And its inverse is:

$$\begin{aligned}
&(\begin{pmatrix} V & -V \\ U & U \end{pmatrix})^{-1} \\
=&\det(P)\begin{pmatrix} U^{-1} & V^{-1} \\ -U^{-1} & V^{-1} \end{pmatrix} \\
=&\frac{1}{2}\begin{pmatrix} U^T & V^T \\ -U^T & V^T \end{pmatrix} && \text{where we use the fact U,V are orthogonal vector}
\end{aligned}$$

Therefore the eigendecomposition for H is:

$$H = \begin{pmatrix} V & -V \\ U & U \end{pmatrix}\begin{pmatrix} \Sigma & \\ & -\Sigma \end{pmatrix}\frac{1}{2}\begin{pmatrix} U^T & V^T \\ -U^T & V^T \end{pmatrix} \tag{1}$$

3 Question 3

Here is a 4×4 A:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Now we know all eigenvalues of A lie in the union of the Gershgorin disks. The union of Gershgorin disks of $A \in \mathbb{C}^{n\times n}$ is of the following:

$$\begin{aligned}
S_i &= \{z \in \mathbb{C} : |z-2| \le 1\} && j = 1, n \\
\cup S_j &= \{z \in \mathbb{C} : |z-2| \le 2\} && j = 2, \cdots, n-1
\end{aligned}$$

Therefore, all the eigenvalues will be between $0 \le \lambda \le 4$. Now I would like to prove it cannot be 0.
Suppose there exists a 0 eigenvalue. Then there must exists at least one non-zero eigenvector V which satisfy the following condition:

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} V = 0$$

We can then formulate n simultaneous equation

$$\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = 0 \\ -x_2 + 2x_3 - x_4 = 0 \\ \quad\vdots \\ -x_{n-1} + x_n = 0 \end{cases}$$

After solving the equation we find out the only solution to them is $x_1 = x_2 = \cdots = x_n = 0$.
Therefore, 0 is not an eigenvector. $\Longrightarrow$ All eigenvector must be strictly bigger than 0.

4 Question 4

4.1 Part 1

Firstly substitute y_k into the differential equation:

$$\begin{aligned}
&-\frac{\mathrm{d}^2 \pm \sin(k\pi x)}{\mathrm{d}x^2} \\
=&-\frac{\mathrm{d} \pm k\pi\cos(k\pi x)}{\mathrm{d}x} \\
=&- \mp k^2\pi^2 \sin(k\pi x) \\
=&\pm k^2\pi^2 \sin(k\pi x) \\
=&(k\pi)^2 y_k
\end{aligned}$$