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CS251 - Computer Organization and Design

Intro to Digital Logic Design - Combinational Logic Design

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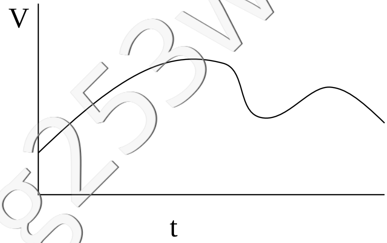
Spring 2023

A Brief Look at Electricity

- Computers work with current/voltage

$$V = IR$$

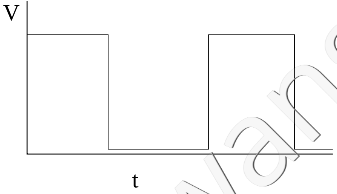
- These quantities are continuous
  Plot: voltage vs time



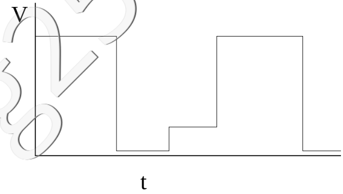
- They can be manipulated, but accuracy is difficult

Digitizing

- Discrete Signal



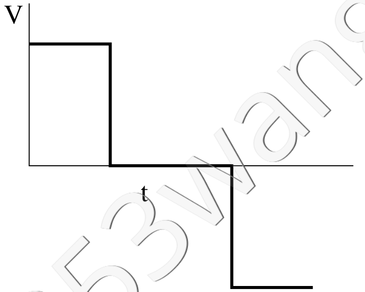
- Signal is either high (1) or low (0)
- Transformation could lead to intermediate values



but these can be “designed out”

Why Binary?

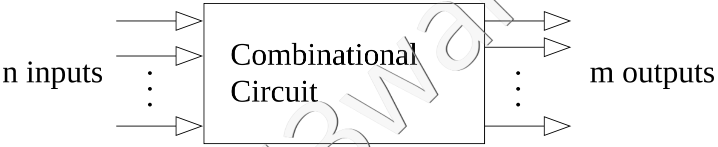
- Could have more levels...



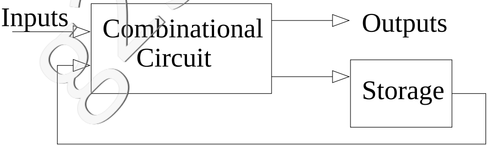
- Plus, Zero, Minus ($+1$, 0, -1 — ternary)
- Two levels are simpler and just as expressive
- Nearly all computers today use binary
- Nearly all computers have similar underlying structure

Logic Blocks

- Inputs and outputs are 1/0
  (High/low voltage, true/false)
- Combinational: without memory



- Sequential: with memory



Specifying input/output behaviour

- Truth table: specifies outputs for each possible input combination

X	Y	Z	F	G
0	0	0	0	1
0	0	1	1	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	0

- Complete description, but big and hard to understand

Compact alternative: Boolean algebra

- Variables (usually A, B, C or X, Y, Z) have values 0 or 1
- OR $(+)$ operator has result 1 iff either operand has value 1
- AND $(\cdot)$ operator has result 1 iff both operands have value 1
  $A \cdot B$ often written AB
- NOT $(\neg)$ operator has result 1 iff operand has value 0
  $\neg A$ usually written $\bar{A}$

OR		
A	B	A+B
0	0	0
0	1	1
1	0	1
1	1	1

AND		
A	B	AB
0	0	0
0	1	0
1	0	0
1	1	1

NOT	
A	¬A
0	1
1	0

Compact alternative: Example using Boolean Algebra

Try this

Write the function F and G in Boolean algebra.

X	Y	Z	F	G
0	0	0	0	1
0	0	1	1	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	0

Solution: Boolean algebra

Try this

Write the function F and G in Boolean algebra.

X	Y	Z	F	G
0	0	0	0	1
0	0	1	1	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	0

- It is clear that $G = \overline{XYZ}$
- However, it is not clear that $F = \bar{X}\bar{Y}Z + X\bar{Y}Z + XY\bar{Z} + XYZ$

Truth Table to Formula Using Minimal Terms

A	B	C	F	$\bar{A}\bar{B}C$	$A\bar{B}C$	ABC	$\bar{A}\bar{B}C + A\bar{B}C + ABC$
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	1
0	1	0	0	0	0	0	0
0	1	1	0	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	0	1	0	1
1	1	0	0	0	0	0	0
1	1	1	1	0	0	1	1

Example: Derive Formula Using Minimal Terms from Truth Table

Try this

Write the function F in Boolean algebra.

X	Y	Z	F
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

Two-Level Representations

- Any Boolean function can be represented as a sum of products (OR of ANDs) of literals
- Each term in sum corresponds to a single line in truth table with value 1
- This can be simplified by hand or by machine
- An alternative representation is product of sums that may also be useful

Example: Derive formula

Consider the following truth table:

X	Y	F
0	0	1
0	1	0
1	0	1
1	1	1

Try this

Write the function F in Boolean algebra using product of sums.

Try this

Write the function F in Boolean algebra using sum of products.

Don't Cares

- Represented as X instead of 0 or 1
- When used in output, indicates that we don't care what output is for that input
- When used in input, indicates outputs are valid for all inputs created by replacing X by 0 or 1 (useful in compressing truth tables)
- Example: The two rows below gives the same output value for F:

A	B	C	F
0	0	0	0
0	0	1	0

The don't care for **input** C simplifies this table

A	B	C	F
0	0	X	0

Becareful with Don't Cares

The following is not a valid truth table.

A	B	C	F
0	0	X	0
0	X	1	1
1	X	X	X

Here is what's wrong:

- 001 appears in two rows with different outputs. A truth table must have exactly one output for each input.
- 010 is missing in the input. Each possible input must be specified exactly once.

Example: Formula for Truth Tables with Don't Cares

Try this

Write the function for F

A	B	C	F
0	0	X	0
0	1	X	1
1	X	X	X

Compressed Truth Tables and Non-Minimal Terms

Try this

Write the function for F

A	B	C	F	$\bar{A}\bar{B}C$	AC	$\bar{A}\bar{B}C + AC$
0	0	0	0	0	0	0
0	0	1	1	1	0	1
0	1	X	0	0	0	0
1	X	0	0	0	0	0
1	X	1	1	0	1	1

Using Overlapping Non-Minimal Terms

Try this

Write the function for F

A	B	C	F	AB	AC	$AB + AC$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	0	0
1	0	0	0	0	0	0
1	0	1	1	0	1	1
1	1	0	1	1	0	1
1	1	1	1	1	1	1

Laws of Boolean Algebra

Rule	Dual Rule	
$\bar{\bar{X}} = X$		
$X + 0 = X$	$X \cdot 1 = X$	(identity)
$X + 1 = 1$	$X \cdot 0 = 0$	(zero/one)
$X + X = X$	$XX = X$	(absorption)
$X + \bar{X} = 1$	$X\bar{X} = 0$	(inverse)
$X + Y = Y + X$	$XY = YX$	(commutative)
$X + (Y + Z) = (X + Y) + Z$	$X(YZ) = (XY)Z$	(associative)
$X(Y + Z) = XY + XZ$	$X + YZ = (X + Y)(X + Z)$	(distributive)
$\overline{X + Y} = \overline{X} \cdot \overline{Y}$	$\overline{XY} = \overline{X} + \overline{Y}$	(DeMorgan)

Formula Simplification Using Laws

- We can use algebraic manipulation (based on laws) to simplify formulas.
- For example:

X	Y	Z	F	G
0	0	0	0	1
0	0	1	1	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	0

$$
\begin{aligned}
F &= \bar{X}\bar{Y}Z + X\bar{Y}Z + XY\bar{Z} + XYZ \\
&= \bar{Y}Z(\bar{X} + X) + XY(\bar{Z} + Z) \\
&= \bar{Y}Z + XY
\end{aligned}
$$

- Difficult even for humans, tricky to automate
- Seems inherently hard to get "simplest" formula
- Is simplest formula the best for implementation?

Example: Simplified Expression for Output Function

Try this

Write a simplified boolean equation for output function F:

A	B	C	F	$\bar{A}\bar{B}C$	$A\bar{B}C$	ABC	$\bar{A}\bar{B}C + A\bar{B}C + ABC$
0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	1
0	1	0	0	0	0	0	0
0	1	1	0	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	0	1	0	1
1	1	0	0	0	0	0	0
1	1	1	1	0	0	1	1

Solution: Using Don't Cares

A	B	C	F	$\bar{A}\bar{B}C$	AC	$\bar{A}\bar{B}C+AC$
0	0	0	0	0	0	0
0	0	1	1	1	0	1
0	1	X	0	0	0	0
1	X	0	0	0	0	0
1	X	1	1	0	1	1

Solution: Further Simplification

The original sum of products expression

$$F = \bar{A}\bar{B}C + A\bar{B}C + ABC$$

was simplified to

$$F = \bar{A}\bar{B}C + AC$$

But this expression looks like it can be simplified further since $\bar{A}$ and A shows up in two minterms.

A	B	C	F	$\bar{A}\bar{B}C$	AC	$\bar{A}\bar{B}C + AC$
0	0	0	0	0	0	0
$\boxed{0}$	0	1	1	1	0	1
0	1	X	0	0	0	0
1	X	0	0	0	0	0
$\boxed{1}$	X	1	1	0	1	1

Solution: Using Laws of Boolean Algebra

Indeed,

$$\begin{aligned} F &= \bar{A}\bar{B}C + A\bar{B}C + ABC \\ &= \bar{A}\bar{B}C + A\bar{B}C + A\bar{B}C + ABC \\ &= (\bar{A} + A)\bar{B}C + A(\bar{B} + \bar{B})C \\ &= \bar{B}C + AC \end{aligned}$$

Thus, if $\bar{X}$ appears in one minterm, and X appears in another, try to simplify further.

Textbook Readings

- Readings: Section A.2 from the textbook