Assignment 1

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1 Question 1

We want to prove set S is convex, then we need to show for any x,y $\in S$, we have segment $[x, y] \in S$.
We know any segment $[x, y]$ to be express as the line $tx + (1-t)y$, where $t \in [0, 1]$.
We now need to show $\|tx + (1-t)y\| \leq 1$ in order to show S is convex.
We have the following:

$$\begin{aligned}
&\|tx + (1-t)y\| \\
\leq&\|tx\| + \|(1-t)y\| && \text{triangle inequalities} \\
=&t\|x\| + (1-t)\|y\| && \text{Norm space property} \\
\leq&t + (1-t) && \text{as x,y are in S, so their norm are} \leq 1 \text{ by definition} = && 1
\end{aligned}$$

Therefore, set S is convex.

2 Question 2

Firstly, let's prove the unit sphere is not convex in this case. If we want to prove the unit ball is convex, we need to show that any point on the line segment is also belong to this unit ball. i.e. $\rho_p(tx + (1-t)y),\ t \in [0, 1]$ is also in the unit ball.
Assume we are in the unit ball, then $p < 1$ and $x_i < 1$:

$$\sum_{i=1}^{n}(x_i)^p > \sum_{i=1}^{n}(x_i) \tag{1}$$

$$(\sum_{i=1}^{n}(x_i)^p)^{1/p} > (\sum_{i=1}^{n}(x_i))^{1/p} \tag{2}$$

Suppose we are in $\mathbb{R}^2$, so it is valid to choose a point $(\frac{1}{2}, \frac{1}{2})$ as a point with the segment $[(1, 0), (0, 1)]$. Then RHS of (2) will be equal to 1. Therefore, $\rho_p(\frac{1}{2}, \frac{1}{2}) > 1$, showing such point is not in the unit ball. We find a counterexample, so the unit ball is not convex.
By using the contra-positive statement of Q1, we have if unit ball is not convex, then we cannot define a norm.

3 Question 3

We know z_a is a solution of $P(a, z)$, and we would like to perturb the result at point $P(a, z_a)$ by Δa_j. Then we have the following:

$$\begin{aligned}
0 =&P(a + \Delta a_j, z_a + \Delta z) \\
0 =&P(a, z_a) + \frac{\partial P}{\partial a_j}\Delta a_j + \frac{\partial P}{\partial z}\Delta z && \text{Taylor expansion at}(a, z_a)\text{, ignore higher order term} \\
0 =&\frac{\partial P}{\partial a_j}\Delta a_j + \frac{\partial P}{\partial z}\Delta z && z_a \text{ is the root of P} \\
\Delta z =&\frac{-\frac{\partial P}{\partial a_j}\Delta a_j}{\frac{\partial P}{\partial z}}
\end{aligned}$$

As $\dfrac{\partial P}{\partial a_j} = \dfrac{\partial(a_j z^j)}{\partial a_j} = z^j$

$$\Delta z = \frac{-z^j \Delta a_j}{\frac{\partial P}{\partial z}}$$

The condition number is the ratio of the relative forward error to the relative backward error, which is the following:

$$
\begin{aligned}
\kappa_{rel} &= \max_{z \in \text{roots}} \frac{\frac{|\Delta z|}{|z|}}{\frac{|\Delta a_j|}{|a_j|}} \\
&= \max_{z \in \text{roots}} \frac{|z^{j-1}||a_j|}{|\frac{\partial P}{\partial z}|}
\end{aligned}
$$

4 Question 4

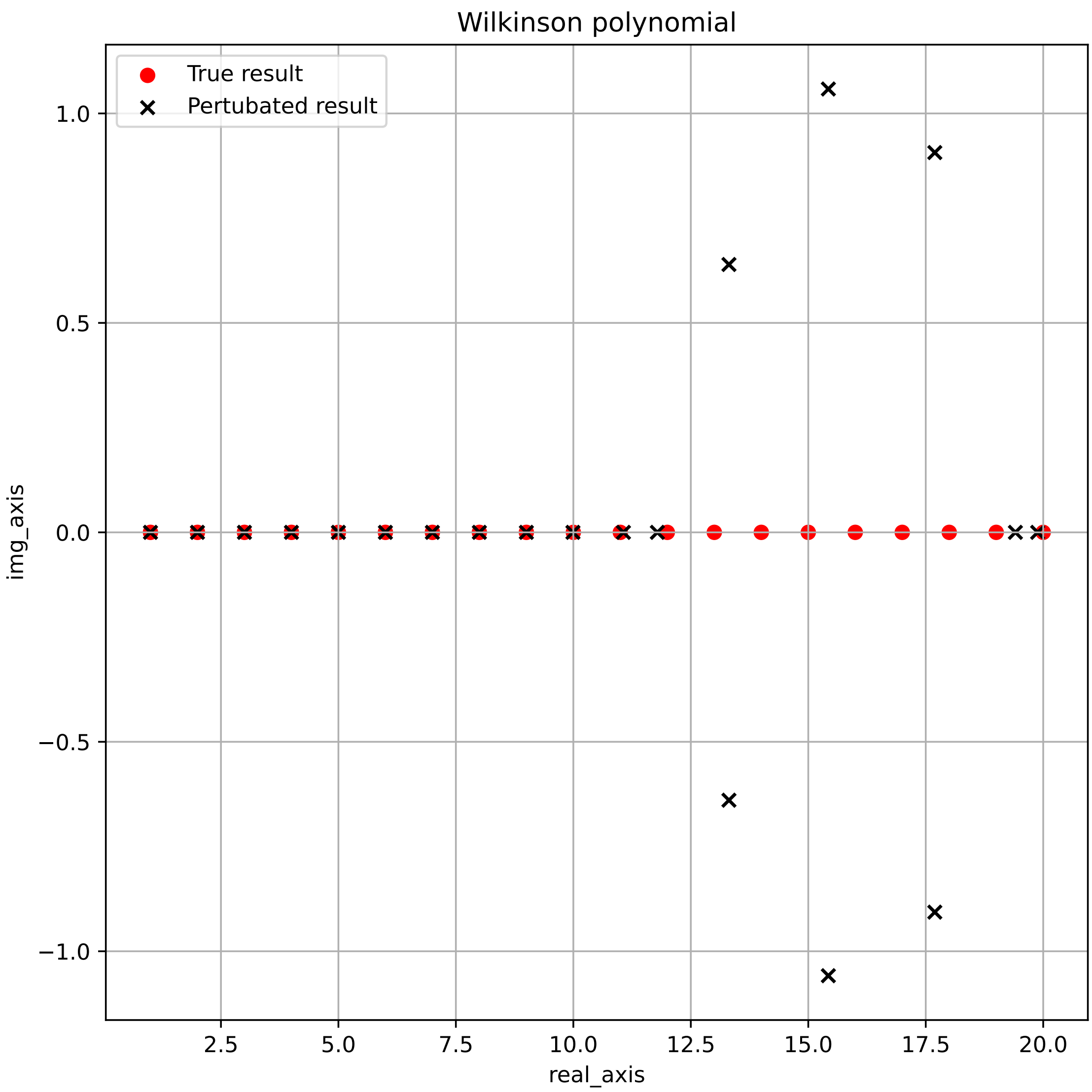

Figure 1: result

****Here is the perturbed zeros I got by using python:**

```
assignment 1.py"
This is the coeficient of the Wilkinson polynomial [1, -210, 20615, -1256850, 53327946, -1672280820, 40171771630, -756111184500, 11310276995381,
-135585182899530, 1307535010540395, -10142299865511450, 63030812099294896, -311333643161390640, 1206647803780373360, -3599979517947607200, 8037811822645051776,
-12870931245150988800, 13803759753640704000, -8752948036761600000, 2432902008176640000]
This is the zeros for the perturbated polynomial [19.88204725+0.j         19.40807686+0.j         17.69268593+0.90659877j
 17.69268593-0.90659877j 15.42739395+1.05836856j 15.42739395-1.05836856j
 13.31160995+0.6394157j  13.31160995-0.6394157j  11.78824114+0.j
 11.06627242+0.j          9.99141375+0.j          9.00042813+0.j
  8.00020165+0.j          6.99992738+0.j          6.00001308+0.j
  4.99999859+0.j          4.00000009+0.j          3.        +0.j
  2.        +0.j          1.        +0.j        ]

[Done] exited with code=0 in 7.152 seconds
```

Figure 2: perturbed solution

5 Question 5

When we expand $P(z)$, we can express it as $\sum_{j=0}^{20} a_j z^j$. Then we can use the result from question 3 to get the magnitude of the conditional number, with $j = 20$ as we are perturbing the z^{20} term. We get

$$\frac{z^{19} a_{20}}{\frac{\partial P(a,z)}{\partial z}} \tag{3}$$

$$= \frac{z^{19} a_{20}}{\sum_{j=1}^{20} j a_j z^{j-1}} \tag{4}$$

$$= \frac{a_{20}}{\sum_{j=1}^{20} j a_j z^{j-20}} \tag{5}$$

As shown in figure 3, we can find that the biggest value of (5) is when $z = 13$, we get 2952105693.37510. We can also guess the conditional number from figure 1. In order for us to see the difference in the plot, as $\epsilon = 10^{-10}$, so the conditional number is at least of magnitude 10^{10} in order for relative forward error to be 1.

```
This is the conditional number for different z: [8.22063524744612e-18, 8.18896278403838e-11, 1.63382421565376e-6, 0.00218962124810537, 0.607741718594045,
58.2485276219030, 2542.41708308882, 59693.4814429554, 840603.365570606, 7492146.60082124, 51136356.6889516, 142179509.569656, 2952105693.37510,
363499746.276071, 651809146.574444, 413931509.249139, 675871702.429876, 333010309.170962, 808521673.431571, 39680104.3407707]
 Hence the maximum is 2952105693.37510, which come from z=13.
```

Figure 3: Relative conditional number for $z \in [1, 20]$