Assignment 2

George Tang

February 2023

1 Question 1

1.1 $x^T y$

As x,y are vectors in $\mathbb{R}^n$, then x and y will have the form $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

Therefore, the dot product operation can be written as $\sum_{i=1}^{n} x_i y_i$.
We have n multiplication as well n-1 addition, so in total, we will have 2n-1 operations. Therefore it is $\mathcal{O}(n)$.

1.2 Ax for $A \in \mathbb{R}^{n\times n}$

Suppose the matrix have the following form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}$$

We can do the matrix multiplication columnwise: $x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$ Then we have n operations for each x to times the vectors and n of such operations. Then there is n-1 plus. Therefore, there is $n^2 + n - 1$ operations, which is $\mathcal{O}(n^2)$
Alternatively we can do the matrix multiplication entry by entry. For each entry y_i, $y_i = \sum_{j=1}^{n} a_{ij} x_j$. We can use the result from part 1, which gives $2n^2 - n$ operations, (**from now on we will use this result**) which is $\mathcal{O}(n^2)$

1.3 Matrix product AB

Two n by n matrix multiply together is equivalent to matrix A multiply each of the column of B. We know Ax have $2n^2 - n$ operations, then n of them will have $2n^3 - n^2$ operations, which is $\mathcal{O}(n^3)$

2 Question 2

From the linear system of equations

$$\begin{pmatrix} D & B \\ C & \hat{A} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

, we can obtain two simultaneous equation.

$$\begin{cases} Dx + By = b_1 \\ Cx + \hat{A}y = b_2 \end{cases} \tag{1}$$

By substitutes x from the top equation to the bottom ones, we can obtain the following:

$$CD^{-1}(b_1 - By) + \hat{A}y = b_2$$
$$(\hat{A} - CD^{-1}B)y = b_2 - CD^{-1}b_1$$

Now I would like to compute the cost of computing the LU Decomposition of $\hat{A} - CD^{-1}B$.

1. let's look at the cost of inverting D. As D is diagonal, inverting the diagonal matrix only cost $\mathcal{O}(n)$.
2. $D^{-1}B$: For normal matrix multiplication, we need to do row times column n^2 times as there are n^2 entry. However, for a diagonalise matrix D^{-1}, (we know the inverse of a diagonalise matrix is also diagonalised) each time we multiply a row in D^{-1} to a column in B, we only multiply one value as other entry in D^{-1} is always 0. Therefore, there will be n^2 operations.
3. The cost of $CD^{-1}B$ will be $2n^3 - n^2$ because it is matrix times matrix which we already computed in Q1.
4. The cost for subtraction of $\hat{A} - CD^{-1}B$ will cost n^2.
5. LU decomposition will take $\frac{2}{3}n^3 + \mathcal{O}(n^2)$, which we already evaluated during the lecture.
6. Therefore, the overall costs for decomposing $\hat{A} - CD^{-1}B$ is $\frac{8}{3}n^3 + \mathcal{O}(n^2)$.

If we apply LU directly to A, we know A is $2n \times 2n$, therefore, the cost will be: $\frac{2}{3}(2n)^3 + \mathcal{O}(2n^2) = \frac{16}{3}n^3 + \mathcal{O}(n^2)$

3 Qeustion 3

$$\begin{aligned}
Hx_0 &= (I - 2vv^T)x_0 \\
&= x_0 - 2vv^T x_0 \\
&= x_0 - 2\frac{(\alpha\|x_0\|_2 e_1 - x_0)(\alpha\|x_0\|_2 e_1 - x_0)^T x_0}{(\alpha\|x_0\|_2 e_1 - x_0)^T(\alpha\|x_0\|_2 e_1 - x_0)} \\
&= x_0 - 2(\alpha\|x_0\|_2 e_1 - x_0)\frac{\alpha\|x_0\|_2 e_1^T x_0 - \|x_0\|_2^2}{2\|x_0\|_2^2 - 2\alpha\|x_0\|_2 e_1^T x_0} && \text{Expand the bottom, notice } \alpha\alpha = 1 \text{ and } e_1^T e_1 = 1 \\
&= x_0 - (x_0 - \alpha\|x_0\|_2 e_1 \\
&= \alpha\|x_0\| e_1
\end{aligned}$$

$$\begin{aligned}
&H^T H \\
&= (I - 2vv^T)^T(I - 2vv^T) && H^T = H \\
&= (I - 2vv^T)(I - 2vv^T) \\
&= I - 4vv^T + 4vv^T vv^T \\
&= I - 4vv^T + 4vv^T && v^T v = 1 \\
&= I
\end{aligned}$$

Therefore, $H^T = H^{-1}$ and we also got: $H^T = H$ which implies $H = H^{-1}$
As we have a symmetric matrix, and we know all real symmetric matrix will contains a basis of orthogonal basis, therefore, it is orthogonal.

4 Question 4a

$$\begin{aligned}
\frac{\partial^2 y}{\partial x^2} &= -1 \\
\frac{\partial y}{\partial x} &= -x + c && \text{where c is a constant} \\
y &= -\frac{1}{2}x^2 + cx + d && \text{where d is a constant}
\end{aligned}$$

As $y(0) = y(1) = 1$, we have

$$\begin{cases} 0 = d \\ 0 = -\frac{1}{2} + c \end{cases}$$

This implies $c = \frac{1}{2}$ and $d = 0$.

$$y = -\frac{1}{2}x^2 + \frac{1}{2}x$$