Build X Algorithms: Cryptography

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Outline

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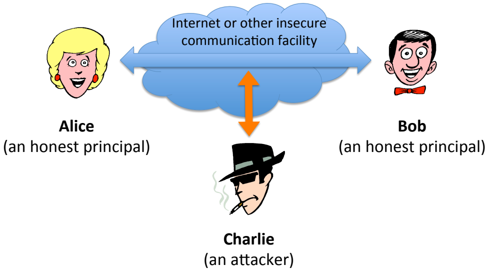
Cryptography in Network Security

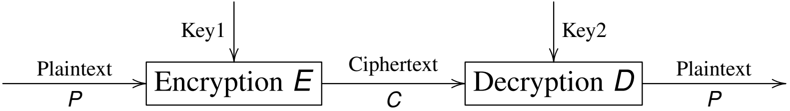


How do we turn an insecure communication facility (like the Internet) into a secure one?

Where security means that one of more security properties (e.g., confidentiality, integrity, authentication, non-repudiation, anonymity, unobservability, timeliness, availability, etc.) are guaranteed.

Cryptography is the enabling technology.

General cryptographic schema



where $E(\text{Key1}, P) = C$ and $D(\text{Key2}, C) = P$.

- **Symmetric algorithms**:
  - Key1 = Key2, or are easily derived from each other.
- **Asymmetric (or public key) algorithms**:
  - Different keys, which cannot be derived from each other.
  - Public key can be published without compromising private key.
- Encryption and decryption should be easy, if keys are known.
- **Security depends only on secrecy of the key, not on the algorithm.**

Encryption/decryption

- $\mathcal{A}$, the alphabet, is a finite set.
- $\mathcal{M} \subseteq \mathcal{A}^*$ is the message space. $M \in \mathcal{M}$ is a plaintext (message).
- $\mathcal{C}$ is the ciphertext space, whose alphabet may differ from $\mathcal{M}$.
- $\mathcal{K}$ denotes the key space of keys.
- Each $e \in \mathcal{K}$ determines a bijective function from $\mathcal{M}$ to $\mathcal{C}$, denoted by E_e. E_e is the encryption function (or transformation).

  Note: we will write $E_e(P) = C$ or, equivalently, $E(e, P) = C$.
- For each $d \in \mathcal{K}$, D_d denotes a bijection from $\mathcal{C}$ to $\mathcal{M}$. D_d is the decryption function.
- Applying E_e (or D_d) is called encryption (or decryption).

Encryption/decryption (cont.)

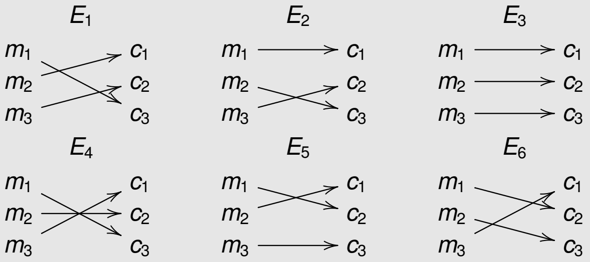
- An encryption scheme (or cipher) consists of a set $\{E_e \mid e \in \mathcal{K}\}$ and a corresponding set $\{D_d \mid d \in \mathcal{K}\}$ with the property that for each $e \in \mathcal{K}$ there is a unique $d \in \mathcal{K}$ such that $D_d = E_e^{-1}$; i.e.,

$$D_d(E_e(m)) = m \qquad \text{for all } m \in \mathcal{M}\,.$$

- The keys e and d above form a key pair, sometimes denoted by (e, d). They can be identical (i.e., the symmetric key).
- To construct an encryption scheme requires fixing a message space $\mathcal{M}$, a ciphertext space $\mathcal{C}$, and a key space $\mathcal{K}$, as well as encryption transformations $\{E_e \mid e \in \mathcal{K}\}$ and corresponding decryption transformations $\{D_d \mid d \in \mathcal{K}\}$.

An example

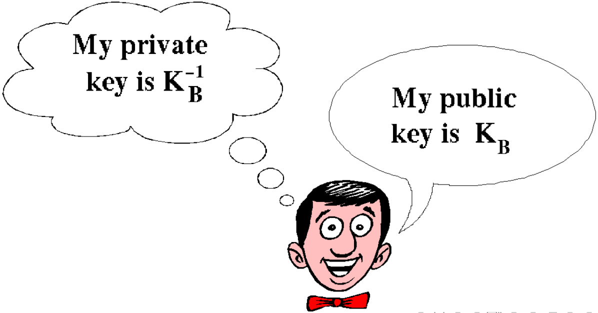
Let $\mathcal{M} = \{m_1, m_2, m_3\}$ and $\mathcal{C} = \{c_1, c_2, c_3\}$.
There are 3! = 6 bijections from $\mathcal{M}$ to $\mathcal{C}$.
The key space $\mathcal{K} = \{1, 2, 3, 4, 5, 6\}$ specifies these transformations.



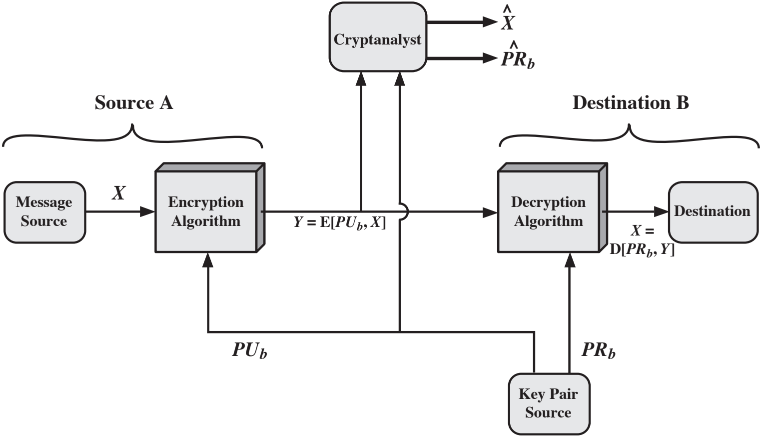
Suppose Alice and Bob agree on the transformation E_1.
To encrypt m_1, Alice computes $E_1(m_1) = c_3$.
Bob decrypts c_3 by reversing the arrows on the diagram for E_1 and observing that c_3 points to m_1.

Public-key cryptography

- Let $\{E_e \mid e \in \mathcal{K}\}$ and $\{D_d \mid d \in \mathcal{K}\}$ form an encryption scheme.
- Consider transformation pairs (E_e, D_d) where knowing E_e it is infeasible, given $c \in \mathcal{C}$, to find an $m \in \mathcal{M}$ such that $E_e(m) = c$.
- This implies it is **infeasible to determine** d **from** e.
- Hence, E_e constitutes a trap-door one-way function with trapdoor d (as explained in more detail later).
- Called **public key** as e can be public information:



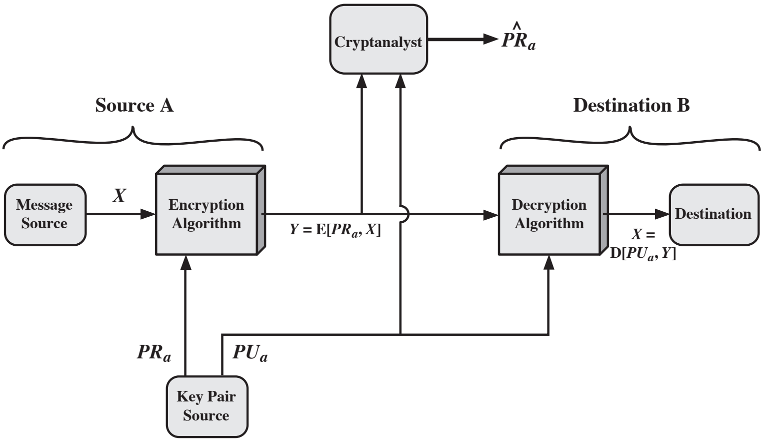
Public-key cryptosystem: secrecy (confidentiality)



Secrecy (confidentiality)

- X is a secret intended for B.
- Only B, who possesses PR_b, can decrypt $Y = E(PU_B, X)$.

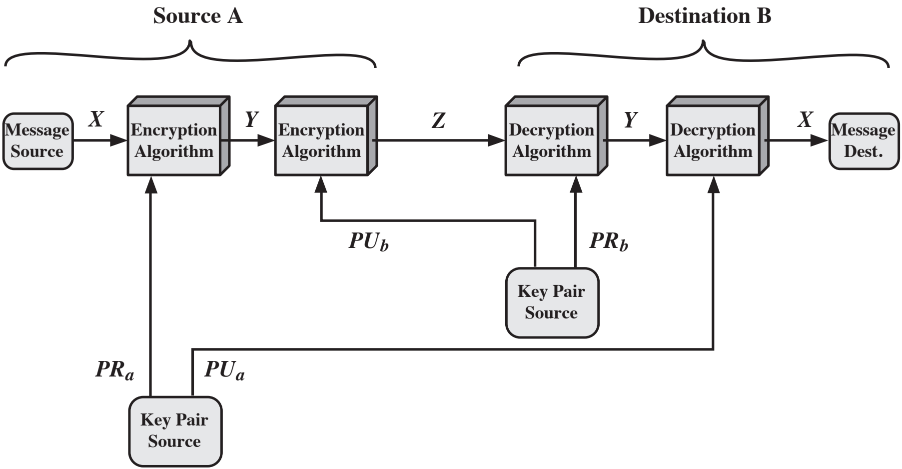
Public-key cryptosystem: authentication



Authentication

- Only A, who possesses PR_a, can have generated $Y = E(PR_A, X)$.
- Note that everybody can decrypt Y (and read X) as PU_a is public.

Public-key cryptosystem: secrecy and authentication



$$Z = E(PU_b, E(PR_a, X)) \quad \text{and} \quad X = D(PU_a, D(PR_b, Z))$$

Requirements for public-key cryptography

1. It is computationally easy for any principal B to generate a pair (public key PU_b, private key PR_b).
2. It is computationally easy for sender A, knowing PU_b and M, to generate
$$C = E(PU_b, M) \,.$$
3. It is computationally easy for receiver B to decrypt C using PR_b to recover M:
$$M = D(PR_b, C) = D(PR_b, E(PU_b, M)) \,.$$
4. It is computationally infeasible for an adversary
   - knowing PU_b to determine PR_b,
   - knowing PU_b and C to recover M.
5. (Useful, but not always necessary) The two keys can be applied in either order:
$$M = D(PU_b, E(PR_b, M)) = D(PR_b, E(PU_b, M)) \,.$$

Requirements for Public-Key Cryptography (cont.)

- These are difficult requirements.
- As a matter of fact only a few algorithms enjoying the above requirements have received widespread acceptance so far, e.g.,

Algorithm	**Encryption/Decryption**	**Digital signature**	**Key exchange**
RSA	Yes	Yes	Yes
Elliptic Curve	Yes	Yes	Yes
Diffie-Hellman	No	No	Yes
DSS	No	Yes	No

- We will focus on RSA and Diffie-Hellman.

One-way function

One-way function

A function $f : X \to Y$ is a **one-way function**, if f is "easy" to compute for all $x \in X$, but f^{-1} is "hard" (or "infeasible") to compute.

- **Easy**: generally, defined to mean a problem that can be solved in polynomial time as a function of input length.
  - If input length is n bits, then time to compute function is proportional to n^a, where a is a fixed constant.
- **Infeasible**: effort to solve problem grows faster than polynomial time as a function of input size.
  - Time to compute function proportional to 2^n for input length n bits.
    - Difficult to determine if a particular algorithm exhibits this complexity.
    - Computational complexity traditionally focuses on worst-case or average-case complexity of an algorithm, but cryptography requires that it be infeasible to invert a function for virtually all inputs.

One-way function: examples

- **Square root.**
  - If you know $x = 512$, $f(x) = x^2 = 512^2 =$

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  - If you know $f(x) = 262144$, $x = \sqrt{x^2} = \sqrt{262144}$ is difficult to compute.

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  - If you know $f(x) = 262144$, $x = \sqrt{x^2} = \sqrt{262144}$ is difficult to compute.
- **Modular cube roots.**
  - Select primes $p = 48611$ and $q = 53993$.
  - Let $n = p \times q = 2624653723$ and $X = \{1, 2, \ldots, n-1\}$.
  - Define $f : X \to \mathbb{N}$ by $f(x) = x^3 \bmod n$.
  - Example: $f(2489991) = 1981394214$.
  - Computing f is easy.
  - Inverting f is hard: find x which is cubed and yields remainder!

One-way function: examples

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This is useful because:

- Encryption is (very) easy whereas decryption is (very) difficult.
- The idea is: "$f(x)$ acts as a public key and x as a private key".

Trapdoor one-way function

- A trapdoor one-way function is easy to calculate in one direction and infeasible to calculate in the other direction unless certain additional information is known.
  - With additional info, inverse can be calculated in polynomial time.

Trapdoor one-way function

A **trapdoor one-way function** is a one-way function $f_k : X \to Y$ where, given extra information k (the trapdoor information) it is feasible to find, for $y \in Image(f)$, an $x \in X$ where $f_k(x) = y$.

- Hence, a *trapdoor one-way function is a family of invertible functions* f_k such that computing

$Y = f_k(X)$	is easy if k and X are known
$X = f_k^{-1}(Y)$	is easy if k and Y are known
$X = f_k^{-1}(Y)$	is infeasible if Y is known but k is not known

- **Example:** Computing modular cube roots is easy when p and q are known (basic number theory).

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Prime factorization

- Numbers: **naturals** $\mathbb{N} = \{0, 1, 2, \ldots\}$, **integers** $\mathbb{Z} = \{0, 1, -1, \ldots\}$, **primes** $\mathcal{P} = \{2, 3, 5, 7, ...\}$.
- To **factor** a number a is to write it as a product of other numbers, e.g., $a = b \times c \times d$.
- Multiplying numbers is easy, factoring numbers appears hard. We cannot factor most numbers with more than 1024 bits.
- The **prime factorization** of a number a amounts to writing it as a product of powers of primes:

$$a = \prod_{p \in \mathcal{P}} p^{a_p} = 2^{a_2} \times 3^{a_3} \times 5^{a_5} \times 7^{a_7} \times 11^{a_{11}} \times \ldots \quad \text{where } a_p \in \mathbb{N}$$

  For any particular value of a, most of the exponents a_p will be 0, e.g.,

$$\begin{aligned} 91 &= 7 \times 13 \\ 3600 &= 2^4 \times 3^2 \times 5^2 \\ 11011 &= 7 \times 11^2 \times 13 \end{aligned}$$

Divisors

$a \neq 0$ **divides** b (written $a \mid b$) if there is an m such that $m \times a = b$.

- Examples: $3 \mid 6$ and $7 \mid 21$.

a **does not divide** b (written $a \nmid b$) if there is no m such that $m \times a = b$.

- Examples: $3 \nmid 7$, $3 \nmid 10$ and $7 \nmid 22$.

Relatively prime numbers & greatest common divisor

Two natural numbers a, b are **relatively prime** if they have no common divisors/factors apart from 1, i.e., if their **greatest common divisor** gcd is equal to 1

$$\gcd(a, b) = 1\,.$$

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  - factors of 15 are 1, 3, 5, 15,
  - and 1 is the only common factor.
- Conversely, we can determine the greatest common divisor by comparing their prime factorizations and using least powers, e.g.
  - $150 = 2^1 \times 3^1 \times 5^2$ and $18 = 2^1 \times 3^2$, thus $\gcd(18, 150) = 2^1 \times 3^1 \times 5^0 = 6$.

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  - $60 = 2^2 \times 3 \times 5$ and $14 = 2 \times 7$, thus $\gcd(60, 14) = 2$.

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Greatest common divisor and Euclid's algorithm

- gcd can be computed quickly using **Euclid's algorithm.**

$$\begin{array}{lcrcl} \mathrm{gcd}(60,14) & : & 60 & = & (4 \times 14) + 4 \\ \mathrm{gcd}(14,4) & : & 14 & = & (3 \times 4) + 2 \\ \mathrm{gcd}(4,2) & : & 4 & = & 2 \times 2 \end{array}$$

- **Extended Euclid's algorithm** computes $x, y \in \mathbb{Z}$ such that

$$\mathrm{gcd}(a,b) = (x \times a) + (y \times b)$$

Here $2 = 14 - 3 \times (60 - (4 \times 14)) = (-3 \times 60) + (13 \times 14)$

Euclid's Algorithm

Euclid's algorithm is based on the theorem

$\gcd(a, b) = \gcd(b, a \bmod b)$ for any nonnegative integer a and any positive integer b.

For example:

- $\gcd(55, 22) = \gcd(22, 55 \bmod 22) = \gcd(22, 11) = 11.$

Euclid's algorithm

Euclid(a, b)
1 **if** $b = 0$
2   **then return** a
3   **else return** Euclid(b, $a \bmod b$)

For example:

- Euclid$(30, 21)$ = Euclid$(21, 9)$ = Euclid$(9, 3)$ = Euclid$(3, 0)$ = 3.

Extended Euclid's Algorithm

Extend Euclid's algorithm to compute integer coefficients x, y such that

$$d = \gcd(a, b) = (a \times x) + (b \times y)$$

Extended Euclid's algorithm

Extended-Euclid(a, b)
1 **if** $b = 0$
2 **then return** $(a, 1, 0)$
3 $(d', x', y') \leftarrow$ Extended-Euclid$(b,\ a \bmod b)$
4 $(d, x, y) \leftarrow (d', y', x' - (\lfloor a/b \rfloor \times y'))$
5 **return** (d, x, y)

where $q = \lfloor a/b \rfloor$ is the **quotient of the division** (for $a = (q \times b) + r$).

Note: the *d* here is the greatest common **d**ivisor, not to be confused with the *d* that is (part of) an RSA private key (discussed later on).

Extended Euclid's Algorithm: example

$\text{Extended-Euclid}(99, 78) = 3 = (99 \times (-11)) + (78 \times 14)$

Extended-Euclid(a, b)
1 **if** $b = 0$
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5 **return** (d, x, y)

a	b	$\lfloor a/b \rfloor$	d	x	y

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a	b	$\lfloor a/b \rfloor$	d	x	y
99	78				

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a	b	$\lfloor a/b \rfloor$	d	x	y
99	78	1			
78	21				

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a	b	$\lfloor a/b \rfloor$	d	x	y
99	78	1			
78	21	3			
21	15				

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99	78	1			
78	21	3			
21	15	1			
15	6	2			
6	3				

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a	b	$\lfloor a/b \rfloor$	d	x	y
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3	0	–			

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a	b	$\lfloor a/b \rfloor$	d	x	y
99	78	1	3		
78	21	3	3		
21	15	1	3		
15	6	2	3		
6	3	2	3	0	1
3	0	–	3	1	0

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99	78	1	3		
78	21	3	3		
21	15	1	3		
15	6	2	3	1	−2
6	3	2	3	0	1
3	0	–	3	1	0

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return (d, x, y)

a	b	$\lfloor a/b \rfloor$	d	x	y
99	78	1	3		
78	21	3	3		
21	15	1	3	-2	3
15	6	2	3	1	-2
6	3	2	3	0	1
3	0	–	3	1	0

Extended Euclid's Algorithm: example

$$\text{Extended-Euclid}(99, 78) = 3 = (99 \times (-11)) + (78 \times 14)$$

Extended-Euclid(a, b)
1 **if** $b = 0$
2 **then return** $(a, 1, 0)$
3 $(d', x', y') \leftarrow$ Extended-Euclid(b, $a \bmod b$)
4 $(d, x, y) \leftarrow (d', y', x' - (\lfloor a/b \rfloor \times y'))$
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Each line shows one level of the recursion.

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Modular arithmetics

Remainder

- $\forall a\, n.\ \exists q\, r.\ (a = (q \times n) + r)$ where $0 \leq r < n$.

  Here r is the **remainder**, which we write as

$$r = a \bmod n\,.$$

Congruent modulo

- $a, b \in \mathbb{Z}$ are **congruent modulo** n, if $a \bmod n = b \bmod n$.

  We write this as

$$a =_n b\,.$$

Modulo operator has following properties (of congruences)

- Reflexivity: $a =_n a$.
- Symmetry: If $a =_n b$ then $b =_n a$.
- Transitivity: If $(a =_n b$ and $b =_n c)$ then $a =_n c$.

Other properties of the modulo operator

$(a \bullet b) =_n (a \bmod n) \bullet (b \bmod n) \qquad \text{for } \bullet \in \{+, -, \times\}$

i.e., $(a \bullet b) \bmod n = [(a \bmod n) \bullet (b \bmod n)] \bmod n$

Example:

$$2 = (5 \times 6) \bmod 4$$

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Example:

$$\begin{aligned} 2 &= (5 \times 6) \bmod 4 \\ &= [(5 \bmod 4) \times (6 \bmod 4)] \bmod 4 \\ &= (1 \times 2) \bmod 4 = 2 \bmod 4 = 2 \end{aligned}$$

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If $a \times b =_n a \times c$ and a relatively prime to n, then $b =_n c$.

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Example:

- $8 \times 4 =_3 8 \times 1$.

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- $8 \times 4 =_3 8 \times 1$.
- 8 is relatively prime to 3.

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If $a \times b =_n a \times c$ and a relatively prime to n, then $b =_n c$.

Example:

- $8 \times 4 =_3 8 \times 1$.
- 8 is relatively prime to 3.
- So: $4 =_3 1$.

If $a_1 =_n b_1$ and $a_2 =_n b_2$, then

$$(a_1 + a_2) =_n (b_1 + b_2) \quad \text{and} \quad (a_1 \times a_2) =_n (b_1 \times b_2)$$

- This can also be expressed as

$$[(a_1 \bmod n) + (a_2 \bmod n)] \bmod n = (a_1 + a_2) \bmod n$$

  and

$$[(a_1 \bmod n) \times (a_2 \bmod n)] \bmod n = (a_1 \times a_2) \bmod n$$

- Example: Let $r_a = a \bmod n$ and $r_b = b \bmod n$.
  Then, there are integers j and k such that

$$a = r_a + jn \text{ and } b = r_b + kn$$

  and we can proceed as follows:

$$(a + b) \bmod n \quad = \quad (r_a + jn + r_b + kn) \bmod n$$

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  and we can proceed as follows:

$$\begin{aligned}
(a+b) \bmod n &= (r_a + jn + r_b + kn) \bmod n \\
&= (r_a + r_b + (j+k)n) \bmod n \\
&= (r_a + r_b) \bmod n \\
&= [(a \bmod n) + (b \bmod n)] \bmod n
\end{aligned}$$

- $a = q \times n + r$ with $q = \lfloor a/n \rfloor$ and $0 \leq r < n$ and $r = a \bmod b$
- For any integer a, we can rewrite this as follows:
  $a = \lfloor a/n \rfloor \times n + (a \bmod n)$

Then, for example:

- $11 \bmod 7 = 4$
- $-11 \bmod 7 = -4$ ($= 3$ when reasoning modulo 7)
- $73 =_{23} 4$
- $21 =_{10} -9$
- $147 =_{220} -73$

- If $a =_n 0$ then $n \mid a$
- $a =_n b$ if $n \mid (a - b)$

- To demonstrate the last point, if $n \mid (a - b)$, then $(a - b) = k \times n$ for some k.

  So we can write $a = b + (k \times n)$.

  Therefore, (a mod n) = (remainder when $b + (k \times n)$ is divided by n) = (remainder when b is divided by n) = (b mod n).
- Then, for example:
  - $23 =_5 8$ because $23 - 8 = 15 = 5 \times 3$
  - $-11 =_8 5$ because $-11 - 5 = -16 = 8 \times (-2)$
  - $81 =_{27} 0$ because $81 - 0 = 81 = 27 \times 3$

Modular arithmetics: two theorems

Theorem

Suppose that $a, b \in \mathbb{Z}$ are relatively prime. There is a $c \in \mathbb{Z}$ satisfying $(b \times c) \bmod a = 1$, i.e., we can compute $b^{-1} \bmod a$.

Proof: From Extended Euclidean Algorithm, there exist $x, y \in \mathbb{Z}$ where

$$1 = (a \times x) + (b \times y)$$

Since $a \mid (a \times x)$, we have $(b \times y) \bmod a = 1$.
Assertion follows with $c = y$.

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Fermat's little theorem

For a and n relatively prime and n prime

$$a^{n-1} =_n 1$$

Example: $4^6 \bmod 7 = (16 \times 16 \times 16) \bmod 7 = (2 \times 2 \times 2) \bmod 7 = 1$.

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Euler Totient Function

- When doing arithmetic modulo n.
- Complete set of **residues** is $0, \ldots, n-1$.
- **Reduced set of residues** consists of those numbers (*residues*) that are relatively prime to n.

  For instance, for $n = 10$:
  - complete set of residues is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
  - reduced set of residues is $\{1, 3, 7, 9\}$.
- Number of elements in reduced set of residues is called the **Euler Totient Function** $\phi(n)$.
  - In other words, $\phi(n)$ **is the number of positive integers less than** n **which are relatively prime to** n, i.e.,
    $\phi(n)$ is the number of $a \in \{1, 2, \ldots, n-1\}$ with $\gcd(a, n) = 1$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8

Euler's Totient Function and Euler's Theorem

Properties:

- $\phi(1) = 1$.
- $\phi(p) = p - 1$ if p is prime.
- $\phi(p \times q) = \phi(p) \times \phi(q) = (p - 1) \times (q - 1)$ if p and q are prime and $p \neq q$.

So that Fermat's little theorem (for a and n relatively prime and n prime) can be rewritten to

Euler's Theorem

$a^{\phi(n)} =_n 1$ for all a, n such that $\gcd(a, n) = 1$.

Examples:

- If $a = 3$ and $n = 10$, then $\phi(10) = 4$ and $3^4 = 81 =_{10} 1$
- If $a = 2$ and $n = 11$, then $\phi(11) = 10$ and $2^{10} = 1024 =_{11} 1$

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Rivest, Shamir, Adleman: RSA Algorithm

- Named after inventors: Rivest, Shamir, Adleman, 1978.
- Published after 1976 challenge by Diffie and Hellman.
- RSA algorithm is a block cipher in which the plaintext and ciphertext are integers between 0 and $n - 1$ for some n.
  - A typical size for n is 1024 bits, or 309 decimal digits.
  - That is, n is less than 2^{1024}.
- Security comes from difficulty of factoring large numbers. Keys are functions of a pairs of large, $\geq$ 100 digits, prime numbers.
- Most popular public-key algorithm. Used in many applications, e.g., PGP, PEM, SSL, ...

RSA algorithm

Ingredients:

	p, q, two prime numbers	private, chosen
	$n = p \times q$ (or pq for short)	public, calculated
	e, with $\gcd(\phi(n), e) = 1$; $1 < e < \phi(n)$	public, chosen
	$d = e^{-1} \bmod \phi(n)$	private, calculated

- Generation of a public/private key pair:
  1. Generate two (large) distinct primes p and q.
  2. Compute $n = p \times q$ and $\phi(n) = (p-1) \times (q-1)$.
  3. Select an e, with $1 < e < \phi(n)$, relatively prime to $\phi(n)$.
  4. Compute $d = e^{-1} \bmod \phi(n)$.
  5. Publish (e, n), keep (d, n) private, discard p and q.
- Encryption with key (e, n)
  1. Break message M into blocks $M_1 M_2 \cdots$ with $M_i < n$
  2. Compute $C_i = M_i^e \bmod n$.
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RSA algorithm: example

- Generation of a public/private key pair:
  1. Generate $p = 47$, $q = 71$.
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  5. Public key $(e, n) = (79, 3337)$, private key $(d, n) = (1019, 3337)$
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  1. Break message M into blocks, e.g., 688 232 687 966 668 ...
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- Encryption with key $(e, n) = (79, 3337)$:
  1. Break message M into blocks, e.g., 688 232 687 966 668 ...
  2. Compute $C_1 = 688^{79} \bmod 3337 = 1570$, $C_2 = \ldots$
- Decryption with key (d, n) :
  1. Compute $M_i = C_i^d \bmod n$.

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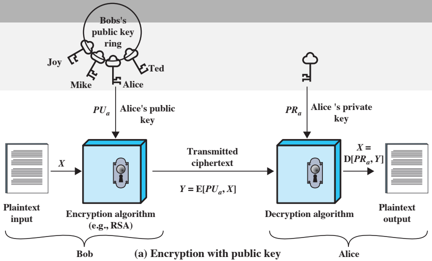
- Encryption with key $(e, n) = (79, 3337)$:
  1. Break message M into blocks, e.g., 688 232 687 966 668 ...
  2. Compute $C_1 = 688^{79} \bmod 3337 = 1570$, $C_2 = \ldots$
- Decryption with key $(d, n) = (1019, 3337)$:
  1. Compute $M_1 = 1570^{1019} \bmod 3337 = 688$, $M_2 = \ldots.$

RSA: another example



Alice generates a public/private key pair.
Bob encrypts using Alice's public key.
Alice decrypts using her private key.

- Keys can be generated as follows:
  - Select two prime numbers, $p = 17$ and $q = 11$.
  - Calculate $n = p \times q = 17 \times 11 = 187$.
  - Calculate $\phi(n) = (p - 1) \times (q - 1) = 16 \times 10 = 160$.
  - Select e such that e is relatively prime to $\phi(n) = 160$ and less than $\phi(n)$; we choose $e = 7$.
  - Determine d (e.g., using Extended Euclid's algorithm) such that $d \times e = 1 \bmod 160$ and $d < 160$.
    The correct value is $d = 23$, because $23 \times 7 = 161 = (1 \times 160) + 1$.

Note: the d here is the private key, not to be confused with the d that is the greatest common **d**ivisor in the Extended Euclid's algorithm.

Resulting keys are public key $PU_a = (e, n) = (7, 187)$ and private key $PR_a = (d, n) = (23, 187)$.

RSA algorithm: another example (cont.)

- We have $n = 187$, $\phi(n) = 160$, $e = 7$.

RSA algorithm: another example (cont.)

- We have $n = 187$, $\phi(n) = 160$, $e = 7$.
- d can be computed using the Extended Euclid algorithm

$$D = \mathrm{gcd}(A, B) = A \times x + B \times y$$

as follows:

RSA algorithm: another example (cont.)

- We have $n = 187$, $\phi(n) = 160$, $e = 7$.
- d can be computed using the Extended Euclid algorithm

$$D = \gcd(A, B) = A \times x + B \times y$$

  as follows:
  - Since d is such that $e \times d =_{\phi(n)} 1$, we can compute

$$1 = \gcd(\phi(n), e) = \phi(n) \times x + e \times d$$

  - It must be $1 < d < \phi(n)$, so when $y < 0$ we simply reason modulo $\phi(n)$.

RSA algorithm: another example (cont.)

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- In this case:

$$1 = \gcd(160, 7) = 160 \times x + 7 \times y$$

A	B	$\lfloor A/B \rfloor$	D	x	y
160	7	22	1	-1	23
7	6	1	1	1	-1
6	1	6	1	0	1
1	0	–	1	1	0

RSA algorithm: another example (cont.)

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That is, $1 = \gcd(160, 7) = 160 \times (-1) + 7 \times 23$.

RSA algorithm: another example (cont.)

- We have $n = 187$, $\phi(n) = 160$, $e = 7$.
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That is, $1 = \mathrm{gcd}(160, 7) = 160 \times (-1) + 7 \times 23$. Check: $7 \times 23 =_{160} 1$.

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- We have $n = 187$, $\phi(n) = 160$, $e = 7$.
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  as follows:
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  That is, $1 = \gcd(160, 7) = 160 \times (-1) + 7 \times 23$. Check: $7 \times 23 =_{160} 1$.
  So, we can pick $d = y = 23$.

RSA algorithm: another example (cont.)

- We have $n = 187$, $\phi(n) = 160$.
- Note that if we had picked $e = 23$, then $d = 7$.
  - Since d is such that $e \times d =_{\phi(n)} 1$, we can compute

$$1 = \gcd(\phi(n), e) = \phi(n) \times x + e \times d$$

- In this case:

$$1 = \gcd(160, 23) = 160 \times x + 23 \times y$$

A	B	$\lfloor A/B \rfloor$	D	x	y
160	23	6	1	-1	7
23	22	1	1	1	-1
22	1	22	1	0	1
1	0	–	1	1	0

That is, $1 = \gcd(160, 23) = 160 \times (-1) + 23 \times 7$. Check: $23 \times 7 =_{160} 1$.

So, we can pick $d = y = 7$.

RSA algorithm: a remark on the computed d

It must be $1 < d < \phi(n)$, so when $y < 0$ we simply reason modulo $\phi(n)$.

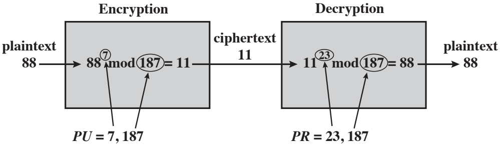
Consider, for example, $\phi(n) = 220$ and $e = 3$:

$$1 = \gcd(\phi(n), e) = \phi(n) \times x + e \times d = \gcd(220, 3) = 220 \times x + 3 \times y$$

A	B	$\lfloor A/B \rfloor$	D	x	y
220	3	73	1	1	-73
3	1	3	1	0	1
1	0	–	1	1	0

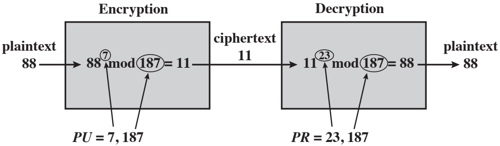
That is, $1 = \gcd(220, 3) = 220 \times 1 + 3 \times (-73) = 220 - 219$.
So, we can pick $d = 147$, i.e., $-73 \mod 220$.

RSA algorithm: another example (cont.)



- Let's continue the previous example.
- To encrypt a plaintext input $M = 88$, we need to calculate $C = M^e \bmod n = 88^7 \bmod 187 = 11$.
- We can do this by exploiting properties of modular arithmetic:
  - $88^7 \bmod 187 = ((88^4 \bmod 187) \times (88^2 \bmod 187) \times (88^1 \bmod 187)) \bmod 187$
  - $88^1 \bmod 187 = 88$
  - $88^2 \bmod 187 = 7744 \bmod 187 = 77$
  - $88^4 \bmod 187 = 59,969,536 \bmod 187 = 132$
  - $88^7 \bmod 187 = (88 \times 77 \times 132) \bmod 187 = 894,432 \bmod 187 = 11$

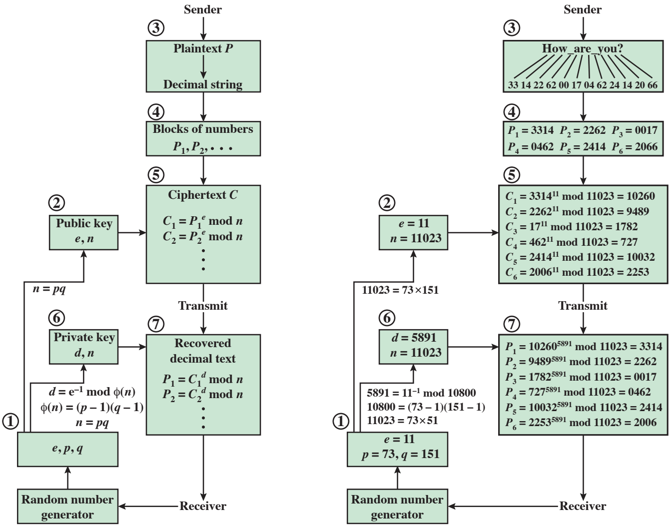
RSA algorithm: another example (cont.)



- For decryption, we calculate $M = C^d \bmod n = 11^{23} \bmod 187$:
  - $11^{23} \bmod 187 = ((11^1 \bmod 187) \times (11^2 \bmod 187) \times (11^4 \bmod 187) \times (11^8 \bmod 187) \times (11^8 \bmod 187)) \bmod 187$
  - $11^1 \bmod 187 = 11$
  - $11^2 \bmod 187 = 121$
  - $11^4 \bmod 187 = 14{,}641 \bmod 187 = 55$
  - $11^8 \bmod 187 = 214{,}358{,}881 \bmod 187 = 33$
  - $11^{23} \bmod 187 = (11 \times 121 \times 55 \times 33 \times 33) \bmod 187 = 79{,}720{,}245 \bmod 187 = 88$

Use of RSA to process multiple blocks of data: example

- In this simple example, plaintext is an alphanumeric string.
- Each plaintext symbol is assigned a unique code of 2 decimal digits (e.g., a = 00, A = 26).
- A plaintext block consists of 4 decimal digits, or 2 alphanumeric characters.
- Circled numbers indicate order in which operations are performed.



(a) General approach (b) Example

RSA Security

- Computation of secret d given (e, n).
  - As difficult as factorization. If we can factor $n = p \times q$ then we can compute $\phi(n) = (p-1) \times (q-1)$ and hence $d = e^{-1} \bmod \phi(n)$.
  - No known polynomial time algorithm.
    But given progress in factoring, n should have at least 1024 bits.
- Computation of M_i, given C_i, and (e, n).
  - Unclear (= no proof) whether it is necessary to compute d, i.e., to factorize n.

Hence: Progress in number theory could make RSA insecure.

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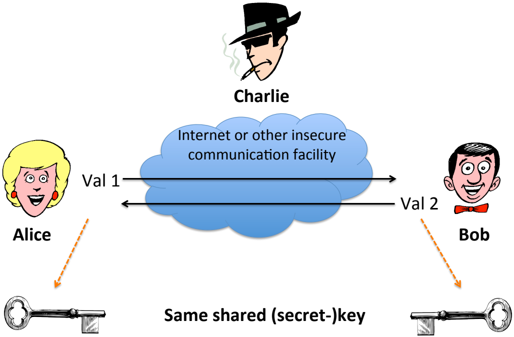
Diffie-Hellman key exchange: in a nutshell



- A simple public-key algorithm that enables two users to establish a secret key using a public-key scheme based on discrete logarithms.
- The protocol is secure only if the authenticity of the two participants can be established.

Background on discrete logarithms

- A **primitive root** s of a prime number p is a number whose powers generate $1, \ldots, p-1$.
  So s^0 mod p, s^1 mod p, s^2 mod p, ..., s^{p-1} mod p are distinct, i.e., a permutation of 1 through $p-1$. Hence:

  $$\forall b \in \mathbb{Z}. \exists i \in \{0, \ldots, p-1\}. \; b = s^i \bmod p$$

  In words: for any integer b and a primitive root s of prime number p, we can find a unique exponent i such that

  $$b = s^i \bmod p$$

  where $0 \leq i \leq (p-1)$.
  i is called the **discrete logarithm** of b for base s, mod p.

- Computing discrete logarithms appears infeasible today.

Diffie-Hellman key exchange

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3. A computes $Y_A = \alpha^{X_A} \bmod q$, B computes $Y_B = \alpha^{X_B} \bmod q$.
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$$K_A \quad = \quad Y_B^{X_A} \bmod q$$

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$$
\begin{aligned}
K_A &= Y_B^{X_A} \bmod q \\
&= (\alpha^{X_B} \bmod q)^{X_A} \bmod q = (\alpha^{X_B})^{X_A} \bmod q \\
&= \alpha^{X_A X_B} \bmod q = (\alpha^{X_A})^{X_B} \bmod q \\
&= (\alpha^{X_A} \bmod q)^{X_B} \bmod q = Y_A^{X_B} \bmod q \;=\; K_B
\end{aligned}
$$

Diffie-Hellman key exchange: ingredients

Global Public Elements

q	prime number
α	$\alpha < q$ and α a primitive root of q

User A Key Generation

Select private X_A	$X_A < q$
Calculate public Y_A	$Y_A = \alpha^{X_A} \bmod q$

User B Key Generation

Select private X_B	$X_B < q$
Calculate public Y_B	$Y_B = \alpha^{X_B} \bmod q$

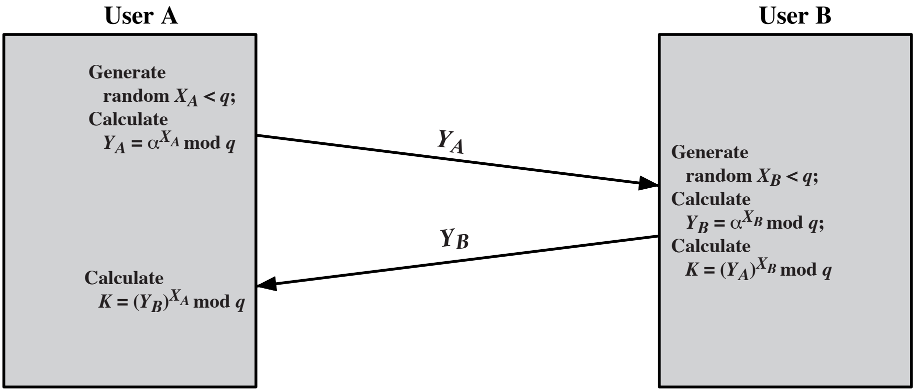
Calculation of Secret Key by User A

$K = (Y_B)^{X_A} \bmod q$

Calculation of Secret Key by User B

$K = (Y_A)^{X_B} \bmod q$

Diffie-Hellman key exchange: figure



Diffie-Hellman key exchange: strengths

- The shared secret key is never transmitted (not even in encrypted form)... it is created “out of nothing”!
  - $Y_A = \alpha^{X_A} \bmod q$ and $Y_B = \alpha^{X_B} \bmod q$ are the public keys.
  - X_A and X_B are the private keys.
  - Because X_A and X_B are private, an adversary C only has the following ingredients to work with: q, α, Y_A and Y_B.
  - Thus, C must take a discrete logarithm to determine the key. For example, to determine the private key of user B, C must compute
  $$X_B = \mathrm{dlog}_{\alpha,q}(Y_B)$$
- Security of Diffie-Hellman key exchange lies in the fact that
  - it is relatively easy to calculate exponentials modulo a prime, but
  - it is very difficult to calculate discrete logarithms (e.g., it is considered infeasible for large primes).

  Security depends on the difficulty of computing discrete logarithms.

Diffie-Hellman: example (calculating the secret key)

- A and B choose prime number $q = 353$ and $\alpha = 3$ (which is one of the primitive roots of 353).

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  - A computes $Y_A = \alpha^{X_A} \bmod q = 3^{97} \bmod 353 = 40$.

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- Each computes its public key:
  - A computes $Y_A = \alpha^{X_A} \bmod q = 3^{97} \bmod 353 = 40$.
  - B computes $Y_B = \alpha^{X_B} \bmod q = 3^{233} \bmod 353 = 248$.

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- After they exchange public keys, each can compute the common secret key K:
  - A computes $K = (Y_B)^{X_A} \bmod 353 = 248^{97} \bmod 353 = 160$.

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- After they exchange public keys, each can compute the common secret key K:
  - A computes $K = (Y_B)^{X_A} \bmod 353 = 248^{97} \bmod 353 = 160$.
  - B computes $K = (Y_A)^{X_B} \bmod 353 = 40^{233} \bmod 353 = 160$.
- Now they case use the symmetric key K to encrypt the messages they want to exchange.

Diffie-Hellman: example (attacking the key)

- Attacker C knows: $q = 353$, $\alpha = 3$, $Y_A = 40$ and $Y_B = 248$.
  - In this simple example, it would be possible by brute force to determine the secret key $K = 160$.
  - In particular, C can determine K by discovering a solution to
    - the equation $3^a \bmod 353 = 40$ or
    - the equation $3^b \bmod 353 = 248$.
  - Brute-force approach: calculate powers of 3 mod 353, stopping when the result equals either 40 or 248.
    - Desired answer is reached with the exponent value of 97, which provides $3^{97} \bmod 353 = 40$.
- With larger numbers, the problem becomes impractical.

Diffie-Hellman key exchange: weakness

Keys are **unauthenticated** and thus Diffie-Hellman key exchange is vulnerable to the following **man-in-the-middle attack**:

0. Attacker C prepares for the attack by generating two random private keys X_{C_1} and X_{C_2} and then computing the corresponding public keys $Y_{C_1} = \alpha^{X_{C_1}} \bmod q$ and $Y_{C_2} = \alpha^{X_{C_2}} \bmod q$ (since α and q are public).

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Now A and B think that they share a secret key,

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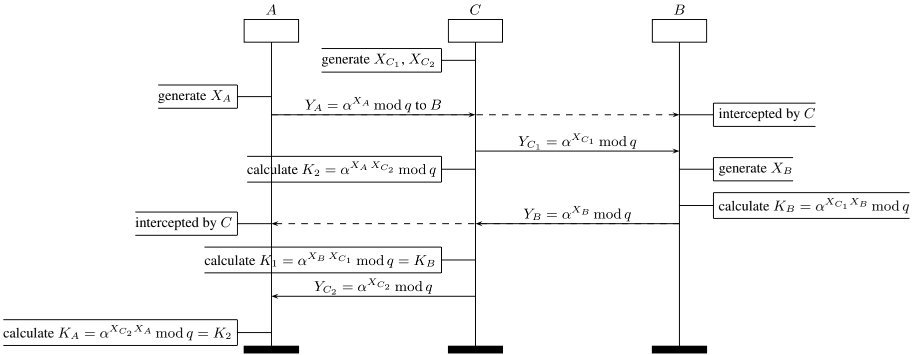
Now A and B think that they share a secret key, but instead A shares secret key $K_A = K_2$ with C and B shares secret key $K_B = K_1$ with C.

DH key exchange: man-in-the-middle attack



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DH key exchange: man-in-the-middle attack

- All future communication between Bob and Alice is compromised in the following way.
  1. A sends an encrypted message M, i.e., $E(K_2, M)$.
  2. C intercepts the encrypted message and decrypts it to recover M.
  3. C sends to Bob either
     - $E(K_1, M)$, if C simply wants to eavesdrop on the communication without altering it, or
     - $E(K_1, M')$, where M' is any message, if C wants to modify the message going to B.

- The Diffie-Hellman key exchange is vulnerable to such an attack because it does not authenticate the participants.
  - This vulnerability can be overcome with the use of **digital signatures and public-key certificates** to achieve mutual authentication between A and B.
  - Typically: add an exchange of digitally signed identification (ID) tokens.

Group Diffie-Hellman (for three or more parties)

Given a Diffie-Hellman group (α, q), three honest parties Alice, Bob and Carol can generate together a secret key $K = \alpha^{X_A X_B X_C} \bmod q$ by:

1. Alice chooses a random large integer X_A and sends to Bob: $Y_A = \alpha^{X_A} \bmod q$
2. Bob chooses a random large integer X_B and sends to Carol $Y_B = \alpha^{X_B} \bmod q$
3. Carol chooses a random large integer X_C and sends to Alice: $Y_C = \alpha^{X_C} \bmod q$
4. Alice sends to Bob $Y'_C = Y_C^{X_A} \bmod q$
5. Bob sends to Carol $Y'_A = Y_A^{X_B} \bmod q$
6. Carol sends to Alice $Y'_B = Y_B^{X_C} \bmod q$
7. Alice computes: $K = {Y'_B}^{X_A} \bmod q$
8. Bob computes: $K = {Y'_C}^{X_B} \bmod q$
9. Carol computes: $K = {Y'_A}^{X_C} \bmod q$

Can be extended to more parties by adding more rounds of computations.

Table of contents I

Idea

What do weapons of mass destruction

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What do weapons of mass destruction, a drink

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What do weapons of mass destruction, a drink and Ali Baba's cave all have in common?

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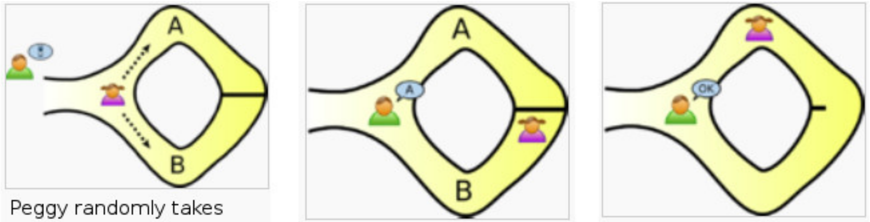
What do weapons of mass destruction, a drink and Ali Baba's cave all have in common?

Zero-knowledge proofs

In zero-knowledge proofs we can usually specify a statement that is being proved.

- Definitely, that statement is revealed to the verifier
- The verifier (or others) should not learn anything else
- Everybody can draw conclusions from everything they learned

Zero-knowledge proofs: Ali Baba’s cave

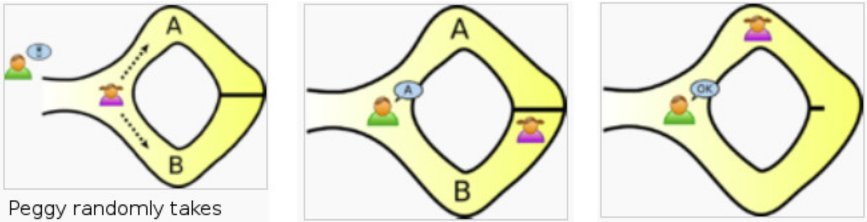


Peggy randomly takes either path A or B, while Victor waits outside

Victor chooses an exit path

Peggy reliably appears at the exit Victor names

- A cave has a door that opens only when a secret word is spoken.
- Peggy (the Prover) wants to convince Victor (the Verifier) that she knows the secret word, but without reveling it!
- If they walk to the door together, Peggy will be able to open it but then Victor will learn the secret word.
- So, they carry out a zero-knowledge protocol.



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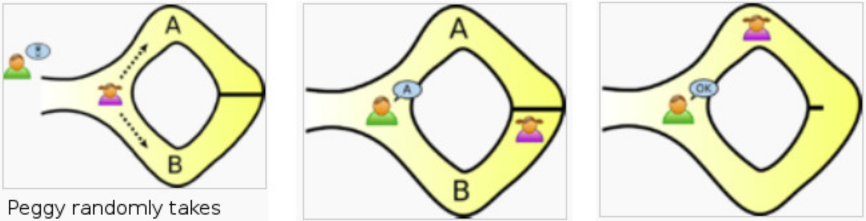
1. Victor stands at the cave's entrance, while Peggy walks to the door.
2. Victor walks to the bifurcation of the cave's paths and shouts to Peggy either to
   - come out of the left path A or
   - come out of the right path B.
3. Peggy complies, using the secret word to open the door, if needed.
4. Peggy and Victor repeat the experiment (steps 1-3) *n* times.



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- Now assume that Peggy doesn't actually know the secret word.
- Then she can only come out the way she went in.
  - After 1 round, she has only 1 chance out of 2 of fooling Victor.
  - After n rounds, she has only 1 chance out of 2^n of fooling Victor.
- So, after a while, Victor will be convinced that Peggy knows the secret.
- In other words: Peggy wins if she passes the test all of the time.
  - The probability that Peggy wins is very low if she does not know the secret word: after n rounds, it is $(1/2)^n = \frac{1}{2^n}$.

Zero-knowledge proofs: the idea

- In a challenge-response protocol, the Prover proves that she knows a secret.
  - If a symmetric cryptosystem is used, then the Verifier also knows the secret.
  - If a public-key signature system is used, then Verifier does not know the secret.
- An example of a zero-knowledge protocol is the Fiat-Shamir Identification Protocol.

Example: Fiat-Shamir Identification Protocol

- Three principals:
  - **Prover Peggy**,
  - **Verifier Victor** and
  - **Trusted Third Party Trent**.
- Setup:
  - Trent chooses two large prime numbers p and q to calculate $n = p \times q$.
  - n is announced to the public, whereas p and q are kept secret.
  - Peggy chooses a **secret number** s between 1 and $n - 1$, and calculates $v = s^2 \bmod n$.
    Peggy keeps s as her **private key** and registers v as her **public key** with the third party.
- Victor knows $v = s^2 \bmod n$, but does not know s.
- Squaring modulo n is easy to compute but square root modulo n is probably not (we believe...).
- **Goal**: Peggy wants to convince Victor that she knows the secret s but Victor should not learn s!

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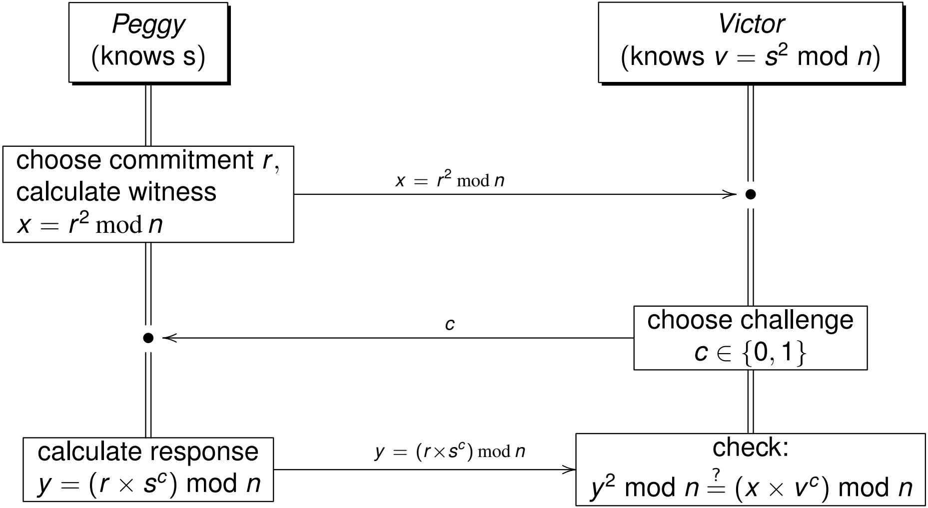
     She claims to be Peggy.
  4. Victor calculates $y^2 \bmod n$ and $(x \times v^c) \bmod n$. If these values are congruent, then Peggy either knows the value of s (she is honest) or she has calculated the value of y in some either ways (dishonest) because in modulo n arithmetic we actually have that

$$y^2 =_n (r \times s^c)^2 =_n r^2 \times s^{2c} =_n r^2 \times (s^2)^c =_n x \times v^c$$

- The 4 steps constitute a **round**.
- The verification is repeated several times with the value of c equal to 0 or 1, chosen randomly.
- Peggy must pass the test in each round to be verified: if she fails one single round, the process is aborted.



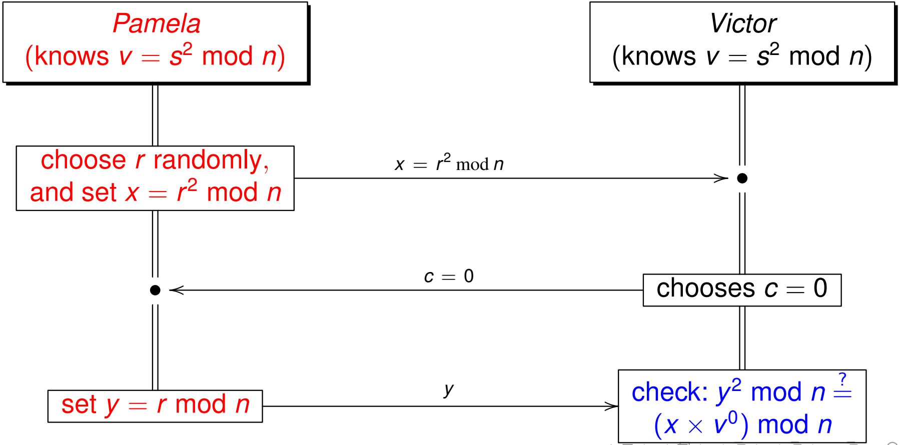
If the check

$$y^2 =_n (r \times s^c)^2 =_n r^2 \times s^{2c} =_n r^2 \times (s^2)^c =_n x \times v^c$$

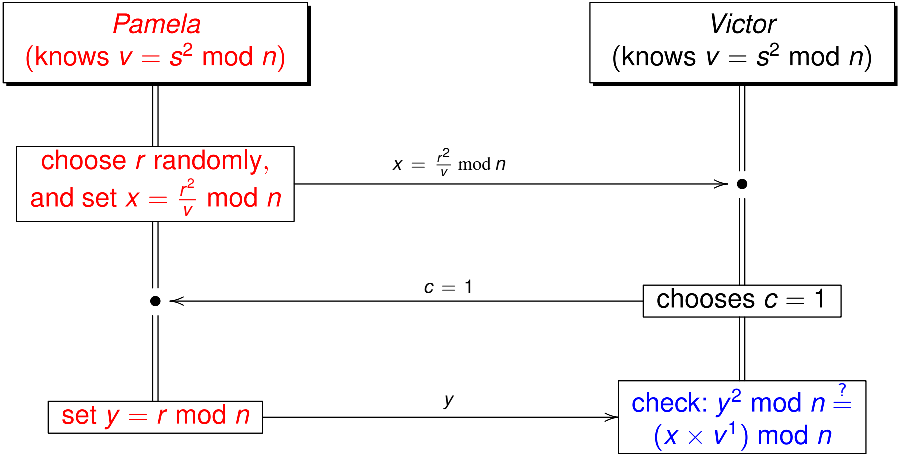
returns a yes, then verification is probable; otherwise, the process is aborted.

Fiat-Shamir: Trying to Cheat

Pamela does not know the secret s, but tries to prove its knowledge.
Pamela guesses that Victor is going to choose $c = 0$ (if she guesses wrong, then she loses).



Pamela guesses that Victor is going to choose $c = 1$ (if she guesses wrong, then she loses).



So, Pamela must find numbers x and y such that $x \times v =_n (x \times s^2)$.
Choose y randomly and then set $\frac{r^2}{v}$ mod n (division modulo n is also easy!).

Fiat-Shamir: Trying to Cheat

- Pamela has a strategy to cheat if $c = 0$:

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- If $c \in \{0, 1\}$ is randomly chosen, Pamela has thus a chance of $\frac{1}{2}$ to cheat.

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- If Victor accepts only after n successful rounds, the chance to cheat is only $\frac{1}{2^n}$.

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- If $c \in \{0, 1\}$ is randomly chosen, Pamela has thus a chance of $\frac{1}{2}$ to cheat.
- If Victor accepts only after n successful rounds, the chance to cheat is only $\frac{1}{2^n}$.
- We can conclude that

Pamela has no strategy to cheat for unpredictable c.

Fiat-Shamir: Curious Victor

Victor would like to learn the secret $x\ldots$
but we can conclude that

Fiat-Shamir: Curious Victor

Victor would like to learn the secret x...
but we can conclude that

Zero-Knowledge Property

Victor learns nothing except the proved statement.

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