Ex 1: $y'(x) + \cos x\, y(x) = \cos x$ → ec. afină scalară

Pas 1: $\bar{y}'(x) + \cos x\, \bar{y}(x) = 0 \iff \bar{y}'(x) = -\cos x\, \bar{y}(x)$

ec. cu var. separabile
$a(x)\, b(\bar{y}(x))$
$b(\bar{y}) = \bar{y}$

$b(\bar{y}) = 0 \Rightarrow$ sol staționare

$b(\bar{y}) \neq 0 \quad \int \frac{\bar{y}'(x)}{b(\bar{y}(x))} dx = \int a(x) dx$

$b(\bar{y}) = \bar{y}$, $b(\bar{y}) = 0 \Rightarrow y = 0$ sol staționară

$y \neq 0$ Căutăm sol netriviale

$$\frac{\bar{y}'(x)}{\bar{y}(x)} = -\cos x \;\Big|\; \int$$

$$\int \frac{\bar{y}'(x)}{\bar{y}(x)} dx = \int -\cos x\, dx$$

$$\ln|\bar{y}(x)| = -\sin x + c_1$$

$$e^{\ln|\bar{y}(x)|} = e^{-\sin x} \cdot e^{c_1}$$

$$\bar{y}(x) = c_2 e^{-\sin x}$$

Pas 2: Aplic metoda variației constantelor.
Caut $y(x)$ de forma?

$$y(x) = c(x) e^{-\sin x} \;\Big|\,' \qquad c(x) = ?$$

$$c'(x) e^{-\sin x} + \cancel{c(x) \cdot (-\cos x) e^{-\sin x}} + \cancel{\cos x\, c(x) \cdot e^{-\sin x}} = \cos x$$

$$c'(x) = \cos x \cdot e^{\sin x} \;\Big|\; \int$$

$$c(x) = \int \cos x \cdot e^{\sin x} dx = e^{\sin x} + \tilde{c}$$

$$y(x) = (e^{\sin x} + \tilde{c}) e^{-\sin x} = 1 + \tilde{c} e^{-\sin x}$$

Ex 2: $y'' - 5y' + 6y = \begin{cases} e^x & (1) \\ e^{2x} & (2) \end{cases}$

(1)

Pas 1: Consider $\bar{y}'' - 5\bar{y}' + 6\bar{y} = 0$

ec. caracteristică asociată:

$$r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0 \Rightarrow$$

$$\Rightarrow r_1 = 2,\ r_2 = 3 \Rightarrow \bar{y}(x) = c_1 e^{2x} + c_2 e^{3x}$$

$y = \bar{y} + y_{particular}$

MVC Caut $y(x) = c_1(x) e^{2x} + c_2(x) e^{3x}$

$c_1(x)$ și $c_2(x)$ se det din sist. $\begin{pmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{pmatrix} \begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ e^x \end{pmatrix}$

$$\begin{cases} e^{2x} c_1'(x) + e^{3x} c_2'(x) = 0 \;|\cdot 2 \\ 2e^{2x} c_1'(x) + 3e^{3x} c_2'(x) = e^x \end{cases}$$

$$c_2(x) = \int e^{-2x} dx = -\frac{1}{2} e^{-2x} + \tilde{c}_1 \Rightarrow c_1'(x) = -e^{-x}$$

$$c_1(x) = \int -e^{-x} dx = e^{-x} + \tilde{c}_2$$

$$y(x) = (e^{-x} + \tilde{c}_2) e^{2x} + \left(-\frac{1}{2} e^{-2x} + \tilde{c}_1\right) e^{3x}$$

$$= e^x - \frac{1}{2} e^x + \tilde{c}_2 \cdot e^{2x} + \tilde{c}_1 e^{3x}$$

$$= \underbrace{\frac{1}{2} e^x}_{\text{sol. particulară}} + \underbrace{\tilde{c}_2 \cdot e^{2x} + \tilde{c}_1 e^{3x}}_{\text{sol. generală a ec. omogene}} \qquad \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$$

Ex 3: $y' + \frac{2}{x} y = x^3$

Pas 1: $\bar{y}' + \frac{2}{x}\bar{y} = 0 \Rightarrow \bar{y}' = -\frac{2}{x}\bar{y}$

$-\bar{y} = 0$ sol. staționară

$\bar{y} \neq 0 \quad \int \frac{\bar{y}'}{\bar{y}} dx = \int -\frac{2}{x} dx \Rightarrow -2\ln|x| = \ln|y| + c_1$

Aplicăm exp $\Rightarrow e^{-2\ln|x|} = e^{\ln|y|} \cdot e^{c_1} \Rightarrow \frac{1}{x^2} = |\bar{y}(x)| \cdot k$

$$\bar{y}(x) = \frac{\tilde{c}}{x^2}$$

MVC Caut $y(x) = \frac{c(x)}{x^2}$

$$\frac{c'(x)}{x^2} + \cancel{c(x) \frac{-2}{x^3}} + \cancel{\frac{2}{x} \cdot \frac{c(x)}{x^2}} = x^3$$

$$\frac{c'(x)}{x^2} = x^3$$

$$c'(x) = x^5 \Rightarrow c(x) = \int x^5 dx = \frac{x^6}{6} + c$$

$$y(x) = \left(\frac{x^6}{6} + c\right) \cdot \frac{1}{x^2} = \frac{x^4}{6} + \frac{c}{x^2},\ c \in \mathbb{R}$$

Ecuații de tip Euler

$$\sum_{i=1}^{n} a_i x^i y^{(i)}(x) = f(x) \qquad a_i = ct$$

Ex: $x^2 y'' + xy' - y = x^2 \qquad x > 0$

↓ SV

$x = e^s$

Fie $v(s) = y(e^s) = y(x)$; $y(x) = v(s) = v(\ln x)$
$s = \ln x$

$$y'(x) = v'(\ln x) \cdot \frac{1}{x} \Rightarrow xy' = v'(s)$$

$$y''(x) = v''(\ln x) \cdot \frac{1}{x} \cdot \frac{1}{x} + \left(-\frac{1}{x^2}\right) \cdot v'(\ln x) =$$

$$= -\frac{1}{x^2} v'(\ln x) + \frac{1}{x^2} v''(\ln x)$$

$$x^2 y'' = -v'(s) + v''(s)$$

(ec) $\iff -\cancel{v'(s)} + v''(s) + \cancel{v'(s)} - v(s) = e^{2s}$

$v''(s) - v(s) = e^{2s}$ ec. lin cu coef. ct.

$$\bar{v}''(s) - \bar{v}(s) = 0$$

$$r^2 - 1 = 0 \Rightarrow r_1 = 1 \quad r_2 = -1.$$

$$\bar{v}(s) = c_1 e^s + c_2 \cdot e^{-s}$$

MVC $v(s) = c_1(s) e^s + c_2(s) e^{-s}$

$$\begin{pmatrix} e^s & e^{-s} \\ e^s & -e^{-s} \end{pmatrix} \begin{pmatrix} c_1'(s) \\ c_2'(s) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{2s} \end{pmatrix}$$

$$\begin{cases} e^s c_1'(s) + e^{-s} c_2'(s) = 0 \\ e^s c_1'(s) - e^{-s} c_2'(s) = e^{2s} \end{cases}$$

$$2e^s c_1'(s) = e^{2s} \Rightarrow c_1'(s) = \frac{1}{2} e^{+s} \Rightarrow$$

$$\Rightarrow c_1(s) = \int \frac{1}{2} e^s ds = \frac{1}{2} e^s + \tilde{c}$$

$$e^s \cdot \frac{1}{2} e^s + e^{-s} c_2'(s) = 0.$$

$$\frac{1}{2} e^{2s} + e^{-s} c_2'(s) = 0.$$

$$c_2'(s) = -\frac{1}{2} e^{3s}. \Rightarrow c_2(s) = -\frac{1}{6} e^{3s} + \tilde{c}_2$$

$$v(s) = \left(\frac{1}{2} e^s + \tilde{c}_1\right) e^s + \left(-\frac{1}{6} e^{3s} + \tilde{c}_2\right) \cdot e^{-s}$$

$$= \frac{1}{3} e^{2s} + \tilde{c}_1 e^s + \tilde{c}_2 \cdot e^{-s}$$

$$\boxed{y(x) = v(s) = \frac{1}{3} x^2 + \tilde{c}_1 x + \tilde{c}_2 \cdot \frac{1}{x}}$$

Ex: Calc deriv. parț. de ordin I pe dom. max de definiție pentru:

a) $f(x,y) = \ln\left(\cos \frac{y}{x}\right)$ $\qquad \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$?

b) $f(x,y) = (x+y^2)^{\sqrt{\frac{x}{y}}}$

c) $f(x,y) = tg(x \arcsin y)$

a) $$\frac{\partial f}{\partial x} = \frac{1}{\cos \frac{y}{x}} \cdot \left(-\sin \frac{y}{x}\right)\left(-\frac{1}{x^2}\right) y$$

$$\frac{\partial f}{\partial y} = \frac{1}{\cos \frac{y}{x}} \cdot \left(-\sin \frac{y}{x}\right) \cdot \frac{1}{x}$$

b) $$f(x,y) = e^{\ln(x+y^2)^{\sqrt{\frac{x}{y}}}} = e^{\frac{\sqrt{x}}{\sqrt{y}} \ln(x+y^2)}$$

V2: $\left(f(x)^{g(x)}\right)' = f(x)^{g(x)} \ln f(x) \cdot g'(x) + g(x) \cdot f(x)^{g(x)-1} \cdot f'(x)$

$$\frac{\partial f}{\partial y} = (x+y^2)^{\sqrt{\frac{x}{y}}} \cdot \ln(x+y^2) \cdot \frac{\partial}{\partial y}\left(\sqrt{\frac{x}{y}}\right) + \sqrt{\frac{x}{y}} (x+y^2)^{\sqrt{\frac{x}{y}} - 1} \cdot$$

$$\sqrt{x} \cdot \left(-\frac{1}{2}\right) \cdot y^{-\frac{3}{2}} \qquad \frac{\partial}{\partial y}(x+y^2) = 2y$$

$$-\frac{1}{2} \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{y^3}}$$