{x_0, x_1, \dots, x_n\}\subset[a,b]$, $x_i\neq x_j$, $0\leq i<j\leq n$, $n\in\mathbb{N}$.

(i) Sn diferența divizată (DD) de ordin 0 a lui f în raport cu x_0

$$\boxed{f[x_0]=f(x_0)}$$

(ii) Sn DD de ordin 1 a lui f în raport cu x_0, x_1

$$\boxed{f[x_0,x_1]=\frac{f[x_1]-f[x_0]}{x_1-x_0}}$$

(iii) Sn DD de ordin k a lui f în raport cu $x_0, x_1, \dots, x_k$, $k\leq n$:

$$\boxed{f[x_0,x_1,..,x_k]=\frac{f[x_1,..,x_k]-f[x_0,..,x_{k-1}]}{x_k-x_0}}$$

PROPOZIȚIA 1:

$$\boxed{\forall n \geq 1 \quad f[x_0, x_1, \dots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{(x_i - x_0)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)}}$$

Dem: Inducție după $n \geq 1$

$n=1$: Cf definiției DD de ordin 1:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} \checkmark$$

$n \longmapsto n+1$: Cf definiției DD de ordin $(n+1)$:

$$f[x_0, x_1, \dots, x_{n+1}] = \frac{f[x_1, \dots, x_{n+1}] - f[x_0, \dots, x_n]}{x_{n+1} - x_0}$$

$$= \frac{1}{x_{n+1} - x_0} \left[\sum_{i=1}^{n+1} \frac{f(x_i)}{(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots\underline{\underline{(x_i - x_{n+1})}}} \right.$$

$$\left. - \sum_{i=0}^{n} \frac{f(x_i)}{\underline{\underline{(x_i - x_0)}}\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)} \right]$$

$$= \frac{1}{x_{n+1}-x_0}\left[\frac{f(x_{n+1})}{(x_{n+1}-x_1)\dots(x_{n+1}-x_n)} - \frac{f(x_0)}{(x_0-x_1)\dots(x_0-x_n)}\right.$$

$$+ \sum_{i=1}^{n} \frac{f(x_i)}{(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} \times$$

$$\left.\times \left(\frac{1}{x_i-x_{n+1}} - \frac{1}{x_i-x_0}\right)\right]$$

$$= \frac{f(x_{n+1})}{(x_{n+1}-x_0)(x_{n+1}-x_1)\dots(x_{n+1}-x_n)}$$

$$+ \frac{f(x_0)}{(x_0-x_1)\dots(x_0-x_n)(x_0-x_{n+1})}$$

$$+ \sum_{i=1}^{n} \frac{f(x_i)}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_{n+1})}$$

$$= \sum_{i=0}^{n+1} \frac{1}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_{n+1})}$$

$\square$

TEOREMA 2 (formula Newton cu DD):

Fie $f:[a,b]\to\mathbb{R}$, $\{x_0,x_1,\dots,x_n\}\subset[a,b]$, $x_i\neq x_j$, $0\le i<j\le n$, $n\in\mathbb{N}$. Atunci:

$$\boxed{\begin{aligned}P_n(x) &= f[x_0]+f[x_0,x_1](x-x_0)\\ &+ f[x_0,x_1,x_2](x-x_0)(x-x_1)+\dots+\\ &+ f[x_0,x_1,\dots,x_n](x-x_0)(x-x_1)\dots(x-x_{n-1})\end{aligned}}$$

Dem: Inducție după $n\ge 0$.

$n=0$: $P_0(x)=f(x_0)=f[x_0]$ ✓

$n\mapsto n+1$:

$$\Big[f[x_0]+f[x_0,x_1](x-x_0)$$
$$+f[x_0,x_1,x_2](x-x_0)(x-x_1)+\dots+$$
$$+f[x_0,x_1,\dots,x_n](x-x_0)\dots(x-x_{n-1})\Big]+$$
$$+f[x_0,x_1,\dots,x_{n+1}](x-x_0)\dots(x-x_n)$$
$$=P_n(x)+f[x_0,x_1,\dots,x_{n+1}](x-x_0)\dots(x-x_n)$$
$$=\sum_{i=0}^{n}L_{n,i}(x)\,f(x_i)+$$

$$+\sum_{i=0}^{n+1} \frac{f(x_i)}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_{n+1})}$$

$$\cdot (x-x_0)(x-x_1)\dots(x-x_n)$$

$$=\sum_{i=0}^{n} \frac{(x-x_0)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x-x_n)} f(x_i)$$

$$+\sum_{i=0}^{n} \frac{(x-x_0)\dots(x-x_{i-1})\overline{\overline{(x-x_i)}}(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)\dots(x_i-x_{i-1})\,(x_i-x_{i+1})\dots(x-x_n)}$$

$$\cdot \frac{f(x_i)}{\underline{\underline{x_i}}-x_{n+1}}$$

$$+\underbrace{\frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(x_{n+1}-x_0)(x_{n+1}-x_1)\dots(x_{n+1}-x_n)}}_{=L_{n+1,n+1}(x)} f(x_{n+1})$$

$$=\sum_{i=0}^{n} \frac{(x-x_0)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} \cdot$$

$$\cdot f(x_i)\underbrace{\left(1+\frac{x-x_i}{x_i-x_{n+1}}\right)}_{=\frac{x-x_{n+1}}{x_i-x_{n+1}}} + L_{n+1,n+1}(x)\, f(x_{n+1})$$

$$= \sum_{i=0}^{n} L_{n+1,i}(x) f(x_i) + L_{n+1,n+1}(x) f(x_{n+1})$$

$$= \sum_{i=0}^{n+1} L_{n+1,i}(x) f(x_i) = P_{n+1}(x)$$

$\square$

<u>ALGORITM</u> (formula lui Newton cu DD):

<u>Date</u>: n; $D_n = \{(x_i, f(x_i)) \mid i=\overline{0,n}\} \subset \mathbb{R}^2$

$(x_i \neq x_j,\ 0 \leq i < j \leq n)$

<u>$k=0$</u>: $c_0 = f(x_0) = f[x_0]$

$P_0(x) = c_0$

<u>$k=\overline{1,n}$</u>: $c_k = f[x_0, x_1, \ldots, x_k]$

$$P_k(x) = P_{k-1}(x) + c_k \prod_{i=0}^{k-1}(x - x_i)$$

<u>Eroarea de interpolare Lagrange cu DD</u>

$f \in C^{n+1}[a,b]$, $a \leq x_0 < x_1 < \ldots < x_n \leq b \Rightarrow$

$\forall x \in [a,b]$, $\exists \xi \in [a,b]$:

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

<u>Dem</u>: EX!