Assignment 2

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1 Question 1

1.1 $x^T y$

As x,y are vectors in $\mathbb{R}^n$, then x and y will have the form $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

Therefore, the dot product operation can be written as $\sum_{i=1}^{n} x_i y_i$.
We have n multiplication as well n-1 addition, so in total, we will have 2n-1 operations. Therefore it is $\mathcal{O}(n)$.

1.2 Ax for $A \in \mathbb{R}^{n \times n}$

Suppose the matrix have the following form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}$$

We can do the matrix multiplication columnwise: $x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$ Then we have n operations for each x to times the vectors and n of such operations. Then there is n-1 plus. Therefore, there is $n^2 + n - 1$ operations, which is $\mathcal{O}(n^2)$
Alternatively we can do the matrix multiplication entry by entry. For each entry y_i, $y_i = \sum_{j=1}^{n} a_{ij} x_j$. We can use the result from part 1, which gives $2n^2 - n$ operations, (**from now on we will use this result**) which is $\mathcal{O}(n^2)$

1.3 Matrix product AB

Two n by n matrix multiply together is equivalent to matrix A multiply each of the column of B. We know Ax have $2n^2 - n$ operations, then n of them will have $2n^3 - n^2$ operations, which is $\mathcal{O}(n^3)$

2 Question 2

From the linear system of equations

$$\begin{pmatrix} D & B \\ C & \hat{A} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

, we can obtain two simultaneous equation.

$$\begin{cases} Dx + By = b_1 \\ Cx + \hat{A}y = b_2 \end{cases} \tag{1}$$

By substitutes x from the top equation to the bottom ones, we can obtain the following:

$$CD^{-1}(b_1 - By) + \hat{A}y = b_2$$
$$(\hat{A} - CD^{-1}B)y = b_2 - CD^{-1}b_1$$

Now I would like to compute the cost of computing the LU Decomposition of $\hat{A} - CD^{-1}B$.

1. let's look at the cost of inverting D. As D is diagonal, inverting the diagonal matrix only cost $\mathcal{O}(n)$.
2. $D^{-1}B$: For normal matrix multiplication, we need to do row times column n^2 times as there are n^2 entry. However, for a diagonalise matrix D^{-1}, (we know the inverse of a diagonalise matrix is also diagonalised) each time we multiply a row in D^{-1} to a column in B, we only multiply one value as other entry in D^{-1} is always 0. Therefore, there will be n^2 operations.
3. The cost of $CD^{-1}B$ will be $2n^3 - n^2$ because it is matrix times matrix which we already computed in Q1.
4. The cost for subtraction of $\hat{A} - CD^{-1}B$ will cost n^2.
5. LU decomposition will take $\frac{2}{3}n^3 + \mathcal{O}(n^2)$, which we already evaluated during the lecture.
6. Therefore, the overall costs for decomposing $\hat{A} - CD^{-1}B$ is $\frac{8}{3}n^3 + \mathcal{O}(n^2)$.

If we apply LU directly to A, we know A is $2n \times 2n$, therefore, the cost will be: $\frac{2}{3}(2n)^3 + \mathcal{O}(2n^2) = \frac{16}{3}n^3 + \mathcal{O}(n^2)$

3 Qeustion 3

$$
\begin{aligned}
Hx_0 &= (I - 2vv^T)x_0 \\
&= x_0 - 2vv^Tx_0 \\
&= x_0 - 2\frac{(\alpha\|x_0\|_2e_1 - x_0)(\alpha\|x_0\|_2e_1 - x_0)^Tx_0}{(\alpha\|x_0\|_2e_1 - x_0)^T(\alpha\|x_0\|_2e_1 - x_0)} \\
&= x_0 - 2(\alpha\|x_0\|_2e_1 - x_0)\frac{\alpha\|x_0\|_2e_1^Tx_0 - \|x_0\|_2^2}{2\|x_0\|_2^2 - 2\alpha\|x_0\|_2e_1^Tx_0} \quad \text{Expand the bottom, notice } \alpha\alpha = 1 \text{ and } e_1^Te_1 = 1 \\
&= x_0 - (x_0 - \alpha\|x_0\|_2e_1 \\
&= \alpha\|x_0\|e_1
\end{aligned}
$$

$$
\begin{aligned}
&H^TH \\
&= (I - 2vv^T)^T(I - 2vv^T) && H^T = H \\
&= (I - 2vv^T)(I - 2vv^T) \\
&= I - 4vv^T + 4vv^Tvv^T \\
&= I - 4vv^T + 4vv^T && v^Tv = 1 \\
&= I
\end{aligned}
$$

Therefore, $H^T = H^{-1}$ and we also got: $H^T = H$ which implies $H = H^{-1}$
As we have a symmetric matrix, and we know all real symmetric matrix will contains a basis of orthogonal basis, therefore, it is orthogonal.

4 Question 4a

$$
\begin{aligned}
\frac{\partial^2 y}{\partial x^2} &= -1 \\
\frac{\partial y}{\partial x} &= -x + c && \text{where c is a constant} \\
y &= -\frac{1}{2}x^2 + cx + d && \text{where d is a constant}
\end{aligned}
$$

As $y(0) = y(1) = 1$, we have

$$
\begin{cases} 0 = d \\ 0 = -\frac{1}{2} + c \end{cases}
$$

This implies $c = \frac{1}{2}$ and $d = 0$.

$$
y = -\frac{1}{2}x^2 + \frac{1}{2}x
$$

5 Question 4b

5.1 Part 1

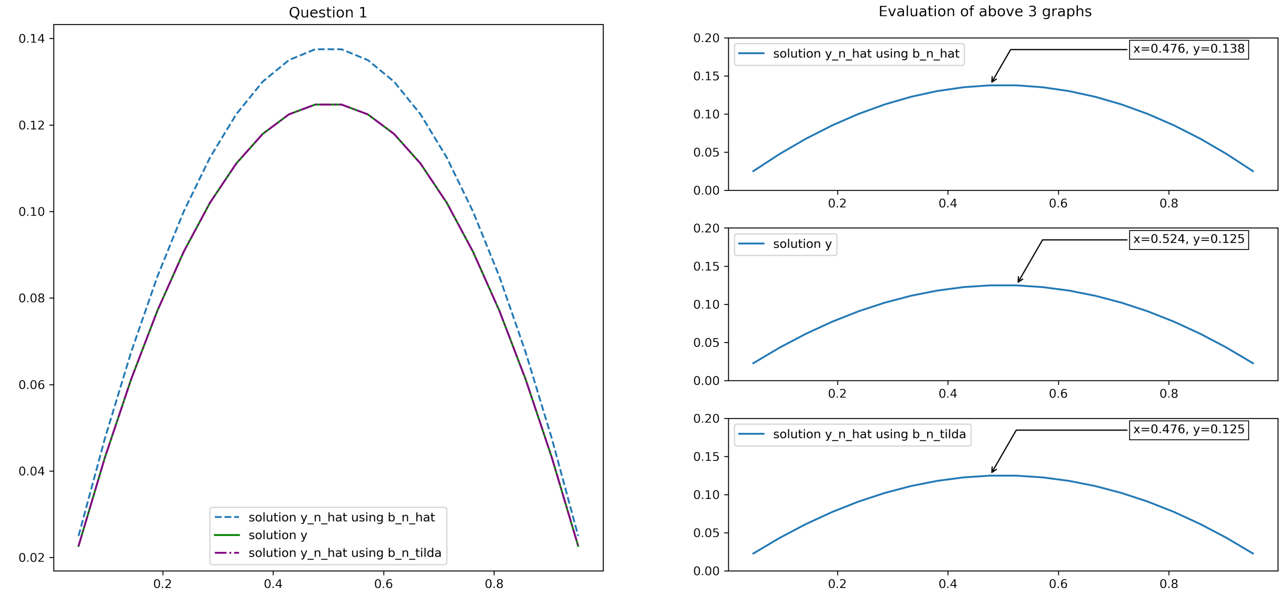


Figure 1: Part 1

5.2 Part 2

Here is the numerical result ($\hat{e}_n$) I got for $n \in \{20, 40, 80, 160\}$.

1. When $n = 20$, we get: en value when n=20 is [[-0.00232426] [-0.0044161] [-0.00627551] [-0.00790249] [-0.00929705] [-0.01045918] [-0.01138889] [-0.01208617] [-0.01255102] [-0.01278345] [-0.01278345] [-0.01255102] [-0.01208617] [-0.01138889] [-0.01045918] [-0.00929705] [-0.00790249] [-0.00627551] [-0.0044161] [-0.00232426]]

2. When $n = 40$, we get: en value when n=40 is [[-0.00060232] [-0.00117452] [-0.00171661] [-0.00222858] [-0.00271044] [-0.00316218] [-0.0035838] [-0.00397531] [-0.0043367] [-0.00466798] [-0.00496914] [-0.00524018] [-0.00548111] [-0.00569192] [-0.00587262] [-0.0060232] [-0.00614366] [-0.00623401] [-0.00629424] [-0.00632436] [-0.00632436] [-0.00629424] [-0.00623401] [-0.00614366] [-0.0060232] [-0.00587262] [-0.00569192] [-0.00548111] [-0.00524018] [-0.00496914] [-0.00466798] [-0.0043367] [-0.00397531] [-0.0035838] [-0.00316218] [-0.00271044] [-0.00222858] [-0.00171661] [-0.00117452] [-0.00060232]]

3. When $n = 80$, we get: en value when n=80 is [[-0.00015337] [-0.0003029] [-0.0004486] [-0.00059047] [-0.0007285] [-0.0008627] [-0.00099306] [-0.00111959] [-0.00124228] [-0.00136114] [-0.00147617] [-0.00158736] [-0.00169472] [-0.00179824] [-0.00189793] [-0.00199379] [-0.00208581] [-0.002174] [-0.00225835] [-0.00233887] [-0.00241555] [-0.0024884] [-0.00255742] [-0.0026226] [-0.00268395] [-0.00274146] [-0.00279514] [-0.00284498] [-0.00289099] [-0.00293317] [-0.00297151] [-0.00300602] [-0.00303669] [-0.00306353] [-0.00308654] [-0.00310571] [-0.00312105] [-0.00313255] [-0.00314022] [-0.00314405] [-0.00314405] [-0.00314022] [-0.00313255] [-0.00312105] [-0.00310571] [-0.00308654] [-0.00306353] [-0.00303669] [-0.00300602] [-0.00297151] [-0.00293317] [-0.00289099] [-0.00284498] [-0.00279514] [-0.00274146] [-0.00268395] [-0.0026226] [-0.00255742] [-0.0024884] [-0.00241555] [-0.00233887] [-0.00225835] [-0.002174] [-0.00208581] [-0.00199379] [-0.00189793] [-0.00179824] [-0.00169472] [-0.00158736] [-0.00147617] [-0.00136114] [-0.00124228] [-0.00111959] [-0.00099306] [-0.0008627] [-0.0007285] [-0.00059047] [-0.0004486] [-0.0003029] [-0.00015337]]

4. When $n = 160$, we get: en value when n=160 is [[-3.86993172e-05] [-7.69148928e-05] [-1.14646727e-04] [-1.51894820e-04] [-1.88659171e-04] [-2.24939781e-04] [-2.60736649e-04] [-2.96049776e-04] [-3.30879162e-04] [-3.65224806e-04] [-3.99086708e-04] [-4.32464869e-04] [-4.65359289e-04] [-4.97769967e-04] [-5.29696904e-04] [-5.61140099e-04] [-5.92099552e-04] [-6.22575265e-04] [-6.52567236e-04] [-6.82075465e-04] [-7.11099953e-04] [-7.39640699e-04] [-7.67697704e-04] [-7.95270968e-04] [-8.22360490e-04] [-8.48966270e-04] [-8.75088309e-04] [-9.00726607e-04] [-9.25881163e-04] [-9.50551978e-04] [-9.74739051e-04] [-9.98442383e-04] [-1.02166197e-03] [-1.04439782e-03] [-1.06664993e-03] [-1.08841830e-03] [-1.10970292e-03] [-1.13050380e-03] [-1.15082094e-03]

[-1.17065434e-03] [-1.19000400e-03] [-1.20886992e-03] [-1.22725210e-03] [-1.24515053e-03] [-1.26256522e-03] [-1.27949617e-03] [-1.29594338e-03] [-1.31190685e-03] [-1.32738658e-03] [-1.34238256e-03] [-1.35689481e-03] [-1.37092331e-03] [-1.38446807e-03] [-1.39752909e-03] [-1.41010637e-03] [-1.42219991e-03] [-1.43380970e-03] [-1.44493575e-03] [-1.45557807e-03] [-1.46573664e-03] [-1.47541147e-03] [-1.48460255e-03] [-1.49330990e-03] [-1.50153351e-03] [-1.50927337e-03] [-1.51652949e-03] [-1.52330187e-03] [-1.52959051e-03] [-1.53539541e-03] [-1.54071656e-03] [-1.54555398e-03] [-1.54990765e-03] [-1.55377758e-03] [-1.55716377e-03] [-1.56006622e-03] [-1.56248493e-03] [-1.56441990e-03] [-1.56587112e-03] [-1.56683860e-03] [-1.56732234e-03] [-1.56732234e-03] [-1.56683860e-03] [-1.56587112e-03] [-1.56441990e-03] [-1.56248493e-03] [-1.56006622e-03] [-1.55716377e-03] [-1.55377758e-03] [-1.54990765e-03] [-1.54555398e-03] [-1.54071656e-03] [-1.53539541e-03] [-1.52959051e-03] [-1.52330187e-03] [-1.51652949e-03] [-1.50927337e-03] [-1.50153351e-03] [-1.49330990e-03] [-1.48460255e-03] [-1.47541147e-03] [-1.46573664e-03] [-1.45557807e-03] [-1.44493575e-03] [-1.43380970e-03] [-1.42219991e-03] [-1.41010637e-03] [-1.39752909e-03] [-1.38446807e-03] [-1.37092331e-03] [-1.35689481e-03] [-1.34238256e-03] [-1.32738658e-03] [-1.31190685e-03] [-1.29594338e-03] [-1.27949617e-03] [-1.26256522e-03] [-1.24515053e-03] [-1.22725210e-03] [-1.20886992e-03] [-1.19000400e-03] [-1.17065434e-03] [-1.15082094e-03] [-1.13050380e-03] [-1.10970292e-03] [-1.08841830e-03] [-1.06664993e-03] [-1.04439782e-03] [-1.02166197e-03] [-9.98442383e-04] [-9.74739051e-04] [-9.50551978e-04] [-9.25881163e-04] [-9.00726607e-04] [-8.75088309e-04] [-8.48966270e-04] [-8.22360490e-04] [-7.95270968e-04] [-7.67697704e-04] [-7.39640699e-04] [-7.11099953e-04] [-6.82075465e-04] [-6.52567236e-04] [-6.22575265e-04] [-5.92099552e-04] [-5.61140099e-04] [-5.29696904e-04] [-4.97769967e-04] [-4.65359289e-04] [-4.32464869e-04] [-3.99086708e-04] [-3.65224806e-04] [-3.30879162e-04] [-2.96049776e-04] [-2.60736649e-04] [-2.24939781e-04] [-1.88659171e-04] [-1.51894820e-04] [-1.14646727e-04] [-7.69148928e-05] [-3.86993172e-05]]

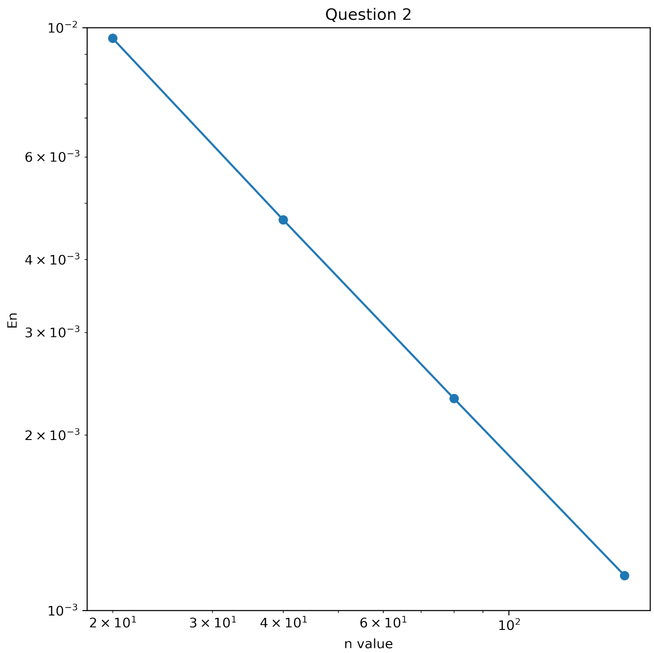
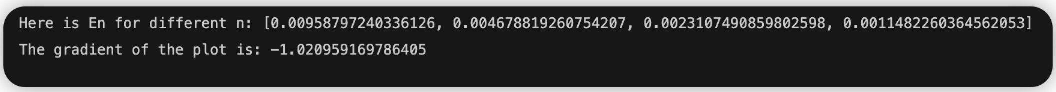


Figure 2: Log scale,Part 2.

Figure 2 shows we have a logarithm decay. We can conclude from the graph that the gradient of the graph is -1. Therefore it is $\mathcal{O}(n^{-1})$



```
Here is En for different n: [0.00958797240336126, 0.004678819260754207, 0.0023107490859802598, 0.0011482260364562053]
The gradient of the plot is: -1.020959169786405
```

Figure 3: En and gradient

5.3 Part 3

Figure 5 shows we have a logarithm growth. We can find out the gradient of the graph is approximately 2 which means $\mathcal{O}(n^2)$

```
Here is the conditional number for different n: [178.06427461086017, 680.6170700217076, 2658.4065019157188, 10504.718944451404]
The gradient of the plot is: 1.960230060929547
```

Figure 4: condition number and gradient

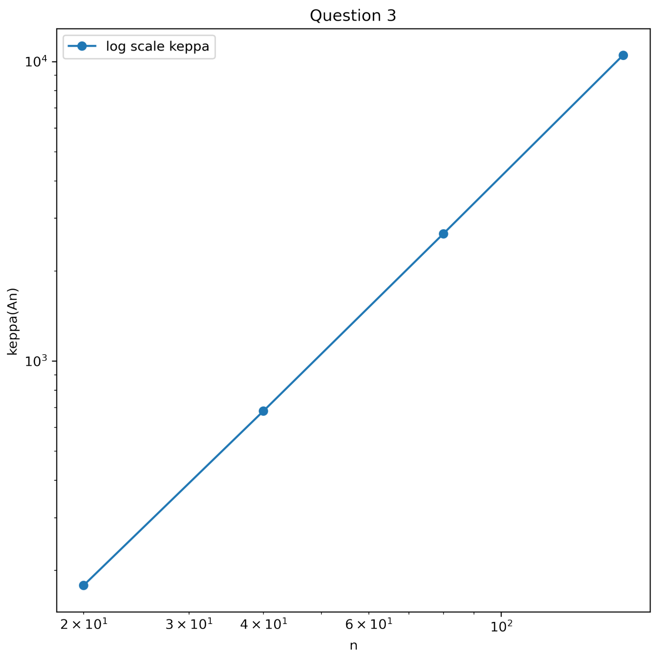


Figure 5: Conditional number.