Assignment 2

George Tang

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1 Question 1

1.1 $x^T y$

As x,y are vectors in $\mathbb{R}^n$, then x and y will have the form $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$
Therefore, the dot product operation can be written as $\sum_{i=1}^{n} x_i y_i$.
We have n multiplication as well n-1 addition, so in total, we will have 2n-1 operations. Therefore it is $\mathcal{O}(n)$.

1.2 Ax for $A \in \mathbb{R}^{n\times n}$

Suppose the matrix have the following form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}$$

We can do the matrix multiplication columnwise: $x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$ Then we have n operations for each x to times the vectors and n of such operations. Then there is n-1 plus. Therefore, there is $n^2 + n - 1$ operations, which is $\mathcal{O}(n^2)$
Alternatively we can do the matrix multiplication entry by entry. For each entry y_i, $y_i = \sum_{j=1}^{n} a_{ij} x_j$. We can use the result from part 1, which gives $2n^2 - n$ operations, (**from now on we will use this result**) which is $\mathcal{O}(n^2)$

1.3 Matrix product AB

Two n by n matrix multiply together is equivalent to matrix A multiply each of the column of B. We know Ax have $2n^2 - n$ operations, then n of them will have $2n^3 - n^2$ operations, which is $\mathcal{O}(n^3)$

2 Question 2

From the linear system of equations

$$\begin{pmatrix} D & B \\ C & \hat{A} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

, we can obtain two simultaneous equation.

$$\begin{cases} Dx + By = b_1 \\ Cx + \hat{A}y = b_2 \end{cases} \tag{1}$$

By substitutes x from the top equation to the bottom ones, we can obtain the following:

$$CD^{-1}(b_1 - By) + \hat{A}y = b_2$$
$$(\hat{A} - CD^{-1}B)y = b_2 - CD^{-1}b_1$$