And its inverse is:

$$
\begin{aligned}
&(\begin{pmatrix} V & -V \\ U & U \end{pmatrix})^{-1} \\
=&\det(P)\begin{pmatrix} U^{-1} & V^{-1} \\ -U^{-1} & V^{-1} \end{pmatrix} \\
=&\frac{1}{2}\begin{pmatrix} U^{T} & V^{T} \\ -U^{T} & V^{T} \end{pmatrix} && \text{where we use the fact U,V are orthogonal vector}
\end{aligned}
$$

Therefore the eigendecomposition for H is:

$$
H = \begin{pmatrix} V & -V \\ U & U \end{pmatrix}\begin{pmatrix} \Sigma & \\ & -\Sigma \end{pmatrix}\frac{1}{2}\begin{pmatrix} U^{T} & V^{T} \\ -U^{T} & V^{T} \end{pmatrix} \tag{1}
$$

3 Question 3

Here is a 4×4 A:

$$
\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}
$$

Now we know all eigenvalues of A lie in the union of the Gershgorin disks. The union of Gershgorin disks of $A \in \mathbb{C}^{n\times n}$ is of the following:

$$
\begin{aligned}
S_i &= \{z \in \mathbb{C} : |z-2| \leq 1\} && j = 1, n \\
\cup S_j &= \{z \in \mathbb{C} : |z-2| \leq 2\} && j = 2, \cdots, n-1
\end{aligned}
$$

Therefore, all the eigenvalues will be between $0 \leq \lambda \leq 4$. Now I would like to prove it cannot be 0.
Suppose there exists a 0 eigenvalue. Then there must exists at least one non-zero eigenvector V which satisfy the following condition:

$$
\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} V = 0
$$

We can then formulate n simultaneous equation

$$
\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = 0 \\ -x_2 + 2x_3 - x_4 = 0 \\ \quad\vdots \\ -x_{n-1} + x_n = 0 \end{cases}
$$

After solving the equation we find out the only solution to them is $x_1 = x_2 = \cdots = x_n = 0$.
Therefore, 0 is not an eigenvector. $\Longrightarrow$ All eigenvector must be strictly bigger than 0.

4 Question 4

4.1 Part 1

Firstly substitute y_k into the differential equation:

$$
\begin{aligned}
&-\frac{\mathrm{d}^2 \pm \sin(k\pi x)}{\mathrm{d}x^2} \\
=&-\frac{\mathrm{d} \pm k\pi\cos(k\pi x)}{\mathrm{d}x} \\
=&- \mp k^2\pi^2 \sin(k\pi x) \\
=&\pm k^2\pi^2 \sin(k\pi x) \\
=&(k\pi)^2 y_k
\end{aligned}
$$

4.2 Part 2

Here is eigenvalue and eigenvectors of the original matrix A:

```
Here is all the eigenvalues: [0.0058684  0.02343915 0.05260915 0.09320722 0.1449951  0.20766889
 0.28086079 0.36414128 0.45702164 0.55895681 0.6693486  0.78754918
 0.9128649  1.04456036 1.18186273 1.32396624 1.470037   1.61921778
 1.77063315 1.92339453 2.07660547 2.22936685 2.38078222 2.529963
 2.67603376 2.81813727 2.95543964 3.0871351  3.21245082 3.3306514
 3.44104319 3.54297836 3.63585872 3.71913921 3.79233111 3.8550049
 3.90679278 3.94739085 3.97656085 3.9941316 ]
```

Figure 1: eigenvalues

Here is all the eigenvectors: [[-0.0169069 0.03371459 -0.05032442 -0.06663893 0.08256238 0.09800132 -0.11286515 0.12706664 -0.14052245 -0.15315362 0.16488602 0.17565081 -0.18538481 -0.1940309 -0.20153833 0.20786306 0.21296797 0.2168231 -0.21940582 -0.22070098 0.22070098 -0.21940582 0.2168231 -0.21296797 0.20786306 0.20153833 -0.1940309 -0.18538481 0.17565081 -0.16488602 0.15315362 0.14052245 -0.12706664 0.11286515 0.09800132 -0.08256238 -0.06663893 0.05032442 0.03371459 0.0169069] [-0.03371459 0.06663893 -0.09800132 -0.12706664 0.15315362 0.17565081 -0.1940309 0.20786306 -0.2168231 -0.22070098 0.21940582 0.21296797 -0.20153833 -0.18538481 -0.16488602 0.14052245 0.11286515 0.08256238 -0.05032442 -0.0169069 -0.0169069 0.05032442 -0.08256238 0.11286515 -0.14052245 -0.16488602 0.18538481 0.20153833 -0.21296797 0.21940582 -0.22070098 -0.2168231 0.20786306 -0.1940309 -0.17565081 0.15315362 0.12706664 -0.09800132 -0.06663893 -0.03371459] [-0.05032442 0.09800132 -0.14052245 -0.17565081 0.20153833 0.2168231 -0.22070098 0.21296797 -0.1940309 -0.16488602 0.12706664 0.08256238 -0.03371459 0.0169069 0.06663893 -0.11286515 -0.15315362 -0.18538481 0.20786306 0.21940582 -0.21940582 0.20786306 -0.18538481 0.15315362 -0.11286515 -0.06663893 0.0169069 -0.03371459 0.08256238 -0.12706664 0.16488602 0.1940309 -0.21296797 0.22070098 0.2168231 -0.20153833 -0.17565081 0.14052245 0.09800132 0.05032442] [-0.06663893 0.12706664 -0.17565081 -0.20786306 0.22070098 0.21296797 -0.18538481 0.14052245 -0.08256238 -0.0169069 -0.05032442 -0.11286515 0.16488602 0.20153833 0.21940582 -0.2168231 -0.1940309 -0.15315362 0.09800132 0.03371459 0.03371459 -0.09800132 0.15315362 -0.1940309 0.2168231 0.21940582 -0.20153833 -0.16488602 0.11286515 -0.05032442 -0.0169069 -0.08256238 0.14052245 -0.18538481 -0.21296797 0.22070098 0.20786306 -0.17565081 -0.12706664 -0.06663893] [-0.08256238 0.15315362 -0.20153833 -0.22070098 0.20786306 0.16488602 -0.09800132 0.0169069 0.06663893 0.14052245 -0.1940309 -0.21940582 0.21296797 0.17565081 0.11286515 -0.03371459 0.05032442 0.12706664 -0.18538481 -0.2168231 0.2168231 -0.18538481 0.12706664 -0.05032442 -0.03371459 -0.11286515 0.17565081 0.21296797 -0.21940582 0.1940309 -0.14052245 -0.06663893 -0.0169069 0.09800132 0.16488602 -0.20786306 -0.22070098 0.20153833 0.15315362 0.08256238] [-0.09800132 0.17565081 -0.2168231 -0.21296797 0.16488602 0.08256238 0.0169069 -0.11286515 0.18538481 0.21940582 -0.20786306 -0.15315362 0.06663893 -0.03371459 -0.12706664 0.1940309 0.22070098 0.20153833 -0.14052245 -0.05032442 -0.05032442 0.14052245 -0.20153833 0.22070098 -0.1940309 -0.12706664 0.03371459 -0.06663893 0.15315362 -0.20786306 0.21940582 0.18538481 -0.11286515 0.0169069 -0.08256238 0.16488602 0.21296797 -0.2168231 -0.17565081 -0.09800132] [-0.11286515 0.1940309 -0.22070098 -0.18538481 0.09800132 -0.0169069 0.12706664 -0.20153833 0.21940582 0.17565081 -0.08256238 0.03371459 -0.14052245 -0.20786306 -0.2168231 0.16488602 0.06663893 -0.05032442 0.15315362 0.21296797 -0.21296797 0.15315362 -0.05032442 -0.06663893 0.16488602 0.2168231 -0.20786306 -0.14052245 0.03371459 0.08256238 -0.17565081 -0.21940582 0.20153833 -0.12706664 -0.0169069 -0.09800132 -0.18538481 0.22070098 0.1940309 0.11286515] [-0.12706664 0.20786306 -0.21296797 -0.14052245 0.0169069 -0.11286515 0.20153833 -0.2168231 0.15315362 0.03371459 0.09800132 0.1940309 -0.21940582 -0.16488602 -0.05032442 -0.08256238 -0.18538481 -0.22070098 0.17565081 0.06663893 0.06663893 -0.17565081 0.22070098 -0.18538481 0.08256238 -0.05032442 0.16488602 0.21940582 -0.1940309 0.09800132 0.03371459 0.15315362 -0.2168231 0.20153833 0.11286515 0.0169069 0.14052245 -0.21296797 -0.20786306 -0.12706664] [-0.14052245 0.2168231 -0.1940309 -0.08256238 -0.06663893 -0.18538481 0.21940582 -0.15315362 0.0169069 -0.12706664 0.21296797 0.20153833 -0.09800132 0.05032442 0.17565081 -0.22070098 -0.16488602 -0.03371459 -0.11286515 -0.20786306 0.20786306 -0.11286515 -0.03371459 0.16488602 -0.22070098 -0.17565081 0.05032442 -0.09800132 0.20153833 -0.21296797 0.12706664 -0.0169069 0.15315362 -0.21940582 -0.18538481 0.06663893 -0.08256238 0.1940309 0.2168231 0.14052245] [-0.15315362 0.22070098 -0.16488602 -0.0169069 -0.14052245 -0.21940582 0.17565081 -0.03371459 -0.12706664 -0.2168231 0.18538481 0.05032442 0.11286515 0.21296797 0.1940309 -0.06663893 0.09800132 0.20786306 -0.20153833 -0.08256238 -0.08256238 0.20153833 -0.20786306 0.09800132 0.06663893 0.1940309 -0.21296797 -0.11286515 -0.05032442 0.18538481 -0.2168231 -0.12706664 -0.03371459 0.17565081 0.21940582 -0.14052245 0.0169069 -0.16488602 -0.22070098 -0.15315362] [-0.16488602 0.21940582 -0.12706664 0.05032442 -0.1940309 -0.20786306 0.08256238 0.09800132 -0.21296797 -0.18538481 0.03371459 -0.14052245 0.22070098 0.15315362 -0.0169069 0.17565081 0.2168231 0.11286515 0.06663893 0.20153833 -0.20153833 0.06663893 0.11286515 -0.2168231