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Partial Differential Equations and Functional Analysis

by

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In this project, we aim to analyze and solve the steady-state and time dependent convection-diffusion equation in the upper right quarter of the unit disk, using *Functional Analysis*. Our goal is first to prove the existence and uniqueness of weak solutions using the symmetric Lax-Milgram theorem, then to construct the associated solution operator and study its properties, including compactness, eigenvalues and regularity.

1^{st}: We prove Lax-Milgram Theorem in the Symmetric Case

i.e. Prove that the problem

$$u \in \mathcal{U}_{ad} = V + g \quad \text{such that} \quad J(u) = \frac{1}{2}A(u,u) - F(u) = \min_{v \in \mathcal{U}_{ad}} J(v)$$

has a unique solution, under the following assumptions.

Assumptions

- $A : V \times V \to \mathbb{R}$ is bilinear.
- A is **coercive**: $\exists\, \alpha > 0$ such that $A(v,v) \geq \alpha \|v\|^2$ for all $v \in V$.
- A is **bounded**: $\exists\, c > 0$ such that $|A(u,v)| \leq c\|u\|\|v\|$ for all $u, v \in V$.
- A is **symmetric**: $A(u,v) = A(v,u)$.
- F is a **bounded linear form**.

What we want to show

There exists a unique solution $u \in \mathcal{U}_{ad}$ such that

$$A(u,v) = F(v) \quad \forall v \in V$$

and this u is the unique minimizer of the functional

$$J(u) = \frac{1}{2}A(u,u) - F(u).$$

Existence for Symmetric Lax-Milgram:

Let $\mathcal{V}$ be a Hilbert space, $A : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ a symmetric, bicountinuous, and coercive bilinear form, $F \in \mathcal{V}$ bounded and linear, and $U_{\text{ad}} = \mathcal{V} + g$ for some fixed $g \in \mathcal{V}$. Then there exists a unique $u \in U_{\text{ad}}$ minimizing:

$$J(u) = \frac{1}{2}A(u,u) - F(u).$$

We begin by shifting to the linear space $\mathcal{V}$. Let $u = w + g$, where $w \in \mathcal{V}$. Substituting into $J(u)$:

$$J(u) = \frac{1}{2}A(w+g, w+g) - F(w+g).$$

Expanding A and separating terms:

$$\begin{aligned} J(u) &= \frac{1}{2}A(w,w) + A(w,g) + \frac{1}{2}A(g,g) - F(w) - F(g) \\ &= \frac{1}{2}A(w,w) - [F(w) - A(w,g)] + \text{constants}. \end{aligned}$$

Define the reduced functional on $\mathcal{V}$:

$$\tilde{J}(w) = \frac{1}{2}A(w,w) - \tilde{F}(w), \quad \text{where } \tilde{F}(w) = F(w) - A(g,w).$$

Apply the Riesz Representation Theorem. Since A is symmetric, bi-continuous, and coercive, it defines an inner product on $\mathcal{V}$ equivalent to the original norm. By the Riesz Representation Theorem, there exists a unique $\tilde{f} \in \mathcal{V}$ such that:

$$\tilde{F}(w) = A(\tilde{f}, w) \quad \forall w \in \mathcal{V}.$$

Solve the variational equation. The minimizer $w \in \mathcal{V}$ satisfies:

$$A(w, v) = A(\tilde{f}, v) \quad \forall v \in \mathcal{V}.$$

By coercivity, and as we proceeded in the proof of uniqueness this implies $w = \tilde{f}$.

Reconstruct $u \in U_{\text{ad}}$. Set $u = \tilde{f} + g$. By construction:

$$A(u, v) = A(\tilde{f} + g, v) = A(\tilde{f}, v) + A(g, v) = \tilde{F}(v) + A(g, v) = F(v) \quad \forall v \in \mathcal{V}. \quad \square \tag{1}$$

Uniqueness

Suppose u_1, u_2 satisfy:

$$A(u_1, v) = F(v), \quad A(u_2, v) = F(v) \quad \forall v \in V.$$

Subtract:

$$A(u_1 - u_2, v) = 0 \quad \forall v \in V.$$

Take $v = u_1 - u_2$:

$$A(u_1 - u_2, u_1 - u_2) = 0.$$

By coercivity:

$$0 = A(u_1 - u_2, u_1 - u_2) \geq \alpha \|u_1 - u_2\|^2 \geq 0 \Rightarrow \|u_1 - u_2\| = 0 \Rightarrow u_1 = u_2.$$

We used **positivity** of the norm as well as the **sandwich theorem**.

Using Symmetry, Bilinearity, and Coercivity to Show $u = v$

Although A depends on u and v, we noticed that J is in terms of only one variable, so we have to eliminate the dependence on one of these variables.
Assume $A(u, v) = A(v, u)$.
Then, taking to the other side we get:

$$A(u - v, v - u) = 0$$

By bilinearity:

$$A(u - v, v - u) = -A(u - v, u - v) = 0$$

So:

$$A(u - v, u - v) = 0$$

By coercivity, and like we proceeded before:

$$A(u - v, u - v) \geq \alpha \|u - v\|^2 \Rightarrow \|u - v\| = 0 \Rightarrow u = v.$$

This enable us to find a $v \in V$ such that $A(v, v) = F(v)$.

Setting. Let V be a real Hilbert space, $A : V \times V \to \mathbb{R}$ a bounded, coercive, *symmetric* bilinear form, and $F \in V'$ a continuous linear functional. By the Lax–Milgram theorem there is a unique $u \in V$ such that

$$A(u, v) = F(v) \qquad \forall v \in V.$$

Energy functional. Define

$$J(v) = \tfrac{1}{2}\, A(v, v) - F(v), \qquad v \in V.$$

Proof that the solution of the PDE is the minimizer:

1. For any $v \in V$ substitute $F(v) = A(u,v)$ (which holds by definition of u):

$$J(v) = \tfrac{1}{2}A(v,v) - A(u,v).$$

Add and subtract $\frac{1}{2}A(u,u)$:

$$J(v) = \tfrac{1}{2}\big[A(v,v) + A(u,u) - 2A(u,v)\big] \ - \ \tfrac{1}{2}A(u,u).$$

2. Because A is symmetric, the bracket is $A(v-u, v-u)$. The term $-\frac{1}{2}A(u,u)$ is exactly $J(u)$ since u is the solution:

$$F(u) = A(u,u) \quad \Longrightarrow \quad J(u) = \tfrac{1}{2}A(u,u) - F(u) = \tfrac{1}{2}A(u,u) - A(u,u) = -\tfrac{1}{2}A(u,u).$$

Hence

$$J(v) = J(u) + \tfrac{1}{2}\,A(v-u, v-u).$$

3. Coercivity gives $A(w,w) \geq \alpha\|w\|_V^2 \geq 0$. With $w = v - u$:

$$J(v) \ = \ J(u) + \tfrac{1}{2}A(w,w) \ \geq \ J(u),$$

and equality holds only when $w = 0$ (i.e. $v = u$). Thus u uniquely minimizes J.

□

We Want to Prove Strict Convexity of J:

Let $\mathcal{V}$ be a Hilbert space, $A : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ a symmetric, bicontinuous, and coercive bilinear form, and $F \in \mathcal{V}$. Then the functional

$$J(w) = \frac{1}{2}A(w,w) - F(w)$$

is strictly convex on ($\mathcal{V}$). □

Proof. A functional J is strictly convex if for all $w_1 \neq w_2 \in \mathcal{V}$ and $t \in (0,1)$:

$$J(tw_1 + (1-t)w_2) < tJ(w_1) + (1-t)J(w_2).$$

Expand $J(tw_1 + (1-t)w_2)$:

$$\begin{aligned} J(tw_1 + (1-t)w_2) &= \frac{1}{2}A(tw_1 + (1-t)w_2, tw_1 + (1-t)w_2) - F(tw_1 + (1-t)w_2) \\ &= \frac{1}{2}\big[t^2A(w_1,w_1) + 2t(1-t)A(w_1,w_2) + (1-t)^2A(w_2,w_2)\big] \\ &\quad - tF(w_1) - (1-t)F(w_2). \end{aligned}$$

Compare to $tJ(w_1) + (1-t)J(w_2)$:

$$tJ(w_1) + (1-t)J(w_2) = \frac{t}{2}A(w_1,w_1) - tF(w_1) + \frac{(1-t)}{2}A(w_2,w_2) - (1-t)F(w_2).$$

Subtract the two expressions:

$$\begin{aligned} &J(tw_1 + (1-t)w_2) - \big[tJ(w_1) + (1-t)J(w_2)\big] \\ &= \frac{1}{2}\big[t^2A(w_1,w_1) + 2t(1-t)A(w_1,w_2) + (1-t)^2A(w_2,w_2)\big] \\ &\quad - \frac{1}{2}\big[tA(w_1,w_1) + (1-t)A(w_2,w_2)\big] \\ &= \frac{t(1-t)}{2}\big[-A(w_1,w_1) + 2A(w_1,w_2) - A(w_2,w_2)\big] \\ &= -\frac{t(1-t)}{2}A(w_1 - w_2, w_1 - w_2). \end{aligned}$$

By coercivity of A:

$$A(w_1 - w_2, w_1 - w_2) \geq \alpha \|w_1 - w_2\|^2 > 0 \quad \text{for } w_1 \neq w_2.$$

Thus:

$$J(tw_1 + (1-t)w_2) - \big[tJ(w_1) + (1-t)J(w_2)\big] < 0,$$

proving strict convexity. □

Minimizer of J as the Solution:

Any minimizer satisfies the variational formulation.

Let $J : U_{\text{ad}} \to \mathbb{R}$ be defined by

$$J(v) := \frac{1}{2}A(v,v) - F(v),$$

where $A : V \times V \to \mathbb{R}$ is bilinear, symmetric, bicontinuous, and coercive, and $F \in V$ is a bounded linear form. Assume $U_{\text{ad}} = V + g$, where $g \in H^1(\Omega)$ is fixed. Suppose $u \in U_{\text{ad}}$ minimizes J, and let $w \in V$ be arbitrary. For all $t \in \mathbb{R}$, the perturbed function $u + tw \in U_{\text{ad}}$. Define

$$\varphi(t) := J(u + tw).$$

Then $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable, and since u is a minimizer, we must have $\varphi'(0) = 0$. Compute:

$$\varphi'(t) = A(u + tw, w) - F(w) \quad \Rightarrow \quad \varphi'(0) = A(u, w) - F(w).$$

Therefore, for all $w \in V$, we have

$$A(u, w) = F(w),$$

which shows that u satisfies the variational formulation. Hence, any minimizer of J over U_{ad} must satisfy $A(u, v) = F(v)$. However, we proved that the minimizer is unique using convexity. We also used coercivity to show that the solution is unique.

Through these analyses, we will be able to use these results extensively in later stages of the project.

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2^{nd}: We will now proceed with the steady-state wave convection equation

Starting with the Problem Statement

Consider the following boundary value problem:

$$\begin{cases} -\operatorname{div}\left[D(x,y)\nabla u + \vec{V}(x,y)u\right] = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ D(x,y)(\nu \cdot \nabla u) + u = h & \text{on } \Gamma_N, \end{cases}$$

With given:

$$\begin{cases} f \in H^1(\Omega), \\ h \in L^2(\Gamma_N), \\ \vec{V} \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}), \\ \nu \cdot \vec{V} = 0 \quad \text{on } \Gamma_N, \\ D \in C(\overline{\Omega}), \quad D(x,y) \geq D_0 > 0. \end{cases}$$

We want to find the variational form:

$$\text{Find } u \in \mathcal{H}_0 \text{ such that } A(u,v) = F(v), \quad \forall v \in \mathcal{H}_0,$$

by defining the bilinear form $A(u,v)$, the linear functional $F(v)$, and the function space $\mathcal{H}_0$. To convert the given PDE into variational form, we follow the following steps:

Start by Defining the space we are working with the Function Space $\mathcal{H}_0$:

The space consists of $H^1(\Omega)$ functions vanishing on the Dirichlet boundary Γ_D:

$$\mathcal{H}_0 = \left\{ v \in H^1(\Omega) \,|v = 0 \text{ on } \Gamma_D \right\}.$$

We Want to Change the PDE to Variational form; thus: Multiply the PDE by a Test Function $v \in \mathcal{H}_0$:

Start with the original PDE:

$$-\text{div}[D\nabla u + \vec{V}u] = f \quad \text{in } \Omega.$$

Multiply by $v \in \mathcal{H}_0$ and integrate over Ω:

$$-\int_\Omega \text{div}[D\nabla u + \vec{V}u]\, v \, d\Omega = \int_\Omega f v \, d\Omega.$$

Apply Green's Identity:

Expand the divergence term and apply Green's identity:

$$\int_\Omega [D\nabla u + \vec{V}u] \cdot \nabla v \, d\Omega - \int_{\partial\Omega} [D\nabla u + \vec{V}u] \cdot \nu \, v \, dS = \int_\Omega f v \, d\Omega.$$

Split the Boundary Integral:

The boundary $\partial\Omega$ consists of Γ_D (Dirichlet) and Γ_N (Neumann).

- On Γ_D, $v = 0$, so the integral over Γ_D vanishes.
- On Γ_N, use the boundary condition $D(\nu \cdot \nabla u) + u = h$:

$$D(\nu \cdot \nabla u) = h - u$$

  and $\nu \cdot \vec{V} = 0$, so $\vec{V}u \cdot \nu = 0$.

The boundary term becomes:

$$\int_{\Gamma_N} (h - u)v \, dS.$$

Rearrange Terms:

Substitute the boundary term back into the equation:

$$\int_\Omega D\nabla u \cdot \nabla v \, d\Omega + \int_\Omega (\vec{V} \cdot \nabla v)u \, d\Omega - \int_{\Gamma_N} (h - u)v \, dS = \int_\Omega fv \, d\Omega.$$

Move the boundary term involving u to the left-hand side: $\int_\Omega D\nabla u \cdot \nabla v \, d\Omega + \int_\Omega (\vec{V} \cdot \nabla v)u \, d\Omega + \int_{\Gamma_N} uv \, dS = \int_\Omega fv \, d\Omega + \int_{\Gamma_N} hv \, dS.$

To be able to apply Lax-Milgram, we have to have restrictions on our parameters such as $h, \vec{V}, D(x, y)$. So, we will start by restricting $\vec{V}$. We notice that only $A(u, v)$ depends on $\vec{V}$, thus, we need to prove:

- Coercivity
- Bilinearity
- Bicontinuity

Identify the Bilinear, Linear Forms, and $\mathcal{H}_0$:

Bilinear Form $A(u, v)$: $A(u, v) = \int_\Omega D\nabla u \cdot \nabla v \, d\Omega + \int_\Omega (\vec{V} \cdot \nabla v)u \, d\Omega + \int_{\Gamma_N} uv \, dS.$
Linear Form $F(v)$: $F(v) = \int_\Omega fv \, d\Omega + \int_{\Gamma_N} hv \, dS.$

$$\mathcal{H}_{\prime} = \left\{ v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_D \right\} \tag{2}$$

Coercivity and the Lax-Milgram Theorem

To establish uniqueness and existence of a solution, we apply the Lax-Milgram theorem. We observe that, in our variational formulation, only the bilinear form $A(u, v)$ depends on the vector field $\vec{V}$.

Coercivity Requirement: The bilinear form $A(u, v)$ must satisfy:

$$A(v, v) \geq \alpha \|v\|^2_{H^1(\Omega)} \quad \text{for some } \alpha > 0 \text{ and for all } v \in \mathcal{H}_0.$$

Expand $A(v, v)$: Substitute $u = v$ into the bilinear form:

$$A(v, v) = \underbrace{\int_\Omega D|\nabla v|^2 \, d\Omega}_{\text{Diffusion}} + \underbrace{\int_\Omega (\vec{V} \cdot \nabla v)v \, d\Omega}_{\text{Convection}} + \underbrace{\int_{\Gamma_N} v^2 \, dS}_{\text{Boundary}}.$$

Analyze the Convection Term: Integrate by parts using Greens:

$$\int_\Omega (\vec{V}\cdot\nabla v)v\,d\Omega = \frac{1}{2}\int_\Omega \vec{V}\cdot\nabla(v^2)\,d\Omega = -\frac{1}{2}\int_\Omega v^2\ \mathrm{div}\,\vec{V}\,d\Omega + \frac{1}{2}\int_{\partial\Omega} v^2(\vec{V}\cdot\nu)\,dS.$$

The boundary term vanishes because:

- On Γ_D, $v = 0$.
- On Γ_N, $\vec{V}\cdot\nu = 0$ (by assumption).

Thus, the convection term simplifies to:

$$\int_\Omega (\vec{V}\cdot\nabla v)v\,d\Omega = -\frac{1}{2}\int_\Omega v^2\ \mathrm{div}\,\vec{V}\,d\Omega.$$

Substitute Back into $A(v,v)$:

$$A(v,v) = \int_\Omega D|\nabla v|^2\,d\Omega - \frac{1}{2}\int_\Omega v^2\ \mathrm{div}\,\vec{V}\,d\Omega + \int_{\Gamma_N} v^2\,dS. \tag{3}$$

Coercivity with an upper bound on $div(\vec{V})$

Assume the divergence of $\vec{V}(x,y)$ is uniformly bounded

$$|div(\vec{V}(x,y))| \ \le\ M \qquad \text{for all } (x,y)\in\Omega,\ \text{for all } \vec{V}(x,y) \tag{4}$$

where

$$M \ :=\ \sup_{(x,y)\in\Omega}\left|div(\vec{V}(x,y))\right| \ \in \mathbb{R}.$$

We start with the definition of the bilinear form:

$$A(u,u) = \int_\Omega D|\nabla u|^2\,dx - \frac{1}{2}\int_\Omega \mathrm{div}(\vec{V})\,u^2\,dx + \int_{\Gamma_N} u^2\,ds.$$

Assuming that $\mathrm{div}(\vec{V}) \le M$, and using the bound $-\mathrm{div}(\vec{V}) \ge -M$, we get:

$$A(u,u)\ \ge\ D_0\|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{2}M\|u\|^2_{L^2(\Omega)}.$$

Now, recall that the H^1-norm is given by:

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)},$$

and by Poincaré's inequality, we have:

$$\|u\|^2_{L^2(\Omega)} \le C_P^2\|\nabla u\|^2_{L^2(\Omega)}.$$

Substituting into the previous inequality:

$$\begin{aligned} A(u,u) &\ge D_0\|\nabla u\|^2_{L^2} - \frac{1}{2}MC_P^2\|\nabla u\|^2_{L^2} \\ &= \left(D_0 - \frac{1}{2}MC_P^2\right)\|\nabla u\|^2_{L^2}. \end{aligned}$$

Since $\|\nabla u\|_{L^2} \le \|u\|_{H^1}$, we can write:

$$A(u,u)\ \ge\ \left(D_0 - \frac{1}{2}MC_P^2\right)\|u\|^2_{H^1(\Omega)},$$

which gives coercivity provided that:

$$\boxed{2D_0C_P^2 > M.}$$

Final coercivity statement. If the upper bound restriction holds, then

$$A(v,v) \;\geq\; \widetilde{D}\,\|\nabla v\|_{L^2}^2 + \int_{\Gamma_N} v^2\,ds \;\geq\; \tilde{D}\,\|v\|_{H^1(\Omega)}^2,$$

Hence $A(\cdot,\cdot)$ is coercive on $H^1(\Omega)$ provided the upper-bound condition is satisfied.

Bi-Continuity for $A(u,v)$

To prove the bilinear form $A(u,v)$ is bi-continuous on $\mathcal{H}_0 \times \mathcal{H}_0$, we must show there exists a constant $C > 0$ such that:

$$|A(u,v)| \leq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \quad \forall u,v \in \mathcal{H}_0.$$

We analyze each term in $A(u,v)$:

1. Diffusion Term: $\int_\Omega D\nabla u \cdot \nabla v\,d\Omega$

- **Assumptions:** $D \in C(\overline{\Omega})$, and $D \geq D_0 > 0$.
- **Boundedness:** Since Ω is compact and D is continuous:

$$\exists\, C_D > 0 \quad \text{such that} \quad |D(x,y)| \leq C_D \quad \forall (x,y) \in \Omega.$$

- **Estimation:**

$$\left|\int_\Omega D\nabla u \cdot \nabla v\,d\Omega\right| \leq C_D \int_\Omega |\nabla u \cdot \nabla v|\,d\Omega \leq C_D\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \leq C_D\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

2. Convection Term: $\int_\Omega (\vec{V} \cdot \nabla v)u\,d\Omega$

- **Assumptions:** $\vec{V} \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$.
- **Boundedness:**

$$\exists\, C_V > 0 \quad \text{such that} \quad \|\vec{V}\|_{L^\infty(\Omega)} \leq C_V.$$

- **Estimation:**

$$\left|\int_\Omega (\vec{V} \cdot \nabla v)u\,d\Omega\right| \leq C_V \int_\Omega |u||\nabla v|\,d\Omega \leq C_V\|u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \leq C_V\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

3. Boundary Term: $\int_{\Gamma_N} uv\,dS$

- **Trace Theorem:** For $u,v \in H^1(\Omega)$, the trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ is bounded:

$$\exists\, C_T > 0 \quad \text{such that} \quad \|u\|_{L^2(\Gamma_N)} \leq C_T\|u\|_{H^1(\Omega)}.$$

- **Estimation:**

$$\left|\int_{\Gamma_N} uv\,dS\right| \leq \|u\|_{L^2(\Gamma_N)}\|v\|_{L^2(\Gamma_N)} \leq C_T^2\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

Combining All Terms: Let $C = C_D + C_V + C_T^2$. Then:

$$|A(u,v)| \leq \left(C_D + C_V + C_T^2\right)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

Thus, $A(u,v)$ is continuous (bi-continuous) on $\mathcal{H}_0 \times \mathcal{H}_0$.

Bilinearity of $A(u, v)$

The bilinear form $A(u, v)$ in the variational problem is **inherently bilinear** under the given assumptions. Specifically:

$$A(u,v) = \int_{\Omega} D\nabla u \cdot \nabla v \, d\Omega + \int_{\Omega} (\vec{V} \cdot \nabla v) u \, d\Omega + \int_{\Gamma_N} uv \, dS$$

1. **Linearity in u:**
   - ∇u is linear in u.
   - $(\vec{V} \cdot \nabla v)u$ is linear in u.
   - uv is linear in u.
2. **Linearity in v:**
   - ∇v is linear in v.
   - $\vec{V} \cdot \nabla v$ is linear in v.
   - uv is linear in v.
3. **No Hidden Nonlinearities:**
   - The coefficients $D(x, y)$ and $\vec{V}(x, y)$ are *given functions* independent of u and v.
   - All involved operators—gradient, dot product, and integration—are linear.

Thus, $A(u, v)$ is **bilinear** by construction.

Conclusion on the admissible divergence of $\vec{V}$

The bilinear form $A(\cdot, \cdot)$ is coercive on $H^1(\Omega)$ as long as the divergence of the velocity field is *bounded above*:

$$\operatorname{div} \vec{V}(x,y) \;\le\; M \quad \text{for all } (x,y) \in \Omega, \qquad M \;<\; \frac{2D_0}{C_P^2}.$$

Equivalently,

$$\alpha = D_0 - \frac{M}{2}\, C_P^2 > 0.$$

First, we assumed $div(\vec{V}) \le 0$, but we found it to be more restrictive than the bound now. However, the earlier restrictive requirement $\operatorname{div} \vec{V} \le 0$ is recovered by choosing $M = 0$ or in general any constant c; the more general bound above covers that special case and all flows whose compression is sufficiently small compared with the diffusion strength D_0.

3^{rd}: Now, we will deal with the operator $\mathcal{T}$

Let us now consider a special case of the wave-convetion equation where: $\vec{V} = 0, h = 0, D(x, y) = 1$:
And we want to prove that the solution exists and is unique: Thus:

Boundedness of the Solution Operator $\mathcal{T}$

Since $\vec{V} = 0$, then $\operatorname{div} \vec{V} = 0 \leq 0$, we can apply the results from the previous part, particularly the Lax-Milgram theorem.

Variational Problem Setup: We consider the problem:

$$A(u, v) = F(v) \quad \forall v \in \mathcal{H}_0,$$

with bilinear form and linear functional given by:

$$A(u, v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega, \qquad F(v) = \int_\Omega f v \, d\Omega.$$

Linearity and Boundedness of $F(v)$:

- **Linearity:** The map $f \mapsto F(v) = \int_\Omega f v \, d\Omega$ is linear in v.
- **Boundedness:** By the Cauchy-Schwarz inequality:

$$|F(v)| = \left| \int_\Omega f v \, d\Omega \right| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}.$$

Existence of Solution Operator $\mathcal{T}$: By the Lax-Milgram theorem, for each $f \in L^2(\Omega)$, there exists a unique $u = \mathcal{T}(f) \in \mathcal{H}_0$ such that:

$$A(u, v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega \quad \forall v \in \mathcal{H}_0.$$

Bound on $\|\mathcal{T}(f)\|_{\mathcal{H}_0}$: Set $v = u$ in the variational formulation:

$$A(u, u) = \int_\Omega |\nabla u|^2 \, d\Omega = \int_\Omega f u \, d\Omega.$$

Using Cauchy-Schwarz:

$$\|\nabla u\|^2_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}.$$

From the previous part, we have already established that the bilinear form is coercive:

$$A(u, u) = \|\nabla u\|^2_{L^2(\Omega)} \geq \alpha \|u\|^2_{H^1(\Omega)}, \quad \text{for some constant } \alpha$$

Also, from the variational problem:

$$A(u, u) = \int_\Omega f u \, d\Omega \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}.$$

Combining both:

$$\alpha \|u\|^2_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} \Rightarrow \|u\|_{H^1(\Omega)} \leq \frac{1}{\alpha} \|f\|_{L^2(\Omega)}.$$

Boundedness of the Operator:

$$\|\mathcal{T}(f)\|_{\mathcal{H}_0} = \|u\|_{H^1(\Omega)} \le \frac{1}{\alpha}\|f\|_{L^2(\Omega)}$$

Taking the supremum over all nonzero $f \in L^2(\Omega)$:

$$\sup_{f\neq 0} \frac{\|\mathcal{T}(f)\|_{\mathcal{H}_0}}{\|f\|_{L^2(\Omega)}} \le \frac{1}{\alpha} \qquad \square$$

Compactness of the Solution Operator $\mathcal{T}$:

We aim to show that the operator $\mathcal{T} : L^2(\Omega) \to L^2(\Omega)$, which maps f to the weak solution u of a PDE, is compact.

Rellich-Kondrachov Compact Embedding Theorem

For a **bounded domain** $\Omega \subset \mathbb{R}^n$ with **Lipschitz boundary**, the embedding:

$$H^1(\Omega) \hookrightarrow L^2(\Omega)$$

is **compact**. That is Every **bounded sequence** $\{u_k\}$ in $H^1(\Omega)$ has a **subsequence** $\{u_{k_n}\}$ that **converges strongly** in $L^2(\Omega)$.

Boundedness of the Solution Operator $\mathcal{T}$

The solution operator $\mathcal{T} : L^2(\Omega) \to H^1(\Omega)$ is **bounded**, meaning:

$$\exists\, C > 0 \text{ such that } \forall f \in L^2(\Omega), \quad \|\mathcal{T}f\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

Proof of Compactness

1. Let $\{f_k\}$ be a bounded sequence in $L^2(\Omega)$. Then $\{u_k = \mathcal{T}f_k\}$ is bounded in $H^1(\Omega)$:

$$\|u_k\|_{H^1(\Omega)} \le C\|f_k\|_{L^2(\Omega)} \le C \cdot M.$$

2. By the Rellich-Kondrachov theorem, a subsequence $\{u_{k_n}\}$ converges in $L^2(\Omega)$:

$$u_{k_n} \to u \quad \text{in } L^2(\Omega).$$

3. Since $u_{k_n} = \mathcal{T}f_{k_n}$, this means $\mathcal{T}f_{k_n} \to u$ in $L^2(\Omega)$. Thus, $\mathcal{T}$ maps bounded sequences to relatively compact ones.

4. Therefore, $\mathcal{T}$ is a compact operator on $L^2(\Omega)$.

Proof of Self-Adjointness of $\mathcal{T}$

Definition: An operator $\mathcal{T}$ on a Hilbert space is self-adjoint if

$$\langle \mathcal{T}f, g\rangle = \langle f, \mathcal{T}g\rangle \quad \forall f, g \in \mathrm{Dom}(\mathcal{T}).$$

Proof: Let $u = \mathcal{T}f$ and $v = \mathcal{T}g$, i.e., solutions in $\mathcal{H}_0$ of:

$$\int_\Omega \nabla u \cdot \nabla\phi\, dx = \int_\Omega f\phi\, dx \quad \forall \phi \in \mathcal{H}_0,$$

$$\int_\Omega \nabla v \cdot \nabla\phi\, dx = \int_\Omega g\phi\, dx \quad \forall \phi \in \mathcal{H}_0.$$

Choosing $\phi = v$ in the first equation and $\phi = u$ in the second gives:

$$\int_\Omega fv\,dx = \int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega gu\,dx.$$

Thus,

$$\langle f, \mathcal{T}g\rangle = \int_\Omega fv\,dx = \int_\Omega gu\,dx = \langle \mathcal{T}f, g\rangle.$$

So $\mathcal{T}$ is self-adjoint. This follows also directly from the fact that $A(.,.)$ is symmetric.

Spectral Properties of $\mathcal{T}$

Since $\mathcal{T} : L^2(\Omega) \to L^2(\Omega)$ is a bounded, compact, and self-adjoint operator, then by the Fredholm Alternative and the spectral theory of compact operators on Hilbert spaces, we obtain the following:

(i) The spectrum $\sigma(\mathcal{T})$ is a countable subset of $\mathbb{R}$, given by:

$$\sigma(\mathcal{T}) = \{\mu_1, \mu_2, \dots\}, \quad \mu_n \to 0.$$

(ii) The spectrum is bounded:

$$\sup_{\lambda \in \sigma(\mathcal{T})} |\lambda| < \infty.$$

(iii) Since $\mathcal{T}$ is self-adjoint, there exists a countable orthonormal basis $\{\phi_n\} \subset L^2(\Omega)$ consisting of eigenfunctions of $\mathcal{T}$ such that:

$$\mathcal{T}\phi_n = \mu_n \phi_n, \quad \langle \phi_i, \phi_j\rangle_{L^2(\Omega)} = \delta_{ij}.$$

The set $\{\phi_n\}$ is complete in $L^2(\Omega)$, i.e., for every $v \in L^2(\Omega)$:

$$\lim_{N\to\infty} \left\| v - \sum_{n=1}^{N} \langle v, \phi_n\rangle \phi_n \right\|_{L^2(\Omega)} = 0.$$

Sturm–Liouville Eigenvalue Problem and the Fredholm Alternative

We consider the Sturm–Liouville eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \nu} + u = 0, & \text{on } \Gamma_N, \end{cases}$$

where ν is the outward unit normal vector on $\partial\Omega$.

since we have Δu, we notice $f = \lambda u$ where

$$u \in H^2 \cap C^1 \subset L^2$$

Our solution operator:

$$T(f) = T(\lambda u) = \lambda T(u) \quad \Rightarrow \quad T(\lambda u) = u \quad \Rightarrow \quad \lambda T(u) = u \quad \Rightarrow \quad T(u) = \frac{1}{\lambda} u$$

Since the solution operator is linear.
Boundedness, and self-adjointness follow from the previous section.
with the given boundary conditions is equivalent to solving the operator equation:

$$T(u) = \frac{1}{\lambda} u.$$

Hence, the eigenvalues of the Sturm–Liouville problem are precisely the reciprocals of the non-zero eigenvalues of $\mathcal{T}$.

Conclusion via the Fredholm Alternative: Since T: $L^2 \to H^2 \subset H^1 \subset\subset L^2$ and it is bounded , then it is compact. Based on the previous part, and by the Fredholm alternative, we get:

- There exists an orthonormal basis $\{u_n\} \subset \mathcal{H}_0$ of eigenfunctions of $\mathcal{T}$,
- Corresponding eigenvalues $\mu_n > 0$ such that $\mu_n \to 0$,
- Therefore, $\lambda_n = \frac{1}{\mu_n} \to \infty$,

so the original Sturm–Liouville problem admits a countably infinite set of solutions:

$$(\lambda_n, u_n)_{n \in \mathbb{N}}, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty.$$

4^{th}: Semi-Variational Form of a Time-Dependent PDE

We are given the following second-order time-dependent PDE problem:

$$\begin{cases} u_{tt}(x,y,t) - \operatorname{div}[\nabla u + \vec{V}(x,y)u] = f(x,y), & (x,y) \in \Omega,\ t > 0, \\ u(x,y,t) = 0, & (x,y) \in \Gamma_D,\ \forall t > 0, \\ \nu \cdot \nabla u + u = 0, & (x,y) \in \Gamma_N,\ \forall t > 0, \\ u(x,y,0) = \phi(x,y), & \text{with } \phi \in L^2(\Omega), \\ u_t(x,y,0) = \psi(x,y), & \text{with } \psi \in L^2(\Omega), \end{cases}$$

We now convert this problem into a semi-variational form of the type:

$$\left\langle \frac{d^2u}{dt^2}, v \right\rangle_H + B(u(t), v) = G(v), \quad \forall v \in V, \quad u(0) = \phi, \quad u_t(0) = \psi.$$

Step-by-Step Derivation:

1. **Multiply the PDE by a test function $v \in V$ and integrate over the domain:**

$$\int_\Omega u_{tt} v\, d\Omega - \int_\Omega \operatorname{div}[\nabla u + \vec{V}u] v\, d\Omega = \int_\Omega f v\, d\Omega.$$

2. **Apply Greens formula to the divergence term:**

$$-\int_\Omega \operatorname{div}[\nabla u + \vec{V}u] v\, d\Omega = \int_\Omega [\nabla u \cdot \nabla v + (\vec{V} \cdot \nabla v)u]\, d\Omega - \int_{\partial\Omega} [\nabla u + \vec{V}u] \cdot \nu\, v\, dS.$$

3. **Apply boundary conditions:**
   - On Γ_D: $v = 0$, so the boundary integral over Γ_D vanishes.
   - On Γ_N: $\nu \cdot \nabla u + u = 0$ and $\nu \cdot \vec{V} = 0$, so:

$$\int_{\Gamma_N} [\nabla u + \vec{V}u] \cdot \nu\, v\, dS = \int_{\Gamma_N} (-u) v\, dS.$$

4. **Combine all terms:**

$$\int_\Omega u_{tt} v\, d\Omega + \int_\Omega \Big(\nabla u \cdot \nabla v + (\vec{V} \cdot \nabla v)u\Big)\, d\Omega + \int_{\Gamma_N} uv\, dS = \int_\Omega f v\, d\Omega.$$

Final Semi-Variational Form:

$$\left\langle \frac{d^2u}{dt^2}, v \right\rangle_H + B(u(t), v) = G(v), \quad \forall v \in V, \quad u(0) = \phi, \quad u_t(0) = \psi,$$

where:

$$\begin{aligned} H &= L^2(\Omega), \\ V &= \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \right\}, \\ B(u,v) &= \int_\Omega \Big(\nabla u \cdot \nabla v + (\vec{V} \cdot \nabla v)u\Big)\, d\Omega + \int_{\Gamma_N} uv\, dS, \\ G(v) &= \int_\Omega f v\, d\Omega. \end{aligned}$$

Proof of Existence and Uniqueness via Lions' Theorem

Lions' Theorem for Second-Order Evolution Problems Before applying Lions' Theorem to our time-dependent PDE, we summarize the theorem and outline how it applies in our setting. **Lions' Theorem (Second-Order in Time):** Let $V \subset H$ be Hilbert spaces with continuous and dense embedding, and let $A(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a bilinear form satisfying:

(i) **Boundedness:** $|A(u,v)| \le M\|u\|_V\|v\|_V$ for all $u, v \in V$,

(ii) **weak coercivity:** $A(v,v) \ge c_0\|v\|_V^2 - c_1\|v\|_H^2$ for all $v \in V$, with $c_0 > 0$,

(iii) $f \in L^2(0,T;H)$, initial data $u(0) \in V$, $u_t(0) \in H$.

Then, the problem:

$$\begin{cases} \langle u_{tt}(t), v\rangle_H + A(u(t), v) = \langle f(t), v\rangle_H, \quad \forall v \in V,\ t \in (0,T], \\ \qquad\qquad u(0) = \varphi \in V, \quad u_t(0) = \psi \in H \end{cases}$$

admits a unique weak solution:

$$u \in C([0,T];V), \quad u_t \in C([0,T];H), \quad u_{tt} \in L^2(0,T;H).$$

Thus, we attack the PDE with Lion's theorem proceeding with the following steps:

We consider the abstract second-order problem:

$$\left\langle \frac{d^2u}{dt^2}, v\right\rangle_H + B(u(t), v) = G(v), \quad \forall v \in V, \quad u(0) = \phi, \quad u_t(0) = \psi,$$

and apply Lions' Theorem to show existence and uniqueness of a weak solution.

Functional Setting:

- Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where $\Gamma_D \cap \Gamma_N = \emptyset$ and $\text{meas}(\Gamma_D) > 0$.
- Define:
$$H := L^2(\Omega), \qquad V := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}.$$
- Inner products:
$$(u,v)_H = \int_\Omega uv\,dx, \qquad (u,v)_V = \int_\Omega \nabla u \cdot \nabla v\,dx + \int_\Omega uv\,dx.$$

Define the Bilinear and Linear Forms:

$$B(u,v) := \int_\Omega \nabla u \cdot \nabla v\,dx + \int_\Omega (\vec{V} \cdot \nabla v)u\,dx + \int_{\Gamma_N} uv\,dS,$$

$$G(v) := \int_\Omega fv\,dx.$$

Step 1: Boundedness of $B(.,.)$ We show $|B(u,v)| \le C\|u\|_V\|v\|_V$ for all $u, v \in V$.

Using Cauchy–Schwarz and Hölder's inequality:

$$\left|\int_\Omega \nabla u \cdot \nabla v\,dx\right| \le \|\nabla u\|\|\nabla v\|, \quad \left|\int_\Omega (\vec{V} \cdot \nabla v)u\,dx\right| \le \|\vec{V}\|_{L^\infty}\|u\|\|\nabla v\|,$$

$$\left|\int_{\Gamma_N} uv\,dS\right| \le C_{\text{tr}}^2\|u\|_V\|v\|_V.$$

Combining:

$$|B(u,v)| \leq C\|u\|_V \|v\|_V.$$

where $C = ||V||_{L_\infty} + C_{tr}^2 + 1$, using Cauchy-Shwartz and trace theorem.

Step 2: Weak Coercivity of $B(u,u)$

We want to verify the weak coercivity of the bilinear form:

$$B(u,u) = \|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega (\vec{V}\cdot\nabla u)u\,dx + \|u\|^2_{L^2(\Gamma_N)}$$

Using Poincaré's inequality:

$$\|u\|^2_{L^2(\Omega)} \leq C_P^2\|\nabla u\|^2_{L^2(\Omega)},$$

we write:

$$B(u,u) \geq \|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega (\vec{V}\cdot\nabla u)u\,dx + \|u\|^2_{L^2(\Gamma_N)} \geq C_P^2\|u\|^2_{H^1(\Omega)} + \int_\Omega (\vec{V}\cdot\nabla u)u\,dx$$

Now integrate the convection term by parts:

$$\int_\Omega (\vec{V}\cdot\nabla u)u\,dx = -\frac{1}{2}\int_\Omega \operatorname{div}(\vec{V})u^2\,dx + \frac{1}{2}\int_{\Gamma_N}(\vec{V}\cdot\vec{\nu})u^2\,dS$$

But since $\vec{V}\cdot\vec{\nu} = 0$ on Γ_N (from boundary conditions), the boundary term vanishes.
Assuming $\operatorname{div}(\vec{V}) \leq M$ uniformly in Ω, we get:

$$\int_\Omega (\vec{V}\cdot\nabla u)u\,dx \geq -\frac{1}{2}M\|u\|^2_{L^2(\Omega)}$$

Putting it all together:

$$B(u,u) \geq C_P^2\|u\|^2_{H^1(\Omega)} - \frac{1}{2}M\|u\|^2_{L^2(\Omega)}$$

Step 3: Bilinearity, Boundedness and, Continuous To proceed with the Lion's theorem. $B(u,v)$ has to be bilinear. Also, f has to be bounded and Continuos.

Step 4: Apply Lions' Theorem All hypotheses are satisfied:

- $V \subset H$ continuously and densely,
- $B(.,.)$ is bounded and coercive (in Gårding sense),
- G is a bounded linear functional on V,
- $\phi \in V$, $\psi \in H$.

Therefore, by Lions' Theorem, the problem admits a unique weak solution:

$$u \in C([0,T];V), \quad u_t \in C([0,T];H), \quad u_{tt} \in L^2(0,T;V'),$$

satisfying:

$$\left\langle \frac{d^2u}{dt^2}, v\right\rangle_H + B(u(t),v) = G(v), \quad \forall v \in V, \quad u(0)=\phi, \quad u_t(0)=\psi.$$

Conclusion:

$$\text{There exists a unique weak solution } u \in C([0,T];V), \quad u_t \in C([0,T];H)$$

to the second-order variational problem.

Energy estimate

Governing equation and notation

Define for any $t \geq 0$ the (total) energy

$$E(t) = \|u_t(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{L^2(\Gamma_N)}. \tag{5}$$

Case I: Pure wave equation ($\mathbf{V} \equiv 0$)

Step 1: Multiply by u_t and integrate. Set $\mathbf{V} \equiv 0$ in PDE. Multiplying by u_t and integrating over Ω gives

$$\int_\Omega u_{tt} u_t \, dx - \int_\Omega \Delta u \, u_t \, dx = \int_\Omega f \, u_t \, dx.$$

Step 2: Convert each term to a time derivative.

- $\displaystyle\int_\Omega u_{tt} u_t \, dx = \frac{1}{2} \frac{d}{dt} \|u_t\|_2^2.$
- For the Laplacian term, integrate by parts and use the boundary conditions:

  $$-\int_\Omega \Delta u \, u_t \, dx = \int_\Omega \nabla u \cdot \nabla u_t \, dx - \int_{\partial\Omega} \partial_{\boldsymbol{\nu}} u \, u_t \, dS.$$

  On Γ_D we have $u = 0 \implies u_t = 0$, so the flux disappears. On Γ_N we use $\partial_{\boldsymbol{\nu}} u = 0$ (since $\mathbf{V} \equiv 0$), obtaining

  $$-\int_\Omega \Delta u \, u_t = \frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2.$$

Collecting, we arrive at

$$\frac{dE}{dt} = 2(f, u_t)_{L^2(\Omega)}. \tag{6}$$

Step 3: Differential inequality and bound. Apply Young's inequality $2ab \leq a^2 + b^2$ with $a = \|f\|_2$, $b = \|u_t\|_2$:

$$\frac{dE}{dt} \leq \|f\|_2^2 + E(t).$$

A standard Grönwall argument yields, for all $t \in [0, T]$,

$$\boxed{E(t) \leq e^t \Big(E(0) + \int_0^t e^{-s} \|f(s)\|_2^2 \, ds \Big)} \tag{7}$$

and energy conservation $E(t) = E(0)$ when $f \equiv 0$.

Case II: Wave–advection equation ($\mathbf{V} \not\equiv 0$)

Step 1: Multiply by u_t and integrate. Starting again from (**??**),

$$\int_\Omega u_{tt} u_t \, dx - \int_\Omega \Delta u \, u_t \, dx + \int_\Omega (\mathbf{V} \cdot \nabla u) \, u_t \, dx = \int_\Omega f \, u_t \, dx.$$

Step 2: Handle the first two terms exactly as before. They give $\frac{1}{2} \frac{d}{dt} \|u_t\|_2^2 + \frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2$.

Step 3: Boundary contribution generated by the Robin term. Using $\partial_{\boldsymbol{\nu}} u = -\frac{1}{2} (\mathbf{V} \cdot \boldsymbol{\nu}) \, u$ on Γ_N, the flux integral becomes

$$-\int_{\Gamma_N} \partial_{\boldsymbol{\nu}} u \, u_t \, dS = \tfrac{1}{2} \int_{\Gamma_N} (\mathbf{V} \cdot \boldsymbol{\nu}) \, u \, u_t \, dS = \frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2(\Gamma_N)} - \frac{1}{2} \int_{\Gamma_N} (\mathbf{V} \cdot \boldsymbol{\nu}) \, u^2 \, dS.$$

Step 4: Exact energy identity. Putting everything together,

$$\frac{dE}{dt} = 2\int_\Omega f\, u_t\, dx + \int_\Omega (\mathbf{V}\cdot\nabla u)\, u_t\, dx + \frac{1}{2}\int_{\Gamma_N} (\mathbf{V}\cdot\boldsymbol{\nu})\, u^2\, dS. \tag{8}$$

Step 5: Estimating the advection terms. Let $C_N := \|\mathbf{V}\cdot\boldsymbol{\nu}\|_{L^\infty(\Gamma_N)}$ and $V_\infty := \|\mathbf{V}\|_{L^\infty(\Omega)}$.

$$\text{Volume term:} \quad \left|\int_\Omega (\mathbf{V}\cdot\nabla u)\, u_t\, dx\right| \le V_\infty \|\nabla u\|_2 \|u_t\|_2 \le \tfrac{1}{2} V_\infty\, E(t),$$

$$\text{Boundary term:} \quad \left|\tfrac{1}{2}\int_{\Gamma_N} (\mathbf{V}\cdot\boldsymbol{\nu}) u^2\, dS\right| \le \tfrac{1}{2} C_N\, \|u\|^2_{L^2(\Gamma_N)} \le \tfrac{1}{2} C_N\, E(t).$$

Step 6: Differential inequality and bound. With $2(f, u_t) \le \|f\|_2^2 + E(t)$ we obtain

$$\frac{dE}{dt} \;\le\; \Big(1 + \tfrac{1}{2} V_\infty + \tfrac{1}{2} C_N\Big)\, E(t) + \|f(t)\|_2^2.$$

Set $C_V := 1 + \frac{1}{2}V_\infty + \frac{1}{2}C_N$. Applying Grönwall on $[0, T]$ yields

$$\boxed{E(t) \le e^{C_V t}\Big(E(0) + \int_0^t e^{-C_V s} \|f(s)\|_2^2\, ds\Big)}, \qquad 0 \le t \le T. \tag{9}$$

In particular, for $f \equiv 0$ we have the explicit growth estimate $E(t) \le E(0)\, e^{C_V t}$.

5^{th}: Picard's theorem

The last stage of this project is to apply a rigorous justification of the first-order formulation of the second-order PDE using Picard's theorem to prove local (and possibly global) existence and uniqueness of solutions in a suitable Banach space.

Rigorous Justification of First–Order Formulation and Picard Theorem

We rewrite the second-order PDE problem

$$u_{tt} - \Delta u = f(x,y), \quad u = 0 \text{ on } \Gamma_D, \quad \partial_\nu u + u = 0 \text{ on } \Gamma_N,$$

as a first-order system suitable for applying Picard's theorem.

Step 1: Choice and Justification of the Space E. Define the state space:

$$E := (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega),$$

with norm:

$$\|(u,w)\|_E^2 := \|u\|_{H^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2.$$

This choice is motivated by the need for:

- Sufficient regularity to define $\Delta u \in L^2(\Omega)$.
- Ensuring the boundary conditions $u|_{\Gamma_D} = 0$ and $\partial_\nu u + u = 0$ on Γ_N hold, which are well-defined for functions in $H^2(\Omega)$.
- Providing a Hilbert structure that simplifies the verification of Lipschitz continuity.

Step 2: Proving E is a Banach Space. Since $H^2(\Omega) \cap H_0^1(\Omega)$ and $L^2(\Omega)$ are Hilbert spaces (closed subspace and standard Hilbert space respectively), the product space:

$$E = (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$$

with norm defined as above is also a Hilbert space (and thus a Banach space). Explicitly, if $\{(u_n, w_n)\}$ is a Cauchy sequence in E, then:

$$\|u_n - u_m\|_{H^2}^2 + \|w_n - w_m\|_{L^2}^2 \to 0 \quad \text{as } n, m \to \infty.$$

Since $H^2(\Omega) \cap H_0^1(\Omega)$ and $L^2(\Omega)$ are complete, there exist limits $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $w \in L^2(\Omega)$ such that:

$$u_n \to u \text{ in } H^2(\Omega), \quad w_n \to w \text{ in } L^2(\Omega),$$

thus $(u,w) \in E$. Hence E is complete and is a Banach space.

Step 3: Defining the Operator F. Introduce the new vector variable:

$$U(t) = \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix}, \quad U_0 = \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

Define the operator $F : E \to E$:

$$F(u,w) = \begin{pmatrix} w \\ \Delta u + f \end{pmatrix}.$$

This formulation exactly reflects the structure of the PDE in first-order form:

$$U_t = F(U), \quad U(0) = U_0.$$

Step 4: Verifying Lipschitz Continuity of F. Let $(u_1, w_1), (u_2, w_2) \in E$. Consider:

$$\|F(u_1, w_1) - F(u_2, w_2)\|_E^2 = \|(w_1 - w_2,\ \Delta(u_1 - u_2))\|_E^2.$$

Expand the definition of norm:

$$= \|w_1 - w_2\|_{L^2}^2 + \|\Delta(u_1 - u_2)\|_{L^2}^2.$$

Since $\Delta : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is linear and bounded, there exists a constant $C_\Delta > 0$ such that:

$$\|\Delta(u_1 - u_2)\|_{L^2} \leq C_\Delta \|u_1 - u_2\|_{H^2}.$$

Thus, we have:

$$\|F(u_1, w_1) - F(u_2, w_2)\|_E^2 \leq \|w_1 - w_2\|_{L^2}^2 + C_\Delta^2 \|u_1 - u_2\|_{H^2}^2.$$

Using the norm definition in E:

$$\|F(u_1, w_1) - F(u_2, w_2)\|_E^2 \leq (1 + C_\Delta^2)(\|u_1 - u_2\|_{H^2}^2 + \|w_1 - w_2\|_{L^2}^2),$$

which yields:

$$\|F(u_1, w_1) - F(u_2, w_2)\|_E \leq \sqrt{1 + C_\Delta^2} \|(u_1, w_1) - (u_2, w_2)\|_E.$$

Hence, F is globally Lipschitz with constant $L = \sqrt{1 + C_\Delta^2}$.

Note: we still need to prove this identity:$\|\Delta(u_1 - u_2)\|_{L^2} \leq C_\Delta \|u_1 - u_2\|_{H^2}$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For every $u_1, u_2 \in H^2(\Omega)$ one has

$$\|\Delta(u_1 - u_2)\|_{L^2(\Omega)} \ \leq \ C_\Delta \, \|u_1 - u_2\|_{H^2(\Omega)}, \qquad C_\Delta = \sqrt{d}.$$

Set $w := u_1 - u_2 \in H^2(\Omega)$. Write the homogeneous H^2–seminorm as

$$|w|_{H^2}^2 := \sum_{|\alpha|=2} \|D^\alpha w\|_{L^2}^2 = \sum_{i,j=1}^{d} \|\partial_{ij} w\|_{L^2}^2.$$

Because $\Delta w = \sum_{i=1}^{d} \partial_{ii} w$, let $a_i := \|\partial_{ii} w\|_{L^2}$ $(i = 1, \dots, d)$. The Cauchy–Schwarz inequality in $\mathbb{R}^d$ gives

$$\sum_{i=1}^{d} a_i \ = \ \mathbf{a} \cdot \mathbf{1} \ \leq \ \|\mathbf{a}\|_{L^2} \, \|\mathbf{1}\|_{L^2} \ = \ \sqrt{d} \Big(\sum_{i=1}^{d} a_i^2 \Big)^{1/2}.$$

Consequently

$$\|\Delta w\|_{L^2} \leq \sqrt{d} \left(\sum_{i=1}^{d} \|\partial_{ii} w\|_{L^2}^2 \right)^{1/2} \leq \sqrt{d} \, |w|_{H^2} \leq \sqrt{d} \, \|w\|_{H^2}.$$

Since $C_0^\infty(\Omega)$ is dense in $H^2(\Omega)$ and both sides are continuous in the H^2–norm, the estimate extends to all $w \in H^2(\Omega)$. Substituting $w = u_1 - u_2$ concludes the proof. $\square$

Step 5: Application of Picard's Theorem. Picard's theorem (Theorem 4 above) states that if X is a Banach space and $F : X \to X$ is globally Lipschitz continuous, then for any initial condition $U_0 \in X$, the abstract initial value problem:

$$U_t(t) = F(U(t)), \quad U(0) = U_0,$$

has a unique solution $U \in C^1([0, +\infty); X)$.

We have verified:

- E is Banach (Hilbert, hence Banach),
- $F : E \to E$ is globally Lipschitz with constant $L = \sqrt{1 + C_\Delta^2}$.

Thus, Picard's theorem ensures a unique solution:

$$U \in C^1([0, +\infty); E).$$

Conclusion (Back to Original Variables). In terms of the original PDE, we recover:

$$u \in C^1([0, +\infty); H^2(\Omega) \cap H_0^1(\Omega)), \quad u_{tt} \in C([0, +\infty); L^2(\Omega)),$$

solving:

$$u_{tt} - \Delta u = f, \quad u|_{\Gamma_D} = 0, \quad \partial_\nu u + u = 0, \quad u(0) = \phi, \quad u_t(0) = \psi.$$

Thus, we have rigorously justified the choice of the space E, verified the Lipschitz continuity of F, and applied Picard's theorem to establish existence and uniqueness.