Name: Georges Hatem

CS 590 Homework Assignment 9: Divide-and-Conquer Reinforcement Exercises

Due Date: April 3, 2022

Problem 11.6.1:

To characterize the recurrence equations using the master theorem, we must apply check which case apply as follows. The master theorem is used for recurrence equations of the form:

$$T(n) = \begin{cases} c & if\ n < d \\ aT\left(\frac{n}{b}\right) + f(n) & if\ n \geq d \end{cases}$$

Where d >= 1 is an integer constant, a >= 1, c > 0, and b > 1 are real constants, and f(n) is a function that is positive for n >= d.

The Master Theorem is defined as below:

Let f(n) and T(n) be defined as above.

1. If there is a small constant ϵ>0, such that f(n) is $O(n^{log_b(a)-\varepsilon})$, then T(n) is $\Theta(n^{log_b(a)})$.
2. If there is a constant k≥0, such that f(n) is $\Theta(n^{log_b(a)} log^k n)$, then T(n) is $\Theta(n^{log_b(a)} log^{(k+1)} n)$
3. If there are small constants ϵ>0 and δ<1, such that f(n) is $\Omega(n^{log_b(a)+\epsilon})$ and $af\left(\frac{n}{b}\right) \leq \delta f(n)$, for n >=d, then T(n) is $\Theta(f(n))$.

Case 1 characterizes the situation where f(n) is polynomially smaller than the special function, $n^{log_b(a)}$. Case 2 characterizes the situation when f(n) is asymptotically close to the special function, and Case 3 characterizes the situation when f(n) is polynomially larger than the special function.

Part A:

Let’s characterize the recurrence equation below using the master theorem:

$$T(n) = \left(2 * T\left(\frac{n}{2}\right)\right) + \log(n)$$

From the given above,

$$T(n) = \left(a * T\left(\frac{n}{b}\right)\right) + f(n) \quad for\ n \geq d$$

From the above, we can deduce the following:

$$f(n) = \log(n)\,, a = 2, and\ b = 2$$

Let’s find what the special function gets us:

$$n^{log_b(a)} = n^{log_2(2)} = n^1 = n$$

From the analysis, we can see that:

$$n > \log(n)$$

Which means that:

$$n^{log_b(a)} > f(n)$$

This characterizes case 1 above. This means that T(n) is $\Theta(n^{log_b(a)})$ based on case 1 above.

So,

$$\boldsymbol{T(n)\ is\ \Theta(n^{log_2(2)})}$$

Which means that

$$\boldsymbol{T(n)\ is\ \Theta(n)}$$

Part B:

Let’s characterize the recurrence equation below using the master theorem:

$$T(n) = 8T\left(\frac{n}{2}\right) + n^2$$

From the given above,

$$T(n) = \left(a * T\left(\frac{n}{b}\right)\right) + f(n) \quad for\ n \geq d$$

From the above, we can deduce the following:

$$f(n) = n^2, a = 8, and\ b = 2$$

Let's find what the special function gets us:

$$n^{log_b(a)} = n^{log_2(8)} = n^3$$

From the analysis, we can see that:

$$n^3 > n^2$$

Which means that:

$$n^{log_b(a)} > f(n)$$

This characterizes case 1 above. This means that T(n) is $\Theta(n^{log_b(a)})$ based on case 1 above.

So,

$$\boldsymbol{T(n)\ is\ \Theta\left(n^{\log_2(8)}\right)}$$

Which means that

$$\boldsymbol{T(n)\ is\ \Theta(n^3)}$$

<u>Part C:</u>

Let's characterize the recurrence equation below using the master theorem:

$$T(n) = 16T\left(\frac{n}{2}\right) + \left(nlog(n)\right)^4$$

From the given above,

$$T(n) = \left(a * T\left(\frac{n}{b}\right)\right) + f(n) \quad for\ n \geq d$$

From the above, we can deduce the following:

$$f(n) = \left(n log(n)\right)^4, a = 16, and\ b = 2$$

Let's find what the special function gets us:

$$n^{log_b(a)} = n^{log_2(16)} = n^4$$

From the analysis, we can see that the special function is asymptotically close to f(n), which means that we are in case 2.

Since we are in case 2 above, this means that T(n) is $\boldsymbol{\Theta(n^{log_b(a)} log^{(k+1)} n)}$ based on case 2 above.

We need to determine k, we can determine it as follows:

From case 2 definition:

$$\text{f(n) is } \Theta(n^{log_b(a)} log^k n)$$

We know from the equation given in the Exercise that f(n) is:

$$\boldsymbol{f(n) = n^4 log^4(n)}$$

And since we have a and b we know that:

$$\boldsymbol{log_b(a) = 4}$$

This means that

$$\text{f(n) is } \Theta(n^4 log^k n)$$

This means that k = 4

Now, back to where we were at:

Since we are in case 2 above, this means that T(n) is $\boldsymbol{\Theta(n^{log_b(a)}log^{(k+1)}n)}$ based on case 2 above.

So,

$$\boldsymbol{T(n)\ is\ \Theta(n^4 log^5(n))}$$

<u>Part D:</u>

Let’s characterize the recurrence equation below using the master theorem:

$$T(n) = \left(7 * T\left(\frac{n}{3}\right)\right) + \mathrm{n}$$

From the given above,

$$T(n) = \left(a * T\left(\frac{n}{b}\right)\right) + f(n) \quad for\ n \geq d$$

From the above, we can deduce the following:

$$f(n) = \mathrm{n}, a = 7, and\ b = 3$$

Let's find what the special function gets us:

$$n^{log_b(a)} = n^{log_3(7)} \cong n^{1.771}$$

From the analysis, we can see that:

$$n^{1.771} > \mathrm{n}$$

Which means that:

$$n^{log_b(a)} > f(n)$$

This characterizes case 1 above. This means that T(n) is $\boldsymbol{\Theta(n^{log_b(a)})}$ based on case 1 above.

So,

$$\boldsymbol{T(n)\ is\ \Theta(n^{log_3(7)})}$$

Part E:

Let's characterize the recurrence equation below using the master theorem:

$$T(n) = \left(9 * T\left(\frac{n}{3}\right)\right) + (\mathrm{n}^3 \log(n))$$

From the given above,

$$T(n) = \left(a * T\left(\frac{n}{b}\right)\right) + f(n) \quad for\ n \geq d$$

From the above, we can deduce the following:

$$f(n) = \mathrm{n}^3 \log{(n)}, a = 9, and\ b = 3$$

Let's find what the special function gets us:

$$n^{log_b(a)} = n^{log_3(9)} = n^2$$

From the analysis, we can see that:

$$n^2 < \mathrm{n}^3 \log{(n)}$$

Which means that:

$$n^{log_b(a)} < f(n)$$

This characterizes case 3 above. This means that T(n) is $\boldsymbol{\Theta(f(n))}$ based on case 3 above.

So,

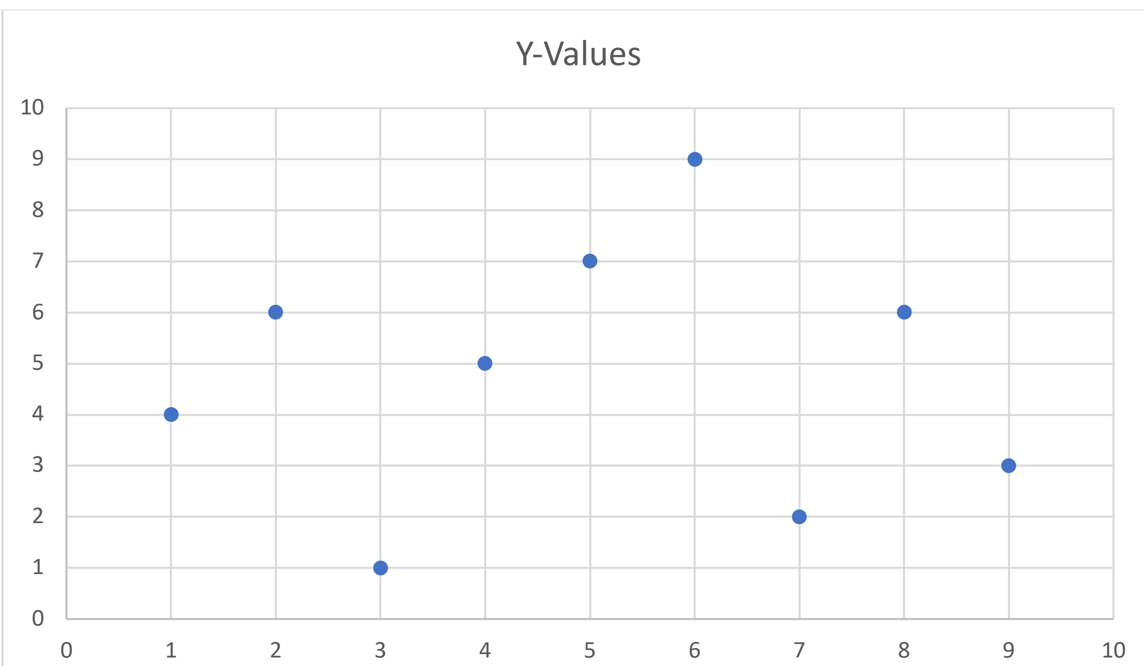
$$\boldsymbol{T(n)\ is\ \Theta(n^3 log(n))}$$

Problem 11.6.5:

Let's get the maxima set from the following set of points:

{(7,2),(3,1),(9,3),(4,5),(1,4),(6,9),(2,6),(5,7),(8,6)}

We will use the knowledge learned from Section 11.1 to determine the maxima. Let's first graph the points in the set above:



From the above graph, we can see the following:

Both x and y values of Point (1,4) are less than the x and y values of Point (2,6). This means that Point (1,4) cannot be in the maxima set and will be eliminated.

Both x and y values of Point (6,9) are higher than both x and y values of Points (2,6), (3,1), (4,5), and (5,7). This means that Points (2,6), (3,1), (4,5), and (5,7) cannot be in the maxima set and will be eliminated.

Both x and y values of Points (8,6) and (9,3) are higher than both x and y values of Point (7,2). This means that Point (7,2) cannot be in the maxima set and will be eliminated.

So, the maxima set is as follows:

{(6,9),(8,6),(9,3)}