Assignment 4

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1 Question 1

Firstly we know the approximated eigenvalue $\tilde{\lambda} = 0.9$. The eigenvector $\tilde{x}$ where $\|x\|_2 = 1$ and $\|(A - \tilde{\lambda}I)\tilde{x}\|_2 \leq 10^{-2}$. Then by the theorem about eigenvalue perturbation. $\|(A - \tilde{\lambda}I)\tilde{x}\|_2 \leq 10^{-2}$ and $\|x\|_2 = 1 \implies \|\Delta A\|_2 \leq 10^{-2}$ where ΔA is the **absolute backward error**.
We also know

$$|\delta\lambda| \leq \frac{\|x\|\|y\|}{|x^H y|}\|\Delta A\|$$

. Therefore, for this question, we know the **absolute forward error** is bounded above by

$$|\delta\lambda| \leq \frac{\|y\|}{|x^H y|}10^{-2}$$

where y is the left eigenvector.
Remark: We haven't been given enough information to determine the value of the conditional number, therefore, that is the best result we can get.
If we have a real symmetric matrix A, then the hermitian form of A is the same as A. Moreover, the left eigenvectors are identical to the right eigenvectors. Since x=y, then we have

$$|\delta\lambda| \leq 10^{-2}$$

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2 Question 2

The SVD of A is given by $A = U\Sigma V^T$. Then H can be expressed as: $\begin{pmatrix} 0 & V\Sigma U^T \\ U\Sigma V^T & 0 \end{pmatrix}$
Using the hint in the question we know if σ is a singular value of A then $\pm\sigma$ are eigenvalues of H.
If Σ is a n by n matrix containing eigenvalues of A. Then

$$\begin{pmatrix} \Sigma & \\ & -\Sigma \end{pmatrix}$$

is the eigenvalues of H. Then if we want to find the eigenvalue of H. We can simply solve the following equation: (let $P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$)

$$\begin{pmatrix} 0 & V\Sigma U^T \\ U\Sigma V^T & 0 \end{pmatrix}\begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}\begin{pmatrix} \Sigma & \\ & -\Sigma \end{pmatrix}$$

where $P_1,\ P_2,\ P_3,\ P_4$ are the eigenvectors of A.
By comparing the LHS and RHS of the equation. We obtain the following simultaneous equation:

$$\begin{cases} VU^T P_3 = P_1 \\ UV^T P_1 = P_3 \\ VU^T P_4 = -P_2 \\ UV^T P_2 = -P_4 \end{cases}$$

Therefore we get

$$\begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} = \begin{pmatrix} V & -V \\ U & U \end{pmatrix}$$

And its inverse is:

$$\begin{aligned}
&(\begin{pmatrix} V & -V \\ U & U \end{pmatrix})^{-1} \\
=&\det(P)\begin{pmatrix} U^{-1} & V^{-1} \\ -U^{-1} & V^{-1} \end{pmatrix} \\
=&\frac{1}{2}\begin{pmatrix} U^T & V^T \\ -U^T & V^T \end{pmatrix} && \text{where we use the fact U,V are orthogonal vector}
\end{aligned}$$

Therefore the eigendecomposition for H is:

$$H = \begin{pmatrix} V & -V \\ U & U \end{pmatrix}\begin{pmatrix} \Sigma & \\ & -\Sigma \end{pmatrix}\frac{1}{2}\begin{pmatrix} U^T & V^T \\ -U^T & V^T \end{pmatrix} \tag{1}$$

3 Question 3

Here is a 4×4 A:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Now we know all eigenvalues of A lie in the union of the Gershgorin disks. The union of Gershgorin disks of $A \in \mathbb{C}^{n\times n}$ is of the following:

$$\begin{aligned}
S_i &= \{z \in \mathbb{C} : |z-2| \leq 1\} && j = 1, n \\
\cup S_j &= \{z \in \mathbb{C} : |z-2| \leq 2\} && j = 2, \cdots, n-1
\end{aligned}$$

Therefore, all the eigenvalues will be between $0 \leq \lambda \leq 4$. Now I would like to prove it cannot be 0.
Suppose there exists a 0 eigenvalue. Then there must exists at least one non-zero eigenvector V which satisfy the following condition:

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} V = 0$$

We can then formulate n simultaneous equation

$$\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = 0 \\ -x_2 + 2x_3 - x_4 = 0 \\ \quad\vdots \\ -x_{n-1} + x_n = 0 \end{cases}$$

After solving the equation we find out the only solution to them is $x_1 = x_2 = \cdots = x_n = 0$.
Therefore, 0 is not an eigenvector. $\implies$ All eigenvector must be strictly bigger than 0.

4 Question 4

4.1 Part 1

Firstly substitute y_k into the differential equation:

$$\begin{aligned}
&-\frac{\mathrm{d}^2 \pm \sin(k\pi x)}{\mathrm{d}x^2} \\
=&-\frac{\mathrm{d} \pm k\pi\cos(k\pi x)}{\mathrm{d}x} \\
=&- \mp k^2\pi^2 \sin(k\pi x) \\
=&\pm k^2\pi^2 \sin(k\pi x) \\
=&(k\pi)^2 y_k
\end{aligned}$$