

Now I would like to compute the cost of computing the LU Decomposition of $\hat{A} - CD^{-1}B$.

1. let's look at the cost of inverting D. As D is diagonal, inverting the diagonal matrix only cost $\mathcal{O}(n)$.
2. $D^{-1}B$: For normal matrix multiplication, we need to do row times column n^2 times as there are n^2 entry. However, for a diagonalise matrix D^{-1} , (we know the inverse of a diagonalise matrix is also diagonalised) each time we multiply a row in D^{-1} to a column in B, we only multiply one value as other entry in D^{-1} is always 0. Therefore, there will be n^2 operations.
3. The cost of $CD^{-1}B$ will be $2n^3 - n^2$ because it is matrix times matrix which we already computed in Q1.
4. The cost for subtraction of $\hat{A} - CD^{-1}B$ will cost n^2 .
5. LU decomposition will take $\frac{2}{3}n^3 + \mathcal{O}(n^2)$, which we already evaluated during the lecture.
6. Therefore, the overall costs for decomposing $\hat{A} - CD^{-1}B$ is $\frac{8}{3}n^3 + \mathcal{O}(n^2)$.

If we apply LU directly to A, we know A is $2n \times 2n$, therefore, the cost will be: $\frac{2}{3}(2n)^3 + \mathcal{O}(2n^2) = \frac{16}{3}n^3 + \mathcal{O}(n^2)$

3 Question 3

$$\begin{aligned}
 Hx_0 &= (I - 2vv^T)x_0 \\
 &= x_0 - 2vv^Tx_0 \\
 &= x_0 - 2\frac{(\alpha\|x_0\|_2e_1 - x_0)(\alpha\|x_0\|_2e_1 - x_0)^Tx_0}{(\alpha\|x_0\|_2e_1 - x_0)^T(\alpha\|x_0\|_2e_1 - x_0)} \\
 &= x_0 - 2(\alpha\|x_0\|_2e_1 - x_0)\frac{\alpha\|x_0\|_2e_1^Tx_0 - \|x_0\|_2^2}{2\|x_0\|_2^2 - 2\alpha\|x_0\|_2e_1^Tx_0} \quad \text{Expand the bottom, notice } \alpha\alpha = 1 \text{ and } e_1^Te_1 = 1 \\
 &= x_0 - (x_0 - \alpha\|x_0\|_2e_1) \\
 &= \alpha\|x_0\|_2e_1
 \end{aligned}$$

$$\begin{aligned}
 H^TH &= (I - 2vv^T)^T(I - 2vv^T) & H^T &= H \\
 &= (I - 2vv^T)(I - 2vv^T) \\
 &= I - 4vv^T + 4vv^Tvv^T \\
 &= I - 4vv^T + 4vv^T & v^Tv &= 1 \\
 &= I
 \end{aligned}$$

Therefore, $H^T = H^{-1}$ and we also got: $H^T = H$ which implies $H = H^{-1}$

As we have a symmetric matrix, and we know all real symmetric matrix will contains a basis of orthogonal basis, therefore, it is orthogonal.

4 Question 4a

$$\begin{aligned}
 \frac{\partial^2 y}{\partial x^2} &= -1 \\
 \frac{\partial y}{\partial x} &= -x + c & \text{where c is a constant} \\
 y &= -\frac{1}{2}x^2 + cx + d & \text{where d is a constant}
 \end{aligned}$$

As $y(0) = y(1) = 1$, we have

$$\begin{cases} 0 = d \\ 0 = -\frac{1}{2} + c \end{cases}$$

This implies $c = \frac{1}{2}$ and $d = 0$.

$$y = -\frac{1}{2}x^2 + \frac{1}{2}x$$