Coordinated Disaster Relief

Strategic Minds Collective
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Scenario

- A country has been affected by a major natural disaster (National Pandemic)
- There are ***n*** provinces in the country
- Each province has their own relief organizations working to benefit their province the most
- The disaster affected each province **equally**
- Provinces have the choice to either focus all resources on themselves or coalesce with other provinces to benefit the country as a whole
- The way funds are added makes this game a ***superadditive coalition game***

Typology

- Superadditive Coalition Game
- Transferable
    - Provinces buy from the same third party contractor
- Characteristic Function Game
    - The supplier has infinite supplies
- Non-excludable
    - Provinces cannot keep other provinces from buying supplies
- Non-rival
    - Purchasing supplies does not deplete supplies for the other provinces

Asymmetric

- Each province has a different budget to start with
- Thus groups of the same size may have different payoffs as well
- This reflects that fact that different provinces have different financial situations
  - Some provinces are poorer, some are richer
- The asymmetry of this scenario is what causes the collective action problem
  - Richer provinces have the potential to gain more resources should they act selfishly
  - The overall country would benefit if ***all*** provinces pooled their money into a grand coalition
  - However, because of the possible outcomes the result could be highly unfair towards poorer provinces

Fairness - Shapley Value

- Different team members contribute to a project's success
- The goal is to calculate a fair share of the reward to each team member based on their contribution to the team
- However, a player's contribution can vary based on when they join the team:
  - Example, if a player is the second to join a team, their contribution will likely be a lot
  - On the other hand, if they are the hundredth player, likely to not be that much
  - But this isn't the player's fault!
- The Shapley value accounts for this by calculating all possible ways a team could be formed
- Collaboration is accounted for so everyone is rewarded for their contribution and not just on their individual work

Fairness - Shapley Value

Mathematically defining the Shapley:

$\varphi_i(v)$ is Shapley value for i-th player given a payoff

n is the number of players

R is the collection of all possible orders of the players

P_i^R is the set of all players that occur before the i-th player in a given order

$$\varphi_i(v) = \frac{1}{n!} \sum_R \left[v(P_i^R \cup \{i\}) - v(P_i^R) \right]$$

Fairness - Shapley Value

- We want to ensure a fair outcome for all provinces.
- We choose the Shapley value to be the “fairest” outcome. Chosen because it depends on everyone’s marginal contributions.
- Four different properties that it (and only it fulfills all at once).
  - **Efficiency**, so the Shapley value for each player summed up is the total value of the grand coalition.
  - **Symmetry**, if two players contribute equal payoffs then their Shapley value is the same.
  - **Null player**, if a player doesn’t make any contribution to any coalition, their value is zero.
  - **Linearity**, the Shapley value given a payoff vector is linear for all players. (This also means we can represent the conversion of a payoff to a Shapley vector by a matrix)

Monte-Carlo for Shapley

Don’t want to consider all possible permutations since this grows very quickly

- Complexity would be O(n*n!)

Can instead treat each player’s Shapley value as a statistical parameter
Essentially:

- Take a random permutation
- Compute the marginal distribution
- Repeat again and again
- Take the average of the recorded marginal distributions as Shapley estimate

Can compute Shapley for much larger games

Monte-Carlo for Shapley

Can use the Hoeffding Inequality to determine how many trials

Let $X_1, \ldots, X_n$ be independent random variables such that $a_i \leq X_i \leq b_i$ almost surely. Consider the sum of these random variables,

$$S_n = X_1 + \cdots + X_n.$$

Then Hoeffding's theorem states that, for all $t > 0$,[3]

$$\mathrm{P}\left(S_n - \mathrm{E}\left[S_n\right] \geq t\right) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

$$\mathrm{P}\left(\left|S_n - \mathrm{E}\left[S_n\right]\right| \geq t\right) \leq 2\exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

In our case, δ is our desired probability bound, t is the lower bound on the distance of our estimate from the real value, and we can take $a_i=0$ and all $b_i=b$ (such as the value of the grand coalition)

Number of trials is roughly $O(\frac{b}{t^2} ln(\frac{1}{\delta}))$

A quick demo! (monte_carlo.jl)

Fairness - Strong Egalitarian Core

Suppose we have an outcome in the core

A player may have some amount of payoff over other players

This player could transfer some of their payoff with other players but not if they end up with less than the other player after the transfer

- Essentially, “I will share if I can...”

Bilateral transfer

Tuple (i, j, α, x) is ***bilateral transfer***, if

$$x_i - \alpha \geq x_j + \alpha.$$

Fairness - Strong Egalitarian Core

Egalitarian core:

- Point is in this if we cannot reach any other point in the core with a single bilateral transfer

Strong Egalitarian core

- We cannot reach any other point in the core with finitely many bilateral transfers
- This turns out to be a single point
- Just the outcome in the core with the minimal euclidean norm

Strong egalitarian core may not seem “fair” in the example
But can help contextualize the Shapley to determine how “equal” it is

Fairness - Pre-kernel

We define the excess as: $e^v(S,x) = v(S) - \sum_{k \in S} x_k$

Essentially saying "How much could I gain by simply breaking off the grand coalition?"

Lower surplus is good, means that total payoff in coalition is close to its value

For a player, coalitions that contain them with lower excess are better

Consider this excess for player *i* vs player *j*, we reach:

$$s_{ij}^v(x) = \max e^v(S,x) : S \in N - \{j\}, i \in S$$

Essentially saying "What's the most I could gain without the help of player *j*?"

Fairness - Pre-kernel

For a pair of players *i* and *j*, *i* will "outweigh" *j* if

- $s^v_{ij}(x) > s^v_{ji}(x)$ and
- $x_j > v(j)$

From the viewpoint of bargaining, "I can get more from leaving you than you can by leaving me and you will lose out anyways if I leave!"
Player *j* would have to submit to the "objection" to the current payoff *x* made by *i*

Imputations are outcomes that are efficient and individually rational

Kernel is the set of all imputations where no player outweighs anyone else, stable!
Kernel is non-empty if imputations are non-empty (our game has this always true)

Fairness - Pre-kernel

We can define the pre-kernel as the set of pre-imputations (efficient outcomes) where the surpluses between all pairs of players match.

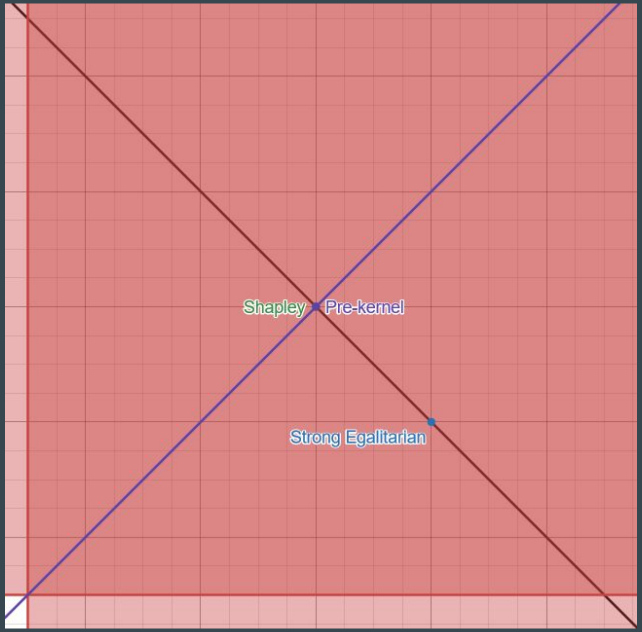
Essentially “All pairs of players are in equilibrium in terms of bargaining power over each other”

This solution:

- Always exists, but may take multiple values
- Imputations in pre-kernel are in the kernel
- Matches the kernel for weakly superadditive games
- Unique for convex games

Alternative measure of fairness for Shapley value

Complexity to find point is not well understood but can be found with iterative methods



{1}=3, {2}=7, {1,2}=20

Fairness - Pre-kernel

- Triangle: Pre-imputations
- Blue set: Core
- Blue center: Shapley
- Red center: Pre-kernel
- Black lines in core: Bargaining ranges

The pre-kernel is intersection of all the midpoints of all bargaining ranges

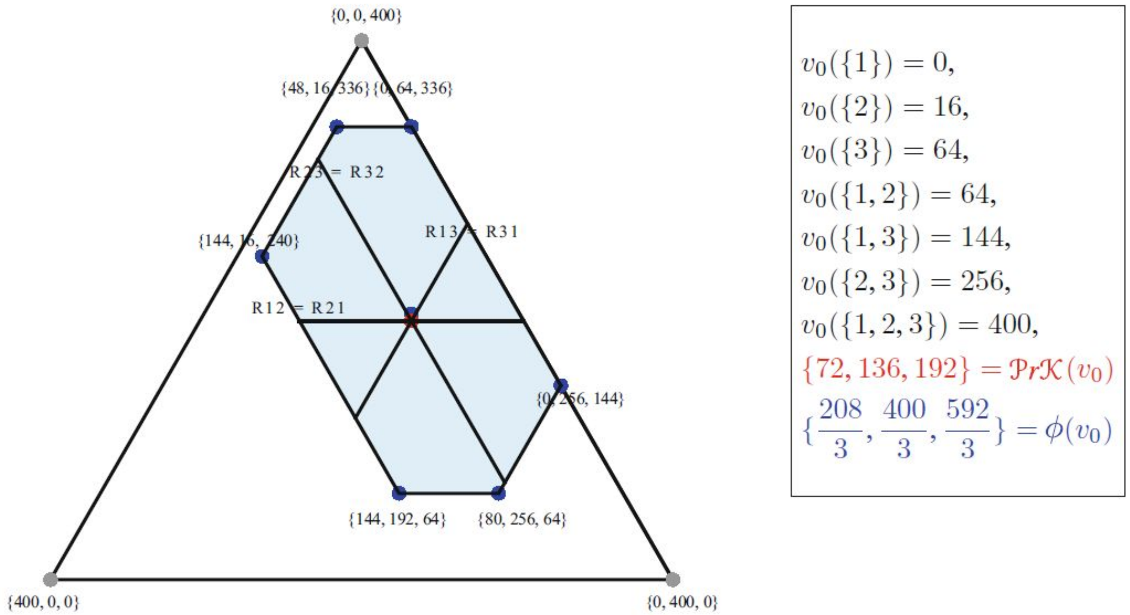


Fig. 3.1 Pre-Kernel solution splits the bargaining ranges in half

Sources

- *The Bargaining Set for Cooperative Games* by Robert Aumann and Michael Maschler
- *The Pre-Kernel as a Tractable Solution for Cooperative Games, An Exercise in Algorithmic Game Theory* by Holger Ingmar Meinhardt
- *The Kernel of a Cooperative Game Theory* by Morton Davis and Michael Maschler
- *Introduction to the Theory of Cooperative Games* by Bezalel Peleg and Peter Sudholter
- Made by Martin Černý: https://kam.mff.cuni.cz/~cerny/data/CG/CG9-fairness_handout.pdf

Without Mechanisms

- Without a Mechanism, the core can contain many unfair outcomes
- Only one province may actually benefit from the coalition
- Or a strong contributor may get almost nothing in return

- The core may not even exist!
- This is a very big detriment since this game is superadditive
- The grand coalition can supply the most goods overall

With Mechanisms

- The first mechanism is meant to ensure there always is a core
- We do this by penalizing provinces who wish to deviate from the grand coalition
- This is essentially an ϵ-core

- The second mechanism is meant to minimize the distance between the distance from the shapley value to any point in the core
- We do this by taking a tax from all provinces and redistributing the earnings from that tax.
- Preserves the value of the grand coalition

Mechanism - Penalty

- Ensures there will always be a core by penalizing provinces that do not cooperate towards a Grand Coalition
- This penalty will be a percent deduction from their original budget

- This should be used carefully since it makes it more difficult for a province to leave the grand coalition
- This can lead to significantly more unfair outcomes

- This should only be used to establish the existence of a core

Mechanism - Tax

- The primary mechanism we plan to implement is a tax based system on all the provinces
- Each province will be taxed an amount depending on some algorithm
  - Flat tax is easy but we end up taking money from poorer provinces only to give it right back
  - Proportional tax can do more with a lower amount of money taken
- This tax forms a pool that will then be redistributed to the provinces
- Poorest province gets enough money to match second poorest
- Then last and second last get enough to match third last and so on
- Any left over is equally distributed to all provinces

- Gives poorer provinces more leverage to avoid bad deals
- This will shift the Shapley value!

Example

Some demos for you!