Q4. a) Solve the equation $u(0) = u(1) = 0$ and

$$-\frac{\partial^2 y}{\partial x^2} = 1.$$

This models the vertical displacement y of an elastic string attached at 0 at $x = 0$ and $x = 1$ and the right hand side models the force acting on the string.

b) Let A_n be the $n \times n$ matrix defined by $a_{jj} = 2$ for $1 \leq j \leq n$, $a_{j,j-1} = -1$ for $2 \leq j \leq n$ and $a_{j,j+1} = -1$ for $1 \leq j \leq n-1$. Hence, A_n is a matrix with the diagonal being constant 2 and the direct upper and lower off-diagonal are constant -1.
Let $\hat{b}_n = [1/n^2, \ldots, 1/n^2]^T \in \mathbb{R}^n$. Let $\hat{x}_n$ be the vector defined by $(\hat{x}_n)_i = i/(n+1)$, $i = 1, \ldots, n$.

1. Using Python solve the system $A_n \hat{y}_n = \hat{b}_n$ for $n = 20$. Plot $\hat{y}_n$ against $\hat{x}_n$. In the same graphic plot the solution y from part a) evaluated in the points $\hat{x}_n$. Consider the same computation using the modified right hand side $\tilde{b}_n = [1/(n+1)^2, \ldots, 1/(n+1)^2]^T \in \mathbb{R}^n$ and plot the solution in the same graphic.
2. For $n \in \{20, 40, 80, 160\}$ solve $A_n \hat{y}_n = \hat{b}_n$, compute $\hat{e}_n \in \mathbb{R}^n$ defined by

$$(\hat{e}_n)_i = y((\hat{x}_n)_i) - (\hat{y}_n)_i, i = 1..., n \text{ and } E_n = n^{-\frac{1}{2}} \|\hat{e}_n\|_2.$$

   Here $y((\hat{x}_n)_i)$ denotes the function found in part (a) evaluated at the position $(\hat{x}_n)_i$. Now plot E_n against n with a log scaling of the axes. Report the result. Can you deduce a relation on the form $E_n = O(n^s)$, from the plot and specify s?
3. For $n \in \{20, 40, 80, 160\}$ compute $c_n = \kappa_2(A_n)$, the conditioning of $A_n \in \mathbb{R}^{n \times n}$. Using log-scaling of the axis plot c_n against n. Can you deduce a relation on the form $c_n = O(n^r)$, from the plot and specify r?