Assignment 4

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1 Question 1

Firstly we know the approximated eigenvalue $\tilde{\lambda} = 0.9$. The eigenvector $\tilde{x}$ where $\|x\|_2 = 1$ and $\|(A - \tilde{\lambda}I)\tilde{x}\|_2 \leq 10^{-2}$. Then by the theorem about eigenvalue perturbation. $\|(A - \tilde{\lambda}I)\tilde{x}\|_2 \leq 10^{-2}$ and $\|x\|_2 = 1 \implies \|\Delta A\|_2 \leq 10^{-2}$ where ΔA is the **absolute backward error**.
We also know

$$|\delta\lambda| \leq \frac{\|x\|\|y\|}{|x^H y|}\|\Delta A\|$$

. Therefore, for this question, we know the **absolute forward error** is bounded above by

$$|\delta\lambda| \leq \frac{\|y\|}{|x^H y|}10^{-2}$$

where y is the left eigenvector.
Remark: We haven't been given enough information to determine the value of the conditional number, therefore, that is the best result we can get.
If we have a real symmetric matrix A, then the hermitian form of A is the same as A. Moreover, the left eigenvectors are identical to the right eigenvectors. Since x=y, then we have

$$|\delta\lambda| \leq 10^{-2}$$

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2 Question 2

The SVD of A is given by $A = U\Sigma V^T$. Then H can be expressed as: $\begin{pmatrix} 0 & V\Sigma U^T \\ U\Sigma V^T & 0 \end{pmatrix}$
Using the hint in the question we know if σ is a singular value of A then $\pm\sigma$ are eigenvalues of H.
If Σ is a n by n matrix containing eigenvalues of A. Then

$$\begin{pmatrix} \Sigma & \\ & -\Sigma \end{pmatrix}$$

is the eigenvalues of H. Then if we want to find the eigenvalue of H. We can simply solve the following equation: (let $P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$)

$$\begin{pmatrix} 0 & V\Sigma U^T \\ U\Sigma V^T & 0 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \begin{pmatrix} \Sigma & \\ & -\Sigma \end{pmatrix}$$

where $P_1,\ P_2,\ P_3,\ P_4$ are the eigenvectors of A.
By comparing the LHS and RHS of the equation. We obtain the following simultaneous equation:

$$\begin{cases} VU^T P_3 = P_1 \\ UV^T P_1 = P_3 \\ VU^T P_4 = -P_2 \\ UV^T P_2 = -P_4 \end{cases}$$

Therefore we get

$$\begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} = \begin{pmatrix} V & -V \\ U & U \end{pmatrix}$$

And its inverse is:

$$
\begin{aligned}
&(\begin{pmatrix} V & -V \\ U & U \end{pmatrix})^{-1} \\
=&\det(P)\begin{pmatrix} U^{-1} & V^{-1} \\ -U^{-1} & V^{-1} \end{pmatrix} \\
=&\frac{1}{2}\begin{pmatrix} U^T & V^T \\ -U^T & V^T \end{pmatrix} && \text{where we use the fact U,V are orthogonal vector}
\end{aligned}
$$

Therefore the eigendecomposition for H is:

$$
H = \begin{pmatrix} V & -V \\ U & U \end{pmatrix}\begin{pmatrix} \Sigma & \\ & -\Sigma \end{pmatrix}\frac{1}{2}\begin{pmatrix} U^T & V^T \\ -U^T & V^T \end{pmatrix} \tag{1}
$$

3 Question 3

Here is a 4×4 A:

$$
\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}
$$

Now we know all eigenvalues of A lie in the union of the Gershgorin disks. The union of Gershgorin disks of $A \in \mathbb{C}^{n \times n}$ is of the following:

$$
\begin{aligned}
S_i &= \{z \in \mathbb{C} : |z-2| \leq 1\} && j = 1, n \\
\cup S_j &= \{z \in \mathbb{C} : |z-2| \leq 2\} && j = 2, \cdots, n-1
\end{aligned}
$$

Therefore, all the eigenvalues will be between $0 \leq \lambda \leq 4$. Now I would like to prove it cannot be 0.
Suppose there exists a 0 eigenvalue. Then there must exists at least one non-zero eigenvector V which satisfy the following condition:

$$
\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} V = 0
$$

We can then formulate n simultaneous equation

$$
\begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = 0 \\ -x_2 + 2x_3 - x_4 = 0 \\ \quad\vdots \\ -x_{n-1} + x_n = 0 \end{cases}
$$

After solving the equation we find out the only solution to them is $x_1 = x_2 = \cdots = x_n = 0$.
Therefore, 0 is not an eigenvector. $\Longrightarrow$ All eigenvector must be strictly bigger than 0.

4 Question 4

4.1 Part 1

Firstly substitute y_k into the differential equation:

$$
\begin{aligned}
&-\frac{\mathrm{d}^2 \pm \sin(k\pi x)}{\mathrm{d}x^2} \\
=&-\frac{\mathrm{d} \pm k\pi\cos(k\pi x)}{\mathrm{d}x} \\
=&- \mp k^2\pi^2 \sin(k\pi x) \\
=&\pm k^2\pi^2 \sin(k\pi x) \\
=&(k\pi)^2 y_k
\end{aligned}
$$

4.2 Part 2

Here is eigenvalue and eigenvectors of the original matrix A:

```
Here is all the eigenvalues: [0.0058684  0.02343915 0.05260915 0.09320722 0.1449951  0.20766889
 0.28086079 0.36414128 0.45702164 0.55895681 0.6693486  0.78754918
 0.9128649  1.04456036 1.18186273 1.32396624 1.470037   1.61921778
 1.77063315 1.92339453 2.07660547 2.22936685 2.38078222 2.529963
 2.67603376 2.81813727 2.95543964 3.0871351  3.21245082 3.3306514
 3.44104319 3.54297836 3.63585872 3.71913921 3.79233111 3.8550049
 3.90679278 3.94739085 3.97656085 3.9941316 ]
```

Figure 1: eigenvalues

Here is all the eigenvectors: [[-0.0169069 0.03371459 -0.05032442 -0.06663893 0.08256238 0.09800132 -
0.11286515 0.12706664 -0.14052245 -0.15315362 0.16488602 0.17565081 -0.18538481 -0.1940309 -0.20153833
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-0.1940309 0.03371459 -0.2168231 -0.11286515 -0.14052245 0.20786306] [-0.20153833 -0.16488602 0.06663893 -0.21940582 -0.11286515 -0.12706664 -0.2168231 -0.05032442 0.17565081 -0.1940309 0.0169069 -0.20786306 -0.15315362 -0.08256238 0.22070098 0.09800132 0.14052245 -0.21296797 -0.03371459 -0.18538481 -0.18538481 0.03371459 0.21296797 0.14052245 -0.09800132 0.22070098 0.08256238 0.15315362 0.20786306 0.0169069 -0.1940309 0.17565081 -0.05032442 -0.2168231 0.12706664 -0.11286515 0.21940582 0.06663893 0.16488602 -0.20153833] [-0.1940309 -0.18538481 0.0169069 -0.20153833 -0.17565081 -0.03371459 -0.20786306 -0.16488602 0.05032442 -0.21296797 -0.15315362 -0.06663893 -0.2168231 0.14052245 0.08256238 0.21940582 -0.12706664 -0.09800132 -0.22070098 0.11286515 -0.11286515 -0.22070098 -0.09800132 0.12706664 0.21940582 -0.08256238 0.14052245 -0.2168231 -0.06663893 0.15315362 0.21296797 -0.05032442 0.16488602 0.20786306 -0.03371459 0.17565081 -0.20153833 -0.0169069 -0.18538481 0.1940309] [-0.18538481 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-0.21296797 0.18538481 0.03371459 0.14052245 0.22070098 -0.15315362 -0.0169069 -0.17565081 0.2168231 -0.11286515 0.06663893 -0.20153833 -0.20153833 -0.06663893 0.11286515 0.2168231 0.17565081 -0.0169069 0.15315362 -0.22070098 -0.14052245 0.03371459 0.18538481 -0.21296797 -0.09800132 0.08256238 -0.20786306 -0.1940309 0.05032442 -0.12706664 0.21940582 -0.16488602] [-0.15315362 -0.22070098 -0.16488602 0.0169069 -0.14052245 0.21940582 0.17565081 0.03371459 -0.12706664 0.2168231 0.18538481 -0.05032442 0.11286515 -0.21296797 0.1940309 0.06663893 0.09800132 -0.20786306 -0.20153833 0.08256238 -0.08256238 -0.20153833 -0.20786306 -0.09800132 0.06663893 -0.1940309 -0.21296797 0.11286515 -0.05032442 -0.18538481 -0.2168231 0.12706664 -0.03371459 -0.17565081 0.21940582 0.14052245 0.0169069 0.16488602 -0.22070098 0.15315362] [-0.14052245 -0.2168231 -0.1940309 0.08256238 -0.06663893 0.18538481 0.21940582 0.15315362 0.0169069 0.12706664 0.21296797 -0.20153833 -0.09800132 -0.05032442 0.17565081 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-0.14052245 0.06663893 -0.0169069 -0.09800132 0.16488602 0.20786306 -0.22070098 -0.20153833 0.15315362 -0.08256238] [-0.06663893 -0.12706664 -0.17565081 0.20786306 0.22070098 -0.21296797 -0.18538481 -0.14052245 -0.08256238 0.0169069 -0.05032442 0.11286515 0.16488602 -0.20153833 0.21940582 0.2168231 -0.1940309 0.15315362 0.09800132 -0.03371459 0.03371459 0.09800132 0.15315362 0.1940309 0.2168231 -0.21940582 -0.20153833 0.16488602 0.11286515 0.05032442 -0.0169069 0.08256238 0.14052245 0.18538481 -0.21296797 -0.22070098 0.20786306 0.17565081 -0.12706664 0.06663893] [-0.05032442 -0.09800132 -0.14052245 0.17565081 0.20153833 -0.2168231 -0.22070098 -0.21296797 -0.1940309 0.16488602 0.12706664 -0.08256238 -0.03371459 -0.0169069 0.06663893 0.11286515 -0.15315362 0.18538481 0.20786306 -0.21940582 -0.21940582 -0.20786306 -0.18538481 -0.15315362 -0.11286515 0.06663893 0.0169069 0.03371459 0.08256238 0.12706664 0.16488602 -0.1940309 -0.21296797 -0.22070098 0.2168231 0.20153833 -0.17565081 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0.03371459 -0.05032442 0.06663893 0.08256238 -0.09800132 -0.11286515 -0.12706664 -0.14052245 0.15315362 0.16488602 -0.17565081 -0.18538481 0.1940309 -0.20153833 -0.20786306 0.21296797 -0.2168231 -0.21940582 0.22070098 0.22070098 0.21940582 0.2168231 0.21296797 0.20786306 -0.20153833 -0.1940309 0.18538481 0.17565081 0.16488602 0.15315362 -0.14052245 -0.12706664 -0.11286515 0.09800132 0.08256238 -0.06663893 -0.05032442 0.03371459 -0.0169069]]

Here is 3rd, 6th and 10th eigenvectors of A scaled by the supremum norm:

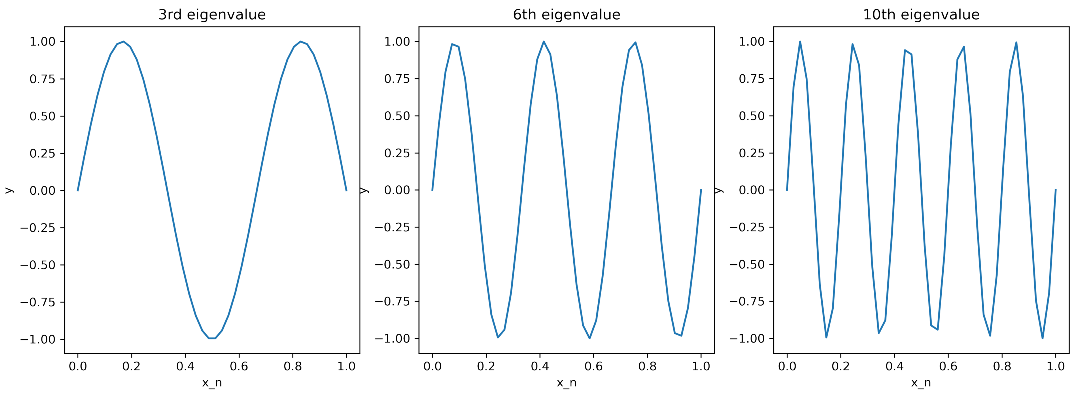

Figure 2: eigenvector of A

We can see the factor the eigenvectors of A coincide with the eigenfunction of the second order ODE we are solving in part A. However, as the eigenvalue getting bigger, the estimated eigenvector is slightly different to the eigenfuctions.

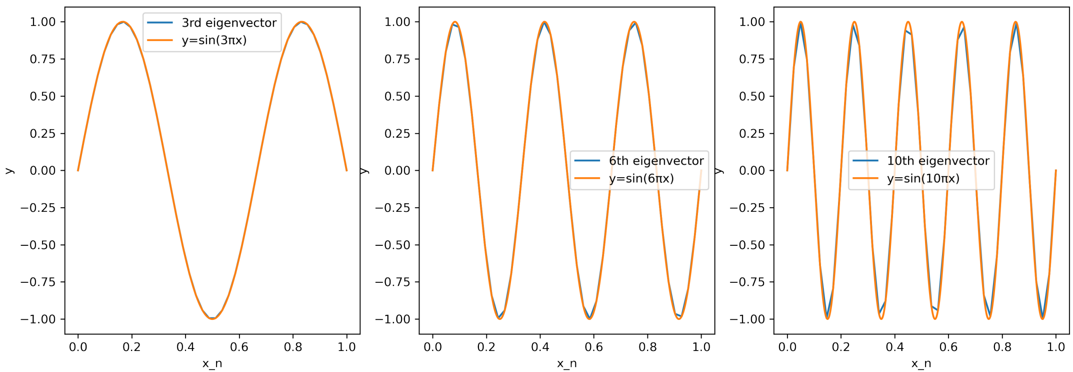

Figure 3: comparation between solution of ODE and A

We can see the error gets bigger as the eigenvalues getting bigger.

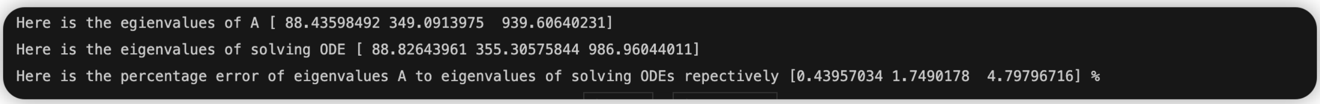

Figure 4: eigenvalue error of 3rd,6th and 10th

Here is the percentage error of all eigenvalues:

```
Here is the egienvalues of A [   9.86477642   39.40121525   88.43598492  156.68132991  243.73675939
  349.0913975   472.12698131  612.1214891   768.25337744  939.60640231
 1125.17499604 1323.87016833 1534.5258969  1755.90597016 1986.71124183
 2225.58725479 2471.13218961 2721.90509099 2976.43432381 3233.22620933
 3490.77379067 3747.56567619 4002.09490901 4252.86781039 4498.41274521
 4737.28875817 4968.09402984 5189.4741031  5400.12983167 5598.82500396
 5784.39359769 5955.74662256 6111.8785109  6251.87301869 6374.9086025
 6480.26324061 6567.31867009 6635.56401508 6684.59878475 6714.13522358]
Here is the eigenvalues of solving ODE [9.869604401089358, 39.47841760435743, 88.82643960980423, 157.91367041742973, 246.74011002723395, 355.3057584392169,
483.61061565337855, 631.6546816697189, 799.437956488238, 986.9604401089358, 1194.2221325318121, 1421.2230337568676, 1667.9631437841017, 1934.4424626135142,
2220.660990245105, 2526.6187266788756, 2852.3156719148246, 3197.751825952952, 3562.9271887932578, 3947.8417604357433, 4352.495540880407, 4776.8885301272485,
5221.02072817627, 5684.89213502747, 6168.502750680849, 6671.852575136407, 7194.941608394141, 7737.769850454057, 8300.33730131615, 8882.64396098042,
9484.689829446872, 10106.474906715503, 10747.999192786312, 11409.262687659299, 12090.265391334462, 12791.007303811808, 13511.488425091331,
14251.708755173031, 15011.668294056912, 15791.367041742973]
Here is the percentage error of eigenvalues A to eigenvalues of solving ODEs repectively [4.89176732e-02 1.95555847e-01 4.39570345e-01 7.80388742e-01
 1.21721216e+00 1.74901780e+00 2.37456209e+00 3.09238467e+00
 3.90081291e+00 4.79796716e+00 5.78176661e+00 6.84993580e+00
 8.00001171e+00 9.22935140e+00 1.05351402e+01 1.19144004e+01
 1.33640006e+01 1.48806649e+01 1.64609837e+01 1.81014234e+01
 1.97983373e+01 2.15479773e+01 2.33465041e+01 2.51899999e+01
 2.70744794e+01 2.89959017e+01 3.09501828e+01 3.29332068e+01
 3.49408387e+01 3.69689360e+01 3.90133605e+01 4.10699905e+01
 4.31347323e+01 4.52035316e+01 4.72723849e+01 4.93373502e+01
 5.13945580e+01 5.34402216e+01 5.54706469e+01 5.74822420e+01] %
```

Figure 5: eigenvalue error